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GLOBAL DISSIPATIVE SOLUTIONS OF THE NOVIKOV EQUATION*

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Abstract. This paper is regarding the continuation of solutions to the Novikov equation beyond wave breaking. Our method is based on the characteristic of establishing new variables, then we transform the Novikov equation to a closed semilinear system on these new variables so that all singularities are resolved due to possible wave breaking. Returning to the original variables, we obtain a semigroup of global dissipative solutions, which depends continuously on the initial data. Note that the nonlinearity of the Novikov equation is higher than the Camassa-Holm equation; this requires us to seek the high-order energy density and another conservative law.

Keywords. Novikov equation; Global dissipative solutions.

AMS subject classifications. 60F10; 60J75; 62P10; 92C37.

1. Introduction

In this paper, we deal with the continuation of solutions for the Novikov equation

$$\begin{cases} u_t - u_{xxt} + 4u^2 u_x - 3u u_x u_{xx} - u^2 u_{xxx} = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R},\ t = 0. \end{cases} \tag{1.1}$$

Motivated by the elegant work for the Camassa-Holm equation in [2,3], we will give a dissipative solution where energy may vanish from the system.

The motivation to study the Novikov equation (1.1) is that it can be regarded as a generalization of the famous Camassa-Holm equation (CH) [5,16]:

$$m_t + c_0 u + u m_x + 2u_x m = 0. \tag{1.2}$$

The Camassa-Holm equation (1.2) was first implicitly contained in a bi-Hamiltonian generalization of the KdV equation by Fuchssteiner and Fokas [22], and later deduced as a model for unidirectional propagation of shallow water over a flat bottom by Camassa and Holm [5]. Analogous to the famous KdV equation, the CH equation also has a bi-Hamiltonian structure [5,22], and the CH equation is completely integrable not only in the sense of the existence of a Lax pair [5], but also (by means of inverse scattering and inverse spectral theory) as an infinite-dimensional Hamiltonian flow that can be linearised in suitable action-angle variables, (cf. [1,9,10,13,18,21]). The orbital stability of solitary waves and the stability of the peakons ($c_0 = 0$) for the CH equation were investigated by Constantin and Strauss [19,20]. The advantage of the CH equation in comparison with the KdV equation is that the CH equation models the special wave breaking phenomena, that is, the solution remains bounded but its slope becomes unbounded in finite time (cf. [6,15]). Moreover, the wave breaking is one of the most interesting aspects of the equation, namely, the travelling wave solutions of greatest height of the governing equations for water waves have a peak at their crest (cf. [11,12]).

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The Novikov equation (1.1) is nothing but an integrable Camassa-Holm-type equation with cubic nonlinearities deduced in [31]. Similar to the CH equation, the Novikov equation also possesses a Lax pair, a bi-Hamiltonian structure, an infinite sequences of conserved quantities, and peakon solutions, as well as the explicit formulae for multi-peakon solutions [27,28]. It is worth pointing out that the peakons are solitons and the characteristic singularity of greatest height and largest amplitude, which arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, cf. [8,14,17,33]. The above properties mean that the peakons can be considered as good approximations to exact solutions of the governing equations for water waves. The local well-posedness for the Novikov equation was investigated in [23, 32, 36]. Furthermore, the global existence of strong solutions was obtained in [35] under some sign conditions, and the blow-up phenomena of the strong solutions were studied in [37]. The global weak solutions for (1.1) were investigated in [30, 34].

Owing to the possible development of singularities in a finite time, many researchers study the behavior of a solution beyond the occurrence of wave breaking. Recently, these topics have been discussed for the Camassa-Holm equation in [2,3], by introducing a new set of independent and dependent variables. The conservative and dissipative solutions of the Camassa-Holm equation were also shown in [24–26], by introducing a coordinate transformation into Lagrangian coordinates. Moreover, in [4], the authors have shown that the conservative solution of the Camassa-Holm equation is unique. It is worth pointing out that the nonlinearity of the Camassa-Holm equation is quadratic, and here we can only use the $H^1(\mathbb{R})\times L^2(\mathbb{R})$-norm conserved quantity to study the conservative and dissipative solutions of the CH equation.

Very recently, Chen et.al. in [7] proved the global existence and uniqueness of conservative solutions. However, the global dissipative solutions to the Novikov equation have not been investigated. We know that the dissipative case is more delicate, because the corresponding O.D.E. now includes a discontinuous non-local source term. Inspired by the dissipative solutions for the Camassa-Holm equation in [3], in the present paper, we will first construct a continuous semigroup of dissipative solutions forward in time by introducing a new set of independent and dependent variables. Then reverting to the original coordinates, we obtain a global dissipative solution of the Novikov equation. Nevertheless, the Novikov equation involves a higher order term u_x^3 in the convolution Q(defined in (2.2)), which cannot be controlled by the conservation law $\int_{\mathbb{R}}\left(u^2+u_x^2\right)dx$. It inspired us to seek another conservation law $\int_{\mathbb{R}}\left(u^4+2u^2u_x^2-\frac{1}{3}u_x^4\right)dx$ and the higher-order energy density $(1+u_x^2)^2$(for the CH it is only $1+u_x^2$). What we show here is that: Our constructive procedure (via coordinate transforations) obtains a unique semigroup of solutions, defined on the entire space $H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})$. The solution $u=u(t,x)$ satisfies the Oleinik-type inequality

$$u_x(t,x)\leq C(1+t^{-1}),\ t>0, \tag{1.3}$$

where the constant C depends only on the norm of the initial data $||\tilde{u}||_{H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})}$.

This paper is organized as follows. In Section 2, we introduce a new set of independent and dependent variables. In view of these new variables, we transfer the Novikov equation (2.1) to the semilinear system (3.5) in Section 3, and then we prove the global existence of solutions to the semilinear system. In Section 4, reverting to the original coordinates we prove the global dissipative solutions to the equation (2.1). In Section 5, a global continuous semigroup of weak dissipative solutions to equation (2.1) is constructed.

2. Preliminaries

2.1. The basic equations. As usual, we can rewrite equation (1.1) as follows:

$$u_t = -u^2 u_x - P_x - Q, \tag{2.1}$$

with the source terms P and Q being defined as a convolution

$$\begin{cases} P \doteq (1-\partial_x^2)^{-1}\left(u^3 + \frac{3}{2}u(\partial_x u)^2\right), \\ Q \doteq \frac{1}{2}(1-\partial_x^2)^{-1}(\partial_x u)^3. \end{cases} \tag{2.2}$$

The initial data is given as

$$u(0,x) = \tilde{u}(x) \in H^1(\mathbb{R}) \cap W^{1,4}(\mathbb{R}). \tag{2.3}$$

The Novikov equation (2.1) has the following two useful conservation laws [7]:

$$E(t) := \int_{\mathbb{R}} (u^2 + u_x^2)(t,x)dx = E(0) \doteq E_0, \tag{2.4}$$

$$G(t) := \int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right)(t,x)dx = G(0) \doteq G_0. \tag{2.5}$$

From the above two conservation laws and the Sobolev inequality $||u||_{L^\infty} \le ||u||_{H^1}^2 = E_0$, we deduce that

$$\begin{aligned} ||u_x||_{L^4}^4 &= 3\int_{\mathbb{R}} (u^4 + 2u^2u_x^2)dx - 3G(t) \\ &\le 3\left(||u||_{L^\infty} \int_{\mathbb{R}} (u^2 + 2u_x^2)dx - G(t)\right) \le 6E_0^2 - 3G_0. \end{aligned}$$

Therefore,

$$||u_x||_{L^3}^3 \le ||u_x||_{L^2} ||u_x||_{L^4}^2 \le \sqrt{3E(0)[2E_0^2 - G_0]} =: K.$$

This estimate and Young's inequality for convolutions guarantee that

$$\begin{aligned} ||P(t)||_{L^\infty},\ ||P_x(t)||_{L^\infty} &\le \left\|\frac{1}{2}e^{-|x|}\right\|_{L^\infty} \left\|\frac{3}{2}uu_x^2 + u^3\right\|_{L^1} = \frac{3}{4}E_0^{3/2}, \\ ||P(t)||_{L^2},\ ||P_x(t)||_{L^2} &\le \left\|\frac{1}{2}e^{-|x|}\right\|_{L^2} \left\|\frac{3}{2}uu_x^2 + u^3\right\|_{L^1} = \frac{3\sqrt{2}}{4}E_0^{3/2}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} ||Q(t)||_{L^\infty},\ ||Q_x(t)||_{L^\infty} &\le \frac{1}{2}\left\|\frac{1}{2}e^{-|x|}\right\|_{L^\infty} \|u_x^3\|_{L^1} = \frac{1}{4}K, \\ ||Q(t)||_{L^2},\ ||Q_x(t)||_{L^2} &\le \frac{1}{2}\left\|\frac{1}{2}e^{-|x|}\right\|_{L^2} \|u_x^3\|_{L^1} = \frac{2\sqrt{2}}{4}K. \end{aligned} \tag{2.7}$$

For such solutions, the energies (2.4) and (2.5) remain constant on $[0,T)$, while we know that the solution of the Novikov equation (2.1) blows up in finite time if and only if (cf. [29, 38])

$$\lim_{t\to T^-} \inf_{x\in\mathbb{R}} \{uu_x(t,x)\} = -\infty.$$

Chen et.al., in [7], proved that the solution can be continued after the breaking time by requiring that the energy remains constant for a.e. $t \geq 0$. In the present paper, we deal with the dissipative solutions for equation (1.1), where wave breaking might induce a partial or even total loss of energy.

Definition 2.1. *By a solution of the Cauchy problem (2.1)-(2.3) on $[t_1,t_2]$, we mean a Hölder continuous function $u(t,x)$ defined on $[t_1,t_2]\times\mathbb{R}$, satisfying the following properties:*

(1) At each fixed t, we have $u(\cdot,t)\in H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})$.

(2) the map $u(\cdot,t)$ is Lipschitiz continuous from $[t_1,t_2]$ to L^2, satisfying equality (2.1) with initial condition (2.3).

Now, we give the definition of dissipative solutions to the Novikov equation (2.1).

Definition 2.2. *A solution of the Cauchy problem (2.1)-(2.3) is called a dissipative solution if it satisfies the inequality (1.3) for some constant C, and moreover the energy $E(t)$ in (2.4) is a nonincreasing function of time.*

2.2. A new set of independent and dependent variables. In order to change the equation into a semilinear hyperbolic system, we now introduce a new set of independent and dependent variables. This was first used in [2,3] to establish global conservative and dissipative solutions for the Camassa-Holm equation. Here, in order to deal with the cubic term, an appropriate modification of energy density will yield the global dissipative solutions.

Given the initial data $\tilde{u}=u_0(x)\in H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})$, consider the energy variable $\omega\in\mathbb{R}$, and let the nondecreasing map $\omega\mapsto\tilde{y}(\omega)$ be defined as follows

$$\int_0^{\tilde{y}(\omega)}\left(1+\tilde{u}_x^2\right)^2 dx=\omega. \tag{2.8}$$

Supposing that the solution u to equation (2.1) is still Lipschitz continuous for $t\in[0,T]$, we now deduce an equivalent system of equations by using the independent variables (t,ω). Let $t\mapsto y(t,\omega)$ be the characteristic beginning at $\tilde{y}(\omega)$, so that

$$\frac{\partial}{\partial t}y(t,\omega)=u^2(t,y(t,\omega)),\quad y(0,\omega)=\tilde{y}(\omega). \tag{2.9}$$

Moreover, we use the following notions

$$u(t,\omega)\doteq u(t,y(t,\omega)),\ u_x(t,\omega)\doteq u_y(t,y(t,\omega)),$$

$$P(t,\omega)\doteq P(t,y(t,\omega)),\ P_x(t,\omega)\doteq P_y(t,y(t,\omega)),$$

$$Q(t,\omega)\doteq Q(t,y(t,\omega)),\ Q_x(t,\omega)\doteq Q_y(t,y(t,\omega)),$$

and define the variables: $v=v(t,\omega)$ and $q=q(t,\omega)$ as

$$v\doteq 2\arctan u_x,\quad q\doteq(1+u_x^2)^2\cdot\frac{\partial y}{\partial\omega}. \tag{2.10}$$

Owing to v being defined up to multiples of 2π, all subsequent equations involving v are invariant under addition of multiples of 2π. From (2.8) and (2.10), we deduce the following identities,

$$q(0,\omega)\equiv 1, \tag{2.11}$$

$$\frac{1}{1+u_x^2}=\cos^2\frac{v}{2},\ \ \frac{u_x}{1+u_x^2}=\frac{1}{2}\sin v,\ \ \frac{u_x^2}{1+u_x^2}=\sin^2\frac{v}{2}, \tag{2.12}$$

$$\frac{\partial y}{\partial \omega}=\frac{q}{(1+u_x^2)^2}=\cos^4\frac{v}{2}\cdot q. \tag{2.13}$$

Equation (2.13) yields

$$y(t,\theta)-y(t,\omega)=\int_\omega^\theta \cos^4\frac{v(t,s)}{2}\cdot q(t,s)ds. \tag{2.14}$$

Moreover, we have

$$\begin{aligned}
P(t,\omega)=&\frac{1}{2}\int_{-\infty}^{\infty}\exp\{-|y(t,\omega)-x|\}\left(u^3+\frac{3}{2}u(u_x)^2\right)dx,\\
Q(t,\omega)=&\frac{1}{4}\int_{-\infty}^{\infty}\exp\{-|y(t,\omega)-x|\}(u_x)^3dx,\\
P_x(t,\omega)=&\frac{1}{2}(\int_{y(t,\omega)}^{\infty}-\int_{-\infty}^{y(t,\omega)})\exp\{-|y(t,\omega)-x|\}\left(u^3+\frac{3}{2}u(u_x)^2\right)dx,\\
Q_x(t,\omega)=&\frac{1}{4}(\int_{y(t,\omega)}^{\infty}-\int_{-\infty}^{y(t,\omega)})\exp\{-|y(t,\omega)-x|\}(u_x)^3dx.
\end{aligned}$$

In the above equalities, we can implement the change of variables $x=y(t,\theta)$, and rewrite the convolutions as an integral over the variable θ. Using the identities (2.12)-(2.14), thus, we get an expression for P,Q and P_x,Q_x in terms of the new variable ω, that is,

$$\begin{aligned}
P(\omega)=&\frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-\left|\int_\omega^\theta\left(q\cdot\cos^4\frac{v}{2}\right)(s)ds\right|\right\}\\
&\cdot\left[q\left(u^3\cos^4\frac{v}{2}+\frac{3}{2}u\sin^2\frac{v}{2}\cos^2\frac{v}{2}\right)\right](\theta)d\theta,
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
P_x(t,\omega)=&\frac{1}{2}\left(\int_\omega^{\infty}-\int_{-\infty}^{\omega}\right)\exp\left\{-\left|\int_\omega^\theta\left(q\cdot\cos^4\frac{v}{2}\right)(s)ds\right|\right\}\\
&\cdot\left[q\left(u^3\cos^4\frac{v}{2}+\frac{3}{2}u\sin^2\frac{v}{2}\cos^2\frac{v}{2}\right)\right](\theta)d\theta,
\end{aligned} \tag{2.16}$$

$$Q(\omega)=\frac{1}{8}\int_{-\infty}^{\infty}\exp\left\{-\left|\int_\omega^\theta\left(q\cdot\cos^4\frac{v}{2}\right)(s)ds\right|\right\}\cdot\left(q\sin v\sin^2\frac{v}{2}\right)(\theta)d\theta, \tag{2.17}$$

$$Q_x(t,\omega)=\frac{1}{8}\left(\int_\omega^{\infty}-\int_{-\infty}^{\omega}\right)\exp\left\{-\left|\int_\omega^\theta\left(q\cdot\cos^4\frac{v}{2}\right)(s)ds\right|\right\}\cdot\left(q\sin v\sin^2\frac{v}{2}\right)(\theta)d\theta. \tag{2.18}$$

In view of (2.1) and (2.9), the evolution equation for u in the new variables (t,ω) has the form

$$\frac{\partial}{\partial t}u(t,\omega)=u_t+u_yy_t=u_t+u^2u_x=-P_x(t,\omega)-Q(t,\omega), \tag{2.19}$$

where P_x and Q are defined in (2.16) and (2.17), respectively. Next, we deduce an evolution equation for the variable q from (2.10)

$$\int_{\omega_1}^{\omega_2} q(t,\omega)d\omega = \int_{y(t,\omega_1)}^{y(t,\omega_2)} \left(1+u_x^2(t,x)\right)^2 dx.$$

Equation (2.9) and $P_{xx} = P - u^3 - \frac{3}{2}uu_x^2$ yield

$$\begin{aligned}\frac{d}{dt}\int_{\omega_1}^{\omega_2} q(t,\omega)d\omega &= \int_{y(t,\omega_1)}^{y(t,\omega_2)} \{[(1+u_x^2)^2]_t + [u^2(1+u_x^2)^2]_x\}dx \\ &= \int_{y(t,\omega_1)}^{y(t,\omega_2)} 4(1+u_x^2)u_x\left(u_{xt}+\frac{1}{2}uu_x^2+u^2u_{xx}+\frac{1}{2}u\right)dx \\ &= \int_{y(t,\omega_1)}^{y(t,\omega_2)} 4(1+u_x^2)u_x\left(\frac{1}{2}u+u^3-P-Q_x\right)dx.\end{aligned}$$

Differentiating the above equality with respect to ω, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}q(t,\omega) &= \left(\frac{1}{2}u+u^3-P-Q_x\right)\frac{4u_x}{1+u_x^2}\cdot q \\ &= \left(u+2u^3-2P-2Q_x\right)\sin v\cdot q. \end{aligned} \tag{2.20}$$

Finally, using (2.9) and (2.10), we get

$$\begin{aligned}\frac{\partial}{\partial t}v(t,\omega) &= \frac{2}{1+u_x^2}(u_{xt}+u^2u_{xx}) \\ &= \frac{2}{1+u_x^2}\left(-\frac{1}{2}u(u_x)^2+u^3-P-Q_x\right) \\ &= 2(u^3-P-Q_x)\cos^2\frac{v}{2} - u\sin^2\frac{v}{2}. \end{aligned} \tag{2.21}$$

In (2.20)and (2.21), the functions P, P_x, Q and Q_x are given by (2.15)-(2.18).

3. Global existence of solutions to the semilinear system

In order to obtain the global dissipative solutions of equations (2.1)-(2.3), it is important to modify them suitably. Assume that along a given characteristic $t \mapsto y(t,\omega)$, the wave breaks at the first time $t = \tau(\omega)$. Recalling our rescaled variable $v = 2\arctan u_x$, this means that $u_x(t,\omega) \to \pm\infty$, as $t \uparrow \tau(\omega)$ by setting $v(t,\omega) \equiv \pm\pi$ for all $t \geq \tau(\omega)$. Then, P, P_x, Q and Q_x in (2.15)-(2.18) can be replaced by

$$\begin{aligned}P(\omega) = \frac{1}{2}\int_{\{-\pi<v(\theta)<\pi\}} &\exp\left\{-\left|\int_{\{s\in[\omega,\theta],-\pi<v(s)<\pi\}} \cos^4\frac{v(s)}{2}\cdot q(s)ds\right|\right\} \\ &\cdot\left[u^3(\theta)\cos^4\frac{v(\theta)}{2}+\frac{3}{2}u(\theta)\sin^2\frac{v(\theta)}{2}\cos^2\frac{v(\theta)}{2}\right]q(\theta)d\theta, \end{aligned} \tag{3.1}$$

$$\begin{aligned}P_x(\omega) = \frac{1}{2}&\left(\int_{\{\theta>\omega,\pi<v(\theta)<\pi\}} - \int_{\{\theta<\omega,-\pi<v(\theta)<\pi\}}\right) \\ &\exp\left\{-\left|\int_{\{s\in[\omega,\theta],-\pi<v(s)<\pi\}} \cos^4\frac{v(s)}{2}\cdot q(s)ds\right|\right\}\end{aligned}$$

$$\cdot\left[u^3(\theta)\cos^4\frac{v(\theta)}{2}+\frac{3}{2}u(\theta)\sin^2\frac{v(\theta)}{2}\cos^2\frac{v(\theta)}{2}\right]q(\theta)d\theta, \tag{3.2}$$

and

$$Q(\omega)=\frac{1}{8}\int_{\{-\pi<v(\theta)<\pi\}}\exp\left\{-\left|\int_{\{s\in[\omega,\theta],-\pi<v(s)<\pi\}}\cos^4\frac{v(s)}{2}\cdot q(s)ds\right|\right\}$$
$$\cdot\left(\sin v(\theta)\sin^2\frac{v(\theta)}{2}\right)q(\theta)d\theta, \tag{3.3}$$

$$Q_x(\omega)=\frac{1}{8}\left(\int_{\{\theta>\omega,-\pi<v(\theta)<\pi\}}-\int_{\{\theta<\omega,-\pi<v(\theta)<\pi\}}\right)$$
$$\exp\left\{-\left|\int_{\{s\in[\omega,\theta],-\pi<v(s)<\pi\}}\cos^4\frac{v(s)}{2}\cdot q(s)ds\right|\right\}\cdot\left(\sin v(\theta)\sin^2\frac{v(\theta)}{2}\right)q(\theta)d\theta. \tag{3.4}$$

Therefore, equations (2.19)-(2.21) are converted into the following form:

$$\begin{cases}\frac{\partial u}{\partial t}=-P_x-Q,\\ \frac{\partial v}{\partial t}=\begin{cases}2(u^3-P-Q_x)\cos^2\frac{v}{2}-u\sin^2\frac{v}{2}, & -\pi<v<\pi,\\ 0, & \text{else},\end{cases}\\ \frac{\partial q}{\partial t}=\begin{cases}(u+2u^3-2P-2Q_x)\sin v\cdot q, & -\pi<v<\pi,\\ 0, & \text{else},\end{cases}\end{cases} \tag{3.5}$$

with the initial condition

$$\begin{cases}u(0,\omega)=\tilde{u}(\tilde{y}(\omega))\\ v(0,\omega)=2\arctan\tilde{u}_x(\tilde{y}(\omega))\\ q(0,\omega)=1.\end{cases} \tag{3.6}$$

System (3.5) can be regarded as an O.D.E. in a Banach space. It is easy to see that the right-hand side of the system (3.5) is discontinuous. The discontinuity appears precisely when $v=\pm\pi$. We observe that v is close to the values $\pm\pi$ transversally, i.e., with derivative $v_t=-u$ by the second equation in (3.5). It is exactly the transversality condition that guarantees the well-posedness of the system (3.5).
Consider here the Cauchy problem for system (3.5) in a more convenient form as

$$\frac{\partial}{\partial t}U(t,\omega)=F(U(t,\omega))+H(\omega,U(t,\cdot))\quad \omega\in\mathbb{R}, \tag{3.7}$$

$$U(0,\omega)=\tilde{U}(\omega), \tag{3.8}$$

where $U=(u,v,q)\in\mathbb{R}^3$, while

$$F(\omega,U(t))=\begin{cases}\left(0,\ u^3(1+\cos v)-u\sin^2\frac{v}{2},\ (u+2u^3)\sin v\cdot q\right), & -\pi<v<\pi,\\ (0,\quad -u,\quad 0), & \text{else}\end{cases} \tag{3.9}$$

and

$$H(\omega,U(t))=\begin{cases}-\left(P_x+Q,\ (P+Q_x)(1+\cos v),\ 2(P+Q_x)\sin v\cdot q\right), & -\pi<v<\pi,\\ (-P_x-Q,\quad 0,\quad 0), & \text{else}.\end{cases} \tag{3.10}$$

The nonlocal operators P,P_x,Q and Q_x were given in (3.1)-(3.4).

We note that if a solution of (3.7)-(3.10) is obtained, the mapping

$$(t,\omega)\to(u(t,\omega),v(t,\omega),q(t,\omega)) \tag{3.11}$$

provides a solution of system (3.5). Notice that the vector field $F:\mathbb{R}^3\to\mathbb{R}^3$ in (3.7) is uniformly bounded and Lipschitz continuous when u is still in a bounded set. Nevertheless, the nonlocal operator H is discontinuous. In fact, at a time τ^* such that $\text{meas}(\{\omega;v(\tau^*,\omega)=\pm\pi\})>0$, the set $\{\omega;\tau(\omega)>t\}=\{\omega;-\pi<v(\omega)<\pi\}$ may suddenly shrink. Therefore, the integral terms P,P_x,Q and Q_x in (3.1)-(3.4) are discontinuous.

To begin with, we obtain the local existence and uniqueness of the solution to the Cauchy problem (2.1). Furthermore, we claim that this local solution can be extended globally in time. The main theorem of this section is stated as follows.

Theorem 3.1. *Given $\tilde{u}\in H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})$, the Cauchy problem (3.7)-(3.10) with initial data $\tilde{U}\doteq(\tilde{u},2\arctan\tilde{u}_x,1)$ has a unique solution in the Banach space $C([0,T],L^\infty(\mathbb{R};\mathbb{R}^3))$ for $T>0$.*

Proof.

Step 1. Establishing the local existence of solutions. Firstly, we get some priori estimates on F and H in (3.7). Suppose that $U=(u,v,q)\in L^\infty(\mathbb{R};\mathbb{R}^3)$ satisfies the following inequalities

$$\|u(w)\|_{L^\infty}\le C,\quad \frac{1}{C}\le q(\omega)\le C,\quad \text{for all } \omega, \tag{3.12}$$

$$\text{meas}(\{\omega;-\pi<v(\omega)<\pi,|v(\omega)|\ge\frac{\pi}{2}\})\le C, \tag{3.13}$$

for some constant C. Therefore, there is a constant r^* depending only on C such that

$$||Q||_{L^\infty(\mathbb{R})}+||Q_x||_{L^\infty(\mathbb{R})}+||P||_{L^\infty(\mathbb{R})}+||P_x||_{L^\infty(\mathbb{R})}\le r^*,$$
$$||F(U)||_{L^\infty(\mathbb{R})}\le r^*,\quad ||H(U)||_{L^\infty(\mathbb{R})}\le r^*.$$

Furthermore, there exists a Lipschitz constant r such that if $U^*=(u^*,v^*,q^*)$ satisfies the same estimates (3.12) and (3.13), then

$$||F(U)-F(U^*)||_{L^\infty(\mathbb{R})}\le r||U-U^*||_{L^\infty(\mathbb{R})}.$$

Assume the initial data $\tilde{u}\in H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})$ to be given. Since $\tilde{u},\tilde{u}_x\in L^2(\mathbb{R})\cap L^4(\mathbb{R})$, the set

$$\{x\in\mathbb{R};|\tilde{u}_x|\ge\epsilon\}$$

has finite measure for every $\epsilon>0$. Then we can find a constant $C>0$ such that

$$||\tilde{u}||_{L^\infty(\mathbb{R})}\le\frac{C}{2},\quad \text{meas}\left(\left\{\omega;|\tilde{v}(\omega)|\ge\frac{\pi}{4}\right\}\right)\le\frac{C}{2}.$$

Choose $\delta>0$ small enough such that $v(\omega)\in(-\pi,-\pi+\delta]$ (or, $v(\omega)\in[\pi-\delta,\pi)$), which means $u_x<0$ (respectively, $u_x>0$). The precise blow-up scenario and the hypothesis (3.12) imply that $0<u(w)\le C$ (respectively, $-C\le u(w)<0$). Furthermore, there exists a constant $M>0$ such that

$$\frac{\partial}{\partial t}v(t,\omega)=(u^3-P-Q_x)(1+\cos v)-u\sin^2\frac{v}{2}\le-\frac{1}{M}.(\text{Respectively, }\frac{\partial}{\partial t}v(t,\omega)\ge\frac{1}{M}). \tag{3.14}$$

Otherwise, $\frac{\partial}{\partial t}v\nearrow 0$ (respectively, $\frac{\partial}{\partial t}v\searrow 0$), which is a contradiction with $v\searrow -\pi$(respectively, $v\nearrow\pi$).

Define the sets

$$\Gamma^\delta \doteq \{\omega\in\mathbb{R};\tilde{v}(\omega)\in(-\pi,\delta-\pi]\},\quad \Gamma'\doteq\mathbb{R}\setminus\Gamma^\delta.$$

By possibly reducing the size of $\delta>0$, we can suppose that

$$\mathrm{meas}(\Gamma^\delta)\le\frac{1}{8rM}.$$

On a suitable domain $\mathcal{D}\subset\mathcal{C}([0,T],L^\infty(\mathbb{R}))$, we will get the solution $t\mapsto U(t)$ as the unique fixed point of a contractive transformation $\mathcal{J}:\mathcal{D}\mapsto\mathcal{D}$. Concretely, for a given $T>0$, we define the domain $\mathcal{D}$ as the set of continuous mappings $t\mapsto U(t)=(u(t),v(t),q(t))$ from $[0,T]$ into $L^\infty(\mathbb{R},\mathbb{R}^3)$ satisfying the following properties,

$$U(0)=\tilde{U},$$

$$||U(t)-U(s)||_{L^\infty(\mathbb{R})}\le 2r^*|t-s|,$$

$$v(t,\omega)-v(s,\omega)\le-\frac{1}{M}(t-s),\quad \omega\in\Gamma^\delta,0\le s<t\le T.$$

Here the operator $\mathcal{J}$ is defined as

$$(\mathcal{J}(U))(t,\omega)=\tilde{U}+\int_0^t[F(U(\tau,\omega))+H(\omega,U(\tau,\cdot))]d\tau.$$

Next, we will show that if we choose $T>0$ small enough, then the mapping $\mathcal{J}$ is contractive. It is clear that $\mathcal{J}$ maps the domain $\mathcal{D}$ into itself. It remains to verify that $\mathcal{J}$ is a strict contraction. Indeed, suppose that $U,U^*\in\mathcal{D}$ and define

$$\eta=\max_{t\in[0,T]}||U(t)-U^*(t)||_{L^\infty(\mathbb{R})},\tag{3.15}$$

and the crossing time

$$\tau(\omega)=\sup\{t\in[0,T];v(t,\omega)>-\pi\}.\text{(Respectively, }\tau(\omega)=\sup\{t\in[0,T];v(t,\omega)<\pi\}.$$

The $\tau^*(\omega)$ is defined in the same way. The definitions of $\tau(\omega)$ and $\tau^*(\omega)$ imply that $||v(\tau(\omega))-v^*(\tau^*(\omega))||\le\eta$. We claim that for each $\omega\in\Gamma^\delta$,

$$|\tau(\omega)-\tau^*(\omega)|\le 2\eta M.$$

Without loss of generality, we set $\tau(\omega)\ge\tau^*(\omega)$, (3.14) and (3.15) give

$$\begin{aligned}|\tau(\omega)-\tau^*(\omega)|&=\tau(\omega)-\tau^*(\omega)\le M(v(\tau^*(\omega))-v(\tau(\omega)))\\&\le M(|v(\tau(\omega))-v^*(\tau^*(\omega))|+|v(\tau^*(\omega))-v^*(\tau^*(\omega))|)\\&\le 2\eta M.\end{aligned}$$

For $t\in[0,T]$ we then have

$$\begin{aligned}&||\mathcal{J}U(t)-\mathcal{J}U^*(t)||_{L^\infty(\mathbb{R})}\\&\le\int_0^t||F(U(\tau))-F(U^*(\tau))||_{L^\infty(\mathbb{R})}d\tau+\int_0^t||H(U(\tau))-H(U^*(\tau))||_{L^\infty(\mathbb{R})}d\tau\end{aligned}$$

$$\leq 2r\int_0^t ||U(\tau)-U^*(\tau)||_{L^\infty(\mathbb{R})}d\tau + r\int_0^t \text{meas}(\{\omega; v(\tau,\omega)>-\pi, v^*(\tau,\omega)\leq -\pi\})d\tau$$
$$+r\int_0^t \text{meas}(\{\omega; v(\tau,\omega)\leq -\pi, v^*(\tau,\omega)>-\pi\})d\tau \leq 2rT\eta + r\int_{\Gamma^\delta} 2|\tau(\omega)-\tau^*(\omega)|d\omega \leq \frac{1}{2}\eta,$$

provided that T is small enough. This yields that $\mathcal{J}$ is a strict contraction when $u(w)>0$ for $v(w)\in(-\pi,\delta-\pi]$. Similarly, when $u(w)<0$ for $v(w)\in[\pi-\delta,\pi)$, we can obtain that $\mathcal{J}$ is also a strict contraction. The desired local solution of the Cauchy problem (3.7)-(3.10) follows from the fact that it has a unique fixed point.

Step 2. Extending the local solutions of the semilinear system (3.5) globally in time. The basic component is a global bound on the total energy,

$$E(t)=\int_{\{-\pi<v(t,\omega)<\pi\}} \left(u^2(t,\omega)\cos^4\frac{v(t,\omega)}{2}+\sin^2\frac{v(t,\omega)}{2}\cos^2\frac{v(t,\omega)}{2}\right)q(t,\omega)d\omega \quad (3.16)$$

and

$$G(t)=\int_{\{-\pi<v(t,\omega)<\pi\}} \left[\left(u^4\cos^4\frac{v}{2}+2u^2\sin^2\frac{v}{2}\cos^2\frac{v}{2}-\frac{1}{3}\sin^4\frac{v}{2}\right)q\right](t,\omega)d\omega. \quad (3.17)$$

We start with presenting the fact that

$$u_\omega = \frac{q}{2}\sin v\cos^2\frac{v}{2}, \quad (3.18)$$

provided the local solution of (3.5) is defined. Indeed, the first equation of (3.5) and the definitions of P_x and Q at (3.2) and (3.3) give

$$u_{\omega t}=\begin{cases}\left(u^3\cos^4\frac{v}{2}+\frac{3}{2}u\sin^2\frac{v}{2}\cos^2\frac{v}{2}-P\cos^4\frac{v}{2}-Q_x\cos^4\frac{v}{2}\right)q, & \text{if } -\pi<v<\pi,\\ 0, & \text{else.}\end{cases}$$

On the other hand, the last two equations in (3.5) imply

$$\begin{aligned}\left(\frac{q}{2}\sin v\cos^2\frac{v}{2}\right)_t &= \frac{q_t}{2}\sin v\cos^2\frac{v}{2}+\frac{q}{2}v_t\cos v\cos^2\frac{v}{2}-\frac{q}{2}v_t\sin v\cos\frac{v}{2}\sin\frac{v}{2}\\ &= \left(u^3\cos^4\frac{v}{2}+\frac{3}{2}u\sin^2\frac{v}{2}\cos^2\frac{v}{2}-P\cos^4\frac{v}{2}-Q_x\cos^4\frac{v}{2}\right)q,\end{aligned}$$

for $-\pi<v<\pi$. While, if $v=\pm\pi$, the last two equations in (3.5) reflect

$$\left(\frac{q}{2}\sin v\cos^2\frac{v}{2}\right)_t=0.$$

Since $\tilde{q}=1$, at the initial time $t=0$, we have

$$\frac{\partial u}{\partial\omega}=\tilde{u}_x\cdot\frac{\partial\tilde{y}}{\partial\omega}=\frac{\tilde{u}_x}{(1+\tilde{u}_x^2)^2}=\frac{\sin v}{2}\cos^2\frac{v}{2}=\frac{q}{2}\sin v\cos^2\frac{v}{2}.$$

Thus, we obtain that (3.18) remains valid for all times $t\geq 0$, provided the solution is defined.

In the following, we prove that the extended energies

$$\tilde{E}(t)=\int_{\mathbb{R}}\left(u^2(t,\omega)\cos^4\frac{v(t,\omega)}{2}+\sin^2\frac{v(t,\omega)}{2}\cos^2\frac{v(t,\omega)}{2}\right)q(t,\omega)d\omega \quad (3.19)$$

and

$$\tilde{G}(t)=\int_{\mathbb{R}}\left[\left(u^4\cos^4\frac{v}{2}+2u^2\sin^2\frac{v}{2}\cos^2\frac{v}{2}-\frac{1}{3}\sin^4\frac{v}{2}\right)q\right](t,\omega)d\omega \tag{3.20}$$

remain constant in time. From (3.5), we obtain that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left(u^2\cos^4\frac{v}{2}+\sin^2\frac{v}{2}\cos^2\frac{v}{2}\right)qd\omega\\&=\int_{\mathbb{R}}\Big[-2u\cos^4\frac{v}{2}(P_x+Q)-\big(2u^2\cos^3\frac{v}{2}\sin\frac{v}{2}-\frac{1}{2}\cos v\sin v\big)\\&\qquad\big(-u\sin^2\frac{v}{2}+2u^3\cos^2\frac{v}{2}-2\cos^2\frac{v}{2}(P+Q_x)\big)\\&\qquad\qquad+\sin v\big(u^2\cos^4\frac{v}{2}+\frac{1}{4}\sin^2 v\big)\big(2u^3+u-2P-2Q_x\big)\Big]qd\omega.\end{aligned} \tag{3.21}$$

On the other hand, the definitions of P,P_x,Q and Q_x in (3.1)-(3.4) yield

$$P_\omega=qP_x\cos^4\frac{v}{2},\ \ Q_=qQ_x\cos^4\frac{v}{2},$$

$$(P_x)_\omega=-(\frac{3}{8}u\sin^2 v+u^3\cos^4\frac{v}{2}-\cos^4\frac{v}{2}P)q,$$

$$(Q_x)_\omega=-\frac{1}{4}\cos^2\frac{v}{2}\sin vq+\cos^4\frac{v}{2}Qq.$$

Notice that on the right-hand of (3.19), we are integrating over the entire real line. Certainly, this does not make a difference because $\cos\frac{v}{2}=0$ whenever $v=\pm\pi$. Using (3.18) and the above equality for P_ω and $(Q_x)_\omega$, we obtain

$$(uP)_\omega=u_\omega P+uP_\omega=q\left(P\cos^3\frac{v}{2}\sin\frac{v}{2}+uP_x\cos^4\frac{v}{2}\right),$$

$$(uQ_x)_\omega=u_\omega Q_x+u(Q_x)_\omega=q\left(Q_x\sin\frac{v}{2}\cos^3\frac{v}{2}+uQ\cos^4\frac{v}{2}-\frac{1}{2}u\sin\frac{v}{2}\cos^3\frac{v}{2}\right).$$

Therefore,

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u^2\cos^4\frac{v}{2}+\sin^2\frac{v}{2}\cos 2\frac{v}{2}\right)qd\omega=\int_{\mathbb{R}}\partial_\omega\left\{u^4-2uP-2uQ_x\right\}d\omega=0.$$

The last equality holds since $\lim\limits_{|\omega|\to\infty}u(\omega)=0$ as $u\in H^1(\mathbb{R})$, while P,P_x,Q and Q_x are uniformly bounded. This proves

$$\tilde{E}(t)=\int_{\mathbb{R}}\left(u^2(t,\omega)\cos^4\frac{v(t,\omega)}{2}+\sin^2\frac{v(t,\omega)}{2}\cos^2\frac{v}{2}\right)q(t,\omega)d\omega=\tilde{E}(0)\doteq E_0, \tag{3.22}$$

along any solution of (3.5).

Similarly, for the conservation of $\tilde{G}$, by computation, we get

$$\frac{d\tilde{G}}{dt}=\int_{\mathbb{R}}\partial_\omega\left\{u^6+\frac{4}{3}(u^3(P+Q_x)+(Q+P_x)^2-(P+Q_x)^2)\right\}d\omega=0.$$

This yields that

$$\tilde{G}(t)=\int_{\mathbb{R}}\left(u^4\cos^4\frac{v}{2}+2u^2\sin^2\frac{v}{2}\cos^2\frac{v}{2}-\frac{1}{3}\sin^4\frac{v}{2}\right)qd\omega=\tilde{G}(0)\doteq G_0. \tag{3.23}$$

If the solution is defined, using (3.18) and (3.22), we obtain the bound

$$\sup_{\omega\in\mathbb{R}}|u^2(t,\omega)|\le 2\int_{\mathbb{R}}|uu_\omega|d\omega\le 2\int_{\mathbb{R}}|u|\cdot\left|\sin\frac{v}{2}\cos^3\frac{v}{2}\right|qd\omega\le E_0. \tag{3.24}$$

This provides a uniform a priori bound on $||u(t)||_{L^\infty(\mathbb{R})}$. From (3.22)-(3.23) and the definitions (3.1)-(3.4), it follows that

$$\begin{aligned}||P(t)||_{L^\infty(\mathbb{R})},||P_x(t)||_{L^\infty(\mathbb{R})}\le C_1,\\ ||Q(t)||_{L^\infty(\mathbb{R})},||Q_x(t)||_{L^\infty(\mathbb{R})}\le C_2,\end{aligned} \tag{3.25}$$

where $C_1=\frac{3}{4}E_0^{\frac{3}{2}}$ and $C_2=\frac{1}{4}\sqrt{6E_0^3-3E_0G_0}$. Observe the third equation in (3.5), as long as the solution is defined, by (3.24)-(3.25), we deduce that

$$|q_t|\le\left(2(C_1+C_2+E_0^{\frac{3}{2}})+E_0^{\frac{1}{2}}\right)q.$$

Since $q(0,\omega)=1$, the above differential inequality implies

$$e^{-Ct}\le q(t)\le e^{Ct}, \tag{3.26}$$

the constant C depends only on E_0 and G_0. From the second equation in (3.5), it is obvious that

$$-\pi\le v(t,\omega)<\pi.$$

Finally, the lower bound on q in (3.26) combines the estimates (3.22) and (3.23), from which we can deduce that, for every $\eta>0$, there exists a constant C_η depending on E_0,G_0 and T such that

$$\text{meas}(\{\omega;|v(t,\omega)|\ge\eta\})\le C_\eta\quad t\in[0,T].$$

All the above analysis, the priori bounds (3.12)-(3.13) which we need to establish a local solution are satisfied with a constant C uniformly valid over any given time interval $[0,T]$. This completes the proof of extending the local solution globally extended for all times $t\ge 0$. □

Next, we will show the continuous dependence of solutions to system (2.1) on the initial data belonging to $H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})$. This needs further estimates. Suppose that $\tilde{u}_n\to\tilde{u}$ in $H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})$, with $v=2\arctan u_x$, then

$$||u_n-\tilde{u}||_{L^\infty(\mathbb{R})}\to 0,||\tilde{v}_n-\tilde{v}||_{L^2(\mathbb{R})}\to 0. \tag{3.27}$$

Generally, $||\tilde{v}_n-\tilde{v}||_{L^\infty(\mathbb{R})}\to 0$ does not converge to zero. By the weaker assumptions (3.27), we prove the following result on continuous dependence of solutions to system (3.5).

THEOREM 3.2. *Given a sequence of initial data $\tilde{u}_n$, such that $||u_n-\tilde{u}||_{H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})}\to 0$. Therefore, for any $T>0$, the corresponding solutions $u_n(t,\omega)$ of (3.5) with initial data (3.6) converge to $u(t,\omega)$ uniformly for $(t,\omega)\in[0,T]\times\mathbb{R}$.*

Proof. The steps of the proof of Theorem 3.2 are similar to the Theorem 2 in [3]. Hence, we omit it here. □

4. Global dissipative solutions to the Equation (2.1)

In present section, we will prove that the global solution of system (3.5) provides a global dissipative solution to equation (2.1), in the original variables (t,x). Let (u,v,q) be a global solution to (3.5) and

$$y(t,\omega) \doteq \tilde{y}(\omega) + \int_0^t u^2(\tau,\omega)d\tau. \tag{4.1}$$

Then for each fixed ω, the function $t \mapsto y(t,\omega)$ yields a solution to the Cauchy problem

$$\frac{\partial}{\partial t} y(t,\omega) = u^2(t,\omega), y(0,\omega) = \tilde{y}(\omega). \tag{4.2}$$

We claim that a solution of (2.1) can be obtain by setting

$$u(t,x) \doteq u(t,\omega) \text{ if } y(t,\omega) = x. \tag{4.3}$$

THEOREM 4.1. *If (u,v,q) is a global solution to the Cauchy problem (3.5) with the initial data (3.6), then the function $u = u(t,x)$ defined by (4.1)-(4.3) gives a solution to the initial value problem (2.1) for the Novikov equation. Moreover, the solution $u = u(t,x)$ has the following property:*

$$||u(t)||^2_{H^1(\mathbb{R})} \leq ||u(t')||^2_{H^1(\mathbb{R})} \quad \text{if} \quad 0 \leq t' \leq t. \tag{4.4}$$

Moreover, there exists a constant C depending only on the total energy $||\tilde{u}||_{H^1}$ such that

$$|u_x(t,x)| \leq C(1+t^{-1}), \quad t>0, \ x \in \mathbb{R}. \tag{4.5}$$

Furthermore, given a sequence of initial data $\tilde{u}_n$, as long as

$$||\tilde{u}_n - \tilde{u}||_{H^1(\mathbb{R}) \cap W^{1,4}(\mathbb{R})} \to 0, \tag{4.6}$$

then, the corresponding solutions $u_n(t,x)$ converge to $u(t,x)$ uniformly for (t,x) in any bounded set.

Proof. The uniform bound follows from equation (3.22)

$$|u^2(t,\omega)| \leq E_0. \tag{4.7}$$

From (4.1), we get the estimate

$$\tilde{y}(\omega) - E_0 t \leq y(t,\omega) \leq \tilde{y}(\omega) + E_0 t, \ t \geq 0. \tag{4.8}$$

The definition of ω in (2.8) yields

$$\lim_{\omega \to \pm\infty} y(t,\omega) = \pm\infty. \tag{4.9}$$

Then the image of the map $(t,\omega) \mapsto (t, y(t,\omega))$ covers the entire half-plane $\mathbb{R}^+ \times \mathbb{R}$. Now we claim

$$y_\omega = q\cos^4\frac{v}{2} \qquad \text{for all } t \geq 0 \text{ and a.e } \omega \in \mathbb{R}. \tag{4.10}$$

Indeed, from (3.5), a straightforward calculation yields

$$\begin{aligned}\frac{\partial}{\partial t}\left(q\cos^4\frac{v}{2}\right)(t,\omega)&=-2qv_t\sin\frac{v}{2}\cdot\cos^3\frac{v}{2}+q_t\cos^4\frac{v}{2}\\&=2uq\sin\frac{v}{2}\cdot\cos^2\frac{v}{2}\\&=2uu_\omega.\end{aligned}$$

On the other hand, (4.1) yields

$$\frac{\partial}{\partial t}y_\omega(t,\omega)=2uu_\omega(t,\omega).$$

The identity (4.10) holds true for almost every ω at $t=0$ because the function $x\mapsto 2\arctan\tilde{u}_x(x)$ is measurable. By the above calculation it remains true for all $t\geq 0$. From (4.10), we get that $y(t,\omega)$ is nondecreasing. Moreover, if $\omega<\theta$ but $y(t,\omega)=y(t,\theta)$, then

$$\int_\omega^\theta y_\omega(t,s)ds=\int_\omega^\theta q(t,s)\cos^4\frac{v}{2}ds=0.$$

Hence $\cos\frac{v}{2}\equiv 0$ throughout the interval of integration. Then, by (3.18), we have

$$u(t,\theta)-u(t,\omega)=\int_\omega^\theta\frac{q(t,s)}{2}\sin v(t,s)\cos^2\frac{v}{2}ds=0.$$

This shows that the map $(t,x)\mapsto u(t,x)$ in (4.3) is well defined for all $t\geq 0$ and $x\in\mathbb{R}$. We have that, for every fixed t

$$\begin{aligned}\text{meas}(\{y(t,\omega);v(t,\omega)=\pm\pi\})&=\int_{\{v(t,\omega)=\pm\pi\}}y_\omega(t,\omega)d\omega\\&=\int_{\{v(t,\omega)=\pm\pi\}}q(t,\omega)\cos^4\frac{v(t,\omega)}{2}d\omega=0.\end{aligned}\tag{4.11}$$

In the following, using (4.11) to change the variable of integration, for every fixed t, in view of (3.22), we obtain

$$\int_{\mathbb{R}}\left(u^2(t,x)+u_x^2(t,x)\right)dx=\int_{\{-\pi<v(t,x)<\pi\}}\left[\left(u^2\cos^4\frac{v}{2}+\sin^2\frac{v}{2}\cos^2\frac{v}{2}\right)q\right](t,\omega)d\omega\leq E_0.\tag{4.12}$$

It is clear that u, as a function of x, is (uniformly) Hölder continuous with the exponent $\frac{1}{2}$ by Sobolev's inequality. In view of the first equation in (3.5) and $||P_x||_{L^\infty(\mathbb{R})}$ and $||Q||_{L^\infty(\mathbb{R})}$ being uniformly bounded, we can conclude that, along every characteristic curve $t\to y(t,\omega)$, the map $t\to u(t,y(t,\omega))$ is uniformly Lipschitz continuous. So, $u=u(t,x)$ is globally Hölder continuous.

We now claim the Lipschitz continuity of $u(t,x)$ with values in $L^2(\mathbb{R})$. Consider any interval $[\tau,\tau+h]$. For a point x, we take $\omega\in\mathbb{R}$ such that the characteristic $t\mapsto y(t,\omega)$ passes through the point (τ,x). From the first equation in (3.5) and $\|u\|^2_{L^\infty}\leq E_0$, it follows that

$$|u(\tau+h,x)-u(\tau,x)|\leq|u(\tau+h,x)-u(\tau+h,y(\tau+h,\omega))|$$

$$+|u(\tau+h,y(\tau+h,\omega))-u(\tau,x)|$$
$$\leq \sup_{|y-x|\leq E_0h}|u(\tau+h,y)-u(\tau+h,x)|+\int_\tau^{\tau+h}|P_x(t,\omega)+Q(t,\omega)|dt.$$

Then, integrating over the whole real line, using the bound of $||P_x||_{L^2(\mathbb{R})}$, $\|Q(x)\|_{L^2}$ and $\|u_x\|_{L^2}$, we have

$$\begin{aligned}&\int_{\mathbb{R}}|u(\tau+h,y)-u(\tau,y)|^2dx\\ \leq&2\int_{\mathbb{R}}\left(\int_{x-E_0h}^{x+E_0h}|u_x(\tau+h,y)|dy\right)^2dx\\ &\qquad+2\int_{\mathbb{R}}\left(\int_\tau^{\tau+h}|P_x(t,\omega)+Q(t,\omega)|dt\right)^2q(\tau,\omega)\cos^4\frac{v(\tau,\omega)}{2}d\omega\\ \leq&4E_0h\int_{\mathbb{R}}\int_{x-E_0h}^{x+E_0h}|u_x(\tau+h,y)|^2dydx+2h\|q\|_{L^\infty}\int_{\mathbb{R}}\int_\tau^{\tau+h}|P_x(t,\omega)+Q(t,\omega)|^2dtd\omega\\ \leq&8E_0^2h^2||u_x(\tau+h)||^2_{L^2(\mathbb{R})}+2h\|q\|_{L^\infty}\int_\tau^{\tau+h}||P_x(t)+Q(t)||^2_{L^2(\mathbb{R})}dt\leq Ch^2.\end{aligned}$$

The above inequality, holding for some constant C depending only on T, shows the Lipschitz continuity of the map $t\mapsto u(t)$, in terms of the x variable.

Since $L^2(\mathbb{R})$ is a reflexive space and the right-hand side of (2.1) also lies in $L^2(\mathbb{R})$, to establish the equality it suffices to prove the following. For every smooth function $\phi\in\mathcal{C}_c^\infty$, at almost every time t, we need to show that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}u(t,x)\phi(x)dx=\int_{\mathbb{R}}\left(-u^2(t,x)u_x(t,x)-P_x(t,x)-Q\right)\phi(x)dx\\ =&\int_{\mathbb{R}}\left(u^3(t,x)\phi'(x)+2u^2(t,x)u_x(t,x)\phi(x)-(P_x(t,x)+Q(t,x))\phi(x)\right)dx.\end{aligned}\tag{4.13}$$

Let us set

$$\tau\doteq\inf\{t>0;v(t)=\pm\pi\}\tag{4.14}$$

for each $\omega\in\mathbb{R}$. Note that, for almost every time $t\geq0$ one has

$$\text{meas}\big(\{\omega;\tau(\omega)=t\}\big)=0.$$

Choosing a time t such that the above equality holds. Recalling (2.12), (2.13) and (3.5), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}u(t,\omega)\phi(y(t,\omega))\left[q(t,\omega)\cos^4\frac{v(t,\omega)}{2}\right]d\omega\\ =&\int_{\mathbb{R}}\left\{u_t\phi q\cos^4\frac{v}{2}+u\phi'y_tq\cos^4\frac{v}{2}+u\phi q_t\cos^4\frac{v}{2}-u\phi qv_t\sin v\cos^2\frac{v}{2}\right\}d\omega\\ =&\int_{\{-\pi<v(t,\omega)<\pi\}}\{-(P_x+Q)\phi q\cos^4\frac{v}{2}+u^3\phi'q\cos^4\frac{v}{2}+u^2\phi q\sin v\cos^2\frac{v}{2}\}d\omega\\ =&\int_{\{-\pi<v(t,\omega)<\pi\}}\{-(P_x+Q)\phi+u^3\phi'+2u^2u_x\phi\}q\cos^4\frac{v}{2}d\omega\end{aligned}$$

$$=\int_{\mathbb{R}}\left(u^3(t,x)\phi'(x)+2u^2(t,x)u_x(t,x)\phi(x)-(P_x(t,x)+Q(t,x))\phi(x)\right)dx.$$

This means (4.13) holds. Up to now, we can conclude that the pair of functions $u(t,x)$ is a solution of (2.1) in the sense of Definition 2.1.

To prove (4.4), for each $\omega\in\mathbb{R}$, we define $\tau(\omega)$ as (4.14). Recalling (3.22) and (4.12), we have

$$\begin{aligned}||u(t)||^2_{H^1(\mathbb{R})}&=\int_{\{-\pi<v(t,\omega)<\pi\}}\left(u^2(t,\omega)\cos^4\frac{v(t,\omega)}{2}+\sin^2\frac{v(t,\omega)}{2}\cos^2\frac{v(t,\omega)}{2}\right)q(t,\omega)d\omega\\
&=E_0-\int_{\{\tau(\omega)\le t\}}\left(u^2(t,\omega)\cos^4\frac{v(t,\omega)}{2}+\sin^2\frac{v(t,\omega)}{2}\cos^2\frac{v(t,\omega)}{2}\right)q(t,\omega)d\omega\\
&=E_0-\int_{\{\tau(\omega)\le t'\}\cup\{t'<\tau(\omega)\le t\}}\sin^2\frac{v(t,\omega)}{2}\cos^2\frac{v(t,\omega)}{2}q(\tau(\omega),\omega)d\omega\\
&\le E_0-\int_{\{\tau(\omega)\le t'\}}\sin^2\frac{v(t,\omega)}{2}\cos^2\frac{v(t,\omega)}{2}q(\tau(\omega),\omega)d\omega\\
&=||u(t')||^2_{H^1(\mathbb{R})}.\end{aligned}$$

We next prove (4.5). Observe that if δ is small enough, then

$$v\in[\pi-\delta,\pi),\quad (or, v\in(-\pi,-\pi+\delta])$$

yields

$$\frac{\partial}{\partial t}v(t,\omega)\le-\frac{1}{M},\quad (respectively, \frac{\partial}{\partial t}v(t,\omega)\ge\frac{1}{M}).$$

Therefore,

$$v(t,\omega)<\min\{\pi-\delta,\pi-\frac{t}{M}\},(respectively, v(t,\omega)<\max\{\delta-\pi,\frac{t}{M}-\pi\}).$$

This yields (4.5) for a suitable constant C. Finally, the convergence of $u_n(t,x)$ is a consequence of Theorem 3.1. This finishes the proof of the Theorem 4.1. ∎

5. A semigroup of the dissipative solution

Since the solution to the Cauchy problem (4.7)-(4.8) is unique, the global dissipative solution of (2.1) can be organized as a semigroup $\{Z_t\}_{t\ge0}$ of applications, that is,

$$Z_t(\tilde{u})=u(t),t\ge0,\tilde{u}\in H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R}).$$

Therefore, we conclude this paper by the following theorem.

THEOREM 5.1. *Given the initial data $\tilde{u}\in H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})$. Let $u(t)=Z_t(\tilde{u})$ be the corresponding global solution of (2.1) which constructed in Theorem 4.1. Then the map $Z:H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})\times[0,\infty)\mapsto H^1(\mathbb{R})\cap W^{1,4}(\mathbb{R})$ is a semigroup.*

Proof. For fixed $\tau>0$ and all $t>0$, we need to show that

$$Z_t(Z_\tau\tilde{u})=Z_{\tau+t}\tilde{u}. \tag{5.1}$$

Let $(t,\omega)\mapsto(u,v,q)(t,\omega)$ be the corresponding solution of (3.5). Set $\hat{u}=Z_\tau\tilde{u}$. Consider a new energy variable σ and define the map $\omega\mapsto\sigma(\omega)$ to be a solution of the ODE.

$$\frac{d}{d\omega}\sigma(\omega)=\begin{cases}q(\tau,\omega) & \text{if } -\pi<v(\tau,\omega)<\pi,\\ 0 & \text{if } v(\tau,\omega)=\pm\pi,\end{cases} \tag{5.2}$$

with initial data

$$\sigma(\omega_0) = 0. \tag{5.3}$$

Here, the value ω_0 is chosen such that $y(\tau,\omega_0) = 0$. We define

$$\begin{cases} \hat{u}(t,\sigma) = u(\tau + t, \omega(\sigma)), \\ \hat{v}(t,\sigma) = u(\tau + t, \omega(\sigma)), \\ \hat{q}(t,\sigma) = \frac{q(\tau+t,\omega(\sigma))}{q(\tau,\omega(\sigma))}, \end{cases} \tag{5.4}$$

where $\sigma \mapsto \omega(\sigma)$ provides an inverse of the map in (5.3)-(5.4), namely,

$$\omega(\bar{\sigma}) \doteq \sup\{s; \sigma(s) \leq \bar{\sigma}\}.$$

We claim that, for every σ

$$\int_0^{y(\tau,\omega)} \left(1 + u_x^2(\tau,x)\right)^2 dx = \sigma(\omega). \tag{5.5}$$

In view of (5.3), this is valid when $\omega = \omega_0, \sigma = 0$. Moreover, recalling the identities (2.12) and (2.13), we have

$$\frac{\partial}{\partial \omega} y(\tau,\omega) \cdot \left(1 + u_x^2(\tau, y(\tau,\sigma(\omega)))\right)^2 = q(\tau,\omega) = \frac{d}{d\omega}\sigma(\omega).$$

By integration, in view of (5.1), one gets (5.5).

To establish the semigroup property, it suffices to prove that the functions (5.4) provide a solution to the system (3.5). To this end, we write the identities

$$q(\tau + t, \omega) d\omega = \hat{q}(t, \sigma(\omega)) q(\tau,\omega) \cdot \frac{d\omega}{d\tau} \cdot d\sigma = \hat{q}(t,\sigma(\omega)) d\sigma.$$

This yields

$$\hat{P}(t,\sigma) = P(\tau + t, \omega(\sigma)), \hat{P}_x(t,\sigma) = P_x(\tau + t, \omega(\sigma)),$$

$$\hat{Q}(t,\sigma) = Q(\tau + t, \omega(\sigma)), \hat{Q}_x(t,\sigma) = Q_x(\tau + t, \omega(\sigma)).$$

As the last equation in (3.5) is linear with respect to the variable q, then we can get the conclusion from (5.4) and the above equality. This constructs a semigroup for the global dissipative solutions of equation (2.1). □

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