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PULLBACK DYNAMICAL BEHAVIORS OF THE NON-AUTONOMOUS MICROPOLAR FLUID FLOWS WITH MINIMALLY REGULAR FORCE AND MOMENT*

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Abstract. In this paper, we investigate the pullback asymptotic behaviors of solutions for the non-autonomous micropolar fluid flows in 2D bounded domains. Firstly, when the force and the moment have a little additional regularity, we make use of the semigroup method and ϵ-regularity method to obtain the existence of a compact pullback absorbing family in $\widehat{H}$ and $\widehat{V}$, respectively. Then, applying the global well-posedness and the estimates of the solutions, we verify the flattening property (also known as the "Condition (C)") of the generated evolution process for the universe of fixed bounded sets and for another universe with a tempered condition in spaces $\widehat{H}$ and $\widehat{V}$, respectively. Further, we show the existence and regularity of the pullback attractors of the evolution process. Compared with the regularity of the force and the moment of [31], here we only need the minimal regularity of the force and the moment.

Keywords. pullback attractor; flattening property; semigroup method; ϵ-regularity method; enstrophy equality.

AMS subject classifications. 35B40; 35B41; 76D07.

1. Introduction

In this paper, we study the following non-autonomous micropolar fluid equations:

$$\begin{cases} \dfrac{\partial u}{\partial t}-(\nu+\nu_r)\Delta u-2\nu_r \mathrm{rot}\omega+(u\cdot\nabla)u+\nabla p=f, \\ \nabla\cdot u=0, \\ \dfrac{\partial \omega}{\partial t}-(c_a+c_d)\Delta\omega+4\nu_r\omega+(u\cdot\nabla)\omega-(c_0+c_d-c_a)\nabla \mathrm{div}\omega-2\nu_r \mathrm{rot}u=\tilde{f}, \end{cases} \tag{1.1}$$

where $u=(u_1,u_2,u_3)$ is the velocity, $\omega=(\omega_1,\omega_2,\omega_3)$ is the microrotation field interpreted as the angular velocity field of rotation of particles, p represents the pressure, $f=(f_1,f_2,f_3)$ and $\tilde{f}=(\tilde{f}_1,\tilde{f}_2,\tilde{f}_3)$ are external force and moments, respectively. The positive parameters ν,ν_r,c_0,c_a,c_d represent viscosity coefficients. Precisely, ν represents the usual Newtonian viscosity and ν_r is the microrotation viscosity. The system (1.1) is introduced in the pioneer work of Eringen [11] in 1966, which describes a class of non-Newtonian fluid motions with micro-rotational effects and inertial force involved. This model takes an important role in the fields of applied and computational mathematics, and we can see more details in [11, 20] and others. Note that when the gyration is neglected, the micropolar fluid flows reduce to the incompressible Navier-Stokes flows.

Due to their physical importance, mathematical complexity and wide range of applications, there are many articles on the mathematical theory of the micropolar fluid Equations (1.1). The well-posedness of solutions for the micropolar fluids has been investigated in [10, 13–15, 20, 21], and more. Moreover, lots of works are devoted to the

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long time behavior of solutions for the micropolar fluids in bounded domain. More precisely, Chen, Chen and Dong proved the existence of H^2-compact global attractors in a bounded domain in [6] and verified the existence of uniform attractors in non-smooth domains in [7]. Chen [8] showed the existence of L^2-pullback attractor for the micropolar fluid flows in a Lipschitz bounded domain with non-homogeneous boundary conditions. Łukaszewicz [21] verified the estimates of Hausdorff and fractal dimension of the L^2-global attractor. Łukaszewicz and Tarasińska [23] proved the existence of H^1-pullback attractor for non-autonomous micropolar fluid equation in a bounded domain. Zhao and Sun [32] established the well-posedness of the weak solution by using Faedo-Galerkin approximation and energy equality, and proved the existence of a pullback attractor via energy method and the Sobolev embedding theorem for the micropolar fluid flows with infinite delays. Later, Zhou, Liu and Sun [33] verified the H^2-boundedness of the pullback attractors obtained in [32].

Finally, we also note that there exist work on the long time behavior of solutions for the micropolar fluid flows on unbounded domains, we refer to [9,27,28,30] and the references therein. In this paper, we will discuss the large-time behavior of the solutions for the micropolar fluid model (1.1) with the minimal regularity of the force and the moment in a 2D smooth bounded domains $\Omega\subseteq\mathbb{R}^2$.

For the sake of simplicity, we assume that the velocity component u_3 in the x_3 direction is zero and the axes of rotation of particles are parallel to the x_3 axis. Then $u,\omega,f,\tilde f$ are of the form $u=(u_1,u_2,0)$, $\omega=(0,0,\omega_3)$, $f=(f_1,f_2,0)$, $\tilde f=(0,0,\tilde f_3)$. Hence, the Equations (1.1) can be reduced to the following two-dimensional non-autonomous dynamical system:

$$\begin{cases}\dfrac{\partial u}{\partial t}-(\nu+\nu_r)\Delta u-2\nu_r\nabla\times\omega+(u\cdot\nabla)u+\nabla p=f(t,x),\\ \dfrac{\partial \omega}{\partial t}-\alpha\Delta\omega+4\nu_r\omega-2\nu_r\nabla\times u+(u\cdot\nabla)\omega=\tilde f(t,x),\\ \nabla\cdot u=0,\quad \text{in}\,(\tau,T)\times\Omega,\end{cases}\tag{1.2}$$

where $\alpha:=c_a+c_d$, $x:=(x_1,x_2)\in\Omega\subseteq\mathbb{R}^2$, $u:=(u_1,u_2)$, $f:=(f_1,f_2)$, ω and $\tilde f$ are scalar functions,

$$\nabla\times u:=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\quad\text{and}\quad \nabla\times\omega:=(\frac{\partial\omega}{\partial x_2},-\frac{\partial\omega}{\partial x_1}).$$

To complete the formulation of the initial boundary value problem to system (1.2), we give the following initial boundary conditions:

$$u=0,\,\omega=0,\quad\text{on}\,(\tau,T)\times\partial\Omega,\tag{1.3}$$

$$w(\tau,x)=(u(\tau,x),\omega(\tau,x))=(u_0(x),\omega_0(x)),\,x\in\Omega,\tau\in\mathbb{R}.\tag{1.4}$$

Before stating our results, we first give some notations used throughout this paper. We denote by $L^p(\Omega)$ and $W^{m,p}(\Omega)$ the usual Lebesgue space and Sobolev space (see [1]) endowed with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively. For example, $\|\varphi\|_{L^p}=(\int_\Omega|\varphi|^p\mathrm{d}x)^{1/p}$ and $\|\varphi\|_{m,p}:=(\sum\limits_{|\beta|\leqslant m}\int_\Omega|D^\beta\varphi|^p\mathrm{d}x)^{1/p}$. Especially, we denote $H^m(\Omega):=W^{m,2}(\Omega)$ and $H_0^1(\Omega)$ the closure of $\{\varphi\in\mathcal{C}_0^\infty(\Omega)\}$ with respect to $H^1(\Omega)$ norm.

Then, we introduce the following function spaces:

$$\mathcal{V}:=\{\varphi\in\mathcal{C}_0^\infty(\Omega)\times\mathcal{C}_0^\infty(\Omega)|\,\varphi=(\varphi_1,\varphi_2),\nabla\cdot\varphi=0\},$$

$H :=$ closure of $\mathcal{V}$ in $L^2(\Omega)\times L^2(\Omega)$, with norm $\|\cdot\|_H$ and dual space H^*,
$V :=$ closure of $\mathcal{V}$ in $H^1(\Omega)\times H^1(\Omega)$, with norm $\|\cdot\|_V$ and dual space V^*,
$\widehat{H} := H\times L^2(\Omega)$ with norm $\|\cdot\|_{\widehat{H}}$ and dual space $\widehat{H}^*$,
$\widehat{V} := V\times H_0^1(\Omega)$ with norm $\|\cdot\|_{\widehat{V}}$ and dual space $\widehat{V}^*$,
$\mathcal{O}_\sigma(B) := \{w\in\widehat{V} : \inf_{v\in B}\|w-v\|_{\widehat{V}}<\sigma\}$.

Here

$$\|(u,v)\|_H := (\|u\|_2^2+\|v\|_2^2)^{1/2}, \qquad \|(u,v)\|_V := (\|u\|_{H^1}^2+\|v\|_{H^1}^2)^{1/2},$$
$$\|(u,v,w)\|_{\widehat{H}} := (\|(u,v)\|_H^2+\|w\|_2^2)^{1/2}, \quad \|(u,v,w)\|_{\widehat{V}} := (\|(u,v)\|_V^2+\|w\|_{H^1}^2)^{1/2}.$$

In the subsequent, we simplify the notations $\|\cdot\|_2$, $\|\cdot\|_H$ and $\|\cdot\|_{\widehat{H}}$ by the same notation $\|\cdot\|$ if there is no confusion. In addition, we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega), H$ or $\widehat{H}$, and $\langle\cdot,\cdot\rangle$ the dual pairing between V and V^* or between $\widehat{V}$ and $\widehat{V}^*$. Further, we denote:

$L^p(I;X) :=$ space of strongly measurable functions on the closed interval I, with values in a Banach space X, endowed with norm

$$\|\varphi\|_{L^p(I;X)} := \Big(\int_I \|\varphi\|_X^p \mathrm{d}t\Big)^{1/p}, \quad \text{for } 1\leqslant p<\infty,$$

$\mathcal{C}(I;X) :=$ space of continuous functions on the interval I, with values in the Banach space X, endowed with the usual norm,

$L^2_{loc}(I;X) :=$ space of locally square integrable functions on the interval I, with values in the Banach space X, endowed with the usual norm.

We also denote $\hookrightarrow\hookrightarrow$ the compact embedding between spaces and use $\mathrm{dist_M(X,Y)}$ to represent the Hausdorff semidistance between $X\subseteq M$ and $Y\subseteq M$ with

$$\mathrm{dist_M(X,Y)} = \sup_{\mathrm{x\in X}}\inf_{\mathrm{y\in Y}} \mathrm{dist_M(x,y)}.$$

Now let us write system (1.2)-(1.4) into an abstract form. For this, we further introduce three operators:

$$\begin{cases} \langle Aw,\phi\rangle := (\nu+\nu_\theta)(\nabla u,\nabla\Phi)+\alpha(\nabla\omega,\nabla\phi_3),\ \forall w=(u,\omega), \phi=(\Phi,\phi_3)\in\widehat{V}, \\ \langle B(u,w),\phi\rangle := ((u\cdot\nabla)w,\phi),\ \forall u\in V, w=(u,\omega)\in\widehat{V},\ \forall\phi\in\widehat{V}, \\ N(w) := (-2\nu_r\nabla\times\omega, -2\nu_r\nabla\times u+4\nu_r\omega),\ \forall w=(u,\omega)\in\widehat{V}, \end{cases} \tag{1.5}$$

then, the system (1.2)-(1.4) can be expressed as the following abstract form:

$$\begin{cases} \dfrac{\partial w}{\partial t}+Aw+B(u,w)+N(w)=F(t,x), \ \text{in } (\tau,+\infty)\times\Omega, \\ \nabla\cdot u=0, \ \text{in } (\tau,+\infty)\times\Omega, \\ w=(u,\omega)=0, \ \text{on } (\tau,+\infty)\times\partial\Omega, \\ w(\tau,x)=(u(\tau,x),\omega(\tau,x))=w_0(x), \ x\in\Omega, \tau\in\mathbb{R}, \end{cases} \tag{1.6}$$

where $F(t,x):=(f(t,x),\tilde{f}(t,x))$. The existence and uniqueness of the weak solutions (for the definition, one can see [21,30]) to system (1.6) has been established in [21], that is,

Lemma 1.1. *Assume $F(t,x)\in L^2_{loc}(\mathbb{R};\widehat{H})$.*

(1) *If $w_\tau\in\widehat{H}$, then system (1.6) has a unique solution w satisfying*

$$w\in L^\infty(\tau,+\infty;\widehat{H})\cap\mathcal{C}([\tau,+\infty);\widehat{H})\cap L^2_{loc}(\tau,+\infty;\widehat{V}),\, w'\in L^2_{loc}(\tau,+\infty;\widehat{V}^*).$$

And w depends continuously on the initial value w_τ with respect to the $\widehat{H}$ norm.

(2) *If $w_\tau\in\widehat{V}$, then problem (1.6) has a unique solution w satisfying*

$$w\in L^\infty(\tau,+\infty;\widehat{V})\cap\mathcal{C}([\tau,+\infty);\widehat{V})\cap L^2_{loc}(\tau,+\infty;D(A)),\, w'\in L^2_{loc}(\tau,+\infty;\widehat{H}).$$

Moreover, the solution w depends continuously on the initial value w_τ with respect to the $\widehat{V}$ norm.

Remark 1.1. We point out here that the prerequisite $F\in L^2_{loc}(\mathbb{R};\widehat{H})$ used to ensure the well-posedness of the weak solutions in space $\widehat{H}$ (see [21]) can be relaxed to $F\in L^2_{loc}(\mathbb{R};\widehat{V}^*)$, which can be verified by the Galerkin method.

At this stage, we introduce a definition and give some relevant conclusions.

Definition 1.1.

(1) *A biparametric family of maps $\{U(t,\tau)\}_{t\geqslant\tau}$ is called a process on X, if it satisfies the following properties:*

- $U(t,\tau):X\mapsto X,$ *for any* $\tau\leqslant t$;
- $U(\tau,\tau)=\text{identity}$;
- $U(t,r)U(r,\tau)=U(t,\tau),$ *for any* $\tau\leqslant r\leqslant t$.

Moreover, $\{U(t,\tau)\}_{t\geqslant\tau}$ is a continuous process on X if for any $t\geqslant\tau, U(t,\tau)$ is continuous on X.

(2) *A process $\{U(t,\tau)\}_{t\geqslant\tau}$ on X is said to be closed if for any $\tau\leqslant t$, and any sequence $\{w_n\}\subseteq X$ with $w_n\to w\in X$ and $U(t,\tau)w_n\to y\in X$, then $U(t,\tau)w=y$.*

Then, we have the following conclusions:

(i) If a process is continuous, then it must be closed.

(ii) On the basis of Lemma 1.1 and Definition 1.1, the map defined by

$$U(t,\tau):\begin{cases} w_\tau\mapsto U(t,\tau)w_\tau=w(t;\tau,w_\tau)\in\widehat{H},\ \forall\tau\leqslant t,\ w_\tau\in\widehat{H},\\ w_\tau\mapsto U(t,\tau)w_\tau=w(t;\tau,w_\tau)\in\widehat{V},\ \forall\tau\leqslant t,\ w_\tau\in\widehat{V},\end{cases}\tag{1.7}$$

generates a continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\widehat{H}$ and $\widehat{V}$, respectively.

This paper will study under the following four assumptions, respectively.

(H1) $F(t,x)\in L^2_{loc}(\mathbb{R};\widehat{V}^*)$ and $\displaystyle\int_{-\infty}^0 e^{\delta_1\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta<+\infty.$

(H2) $F(t,x)\in L^2_{loc}(\mathbb{R};\widehat{H})$ and $\displaystyle\int_{-\infty}^0 e^{\delta_1\theta}\|F(\theta)\|^2\mathrm{d}\theta<+\infty.$

(H3) $F(t,x)\in L^p_{loc}(\mathbb{R};\widehat{V}^*)$ for some $p>2$ and $\displaystyle\int_{-\infty}^0 e^{\delta_1\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta<+\infty.$

(H4) $F(t,x)\in L^p_{loc}(\mathbb{R};\widehat{H})$ for some $p>2$ and $\displaystyle\int_{-\infty}^0 e^{\delta_1\theta}\|F(\theta)\|^2\mathrm{d}\theta<+\infty.$

In order to facilitate the discussion, we denote by X the space $\widehat{H}$ or $\widehat{V}$, and by $\mathcal{P}(X)$ the family of all nonempty subsets of X. Let $\mathcal{D}$ be a nonempty class of families parameterized in time, i.e., each element of $\mathcal{D}$ is of the form $\widehat{D}=\{D(t):t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$, which will be called a universe in $\mathcal{P}(X)$. Based on these notations, we introduce the following definitions concerning the pullback attractors. One can refer to [4,16,23,24,31] for general definitions and theories. Note that $U(t,\tau)D(\tau):=U(t,\tau)[D(\tau)]$ is the image of $D(\tau)$ under $U(t,\tau)$.

Definition 1.2.

(1) *A family of sets* $\widehat{D}_0=\{D_0(t)|t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ *is called pullback* $\mathcal{D}$*-absorbing for the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *in* X *if for any* $t\in\mathbb{R}$ *and any* $\widehat{D}=\{D(t)|t\in\mathbb{R}\}\in\mathcal{D}$*, there exists a* $\tau_0(t,\widehat{D})\leqslant t$ *such that* $U(t,\tau)D(\tau)\subseteq D_0(t)$ *for all* $\tau\leqslant\tau_0(t,\widehat{D})$.

(2) *The process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *is said to be pullback* $\widehat{D}_0$*-asymptotically compact in* X *if for any* $t\in\mathbb{R}$*, any sequences* $\{\tau_n\}\subseteq(-\infty,t]$ *and* $\{x_n\}\subseteq X$ *satisfying* $\tau_n\to-\infty$ *as* $n\to\infty$ *and* $x_n\in D_0(\tau_n)$ *for all* n*, the sequence* $\{U(t,\tau_n;x_n)\}$ *is relatively compact in* X. $\{U(t,\tau)\}_{t\geqslant\tau}$ *is called pullback* $\mathcal{D}$*-asymptotically compact in* X *if it is pullback* $\widehat{D}$*-asymptotically compact for any* $\widehat{D}\in\mathcal{D}$.

(3) *A family of sets* $\widehat{\mathcal{A}}_{\mathcal{D}}=\{\mathcal{A}_{\mathcal{D}}(t)|t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ *is called a pullback* $\mathcal{D}$*-attractor of the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *on* X *if it has the following properties:*

Compactness*: for any* $t\in\mathbb{R},\mathcal{A}_{\mathcal{D}}(t)$ *is a nonempty compact subset of* X*;*

Invariance*:* $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)=\mathcal{A}_{\mathcal{D}}(t),\forall t\geqslant\tau$*;*

Pullback attracting*:* $\widehat{\mathcal{A}}_{\mathcal{D}}$ *is pullback* $\mathcal{D}$*-attracting in the following sense:*

$$\lim_{\tau\to-\infty}\mathrm{dist}_X\left(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}}(t)\right)=0,\forall\widehat{D}=\{D(s)|s\in\mathbb{R}\}\in\mathcal{D},t\in\mathbb{R};$$

Minimality*: the family of sets* $\widehat{\mathcal{A}}_{\mathcal{D}}$ *is minimal in the sense that if* $\widehat{O}=\{O(t)|t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ *is another family of closed sets satisfying*

$$\lim_{\tau\to-\infty}\mathrm{dist}_X(U(t,\tau)D(\tau),O(t))=0,\quad\forall\widehat{D}=\{D(t)|t\in\mathbb{R}\}\in\mathcal{D};$$

then $\mathcal{A}_{\mathcal{D}}(t)\subseteq O(t)$ *for* $t\in\mathbb{R}$.

From now on, we denote by $\mathcal{D}^{\widehat{H}}$ the class of all families of nonempty subset $\widehat{D}=\{D(t)|t\in\mathbb{R}\}\subseteq\mathcal{P}(\widehat{H})$ satisfying

$$\lim_{\tau\to-\infty}(e^{\delta_1\tau}\sup_{w\in D(\tau)}\|w\|^2)=0. \tag{1.8}$$

And, we use $\mathcal{D}_F^{\widehat{H}}$ to denote the class of families $\widehat{D}=\{D(t)=D|t\in\mathbb{R}\}$ with D a fixed nonempty bounded subset of $\widehat{H}$. Evidently, it holds that $\mathcal{D}_F^{\widehat{H}}\subseteq\mathcal{D}^{\widehat{H}}$ and $\mathcal{D}^{\widehat{H}}$ is inclusion-closed, i.e, if $\widehat{D}\in\mathcal{D}^{\widehat{H}}$ and $\widehat{D'}=\{D'(t)|t\in\mathbb{R}\}\subseteq\mathcal{P}(\widehat{H})$ with $D'(t)\subseteq D(t)$ for all t, then $\widehat{D'}\in\mathcal{D}^{\widehat{H}}$. In addition, we denote by $\mathcal{D}^{\widehat{H},\widehat{V}}$ the class of all families $\widehat{D}_{\widehat{V}}$ of elements of $\mathcal{P}(\widehat{V})$, here

$$\widehat{D}_{\widehat{V}}=\{D_{\widehat{V}}(t)=D(t)\cap\widehat{V}|t\in\mathbb{R}\}\text{ with }\widehat{D}=\{D(t)|t\in\mathbb{R}\}\in\mathcal{D}^{\widehat{H}}.$$

At the same time, we use $\mathcal{D}_F^{\widehat{V}}$ to denote the universe of fixed nonempty bounded subsets of $\widehat{V}$. It is easy to find that both classes $\mathcal{D}^{\widehat{H},\widehat{V}}$ and $\mathcal{D}_F^{\widehat{V}}$ are universe in $\mathcal{P}(\widehat{V})$ and that $\mathcal{D}_F^{\widehat{V}} \subseteq \mathcal{D}^{\widehat{H},\widehat{V}} \subseteq \mathcal{D}^{\widehat{H}}$. Moreover, $\mathcal{D}^{\widehat{H},\widehat{V}}$ is inclusion-closed.

The first purpose of this work is to prove the existence of a compact absorbing family in two different spaces $\widehat{H}$ and $\widehat{V}$. That is:

THEOREM 1.1. *Assume* (H3) *holds, then there exists a compact pullback absorbing family in* $\widehat{H}$*.*

THEOREM 1.2. *Under the conditions of* (H4)*, there exists a compact pullback absorbing family in* $\widehat{V}$*.*

The second objective is to show the existence and regularity of pullback $\mathcal{D}$- attractors for the generated evolution process for the universe of fixed bounded sets and for another universe with a tempered condition in spaces $\widehat{H}$ and $\widehat{V}$, with minimally regular force and moment, respectively. We have the following theorem.

THEOREM 1.3. *Assume* (H1) *hold, then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *defined by* (1.7) *possesses the minimal pullback* $\mathcal{D}_F^{\widehat{H}}$*- and* $\mathcal{D}^{\widehat{H}}$*- attractors*

$$\widehat{\mathcal{A}}_{\mathcal{D}_F^{\widehat{H}}} = \{\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)|t\in\mathbb{R}\} \quad \text{and} \quad \widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}} = \{\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)|t\in\mathbb{R}\}\in\mathcal{D}^{\widehat{H}},$$

respectively. Furthermore,

$$\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)\subseteq\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t),\ \ \forall t\in\mathbb{R}.$$

Moreover, with a little additional regularity on F, it holds that the following regularity of the pullback $\mathcal{D}$ - attractors obtained in Theorem 1.3.

THEOREM 1.4. *Under the conditions of* (H2)*, the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *defined by* (1.7) *possesses the minimal pullback* $\mathcal{D}_F^{\widehat{V}}$*- attractors*

$$\widehat{\mathcal{A}}_{\mathcal{D}_F^{\widehat{V}}} = \{\mathcal{A}_{\mathcal{D}_F^{\widehat{V}}}(t)|t\in\mathbb{R}\},$$

and the minimal pullback $\mathcal{D}^{\widehat{H},\widehat{V}}$*-attractors*

$$\hat{\mathcal{A}}_{\mathcal{D}^{\widehat{H},\widehat{V}}} = \{\mathcal{A}_{\mathcal{D}^{\widehat{H},\widehat{V}}}(t):t\in\mathbb{R}\}.$$

Moreover, the following statements hold:

(1) For any $t\in\mathbb{R}$*, we have*

$$\mathcal{A}_{\mathcal{D}_F^{\widehat{V}}}(t)\subseteq\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)\subseteq\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)=\mathcal{A}_{\mathcal{D}^{\widehat{H},\widehat{V}}}(t), \tag{1.9}$$

where $\widehat{\mathcal{A}}_{\mathcal{D}_F^{\widehat{H}}} = \{\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)|t\in\mathbb{R}\}$ *and* $\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}} = \{\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)|t\in\mathbb{R}\}$ *are the minimal pullback* $\mathcal{D}_F^{\widehat{H}}$*- and* $\mathcal{D}^{\widehat{H}}$*-attractors of* $\{U(t,\tau)\}_{t\geqslant\tau}$ *in space* $\widehat{H}$*, which are obtained in Theorem 1.3.*

(2) For any $t\in\mathbb{R}$ *and* $\widehat{D}\in\mathcal{D}^{\widehat{H}}$*, there holds*

$$\lim_{\tau\to-\infty}\operatorname{dist}_{\widehat{V}}\big(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)\big)=0. \tag{1.10}$$

(3) Suppose F satisfies

$$\sup_{s\leqslant 0}\left(\mathrm{e}^{-\delta_1 s}\int_{-\infty}^{s}\mathrm{e}^{\delta_1\theta}\|F(\theta)\|^2\mathrm{d}\theta\right)<+\infty. \tag{1.11}$$

Then, for any $t\in\mathbb{R}$ and fixed bounded subset $\mathcal{B}$ of $\widehat{H}$, we have

$$\mathcal{A}_{\mathcal{D}_F^{\widehat{V}}}(t)=\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)=\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)=\mathcal{A}_{\mathcal{D}^{\widehat{H},\widehat{V}}}(t) \tag{1.12}$$

and

$$\lim_{\tau\to-\infty}\mathrm{dist}_{\widehat{V}}\left(U(t,\tau)\mathcal{B},\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)\right)=0. \tag{1.13}$$

Remark 1.2. The existence of the pullback attractors in spaces $\widehat{H}$ and $\widehat{V}$ has been proved by showing the existence of the pullback absorbing set and the asymptotic compactness of the generated evolution process with the force $f\in L^2_{loc}(\mathbb{R};\widehat{H})$ and the moment $\tilde{f}\in L^2_{loc}(\mathbb{R};\widehat{H})$ in [31]. However, here we only need the minimal regularity of the force and the moment. Moreover, we verify the flattening property of the generated evolution process, to show the existence and regularity of the pullback attractors of the evolution process. This argument is essentially different from [31]. In fact, the flattening property (a Fourier splitting technique) makes the analysis significantly simpler.

The idea of the proof is outlined as follows. Firstly, borrowed the ideas and arguments in [17], the existence of compact pullback absorbing family can be proved by the semigroup approach raised by Fujita and Kato [12] and the ϵ-regularity theory developed by Arrita and Carvalho [2]. Here, we emphasize that, compared with the Navier-Stokes equations ($w=0,\nu_r=0$), the micropolar fluid flow consists of the angular velocity field ω of the micropolar particles, which leads to a different nonlinear term $B(u,w)$ and an additional term $N(u)$ in the abstract equation. Therefore, we have to obtain more delicate estimates and analysis for the solutions. Next, we verify the flattening property (also known as the "Condition (C)") of the generated evolution process for the universe of fixed bounded sets and for another universe with a tempered condition in spaces $\widehat{H}$ and $\widehat{V}$, respectively. Further, we show the existence and regularity of the pullback attractors of the evolution process. The main point is to establish several key estimates, which will play an important role in verifying the flattening property of the process. The method has been used in [25] as their "Condition (C)", and in [18] as "the flattening property". Due to the minimal regularity of the force and the moment, we can not use the arguments in [31]. The lower regularity of $F\in L^2_{loc}(\mathbb{R};\widehat{V}^*)$ than [31] with $F\in L^2_{loc}(\mathbb{R};\widehat{H})$ results in the loss of the uniform estimate in $[t-2,t]$ of $\|w(\cdot)\|_{\widehat{V}}$, which forces us to prove the Lemmas 4.1-4.3 to obtain the flattening property of the process $\{U(t,\tau)\}_{t\geqslant\tau}$.

The rest of the paper is organized as follows. In Section 2, we make some necessary preliminaries. That is, we introduce some definitions and give some useful estimates with respect to the operators and recall some known results concerning the micropolar fluid model. Section 3 is devote to show Theorem 1.1 and 1.2. That is, basing on the global well-posedness and the estimates of the solutions, we apply the semigroup method and ϵ- regularity, combining with some Sobolev inequalities, to testify the existence of the compact pullback absorbing family (not only asymptotic compactness) in spaces $\widehat{H}$ and $\widehat{V}$, respectively. In Section 4, we first concentrate on verifying the flattening property of the process. Then, using the flattening property of the process, we prove the existence of the pullback attractors for the universe of fixed bounded sets and for another universe with a tempered condition in spaces $\widehat{H}$ and $\widehat{V}$, respectively. Furthermore, we reveal the regularity result of the pullback attractors by showing that these attractors coincide with each other.

2. Preliminaries

In this section, we make some necessary preliminaries. To begin with, we give some useful estimates with respect to those operators (1.5) in the following lemmas.

LEMMA 2.1. *The operator A is a linear continuous operator both from V to V^* and from $D(A):=V\cap\left(H^2(\Omega)\right)^3$ to H. Indeed, $A=-\mathbb{P}\Delta$, where $\mathbb{P}$ is the Leray projector from $\mathbb{L}^2(\Omega)$ to H. The operator $B(\cdot,\cdot)$ is continuous from $V\times V$ to V^*. Moreover, for any $u\in V, w\in V$, it holds that*

$$\langle B(u,w),\varphi\rangle=-\langle B(u,\varphi),w\rangle. \tag{2.1}$$

Proof. The linearity and continuity of the operator A can be deduced directly from its definition. Similarly, the continuity of the operator $B(\cdot,\cdot)$ can be obtained easily from its definition. We only need to verify (2.1). In fact, for any $u\in V, w\in\widehat{V}$, we have

$$\begin{aligned}
&\langle B(u,w),w\rangle=((u\cdot\nabla)w,w)=\int_\Omega(u_1\frac{\partial}{\partial x_1}+u_2\frac{\partial}{\partial x_2})(w_1,w_2,w_3)(w_1,w_2,w_3)\mathrm{d}x\\
&=\sum_{j=1}^3\sum_{i=1}^2\int_\Omega u_i\frac{\partial w_j}{\partial x_i}w_j\mathrm{d}x=\sum_{j=1}^3\sum_{i=1}^2\frac{1}{2}\int_\Omega u_i\frac{\partial w_j^2}{\partial x_i}\mathrm{d}x=\frac{1}{2}\sum_{j=1}^3\sum_{i=1}^2(u_iw_j^2|_{\partial\Omega}-\int_\Omega w_j^2D_iu_i\mathrm{d}x)\\
&=-\frac{1}{2}\sum_{j=1}^3\sum_{i=1}^2\int_\Omega w_j^2D_iu_i\mathrm{d}x=-\frac{1}{2}\sum_{j=1}^3\int_\Omega w_j^2(\nabla\cdot u)\mathrm{d}x=0.
\end{aligned}\tag{2.2}$$

Hence, the identity (2.1) is valid as a consequence of (2.2). This completes the proof. □

LEMMA 2.2 (see [21,22,30]).
(1) *There are two positive constants c_1 and c_2 such that*

$$c_1\langle Aw,w\rangle\leqslant\|w\|_{\widehat{V}}^2\leqslant c_2\langle Aw,w\rangle,\ \ \forall w\in\widehat{V}. \tag{2.3}$$

(2) *There exists a positive constant λ which depends only on Ω, such that for any $(u,w,\varphi)\in V\times\widehat{V}\times\widehat{V}$, it holds that*

$$|\langle B(u,w),\varphi\rangle|\leqslant\lambda\|u\|^{\frac12}\|\nabla u\|^{\frac12}\|w\|^{\frac12}\|\nabla w\|^{\frac12}\|\nabla\varphi\|. \tag{2.4}$$

Furthermore, for $(u,w,\varphi)\in V\times D(A)\times\widehat{V}$ and $(u,w,\varphi)\in V\times D(A)\times D(A)$, we have

$$|\langle B(u,w),\varphi\rangle|\leqslant\lambda\|u\|^{\frac12}\|\nabla u\|^{\frac12}\|\nabla w\|^{\frac12}\|Aw\|^{\frac12}\|\varphi\|, \tag{2.5}$$

$$|\langle B(u,w),A\varphi\rangle|\leqslant\lambda\|u\|^{\frac12}\|\nabla u\|^{\frac12}\|\nabla w\|^{\frac12}\|Aw\|^{\frac12}\|A\varphi\|. \tag{2.6}$$

(3) *There exist two positive constants $c(\nu_r)$ and $\delta_1:=\min\{\nu,\alpha\}$ such that*

$$\|N(w)\|\leqslant c(\nu_r)\|w\|_{\widehat{V}},\ \ \forall w\in\widehat{V}, \tag{2.7}$$

$$\langle Aw,w\rangle+\langle N(w),w\rangle\geqslant\delta_1\|w\|_{\widehat{V}}^2,\ \ \forall w\in\widehat{V}. \tag{2.8}$$

REMARK 2.1. According to the definition of operator A and the classical spectral theory of elliptic operators (see [3]), there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ (formed by the eigenvalues of A) satisfying

$$0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n\leqslant\cdots,\quad \lambda_n\to+\infty \text{ as } n\to\infty,$$

and a sequence of elements $\{v_n\}_{n=1}^{\infty}\subseteq D(A)$, which forms a orthonormal basis of $\widehat{H}$, so that $\mathrm{span}\{v_1,v_2,\cdots,v_n,\cdots\}$ is dense in $\widehat{V}$, and $Av_n=\lambda_n v_n$ for $\forall n\in\mathbb{N}$.

Next, we recall a basic result (see Theorem 3.11 in [16]).

PROPOSITION 2.1. *Assume $\{U(t,\tau)\}_{t\geqslant\tau}$ is a closed process, $\mathcal{D}$ is a universe in $\mathcal{P}(X)$, $\widehat{D}_0=\{D_0(t)|t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ is pullback $\mathcal{D}$- absorbing for the process, and $\{U(t,\tau)\}_{t\geqslant\tau}$ is pullback $\widehat{D}_0$- asymptotically compact. Then, the family $\widehat{\mathcal{A}}_{\mathcal{D}}=\{\mathcal{A}_{\mathcal{D}}(t)|t\in\mathbb{R}\}$ defined by*

$$\mathcal{A}_{\mathcal{D}}(t)=\overline{\bigcup_{\widehat{D}\in\mathcal{D}}\Lambda(\widehat{D},t)}^{X} \text{ with } \Lambda(\widehat{D},t):=\bigcap_{s\leqslant t}\overline{\bigcup_{\tau\leqslant s}U(t,\tau)D(\tau)}^{X},\ \forall t\in\mathbb{R}, \tag{2.9}$$

satisfies the following properties:

- *Compactness: for any $t\in\mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and $\mathcal{A}_{\mathcal{D}}(t)\subseteq\Lambda(\widehat{D}_0,t)$;*
- *Invariance: $\widehat{\mathcal{A}}_{\mathcal{D}}$ is invariant, i.e. $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)=\mathcal{A}_{\mathcal{D}}(t)$, for all $\tau\leqslant t$;*
- *Pullback attracting: $\widehat{\mathcal{A}}_{\mathcal{D}}$ is pullback $\mathcal{D}$- attracting, that is*

$$\lim_{\tau\to-\infty}\mathrm{dist}_X(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}}(t))=0,\ \textit{for all}\,\widehat{D}\in\mathcal{D},t\in\mathbb{R};$$

- *Minimality: the family $\widehat{\mathcal{A}}_{\mathcal{D}}$ is minimal;*
- *if $\widehat{D}_0\in\mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t)=\Lambda(\widehat{D}_0,t)\subseteq\overline{D_0(t)}^{X}$, for all $t\in\mathbb{R}$.*

REMARK 2.2. *If $\widehat{\mathcal{A}}_{\mathcal{D}}\in\mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{P}$. Furthermore, the sufficient conditions for $\widehat{\mathcal{A}}_{\mathcal{D}}\in\mathcal{D}$ are that*
- *$\widehat{D}_0\in\mathcal{D}$,*
- *the set $D_0(t)$ is closed for all $t\in\mathcal{R}$,*
- *the universe $\mathcal{D}$ is inclusion-closed, that is, if $\widehat{D}\in\mathcal{D}$ and $\widehat{D}'=\{D'(t):t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ with $D'(t)\subseteq D(t)$ for all t, then $\widehat{D}'\in\mathcal{D}$.*

Finally, we introduce a notion called "flattening property" (see [17, 18]), which is also known as "Condition (C)" in [25].

DEFINITION 2.1. *Assume that X is a Banach space with norm $\|\cdot\|_X$, and $\widehat{D}_0=\{D_0(t)|t\in\mathbb{R}\}$ is a given family. We say that the process $\{U(t,\tau)\}_{t\geqslant\tau}$ on X satisfies the pullback $\widehat{D}_0$-flattening property if, for any $t\in\mathbb{R}$ and $\epsilon>0$, there exist $\tau(\epsilon,t,\widehat{D}_0)<t$, a finite dimensional subspace $X(\epsilon,t,\widehat{D}_0)$ of X, and a map $\mathrm{P}(\epsilon,t,\widehat{D}_0):X\mapsto X(\epsilon,t,\widehat{D}_0)$, such that*

$$\{\mathrm{P}U(t,\tau)w_\tau|\tau\leqslant\tau(\epsilon,t,\widehat{D}_0),w_\tau\in D_0(\tau)\}\,\textit{is bounded in}\,X.$$

and

$$\|(I-\mathrm{P})U(t,\tau)w_\tau\|_X<\epsilon,\ \textit{for any}\,\tau\leqslant\tau(\epsilon,t,\widehat{D}_0),w_\tau\in D_0(\tau).$$

REMARK 2.3. García-Luengo, Marín-Rubio and Real [17, Proposition 9] pointed out that, to ensure a process $\{U(t,\tau)\}_{t\geqslant\tau}$ is pullback $\widehat{D}_0$- asymptotically compact, it is enough to show the process satisfies the pullback $\widehat{D}_0$-flattening property.

3. Existence of compact pullback absorbing family

In this section, using the semigroup method (see [12]) and ϵ-regularity method (see [2]), we show the existence of a compact pullback absorbing family (not only asymptotic compactness) in $\widehat{H}$ and $\widehat{V}$, respectively. To do this, let us first define the fractional powers spaces $D(A^\alpha)$ as the domains of operators A^α and analytic semigroup e^{-At}.

DEFINITION 3.1. *For $\alpha>0$, let $D(A^\alpha)=\{w\in\widehat{H}|\sum_{n=1}^{\infty}\lambda_n^{2\alpha}(w,v_n)^2<+\infty\}$. In particular, we also write $u\in D(A^\alpha)$ if $w=(u,\omega)\in D(A^\alpha)$. And $A^\alpha w=\sum_{n=1}^{\infty}\lambda_n^{\alpha}(w,v_n)v_n\in\widehat{H}$, $\forall w\in D(A^\alpha)$, here $\{v_n\}_{n\geqslant1}$ is given in Remark 2.1.*

Note that for all $\alpha>0$, $D(A^\alpha)$ is a Hilbert space with the inner product $(w,\varphi)_{D(A^\alpha)}=(A^\alpha w,A^\alpha\varphi)$ and $D(A^{-\alpha}):=$ the dual space of $D(A^\alpha)$. Particularly, $D(A^0)=\widehat{H}$, $D(A^{\frac12})=\widehat{V}$, $D(A^{-\frac12})=\widehat{V}^*$. For convenience, we write $\|\varphi\|_\alpha:=\|A^\alpha\varphi\|$.

DEFINITION 3.2. *For any $w\in\widehat{H}$ and $t\geqslant0$, define $e^{-At}w=\sum_{n=1}^{\infty}e^{-\lambda_n t}(w,v_n)v_n\in\widehat{H}$.*

3.1. Existence of a compact pullback absorbing sets in $\widehat{H}$.

In this subsection, we focus on the proof of Theorem 1.1. To begin with, the following results can be verified in the same way as Lemmas 3.2-3.3 in our previous work [31], we omit the details here.

LEMMA 3.1. *Assume $F\in L^2_{loc}(\mathbb{R};\widehat{V}^*)$ and w is the solution to system (1.6) with initial value $w_\tau\in\widehat{H}$. Then,*

(1) *it holds that*

$$\|w(t)\|^2\leqslant e^{-\delta_1(t-\tau)}\|w_\tau\|^2+\frac{e^{-\delta_1 t}}{\delta_1}\int_\tau^t e^{\delta_1\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta. \tag{3.1}$$

(2) *for any $t\in\mathbb{R}$ and $\widehat{D}=\{D(t)|t\in\mathbb{R}\}\in\mathcal{D}^{\widehat{H}}$, there exists a $\tau_0(\widehat{D},t)<t-2$ such that, for any $\tau\leqslant\tau_0(\widehat{D},t)$ and $w_\tau\in D(\tau)$,*

$$\|w(r;\tau,w_\tau)\|^2\leqslant\rho_1(t),\quad \forall r\in[t-2,t], \tag{3.2}$$

$$\int_{r-1}^r\|w(r;\tau,w_\tau)\|^2_{\widehat{V}}\mathrm{d}\theta\leqslant\rho_2(t),\quad \forall r\in[t-1,t], \tag{3.3}$$

$$\int_{r-1}^r\|w'(r;\tau,w_\tau)\|^2_{\widehat{V}^*}\mathrm{d}\theta\leqslant\rho_3(t),\quad \forall r\in[t-1,t], \tag{3.4}$$

where

$$\rho_1(t):=1+\frac{e^{-\delta_1(t-2)}}{\delta_1}\int_{-\infty}^t e^{\delta_1\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta,\ \rho_2(t):=\frac{1}{\delta_1}\rho_1(t)+\frac{1}{\delta_1^2}\int_{t-2}^t\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta,$$

$$\rho_3(t):=3(c_1^{-1}+c(\nu_r))^2\rho_2(t)+3\lambda^2\rho_1(t)\rho_2(t)+3\int_{t-2}^t\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta.$$

Invoking (3.2), we can obtain

LEMMA 3.2. *Under the conditions of Lemma 3.1, the family* $\widehat{D}_0=\{D_0(t)|t\in\mathbb{R}\}$ *with* $D_0(t)=\bar{\mathcal{B}}_{\widehat{H}}(0,\mathcal{R}_{\widehat{H}}(t))$ *is pullback* $\mathcal{D}^{\widehat{H}}$*- absorbing for the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *in* $\widehat{H}$*, where*

$$\bar{\mathcal{B}}_{\widehat{H}}(0,\mathcal{R}_{\widehat{H}}(t))=\{w\in\widehat{H}\,|\,\|w\|^2\leqslant\mathcal{R}_{\widehat{H}}(t)\}\ \textit{with}\ \mathcal{R}_{\widehat{H}}(t):=1+\frac{e^{-\delta_1(t-2)}}{\delta_1}\int_{-\infty}^{t}e^{\delta_1\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta$$

is a closed ball in $\widehat{H}$*.*

Then, let us verify a important smoothing estimate.

LEMMA 3.3. *For any* $\beta>0$*, it holds that*

$$\|A^\beta e^{-At}w\|\leqslant c_\beta t^{-\beta}\|w\|,\ \forall w\in\widehat{H}. \tag{3.5}$$

Proof. Since, for any $w\in\widehat{H}$, $w=\sum\limits_{n=1}^{\infty}(w,v_n)v_n$. Then, we have

$$\begin{aligned}\|A^\beta e^{-At}w\|&=\|\sum_{n=1}^{\infty}\lambda_n^\beta e^{-\lambda_n t}(w,v_n)v_n\|=\|t^{-\beta}\sum_{n=1}^{\infty}(\lambda_n t)^\beta e^{-\lambda_n t}(w,v_n)v_n\|\\&\leqslant t^{-\beta}\sup_{\gamma\in[0,+\infty)}\gamma^\beta e^{-\gamma}\|\sum_{n=1}^{\infty}(w,v_n)v_n\|=c_0t^{-\beta}\|w\|,\end{aligned}$$

where $c_\beta:=\sup\limits_{\gamma\in[0,\infty)}\gamma^\beta e^{-\gamma}$. This completes the proof. ◻

Next, we investigate the estimates about operators B and N.

LEMMA 3.4. *For any* $0<\epsilon<\frac{1}{4}$*, there exist constants* $\bar{c}_\epsilon$ *and* $\tilde{c}_\epsilon$ *such that*

$$\begin{aligned}B:\ D(A^\epsilon)\times D(A^\epsilon)\mapsto D(A^{-(1-2\epsilon)})\ \textit{and}\ \|B(u,w)\|_{-(1-2\epsilon)}\leqslant\bar{c}_\epsilon\|w\|^2_\epsilon,\\N:\ D(A^\epsilon)\mapsto D(A^{-(1-2\epsilon)})\ \textit{and}\ \|N(w)\|_{-(1-2\epsilon)}\leqslant\tilde{c}_\epsilon\|w\|_\epsilon.\end{aligned}$$

Proof. Note that for any $\epsilon\in(0,\frac{1}{4})$, it holds that (see [1])

$$D(A^{1-2\epsilon})\hookrightarrow W^{2-4\epsilon,2}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)\hookrightarrow W^{1,\frac{1}{2\epsilon}}(\Omega),\ D(A^\epsilon)\hookrightarrow W^{2\epsilon,2}(\Omega)\hookrightarrow L^{\frac{2}{1-2\epsilon}}(\Omega).$$

Then, for any $\varphi\in D(A^{1-2\epsilon})$, it follows from the above embedding and Hölder inequality that

$$\begin{aligned}\big|\langle B(u,w),\varphi\rangle\big|&=\big|\langle B(u,\varphi),w\rangle\big|\leqslant\|u\|_{L^{\frac{2}{1-2\epsilon}}(\Omega)}\|w\|_{L^{\frac{2}{1-2\epsilon}}(\Omega)}\|\nabla\varphi\|_{L^{\frac{1}{2\epsilon}}(\Omega)}\\&\leqslant\|w\|^2_{L^{\frac{2}{1-2\epsilon}}(\Omega)}\|\nabla\varphi\|_{L^{\frac{1}{2\epsilon}}(\Omega)}\leqslant\bar{c}_\epsilon\|w\|^2_\epsilon\|\varphi\|_{1-2\epsilon},\quad\forall(u,w)\in D(A^\epsilon)\times D(A^\epsilon),\end{aligned}$$

and

$$\begin{aligned}\big|\langle N(w),\varphi\rangle\big|&=\Big|\int_\Omega-2\nu_r\Big(\frac{\partial\omega}{\partial x_2},-\frac{\partial\omega}{\partial x_1},\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}+2\omega\Big)\cdot(\varphi_1,\varphi_2,\varphi_3)\mathrm{d}x\Big|\\&\leqslant c\Big|\int_\Omega\nabla w\cdot\varphi\mathrm{d}x\Big|\leqslant c\int_\Omega|w|\cdot|\nabla\varphi|\mathrm{d}x\leqslant c\|w\|_{L^{\frac{1}{1-2\epsilon}}(\Omega)}\|\nabla\varphi\|_{L^{\frac{1}{2\epsilon}}(\Omega)}\end{aligned}$$

$$\leqslant \tilde{c}_\epsilon \|w\|_\epsilon \|\varphi\|_{1-2\epsilon}, \quad \forall w \in D(A^\epsilon),$$

where $\bar{c}_\epsilon$ and $\tilde{c}_\epsilon$ are positive constants. This completes the proof. ∎

Now, let us show the boundness of $\{U(t,\tau)D(\tau)\}$ in $D(A^\epsilon)$ norm.

LEMMA 3.5. *Assume that* (H3) *holds, then, for any* $\epsilon < \min\{\frac{1}{4}, \frac{1}{2} - \frac{1}{p}\}, t \in \mathbb{R}$ *and* $\widehat{D} \in \mathcal{D}^{\widehat{H}}$,

$$\{U(t,\tau)w_\tau \,|\, w_\tau \in D(\tau), \tau \leqslant \tau_0(\widehat{D},t)\} \text{ is bounded in } D(A^\epsilon),$$

where $\tau_0(\widehat{D},t)$ *is given in Lemma 3.1.*

Proof. First, fix $t \in \mathbb{R}, \widehat{D} \in \mathcal{D}^{\widehat{H}}, \tau \leqslant \tau_0(\widehat{D},t), w_\tau \in D(t)$ and write $w_\sigma(s) = w(\sigma + s; \tau, w_\tau)$, $F_\sigma(s) = F(\sigma + s)$. Noting that any weak solution w to system (1.6) satisfies the variation of constants formula

$$w(t) = e^{-A(t-s)} w(s) + \int_s^t e^{-A(t-\theta)} [F(\theta) - B(u(\theta), w(\theta)) - N(w(\theta))] \mathrm{d}\theta, \ \forall s \in [\tau, t]. \tag{3.6}$$

In particular, for some $\sigma > \tau$ (specified later), we have

$$w_\sigma(s) = e^{-As} w_\sigma(0) + \int_0^s e^{-A(s-\theta)} [F_\sigma(\theta) - B(u_\sigma(\theta), w_\sigma(\theta)) - N(w_\sigma(\theta))] \mathrm{d}\theta, \ \forall s \in [0, t-\sigma]. \tag{3.7}$$

From (3.2) and (3.3), we see that, for any $\sigma \in [t-1, t], s \in [0, t-\sigma]$,

$$\|w_\sigma(s)\| \leqslant \rho_1^{\frac{1}{2}}(t) \text{ and } \int_0^s \|w_\sigma(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta \leqslant \rho_2(t). \tag{3.8}$$

Hence, on the basis of Lemma 3.3 and Lemma 3.4, choosing $\sigma \in [t-1, t]$ and letting $s \leqslant t - \sigma$, then taking the norm of (3.7) in $D(A^\epsilon)$ and multiplying by s^ϵ, we obtain

$$\begin{aligned} s^\epsilon \|w_\sigma(s)\|_\epsilon \leqslant & s^\epsilon \|e^{-As} w_\sigma(0)\|_\epsilon + s^\epsilon \int_0^s \|e^{-A(s-\theta)} [F_\sigma(\theta) - B(u_\sigma(\theta), w_\sigma(\theta)) - N(w_\sigma(\theta))]\|_\epsilon \mathrm{d}\theta \\ \leqslant & c_\epsilon \|w_\sigma(0)\| + c_{1-\epsilon} s^\epsilon \int_0^s (s-\theta)^{-(1-\epsilon)} \big[\|B(u_\sigma(\theta), w_\sigma(\theta))\|_{-(1-2\epsilon)} \\ & + \|N(w_\sigma(\theta))\|_{-(1-2\epsilon)}\big] \mathrm{d}\theta + c_{\frac{1}{2}+\epsilon} s^\epsilon \int_0^s (s-\theta)^{-\frac{1}{2}-\epsilon} \|F_\sigma(\theta)\|_{\widehat{V}^*} \mathrm{d}\theta \\ \leqslant & L_1 + L_2 + L_3, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} L_1 &:= c_\epsilon \rho_1^{\frac{1}{2}}(t) + c_{\frac{1}{2}+\epsilon} s^\epsilon \int_0^s (s-\theta)^{-\frac{1}{2}-\epsilon} \|F_\sigma(\theta)\|_{\widehat{V}^*} \mathrm{d}\theta, \\ L_2 &:= c_{1-\epsilon} \bar{c}_\epsilon s^\epsilon \int_0^s (s-\theta)^{-(1-\epsilon)} \|w_\sigma(\theta)\|_\epsilon^2 \mathrm{d}\theta, \\ L_3 &:= c_{1-\epsilon} \tilde{c}_\epsilon s^\epsilon \int_0^s (s-\theta)^{-(1-\epsilon)} \|w_\sigma(\theta)\|_\epsilon \mathrm{d}\theta. \end{aligned}$$

In order to find an upperbound of $s^\epsilon \|w_\sigma(s)\|_\epsilon$. In the following, we estimate L_1, L_2 and L_3 respectively. First, due to $0 < \epsilon < \frac{1}{2} - \frac{1}{p}$, then, for any $s \in [0, t-\sigma] \subseteq [0,1]$, it is easy to see that

$$L_1 \leqslant c_\epsilon \rho_1^{\frac{1}{2}}(t) + c_{\frac{1}{2}+\epsilon} \Big(\int_0^s (s-\theta)^{\frac{-\frac{1}{2}-\epsilon}{1-\frac{1}{p}}} \mathrm{d}\theta\Big)^{1-\frac{1}{p}} \Big(\int_0^s \|F_\sigma(\theta)\|_{\widehat{V}^*}^p \mathrm{d}\theta\Big)^{\frac{1}{p}}$$

$$\leqslant c_\epsilon \rho_1^{\frac{1}{2}}(t)+c_{\frac{1}{2}+\epsilon}\Big(\int_0^s r^{\frac{-\frac{1}{2}-\epsilon}{1-\frac{1}{p}}}\mathrm{d}r\Big)^{1-\frac{1}{p}}\Big(\int_{t-1}^t \|F(r)\|_{\widehat{V}^*}^p\mathrm{d}r\Big)^{\frac{1}{p}}=:\varphi_1(t), \tag{3.10}$$

where $\varphi_1(t)$ is bounded under the assumption (H3) for any $t\in\mathbb{R}$. Next, consider the following interpolation inequality (see [29], (2.24))

$$\|w(\theta)\|_\epsilon^{\frac{1}{2}}\leqslant\|w(\theta)\|^{\frac{1}{2}-\epsilon}\|w(\theta)\|_{\widehat{V}}^{\epsilon},\ \text{for any}\ w\in\widehat{H},\epsilon\in(0,\frac{1}{4}],$$

which together with (3.8) and Hölder inequality gives

$$\begin{aligned}
L_2\leqslant&\bar{c}_\epsilon c_{1-\epsilon}s^\epsilon\int_0^s(s-\theta)^{-(1-\epsilon)}\|w_\sigma(\theta)\|_\epsilon^{\frac{3}{2}}\|w_\sigma(\theta)\|^{\frac{1}{2}-\epsilon}\|w_\sigma(\theta)\|_{\widehat{V}}^{\epsilon}\mathrm{d}\theta\\
\leqslant&\bar{c}_\epsilon c_{1-\epsilon}s^\epsilon\rho_1^{\frac{1-2\epsilon}{4}}(t)\Big(\int_0^s\|w_\sigma(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\Big)^{\frac{\epsilon}{2}}\Big(\int_0^s(s-\theta)^{-\frac{2(1-\epsilon)}{2-\epsilon}}\|w_\sigma(\theta)\|_\epsilon^{\frac{3}{2-\epsilon}}\mathrm{d}\theta\Big)^{1-\frac{\epsilon}{2}}\\
\leqslant&\bar{c}_\epsilon c_{1-\epsilon}\rho_1^{\frac{1-2\epsilon}{4}}(t)\rho_2^{\frac{\epsilon}{2}}(t)s^\epsilon\Big(\int_0^s(s-\theta)^{-\frac{2(1-\epsilon)}{2-\epsilon}}\|w_\sigma(\theta)\|_\epsilon^{\frac{3}{2-\epsilon}}\mathrm{d}\theta\Big)^{1-\frac{\epsilon}{2}}\\
=:&\varphi_2(t)s^\epsilon\Big(\int_0^s(s-\theta)^{-\frac{2(1-\epsilon)}{2-\epsilon}}\|w_\sigma(\theta)\|_\epsilon^{\frac{3}{2-\epsilon}}\mathrm{d}\theta\Big)^{1-\frac{\epsilon}{2}}.
\end{aligned}\tag{3.11}$$

Similarly, we have

$$\begin{aligned}
L_3\leqslant&\tilde{c}_\epsilon c_{1-\epsilon}s^\epsilon\rho_1^{\frac{1-2\epsilon}{4}}\Big(\int_0^s\|w_\sigma(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\Big)^{\frac{\epsilon}{2}}\Big(\int_0^s(s-\theta)^{-\frac{2(1-\epsilon)}{2-\epsilon}}\|w_\sigma(\theta)\|_\epsilon^{\frac{1}{2-\epsilon}}\mathrm{d}\theta\Big)^{1-\frac{\epsilon}{2}}\\
\leqslant&\varphi_2(t)s^\epsilon\Big(\int_0^s(s-\theta)^{-\frac{2(1-\epsilon)}{2-\epsilon}}\|w_\sigma(\theta)\|_\epsilon^{\frac{1}{2-\epsilon}}\mathrm{d}\theta\Big)^{1-\frac{\epsilon}{2}}.
\end{aligned}\tag{3.12}$$

Now, letting $M(s)=s^\epsilon\|w_\sigma(s)\|_\epsilon$ and substituting (3.10)-(3.12) into (3.9), yields

$$\begin{aligned}
M(s)\leqslant\varphi_1(t)+\varphi_2(t)s^\epsilon\Big[\Big(&\int_0^s(s-\theta)^{-\frac{2(1-\epsilon)}{2-\epsilon}}\theta^{-\frac{3\epsilon}{2-\epsilon}}M^{\frac{3}{2-\epsilon}}(\theta)\mathrm{d}\theta\Big)^{1-\frac{\epsilon}{2}}\\
&+\Big(\int_0^s(s-\theta)^{-\frac{2(1-\epsilon)}{2-\epsilon}}\theta^{-\frac{\epsilon}{2-\epsilon}}M^{\frac{1}{2-\epsilon}}(\theta)\mathrm{d}\theta\Big)^{1-\frac{\epsilon}{2}}\Big].
\end{aligned}$$

Finally, with the help of the following proposition (one can refer to [17] for detailed proof):

Proposition 3.1. *If, for all $s\in[0,t-\sigma]$, there exists a continuous function $Q(s)$ with $Q(0)>M(0)$ satisfying*

$$\begin{aligned}
Q(s)\geqslant\varphi_1(t)+\varphi_2(t)s^\epsilon\Big[\Big(&\int_0^s(s-\theta)^{-\frac{2(1-\epsilon)}{2-\epsilon}}\theta^{-\frac{3\epsilon}{2-\epsilon}}Q^{\frac{3}{2-\epsilon}}(\theta)\mathrm{d}\theta\Big)^{1-\frac{\epsilon}{2}}\\
&+\Big(\int_0^s(s-\theta)^{-\frac{2(1-\epsilon)}{2-\epsilon}}\theta^{-\frac{\epsilon}{2-\epsilon}}Q^{\frac{1}{2-\epsilon}}(\theta)\mathrm{d}\theta\Big)^{1-\frac{\epsilon}{2}}\Big],
\end{aligned}$$

then $M(s)\leqslant Q(s)$, $\forall s\in[0,t-\sigma]$. Particularly, for any fixed $k_0>1$, $k_0\varphi_1(t)\geqslant M(s)$ for all $s\in[0,t-\sigma]$.

We conclude that, for all $w_\tau\in D(\tau),\tau\leqslant\tau_0(\widehat{D},t)$,

$$\|A^\epsilon w(t;\tau,w_\tau)\|=(t-\sigma)^{-\epsilon}M(t-\sigma)\leqslant k_0(t-\sigma)^{-\epsilon}\varphi_1(t),$$

which implies Lemma 3.5 holds. ∎

Proof. **(The proof of Theorem 1.1).** Based on Lemma 3.5 and the compact embedding $D(A^\epsilon)\hookrightarrow\hookrightarrow\widehat{H}$, the existence of a compact pullback absorbing family in $\widehat{H}$ follows immediately. This completes the proof. ∎

3.2. Existence of a compact pullback absorbing family in $\widehat{V}$. The goal of this subsection is to prove the existence of a compact pullback absorbing family in $\widehat{V}$. First, we improve the estimates of the solutions.

LEMMA 3.6. *Assume* (H2) *hold, then for any $t\in\mathbb{R}$ and $\widehat{D}=\{D(t)|t\in\mathbb{R}\}\in\mathcal{D}^{\widehat{H}}$, there exists a $\tau_0'(\widehat{D},t)<t-2$, such that for any $\tau\leqslant\tau_0'(\widehat{D},t)$ and $w_\tau\in D(\tau)$, it holds that*

$$\|w(r;\tau,w_\tau)\|^2\leqslant\rho_4(t),\quad \forall r\in[t-2,t], \tag{3.13}$$

$$\|w(r;\tau,w_\tau)\|_{\widehat{V}}^2\leqslant\rho_5(t),\quad \forall r\in[t-1,t], \tag{3.14}$$

$$\int_{t-1}^{t}\|Aw(\theta;\tau,w_\tau)\|^2\mathrm{d}\theta\leqslant\rho_6(t), \tag{3.15}$$

where

$$\rho_4(t)=1+\frac{e^{-\delta_1(t-2)}}{\delta_1}\int_{-\infty}^{t}e^{\delta_1\theta}\|F(\theta)\|^2\mathrm{d}\theta,\ \rho_6(t):=c_5(2\rho_5(t)+\rho_4\rho_5^2(t)+\int_{t-2}^{t}\|F(\theta)\|^2\mathrm{d}\theta),$$

$$\rho_5(t)=c_3(\rho_4(t)+\int_{t-2}^{t}\|F(\theta)\|^2\mathrm{d}\theta)\times\exp\big\{c_4[(\rho_4(t)+\int_{t-2}^{t}\|F(\theta)\|^2\mathrm{d}\theta)^2+1]\big\}$$

with $c_3=2c_2+\frac{c_2}{c_1\delta_1}, c_4=\max\big\{2c_2\lambda^4\cdot\max\{1,\delta_1^{-2}\},4c_2c^2(\nu_r)\big\}$ and $c_5=\max\{4\lambda^4,4c^2(\nu_r),2c_1^{-1},4\}$.

Proof. The estimate of (3.13) is similar to that of (3.2). Moreover, the estimates (3.14) and (3.15) can be proved similarly to (3.8) and (3.9) in Lemma 3.3 of [31], respectively. Hence, we can omit the details here. ∎

Noting that $\{\bar{\mathcal{B}}_{\widehat{H}}(0,\rho_4(t))|t\in\mathbb{R}\}:=\{w\in\widehat{H}|\,\|w(t)\|^2\leqslant\rho_4(t)\}\in\mathcal{D}^{\widehat{H}}$, based on Lemma 3.6, we immediately have

LEMMA 3.7. *Assume that* (H2) *hold, then the family of sets*

$$\widehat{D}_{0,\widehat{V}}:=\{\widehat{D}_{0,\widehat{V}}(t)=\bar{\mathcal{B}}_{\widehat{H}}(0,\rho_4(t))\cap\widehat{V}|t\in\mathbb{R}\}\in\mathcal{D}^{\widehat{H},\widehat{V}}.$$

Moreover, for any $t\in\mathbb{R},\widehat{D}\in\mathcal{D}^{\widehat{H}}$, and there exists a $\tau_0'(\widehat{D},t)<t$ such that

$$U(t,\tau)D(\tau)\subseteq\widehat{D}_{0,\widehat{V}}(t),\ \forall\tau\leqslant\tau_0'(\widehat{D},t).$$

Particularly, $\widehat{D}_{0,\widehat{V}}$ is pullback $\mathcal{D}^{\widehat{H},\widehat{V}}$- absorbing for process $\{U(t,\tau)\}_{t\geqslant\tau}$.

Then, we verify the following estimates about the operators B and N.

LEMMA 3.8. *For any $\epsilon>0$, there exist constants $\bar{\bar{c}}_\epsilon$ and $\tilde{\tilde{c}}_\epsilon$ such that*

$$B:\ V\times\widehat{V}\mapsto D(A^{-\epsilon})\quad \textit{and}\quad \|B(u,w)\|_{-\epsilon}\leqslant\bar{\bar{c}}_\epsilon\|w\|_{\widehat{V}}^2,\ \forall(u,w)\in V\times\widehat{V}.$$

$$N:\ \widehat{V}\mapsto D(A^{-\epsilon})\quad \textit{and}\quad \|N(w)\|_{-\epsilon}\leqslant\tilde{\tilde{c}}_\epsilon\|w\|_{\widehat{V}},\ \forall w\in\widehat{V}.$$

Proof. First, for any $\epsilon>0$, it holds that

$$W^{1,2}(\Omega)\hookrightarrow L^{\frac{1}{\epsilon}}(\Omega),\quad D(A^\epsilon)\hookrightarrow L^{\frac{2}{1-2\epsilon}}(\Omega),\quad L^2(\Omega)\hookrightarrow L^{\frac{2}{1+2\epsilon}}(\Omega),$$

which together with Hölder inequality leads to that, for any $\phi\in D(A^\epsilon)$,

$$\big|\langle B(u,w),\phi\rangle\big|\leqslant\|u\|_{L^{\frac{1}{\epsilon}}(\Omega)}\|\nabla w\|\|\phi\|_{L^{\frac{2}{1-2\epsilon}}(\Omega)}\leqslant\|w\|_{L^{\frac{1}{\epsilon}}(\Omega)}\|\nabla w\|\|\phi\|_\epsilon$$

$$\leqslant \bar{\bar{c}}_\epsilon \|w\|_{\widehat{V}}^2 \|\phi\|_\epsilon, \quad \forall (u,w) \in V \times \widehat{V},$$

and

$$\begin{aligned}
\big|\langle N(w),\phi\rangle\big| &= \Big|\int_\Omega -2\nu_r\Big(\frac{\partial\omega}{\partial x_2}, -\frac{\partial\omega}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} + 2\omega\Big)\cdot(\phi_1,\phi_2,\phi_3)\mathrm{d}x\Big| \\
&\leqslant c\int_\Omega |\nabla w|\cdot|\phi|\mathrm{d}x \leqslant c\|\nabla w\|_{L^{\frac{2}{1+2\epsilon}}(\Omega)}\|\phi\|_{L^{\frac{2}{1-2\epsilon}}(\Omega)} \leqslant \tilde{\tilde{c}}_\epsilon\|w\|_{\widehat{V}}\|\phi\|_\epsilon, \quad \forall w \in \widehat{V}.
\end{aligned}$$

This completes the proof. ∎

With Lemma 3.7, Lemma 3.3 and Lemma 3.8 in hand, we immediately have

LEMMA 3.9. *Let* (H4) *hold and take* $\delta < \min\{\frac{1}{4}, \frac{1}{2} - \frac{1}{p}\}$, *then, for any* $t \in \mathbb{R}$ *and* $\widehat{D} \in \mathcal{D}^{\widehat{H}}$, $\{U(t,\tau)D(\tau) | \tau \leqslant \tau_0'(\widehat{D},t)\}$ *is bounded in* $D(A^{\frac{1}{2}-\frac{1}{p}})$.

Proof. Taking the norm of (3.6) in $D(A^{\frac{1}{2}+\delta})$, choosing ϵ such that $\delta+\epsilon<\frac{1}{2}$, and using Hölder inequality, we have

$$\begin{aligned}
\|w(t)\|_{\frac{1}{2}+\delta} &\leqslant \|e^{-A}w(t-1)\|_{\frac{1}{2}+\delta} + \int_{t-1}^t \|e^{-A(t-\theta)}F(\theta)\|_{\frac{1}{2}+\delta}\mathrm{d}\theta \\
&\quad + \int_{t-1}^t \|e^{-A(t-\theta)}[B(u(\theta),w(\theta)) + N(w(\theta))]\|_{\frac{1}{2}+\delta}\mathrm{d}\theta \\
&\leqslant c_\delta\|w(t-1)\|_{\widehat{V}} + c_{\frac{1}{2}+\delta+\epsilon}\int_{t-1}^t (t-\theta)^{-\frac{1}{2}-\delta-\epsilon}\|B(u(\theta),w(\theta))\|_{-\epsilon}\mathrm{d}\theta \\
&\quad + c_{\frac{1}{2}+\delta+\epsilon}\int_{t-1}^t (t-\theta)^{-\frac{1}{2}-\delta-\epsilon}\|N(w(\theta))\|_{-\epsilon}\mathrm{d}\theta + c_{\frac{1}{2}+\delta}\int_{t-1}^t (t-\theta)^{-\frac{1}{2}-\delta}\|F(\theta)\|\mathrm{d}\theta \\
&\leqslant c_\delta\rho_5^{\frac{1}{2}}(t) + c_{\frac{1}{2}+\delta+\epsilon}\bar{\bar{c}}_\epsilon\rho_5(t)\int_{t-1}^t (t-\theta)^{-\frac{1}{2}-\delta-\epsilon}\mathrm{d}\theta + c_{\frac{1}{2}+\delta+\epsilon}\tilde{\tilde{c}}_\epsilon\rho_5^{\frac{1}{2}}(t)\int_{t-1}^t (t-\theta)^{-\frac{1}{2}-\delta-\epsilon}\mathrm{d}\theta \\
&\quad + c_{\frac{1}{2}+\delta}\Big[\int_{t-1}^t (t-\theta)^{-\frac{p(\frac{1}{2}+\delta)}{p-1}}\mathrm{d}\theta\Big]^{\frac{p-1}{p}}\Big[\int_{t-1}^t \|F(\theta)\|^p\mathrm{d}\theta\Big]^{\frac{1}{p}} \\
&\leqslant \Big(c_\delta + \frac{2c_{\frac{1}{2}+\delta+\epsilon}\tilde{\tilde{c}}_\epsilon}{1-2\delta-2\epsilon}\Big)\rho_5^{\frac{1}{2}}(t) + \frac{2c_{\frac{1}{2}+\delta+\epsilon}\bar{\bar{c}}_\epsilon}{1-2\delta-2\epsilon}\rho_5(t) + c_{\frac{1}{2}+\delta}\Big[\frac{2(p-1)}{(1-2\delta)p-2}\Big]^{\frac{p-1}{p}}\Big[\int_{t-1}^t \|F(\theta)\|^p\mathrm{d}\theta\Big]^{\frac{1}{p}}.
\end{aligned}$$

Obtaining the desired result. ∎

Finally, let us give the proof of Theorem 1.2.

Proof. (The proof of Theorem 1.2). From Lemma 3.9, the existence of the compact pullback absorbing family in $\widehat{V}$ is a consequence of the compact embedding $D(A^{\frac{1}{2}+\delta}) \hookrightarrow\hookrightarrow \widehat{V}$. This completes the proof. ∎

4. Existence and regularity of pullback attractors

In this section, we are going to show Theorem 1.3 and Theorem 1.4. First, basing on the estimates of solutions obtained in Section 3, we verify the flattening property of the process. Then, using the flattening property of the process, we prove the existence of the pullback attractors for the universe of fixed bounded sets and for another universe with a tempered condition in spaces $\widehat{H}$ and $\widehat{V}$, respectively. Further, we reveal the regularity result of the pullback attractors by showing that these attractors coincide with each other.

4.1. Existence of pullback attractors in space $\widehat{H}$. In this subsection, we concentrate on proving the existence of pullback attractors. According to Remark 2.3, it suffices to show that the process $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses pullback $\mathcal{D}^{\widehat{H}}$- absorbing and satisfies pullback $\mathcal{D}^{\widehat{H}}$- flattening property.

Based on Lemma 3.2, it is sufficient to verify the pullback $\mathcal{D}^{\widehat{H}}$- flattening property for the process $U(t,\tau)$ on $\widehat{H}$. We need to prove three auxiliary results.

LEMMA 4.1. *Under the conditions of Lemma 3.2, for any $t\in\mathbb{R}, \widehat{D}\in\mathcal{D}^{\widehat{H}}$, $\{\tau_n\}\subseteq(-\infty,t-1]$ and $\{w_{\tau_n}\}\subseteq\widehat{H}$ satisfying $\tau_n\to-\infty$ as $n\to\infty$ and $w_{\tau_n}\in D(\tau_n)$ for all n, it holds that the sequence $\{w(\cdot;\tau_n,w_{\tau_n})\}$ is relatively compact in $\mathcal{C}([t-1,t];\widehat{H})$.*

Proof. Let $w^{(n)}(\cdot):=w(\cdot;\tau_n,w_{\tau_n})=U(\cdot,\tau_n;w_{\tau_n})$ be the solution to (1.6). Then, according to Lemma 3.1, we conclude that there exists a $\tau_0(\widehat{D},t)<t-2$ such that the subsequence (relabelled the same) $\{w^{(n)}(\cdot)\big|\tau_n\leqslant\tau_0(\widehat{D},t)\}\subseteq\{w^{(n)}(\cdot)\}$ is uniformly bounded in $L^\infty(t-2,t;\widehat{H})\cap L^2(t-2,t;\widehat{V})$ and $\{(w^{(n)})'(\cdot)\}$ is uniformly bounded in $L^2(t-2,t;\widehat{V}^*)$. Furthermore, from the standard diagonal procedure, there exists a function $w(\cdot)$ such that

$$\begin{cases} w^{(n)}(\cdot)\rightharpoonup^* w(\cdot) \text{ weakly star in } L^\infty(t-2,t;\widehat{H}),\\ w^{(n)}(\cdot)\rightharpoonup w(\cdot) \text{ weakly in } L^2(t-2,t;\widehat{V}),\\ (w^{(n)})'(\cdot)\rightharpoonup w'(\cdot) \text{ weakly in } L^2(t-2,t;\widehat{V}^*). \end{cases}\tag{4.1}$$

Therefore, it follows from Aubin-Lions theorem (i.e., refer to [5,29]) and the embedding $\widehat{V}\hookrightarrow\hookrightarrow\widehat{H}\hookrightarrow\widehat{V}^*$ that

$$w^{(n)}(\cdot)\to w(\cdot) \text{ strongly in } L^2(t-2,t;\widehat{H}).$$

Further, it holds that

$$w^{(n)}(\cdot)\to w(\cdot) \text{ strongly in } \widehat{H},\ a.e. \text{on } [t-2,t].$$

Again from (4.1), we have

$$w^{(n)}(\cdot)\in\mathcal{C}([t-2,t];\widehat{H}),\ w(\cdot)\in\mathcal{C}([t-2,t];\widehat{H}).$$

Now, since

$$w^{(n)}(s_2)-w^{(n)}(s_1)=\int_{s_1}^{s_2}(w^{(n)})'(\theta)\mathrm{d}\theta \text{ in } \widehat{V}^*,\ \forall s_1,s_2\in[t-2,t],$$

and $\{(w^{(n)})'\}$ is uniformly bounded in $L^2(t-2,t;\widehat{V}^*)$, we conclude by Ascoli-Arzelá theorem that

$$w^{(n)}(\cdot)\to w(\cdot) \text{ strongly in } \mathcal{C}([t-2,t];\widehat{V}^*).$$

Hence, for any sequence $\{s_n\}\subseteq[t-2,t]$ with $s_n\to s_*$ as $n\to\infty$, we know that

$$w^{(n)}(s_n)\rightharpoonup w(s_*) \text{ weakly in } \widehat{H}.$$

Indeed, we claim that

$$w^{(n)}(\cdot)\to w(\cdot) \text{ strongly in } \mathcal{C}([t-1,t];\widehat{H}),\tag{4.2}$$

which implies the sequence $\{\cdot;\tau_n,w_{\tau_n}\}$ is relatively compact in $\mathcal{C}([t-1,t];\widehat{H})$. The proof of (4.2) is similar to that of (3.49) in Lemma 3.8 of [31], we omit here. This completes the proof. □

LEMMA 4.2. *Assume that* (H1) *holds, then for any* $\epsilon>0, t\in\mathbb{R}$ *and* $\widehat{D}\in\mathcal{D}^{\widehat{H}}$, *there exists* $\delta=\delta(\epsilon,t,\widehat{D})\in(0,1)$ *such that*

$$\left|\|w(t;\tau,w_\tau)\|^2-\|w(t-s;\tau,w_\tau)\|^2\right|<\epsilon,\quad \forall s\in[0,\delta], \tau\leqslant\tau_0(\widehat{D},t), w_\tau\in D(\tau), \tag{4.3}$$

where $\tau_0(\widehat{D},t)$ *comes from Lemma 3.1.*

Proof. We verify the above assertion by a contradiction argument. Indeed, if (4.3) were not true, then, for any $\delta\in(0,1)$, there exist an $\epsilon_0>0, t\in\mathbb{R}, \widehat{D}\in\mathcal{D}^{\widehat{H}}$ and three sequences $\{\tau_n\}\subseteq(-\infty,t-1]$ with $\tau_n\to-\infty$ as $n\to\infty$, $\{w_{\tau_n}\}$ with $w_{\tau_n}\in D(\tau_n)$, and $\{s_n\}$ with $0\leqslant s_n\leqslant\frac{1}{n}$ such that

$$\left|\|w(t;\tau_n,w_{\tau_n})\|^2-\|w(t-s_n;\tau_n,w_{\tau_n})\|^2\right|\geqslant\epsilon_0,\quad \text{for all } n\geqslant1. \tag{4.4}$$

However, from (4.2), we see that

$$\|w(t;\tau_n,w_{\tau_n})\|\to\|w(t)\| \text{ and } \|w(t-s_n;\tau_n,w_{\tau_n})\|\to\|w(t)\|, \text{ as } n\to\infty,$$

which implies (4.4) is absurd. □

As a consequence of Lemma 4.2, we have

LEMMA 4.3. *Under the conditions of Lemma 4.2, then, for any* $\epsilon>0, t\in\mathbb{R}$ *and* $\widehat{D}\in\mathcal{D}^{\widehat{H}}$, *there exists a* $\delta(\epsilon,t,\widehat{D})\in(0,1)$ *such that*

$$\int_{t-\delta}^{t}\|w(\theta);\tau,w_\tau\|^2\mathrm{d}\theta<\epsilon,\ \forall\tau\leqslant\tau_0(\widehat{D},t),\ w_\tau\in D(\tau). \tag{4.5}$$

Proof. Testing $(1.6)_1$ by $w(t)$ and using (2.2), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\|w(\theta)\|^2+\langle Aw(\theta),w(\theta)\rangle+\langle N(w(\theta)),w(\theta)\rangle=\langle F(\theta),w(\theta)\rangle,$$

which together with (2.8), Schwartz inequality and Young's inequality implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\|w(\theta)\|^2+2\delta_1\|w(\theta)\|^2\leqslant&\frac{\mathrm{d}}{\mathrm{d}\theta}\|w(\theta)\|^2+2\delta_1\|w(\theta)\|^2_{\widehat{V}}\\ \leqslant&2\langle F(\theta),w(\theta)\rangle\leqslant\delta_1\|w(\theta)\|^2_{\widehat{V}}+\frac{1}{\delta_1}\|F(\theta)\|^2_{\widehat{V}^*}.\end{aligned}$$

Integrating the above inequality with respect to time variable from $t-\delta$ to t, we obtain

$$\delta_1\int_{t-\delta}^{t}\|w(\theta)\|^2\mathrm{d}\theta\leqslant\|w(t-\delta)\|^2-\|w(t)\|^2+\frac{1}{\delta_1}\int_{t-\delta}^{t}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta,$$

which together with $F(t,x)\in L^2_{loc}(\mathbb{R};\widehat{V}^*)$ and Lemma 4.2 yields (4.5). This completes the proof. □

Now, we are ready to prove the pullback $\mathcal{D}^{\widehat{H}}$- flattening property for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ on space $\widehat{H}$.

LEMMA 4.4. *Let* (H1) *hold, then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *on* $\widehat{H}$ *satisfies the pullback* $\widehat{D}$*- flattening property for any* $\widehat{D}\in\mathcal{D}^{\widehat{H}}$.

Proof. According to Definition 2.1, we should verify that, for any $\epsilon>0, t\in\mathbb{R}$ and $\widehat{D}\in\mathcal{D}^{\widehat{H}}$, there exists an $m=m(\epsilon,t,\widehat{D})\in\mathbb{N}$ such that the projection $\mathrm{P_m}:\widehat{H}\mapsto\widehat{H}_m$ with $\widehat{H}_m=\mathrm{span}\{v_1,v_2,\cdots,v_m\}$ ($\{v_n\}_{n\geqslant 1}$ is given in Remark 2.1) satisfies the following two properties:

(i) $\{\mathrm{P_m}U(t,\tau)D(\tau):\tau\leqslant\tau_0(\widehat{D},t)\}$ is bounded in $\widehat{H}$,

(ii) $\|(I-\mathrm{P_m})U(t,\tau)w(\tau)\|<\epsilon$, for any $\tau\leqslant\tau_0(\widehat{D},t), w_\tau\in D(\tau)$,

where $\tau_0(\widehat{D},t)$ is given in Lemma 3.1. Observe that $\|\mathrm{P_m}w(t)\|\leqslant\|w(t)\|$, which together with (3.2) implies property (i).

Next, we verify property (ii). Under the condition (H1), from Lemma 12 in [19], we have

$$\lim_{c\to\infty} e^{-ct}\int_{-\infty}^{t} e^{c\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta=0, \text{ for any } t\in\mathbb{R}. \tag{4.6}$$

Consider fixed $\tau\leqslant\tau_0(\widehat{D},t), w_\tau\in D(\tau)$ and let $q_m(\theta):=w(\theta)-\mathrm{P_m}w(\theta)$, then it holds the following Poincaré style inequality

$$\lambda_{m+1}\|q_m(\theta)\|^2\leqslant\|\nabla q_m(\theta)\|^2\leqslant\|q_m(\theta)\|^2_{\widehat{V}}, \tag{4.7}$$

where λ_{m+1} is defined in Remark 2.1. Since $(\mathrm{P_m}w(\theta),w(\theta)-\mathrm{P_m}w(\theta))=0$, we deduce that by taking scalar product in $(1.6)_1$ with $q_m(\theta)$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\|q_m(\theta)\|^2+\langle Aq_m(\theta),q_m(\theta)\rangle+\langle B(u,w),q_m(\theta)\rangle+\langle N(w),q_m(\theta)\rangle=\langle F(\theta),q_m(\theta)\rangle. \tag{4.8}$$

In the following, we estimate the terms of (4.8) one by one. First, from (2.4) and the facts

$$\|u\|^2\leqslant\|w\|^2,\ \|\nabla u\|^2\leqslant\|\nabla w\|^2\leqslant\|w\|^2_{\widehat{V}}, \tag{4.9}$$

we get

$$\begin{aligned}&|\langle B(u(\theta),w(\theta)),q_m(\theta)\rangle|\leqslant\lambda\|u(\theta)\|^{\frac{1}{2}}\|\nabla u(\theta)\|^{\frac{1}{2}}\|w(\theta)\|^{\frac{1}{2}}\|\nabla w(\theta)\|^{\frac{1}{2}}\|\nabla q_m(\theta)\|\\ \leqslant&\lambda\|w(\theta)\|\|w(\theta)\|_{\widehat{V}}\|q_m(\theta)\|_{\widehat{V}}\leqslant c_2\lambda^2\|w(\theta)\|^2\|w(\theta)\|^2_{\widehat{V}}+\frac{1}{4c_2}\|q_m(\theta)\|^2_{\widehat{V}}.\end{aligned}$$

Then, it follows from (2.7) that

$$\begin{aligned}|\langle N(w(\theta)),q_m(\theta)\rangle|\leqslant\|N(w(\theta))\|\|q_m(\theta)\|&\leqslant c(\nu_r)\|w(\theta)\|_{\widehat{V}}\|q_m(\theta)\|_{\widehat{V}}\\ &\leqslant 2c_2c^2(\nu_r)\|w(\theta)\|^2_{\widehat{V}}+\frac{1}{8c_2}\|q_m(\theta)\|^2_{\widehat{V}}.\end{aligned}$$

Moreover, it is easy to get

$$\langle F(\theta),q_m(\theta)\rangle\leqslant\|F(\theta)\|_{\widehat{V}^*}\|q_m(\theta)\|_{\widehat{V}}\leqslant 2c_2\|F(\theta)\|^2_{\widehat{V}^*}+\frac{1}{8c_2}\|q_m(\theta)\|^2_{\widehat{V}}.$$

Finally, taking (2.3), (4.8) and the above three inequalities into account, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\|q_m(\theta)\|^2+\frac{1}{c_2}\|q_m(\theta)\|_{\widehat{V}}^2\leqslant\langle F(\theta),q_m(\theta)\rangle-\langle B(u(\theta),w(\theta)),q_m(\theta)\rangle-\langle N(w(\theta)),q_m(\theta)\rangle$$
$$\leqslant 2c_2\|F(\theta)\|_{\widehat{V}^*}^2+\frac{1}{8c_2}\|q_m(\theta)\|_{\widehat{V}}^2+c_2\lambda^2\|w(\theta)\|^2\|w(\theta)\|_{\widehat{V}}^2+\frac{1}{4c_2}\|q_m(\theta)\|_{\widehat{V}}^2$$
$$+2c_2c^2(\nu_r)\|w(\theta)\|_{\widehat{V}}^2+\frac{1}{8c_2}\|q_m(\theta)\|_{\widehat{V}}^2,$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\|q_m(\theta)\|^2+\frac{1}{c_2}\|q_m(\theta)\|_{\widehat{V}}^2\leqslant 4c_2\|F(\theta)\|_{\widehat{V}^*}^2+2c_2\lambda^2\|w(\theta)\|^2\|w(\theta)\|_{\widehat{V}}^2+4c_2c^2(\nu_r)\|w(\theta)\|_{\widehat{V}}^2.$$

which together with (4.7) leads to

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\|q_m(\theta)\|^2+\frac{\lambda_{m+1}}{c_2}\|q_m(\theta)\|^2\leqslant 4c_2\|F(\theta)\|_{\widehat{V}^*}^2+2c_2\lambda^2\|w(\theta)\|^2\|w(\theta)\|_{\widehat{V}}^2+4c_2c^2(\nu_r)\|w(\theta)\|_{\widehat{V}}^2.$$

Multiplying the above inequality by $e^{c_2^{-1}\lambda_{m+1}\theta}$ and integrating the resultant inequality over $[t-1,t]$, one has

$$e^{c_2^{-1}\lambda_{m+1}t}\|q_m(t)\|^2-e^{c_2^{-1}\lambda_{m+1}(t-1)}\|q_m(t-1)\|^2$$
$$\leqslant 4c_2\int_{t-1}^{t}e^{c_2^{-1}\lambda_{m+1}\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta+2c_2\lambda^2\int_{t-1}^{t}e^{c_2^{-1}\lambda_{m+1}\theta}\|w(\theta)\|^2\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta$$
$$+4c_2c^2(\nu_r)\int_{t-1}^{t}e^{c_2^{-1}\lambda_{m+1}\theta}\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta,$$

which together with (3.2) gives

$$\begin{aligned}\|q_m(t)\|^2\leqslant&e^{-c_2^{-1}\lambda_{m+1}}\|q_m(t-1)\|^2+4c_2c^2(\nu_r)e^{-c_2^{-1}\lambda_{m+1}t}\int_{t+1}^{t}e^{c_2^{-1}\lambda_{m+1}\theta}\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\&+2c_2\lambda^2e^{-c_2^{-1}\lambda_{m+1}t}\int_{t-1}^{t}e^{c_2^{-1}\lambda_{m+1}\theta}\|w(\theta)\|^2\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\&+4c_2e^{-c_2^{-1}\lambda_{m+1}t}\int_{t-1}^{t}e^{c_2^{-1}\lambda_{m+1}\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta\\\leqslant&e^{-c_2^{-1}\lambda_{m+1}}\|q_m(t-1)\|^2+c_3(1+\rho_1(t))e^{-c_2^{-1}\lambda_{m+1}t}\int_{t-1}^{t}e^{c_2^{-1}\lambda_{m+1}\theta}\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\&+4c_2e^{-c_2^{-1}\lambda_{m+1}t}\int_{t-1}^{t}e^{c_2^{-1}\lambda_{m+1}\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta,\end{aligned}\tag{4.10}$$

where $c_3:=\max\{4c_2c^2(\nu_r),2c_2\lambda^2\}$.

Next, we give a further estimate for (4.10). First, since

$$\|q_m(t-1)\|^2\leqslant\|w(t-1)\|^2\leqslant\rho_1(t),\ \text{and}\ \lambda_{m+1}\to\infty,\,\text{as}\,m\to\infty,$$

we conclude that there exists a $m_1=m_1(\epsilon,t,\widehat{D})$ such that, for any $m\geqslant m_1$,

$$e^{-c_2^{-1}\lambda_{m+1}}\|q_m(t-1)\|^2<\frac{\epsilon^2}{3}.\tag{4.11}$$

Then, for any $\delta\in(0,1)$, from (3.3), it holds that

$$\begin{aligned}
&e^{-c_2^{-1}\lambda_{m+1}t}\int_{t-1}^{t} e^{c_2^{-1}\lambda_{m+1}\theta}\|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta\\
=&e^{-c_2^{-1}\lambda_{m+1}t}\int_{t-1}^{t-\delta} e^{c_2^{-1}\lambda_{m+1}\theta}\|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta+e^{-c_2^{-1}\lambda_{m+1}t}\int_{t-\delta}^{t} e^{c_2^{-1}\lambda_{m+1}\theta}\|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta\\
\leqslant&e^{-c_2^{-1}\lambda_{m+1}\delta}\int_{t-1}^{t}\|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta+\int_{t-\delta}^{t}\|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta\\
\leqslant&e^{-c_2^{-1}\lambda_{m+1}\delta}\rho_2(t)+\int_{t-\delta}^{t}\|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta,
\end{aligned}$$

which combines with Lemma 4.3 implies that there exist $\delta^*\in(0,1)$ and $m_2=m_2(\epsilon,t,\widehat{D},\delta^*)$ such that, for any $m\geqslant m_2$,

$$\begin{aligned}
e^{-c_2^{-1}\lambda_{m+1}t}\int_{t-1}^{t} e^{c_2^{-1}\lambda_{m+1}\theta}\|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta\leqslant& e^{c_2^{-1}\lambda_{m+1}\delta^*}\rho_2(t)+\int_{t-\delta^*}^{t}\|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta\\
<&\frac{\epsilon^2}{3c_3(1+\rho_1(t))},\quad \forall\tau\leqslant\tau_0(\widehat{D},t),w_\tau\in D(\tau).
\end{aligned}\tag{4.12}$$

Finally, because of $\lambda_{m+1}\to\infty$ as $m\to\infty$, we can deduce from (4.6) that there exists an $m_3=m_3(\epsilon,t)$ such that, for any $m\geqslant m_3$,

$$e^{-c_2^{-1}\lambda_{m+1}t}\int_{t-1}^{t} e^{c_2^{-1}\lambda_{m+1}\theta}\|F(\theta)\|_{\widehat{V}^*}^2 \mathrm{d}\theta<\frac{\epsilon^2}{12c_2}.\tag{4.13}$$

Substituting (4.11)-(4.13) into (4.10) and taking $m:=\max\{m_1,m_2,m_3\}$, we get

$$\|q_m(t)\|^2<\epsilon^2,\ \text{for any}\ \tau\leqslant\tau_0(\widehat{D},t), w_\tau\in D(\tau),$$

which is property (ii). This completes the proof. □

At this stage, we can give the proof of the main result of this subsection.

Proof. (The proof of Theorem 1.3). According to Definition 1.2, the existence of $\widehat{\mathcal{A}}_{\mathcal{D}_F^{\widehat{H}}}$ and $\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}}$ is a consequence of Proposition 2.1, Remark 2.3, Lemma 3.2 and Lemma 4.4. Furthermore, from Lemma 3.2 and the fact $\mathcal{D}_F^{\widehat{H}}\subseteq\mathcal{D}^{\widehat{H}}$, we obtain $\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)\subseteq\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)$ for $\forall t\in\mathbb{R}$. Obtaining the desired result. □

4.2. Existence and regularity of the pullback attractors. The goal of this subsection is to give the proof of Theorem 1.4. For this, let us verify the pullback $\mathcal{D}^{\widehat{H},\widehat{V}}$- flattening property for the process in $\widehat{V}$, that is, the following lemma.

LEMMA 4.5. *Assume* (H2) *hold, then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *on* $\widehat{V}$ *satisfies the pullback* $\widehat{D}_{\widehat{V}}$*- flattening property for any* $\widehat{D}_{\widehat{V}}\in\mathcal{D}^{\widehat{H},\widehat{V}}$.

Proof. It suffices to show that for any $\epsilon>0, t\in\mathbb{R}$ and $\widehat{D}\in\mathcal{D}^{\widehat{H},\widehat{V}}$, there exists $m=m(t,\epsilon,\widehat{D})\in\mathbb{N}$ such that the projection $\mathrm{P_m}:\widehat{V}\to\widehat{V}_m$ with $\widehat{V}_m=\mathrm{span}\{v_1,v_2,\cdots,v_m\}$ ($\{v_n\}_{n\geqslant1}$ is given in Remark 2.1) satisfies the following properties:

(i) $\{\mathrm{P_m}U(t,\tau)D(\tau):\tau\leqslant\tau_0'(\widehat{D},t)\}$ is bounded in $\widehat{V}$,

(ii) $\|(I-\mathrm{P_m})U(t,\tau)w(\tau)\|_{\widehat{V}}<\epsilon$, for any $\tau\leqslant\tau_0'(\widehat{D},t), w_\tau\in D(\tau)$,
where $\tau_0'(\widehat{D},t)$ is given in Lemma 3.6.

The property (i) follows directly from (3.14) and the fact $\|\mathrm{P_m}w(t)\|_{\widehat{V}}\leqslant\|w(t)\|_{\widehat{V}}$.

To prove property (ii), let us fix $\tau\leqslant\tau_0'(\widehat{D},t), w_\tau\in D(\tau)$ and set $q_m(\theta)=w(\theta)-\mathrm{P_m}w(\theta)$. Similar to (4.8), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\langle q_m(\theta),Aq_m(\theta)\rangle+\|Aq_m(\theta)\|^2+\langle B(u(\theta),w(\theta)),Aq_m(\theta)\rangle \\ +\langle N(w(\theta)),Aq_m(\theta)\rangle=\langle F(\theta),Aq_m(\theta)\rangle. \tag{4.14}$$

Invoking (2.6) and (4.9), one has

$$\begin{aligned}\left|\langle B(u(\theta),w(\theta)),Aq_m(\theta)\rangle\right|&\leqslant\lambda\|u(\theta)\|^{\frac{1}{2}}\|\nabla u(\theta)\|^{\frac{1}{2}}\|\nabla w(\theta)\|^{\frac{1}{2}}\|Aw(\theta)\|^{\frac{1}{2}}\|Aq_m(\theta)\|\\&\leqslant\lambda\|w(\theta)\|^{\frac{1}{2}}\|w(\theta)\|_{\widehat{V}}\|Aw(\theta)\|^{\frac{1}{2}}\|Aq_m(\theta)\|\\&\leqslant\lambda^2\|w(\theta)\|\|w(\theta)\|_{\widehat{V}}^2\|Aw(\theta)\|+\frac{1}{4}\|Aq_m(\theta)\|^2.\end{aligned}$$

From (2.7) and Cauchy inequality, it holds that

$$\left|\langle N(w(\theta)),Aq_m(\theta)\rangle\right|\leqslant\|N(w(\theta))\|\|Aq_m(\theta)\|\leqslant 2c^2(\nu_r)\|w(\theta)\|_{\widehat{V}}^2+\frac{1}{8}\|Aq_m(\theta)\|^2.$$

Moreover, we have

$$\langle F(\theta),Aq_m(\theta)\rangle\leqslant 2\|F(\theta\|^2+\frac{1}{8}\|Aq_m(\theta)\|^2.$$

Substituting the above three inequalities into (4.14), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aq_m(\theta),q_m(\theta)\rangle+\frac{1}{2}\|Aq_m(\theta)\|^2\\&\leqslant\lambda^2\|w(\theta)\|\|w(\theta)\|_{\widehat{V}}^2\|Aw(\theta)\|+2c^2(\nu_r)\|w(\theta)\|_{\widehat{V}}^2+2\|F(\theta)\|^2,\end{aligned}$$

which together with (2.3) and the inequality $\lambda_{m+1}\|q_m(\theta)\|_{\widehat{V}}^2\leqslant\|Aq_m(\theta)\|^2$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\|q_m(\theta)\|_{\widehat{V}}^2+c_2\lambda_{m+1}\|q_m(\theta)\|_{\widehat{V}}^2\leqslant 2c_2\lambda^2\|w(\theta)\|\|w(\theta)\|_{\widehat{V}}^2\|Aw(\theta)\| \\ +4c_2c^2(\nu_r)\|w(\theta)\|_{\widehat{V}}^2+4c_2\|F(\theta)\|^2,$$

where λ_{m+1} is defined in Remark 2.1. Multiplying the above inequality by $e^{c_2\lambda_{m+1}\theta}$ and integrating the resultant inequality with respect to θ over $[t-1,t]$, then applying Lemma 3.6, we conclude

$$\begin{aligned}\|q_m(t)\|_{\widehat{V}}^2\leqslant{}& e^{-c_2\lambda_{m+1}}\|q_m(t-1)\|_{\widehat{V}}^2+2c_2\lambda^2e^{-c_2\lambda_{m+1}t}\int_{t-1}^t e^{c_2\lambda_{m+1}\theta}\|w(\theta)\|\|w(\theta)\|_{\widehat{V}}^2\|Aw(\theta)\|\mathrm{d}\theta\\&+4c_2c^2(\nu_r)e^{-c_2\lambda_{m+1}t}\int_{t-1}^t e^{c_2\lambda_{m+1}\theta}\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta+4c_2e^{-c_2\lambda_{m+1}t}\int_{t-1}^t e^{c_2\lambda_{m+1}\theta}\|F(\theta)\|^2\mathrm{d}\theta\\\leqslant{}& e^{-c_2\lambda_{m+1}}\|w_m(t-1)\|_{\widehat{V}}^2\\&+2c_2\lambda^2\rho_4^{\frac{1}{2}}(t)\rho_5(t)e^{-c_2\lambda_{m+1}t}\Big(\int_{t-1}^t e^{2c_2\lambda_{m+1}\theta}\mathrm{d}\theta\Big)^{\frac{1}{2}}\Big(\int_{t-1}^t\|Aw(\theta)\|^2\mathrm{d}\theta\Big)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&+4c_2c^2(\nu_r)\rho_5(t)e^{-c_2\lambda_{m+1}t}\int_{t-1}^{t}e^{c_2\lambda_{m+1}\theta}\mathrm{d}\theta+4c_2e^{-c_2\lambda_{m+1}t}\int_{t-1}^{t}e^{c_2\lambda_{m+1}\theta}\|F(\theta)\|^2\mathrm{d}\theta\\
\leqslant& e^{-c_2\lambda_{m+1}}\rho_5(t)+\Big(\frac{2c_2}{\lambda_{m+1}}\Big)^{\frac{1}{2}}\lambda^2\rho_4^{\frac{1}{2}}(t)\rho_5(t)\rho_6^{\frac{1}{2}}(t)+\frac{4c^2(\nu_r)\rho_5(t)}{\lambda_{m+1}}\\
&+4c_2e^{-c_2\lambda_{m+1}t}\int_{t-1}^{t}e^{c_2\lambda_{m+1}\theta}\|F(\theta)\|^2\mathrm{d}\theta.
\end{aligned}
\tag{4.15}
$$

On one hand, since $\lambda_{m+1}\to\infty$ as $m\to\infty$, it clear that there exists an $m_1'(\epsilon,t,\widehat{D})$ such that, for any $m\geqslant m_1'(\epsilon,t,\widehat{D})$, we have

$$
e^{-c_2\lambda_{m+1}}\rho_2(t)<\frac{\epsilon^2}{4},\ \Big(\frac{2c_2}{\lambda_{m+1}}\Big)^{\frac{1}{2}}\lambda^2\rho_4^{\frac{1}{2}}(t)\rho_5(t)\rho_6^{\frac{1}{2}}(t)<\frac{\epsilon^2}{4},\ \frac{4c^2(\nu_r)\rho_5(t)}{\lambda_{m+1}}<\frac{\epsilon^2}{4}.
$$

On the other hand, similar to (4.6), under the assumption (H2), we have

$$
\lim_{c\to\infty}e^{-ct}\int_{-\infty}^{t}e^{c\theta}\|F(\theta)\|^2\mathrm{d}\theta=0,\ \text{for any}\ t\in\mathbb{R},
$$

which implies there exists $m_2'(\epsilon,t,\widehat{D})$ such that

$$
4c_2e^{-c_2\lambda_{m+1}t}\int_{t-1}^{t}e^{c_2\lambda_{m+1}\theta}\|F(\theta)\|^2\mathrm{d}\theta<\frac{\epsilon^2}{4},\ \text{for any}\ m\geqslant m_2'(\epsilon,t,\widehat{D}).
$$

Choosing $m=\max\{m_1'(\epsilon,t,\widehat{D}),m_2'(\epsilon,t,\widehat{D})\}$, we obtain

$$
\|q_m(t)\|_{\widehat{V}}<\epsilon,\ \text{for any}\ \tau\leqslant\tau_0'(\widehat{D},t),w_\tau\in D(\tau),
$$

that is, the property (ii). □

Now, let us give the proof of Theorem 1.4.

Proof. (The proof of Theorem 1.4). According to Definition 1.2, the existences of $\widehat{\mathcal{A}}_{\mathcal{D}_F^{\widehat{V}}}$ and $\hat{\mathcal{A}}_{\mathcal{D}^{\widehat{H},\widehat{V}}}$ are direct consequences of Proposition 2.1, Remark 2.3, Lemma 3.7 and Lemma 4.5.

From the fact $\mathcal{D}_F^{\widehat{V}}\subseteq\mathcal{D}^{\widehat{H},\widehat{V}}\subseteq\mathcal{D}^{\widehat{H}}$, it follows that $\mathcal{A}_{\mathcal{D}_F^{\widehat{V}}}(t)\subseteq\mathcal{A}_{\mathcal{D}^{\widehat{H},\widehat{V}}}(t)\subseteq\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)$. Further, by Lemma 3.7, we can obtain $\mathcal{A}_{\mathcal{D}^{\widehat{H},\widehat{V}}}(t)=\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)$. In addition, the relation $\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)\subseteq\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)$ is a conclusion of Theorem 1.3, and evidently, $\mathcal{A}_{\mathcal{D}_F^{\widehat{V}}}(t)\subseteq\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)$. Therefore, (1.9) is valid, which implies (1.10).

Since the set $\bigcup_{t\leqslant T}D_{0,\widehat{V}}$ is a bounded set of $\widehat{V}$, then under the assumption (1.11), we can obtain that $\mathcal{A}_{\mathcal{D}_F^{\widehat{V}}}(t)=\mathcal{A}_{\mathcal{D}^{\widehat{H},\widehat{V}}}(t)$ (see [26]). Therefore, (1.12) and (1.13) are valid. This completes the proof. □

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