COMMUN. MATH. SCI.
Vol. 16, No. 8, pp. 2147–2165

INITIAL-BOUNDARY VALUE PROBLEM FOR 2D MICROPOLAR EQUATIONS WITHOUT ANGULAR VISCOSITY*

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Abstract. This paper concerns the initial-boundary value problem for 2D micropolar equations without angular viscosity in a smooth bounded domain. It is shown that such a system admits a unique and global strong solution. The main contribution of this paper is to fully exploit the structure of this system and establish high order estimates via introducing an auxiliary field which is at the energy level of one order lower than micro-rotation.

Keywords. Initial-boundary value problem; 2D micropolar equations; angular viscosity.

AMS subject classifications. 35Q35; 76D03.

1. Introduction and main results

This paper is devoted to the initial-boundary value problem for the two-dimensional (2D) micropolar equations without angular viscosity. The micropolar equations were introduced in 1965 by C.A. Eringen to model micropolar fluids (see, e.g. [6]). Micropolar fluids are fluids with microstructure. Certain anisotropic fluids, e.g. liquid crystals which are made up of dumbbell molecules, are of this type. The standard 3D micropolar equations are given by

$$\begin{cases} \mathbf{u}_t-(\nu+\kappa)\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=2\kappa\nabla\times\mathbf{w}, \\ \mathbf{w}_t-\gamma\Delta\mathbf{w}+4\kappa\mathbf{w}-(\alpha+\beta)\nabla\nabla\cdot\mathbf{w}+\mathbf{u}\cdot\nabla\mathbf{w}=2\kappa\nabla\times\mathbf{u}, \\ \nabla\cdot\mathbf{u}=0, \end{cases} \tag{1.1}$$

where $\mathbf{u}=\mathbf{u}(x,t)$ denotes the fluid velocity, $\pi(x,t)$ the scalar pressure, $\mathbf{w}(x,t)$ the micro-rotation field (angular velocity of the rotation of the particles of the fluid), and the parameter $\nu\geq 0$ represents the Newtonian kinematic viscosity, $\kappa>0$ the micro-rotation viscosity, $\alpha,\beta,\gamma\geq 0$ the angular viscosities.

Roughly speaking, they belong to a class of non-Newtonian fluids with nonsymmetric stress tensor (called polar fluids) and include, as a special case, the classical fluids modeled by the Navier-Stokes equations. In fact, when the micro-rotation effects are neglected, namely $\mathbf{w}=0$, (1.1) reduces to the incompressible Navier-Stokes equations. The micropolar equations are significant generalizations of the Navier-Stokes equations and cover many more phenomena such as fluids consisting of particles suspended in a viscous medium. In particular, the dynamic micro-rotation viscosity $\kappa>0$ is essential for the micropolar fluid flows, otherwise the velocity and the micro-rotation are uncoupled and the global motion is unaffected by the micro-rotation. Just because of this, the micropolar equations have been extensively applied and studied by many engineers and physicists.

In particular, the well-posedness problem on the micropolar equations have attracted considerable attention recently from the community of mathematical fluids [1, 2, 7, 16]. Lukaszewicz, in his monograph [16], studied the well-posedness problem

*Received: February 2, 2018; Accepted (in revised form): August 13, 2018. Communicated by Yaguang Wang.

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on the 3D stationary model as well as the time-dependent micropolar equations. Yamaguchi [24] investigated the 3D time-dependent model with small initial data. In spite of previous progress on the 3D case, just like the 3D Navier-Stokes equations, the problem of global regularity or finite-time singularity for strong solutions of 3D micropolar equations is still widely open. Therefore, more attention is focused on the 2D micropolar equations, which are a special case of the 3D micropolar equations. In the special case when

$$\mathbf{u}=(u_1(x_1,x_2,t),u_2(x_1,x_2,t),0),\ \pi=\pi(x_1,x_2,t),\ \mathbf{w}=(0,0,w(x_1,x_2,t)),$$

the 3D micropolar equations reduce to the 2D micropolar equations,

$$\begin{cases} \mathbf{u}_t-(\nu+\kappa)\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=-2\kappa\nabla^{\perp}w, \\ w_t-\gamma\Delta w+4\kappa w+\mathbf{u}\cdot\nabla w=2\kappa\nabla^{\perp}\cdot\mathbf{u}, \\ \nabla\cdot\mathbf{u}=0. \end{cases} \tag{1.2}$$

Here $\mathbf{u}=(u_1,u_2)$ is a 2D vector with the corresponding scalar vorticity Φ given by

$$\Phi\equiv\nabla^{\perp}\cdot\mathbf{u}=\partial_1u_2-\partial_2u_1,$$

while ω represents a scalar function with

$$\nabla^{\perp}w=(-\partial_2w,\partial_1w).$$

In [3], Dong and Chen obtained the global existence and uniqueness, and sharp algebraic time-decay rates for the 2D micropolar Equations (1.2). Besides, the system (1.2) with periodic boundary conditions has been extensively analyzed by Szopa [21]. Despite all this, the global regularity problem for the 2D inviscid micropolar equations is currently out of reach. Therefore, more recent efforts are focused on the 2D micropolar equations with partial viscosity, which naturally bridge the inviscid micropolar equations and the micropolar equations with full viscosity. One case for partial viscosity, (1.2) with $\nu=0,\gamma>0,\kappa>0$ and $\kappa\neq\gamma$, was examined by Xue, who was able to obtain the global well-posedness in the frame work of Besov spaces [23]. Recently, for another case when (1.2) involves only the angular viscosity, i.e.,

$$\begin{cases} \mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=-2\kappa\nabla^{\perp}w, \\ w_t-\gamma\Delta w+4\kappa w+\mathbf{u}\cdot\nabla w=2\kappa\nabla^{\perp}\cdot\mathbf{u}, \\ \nabla\cdot\mathbf{u}=0, \end{cases} \tag{1.3}$$

Dong et al. [4] and Jiu et al. [13] proved the global (in time) regularity for the Cauchy problem and initial-boundary value problem respectively.

Nevertheless, there is one model, namely, taking $\alpha=\beta=\gamma=0$ in (1.1), that is more interesting in Physics. To be specific, for this model, the stress momentum is lost in the rotation of the particles, the microstructure plays an important role as it usually increases the load capacity and stabilizes the flows, this sort of micropolar fluid is less prone to instability than a classical fluid [9, 20]. Some polymeric fluids and fluids containing certain additives in narrow films may be represented by this mathematical model (see Section 1 and Section 6 in Eringen [6]). Moreover, the experiments with the fluids containing extremely small amount of polymeric additives indicate that the skin friction near a rigid body in such fluids is considerably lower (up to 30-50%) than for the same fluids without additives (see [19]). Due to its physical applications and mathematical significance, in [5], Dong and Zhang examined (1.1) with the micro-rotation viscosities $\alpha=\beta=\gamma=0$ in $\mathbb{R}^2$, namely

$$\begin{cases} \mathbf{u}_t-(\nu+\kappa)\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=-2\kappa\nabla^{\perp}w, \\ w_t+4\kappa w+\mathbf{u}\cdot\nabla w=2\kappa\nabla^{\perp}\cdot\mathbf{u}, \\ \nabla\cdot\mathbf{u}=0, \end{cases} \tag{1.4}$$

and established the global regularity of system (1.4). However, the initial-boundary value problem of model (1.4) is still *open*. As a matter of fact, in many real-world applications, the flows are often restricted to bounded domains with suitable constraints imposed on the boundaries and these applications naturally lead to the studies of the initial-boundary value problems. In addition, solutions of the initial-boundary value problems may exhibit much richer phenomena than those of the whole-space counterpart.

In this paper, we will investigate the initial-boundary value problem for the system (1.4) with physical boundary conditions

$$\mathbf{u}|_{\partial\Omega}=0, \tag{1.5}$$

and initial conditions

$$(\mathbf{u},w)(x,0)=(\mathbf{u}_0,w_0)(x), \quad \text{in}\,\Omega, \tag{1.6}$$

where $\Omega\subset\mathbb{R}^2$ represents a bounded domain with smooth boundary.

The aim of this paper is to establish the global existence and uniqueness of strong solutions to the system (1.4)-(1.6). As a result, we obtain the following result, i.e. Theorem 1.1. It should be especially noted that the hypothesis on the initial data could be a technical limitation of our proof and not a sharp requirement.

THEOREM 1.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary. Assume $(\mathbf{u}_0,w_0)$ satisfies*

$$\mathbf{u}_0\in H_0^1(\Omega)\cap H^2(\Omega), \quad w_0\in W^{1,4}(\Omega).$$

Then there exists a unique strong solution $(\mathbf{u},w)$ to the system (1.4)-(1.6) globally in time such that

$$\mathbf{u}\in L^{\infty}(0,T;H^2(\Omega))\cap L^2(0,T;W^{2,4}(\Omega)), \quad w\in L^{\infty}(0,T;W^{1,4}(\Omega)) \tag{1.7}$$

for any $T>0$.

We remark that the initial-boundary value problem on (1.4) is not trivial and quite different from the Cauchy problem. The difficulty is due to the dynamic micro-rotational term $-2\kappa\nabla^{\perp}w$ in the equations of the velocity, which prevents us from obtaining any high order estimates except the basic energy estimates. For the Cauchy problem, the corresponding equation satisfied by vorticity Φ is

$$\Phi_t-(\nu+\kappa)\Delta\Phi+\mathbf{u}\cdot\nabla\Phi+2\kappa\Delta w=0, \tag{1.8}$$

which is a transport-diffusion equation with forcing term $-2\kappa\Delta w$ and therefore high order estimates are available due to its non-boundary conditions. To overcome this difficulty, the authors in [5] observe that the sum of vorticity and micro-rotation angular velocity

$$Z=\Phi-\frac{2\kappa}{\nu+\kappa}w$$

satisfies the transport-diffusion equation

$$\partial_t Z-(\nu+\kappa)\Delta Z+\mathbf{u}\cdot\nabla Z=\left(\frac{8\kappa^2}{\nu+\kappa}-\frac{8\kappa^3}{(\nu+\kappa)^2}\right)w-\frac{4\kappa^2}{\nu+\kappa}Z, \tag{1.9}$$

which helps them to obtain the global bound of $\|\Phi(t)\|_{L^\infty(\mathbb{R}^2)}$ via the global bound of $\|Z(t)\|_{L^\infty(\mathbb{R}^2)}$, and therefore the desired high order estimates are established.

However, for the initial-boundary value problem, this method does not work. This is due to the presence of no-slip boundary conditions for $\mathbf{u}$, and hence the transport-diffusion Equations (1.8) and (1.9) satisfied by Φ and Z would not work any more. To overcome the difficulty caused by the term $-2\kappa\nabla^\perp w$, our strategy is to utilize an auxiliary field $\mathbf{v}$ which is at the energy level of one order lower than w and implement appropriate boundary conditions for $\mathbf{v}$. Keep this in mind, we then introduce the vector field $\mathbf{v}=-\frac{2\kappa}{\nu+\kappa}A^{-1}\nabla^\perp w$ to be the unique solution of the stationary Stokes system with source term $-\frac{2\kappa}{\nu+\kappa}\nabla^\perp w$ as follows,

$$\begin{cases}-\Delta\mathbf{v}+\nabla\pi=-\frac{2\kappa}{\nu+\kappa}\nabla^\perp w & \text{in }\Omega,\\ \nabla\cdot\mathbf{v}=0 & \text{in }\Omega,\\ \mathbf{v}=0 & \text{on }\partial\Omega,\end{cases} \tag{1.10}$$

which also solves, after taking the operator $A^{-1}\nabla^\perp$ on $(1.4)^2$, that

$$\partial_t\mathbf{v}+4\kappa\mathbf{v}-2\kappa A^{-1}\nabla^\perp(\nabla^\perp\cdot\mathbf{u})+A^{-1}\nabla^\perp(\mathbf{u}\cdot\nabla w)=0. \tag{1.11}$$

Then, according to (1.4) and (1.10), we further discover that the new field $\mathbf{g}=\mathbf{u}-(\nu+\kappa)\mathbf{v}$ satisfies the system

$$\begin{cases}\partial_t\mathbf{g}-(\nu+\kappa)\Delta\mathbf{g}+\nabla p=\mathbf{Q} & \text{in }\Omega,\\ \nabla\cdot\mathbf{g}=0 & \text{in }\Omega,\\ \mathbf{g}=0 & \text{on }\partial\Omega,\end{cases} \tag{1.12}$$

where $\mathbf{Q}=-\mathbf{u}\cdot\nabla\mathbf{u}-A^{-1}\nabla^\perp(\mathbf{u}\cdot\nabla w)+2\kappa A^{-1}\nabla^\perp(\nabla^\perp\cdot\mathbf{u})-4\kappa\mathbf{v}$. The obvious advantage of doing so lies in that it provides us the cornerstone of establishing high order estimates of velocity $\mathbf{u}$, which naturally overcome the difficulty caused by the dynamic micro-rotational term $-2\kappa\nabla^\perp w$. As a result, after noticing that $\mathbf{v}$ is at the energy level of one order lower than w and some delicate *a priori* estimates for $\mathbf{g}$, we then successfully establish the desired high order estimates, which guarantee the global existence and uniqueness of strong solution to the system (1.4)-(1.6).

The remainder of this paper is organized in four sections. The second section serves as a preparation and presents a list of facts and tools for bounded domains, such as embedding inequalities and logarithmic type interpolation inequalities. Section 3 establishes the *a priori* estimates, which are *essential* in the proof of Theorem 1.1. Section 4 completes the proof of Theorem 1.1.

2. Preliminaries

This section serves as a preparation. We list a few basic tools for bounded domains to be used in the subsequent sections. In particular, we provide the Gagliardo-Nirenberg interpolation inequalities, the logarithmic type interpolation inequalities and regularization estimates for the elliptic equations and Stokes system in bounded domains. These estimates will also be handy for future studies on PDEs in bounded domains.

We start with the well-known Gagliardo-Nirenberg interpolation inequality for bounded domains (see, e.g. [18]).

LEMMA 2.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary. Let $1\le p,q,r\le\infty$ be real numbers and $j\le m$ be non-negative integers. If a real number α satisfies*

$$\frac{1}{p}-\frac{j}{n}=\alpha\left(\frac{1}{r}-\frac{m}{n}\right)+(1-\alpha)\frac{1}{q},\qquad \frac{j}{m}\le\alpha\le 1,$$

then

$$\|\mathrm{D}^j f\|_{L^p(\Omega)}\le C_1\|\mathrm{D}^m f\|^{\alpha}_{L^r(\Omega)}\|f\|^{1-\alpha}_{L^q(\Omega)}+C_2\|f\|_{L^s(\Omega)},$$

where $s>0$, and the constants C_1 and C_2 depend upon Ω and the indices p,q,r,m,j,s only.

Especially, the following special cases will be used.

COROLLARY 2.1. *Suppose $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary, then*

(1) $\|f\|_{L^4(\Omega)}\le C(\|f\|^{\frac12}_{L^2(\Omega)}\|\nabla f\|^{\frac12}_{L^2(\Omega)}+\|f\|_{L^2(\Omega)})$, $\forall f\in H^1(\Omega)$;

(2) $\|\nabla f\|_{L^4(\Omega)}\le C(\|f\|^{\frac14}_{L^2(\Omega)}\|\nabla^2 f\|^{\frac34}_{L^2(\Omega)}+\|f\|_{L^2(\Omega)})$, $\forall f\in H^2(\Omega)$;

(3) $\|f\|_{L^\infty(\Omega)}\le C(\|f\|^{\frac12}_{L^2(\Omega)}\|\nabla^2 f\|^{\frac12}_{L^2(\Omega)}+\|f\|_{L^2(\Omega)})$, $\forall f\in H^2(\Omega)$;

(4) $\|f\|_{L^\infty(\Omega)}\le C(\|f\|^{\frac23}_{L^2(\Omega)}\|\nabla^3 f\|^{\frac13}_{L^2(\Omega)}+\|f\|_{L^2(\Omega)})$, $\forall f\in H^3(\Omega)$.

The next lemmas state the regularization estimates for the elliptic equations and Stokes system defined on bounded domains (see, e.g. [8,10,12,14,22]).

LEMMA 2.2. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the elliptic boundary value problem*

$$\begin{cases}-\Delta f=g & in\,\Omega,\\ f=0 & on\,\partial\Omega.\end{cases}\tag{2.1}$$

If, for $p\in(1,\infty)$ and an integer $m\ge -1$, $g\in W^{m,p}(\Omega)$, then the system (2.1) has a unique solution f satisfying

$$\|f\|_{W^{m+2,p}(\Omega)}\le C\|g\|_{W^{m,p}(\Omega)},$$

where the constant C depends only on Ω,m and p.

LEMMA 2.3. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the stationary Stokes system*

$$\begin{cases}-\Delta\mathbf{u}+\nabla p=\mathbf{f} & in\,\Omega,\\ \nabla\cdot\mathbf{u}=0 & in\,\Omega,\\ \mathbf{u}=0 & on\,\partial\Omega.\end{cases}\tag{2.2}$$

If, for $q\in(1,\infty)$, $\mathbf{f}\in L^q(\Omega)$, then there exists a unique solution $\mathbf{u}\in W^{1,q}_0(\Omega)\cap W^{2,q}(\Omega)$ of system (2.2) satisfying

$$\|\mathbf{u}\|_{W^{2,q}(\Omega)}+\|\nabla p\|_{L^q(\Omega)}\le C\|\mathbf{f}\|_{L^q(\Omega)}.\tag{2.3}$$

If $\mathbf{f}=\nabla\cdot F$ with $F\in L^q(\Omega)$, then

$$\|\mathbf{u}\|_{W^{1,q}(\Omega)}\le C\|F\|_{L^q(\Omega)}.\tag{2.4}$$

Besides, if $\mathbf{f}=\nabla\cdot F$ *with* $F_{ij}=\partial_k H_{ij}^k$ *and* $H_{ij}^k\in W_0^{1,q}(\Omega)$ *for* $i,j,k=1,\dots,n$*, then*

$$\|\mathbf{u}\|_{L^q(\Omega)}\le C\|H\|_{L^q(\Omega)}. \tag{2.5}$$

Here, all the above constants C *depend only on* Ω *and* q.

LEMMA 2.4. *Let* $\Omega\subset\mathbb{R}^n$ *be a bounded domain with smooth boundary and* $\mathbf{f}=\nabla\cdot F$ *be the same as in system* (2.2)*, then for* $F\in W^{1,q}(\Omega)$ *with* $q\in(2,\infty)$*, the solution* $\mathbf{u}$ *of system* (2.2) *satisfies*

$$\|\nabla\mathbf{u}\|_{L^\infty(\Omega)}\le C(1+\|F\|_{L^\infty(\Omega)})\ln(e+\|\nabla F\|_{L^q(\Omega)}), \tag{2.6}$$

where the constant C *depends only on* Ω.

PROPOSITION 2.1. *Let* $\Omega\subset\mathbb{R}^n$ *be a bounded domain with smooth boundary,* $1<p,q<\infty$*, and assume that* $f\in L^p(0,T;L^q(\Omega))$*,* $u_0\in D_q^{1-\frac{1}{p},p}$*. If* $(\mathbf{u},p)$ *is a solution of the Stokes system*

$$\begin{cases}\partial_t\mathbf{u}-\Delta\mathbf{u}+\nabla p=\mathbf{f} & in\,\Omega,\\ \nabla\cdot\mathbf{u}=0 & in\,\Omega,\\ \mathbf{u}=0 & on\,\partial\Omega,\\ \mathbf{u}(x,0)=\mathbf{u}_0(x) & in\,\Omega,\end{cases} \tag{2.7}$$

then there exists a constant C *depending only on* p*,* q *and* Ω *such that*

$$\|\partial_t\mathbf{u},\nabla^2\mathbf{u},\nabla p\|_{L^p(0,T;L^q(\Omega))}\le C(\|\mathbf{f}\|_{L^p(0,T;L^q(\Omega))}+\|\mathbf{u}_0\|_{D_q^{1-\frac{1}{p},p}}). \tag{2.8}$$

Here, the space $D_q^{\alpha,s}$ *is defined as follows:*

$$D_q^{\alpha,s}\overset{\text{def}}{=}\left\{\mathbf{v}\in L^q_\sigma(\Omega):\|\mathbf{v}\|_{D_q^{\alpha,s}}=\|\mathbf{v}\|_{L^q(\Omega)}+\left(\int_0^\infty\|t^{1-\alpha}Ae^{-tA}\mathbf{v}\|^s_{L^q(\Omega)}\frac{dt}{t}\right)^{\frac{1}{s}}<+\infty\right\},$$

where A *is the Stokes operator,* $\boldsymbol{n}$ *is the unit outward normal vector and* $L^q_\sigma(\Omega)\overset{\text{def}}{=}\{\mathbf{v}\in L^q(\Omega)^n:\nabla\cdot\mathbf{v}=0\text{ in }\Omega,\ \mathbf{v}\cdot\mathbf{n}=0\text{ on }\partial\Omega\}$.

LEMMA 2.5. *Let* α*,* s*,* $1<p\le q<\infty$ *and* n *be as in Proposition* 2.1*, then for* $k=1,2$ *and* $0<\alpha<\frac{k}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})$*, there exists an absolute constant* C *such that*

$$\|\mathbf{v}\|_{D_q^{\alpha,s}}\le C\|\mathbf{v}\|_{W^{k,p}(\Omega)}. \tag{2.9}$$

Proof. To avoid repetition, we only provide a simple proof for the case $k=1$. According to the decay estimates for the Stokes semigroup in bounded domain (see [11] for details), the inequality

$$\|A^se^{-tA}\mathbf{v}\|_{L^q(\Omega)}\le Ct^{-s-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\mathbf{v}\|_{L^p(\Omega)} \tag{2.10}$$

holds for $1<p\le q<\infty$ and $s\in[0,1]$. Thanks to (2.10), for $\alpha<\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})$, we have

$$\int_0^1\|t^{1-\alpha}Ae^{-tA}\mathbf{v}\|^s_{L^q(\Omega)}\frac{dt}{t}\le C\|\nabla\mathbf{v}\|^s_{L^p(\Omega)}\int_0^1 t^{-1+\left(\frac{1}{2}-\alpha-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})\right)s}dt\le C\|\mathbf{v}\|^s_{W^{1,p}(\Omega)},$$

and for $\alpha>0$, we have

$$\int_1^\infty \|t^{1-\alpha}Ae^{-tA}\mathbf{v}\|_{L^q(\Omega)}^s\frac{dt}{t}\le C\|\mathbf{v}\|_{L^p(\Omega)}^s\int_1^\infty t^{-1-\left(\alpha+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})\right)s}dt\le C\|\mathbf{v}\|_{L^p(\Omega)}^s.$$

Thus, by summing up the above two inequalities and applying the definition of the space $D_q^{\alpha,s}$, we finally complete the proof. ∎

Thanks to Lemma 2.5, for $1<p\le q<\infty$, by taking $\alpha=1-\frac{1}{p}$, $s=p$ and $k=2$, we can then update Proposition 2.1 as follows.

LEMMA 2.6. *Let $1<p\le q<\infty$, and suppose that $f\in L^p(0,T;L^q(\Omega))$, $u_0\in W^{2,p}(\Omega)$. If $(\mathbf{u},p)$ is a solution of the Stokes system*

$$\begin{cases}\partial_t\mathbf{u}-\Delta\mathbf{u}+\nabla p=\mathbf{f} & in\,\Omega,\\ \nabla\cdot\mathbf{u}=0 & in\,\Omega,\\ \mathbf{u}=0 & on\,\partial\Omega,\\ \mathbf{u}(x,0)=\mathbf{u}_0(x) & in\,\Omega,\end{cases}\tag{2.11}$$

then there holds that

$$\|\partial_t\mathbf{u},\nabla^2\mathbf{u},\nabla p\|_{L^p(0,T;L^q(\Omega))}\le C(\|\mathbf{f}\|_{L^p(0,T;L^q(\Omega))}+\|\mathbf{u}_0\|_{W^{2,p}(\Omega)}),\tag{2.12}$$

where the constant C depends only on p, q and Ω.

3. A priori estimates

This section is devoted to establishing the *a priori* estimates for the system (1.4)-(1.6), which is an important step in the proof of Theorem 1.1. To be more precise, we first introduce the definition of weak solutions of system (1.4)-(1.6) and then state the main result of this section as a proposition.

DEFINITION 3.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary. A pair of measurable functions $(\mathbf{u},w)$ is called a weak solution of system (1.4)-(1.6) if*

(1) $\mathbf{u}\in C(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, $w\in C(0,T;L^4(\Omega))$;

(2) $$\int_\Omega \mathbf{u}_0\cdot\boldsymbol{\varphi}_0dx+\int_0^T\int_\Omega\left[\mathbf{u}\cdot\boldsymbol{\varphi}_t-(\nu+\kappa)\nabla\mathbf{u}:\nabla\boldsymbol{\varphi}+\mathbf{u}\cdot\nabla\boldsymbol{\varphi}\cdot\mathbf{u}+2\kappa w\nabla^\perp\cdot\boldsymbol{\varphi}\right]dxdt=0,$$

$$\int_\Omega w_0\psi_0dx+\int_0^T\int_\Omega\left[w\psi_t-4\kappa w\psi+\mathbf{u}\cdot\nabla\psi w-2\kappa\mathbf{u}\cdot\nabla^\perp\psi\right]dxdt=0;$$

(3) At each time $0\le t<T$, $\int_\Omega\mathbf{u}\cdot\nabla\phi dx=0$;

hold for any test vector field $\boldsymbol{\varphi}\in C_0^\infty([0,T)\times\Omega)^2$ with $\nabla\cdot\boldsymbol{\varphi}=0$, any test functions $\psi\in C_0^\infty([0,T)\times\Omega)$ and $\phi\in C_0^\infty(\Omega)$, where $A:B$ denotes the matrix product $\sum_{i,j}a_{ij}b_{ij}$.

The main result of this section is stated in the following proposition.

PROPOSITION 3.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary and $(\mathbf{u},w)$ be the smooth solution of system (1.4)-(1.6). Assume $\mathbf{u}_0\in H_0^1(\Omega)\cap H^2(\Omega)$ and $w_0\in W^{1,4}(\Omega)$, then there holds that*

$$\|\mathbf{u}\|_{L^\infty(0,T;H^2(\Omega))}+\|\mathbf{u}\|_{L^2(0,T;W^{2,4}(\Omega))}+\|w\|_{L^\infty(0,T;W^{1,4}(\Omega))}\le C,\tag{3.1}$$

where C depends only on Ω, T, $\|\mathbf{u}_0\|_{H^2(\Omega)}$ and $\|w_0\|_{W^{1,4}(\Omega)}$.

The proof of this proposition relies on the following basic energy estimates.

PROPOSITION 3.2. *Suppose $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary and $(\mathbf{u},w)$ be the smooth solution of system (1.4)-(1.6). If, in addition, $\mathbf{u}_0\in L^2(\Omega)$ and $w_0\in L^2(\Omega)$, then it holds that*

$$\|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}+\|\mathbf{u}\|_{L^2(0,T;H_0^1(\Omega))}+\|w\|_{L^\infty(0,T;L^2(\Omega))}\le C,$$

where C depends only on T, $\|\mathbf{u}_0\|_{L^2(\Omega)}$ and $\|w_0\|_{L^2(\Omega)}$.

Proof. We start with the global L^2-bound. Taking the inner product of system (1.4) with $(\mathbf{u},w)$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|_{L^2(\Omega)}^2+\|w\|_{L^2(\Omega)}^2)+(\nu+\kappa)\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2+4\kappa\|w\|_{L^2(D)}^2\\&=-2\kappa\int_D\nabla^\perp w\cdot\mathbf{u}dx+2\kappa\int_\Omega\nabla^\perp\cdot\mathbf{u}wdx.\end{aligned}$$

Noticing that $\nabla^\perp\cdot\mathbf{u}=\partial_1u_2-\partial_2u_1$ and $\nabla^\perp w=(-\partial_2w,\partial_1w)$, we have

$$-\nabla^\perp w\cdot\mathbf{u}=u_1\partial_2w-u_2\partial_1w=\partial_2(u_1w)-\partial_1(u_2w)+\nabla^\perp\cdot\mathbf{u}w.$$

Integrating by parts and applying the boundary condition (1.5) for $\mathbf{u}$, we have

$$\begin{aligned}&-2\kappa\int_\Omega\nabla^\perp w\cdot\mathbf{u}\,dx+2\kappa\int_\Omega\nabla^\perp\cdot\mathbf{u}w\,dx\\&=4\kappa\int_\Omega\nabla^\perp\cdot\mathbf{u}w\,dx-2\kappa\int_{\partial\Omega}\mathbf{u}\cdot\boldsymbol{n}^\perp w ds\\&=4\kappa\int_\Omega\nabla^\perp\cdot\mathbf{u}w\,dx\\&\le\frac{(\nu+\kappa)}{2}\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2+C\|w\|_{L^2(\Omega)}^2,\end{aligned}\tag{3.2}$$

where $\mathbf{n}^\perp=(-n_2,n_1)$. It then follows, after integration in time, that

$$\begin{aligned}&\|\mathbf{u}\|_{L^2(\Omega)}^2+\|w\|_{L^2(\Omega)}^2+(\nu+\kappa)\int_0^T\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2dt+8\kappa\int_0^T\|w\|_{L^2(\Omega)}^2dt\\&\le e^{CT}(\|\mathbf{u}_0\|_{L^2(\Omega)}^2+\|w_0\|_{L^2(\Omega)}^2)\equiv A_1(T,\|\mathbf{u}_0,w_0\|_{L^2}),\end{aligned}\tag{3.3}$$

where $C=C(\nu,\kappa)$. This completes the proof of Proposition 3.2. ∎

Our next goal is to establish the global bound of $\|\mathbf{u}\|_{H^1(\Omega)}$. As stated in the introduction, $\mathbf{v}$ is at the energy level of one order lower than w. Then by recalling the system (1.10) and setting

$$F=\frac{2\kappa}{\nu+\kappa}\begin{pmatrix}0&w\\-w&0\end{pmatrix},$$

we can then invoke (2.4) in Lemma 2.3 to build up the estimates

$$\|\mathbf{v}\|_{W^{1,q}(\Omega)}\le C\|w\|_{L^q(\Omega)}\tag{3.4}$$

holding for any $q\in(1,\infty)$, which further yields, after applying Proposition 3.2, that

$$\|\mathbf{v}\|_{L^\infty(0,T;H^1(\Omega))}\le C\|w\|_{L^\infty(0,T;L^2(\Omega))}\le C. \tag{3.5}$$

Therefore, to establish the $H^1(\Omega)$ estimates of velocity $\mathbf{u}$, it suffices to do the $H^1(\Omega)$ estimates of $\mathbf{g}=\mathbf{u}-(\nu+\kappa)\mathbf{v}$ as below.

LEMMA 3.1. *Under the assumptions of Proposition 3.2, we further assume $\mathbf{u}_0\in H_0^1(\Omega)$ and $w_0\in L^4(\Omega)$, then there holds that*

$$\|\nabla\mathbf{g}\|_{L^\infty(0,T;L^2(\Omega))}+\|\Delta\mathbf{g}\|_{L^2(0,T;L^2(\Omega))}+\|w\|_{L^\infty(0,T;L^4(\Omega))}+\|w\|_{L^2(0,T;L^4(\Omega))}\le C,$$

where C depends only on Ω, T, $\|\mathbf{u}_0\|_{H^1(\Omega)}$ and $\|w_0\|_{L^4(\Omega)}$.

Proof. Taking inner product of $(1.12)^1$ with $-\Delta\mathbf{g}$, and applying the boundary conditions $\mathbf{g}|_{\partial\Omega}=0$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2+(\nu+\kappa)\|\Delta\mathbf{g}\|_{L^2(\Omega)}^2\\&=-\int_\Omega\mathbf{Q}\cdot\Delta\mathbf{g}\,dx\\&\le\|\mathbf{Q}\|_{L^2(\Omega)}\|\Delta\mathbf{g}\|_{L^2(\Omega)}\\&\le\frac{(\nu+\kappa)}{8}\|\Delta\mathbf{g}\|_{L^2(\Omega)}^2+C\|\mathbf{Q}\|_{L^2(\Omega)}^2,\end{aligned} \tag{3.6}$$

with

$$\begin{aligned}\|\mathbf{Q}\|_{L^2(\Omega)}^2\le&\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2(\Omega)}^2+\|A^{-1}\nabla^\perp(\mathbf{u}\cdot\nabla w)\|_{L^2(\Omega)}^2\\&+C\|A^{-1}\nabla^\perp(\nabla^\perp\cdot\mathbf{u})\|_{L^2(\Omega)}^2+C\|\mathbf{v}\|_{L^2(\Omega)}^2\\=&\sum_{i=1}^4 I^i.\end{aligned} \tag{3.7}$$

Next, we will estimate the four terms one by one. By applying Hölder's inequality, Corollary 2.1, (3.4), Lemma 2.2 and Young's inequality, it follows that

$$\begin{aligned}I_1\le&C\|\mathbf{u}\cdot\nabla\mathbf{g}\|_{L^2(\Omega)}^2+C\|\mathbf{u}\cdot\nabla\mathbf{v}\|_{L^2(\Omega)}^2\\\le&C\|\mathbf{u}\|_{L^4(\Omega)}^2\|\nabla\mathbf{g}\|_{L^4(\Omega)}^2+C\|\mathbf{u}\|_{L^4(\Omega)}^2\|\nabla\mathbf{v}\|_{L^4(\Omega)}^2\\\le&C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}(\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2+\|\nabla\mathbf{g}\|_{L^2(\Omega)}\|\Delta\mathbf{g}\|_{L^2(\Omega)})\\&+C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^4(\Omega)}^2\\\le&C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}(\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2+\|\nabla\mathbf{g}\|_{L^2(\Omega)}\|\Delta\mathbf{g}\|_{L^2(\Omega)})\\&+C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|w\|_{L^4(\Omega)}^2\\\le&\frac{(\nu+\kappa)}{8}\|\Delta\mathbf{g}\|_{L^2(\Omega)}^2+C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2\\&+C\|\mathbf{u}\|_{L^2(\Omega)}^2\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2+C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|w\|_{L^4(\Omega)}^2.\end{aligned} \tag{3.8}$$

Regarding the remaining terms, thanks to the incompressible condition $\nabla\cdot\mathbf{u}=0$ and the boundary conditions $\mathbf{u}|_{\partial\Omega}=0$, we can infer that $\mathbf{u}\cdot\nabla w=\nabla\cdot(\mathbf{u}w)$ and $\mathbf{u}w|_{\partial\Omega}=0$. Therefore, by using Hölder's inequality, Corollary 2.1, Lemma 2.3 and (3.4), we obtain

$$I_2+I_3+I_4$$

$$
\begin{aligned}
&\leq C\|A^{-1}\nabla^{\perp}\nabla(\mathbf{u}w)\|_{L^2(\Omega)}^2+C\|A^{-1}\nabla^{\perp}(\nabla^{\perp}\cdot\mathbf{u})\|_{L^2(\Omega)}^2+C\|\mathbf{v}\|_{L^2(\Omega)}^2\\
&\leq C\|\mathbf{u}w\|_{L^2(\Omega)}^2+C\|\mathbf{u}\|_{L^2(\Omega)}^2+C\|w\|_{L^2(\Omega)}^2\\
&\leq C\|\mathbf{u}\|_{L^4(\Omega)}^2\|w\|_{L^4(\Omega)}^2+C\|\mathbf{u}\|_{L^2(\Omega)}^2+C\|w\|_{L^2(\Omega)}^2\\
&\leq C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|w\|_{L^4(\Omega)}^2+C(\|\mathbf{u}\|_{L^2(\Omega)}^2+\|w\|_{L^2(\Omega)}^2).
\end{aligned}
\tag{3.9}
$$

Finally, we add up the estimates from (3.6) to (3.9), which yields that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2+\frac{3(\nu+\kappa)}{4}\|\Delta\mathbf{g}\|_{L^2(\Omega)}^2\\
&\leq C(\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}+\|\mathbf{u}\|_{L^2(\Omega)}^2\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2)\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2\\
&\quad+C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|w\|_{L^4(\Omega)}^2+C(\|\mathbf{u}\|_{L^2(\Omega)}^2+\|w\|_{L^2(\Omega)}^2).
\end{aligned}
\tag{3.10}
$$

Clearly, (3.10) is not a closed estimate still because the bound of $\|w\|_{L^4(\Omega)}$ is unknown. However, we discover that, the estimate of $\|w\|_{L^4(\Omega)}$ can be bounded in turn by $\|\nabla\mathbf{g}\|_{L^2(\Omega)}$ and $\|\Delta\mathbf{g}\|_{L^2(\Omega)}$. This motivates us to search for the closed estimates of $\|\nabla\mathbf{g}\|_{L^\infty(0,T;L^2(\Omega))}^2+\|w\|_{L^\infty(0,T;L^4(\Omega))}^2$. To start with, by multiplying $(1.4)^2$ with $|w|^3w$ and integrating on Ω, we have

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\|w\|_{L^4(\Omega)}^4+4\kappa\|w\|_{L^4(\Omega)}^4\\
&=2\kappa\int_\Omega\nabla^{\perp}\cdot\mathbf{u}|w|^3w\,dx\\
&\leq C\|\nabla\mathbf{u}\|_{L^4(\Omega)}\|w\|_{L^4(\Omega)}^3\\
&\leq C(\|\nabla\mathbf{g}\|_{L^4(\Omega)}+\|\nabla\mathbf{v}\|_{L^4(\Omega)})\|w\|_{L^4(\Omega)}^3\\
&\leq C(\|\nabla\mathbf{g}\|_{L^2(\Omega)}+\|\nabla\mathbf{g}\|_{L^2(\Omega)}^{\frac12}\|\Delta\mathbf{g}\|_{L^2(\Omega)}^{\frac12}+\|w\|_{L^4(\Omega)})\|w\|_{L^4(\Omega)}^3,
\end{aligned}
\tag{3.11}
$$

which further implies, after dividing $\|w\|_{L^4(\Omega)}^2$ on both sides, that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|_{L^4(\Omega)}^2+4\kappa\|w\|_{L^4(\Omega)}^2\\
&\leq C(\|\nabla\mathbf{g}\|_{L^2(\Omega)}+\|\nabla\mathbf{g}\|_{L^2(\Omega)}^{\frac12}\|\Delta\mathbf{g}\|_{L^2(\Omega)}^{\frac12}+\|w\|_{L^4(\Omega)})\|w\|_{L^4(\Omega)}\\
&\leq\frac{(\nu+\kappa)}{4}\|\Delta\mathbf{g}\|_{L^2(\Omega)}^2+C(\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2+\|w\|_{L^4(\Omega)}^2).
\end{aligned}
\tag{3.12}
$$

Subsequently, by summing up the estimates (3.10) and (3.12), we finally obtain, after some basic calculations, that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2+\|w\|_{L^4(\Omega)}^2)+\frac{(\nu+\kappa)}{2}\|\Delta\mathbf{g}\|_{L^2(\Omega)}^2+4\kappa\|w\|_{L^4(\Omega)}^2\\
&\leq C(1+\|\mathbf{u}\|_{L^2(\Omega)}^2\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2)(\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2+\|w\|_{L^4(\Omega)}^2)+C(\|\mathbf{u}\|_{L^2(\Omega)}^2+\|w\|_{L^2(\Omega)}^2).
\end{aligned}
\tag{3.13}
$$

This, together with Grönwall's inequality and (3.3), then yields the following bound

$$
\begin{aligned}
&\|\nabla\mathbf{g}\|_{L^2(\Omega)}^2+\|w\|_{L^4(\Omega)}^2+(\nu+\kappa)\int_0^T\|\Delta\mathbf{g}\|_{L^2(\Omega)}^2dt+8\kappa\int_0^T\|w\|_{L^4(\Omega)}^2dt\\
&\leq C_1e^{C_2T}(\|\nabla\mathbf{u}_0,w_0\|_{L^2(\Omega)}^2+\|w_0\|_{L^4(\Omega)}^2+TA_1(T,\|\mathbf{u}_0,w_0\|_{L^2}))\equiv A_2(T),
\end{aligned}
\tag{3.14}
$$

where $C_1 = C_1(\nu,\kappa)$, $C_2 = C_2(\nu,\kappa,\|\mathbf{u}_0,w_0\|_{L^2(\Omega)})$. This completes the proof of Lemma 3.1. ◻

Although we have derived the estimate of $\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega))}$, to prove the global existence of strong solutions, we need the estimate of $\|\mathbf{u}\|_{L^2(0,T;H^2(\Omega))}$. Therefore, even with the help of the estimate of $\|\mathbf{g}\|_{L^2(0,T;H^2(\Omega))}$, we still need the global bound of $\|\mathbf{v}\|_{L^2(0,T;H^2(\Omega))}$. Namely, we should prove that $\|w\|_{L^2(0,T;H^1(\Omega))}$ is globally bound according to Lemma 2.3. To achieve this, we first establish the bound of $\|w\|_{L^\infty(0,T;L^q(\Omega))}$.

Proposition 3.3. *In addition to the conditions in Lemma 3.1, if we further assume $w_0 \in L^p(\Omega)$ for any $2 \le p \le \infty$, then the micro-rotation w obeys the global bound*

$$\|w\|_{L^\infty(0,T;L^q(\Omega))} \le C,$$

where C depends only on Ω, T, $\|\mathbf{u}_0\|_{H^1(\Omega)}$ and $\|w_0\|_{L^q(\Omega)}$.

Proof. We start with the equation of w, namely $(1.4)^2$. For any $2 \le q < \infty$, multiplying $(1.4)^2$ with $|w|^{q-2}w$ and integrating on Ω, we obtain

$$\frac{1}{q}\frac{d}{dt}\|w\|_{L^q(\Omega)}^q + 4\kappa\|w\|_{L^q(\Omega)}^q \le 2\kappa\|\nabla\mathbf{u}\|_{L^q(\Omega)}\|w\|_{L^q(\Omega)}^{q-1},$$

i.e.,

$$\frac{d}{dt}\|w\|_{L^q(\Omega)} + 4\kappa\|w\|_{L^q(\Omega)} \le 2\kappa\|\nabla\mathbf{u}\|_{L^q(\Omega)}.$$

Then, by employing the definition of $\mathbf{g}$, (3.4) and the Sobolev embedding inequalities, we further have

$$\begin{aligned}
&\frac{d}{dt}\|w\|_{L^q(\Omega)} + 4\kappa\|w\|_{L^q(\Omega)} \\
&\le C\|\nabla\mathbf{g}\|_{L^q(\Omega)} + C\|\nabla\mathbf{v}\|_{L^q(\Omega)} \\
&\le C\|\mathbf{g}\|_{W^{1,q}(\Omega)} + C\|w\|_{L^q(\Omega)} \\
&\le C\|\mathbf{g}\|_{H^2(\Omega)} + C\|w\|_{L^q(\Omega)} \\
&\le C\|\mathbf{g}\|_{L^2(\Omega)} + C\|\Delta\mathbf{g}\|_{L^2(\Omega)} + C\|w\|_{L^q(\Omega)} \\
&\le C\|\mathbf{u}\|_{L^2(\Omega)} + C\|\mathbf{v}\|_{L^2(\Omega)} + C\|\Delta\mathbf{g}\|_{L^2(\Omega)} + C\|w\|_{L^q(\Omega)} \\
&\le C\|\mathbf{u}\|_{L^2(\Omega)} + C\|w\|_{L^2(\Omega)} + C\|\Delta\mathbf{g}\|_{L^2(\Omega)} + C\|w\|_{L^q(\Omega)},
\end{aligned}$$

which, according to Grönwall's inequality, Proposition 3.2 and Lemma 3.1 further implies

$$\begin{aligned}
&\|w\|_{L^q(\Omega)} + 4\kappa\int_0^T \|w\|_{L^q(\Omega)}dt \\
&\le e^{CT}\left[\|w_0\|_{L^q(\Omega)} + \int_0^T \left(\|\mathbf{u}\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} + \|\Delta\mathbf{g}\|_{L^2(\Omega)}\right)dt\right] \\
&\le C(T).
\end{aligned}$$

Noting that the constant C is independent of q, we then derive, after letting $q \to \infty$, that

$$\|w\|_{L^\infty(\Omega)} + 4\kappa\int_0^T \|w\|_{L^\infty(\Omega)}dt$$

$$\leq e^{CT}\left[\|w_0\|_{L^\infty(\Omega)}+\int_0^T\left(\|\mathbf{u}\|_{L^2(\Omega)}+\|w\|_{L^2(\Omega)}+\|\Delta\mathbf{g}\|_{L^2(\Omega)}\right)dt\right]$$
$$\leq C(T).$$

This completes the proof of Proposition 3.3. ∎

We now move on to the next lemma asserting the bound of $\|\mathbf{g}\|_{L^2(0,T;W^{2,q}(\Omega))}$.

Lemma 3.2. *Under the assumptions of Proposition 3.3, if in addition, $\mathbf{u}_0\in H^2(\Omega)$ and $w_0\in H^1(\Omega)$, then the inequality*

$$\|\mathbf{g}\|_{L^2(0,T;W^{2,q}(\Omega))}\leq C$$

holds for any $2\leq q<\infty$, where C depends only on Ω, T, $\|\mathbf{u}_0\|_{H^2(\Omega)}$ and $\|w_0\|_{H^1(\Omega)}$.

Proof. Initially, by applying Lemma 2.6 to (1.12) and Lemma 2.3, it is clear that

$$\begin{aligned}&\|\nabla^2\mathbf{g}\|_{L^2(0,T;L^q(\Omega))}\leq C(\|\mathbf{Q}\|_{L^2(0,T;L^q(\Omega))}+\|\mathbf{g}_0\|_{H^2(\Omega)})\\&\leq C(\|\mathbf{Q}\|_{L^2(0,T;L^q(\Omega))}+\|\mathbf{u}_0\|_{H^2(\Omega)}+\|\mathbf{v}_0\|_{H^2(\Omega)})\\&\leq C(\|\mathbf{Q}\|_{L^2(0,T;L^q(\Omega))}+\|\mathbf{u}_0\|_{H^2(\Omega)}+\|w_0\|_{H^1(\Omega)}),\end{aligned}\tag{3.15}$$

with

$$\begin{aligned}\|\mathbf{Q}\|_{L^2(0,T;L^q(\Omega))}&\leq\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2(0,T;L^q(\Omega))}+\|A^{-1}\nabla^\perp(\mathbf{u}\cdot\nabla w)\|_{L^2(0,T;L^q(\Omega))}\\&\quad+\|A^{-1}\nabla^\perp(\nabla^\perp\cdot\mathbf{u})\|_{L^2(0,T;L^q(\Omega))}+\|\mathbf{v}\|_{L^2(0,T;L^q(\Omega))}\\&=\sum_{i=1}^4 I^i.\end{aligned}\tag{3.16}$$

For the first term, by employing Hölder's inequality, the Sobolev embedding inequalities, (3.4), Proposition 3.2, Lemma 3.1 and Proposition 3.3, we have

$$\begin{aligned}&I_1\\&\leq\|\mathbf{u}\|_{L^\infty(0,T;L^{2q}(\Omega))}\|\nabla\mathbf{u}\|_{L^2(0,T;L^{2q}(\Omega))}\\&\leq C\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega))}(\|\nabla\mathbf{g}\|_{L^2(0,T;L^{2q}(\Omega))}+\|\nabla\mathbf{v}\|_{L^2(0,T;L^{2q}(\Omega))})\\&\leq C(\|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}+\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))})(\|\mathbf{g}\|_{L^2(0,T;H^2(\Omega))}+\|w\|_{L^2(0,T;L^{2q}(\Omega))})\\&\leq C(1+\|\nabla\mathbf{g}\|_{L^\infty(0,T;L^2(\Omega))}+\|\nabla\mathbf{v}\|_{L^\infty(0,T;L^2(\Omega))})(\|\mathbf{g}\|_{L^2(0,T;L^2(\Omega))}+\|\Delta\mathbf{g}\|_{L^2(0,T;L^2(\Omega))}+1)\\&\leq C(1+\|w\|_{L^\infty(0,T;L^2(\Omega))})(\|\mathbf{u}\|_{L^2(0,T;L^2(\Omega))}+\|w\|_{L^2(0,T;L^2(\Omega))}+1)\\&\leq C(T).\end{aligned}\tag{3.17}$$

As for the remaining terms, by using the equality $\mathbf{u}\cdot\nabla w=\nabla\cdot(\mathbf{u}w)$ and the same tools used for estimating (3.17), it follows that

$$\begin{aligned}&I_2+I_3+I_4\\&\leq\|A^{-1}\nabla^\perp\nabla(\mathbf{u}w)\|_{L^2(0,T;L^q(\Omega))}+C\|A^{-1}\nabla^\perp(\nabla^\perp\cdot\mathbf{u})\|_{L^2(0,T;L^q(\Omega))}+C\|\mathbf{v}\|_{L^2(0,T;L^q(\Omega))}\\&\leq C\|\mathbf{u}w\|_{L^2(0,T;L^q(\Omega))}+C\|\mathbf{u}\|_{L^2(0,T;L^q(\Omega))}+C\|w\|_{L^2(0,T;L^q(\Omega))}\\&\leq C\|\mathbf{u}\|_{L^2(0,T;L^{2q}(\Omega))}\|w\|_{L^\infty(0,T;L^{2q}(\Omega))}+C\|\mathbf{u}\|_{L^2(0,T;H^1(\Omega))}+C\|w\|_{L^2(0,T;L^q(\Omega))}\\&\leq C\|\mathbf{u}\|_{L^2(0,T;H^1(\Omega))}\|w\|_{L^\infty(0,T;L^{2q}(\Omega))}+C\|\mathbf{u}\|_{L^2(0,T;H^1(\Omega))}+C\|w\|_{L^2(0,T;L^q(\Omega))}\\&\leq C(T).\end{aligned}\tag{3.18}$$

Thus, through summing up the estimates from (3.15) to (3.18) and applying Proposition 3.2 again, we finally prove that $\|\mathbf{g}\|_{L^2(0,T;W^{2,q}(\Omega))} \leq C(T)$. ∎

Finally, to guarantee both the global existence and the uniqueness of strong solutions, we further need the global bound of $\|\nabla w\|_{L^\infty(0,T;L^4(\Omega))}$. And now, we get to work on it.

PROPOSITION 3.4. *In addition to the conditions in Lemma 3.2, we further assume $\nabla w_0 \in L^q(\Omega)$ for any $2 \leq q < \infty$, we then derive the global bound*

$$\|\nabla w\|_{L^\infty(0,T;L^q(\Omega))} \leq C,$$

where C depends only on Ω, T, $\|\mathbf{u}_0\|_{H^2(\Omega)}$, $\|w_0\|_{H^1(\Omega)}$ and $\|\nabla w_0\|_{L^q(\Omega)}$.

Proof. Taking the first-order partial ∂_i on $(1.4)^2$ yields,

$$\partial_i w_t + 4\kappa \partial_i w + \mathbf{u} \cdot \nabla \partial_i w + \partial_i \mathbf{u} \cdot \nabla w = 2\kappa \partial_i \nabla^\perp \cdot \mathbf{u}. \tag{3.19}$$

Then, for any $2 \leq q < \infty$, multiplying (3.19) with $|\partial_i w|^{q-2} \partial_i w$, summing over i and integrating on Ω, we obtain

$$\frac{1}{q}\frac{d}{dt}\|\nabla w\|^q_{L^q(\Omega)} + 4\kappa \|\nabla w\|^q_{L^q(\Omega)} \leq \|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \|\nabla w\|^q_{L^q(\Omega)} + 2\kappa \|\nabla^2 \mathbf{u}\|_{L^q(\Omega)} \|\nabla w\|^{q-1}_{L^q(\Omega)},$$

i.e.,

$$\frac{d}{dt}\|\nabla w\|_{L^q(\Omega)} + 4\kappa \|\nabla w\|_{L^q(\Omega)} \leq \|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \|\nabla w\|_{L^q(\Omega)} + 2\kappa \|\nabla^2 \mathbf{u}\|_{L^q(\Omega)}.$$

Next, by employing Lemma 2.4 for the system (1.10), it clearly holds

$$\|\nabla \mathbf{v}\|_{L^\infty(\Omega)} \leq C(1 + \|w\|_{L^\infty(\Omega)}) \ln(e + \|\nabla w\|_{L^q(\Omega)}) \tag{3.20}$$

for any $q \in (2,\infty)$. Subsequently, by recalling the definition of $\mathbf{g}$, applying Lemma 2.3, (3.20) and the Sobolev embedding inequalities, we further deduce that

$$\begin{aligned}
&\frac{d}{dt}\|\nabla w\|_{L^q(\Omega)} + 4\kappa \|\nabla w\|_{L^q(\Omega)} \\
&\leq \|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \|\nabla w\|_{L^q(\Omega)} + 2\kappa \|\nabla^2 \mathbf{u}\|_{L^q(\Omega)} \\
&\leq \|\nabla \mathbf{g}\|_{L^\infty(\Omega)} \|\nabla w\|_{L^q(\Omega)} + (\nu + \kappa) \|\nabla \mathbf{v}\|_{L^\infty(\Omega)} \|\nabla w\|_{L^q(\Omega)} + 2\kappa \|\nabla^2 \mathbf{g}\|_{L^q(\Omega)} \\
&\quad + 2\kappa \|\nabla^2 \mathbf{v}\|_{L^q(\Omega)} \\
&\leq C\|\mathbf{g}\|_{W^{2,q}(\Omega)} \|\nabla w\|_{L^q(\Omega)} + C(1 + \|w\|_{L^\infty(\Omega)}) \ln(e + \|\nabla w\|_{L^q(\Omega)}) \|\nabla w\|_{L^q(\Omega)} \\
&\quad + C\|\mathbf{g}\|_{W^{2,q}(\Omega)} + C\|\nabla w\|_{L^q(\Omega)} \\
&\leq C\varphi(t)(1 + \|\nabla w\|_{L^q(\Omega)}) \ln(e + \|\nabla w\|_{L^q(\Omega)}),
\end{aligned}$$

where $\varphi(t) = (1 + \|w\|_{L^\infty(\Omega)})(1 + \|\mathbf{g}\|_{W^{2,q}(\Omega)})$. According to Proposition 3.3 and Lemma 3.2, it is clear that $\varphi(t) \in L^1(0,T)$. This, together with Grönwall's inequality yields that

$$\|\nabla w\|_{L^q(\Omega)} + 4\kappa \int_0^T \|\nabla w\|_{L^q(\Omega)} dt \leq C(T).$$

This completes the proof of Proposition 3.4. ∎

PROPOSITION 3.5. *Assume that* $(\mathbf{u}_0, w_0)$ *satisfies the conditions stated in Theorem 1.1 and* $(\mathbf{u}, w)$ *be the smooth solution of system (1.4)-(1.6). Then, it holds that*

$$\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega))} + \|\mathbf{u}\|_{L^2(0,T;W^{2,4}(\Omega))} + \|w\|_{L^\infty(0,T;W^{1,4}(\Omega))} \le C, \tag{3.21}$$

where the constant C *depends only on* Ω, T, $\|\mathbf{u}_0\|_{H^2(\Omega)}$ *and* $\|w_0\|_{W^{1,4}(\Omega)}$.

Proof. According to the assumptions on the initial data, Proposition 3.3 and Proposition 3.4, it is clear that $\|w\|_{L^\infty(0,T;W^{1,4}(\Omega))} \le C$. Then, by the definition of $\mathbf{g}$ and Lemma 2.3, we have

$$\begin{aligned}
&\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega))} + \|\mathbf{u}\|_{L^2(0,T;W^{2,4}(\Omega))} \\
\le &\|\mathbf{g}\|_{L^\infty(0,T;H^1(\Omega))} + \|\mathbf{g}\|_{L^2(0,T;W^{2,4}(\Omega))} + (\nu+\kappa)\left[\|\mathbf{v}\|_{L^\infty(0,T;H^1(\Omega))} + \|\mathbf{v}\|_{L^2(0,T;W^{2,4}(\Omega))}\right] \\
\le &\|\mathbf{g}\|_{L^\infty(0,T;H^1(\Omega))} + \|\mathbf{g}\|_{L^2(0,T;W^{2,4}(\Omega))} + (\nu+\kappa)\left[\|w\|_{L^\infty(0,T;L^2(\Omega))} + \|w\|_{L^2(0,T;W^{1,4}(\Omega))}\right].
\end{aligned}$$

The terms $\|w\|_{L^\infty(0,T;L^2(\Omega))}, \|\mathbf{g}\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|\mathbf{g}\|_{L^2(0,T;W^{2,4}(\Omega))}$ are globally bounded due to Proposition 3.2, Lemma 3.1 and Lemma 3.2 respectively. To bound the term $\|w\|_{L^2(0,T;W^{1,4}(\Omega))}$, it suffices to apply Proposition 3.3, Proposition 3.4 with $q=4$ and Hölder's inequality. This completes the proof of Proposition 3.5. ∎

Proof. (**Proof of Proposition 3.1.**) In terms of Proposition 3.5, it suffices to establish the estimate of $\|\mathbf{u}\|_{L^\infty(0,T;H^2(\Omega))}$. To this end, we first estimate $\|w_t\|_{L^2(\Omega)}$. Multiplying the equation of w in (1.4) with w_t and integrating on Ω, we have

$$\begin{aligned}
\|w_t\|_{L^2(\Omega)}^2 &= -4\kappa\int_\Omega w w_t\,dx - \int_\Omega \mathbf{u}\cdot\nabla w w_t\,dx + 2\kappa\int_\Omega \nabla^\perp\cdot\mathbf{u} w_t\,dx \\
&\le \frac{1}{2}\|w_t\|_{L^2(\Omega)}^2 + C\|w\|_{L^2(\Omega)}^2 + C\|\mathbf{u}\cdot\nabla w\|_{L^2(\Omega)}^2 + C\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2 \\
&\le \frac{1}{2}\|w_t\|_{L^2(\Omega)}^2 + C\|w\|_{L^2(\Omega)}^2 + C\|\mathbf{u}\|_{L^4(\Omega)}^2\|\nabla w\|_{L^4(\Omega)}^2 + C\|\mathbf{u}\|_{H^1(\Omega)}^2 \\
&\le \frac{1}{2}\|w_t\|_{L^2(D)}^2 + C\|w\|_{L^2(\Omega)}^2 + C\|\mathbf{u}\|_{H^1(\Omega)}^2\|\nabla w\|_{L^4(\Omega)}^2 + C\|\mathbf{u}\|_{H^1(\Omega)}^2.
\end{aligned}$$

The global bounds in Proposition 3.2 and Proposition 3.5 then imply

$$\|w_t\|_{L^\infty(0,T;L^2(\Omega))} \le C. \tag{3.22}$$

To estimate $\|\mathbf{u}_t\|_{L^2(\Omega)}$, we take the temporal derivative of the equations of $\mathbf{u}$ in (1.4) to get

$$\mathbf{u}_{tt} - (\nu+\kappa)\Delta\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}_t + \mathbf{u}_t\cdot\nabla\mathbf{u} + \nabla\pi_t = -2\kappa\nabla^\perp w_t. \tag{3.23}$$

Taking the dot product of (3.23) with $\mathbf{u}_t$ and integrating on Ω, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|_{L^2(\Omega)}^2 + (\nu+\kappa)\|\nabla\mathbf{u}_t\|_{L^2(\Omega)}^2 \\
&= -\int_\Omega \mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\,dx - 2\kappa\int_\Omega \nabla^\perp w_t\cdot\mathbf{u}_t\,dx.
\end{aligned}\tag{3.24}$$

By integration by parts, Hölder's inequality and Young's inequality, we obtain

$$-\int_\Omega \mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\,dx - 2\kappa\int_\Omega \nabla^\perp w_t\cdot\mathbf{u}_t\,dx$$

$$
\begin{aligned}
&= \int_\Omega \mathbf{u}_t\cdot\nabla\mathbf{u}_t\cdot\mathbf{u}\,dx + 2\kappa\int_\Omega w_t\nabla^\perp\cdot\mathbf{u}_t\,dx\\
&\le C\|\mathbf{u}_t\|_{L^4(\Omega)}\|\nabla\mathbf{u}_t\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)} + C\|\nabla\mathbf{u}_t\|_{L^2(\Omega)}\|w_t\|_{L^2(\Omega)}\\
&\le C\|\mathbf{u}_t\|_{L^2(\Omega)}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2(\Omega)}^{\frac32}\|\mathbf{u}\|_{L^2(\Omega)}^{\frac12}\|\nabla\mathbf{u}\|_{L^2(\Omega)}^{\frac12} + C\|\nabla\mathbf{u}_t\|_{L^2(\Omega)}\|w_t\|_{L^2(\Omega)}\\
&\le \frac{\nu+\kappa}{2}\|\nabla\mathbf{u}_t\|_{L^2(\Omega)}^2 + C\|\mathbf{u}\|_{L^2(\Omega)}^2\|\mathbf{u}\|_{H^1(\Omega)}^2\|\mathbf{u}_t\|_{L^2(\Omega)}^2 + C\|w_t\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Combining the estimate above with (3.24), applying (3.22), and invoking Proposition 3.5, we conclude that

$$\|\mathbf{u}_t\|_{L^\infty(0,T;L^2(\Omega))}\le C. \tag{3.25}$$

By Lemma 2.3,

$$\|\mathbf{u}\|_{H^2(\Omega)}\le C_1\left(\|\mathbf{u}_t\|_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)}\right), \tag{3.26}$$

where C_1 depends only on Ω. Then by Hölder's inequality and Corollary 2.1, we have

$$
\begin{aligned}
&\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2(\Omega)}\le\|\mathbf{u}\|_{L^4(\Omega)}\|\nabla\mathbf{u}\|_{L^4(\Omega)}\\
&\le C_2\|\mathbf{u}\|_{L^2(\Omega)}^{\frac12}\|\nabla\mathbf{u}\|_{L^2(\Omega)}^{\frac12}(\|\nabla\mathbf{u}\|_{L^2(\Omega)}^{\frac12}\|\nabla^2\mathbf{u}\|_{L^2(\Omega)}^{\frac12} + \|\nabla\mathbf{u}\|_{L^2(\Omega)})\\
&\le 2C_2\|\mathbf{u}\|_{H^1(\Omega)}^{\frac32}\|\mathbf{u}\|_{H^2(\Omega)}^{\frac12}\\
&\le \frac{1}{2C_1}\|\mathbf{u}\|_{H^2(\Omega)} + 2C_1C_2^2\|\mathbf{u}\|_{H^1(\Omega)}^3,
\end{aligned}
$$

where C_2 also depends only on Ω. Together with (3.26), (3.25) and Proposition 3.5, we obtain the desired global bound. This completes the proof of Proposition 3.1. □

4. Proof of Theorem 1.1

The goal of this section is to complete the proof of Theorem 1.1. To do so, we first establish the global existence of weak solutions by Schauder's fixed point theorem. Moreover, due to the global bounds derived in (3.1), these solutions are actually strong solutions. Then the *a priori* estimates obtained in the previous sections for $\mathbf{u}$ and w allow us to prove the uniqueness of strong solutions.

Proof. (**Existence.**) The proof is a consequence of Schauder's fixed point theorem. We shall only provide the sketches.

To define the functional setting, we fix $T>0$ and R_0 to be specified later. For notational convenience, we write

$$X\equiv C(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$$

with $\|g\|_X\equiv\|g\|_{C(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;H_0^1(\Omega))}^2$, and define

$$B=\{g\in X\,|\,\|g\|_X\le R_0\}.$$

Clearly, $B\subset X$ is closed and convex.

We fix $\epsilon\in(0,1)$ and define a continuous map on B. For any $\mathbf{v}\in B$, we regularize it and the initial data $(\mathbf{u}_0,w_0)$ via the standard mollifying process,

$$\mathbf{v}^\epsilon=\rho^\epsilon*\mathbf{v},\quad \mathbf{u}_0^\epsilon=\rho^\epsilon*\mathbf{u}_0,\quad w_0^\epsilon=\rho^\epsilon*w_0,$$

where ρ^ϵ is the standard mollifier. Initially, the transport equation with smooth external forcing term $2\kappa\nabla^\perp\cdot\mathbf{v}^\epsilon$ and smooth initial data w_0^ϵ

$$\begin{cases} w_t+\mathbf{v}^\epsilon\cdot\nabla w+4\kappa w=2\kappa\nabla^\perp\cdot\mathbf{v}^\epsilon, \\ w(x,0)=w_0^\epsilon(x), \end{cases} \tag{4.1}$$

has a unique solution w^ϵ. We then solve the nonhomogeneous (linearized) Navier-Stokes equations with smooth initial data $\mathbf{u}_0^\epsilon$

$$\begin{cases} \mathbf{u}_t+\mathbf{v}^\epsilon\cdot\nabla\mathbf{u}-(\nu+\kappa)\Delta\mathbf{u}+\nabla\pi=-2\kappa\nabla^\perp w^\epsilon, \\ \nabla\cdot\mathbf{u}=0, \quad \mathbf{u}|_{\partial\Omega}=0, \\ \mathbf{u}(x,0)=\mathbf{u}_0^\epsilon(x), \end{cases} \tag{4.2}$$

and denote the solution by $\mathbf{u}^\epsilon$. This process allows us to define the map

$$F^\epsilon(\mathbf{v})=\mathbf{u}^\epsilon.$$

We then apply Schauder's fixed point theorem to construct a sequence of approximate solutions to the system (1.4)-(1.6). It suffices to show that, for any fixed $\epsilon\in(0,1)$, $F^\epsilon:B\to B$ is continuous and compact. More precisely, we need to show

(a) $\|\mathbf{u}^\epsilon\|_B\le R_0$;

(b) $\|\mathbf{u}^\epsilon\|_{C(0,T;H_0^1(\Omega))}+\|\mathbf{u}^\epsilon\|_{L^2(0,T;H^2(\Omega))}\le C$;

(c) $\|F^\epsilon(\mathbf{v}_1)-F^\epsilon(\mathbf{v}_2)\|_B\le C\|\mathbf{v}_1-\mathbf{v}_2\|_B$ for C independent of ϵ and any $\mathbf{v}_1,\mathbf{v}_2\in B$.

We verify (a) first. A simple L^2-estimate on (4.1) leads to

$$\begin{aligned} \|w^\epsilon\|_{L^2(\Omega)}^2+4\kappa\int_0^T\|w^\epsilon\|_{L^2(\Omega)}^2dt &\le \|w_0^\epsilon\|_{L^2(\Omega)}^2+4\kappa\int_0^T\|\nabla\mathbf{v}^\epsilon\|_{L^2(\Omega)}^2dt \\ &\le \|w_0\|_{L^2(\Omega)}^2+4\kappa\int_0^T\|\nabla\mathbf{v}\|_{L^2(\Omega)}^2dt \\ &\le \|w_0\|_{L^2(\Omega)}^2+4\kappa R_0. \end{aligned}$$

Then by taking inner product of (4.2) with $\mathbf{u}^\epsilon$ and after some basic calculations, we have

$$\|\mathbf{u}^\epsilon\|_{L^2(\Omega)}^2+(\nu+\kappa)\int_0^T\|\nabla\mathbf{u}^\epsilon\|_{L^2(\Omega)}^2dt\le\|\mathbf{u}_0\|_{L^2(\Omega)}^2+\frac{4\kappa^2}{\nu+\kappa}\int_0^T\|w^\epsilon\|_{L^2(\Omega)}^2dt.$$

In order for F^ϵ maps B to B, it suffices for the right-hand side to be bounded by R_0. Invoking the bound for $\|w^\epsilon\|_{L^2(\Omega)}$, we obtain a condition for T and R_0, namely

$$\|\mathbf{u}_0\|_{L^2(\Omega)}^2+CT(\|w_0\|_{L^2(\Omega)}^2+R_0)\le R_0, \tag{4.3}$$

where the constant C depends only on the parameters ν and κ. It is not difficult to see that, if $CT<1$ and $R_0>>\|\mathbf{u}_0\|_{L^2(\Omega)}^2+\|w_0\|_{L^2(\Omega)}^2$, (4.3) would hold. Similarly, we can verify (c) under the condition that T is sufficiently small. Besides, (b) can be verified by the similar way as estimating (3.21). Schauder's fixed point theorem then allows us to conclude the existence of a solution on a finite time interval $[0,T]$. These uniform estimates would allow us to pass to the limit to obtain a weak solution $(\mathbf{u},w)$.

We remark that the local solution obtained by Schauder's fixed point theorem can be easily extended into a global solution via Picard-type extension theorem due to the global bounds obtained in (3.21). This allows us to obtain the desired global weak solutions. ∎

As mentioned in the beginning of this section, the solutions established in the previous step are actually strong solutions due to their global bounds (3.1). Now, we are in the position to prove the uniqueness of strong solutions. To be more precise, we will consider the difference between two strong solutions and then establish the energy estimates for the resulting system of the difference at the level of basic energy.

Proof. (**Uniqueness.**) Assume $(\mathbf{u},w,\pi)$ and $(\widetilde{\mathbf{u}},\widetilde{w},\widetilde{\pi})$ are two strong solutions of the system (1.4)-(1.6) with the regularity specified in (1.7). Consider their difference

$$\mathbf{U}=\mathbf{u}-\widetilde{\mathbf{u}},\ W=w-\widetilde{w},\ \Pi=\pi-\widetilde{\pi},$$

which solves the following initial-boundary value problem

$$\begin{cases}\mathbf{U}_t-(\nu+\kappa)\Delta\mathbf{U}+\mathbf{u}\cdot\nabla\mathbf{U}+\mathbf{U}\cdot\nabla\widetilde{\mathbf{u}}+\nabla\Pi=-2\kappa\nabla^{\perp}W,\\ W_t+4\kappa W+\mathbf{u}\cdot\nabla W+\mathbf{U}\cdot\nabla\widetilde{w}=2\kappa\nabla^{\perp}\cdot\mathbf{U},\\ \nabla\cdot\mathbf{U}=0,\ \mathbf{U}|_{\partial\Omega}=0,\\ (\mathbf{U},W)(x,0)=0.\end{cases}\tag{4.4}$$

Taking the dot product of the first two equations of (4.4) with $(\mathbf{U},W)$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\mathbf{U}\|_{L^2(\Omega)}^2+\|W\|_{L^2(\Omega)}^2)+(\nu+\kappa)\|\nabla\mathbf{U}\|_{L^2(\Omega)}^2+4\kappa\|W\|_{L^2(\Omega)}^2\\ =&-\int_\Omega\mathbf{U}\cdot\nabla\widetilde{\mathbf{u}}\cdot\mathbf{U}dx-\int_\Omega\mathbf{U}\cdot\nabla\widetilde{w}Wdx-2\kappa\int_\Omega\nabla^{\perp}W\cdot\mathbf{U}dx\\ &+2\kappa\int_\Omega\nabla^{\perp}\cdot\mathbf{U}Wdx.\end{aligned}\tag{4.5}$$

Then by the divergence theorem and boundary conditions $\mathbf{U}|_{\partial\Omega}=0$,

$$\begin{aligned}&-2\kappa\int_\Omega\nabla^{\perp}W\cdot\mathbf{U}dx+2\kappa\int_\Omega\nabla^{\perp}\cdot\mathbf{U}Wdx\\ =&4\kappa\int_\Omega\nabla^{\perp}\cdot\mathbf{U}Wdx\\ \le&\frac{(\nu+\kappa)}{4}\|\nabla\mathbf{U}\|_{L^2(\Omega)}^2+C\|W\|_{L^2(\Omega)}^2.\end{aligned}$$

To bound the first and second terms on the right-hand side of (4.5), we invoke the Hölder's inequality, Corollary 2.1 and Young's inequality to obtain

$$\begin{aligned}&-\int_\Omega\mathbf{U}\cdot\nabla\widetilde{\mathbf{u}}\cdot\mathbf{U}dx-\int_\Omega\mathbf{U}\cdot\nabla\widetilde{w}Wdx\\ \le&\|\nabla\widetilde{\mathbf{u}}\|_{L^2(\Omega)}\|\mathbf{U}\|_{L^4(\Omega)}^2+\|\nabla\widetilde{w}\|_{L^4(\Omega)}\|\mathbf{U}\|_{L^4(\Omega)}\|W\|_{L^2(\Omega)}\\ \le&C\|\nabla\widetilde{\mathbf{u}}\|_{L^2(\Omega)}\|\mathbf{U}\|_{L^2(\Omega)}\|\nabla\mathbf{U}\|_{L^2(\Omega)}+C\|\nabla\widetilde{w}\|_{L^4(\Omega)}\|\mathbf{U}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\mathbf{U}\|_{L^2(\Omega)}^{\frac{1}{2}}\|W\|_{L^2(\Omega)}\\ \le&\frac{(\nu+\kappa)}{4}\|\nabla\mathbf{U}\|_{L^2(\Omega)}^2+C(1+\|\nabla\widetilde{\mathbf{u}}\|_{L^2(\Omega)}^2+\|\nabla\widetilde{w}\|_{L^4(\Omega)}^2)(\|\mathbf{U}\|_{L^2(\Omega)}^2+\|W\|_{L^2(\Omega)}^2).\end{aligned}$$

Inserting the above estimates into (4.5) yields

$$\frac{d}{dt}(\|\mathbf{U}\|_{L^2(\Omega)}^2+\|W\|_{L^2(\Omega)}^2)$$

$$\leq C(1+\|\nabla\widetilde{\mathbf{u}}\|^2_{L^2(\Omega)}+\|\nabla\widetilde{w}\|^2_{L^4(\Omega)})(\|\mathbf{U}\|^2_{L^2(\Omega)}+\|W\|^2_{L^2(\Omega)}).$$

By Grönwall's inequality, we obtain

$$\begin{aligned}&\|\mathbf{U}(t)\|^2_{L^2(\Omega)}+\|W(t)\|^2_{L^2(\Omega)}\\ &\leq e^{C\int_0^t\left(1+\|\nabla\widetilde{\mathbf{u}}\|^2_{L^2(\Omega)}+\|\nabla\widetilde{w}\|^2_{L^4(\Omega)}\right)d\tau}(\|\mathbf{U}_0\|^2_{L^2(\Omega)}+\|W_0\|^2_{L^2(\Omega)})\end{aligned}$$

for any $t\in(0,T)$. According to Proposition 3.2, Proposition 3.4 and noting that $\mathbf{U}_0=W_0=0$, we obtain the desired uniqueness $\mathbf{U}=W\equiv 0$. This finishes the proof of Theorem 1.1. ∎

Acknowledgments. The authors would like to thank the anonymous referees for their valuable suggestions in improving the original manuscript. J. Liu is supported by National Natural Science Foundation of China (No. 11801018), Beijing Natural Science Foundation (No. 1192001), Youth Backbone Individual Program of the organization department of Beijing (No. 2017000020124G052) and the Basic Research Fund of Beijing University of Technology (No. 006000546318506). S. Wang is supported by National Natural Science Foundation of China (No. 11531010, No. 11771031, No. 11831003), and Qinghai Provincial Natural Science Foundation of China (No. 2017-ZJ-908).

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