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THE 3D NONLINEAR DISSIPATIVE SYSTEM MODELING ELECTRO-DIFFUSION WITH BLOW-UP IN ONE DIRECTION*

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Abstract. This paper establishes a sufficient condition for the breakdown of local smooth solutions, to the Cauchy problem of the 3D Navier–Stokes/Poisson–Nernst–Planck system modeling electro-diffusion, via one directional derivative of the horizontal component of the velocity field (i.e., $(\partial_i u_1, \partial_j u_2, 0)$ where $i,j \in \{1,2,3\}$) in the framework of the anisotropic Lebesgue spaces. More precisely, let $T_* > 0$ be the finite and maximum existence time of local smooth solution. Then

$$\int_0^{T_*} \Big(\Big\| \|\partial_i u_1(t)\|_{L^{\alpha}_{x_i}} \Big\|^q_{L^{\beta}_{x_{\hat{i}} x_{\tilde{i}}}} + \Big\| \|\partial_j u_2(t)\|_{L^{\alpha}_{x_j}} \Big\|^q_{L^{\beta}_{x_{\hat{j}} x_{\tilde{j}}}} \Big) \mathrm{d}t = +\infty,$$

with $\frac{2}{q} + \frac{1}{\alpha} + \frac{2}{\beta} = m \in [1, \frac{3}{2})$ and $\frac{3}{m} < \alpha \le \beta \le \frac{1}{m-1}$, where $(i,\hat{i},\tilde{i})$ and $(j,\hat{j},\tilde{j})$ belong to the permutation group on the set $\mathbb{S}_3 := \{1,2,3\}$. This reveals that the horizontal component of the velocity field plays a more dominant role than the density functions of charged particles in the blow-up theory of the system.

Keywords. Navier–Stokes/Poisson–Nernst–Planck system; blow-up; anisotropic Lebesgue spaces.

AMS subject classifications. 35B44; 35K55; 35Q35; 76W05.

1. Introduction

In this paper, we consider sufficient conditions for the breakdown of local smooth solutions to the Cauchy problem of the following 3D Navier–Stokes/Poisson–Nernst–Planck system modeling electro-diffusion, which is governed by nonlinear coupling between the conventional Navier–Stokes equations of an incompressible fluid and the transported Poisson–Nernst–Planck equations of a binary diffuse charge densities:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \mu\Delta u + \nabla P = \varepsilon \nabla\cdot\sigma, & (x,t)\in\mathbb{R}^3\times\mathbb{R}_+, \\ \nabla\cdot u = 0, & (x,t)\in\mathbb{R}^3\times\mathbb{R}_+, \\ \partial_t v + (u\cdot\nabla)v = \nabla\cdot(D_1\nabla v - \nu_1 v\nabla\Psi), & (x,t)\in\mathbb{R}^3\times\mathbb{R}_+, \\ \partial_t w + (u\cdot\nabla)w = \nabla\cdot(D_2\nabla w + \nu_2 w\nabla\Psi), & (x,t)\in\mathbb{R}^3\times\mathbb{R}_+, \\ \Delta\Psi = v - w, & (x,t)\in\mathbb{R}^3\times\mathbb{R}_+, \\ (u,v,w)|_{t=0} = (u_0,v_0,w_0), & x\in\mathbb{R}^3. \end{cases} \tag{1.1}$$

Here, $u = (u_1,u_2,u_3)$ is the velocity field, P is the pressure, Ψ is the electric potential, v and w are the densities of binary diffuse negative and positive charges (e.g., ions), respectively. The electric stress σ stems from the balance of kinetic energy with electrostatic energy via the least action principle (cf. [30]), and is given by

$$[\sigma]_{ij} = \big(\nabla\Psi\otimes\nabla\Psi - \frac{1}{2}|\nabla\Psi|^2 I\big)_{ij} = \partial_i\Psi\partial_j\Psi - \frac{1}{2}|\nabla\Psi|^2\delta_{ij} \quad \text{for} \quad i,j = 1,2,3, \tag{1.2}$$

where $\otimes$ denotes the tensor product, I is 3×3 identity matrix and δ_{ij} is the Kronecker symbol. μ is the kinematic viscosity, ε is the dielectric constant of the fluid, known as the Debye length, related to vacuum permittivity, the relative permittivity and characteristic charge density. $D_1 = \frac{kT_0\nu_1}{e}$, $D_2 = \frac{kT_0\nu_2}{e}$, ν_1 and ν_2 are the diffusion and mobility

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coefficients of the charges[1]. Since the concrete values of the constants $\mu,\varepsilon,D_1,D_2,\nu_1$ and ν_2 play no roles in our discussion, for simplicity, we shall assume them to be all equal to one throughout the paper.

System (1.1), first proposed by Rubinstein [28], is capable of describing electrochemical and fluid-mechanical transport throughout the cellular environment, we also refer the readers to [1, 12, 20, 30, 33] and the related references therein for more details of physical background and applied aspects about this model. For mathematical analysis, based on Kato's semigroup framework, Jerome [19] established the local existence of smooth solutions to (1.1). The global existence of weak solutions of the initial/boundary-value problem to (1.1) has been established by Jerome–Sacco [21], Ryham [29] and Schmuck [33]. Recently, Bothe–Fischer–Saal [5] proved the existence of unique local strong solutions in bounded domains $\Omega\subset\mathbb{R}^n$ for any $n\geq 2$, as well as the existence of unique global strong solutions and exponential convergence to uniquely determined steady states in two dimensions; moreover, based on the intrinsic energy structure, Aubin–Simon's compactness arguments, and maximal L^p-regularity, Fischer–Saal [16] further established global existence of weak solutions in a three-dimensional bounded domain. For the Cauchy problem, the small data global existence and large data local existence of strong solutions in various scaling invariant spaces have been studied by [11,35,38–40]. Notice that the Navier–Stokes (N-S) equations is a subsystem of (1.1) (i.e., $v=w=\Psi=0$), one can not expect better results than for the N-S equations. Hence, in the case of three dimensional space, the regularity and uniqueness of global weak solutions or global existence of smooth solutions to system (1.1) are still challenging open problems. Some regularity and uniqueness issues have been studied by [13, 15] even for more general system for the electro–kinetic fluid model.

In the present paper, we are interested in the blow-up issue for the short time smooth solution of system (1.1). It is well-known that if the divergence-free initial velocity $u_0\in H^3(\mathbb{R}^3)$, initial charged densities $v_0,w_0\in L^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)$ and $v_0,w_0\geq 0$, then there exists a time $T_*=T_*(u_0,v_0,w_0)>0$ such that system (1.1) admits a unique local solution $(u,v,w)\in\mathbb{R}^3\times[0,T_*)$ satisfying (cf. [37])

$$u\in C([0,T];H^3(\mathbb{R}^3))\cap L^\infty(0,T;H^3(\mathbb{R}^3))\cap L^2(0,T;H^4(\mathbb{R}^3)), \tag{1.3}$$

and

$$v,w\in C([0,T];H^2(\mathbb{R}^3))\cap L^\infty(0,T;H^2(\mathbb{R}^3))\cap L^2(0,T;H^3(\mathbb{R}^3)), \tag{1.4}$$

for all $0<T<T_*$. Moreover, it holds that $v\geq 0$ and $w\geq 0$ a.e. in $\mathbb{R}^3\times[0,T_*)$. Here we emphasize that such an existence theorem gives no indication as to whether solutions actually lose their regularity or the manner in which they may do so. Assume that T_* is the maximum value such that (1.3) and (1.4) hold, the purpose of this paper is to characterize such a T_*.

To illuminate the motivations of our paper in detail, let us recall the well-known results for the 3D N-S equations, after the celebrated works of Leray [24] and Hopf [18] on the global existence of weak solutions, the global regularity issue has been extensively investigated and many important regularity criteria have been established (e.g., [3, 4, 6, 7, 10, 14, 17, 22, 23, 25–27, 31, 32, 34, 41] and the references therein). The well-known Prodi–Serrin's conditions (see [10,26,34]) state that if $0<T_*<\infty$ is the first

[1]Here T_0 is the ambient temperature, k is the Boltzmann constant, and e is the charge mobility.

finite singular time of local smooth solution u, then

$$\int_0^{T_*} \|u(t)\|_{L^p}^q \mathrm{d}t = +\infty \quad \text{for all } \frac{2}{q}+\frac{3}{p}\le 1, 2<q\le\infty \text{ and } 3\le p<\infty. \tag{1.5}$$

Beirão da Veiga [3] established another Prodi–Serrin-type criterion by replacing (1.5) as

$$\int_0^{T_*} \|\nabla u(t)\|_{L^p}^q \mathrm{d}t = +\infty \quad \text{for all } \frac{2}{q}+\frac{3}{p}\le 2, 1\le q<\infty \text{ and } \frac{3}{2}<p\le\infty. \tag{1.6}$$

Beale–Kato–Majda in [2] proved that the vorticity $\omega = \nabla\times u$ will break down at the first finite singular time T_*, i.e.,

$$\int_0^{T_*} \|\omega(t)\|_{L^\infty} \mathrm{d}t = +\infty. \tag{1.7}$$

Further regularity criteria via only one velocity component or one of the entries of the velocity gradient tensor or the pressure can be found in [6–9,14,27,41] and the references therein. Here we would like to mention that Zhou–Pokorný [41] established that the local smooth solution u to the 3D N-S equations can be continued past any time $T>0$ provided that there holds

$$\int_0^{T} \|u_3(t)\|_{L^p}^q \mathrm{d}t < +\infty \quad \text{with } \frac{2}{q}+\frac{3}{p}\le \frac{3}{4}+\frac{1}{2p} \text{ and } \frac{10}{3}<p<\infty, \tag{1.8}$$

or

$$\int_0^{T} \|\nabla u_3(t)\|_{L^p}^q \mathrm{d}t < +\infty \quad \text{with } \frac{2}{q}+\frac{3}{p}\le \begin{cases} \frac{19}{12}+\frac{1}{2p}, & p\in(\frac{30}{19},3],\\ \frac{3}{2}+\frac{3}{4p}, & p\in(3,\infty]. \end{cases} \tag{1.9}$$

Notice that the property of scaling invariance plays an important role in studying the regularity theory of the solution, namely, if u solves the 3D N-S equations, then so does u_λ for all real numbers $\lambda>0$, where $u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t)$. Considering from this view of point, we can find that criteria (1.8) and (1.9) obtained by Zhou–Pokorný are away from the critical scale. Later, Chemin–Zhang [8] and Chemin–Zhang–Zhang [9] established that for $u_0\in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with $\nabla\times u_0\in L^{\frac{3}{2}}(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$, if the Leray–Hopf weak solution u to the 3D N-S equations satisfies

$$\int_0^{T} \|u_3(t)\|_{\dot{H}^{\frac{1}{2}-\frac{2}{q}}}^q \mathrm{d}t < \infty \quad \text{with } 4<q<\infty, \tag{1.10}$$

then u is regular on $\mathbb{R}^3\times(0,T)$. Since the Sobolev embedding theorem yields that if

$$\int_0^{T} \|\nabla u_3(t)\|_{L^p}^q \mathrm{d}t < \infty \quad \text{with } \frac{2}{q}+\frac{3}{p}=2, 4<q<\infty \text{ and } \frac{3}{2}<p<2, \tag{1.11}$$

then (1.10) holds. Hence, criterion (1.10) implies that the Leray–Hopf weak solution u of the 3D N-S equations satisfying (1.11) is regular on $\mathbb{R}^3\times(0,T)$. In a recent paper [27], Qian proved the regularity criteria in terms of only one of the nine components of the gradient of velocity field in the framework of anisotropic Lebesgue spaces, precisely, by using the method introduced in Cao–Titi [6,7]. The author established that if the local smooth solution u of the 3D N-S equations satisfies

$$\int_0^T \Big\| \|\partial_i u_j(t)\|_{L^\alpha_{x_i}} \Big\|^q_{L^\beta_{x_j x_k}} \,\mathrm{d}t < +\infty \quad \text{with } i,j=1,2,3 \text{ and } i\neq j, \tag{1.12}$$

where $\frac{2}{q}+\frac{1}{\alpha}+\frac{2}{\beta}\le\frac{2\alpha\beta+5\beta+\alpha}{4\alpha\beta}$, $1\le\alpha\le\beta$ and $\frac{7\alpha}{2\alpha+1}<\beta<\infty$, or

$$\int_0^T \Big\| \|\partial_j u_j(t)\|_{L^\alpha_{x_i}} \Big\|^q_{L^\beta_{x_j x_k}} \,\mathrm{d}t < +\infty \quad \text{with } j=1,2,3, \tag{1.13}$$

where $\frac{2}{q}+\frac{1}{\alpha}+\frac{2}{\beta}\le\frac{3\alpha\beta+4\beta+2\alpha}{4\alpha\beta}$, $1\le\alpha\le\beta$ and $2<\beta\le\infty$, then u is smooth up to time T.

As for system (1.1), Zhao–Bai [37] established that the Prodi–Serrin's criteria (1.5), (1.6) and the Beale–Kato–Majda's criterion (1.7) still hold for the local smooth solutions to (1.1). Moreover, the authors also showed that if $0<T_*<\infty$ is the first finite singular time of the smooth solution (u,v,w), then it holds that

$$\int_0^{T_*} \|\nabla_h u_h(t)\|_{\dot{B}^0_{\infty,\infty}}\,dt = +\infty, \tag{1.14}$$

where $\nabla_h \triangleq (\partial_1,\partial_2)$, $u^h\triangleq(u_1,u_2,0)$ is the horizontal component of the velocity field u, and $\dot{B}^0_{\infty,\infty}$ is the homogeneous Besov space. Recently, Zhao [36] established the following logarithmic Beale–Kato–Majda-type criterion, i.e.,

$$\int_0^{T_*} \frac{\|\omega(t)\|^{\frac{2}{2-\alpha}}_{\dot{B}^{-\alpha}_{\infty,\infty}}}{1+\ln(e+\|\omega(t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}})}\mathrm{d}t = +\infty \quad \text{for all } 0<\alpha<2.$$

These results reveal an important fact that the velocity field u plays a more dominant role than the charge densities v and w in the blow-up theory for local smooth solutions to system (1.1). Motivated by the papers cited above for the N-S equations and for system (1.1), the purpose of this paper is to establish a sufficient condition, which is in terms of one directional derivative of the horizontal component of the velocity field (i.e., $(\partial_i u_1, \partial_j u_2, 0)$ with $i,j\in\{1,2,3\}$), to control the breakdown of local smooth solutions of the system (1.1) in the framework of anisotropic Lebesgue spaces. Before stating our main result, let us first recall the following definition of the anisotropic Lebesgue spaces:

Definition 1.1. *Let $1\le p,q,r\le\infty$. We say that a function f belongs to $L^p(\mathbb{R}_{x_1};L^q(\mathbb{R}_{x_2};\ L^r(\mathbb{R}_{x_3})))$ if f is measurable on $\mathbb{R}^3$ and the following norm is finite:*

$$\Big\|\Big\|\|f\|_{L^p_{x_1}}\Big\|_{L^q_{x_2}}\Big\|_{L^r_{x_3}} := \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(x_1,x_2,x_3)|^p\,dx_1\right)^{\frac{q}{p}}dx_2\right)^{\frac{r}{q}}dx_3\right)^{\frac{1}{r}}$$

with the usual change as $p=\infty$ or $q=\infty$ or $r=\infty$.

Now, we state our main result as follows:

Theorem 1.1. *For $u_0\in H^3(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$, $(v_0,w_0)\in L^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)$ and $v_0,w_0\ge 0$, let $T_*>0$ be the finite and maximum value such that the 3D Navier–Stokes/Poisson–Nernst–Planck system (1.1) has a unique local smooth solution (u,v,w) on $(0,T_*)$. Then*

$$\int_0^{T_*}\left(\Big\|\|\partial_i u_1(t)\|_{L^\alpha_{x_i}}\Big\|^q_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}} + \Big\|\|\partial_j u_2(t)\|_{L^\alpha_{x_j}}\Big\|^q_{L^\beta_{x_{\hat{j}}x_{\tilde{j}}}}\right)dt = +\infty, \tag{1.15}$$

where

$$\frac{2}{q}+\frac{1}{\alpha}+\frac{2}{\beta}=m\in[1,\frac{3}{2})\ \text{and}\ \frac{3}{m}<\alpha\le\beta\le\frac{1}{m-1}. \tag{1.16}$$

Here, $(i,\hat{i},\tilde{i})$ *and* $(j,\hat{j},\tilde{j})$ *belong to the permutation group of* $\mathbb{S}_3:=\{1,2,3\}$.

REMARK 1.1. By Theorem 1.1, one obtains that if there exists a finite constant $M>0$ such that the corresponding velocity u satisfies

$$\int_0^{T_*}\left(\Big\|\|\partial_i u_1(t)\|_{L^\alpha_{x_i}}\Big\|^q_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}+\Big\|\|\partial_j u_2(t)\|_{L^\alpha_{x_j}}\Big\|^q_{L^\beta_{x_{\hat{j}}x_{\tilde{j}}}}\right)\mathrm{d}t\le M, \tag{1.17}$$

with $\frac{2}{q}+\frac{1}{\alpha}+\frac{2}{\beta}=m\in[1,\frac{3}{2})$ and $\frac{3}{m}<\alpha\le\beta\le\frac{1}{m-1}$, then the local smooth solution (u,v,w) to system (1.1) can be extended beyond the time T_*.

REMARK 1.2. We emphasize that if $i=j=3$, the blow-up criterion (1.15) in Theorem 1.1 becomes

$$\begin{gathered}\int_0^{T_*}\Big\|\|\partial_3 u_h(t)\|_{L^\alpha_{x_3}}\Big\|^q_{L^\beta_{x_1x_2}}\mathrm{d}t=+\infty,\\ \text{with}\ \frac{2}{q}+\frac{1}{\alpha}+\frac{2}{\beta}=m\in[1,\frac{3}{2})\ \text{and}\ \frac{3}{m}<\alpha\le\beta\le\frac{1}{m-1},\end{gathered} \tag{1.18}$$

where $u_h=(u_1,u_2,0)$ is the horizontal component of u. Furthermore, when we fix $\alpha=\beta$, then (1.18) becomes

$$\begin{gathered}\int_0^{T_*}\|\partial_3 u_h(t)\|^q_{L^\beta}\mathrm{d}t=+\infty,\\ \text{with}\ \frac{2}{q}+\frac{3}{\beta}=m\in[1,\frac{3}{2})\ \text{and}\ \frac{3}{m}<\beta\le\frac{1}{m-1}.\end{gathered}$$

Hence, Theorem 1.1 can be viewed as a generalization of (1.14) obtained by Zhao–Bai [37].

REMARK 1.3. When $v=w=\Psi=0$, system (1.1) becomes the 3D N-S equations. From Theorem 1.1 (see also Remark 1.1), one finds that for $u_0\in H^1(\mathbb{R}^3)$ with $\operatorname{div}u_0=0$, assume that u is the corresponding local smooth solution to the 3D N-S equations on $[0,T_*)$ for some $0<T_*<\infty$, if u satisfies

$$\int_0^{T_*}\left(\Big\|\|\partial_i u_1(t)\|_{L^\alpha_{x_i}}\Big\|^q_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}+\Big\|\|\partial_j u_2(t)\|_{L^\alpha_{x_j}}\Big\|^q_{L^\beta_{x_{\hat{j}}x_{\tilde{j}}}}\right)\mathrm{d}t\le M,$$

for some $M>0$, with $\frac{2}{q}+\frac{1}{\alpha}+\frac{2}{\beta}=m\in[1,\frac{3}{2})$ and $\frac{3}{m}<\alpha\le\beta\le\frac{1}{m-1}$, then u can be extended beyond time T_*. This result can be viewed as a generalization of [6,25] on the N-S equations.

We shall present the proof of Theorem 1.1 in the next section. Throughout the paper, we denote by C a harmless positive constant, which may depend on the initial data and T_*, and its value may change from line to line. The norms of the usual Lebesgue spaces $L^p(\mathbb{R}^3)$ (with $1\le p\le\infty$) are denoted by $\|\cdot\|_{L^p}$, while the directional derivatives of a function f are denoted by $\partial_i f=\frac{\partial f}{\partial x_i}$ with $i=1,2,3$.

2. Proof of Theorem 1.1

In this section, we shall give the proof of Theorem 1.1. Before doing it, let us recall the following useful inequality, which can be viewed as a generalization of the Sobolev-type embedding inequality.

LEMMA 2.1. *Let* $1\le\alpha,\beta,\xi,a,t\le+\infty, 1<r\le+\infty$, *and* $0\le\theta\le1$ *such that*

$$\frac{1}{a}+\frac{1}{t}=\frac{\beta-1}{\beta} \tag{2.1}$$

and

$$\frac{1}{(2r-1)\alpha}+\frac{\theta}{\alpha}=\frac{1-\theta}{\xi(\alpha-1)}. \tag{2.2}$$

Then there exists a positive generic constant C *such that for all* $f,g\in C_0^\infty(\mathbb{R}^3)$, *it holds that*

$$\begin{aligned}\left|\int_{\mathbb{R}^3}|f|^2|g|^2dx\right|\le& C\Big\|\|\partial_i f\|_{L^\alpha_{x_i}}\Big\|^{\frac{1}{r}}_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}\Big\|\|\partial_i f\|_{L^\alpha_{x_i}}\Big\|^{\frac{\theta(2r-1)}{r}}_{L^{\theta(2r-1)t}_{x_{\hat{i}}x_{\tilde{i}}}}\Big\|\|f\|_{L^\xi_{x_i}}\Big\|^{\frac{(1-\theta)(2r-1)}{r}}_{L^{(1-\theta)(2r-1)a}_{x_{\hat{i}}x_{\tilde{i}}}}\\ &\times\|g\|^{\frac{2(r-1)}{r}}_{L^2}\|(\partial_{\hat{i}},\partial_{\tilde{i}})g\|^{\frac{2}{r}}_{L^2}.\end{aligned} \tag{2.3}$$

Here $(i,\hat{i},\tilde{i})$ *belongs to the permutation group of* $\mathbb{S}_3:=\mathrm{span}\{1,2,3\}$.

Proof. The proof of (2.3) is standard, here we give a proof for the reader's convenience. Notice that direct calculus yields that

$$\begin{aligned}|f(x_1,x_2,x_3)|^{2r}\le& C\int_{-\infty}^{x_1}|f(\tau,x_2,x_3)|^{2r-1}|\partial_1 f(\tau,x_2,x_3)|\mathrm{d}\tau\\ \le& C\int_{\mathbb{R}}|f(\tau,x_2,x_3)|^{2r-1}|\partial_1 f(\tau,x_2,x_3)|\mathrm{d}\tau,\\ |f(x_1,x_2,x_3)|^{2r}\le& C\int_{-\infty}^{x_2}|f(x_1,\tau,x_3)|^{2r-1}|\partial_2 f(x_1,\tau,x_3)|\mathrm{d}\tau\\ \le& C\int_{\mathbb{R}}|f(x_1,\tau,x_3)|^{2r-1}|\partial_2 f(x_1,\tau,x_3)|\mathrm{d}\tau,\\ |f(x_1,x_2,x_3)|^{2r}\le& C\int_{-\infty}^{x_3}|f(x_1,x_2,\tau)|^{2r-1}|\partial_3 f(x_1,x_2,\tau)|\mathrm{d}\tau\\ \le& C\int_{\mathbb{R}}|f(x_1,x_2,\tau)|^{2r-1}|\partial_3 f(x_1,x_2,\tau)|\mathrm{d}\tau.\end{aligned}$$

By using these facts above together with Hölder's inequality, one has

$$\begin{aligned}\int_{\mathbb{R}^3}|f|^2|g|^2\mathrm{d}x\le&\int_{\mathbb{R}^2}\left(\max_{x_i\in\mathbb{R}}|f|^2\cdot\int_{\mathbb{R}}|g|^2\mathrm{d}x_i\right)\mathrm{d}x_{\hat{i}}\mathrm{d}x_{\tilde{i}}\\ \le& C\left(\int_{\mathbb{R}^2}\max_{x_i\in\mathbb{R}}|f|^{2r}\mathrm{d}x_{\hat{i}}\mathrm{d}x_{\tilde{i}}\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}|g|^2\mathrm{d}x_i\right)^{\frac{r}{r-1}}\mathrm{d}x_{\hat{i}}\mathrm{d}x_{\tilde{i}}\right)^{\frac{r-1}{r}}\\ \le& C\left(\int_{\mathbb{R}^3}|f|^{2r-1}|\partial_i f|\mathrm{d}x\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}|g|^2\mathrm{d}x_i\right)^{\frac{r}{r-1}}\mathrm{d}x_{\hat{i}}\mathrm{d}x_{\tilde{i}}\right)^{\frac{r-1}{r}}.\end{aligned} \tag{2.4}$$

By using Hölder's inequality and interpolation inequality, one gets

$$\begin{aligned}\int_{\mathbb{R}^3}|f|^{2r-1}|\partial_i f|\mathrm{d}x=&\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}|f|^{2r-1}|\partial_i f|\mathrm{d}x_i\right)\mathrm{d}x_{\hat{i}}\mathrm{d}x_{\tilde{i}}\le C\int_{\mathbb{R}^2}\|\partial_i f\|_{L^\alpha_{x_i}}\|f\|^{2r-1}_{L_{x_i}^{\frac{(2r-1)\alpha}{\alpha-1}}}\mathrm{d}x_{\hat{i}}\mathrm{d}x_{\tilde{i}}\\ \le& C\int_{\mathbb{R}^2}\|\partial_i f\|_{L^\alpha_{x_i}}\|\partial_i f\|^{(2r-1)\theta}_{L^\alpha_{x_i}}\|f\|^{(2r-1)(1-\theta)}_{L^\xi_{x_i}}\mathrm{d}x_{\hat{i}}\mathrm{d}x_{\tilde{i}}\\ \le& C\left\|\|\partial_i f\|_{L^\alpha_{x_i}}\right\|_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}\left\|\|\partial_i f\|^{(2r-1)\theta}_{L^\alpha_{x_i}}\|f\|^{(2r-1)(1-\theta)}_{L^\xi_{x_i}}\right\|_{L^{\frac{\beta}{\beta-1}}_{x_{\hat{i}}x_{\tilde{i}}}}\\ \le& C\left\|\|\partial_i f\|_{L^\alpha_{x_i}}\right\|_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}\left\|\|\partial_i f\|_{L^\alpha_{x_i}}\right\|^{(2r-1)\theta}_{L^{(2r-1)\theta t}_{x_{\hat{i}}x_{\tilde{i}}}}\left\|\|f\|_{L^\xi_{x_i}}\right\|^{(2r-1)(1-\theta)}_{L^{(2r-1)(1-\theta)a}_{x_{\hat{i}}x_{\tilde{i}}}},\end{aligned}\tag{2.5}$$

where α,β,ξ,a,t and θ satisfy (2.1) and (2.2). On the other hand, by using Minkowski's inequality, Hölder's inequality and interpolation inequality, one obtains

$$\begin{aligned}\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}|g|^2\mathrm{d}x_i\right)^{\frac{r}{r-1}}\mathrm{d}x_{\hat{i}}\mathrm{d}x_{\tilde{i}}\right)^{\frac{r-1}{r}}\le&\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}|g|^{\frac{2r}{r-1}}\mathrm{d}x_{\hat{i}}\mathrm{d}x_{\tilde{i}}\right)^{\frac{r-1}{r}}\mathrm{d}x_i\\ \le&\int_{\mathbb{R}}\|g\|^{\frac{2(r-1)}{r}}_{L^2_{x_{\hat{i}}x_{\tilde{i}}}}\|(\partial_{\hat{i}},\partial_{\tilde{i}})g\|^{\frac{2}{r}}_{L^2_{x_{\hat{i}}x_{\tilde{i}}}}\mathrm{d}x_i\\ \le&\|g\|^{\frac{2(r-1)}{r}}_{L^2}\|(\partial_{\hat{i}},\partial_{\tilde{i}})g\|^{\frac{2}{r}}_{L^2}.\end{aligned}\tag{2.6}$$

Inserting (2.5) and (2.6) into (2.4), one obtains (2.1), and this completes the proof of Lemma 2.1. □

By using Lemma 2.1 above, let us give the proof of Theorem 1.1.

Proof of Theorem 1.1. We shall prove Theorem 1.1 by contradiction. By [37], we know that, under the assumptions of Theorem 1.1, there exists a local smooth solution (u,v,w) to system (1.1) such that (1.3) and (1.4) hold. Assume that $[0,T_*)$ is the maximal existence interval of the local smooth solution (u,v,w), and (1.15) is not true, i.e., there is a finite number $M>0$ such that

$$\int_0^{T_*}\left(\left\|\|\partial_i u_1(\tau)\|_{L^\alpha_{x_i}}\right\|^q_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}+\left\|\|\partial_j u_2(\tau)\|_{L^\alpha_{x_j}}\right\|^q_{L^\beta_{x_{\hat{j}}x_{\tilde{j}}}}\right)\mathrm{d}\tau\le M,\tag{2.7}$$

where $\frac{2}{q}+\frac{1}{\alpha}+\frac{2}{\beta}=m\in[1,\frac{3}{2})$ and $\frac{3}{m}<\alpha\le\beta\le\frac{1}{m-1}$. In what follows, we shall show that

$$\sup_{0\le t\le T_*}(\|u(t)\|_{H^3}+\|(v,w)(t)\|_{H^2})\le C,\tag{2.8}$$

for some positive constant C depending only on u_0,v_0,w_0,T_* and M. The above estimate (2.8) is enough to ensure that the local smooth solution (u,v,w) can be extended beyond the time T_*, which leads to a contradiction as T_* is the maximum existence time.

Before doing it, let us first notice that by the maximum principle, one can deduce that if v_0 and w_0 are non-negative, then we have

$$v\ge0 \text{ and } w\ge0\qquad \text{a.e. } (x,t)\in\mathbb{R}^3\times(0,T_*).$$

We refer the readers to [33] for more details.

Step 1. L^2-bound of (u,v,w). Exactly as the same arguments of Zhao–Bai [37], we have

$$\|v(t)\|^2_{L^2}+\|w(t)\|^2_{L^2}+2\int_0^t\left(\|\nabla v(\tau)\|^2_{L^2}+\|\nabla w(\tau)\|^2_{L^2}\right)d\tau$$

$$\leq \|v_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 := C_0 \quad \text{for all } 0 < t \leq T_*, \tag{2.9}$$

and

$$\|u(t)\|_{L^2}^2 + \|\nabla\Psi(t)\|_{L^2}^2 + 2\int_0^t \left(\|\nabla u(\tau)\|_{L^2}^2 + \|\Delta\Psi(\tau)\|_{L^2}^2\right) d\tau \leq C_1 \tag{2.10}$$

for all $0 < t \leq T_*$, where C_1 is a constant depending only on $\|u_0\|_{L^2}^2$ and $\|(v_0, w_0)\|_{L^1 \cap L^2}$.

Step 2. H^1-bound of (u,v,w). Due to (1.2), one can rewrite $(1.1)_1$ as

$$\partial_t u + (u\cdot\nabla)u - \Delta u + \nabla P = \Delta\Psi\nabla\Psi. \tag{2.11}$$

Multiplying (2.11) by Δu, and integrating over $\mathbb{R}^3$, after integration by parts, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 &= -\int_{\mathbb{R}^3} u\cdot\nabla u\cdot\Delta u dx + \int_{\mathbb{R}^3} \Delta\Psi\nabla\Psi\cdot\Delta u dx \\ &:= I_1 + I_2.\end{aligned} \tag{2.12}$$

By using Hölder's inequality, interpolation inequality, Young's inequality, $(1.1)_5$, (2.9) and (2.10), one can bound I_2 as

$$\begin{aligned} I_2 &\leq C\|\nabla\Psi\|_{L^4}\|\Delta\Psi\|_{L^4}\|\Delta u\|_{L^2} \leq \frac{1}{8}\|\Delta u\|_{L^2}^2 + C\|(v,w)\|_{L^4}^2\|\nabla\Psi\|_{L^4}^2 \\ &\leq \frac{1}{8}\|\Delta u\|_{L^2}^2 + C\|(v,w)\|_{L^2}^{\frac{1}{2}}\|(\nabla v,\nabla w)\|_{L^2}^{\frac{3}{2}}\|\nabla\Psi\|_{L^2}^{\frac{1}{2}}\|\Delta\Psi\|_{L^2}^{\frac{3}{2}} \\ &\leq \frac{1}{8}\|\Delta u\|_{L^2}^2 + C\|(v,w)\|_{L^2}^2\|(\nabla v,\nabla w)\|_{L^2}^{\frac{3}{2}}\|\nabla\Psi\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{1}{8}\|\Delta u\|_{L^2}^2 + C\|(v,w)\|_{L^2}^2\|(\nabla v,\nabla w)\|_{L^2}^2 + C\|(v,w)\|_{L^2}^2\|\nabla\Psi\|_{L^2}^2 \\ &\leq \frac{1}{8}\|\Delta u\|_{L^2}^2 + C(\|(\nabla v,\nabla w)\|_{L^2}^2 + 1), \end{aligned} \tag{2.13}$$

while I_1 can be rewritten as

$$\begin{aligned} I_1 = &\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i u_j\partial_k^2 u_j dx = \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i u_j\partial_k^2 u_j dx + \sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_3\partial_3 u_j\partial_k^2 u_j dx \\ &+ \sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3\partial_3 u_3\partial_k^2 u_3 dx + \sum_{j=1}^{3}\int_{\mathbb{R}^3} u_3\partial_3 u_j\partial_3^2 u_j dx \\ = &\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i u_j\partial_k^2 u_j dx - \sum_{j,k=1}^{2}\int_{\mathbb{R}^3} \partial_k u_3\partial_3 u_j\partial_k u_j dx - \sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_3\partial_3\partial_k u_j\partial_k u_j dx \\ &- \sum_{k=1}^{2}\int_{\mathbb{R}^3} \partial_k u_3\partial_3 u_3\partial_k u_3 dx - \sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3\partial_3\partial_k u_3\partial_k u_3 dx + \sum_{j=1}^{3}\int_{\mathbb{R}^3} u_3\partial_3 u_j\partial_3^2 u_j dx. \end{aligned} \tag{2.14}$$

Notice that the divergence-free condition $(1.1)_2$ yields that

$$\partial_1 u_1 + \partial_2 u_2 = -\partial_3 u_3,$$

from which, one can deduce that

$$\begin{aligned}
&\sum_{j,k=1}^{2}\left(\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_j\partial_k u_j \mathrm{d}x+\int_{\mathbb{R}^3}u_3\partial_3\partial_k u_j\partial_k u_j \mathrm{d}x\right)\\
=&-\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_j\partial_3\partial_k u_3\partial_k u_j \mathrm{d}x-\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_j\partial_k u_3\partial_3\partial_k u_j \mathrm{d}x-\frac{1}{2}\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}\partial_3 u_3\partial_k u_j\partial_k u_j \mathrm{d}x\\
=&-\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_j\partial_3\partial_k u_3\partial_k u_j \mathrm{d}x-\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_j\partial_k u_3\partial_3\partial_k u_j \mathrm{d}x+\frac{1}{2}\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}\partial_i u_i\partial_k u_j\partial_k u_j \mathrm{d}x\\
=&-\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_j\partial_3\partial_k u_3\partial_k u_j \mathrm{d}x-\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_j\partial_k u_3\partial_3\partial_k u_j \mathrm{d}x-\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}u_i\partial_i\partial_k u_j\partial_k u_j \mathrm{d}x\\
\leq& C\left(\int_{\mathbb{R}^3}|u_1||\nabla u||\nabla^2 u|\mathrm{d}x+\int_{\mathbb{R}^3}|u_2||\nabla u||\nabla^2 u|\mathrm{d}x\right)
\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}
&\sum_{k=1}^{2}\left(\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_3\partial_k u_3 \mathrm{d}x-\int_{\mathbb{R}^3}u_3\partial_3\partial_k u_3\partial_k u_3 \mathrm{d}x\right)+\sum_{j=1}^{3}\int_{\mathbb{R}^3}u_3\partial_3 u_j\partial_3^2 u_j \mathrm{d}x\\
=&\frac{1}{2}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_3\partial_k u_3 \mathrm{d}x+\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_3\partial_3 u_j\partial_3 u_j \mathrm{d}x\\
=&-\frac{1}{2}\sum_{i,k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_3\partial_i u_i\partial_k u_3 \mathrm{d}x-\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_i u_i\partial_3 u_j\partial_3 u_j \mathrm{d}x\\
=&\sum_{i,k=1}^{2}\int_{\mathbb{R}^3}u_i\partial_i\partial_k u_3\partial_k u_3 \mathrm{d}x+\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}u_i\partial_3 u_j\partial_3\partial_i u_j \mathrm{d}x\\
\leq& C\left(\int_{\mathbb{R}^3}|u_1||\nabla u||\nabla^2 u|\mathrm{d}x+\int_{\mathbb{R}^3}|u_2||\nabla u||\nabla^2 u|\mathrm{d}x\right).
\end{aligned}\tag{2.16}$$

Combining (2.14)–(2.16) together, it follows that

$$I_1\leq C\left(\int_{\mathbb{R}^3}|u_1||\nabla u||\nabla^2 u|\mathrm{d}x+\int_{\mathbb{R}^3}|u_2||\nabla u||\nabla^2 u|\mathrm{d}x\right):=I_{11}+I_{12}.\tag{2.17}$$

In what follows, we shall estimate the two terms I_{11} and I_{12} on the right-hand side of (2.17). By using Young's inequality and Lemma 2.1, one can estimate I_{11} as follows

$$\begin{aligned}
I_{11}=&\frac{1}{16}\int_{\mathbb{R}^3}|\nabla^2 u|^2\mathrm{d}x+C\int_{\mathbb{R}^3}|u_1|^2|\nabla u|^2\mathrm{d}x\\
\leq&\frac{1}{16}\|\nabla^2 u\|_{L^2}^2+C\left\|\|\partial_i u_1\|_{L^{\alpha}_{x_i}}\right\|_{L^{\beta}_{x_{\hat{i}}x_{\tilde{i}}}}^{\frac{1}{r}}\left\|\|\partial_i u_1\|_{L^{\alpha}_{x_i}}\right\|_{L^{(2r-1)\theta t}_{x_{\hat{i}}x_{\tilde{i}}}}^{\frac{(2r-1)\theta}{r}}\\
&\cdot\left\|\|u_1\|_{L^{\xi}_{x_i}}\right\|_{L^{(2r-1)(1-\theta)a}_{x_{\hat{i}}x_{\tilde{i}}}}^{\frac{(2r-1)(1-\theta)}{r}}\|\nabla u\|_{L^2}^{\frac{2(r-1)}{r}}\|\nabla^2 u\|_{L^2}^{\frac{2}{r}},
\end{aligned}$$

where $1<\alpha\leq\beta\leq+\infty$, $\xi,a,t\in[1,+\infty]$, $\theta\in[0,1]$ and $1<r\leq+\infty$ satisfy (2.1) and (2.2). By selecting

$$a=\frac{\xi}{(2r-1)(1-\theta)}\ \text{and}\ t=\frac{\beta}{(2r-1)\theta},\tag{2.18}$$

then one obtains that

$$\begin{aligned}
I_{11} \leq & \frac{1}{16}\|\nabla^2 u\|_{L^2}^2 + C\Big\|\|\partial_i u_1\|_{L^{\alpha}_{x_i}}\Big\|^{\frac{1}{r}}_{L^{\beta}_{x_{\hat{i}}x_{\tilde{i}}}} \Big\|\|\partial_i u_1\|_{L^{\alpha}_{x_i}}\Big\|^{\frac{(2r-1)\theta}{r}}_{L^{(2r-1)\theta t}_{x_{\hat{i}}x_{\tilde{i}}}} \\
& \cdot \Big\|\|u_1\|_{L^{\xi}_{x_i}}\Big\|^{\frac{(2r-1)(1-\theta)}{r}}_{L^{(2r-1)(1-\theta)a}_{x_{\hat{i}}x_{\tilde{i}}}} \|\nabla u\|_{L^2}^{\frac{2(r-1)}{r}} \|\nabla^2 u\|_{L^2}^{\frac{2}{r}} \\
= & \frac{1}{16}\|\nabla^2 u\|_{L^2}^2 + C\Big\|\|\partial_i u_i\|_{L^{\alpha}_{x_i}}\Big\|^{\frac{1+(2r-1)\theta}{r}}_{L^{\beta}_{x_{\hat{i}}x_{\tilde{i}}}} \|u\|_{L^{\xi}}^{\frac{(2r-1)(1-\theta)}{r}} \|\nabla u\|_{L^2}^{\frac{2(r-1)}{r}} \|\nabla^2 u\|_{L^2}^{\frac{2}{r}} \\
\leq & \frac{1}{8}\|\nabla^2 u\|_{L^2}^2 + C\Big\|\|\partial_i u_1\|_{L^{\alpha}_{x_i}}\Big\|^{\frac{1+(2r-1)\theta}{r-1}}_{L^{\beta}_{x_{\hat{i}}x_{\tilde{i}}}} \|u\|_{L^{\xi}}^{\frac{(2r-1)(1-\theta)}{r-1}} \|\nabla u\|_{L^2}^2,
\end{aligned}$$

where we have used Young's inequality in the last inequality, and $1<\alpha\leq\beta\leq+\infty$, $\xi\in[1,+\infty]$, $\theta\in[0,1]$ and $1<r\leq+\infty$ satisfying

$$\begin{cases} \frac{1}{(2r-1)\alpha}+\frac{\theta}{\alpha}=\frac{1-\theta}{\xi(\alpha-1)}, \\ \frac{(2r-1)\theta}{\beta}+\frac{(2r-1)(1-\theta)}{\xi}=\frac{\beta-1}{\beta}. \end{cases} \tag{2.19}$$

By setting

$$r=\frac{(\alpha-1)\beta\xi+\alpha\beta}{2(\alpha+\alpha\beta-\beta)}, \quad \xi=\frac{2r(\alpha+\alpha\beta-\beta)-\alpha\beta}{(\alpha-1)\beta} \text{ and } \theta=\frac{(2r-1)\alpha-\xi(\alpha-1)}{(2r-1)(\xi(\alpha-1)+\alpha)}\in[0,1], \tag{2.20}$$

it is easy to see that r,ξ and θ satisfy (2.19). Furthermore, it is easy to check that the selected $\alpha,\beta,\xi,r,\theta,a$ and t above satisfy all assumptions of (2.3), thus we have

$$\begin{aligned}
I_{11} \leq & \frac{1}{8}\|\nabla^2 u\|_{L^2}^2 + C\Big\|\|\partial_i u_1\|_{L^{\alpha}_{x_i}}\Big\|^{\frac{1+(2r-1)\theta}{r-1}}_{L^{\beta}_{x_{\hat{i}}x_{\tilde{i}}}} \|u\|_{L^{\xi}}^{\frac{(2r-1)(1-\theta)}{r-1}} \|\nabla u\|_{L^2}^2 \\
= & \frac{1}{8}\|\nabla^2 u\|_{L^2}^2 + C\Big\|\|\partial_i u_1\|_{L^{\alpha}_{x_i}}\Big\|^{\frac{2r\alpha}{(r-1)(\xi(\alpha-1)+\alpha)}}_{L^{\beta}_{x_{\hat{i}}x_{\tilde{i}}}} \|u\|_{L^{\xi}}^{\frac{2r\xi(\alpha-1)}{(r-1)(\xi(\alpha-1)+\alpha)}} \|\nabla u\|_{L^2}^2.
\end{aligned}$$

Now, for $m\in[1,\frac{3}{2})$ and $\frac{3}{m}<\alpha\leq\beta\leq\frac{1}{m-1}$, by selecting

$$r=\frac{(\frac{5}{2}-m)\alpha\beta}{\alpha+\alpha\beta-\beta}=\frac{(5-2m)\alpha\beta}{2(\alpha+\alpha\beta-\beta)}, \tag{2.21}$$

then we have from (2.20) that

$$\xi=\frac{2\alpha(2-m)}{\alpha-1}. \tag{2.22}$$

Hence

$$\begin{aligned}
I_{11} \leq & \frac{1}{8}\|\nabla^2 u\|_{L^2}^2 + C\Big\|\|\partial_i u_1\|_{L^{\alpha}_{x_i}}\Big\|^{\frac{2r\alpha}{(r-1)(\xi(\alpha-1)+\alpha)}}_{L^{\beta}_{x_{\hat{i}}x_{\tilde{i}}}} \|u\|_{L^{\xi}}^{\frac{2r\xi(\alpha-1)}{(r-1)(\xi(\alpha-1)+\alpha)}} \|\nabla u\|_{L^2}^2 \\
= & \frac{1}{8}\|\nabla^2 u\|_{L^2}^2 + C\Big\|\|\partial_i u_1\|_{L^{\alpha}_{x_i}}\Big\|^{\frac{2r}{(r-1)(5-2m)}}_{L^{\beta}_{x_{\hat{i}}x_{\tilde{i}}}} \|u\|_{L^{\xi}}^{\frac{4r(2-m)}{(r-1)(5-2m)}} \|\nabla u\|_{L^2}^2 \\
= & \frac{1}{8}\|\nabla^2 u\|_{L^2}^2 + C\Big\|\|\partial_i u_1\|_{L^{\alpha}_{x_i}}\Big\|^{\frac{2\alpha\beta}{(3-2m)\alpha\beta-2\alpha+2\beta}}_{L^{\beta}_{x_{\hat{i}}x_{\tilde{i}}}} \|u\|_{L^{\xi}}^{\frac{4\alpha\beta(2-m)}{(3-2m)\alpha\beta-2\alpha+2\beta}} \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{2.23}$$

Applying Hölder's inequality with

$$\frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}+\frac{3(1-m)\alpha\beta+3\beta}{(3-2m)\alpha\beta-2\alpha+2\beta}=1,$$
$$\text{with } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}\in(0,1] \quad \text{by (1.16).}$$

Then (2.23) becomes

$$I_{11}\le\frac{1}{8}\|\nabla^2u\|_{L^2}^2+\begin{cases} C\Big(\Big\|\|\partial_iu_1\|_{L^\alpha_{x_i}}\Big\|_{L^\beta_{x_{\hat i}x_{\tilde i}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}}+\|u\|_{L^\xi}^{\frac{4\alpha(2-m)}{3(1-m)\alpha+3}}\Big)\|\nabla u\|_{L^2}^2 \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}\in(0,1), \text{ i.e., } \frac{3}{m}<\alpha\le\beta<\frac{1}{m-1}, \\ C\Big\|\|\partial_iu_1\|_{L^\alpha_{x_i}}\Big\|_{L^\beta_{x_{\hat i}x_{\tilde i}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}}\|u\|_{L^2}^{\frac{4\alpha\beta(2-m)}{m\alpha\beta-\beta-2\alpha}}\|\nabla u\|_{L^2}^2 \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}=1, \text{ i.e., } \alpha=\beta=\frac{1}{m-1}. \end{cases}$$

The term I_{12} can be estimated in a similar way. Hence

$$I_1\le\frac{1}{4}\|\Delta u\|_{L^2}^2+\begin{cases} C\Big(\mathcal{G}(t)+\|u\|_{L^\xi}^{\frac{4\alpha(2-m)}{3(1-m)\alpha+3}}\Big)\|\nabla u\|_{L^2}^2 \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}\in(0,1), \text{ i.e., } \frac{3}{m}<\alpha\le\beta<\frac{1}{m-1}, \\ C\mathcal{G}(t)\|u\|_{L^2}^{\frac{4\alpha\beta(2-m)}{m\alpha\beta-\beta-2\alpha}}\|\nabla u\|_{L^2}^2 \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}=1, \text{ i.e., } \alpha=\beta=\frac{1}{m-1}, \end{cases} \tag{2.24}$$

where we have used the identity $\|\nabla^2u\|_{L^2}^2=\|\Delta u\|_{L^2}^2$, and

$$\mathcal{G}(t)=\Big\|\|\partial_iu_1\|_{L^\alpha_{x_i}}\Big\|_{L^\beta_{x_{\hat i}x_{\tilde i}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}}+\Big\|\|\partial_ju_2\|_{L^\alpha_{x_j}}\Big\|_{L^\beta_{x_{\hat j}x_{\tilde j}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}}, \tag{2.25}$$

with $\frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}\in(0,1]$, i.e., $\frac{3}{m}<\alpha\le\beta\le\frac{1}{m-1}$. Plugging (2.13) and (2.24) into (2.12), one gets

$$\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2+\|\Delta u\|_{L^2}^2$$
$$\le\begin{cases} C\Big(\mathcal{G}(t)+\|u(t)\|_{L^\xi}^{\frac{4\alpha(2-m)}{3(1-m)\alpha+3}}+\|(\nabla v,\nabla w)(t)\|_{L^2}^2+1\Big)(e+\|\nabla u\|_{L^2}^2) \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}\in(0,1), \text{ i.e., } \frac{3}{m}<\alpha\le\beta<\frac{1}{m-1}, \\ C\Big(\mathcal{G}(t)+\|(\nabla v,\nabla w)(t)\|_{L^2}^2+1\Big)\|u\|_{L^2}^{\frac{4\alpha\beta(2-m)}{m\alpha\beta-\beta-2\alpha}}(e+\|\nabla u\|_{L^2}^2) \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}=1, \text{ i.e., } \alpha=\beta=\frac{1}{m-1}. \end{cases} \tag{2.26}$$

Notice that from (2.9), (2.10) and the standard interpolation inequality, it follows that

$$(u,v,w)\in L^a(0,T_*;L^b(\mathbb{R}^3)) \quad \text{with } \frac{2}{a}+\frac{3}{b}=\frac{3}{2} \text{ and } 2\le b\le6.$$

On the other hand, it is easy to see that

$$2<\xi=\frac{2\alpha(2-m)}{\alpha-1}<6 \quad \text{if } \frac{3}{m}<\alpha\le\beta<\frac{1}{m-1}$$

and then

$$\frac{3(1-m)\alpha+3}{2\alpha(2-m)}+\frac{3}{\xi}=\frac{3(1-m)\alpha+3}{2\alpha(2-m)}+\frac{3(\alpha-1)}{2\alpha(2-m)}=\frac{3}{2}.$$

Thus, one obtains that

$$(u,v,w)\in L^{\frac{3(1-m)\alpha+3}{2\alpha(2-m)}}(0,T_*;L^{\xi}(\mathbb{R}^3)) \quad \text{for } \frac{3}{m}<\alpha\le\beta<\frac{1}{m-1}. \tag{2.27}$$

When $\alpha=\beta=\frac{1}{m-1}$, we can derive from (2.22) that $\xi=2$, and then from energy inequality (2.10) that

$$\|u(t)\|_{L^2}\le C_1 \text{ for all } 0\le t\le T_*. \tag{2.28}$$

By using these facts above, (2.9) and the assumption (2.7)[2], one can apply Grönwall's inequality on (2.26) to get that

$$\begin{aligned}
&\sup_{0\le t\le T_*}\|\nabla u(t)\|_{L^2}^2+\int_0^{T_*}\|\Delta u\|_{L^2}^2\mathrm{d}t\\
\le&\left(e+\|\nabla u_0\|_{L^2}^2\right)\times\begin{cases}
\exp\left\{C\int_0^{T_*}\left(\mathcal{G}(\tau)+\|(\nabla v(\tau),\nabla w(\tau))\|_{L^2}^2+1\right)d\tau\right\}\\
\qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}\in(0,1), \text{ i.e., } \frac{3}{m}<\alpha\le\beta<\frac{1}{m-1}\\
\exp\left\{C\int_0^{T_*}\left(\mathcal{G}(\tau)+\|(\nabla v,\nabla w)\|_{L^2}^2+1\right)\|u\|_{L^2}^{\frac{4\alpha\beta(2-m)}{m\alpha\beta-\beta-2\alpha}}\mathrm{d}\tau\right\}\\
\qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta}=1, \text{ i.e., } \alpha=\beta=\frac{1}{m-1}
\end{cases}\\
\le&C_2,
\end{aligned} \tag{2.29}$$

where C_2 is a positive constant depending only on M, T_*, $\|u_0\|_{H^1}$ and $\|(v_0,w_0)\|_{L^1\cap L^2}$.

To get H^1-bound of (v,w), we multiply Δv to $(1.1)_3$, and integrate it over $\mathbb{R}^3$, it can be seen that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2+\|\Delta v\|_{L^2}^2=&-\int_{\mathbb{R}^3}(u\cdot\nabla)v\Delta v\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\cdot(v\nabla\Psi)\Delta v\mathrm{d}x\\
:=&I_3+I_4.
\end{aligned} \tag{2.30}$$

Applying Hölder's inequality, Young's inequality, (2.9) and (2.29), the two terms I_3 and I_4 on the right-hand side of (2.30) can be estimated as

$$\begin{aligned}
I_3\le&C\|u\|_{L^6}\|\nabla v\|_{L^3}\|\Delta v\|_{L^2}\le C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\Delta v\|_{L^2}^{\frac{3}{2}}\\
\le&\frac{1}{8}\|\Delta v\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|\nabla v\|_{L^2}^2\\
\le&\frac{1}{8}\|\Delta v\|_{L^2}^2+C\|\nabla v\|_{L^2}^2
\end{aligned}$$

[2]We notice that from (2.7) and the definition of $\mathcal{G}$ on (2.25), one gets $\int_0^{T_*}\mathcal{G}(\tau)\mathrm{d}\tau\le M$.

and

$$\begin{aligned}I_4 \leq& C\|\Delta v\|_{L^2}\|\nabla\cdot(v\nabla\Psi)\|_{L^2} \leq C\|\Delta v\|_{L^2}\big(\|\nabla v\|_{L^3}\|\nabla\Psi\|_{L^6}+\|v\|_{L^4}\|\Delta\Psi\|_{L^4}\big)\\ \leq& \frac{1}{16}\|\Delta v\|_{L^2}^2+C\big(\|\nabla v\|_{L^3}^2\|\nabla\Psi\|_{L^6}^2+\|(v,w)\|_{L^4}^4\big)\\ \leq& \frac{1}{16}\|\Delta v\|_{L^2}^2+C\big(\|(v,w)\|_{L^2}^2\|\nabla v\|_{L^2}\|\Delta v\|_{L^2}+\|(v,w)\|_{L^2}^{\frac{5}{2}}\|(\Delta v,\Delta w)\|_{L^2}^{\frac{3}{2}}\big)\\ \leq& \frac{1}{8}\|(\Delta v,\Delta w)\|_{L^2}^2+C(1+\|(\nabla v,\nabla w)\|_{L^2}^2).\end{aligned}$$

Inserting estimates of I_3 and I_4 above into (2.30), one obtains

$$\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2+2\|\Delta v\|_{L^2}^2 \leq \frac{1}{2}\|(\Delta v,\Delta w)\|_{L^2}^2+C(1+\|(\nabla v,\nabla w)\|_{L^2}^2).$$

Similar estimate still holds for w. Hence, one obtains that

$$\begin{aligned}&\frac{d}{dt}(\|\nabla v(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2)+\frac{3}{2}(\|\Delta v\|_{L^2}^2+\|\Delta w\|_{L^2}^2)\\ &\leq C(1+\|(\nabla v,\nabla w)\|_{L^2}^2),\end{aligned}\tag{2.31}$$

which together with Grönwall's inequality yields that

$$\begin{aligned}&\sup_{0\leq t\leq T_*}(\|\nabla v(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2)+\int_0^{T_*}(\|\Delta v(t)\|_{L^2}^2+\|\Delta w(t)\|_{L^2}^2)\mathrm{d}t\\ \leq& C(1+\|(\nabla v_0,\nabla w_0)\|_{L^2}^2)\leq C_3,\end{aligned}\tag{2.32}$$

where C_3 is a positive constant depending only on T_*, C_0, C_1, C_2 and $\|(v_0,w_0)\|_{H^1}$.

Step 3. H^2-bound of u. Taking Δ on (2.11), then multiplying the resulting equation with Δu and integrating over $\mathbb{R}^3$, by the condition $\nabla\cdot u=0$, we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta u(t)\|_{L^2}^2+\|\nabla\Delta u\|_{L^2}^2=&\int_{\mathbb{R}^3}\nabla((u\cdot\nabla)u)\cdot\nabla\Delta u\mathrm{d}x-\int_{\mathbb{R}^3}\nabla(\Delta\Psi\nabla\Psi)\cdot\nabla\Delta u\mathrm{d}x\\ :=& I_5+I_6.\end{aligned}\tag{2.33}$$

By using Hölder's inequality, interpolation inequality, Young's inequality and (2.29) again, the terms I_5 and I_6 on the right-hand side of (2.33) can be bounded as

$$\begin{aligned}I_5\leq& C\|\nabla\Delta u\|_{L^2}(\|\nabla u\|_{L^4}^2+\|u\|_{L^6}\|\nabla^2 u\|_{L^3})\\ \leq& C\|\nabla\Delta u\|_{L^2}^2(\|\nabla u\|_{L^2}^{\frac{5}{4}}\|\nabla\Delta u\|_{L^2}^{\frac{3}{4}}+\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta u\|_{L^2}^{\frac{1}{2}})\\ \leq& \frac{1}{8}\|\nabla\Delta u\|_{L^2}^2+C(\|\nabla u\|_{L^2}^{10}+\|\nabla u\|_{L^2}^4\|\Delta u\|_{L^2}^2)\\ \leq& \frac{1}{8}\|\nabla\Delta u\|_{L^2}^2+C(\|\Delta u\|_{L^2}^2+1)\end{aligned}$$

and

$$\begin{aligned}I_6=&-\int_{\mathbb{R}^3}\nabla((v-w)\nabla\Psi)\cdot\nabla\Delta u\mathrm{d}x\\ \leq& C\|\nabla\Delta u\|_{L^2}^2\Big(\|(\nabla v,\nabla w)\|_{L^3}\|\nabla\Psi\|_{L^6}+\|(v,w)\|_{L^4}\|\nabla^2\Psi\|_{L^4}\Big)\end{aligned}$$

$$
\begin{aligned}
&\le C\|\nabla\Delta u\|_{L^2}^2\Big(\|(\nabla v,\nabla w)\|_{L^3}\|(v,w)\|_{L^2}+\|(v,w)\|_{L^4}^2\Big)\\
&\le C\|\nabla\Delta u\|_{L^2}\Big(\|(v,w)\|_{L^2}\|(\nabla v,\nabla w)\|_{L^2}^{\frac12}\|(\Delta v,\Delta w)\|_{L^2}^{\frac12}+\|(v,w)\|_{L^2}^{\frac54}\|(\Delta v,\Delta w)\|_{L^2}^{\frac34}\Big)\\
&\le \frac18\|\nabla\Delta u\|_{L^2}^2+\frac14\|(\Delta v,\Delta w)\|_{L^2}^2+C\Big(1+\|(\nabla v,\nabla w)\|_{L^2}^2\Big).
\end{aligned}
$$

Inserting estimates of I_5 and I_6 into (2.33), one obtains that

$$
\frac{d}{dt}\|\Delta u(t)\|_{L^2}^2+\|\nabla\Delta u\|_{L^2}^2\le\frac14\|(\Delta v,\Delta w)\|_{L^2}^2+C\Big(1+\|(\nabla v,\nabla w)\|_{L^2}^2+\|\Delta u\|_{L^2}^2\Big),
$$

which together with (2.31) yields that

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\Delta u(t)\|_{L^2}^2+\|(\nabla v,\nabla w)(t)\|_{L^2}^2\big)+\big(\|\nabla\Delta u\|_{L^2}^2+\|(\Delta v,\Delta w)\|_{L^2}^2\big)\\
\le& C\big(1+\|(\nabla v,\nabla w)\|_{L^2}^2+\|\Delta u\|_{L^2}^2\big).
\end{aligned}
$$

By using Grönwall's inequality again, it follows that

$$
\sup_{0\le t\le T_*}\|\Delta u(t)\|_{L^2}^2+\int_0^{T_*}\|\nabla\Delta u(t)\|_{L^2}^2\mathrm{d}t\le C_4, \tag{2.34}
$$

where C_4 is a positive constant depending only on T_*, C_0, C_1, C_2, $\|u_0\|_{H^2}$ and $\|(v_0,w_0)\|_{H^1}$.

Step 4. Proof of (2.8)**.** Taking $\nabla\Delta$ on (2.11), then multiplying the resulting equation with $\nabla\Delta u$ and integrating over $\mathbb{R}^3$, one obtains

$$
\begin{aligned}
\frac12\frac{d}{dt}\|\nabla\Delta u(t)\|_{L^2}^2+\|\Delta^2 u\|_{L^2}^2=&-\int_{\mathbb{R}^3}\nabla\Delta((u\cdot\nabla)u)\cdot\nabla\Delta u\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\Delta(\Delta\Psi\nabla\Psi)\cdot\nabla\Delta u\mathrm{d}x\\
:=&I_7+I_8.
\end{aligned}\tag{2.35}
$$

To bound I_7, we need to use the following inequality (cf., [22])

$$
\|\Lambda^s(fg)-f\Lambda^s g\|_{L^p}\le C(\|\Lambda f\|_{L^\infty}\|\Lambda^{s-1}g\|_{L^p}+\|\Lambda^s f\|_{L^p}\|g\|_{L^\infty})\text{ for }s>0\text{ and }1<p<\infty,
$$

where $\Lambda=(-\Delta)^{\frac12}$. Then, by using the divergence-free condition $\nabla\cdot u=0$ and interpolation inequality, one has

$$
\begin{aligned}
I_7=&-\int_{\mathbb{R}^3}\big[\nabla\Delta((u\cdot\nabla)u)-(u\cdot\nabla)\nabla\Delta u\big]\cdot\nabla\Delta u\mathrm{d}x\\
\le& C\|\nabla\Delta((u\cdot\nabla)u)-(u\cdot\nabla)\nabla\Delta u\|_{L^{\frac65}}\|\nabla\Delta u\|_{L^6}\\
\le& C\|\Delta^2 u\|_{L^2}\|\nabla u\|_{L^\infty}\|\nabla\Delta u\|_{L^{\frac65}}\\
\le& C\|\Delta^2 u\|_{L^2}(\|\nabla u\|_{L^2}^{\frac12}\|\Delta^2 u\|_{L^2}^{\frac12})(\|\nabla u\|_{L^2}^{\frac12}\|\nabla\Delta u\|_{L^2}^{\frac12})\\
\le&\frac14\|\Delta^2 u\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|\nabla\Delta u\|_{L^2}^2\le\frac14\|\Delta^2 u\|_{L^2}^2+C\|\nabla\Delta u\|_{L^2}^2.
\end{aligned}
$$

For I_8, by using $(1.1)_5$, Hölder's inequality, Young's inequality, (2.9) and (2.32), one has

$$
I_8=-\int_{\mathbb{R}^3}\Delta((v-w)\nabla\Psi)\cdot\Delta^2 u\mathrm{d}x\le C\|\Delta^2 u\|_{L^2}^2\|\Delta((v-w)\nabla\Psi)\|_{L^2}^2
$$

$$
\begin{aligned}
&\le C\|\Delta^2 u\|_{L^2}\Big(\|(\Delta v,\Delta w)\|_{L^3}\|\nabla\Psi\|_{L^6}+\|(\nabla v,\nabla w)\|_{L^6}\|\nabla^2\Psi\|_{L^3}+\|(v,w)\|_{L^3}\|\nabla\Delta\Psi\|_{L^6}\Big)\\
&\le C\|\Delta^2 u\|_{L^2}\Big(\|(\Delta v,\Delta w)\|_{L^3}\|\Delta\Psi\|_{L^2}+\|(v,w)\|_{L^3}\|(\nabla v,\nabla w)\|_{L^6}\Big)\\
&\le C\|\Delta^2 u\|_{L^2}^2\Big(\|(\Delta v,\Delta w)\|_{L^2}^{\frac12}\|(\nabla\Delta v,\nabla\Delta w)\|_{L^2}^{\frac12}+\|(\nabla v,\nabla w)\|_{L^2}^{\frac12}\|(\Delta v,\Delta w)\|_{L^2}^{2}\Big)\\
&\le \frac14\|\Delta^2 u\|_{L^2}^2+\frac18\|(\nabla\Delta v,\nabla\Delta w)\|_{L^2}^2+C\Big(1+\|(\nabla v,\nabla w)\|_{L^2}^2\Big)\|(\Delta v,\Delta w)\|_{L^2}^2\\
&\le \frac14\|\Delta^2 u\|_{L^2}^2+\frac18\|(\nabla\Delta v,\nabla\Delta w)\|_{L^2}^2+C\|(\Delta v,\Delta w)\|_{L^2}^2.
\end{aligned}
$$

Inserting the above two estimates of I_7 and I_8 into (2.35), it follows that

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\Delta u(t)\|_{L^2}^2+\|\Delta^2 u\|_{L^2}^2\\
\le&\frac14\|(\nabla\Delta v,\nabla\Delta w)\|_{L^2}^2+C\Big(\|\nabla\Delta u\|_{L^2}^2+\|(\Delta v,\Delta w)\|_{L^2}^2\Big).
\end{aligned}\tag{2.36}
$$

Taking Δ to $(1.1)_3$, then multiplying the resulting equations by Δv and integrating over $\mathbb{R}^3$, after integration by parts, we deduce that

$$
\begin{aligned}
\frac12\frac{d}{dt}\|\Delta v(t)\|_{L^2}^2+\|\nabla\Delta v\|_{L^2}^2=&\int_{\mathbb{R}^3}\nabla((u\cdot\nabla)v)\cdot\nabla\Delta v\mathrm{d}x+\int_{\mathbb{R}^3}\Delta(v\nabla\Psi)\cdot\nabla\Delta v\mathrm{d}x\\
:=&I_9+I_{10}.
\end{aligned}\tag{2.37}
$$

Applying (2.9), (2.29) (2.32) and (2.34) again,

$$
\begin{aligned}
I_9\le&C\|\nabla\Delta v\|_{L^2}\big(\|\nabla u\|_{L^4}\|\nabla v\|_{L^4}+\|u\|_{L^6}\|\Delta v\|_{L^3}\big)\\
\le&C\|\nabla\Delta v\|_{L^2}\big((\|\nabla u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac34})^{\frac13}(\|\nabla u\|_{L^2}^{\frac58}\|\nabla\Delta u\|_{L^2}^{\frac38})^{\frac23}\|\nabla v\|_{L^2}^{\frac14}\|\Delta v\|_{L^2}^{\frac34}\\
&+\|\nabla u\|_{L^2}\|\Delta v\|_{L^2}^{\frac12}\|\nabla\Delta v\|_{L^2}^{\frac12}\big)\\
\le&\frac18\|\nabla\Delta v\|_{L^2}^2+C\big(\|\nabla u\|_{L^2}^4\|\Delta v\|_{L^2}^2+\|\nabla u\|_{L^2}^2\|\Delta v\|_{L^2}^2+\|\Delta u\|_{L^2}^2\|\nabla\Delta u\|_{L^2}^2\big)\\
\le&\frac18\|\nabla\Delta v\|_{L^2}^2+C\big(\|\nabla\Delta u\|_{L^2}^2+\|\Delta v\|_{L^2}^2\big)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{10}\le&C\|\nabla\Delta v\|_{L^2}^2\|\Delta(v\nabla\Psi)\|_{L^2}\\
\le&C\|\nabla\Delta v\|_{L^2}\big(\|\Delta v\|_{L^3}\|\nabla\Psi\|_{L^6}+\|\nabla v\|_{L^6}\|\nabla^2\Psi\|_{L^3}+\|v\|_{L^3}\|\nabla(v-w)\|_{L^6}\big)\\
\le&C|\nabla\Delta v\|_{L^2}\big(\|\Delta v\|_{L^3}\|(v,w)\|_{L^2}+\|\Delta v\|_{L^2}\|(v,w)\|_{L^3}+\|v\|_{L^3}\|(\Delta v,\Delta w)\|_{L^2}\big)\\
\le&C\|\nabla\Delta v\|_{L^2}\big(\|\Delta v\|_{L^2}^{\frac12}\|\nabla\Delta v\|_{L^2}^{\frac12}+\|(\nabla v,\nabla w)\|_{L^2}^{\frac12}\|(\Delta v,\Delta w)\|_{L^2}\big)\\
\le&\frac18\|\nabla\Delta v\|_{L^2}^2+C\big(1+\|(\Delta v,\Delta w)\|_{L^2}^2\big).
\end{aligned}
$$

Inserting the two estimates of I_9 and I_{10} into (2.37), one gets

$$
\frac{d}{dt}\|\Delta v(t)\|_{L^2}^2+\frac74\|\nabla\Delta v\|_{L^2}^2\le C\big(1+\|\nabla\Delta u\|_{L^2}^2+\|(\Delta v,\Delta w)\|_{L^2}^2\big).
$$

Similar estimate still holds for w. Thus we have

$$
\frac{d}{dt}\|(\Delta v(t),\Delta w(t))\|_{L^2}^2+\frac74\|(\nabla\Delta v,\nabla\Delta w)\|_{L^2}^2\le C\big(1+\|\nabla\Delta u\|_{L^2}^2+\|(\Delta v,\Delta w)\|_{L^2}^2\big),
$$

which together with (2.36) yields that

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\Delta u(t)\|_{L^2}^2+\|(\Delta v,\Delta w)(t)\|_{L^2}^2)+(\|\Delta^2 u\|_{L^2}^2+\|(\nabla\Delta v,\nabla\Delta w)\|_{L^2}^2)\\ \le& C\big(1+\|\nabla\Delta u\|_{L^2}^2+\|(\Delta v,\Delta w)\|_{L^2}^2\big).\end{aligned}$$

Applying Grönwall's inequality again, one can derive that

$$\begin{aligned}&\sup_{0\le t\le T_*}(\|\nabla\Delta u(t)\|_{L^2}^2+\|(\Delta v,\Delta w)(t)\|_{L^2}^2)\\ &+\int_0^{T_*}(\|\Delta^2 u(t)\|_{L^2}^2+\|(\nabla\Delta v,\nabla\Delta w)(t)\|_{L^2}^2)\mathrm{d}t\le C_5,\end{aligned}$$

where C_5 is a positive constant depending only on T_*, C_i $(i=0,1,2,3,4)$, $\|u_0\|_{H^3}$ and $\|(v_0,w_0)\|_{L^1\cap H^2}$. The above inequality together with (2.9) and (2.10) implies (2.8), and this completes the proof of Theorem 1.1. □

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