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GENERALIZATION OF KREISS THEORY TO HYPERBOLIC PROBLEMS WITH BOUNDARY-TYPE EIGENMODES*

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Dedicated to the memory of Heinz-Otto Kreiss

Abstract. The Kreiss symmetrizer technique gives sharp estimates of the solutions of the first-order hyperbolic initial-boundary value problems both in the interior and at the boundary of the domain. Such estimates imply robustness and strong well-posedness in the generalized sense, and the corresponding problems are called strongly boundary stable, satisfying the Kreiss eigenvalue condition. There are however problems that are not strongly boundary stable and yet are well-posed and robust. For such problems sharp estimates of the solution can be obtained only in the interior and not at the boundary. We refer to this class of problems as well-posed in the generalized sense. Examples include hyperbolic problems, governed by elastic and Maxwell's equations, that describe boundary-type wave phenomena, such as surface waves and glancing waves. We introduce the notion of boundary-type generalized eigenvalues and obtain a sufficient algebraic condition for well-posedness in the generalized sense, thereby relaxing the Kreiss eigenvalue condition. Despite the utilization of the Laplace-Fourier mode analysis, since the proofs are based on the construction of smooth Kreiss-type symmetrizers, the developed theory can be applied to problems with variables coefficients in both first-order and second-order forms.

Keywords. Kreiss theory; hyperbolic systems; well-posedness; Kreiss symmetrizers; boundary stability; boundary phenomena; surface waves; glancing waves.

AMS subject classifications. 35L50; 35A02; 35C07; 35Q60; 35Q86.

1. Introduction

The theory of first order linear hyperbolic initial-boundary value problems (IBVPs) is well developed for two classes of problems: 1) the Friedrichs theory for symmetric systems with maximally dissipative boundary conditions, and 2) the Kreiss theory for hyperbolic systems with boundary conditions satisfying the uniform Kreiss eigenvalue condition. For symmetric hyperbolic systems with maximally dissipative boundary conditions, energy estimates can be derived using integration by parts; see e.g. [3–5, 19]. When the system is not symmetric or when the boundary conditions are not maximally dissipative, the Kreiss symmetrizer technique needs to be devised; see e.g. [11, 14, 15, 18, 26]. This technique is based on the principle of frozen coefficients, Fourier and Laplace transforms, construction of Kreiss-type symmetrizers, and the theory of pseudo-differential operators. It gives a necessary and sufficient algebraic condition, known as Kreiss eigenvalue condition, that implies robustness and strong well-posedness in the generalized sense. Problems that satisfy the Kreiss condition are called *strongly boundary stable*; we can obtain sharp solution estimates at the boundary in terms of the boundary data.

There are however hyperbolic problems that are neither of Friedrichs type nor of Kreiss type and yet are well-posed and robust. For such problems sharp estimates of the solution can be obtained only in the interior of the domain and not at the boundary. Examples include hyperbolic problems, governed by elastic and Maxwell's equations, that describe boundary-type wave phenomena, such as surface waves and

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glancing waves. We refer to this class of problems as *well-posed in the generalized sense* and characterize them by boundary-type eigenmodes. The main purpose of the present paper is to develop a theory for this class of problems. Through the construction of smooth Kreiss-type symmetrizers, we obtain a sufficient algebraic condition for well-posedness in the generalized sense, thereby relaxing the Kreiss eigenvalue condition. A major importance of the construction of smooth symmetrizers is that it enables us to use the theory of pseudo-differential operators and treat problems with variable coefficients in general smooth domains.

Well-posedness in the generalized sense was first studied in [23], where a simple two-dimensional model problem was considered and treated. The present paper further extends this original work to more general first-order hyperbolic problems. We note that since the general theory is based on the theory of pseudo-differential operators, it can also be applied to the second-order hyperbolic systems. In fact a second-order system of differential equations can always be written as a first order system of pseudo-differential operators [16]. It is to be noted that similar concepts of well-posedness for second-order hyperbolic systems have been studied in [17], where the authors derive interior estimates by first applying the Laplace-Fourier transforms and then directly solving the transformed equations. The derivation of interior estimates in [17] does not however rely on the construction of Kreiss-type symmetrizers and hence cannot be applied to problems with variable coefficients. A main contribution and importance of the present paper is the development of the theory through the construction of smooth symmetrizers, rather than directly obtaining estimates by employing Laplace-Fourier transforms. The developed theory can hence be applied to problems with variable coefficients in both first-order and second-order forms.

The remainder of the paper is organized as follows. In Section 2 we provide a short, and yet comprehensive, review of the Kreiss theory for first-order hyperbolic IBVPs. We then extend the Kreiss theory to hyperbolic problems with boundary-type eigenmodes in Section 3. We divide this section into two parts. We first define the boundary-type eigenmodes and formulate the main theorem for well-posedness in the generalized sense. We then study a simple model problem that serves as an illustrative example shedding light on the construction of smooth symmetrizers for more general problems. The proof of the main theorem and the construction of smooth symmetrizers for general systems will be presented in Section 4. A number of auxiliary lemmas are collected in the Appendix.

2. Kreiss theory: strong boundary stability

This section provides a short, and yet comprehensive, review of the Kreiss theory for first-order hyperbolic IBVPs. The theory was introduced by Kreiss [11] for strictly hyperbolic systems. It was later extended to systems with constant multiplicity [2] and to a special class of systems with variable multiplicity [22].

2.1. Cauchy problems.

Consider the Cauchy problem for a first-order system of partial differential equations

$$\partial_t \mathbf{u} = P(\partial_{\mathbf{x}})\mathbf{u} + \mathbf{f}(\mathbf{x},t), \qquad P(\partial_{\mathbf{x}}) = A\,\partial_{x_1} + \sum_{j=2}^{m} B_j\,\partial_{x_j}, \qquad \mathbf{x}\in\mathbb{R}^m, \qquad t\geq 0, \tag{2.1}$$

with the initial condition

$$\mathbf{u}(\mathbf{x},0) = \mathbf{h}(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^m. \tag{2.2}$$

Here $\mathbf{u}(\mathbf{x},t)=\big(u_1(\mathbf{x},t),\dots,u_n(\mathbf{x},t)\big)^\top$ is a complex-valued n-dimensional vector function of real variables t and $\mathbf{x}=(x_1,\dots,x_m)\in\mathbb{R}^m$, and the complex coefficient matrices $A,B_j\in\mathbb{C}^{n\times n}$, with $j=2,\dots,m$, are constant.

A fundamental concept in the theoretical study of differential equations is well-posedness: the existence of a unique solution (here $\mathbf{u}$) depending continuously on the given data (here $\mathbf{f}$ and $\mathbf{h}$). A related concept in the numerical study of differential equations is numerical stability: the boundedness of the numerical solution in some sense; see e.g. [6,31]. A satisfactory definition of well-posedness for the Cauchy problem (2.1)-(2.2) that can also be modified to study numerical stability is the one formulated in the semigroup sense as follows. Let the L^2-norm in $\mathbb{R}^m$ be defined by

$$||\mathbf{u}||^2_{L^2(\mathbb{R}^m)}=(\mathbf{u},\mathbf{u}),\qquad (\mathbf{u},\mathbf{v})=\int_{\mathbb{R}^m}\mathbf{u}^*\mathbf{v}\,d\mathbf{x},$$

where $\mathbf{u}^*$ is the adjoint of $\mathbf{u}$.

Definition 2.1. *The Cauchy problem* (2.1)-(2.2) *is called well-posed in the semigroup sense, if*

(1) for a dense set of smooth data, there is a smooth solution; and

(2) the solution of the homogeneous system ($\mathbf{f}\equiv\mathbf{0}$) *satisfies the energy estimate*

$$||\mathbf{u}(.,t)||_{L^2(\mathbb{R}^m)}\le Ke^{\alpha t}||\mathbf{h}||_{L^2(\mathbb{R}^m)},\qquad \forall t\ge 0 \tag{2.3}$$

where $K>0$ and $\alpha\in\mathbb{R}$ are two constants.

We note that it is enough to consider homogeneous problems (with $\mathbf{f}\equiv 0$). The solutions of inhomogeneous systems (with $\mathbf{f}\not\equiv 0$) can then be determined and estimated using Duhamel's principle; see e.g. Section 2.6.3 of [15]:

$$||\mathbf{u}(.,t)||_{L^2(\mathbb{R}^m)}\le C(t)\left(||\mathbf{h}||_{L^2(\mathbb{R}^m)}+\int_0^t||\mathbf{f}(.,\tau)||_{L^2(\mathbb{R}^m)}\,d\tau\right). \tag{2.4}$$

Let us now consider the symbol $P(i\boldsymbol{\omega})$ of the differential operator $P(\partial_\mathbf{x})$, with $\boldsymbol{\omega}=(\omega_1,\boldsymbol{\omega}_-)\in\mathbb{R}^m$ and $\boldsymbol{\omega}_-=(\omega_2,\dots,\omega_m)$, formally obtained by the substitution of $i\omega_j$ for ∂_{x_j}:

$$P(i\boldsymbol{\omega})=iA\omega_1+iB(\boldsymbol{\omega}_-),\qquad B(\boldsymbol{\omega}_-)=\sum_{j=2}^m B_j\omega_j. \tag{2.5}$$

For a complex matrix $Q\in\mathbb{C}^{n\times n}$, let the matrix norm be denoted by $||Q||:=\max_{\mathbf{v}\in\mathbb{C}^{n\times 1},|\mathbf{v}|=1}|Q\mathbf{v}|$, where $|\mathbf{v}|:=\mathbf{v}^*\mathbf{v}$ denotes the magnitude of a complex vector $\mathbf{v}\in\mathbb{C}^{n\times 1}$. There is an equivalent definition of well-posedness in terms of the symbol matrix as follows.

Definition 2.2. *The homogeneous Cauchy problem* (2.1)-(2.2) *is called well-posed in the semigroup sense, if there are constants α and K such that the symbol* (2.5) *satisfies*

$$||e^{P(i\boldsymbol{\omega})t}||\le Ke^{\alpha t},\qquad \forall t\ge 0,\quad \forall\boldsymbol{\omega}=(\omega_1,\boldsymbol{\omega}_-)\in\mathbb{R}^m. \tag{2.6}$$

The equivalence of the above two definitions can easily be seen by taking the Fourier transform of (2.1)-(2.2), with $\mathbf{f}\equiv 0$, with respect to the spatial variables $\mathbf{x}$. We then obtain a system of homogeneous ordinary differential equations with the solution

$$\hat{\mathbf{u}}(\boldsymbol{\omega},t)=e^{P(i\boldsymbol{\omega})t}\hat{\mathbf{h}}(\boldsymbol{\omega}),$$

where $\hat{\mathbf{u}}(\boldsymbol{\omega},t)=\mathcal{F}\mathbf{u}(\mathbf{x},t)=\int_{\mathbb{R}^m} e^{-i\langle\boldsymbol{\omega},\mathbf{x}\rangle}\mathbf{u}\,d\mathbf{x}$ and $\hat{\mathbf{h}}(\boldsymbol{\omega})=\mathcal{F}\mathbf{h}(\mathbf{x})$ are the Fourier transforms of $\mathbf{u}$ and $\mathbf{h}$ with respect to $\mathbf{x}$, respectively. Here $\langle\boldsymbol{\omega},\mathbf{x}\rangle:=\sum_{j=1}^m \omega_j x_j$ for $\mathbf{x},\boldsymbol{\omega}\in\mathbb{R}^m$. By Parseval's identity we have

$$||\mathbf{u}(.,t)||_{L^2(\mathbb{R}^m)}=||\hat{\mathbf{u}}(\boldsymbol{\omega},t)||=||e^{P(i\boldsymbol{\omega})t}\hat{\mathbf{h}}(\boldsymbol{\omega})||\leq Ke^{\alpha t}||\hat{\mathbf{h}}(\boldsymbol{\omega})||=Ke^{\alpha t}||\mathbf{h}||_{L^2(\mathbb{R}^m)}.$$

For a rigorous proof of the equivalence of the above two definitions see Theorem 2.2.2 in [15].

Thanks to Kreiss Matrix Theorem (see e.g. Theorem 2.3.2 in [15] and Theorem 9.4.1 in [31]), Definition 2.2 can be utilized to characterize all systems of form (2.1) for which the Cauchy problem is well-posed.

THEOREM 2.1 (Theorem 2.4.1 in [15]). *The Cauchy problem* (2.1)-(2.2) *is well-posed in the semigroup sense if and only if for every real* $\boldsymbol{\omega}=(\omega_1,\boldsymbol{\omega}_-)$, $\boldsymbol{\omega}_-=(\omega_2,\dots,\omega_m)$ *with* $|\boldsymbol{\omega}|=1$ *the following two conditions hold:*

(1) The symbol (2.5) *has purely imaginary eigenvalues.*

(2) The symbol (2.5) *can be transformed into diagonal form by a transformation* $T(\boldsymbol{\omega})\in\mathbb{C}^{n\times n}$ *with* $|T(\boldsymbol{\omega})|+|T^{-1}(\boldsymbol{\omega})|\leq K$, *where* K *is a constant independent of* $\boldsymbol{\omega}$.

The problems characterized by the conditions of Theorem 2.1 are referred to as *strongly hyperbolic*. It is to be noted that Theorem 2.1 is true only for problems with constant coefficients. For the case of variable coefficients, the symbol should in addition be smoothly symmetrizable, requiring smooth dependence of $T(\boldsymbol{\omega})$ on $\boldsymbol{\omega}$, in order for the Cauchy problem to be well-posed in the semigroup sense [8]. The proof is technical and uses the theory of pseudo-differential operators [24]; see also Theorem 6.2.2 in [15].

An important property of well-posedness in the semigroup sense is that it implies *robustness*, that is, stability against lower order perturbations. If we add a zeroth-order term to the problem (2.1), i.e. if we replace $P(\partial\mathbf{x})$ by $Q(\partial\mathbf{x})=P(\partial\mathbf{x})+D$, where $D\in\mathbb{C}^{n\times n}$, then one can use Kreiss Matrix Theorem and show that the Cauchy problem for $\partial_t\mathbf{u}=Q(\partial\mathbf{x})\mathbf{u}$ is well-posed in the semigroup sense if and only if the Cauchy problem for $\partial_t\mathbf{u}=P(\partial\mathbf{x})\mathbf{u}$ is well-posed in the semigroup sense; see Lemma 2.3.5 and Section 2.6.1 in [15].

REMARK 2.1 (Well-posedness in the sense of Hadamard). Well-posedness in the semigroup sense requires conditions on both eigenvalues and eigenvectors of the symbol. There is a weaker definition of well-posedness that requires conditions only on eigenvalues. The Cauchy problem is said to be well-posed in the sense of Hadamard (or weakly well-posed) [7, 25] if the estimate (2.3) is replaced by

$$||\mathbf{u}(.,t)||_{L^2(\mathbb{R}^m)}\leq Ke^{\alpha t}||\mathbf{h}||_{H^q(\mathbb{R}^m)},\quad ||\mathbf{h}||^2_{H^q(\mathbb{R}^m)}:=\sum_{|\boldsymbol{\nu}|\leq q}||\frac{\partial^{|\boldsymbol{\nu}|}\mathbf{h}}{\partial x_1^{\nu_1}\cdots\partial x_m^{\nu_m}}||^2_{L^2(\mathbb{R}^m)},\quad q\geq 1, \tag{2.7}$$

or equivalently if (2.6) is replaced by

$$|e^{P(i\boldsymbol{\omega})t}|\leq K(1+|\boldsymbol{\omega}|^q)e^{\alpha t},\qquad \forall t\geq 0,\quad \forall\boldsymbol{\omega}\in\mathbb{R}^m. \tag{2.8}$$

Here, $\boldsymbol{\nu}=(\nu_1,\dots,\nu_n)$ is a multi-index, and $|\boldsymbol{\nu}|=\nu_1+\ldots+\nu_n$. One can show that the Cauchy problem is well-posed in the sense of Hadamard if and only if the eigenvalues of the symbol are purely imaginary. A major problem with this weaker definition of well-posedness is that it does not imply robustness. In other words, if the solution of the homogeneous system satisfies only (2.7), but not (2.3), the well-posedness can be destroyed by lower order terms; see e.g. Section 2.2.3 in [15].

Henceforth, we assume that the system (2.1) is *strictly hyperbolic*, i.e., for all real $\boldsymbol{\omega}\in\mathbb{R}^m$ with $|\boldsymbol{\omega}|=1$, the eigenvalues of the symbol (2.5) are purely imaginary and distinct. By Theorem 2.1, therefore, the Cauchy problem for (2.1) is well-posed in the semigroup sense. We further assume, for simplicity and without restriction, that the coefficient matrix A in (2.1) is nonsingular and has the form

$$A=\begin{pmatrix}-A_I & \\ & A_{II}\end{pmatrix}, \qquad A_I\in\mathbb{R}^{r\times r}, \quad A_{II}\in\mathbb{R}^{(n-r)\times(n-r)}, \tag{2.9}$$

where A_I and A_{II} are real positive definite diagonal matrices of order r and $n-r$, respectively. For the singular case see [21].

2.2. Initial-boundary value problems. Let R_0 denote the half-space

$$R_0=\{\mathbf{x}=(x_1,\mathbf{x}_-): x_1\geq 0, \mathbf{x}_-=(x_2,\ldots,x_m)\in R_-\},$$

where R_- is the $(m-1)$-dimensional space tangential to the boundary $x_1=0$ of R_0:

$$R_-=\{\mathbf{x}_-: -\infty<x_j<\infty, j=2,\ldots,m\}.$$

Consider the IBVP for the system (2.1) in the half-space

$$\partial_t\mathbf{u}=P(\partial_\mathbf{x})\mathbf{u}+\mathbf{f}(\mathbf{x},t), \qquad P(\partial_\mathbf{x})=A\,\partial_{x_1}+\sum_{j=2}^{m}B_j\,\partial_{x_j}, \qquad \mathbf{x}\in R_0, \qquad t\geq 0, \tag{2.10}$$

with the initial condition

$$\mathbf{u}(\mathbf{x},0)=\mathbf{h}(\mathbf{x}), \qquad \mathbf{x}\in R_0, \tag{2.11}$$

and the boundary condition at $x_1=0$,

$$\mathbf{u}_I(0,\mathbf{x}_-,t)=S\,\mathbf{u}_{II}(0,\mathbf{x}_-,t)+\mathbf{g}(\mathbf{x}_-,t), \qquad \mathbf{x}_-\in R_-. \tag{2.12}$$

Here $\mathbf{u}_I=(u_1,\ldots,u_r)^\top$ and $\mathbf{u}_{II}=(u_{r+1},\ldots,u_n)^\top$ correspond to the partitions A_I and A_{II}, respectively, and $S\in\mathbb{C}^{r\times(n-r)}$ is a rectangular matrix. We assume that all data are smooth, compactly supported, and compatible. We refer to the IBVP (2.10)-(2.12) as the *half-space problem*. It is to be noted that other types of boundary conditions, such as Neumann and Sommerfeld conditions, can be treated similarly and will not be addressed here.

In the remainder of this section, we first review four different definitions of well-posedness for the half-space problem, including the strong well-posedness in the generalized sense (Definition 2.5), introduced by Kreiss [11]. The goal is to give a clearer picture of well-posedness in the sense of Kreiss compared to other definitions of well-posedness. We then discuss necessary and sufficient conditions for well-posedness in the sense of Kreiss.

2.2.1. Different types of well-posedness. In general, a linear IBVP with smooth coefficients and smooth boundary is called well-posed if for all smooth, compatible data there is a unique smooth solution that can be estimated in terms of the data. Let us first specify the norms that appear in the estimates. For a complex-valued n-dimensional vector function $\mathbf{u}=\mathbf{u}(x_1,\mathbf{x}_-,t)$ of real variables $x_1\geq 0$, $\mathbf{x}_-\in R_-$, and $t\geq 0$, we use the following norms:

$$||\mathbf{u}(.,t)||_{R_0}^2:=\int_{R_0}|\mathbf{u}|^2\,d\mathbf{x}, \qquad ||\mathbf{u}(x_1,.,t)||_{R_-}^2:=\int_{R_-}|\mathbf{u}|^2\,d\mathbf{x}_-, \qquad |\mathbf{u}|^2:=\mathbf{u}^*\mathbf{u}.$$

We can formulate four "satisfactory" definitions for well-posedness (numbered I–IV below) in the sense that "no derivatives are lost", that is, we can control the norm of the solution in terms of the norm of the data. In such estimates, the norm of the derivatives of the data do not appear, and hence there is no loss of derivatives.

I. Well-posedness in the semigroup sense. By a suitable change of variables we can make the boundary conditions homogeneous ($\mathbf{g} \equiv \mathbf{0}$). For instance, we can replace $\mathbf{u}$ by $\mathbf{v} = \mathbf{u} - \mathbf{w}$ where $\mathbf{w}$ satisfies the boundary condition (2.12). We can therefore extend Definition 2.1 for the Cauchy problem (2.1)-(2.2) to the IBVP (2.10)-(2.12).

DEFINITION 2.3. *Consider the half-space problem* (2.10)-(2.12) *with* $\mathbf{g} \equiv 0$. *We call the problem well-posed in the semigroup sense if for all smooth, compactly supported, and compatible data* $\mathbf{f}$ *and* $\mathbf{h}$ *there exists a unique solution* $\mathbf{u}$ *that satisfies*

$$||\mathbf{u}(.,t)||^2_{R_0} \le C(t)\left(||\mathbf{h}||^2_{R_0} + \int_0^t ||\mathbf{f}(.,\tau)||^2_{R_0}\, d\tau\right), \tag{2.13}$$

where $C(t) > 0$ *is a function of* $t \ge 0$ *and independent of the data.* There are two restrictive problems in working with this definition of well-posedness. First, the estimate (2.13) can be derived (by integration by parts) when the system (2.10) is symmetric hyperbolic and the boundary conditions are of Friedrichs' type. For general hyperbolic systems and boundary conditions we may not be able to derive the desired estimates and will therefore need to consider other definitions of well-posedness. Secondly, even if we obtain desired estimates for the solution, we will face technical difficulties in obtaining estimates of derivatives of the solution in terms of derivatives of data. Such estimates require the differentiation of the PDE and the boundary condition to obtain problems of similar type for the derivatives. If the homogeneous boundary condition involves variable coefficients, this process will introduce inhomogeneous boundary terms. Although the inhomogeneity can again be subtracted out by other functions $\mathbf{w}$, the derivatives of $\mathbf{w}$ appear as inhomogeneous terms in the PDE, and hence we "lose" derivatives in the estimates.

II. Strong well-posedness. By generalizing the estimate (2.13), we can give a strong definition for well-posedness involving all data as follows.

DEFINITION 2.4. *We call the half-space problem* (2.10)-(2.12) *strongly well-posed if for all smooth, compactly supported, and compatible data* $\mathbf{f}$, $\mathbf{h}$, *and* $\mathbf{g}$, *there exists a unique solution* $\mathbf{u}$ *that satisfies*

$$||\mathbf{u}(.,t)||^2_{R_0} + \int_0^t ||\mathbf{u}(0,.,\tau)||^2_{R_-}\, d\tau \le C(t)\Big(||\mathbf{h}||^2_{R_0} + \int_0^t ||\mathbf{f}(.,\tau)||^2_{R_0} d\tau + \int_0^t ||\mathbf{g}(.,\tau)||^2_{R_-}\, d\tau\Big), \tag{2.14}$$

where $C(t) > 0$ *is a function of* $t \ge 0$ *and independent of the data.*

The above estimate holds for symmetric hyperbolic systems with maximally dissipative boundary conditions; see e.g. [4,19]. For more general problems other definitions of well-posedness need to be devised. Moreover, one can show through examples that the estimation of the boundary term $\int_0^t ||\mathbf{u}(0,.,\tau)||^2_{R_-}\, d\tau$ in more than one space dimension may be impossible, independent of the technique employed; see e.g. Section 7.3 in [15].

III. Strong well-posedness in the generalized sense. Instead of transforming the half-space problem (2.10)-(2.12) into a problem with homogeneous boundary condition ($\mathbf{g} \equiv \mathbf{0}$), as in well-posedness in the semigroup sense, we can consider transformation into homogeneous initial data ($\mathbf{h} \equiv \mathbf{0}$). This can for instance be done by a suitable change of variables, e.g. by replacing $\mathbf{u}$ with $\mathbf{v} = \mathbf{u} - e^{-t}\mathbf{h}(\mathbf{x})$.

DEFINITION 2.5. *Let $\mathbf{h} \equiv \mathbf{0}$. We call the half-space problem* (2.10)-(2.12) *strongly well-posed in the generalized sense if there is a constant $K > 0$ such that for all smooth, compactly supported, and compatible data $\mathbf{f}$ and $\mathbf{g}$ a unique solution $\mathbf{u}$ exists that satisfies*

$$\int_0^\infty e^{-2\eta t}\left(\eta||\mathbf{u}(.,t)||_{R_0}^2 + ||\mathbf{u}(0,.,t)||_{R_-}^2\right)dt$$
$$\leq K\int_0^\infty e^{-2\eta t}\left(\frac{1}{\eta}||\mathbf{f}(.,t)||_{R_0}^2 dt + ||\mathbf{g}(.,t)||_{R_-}^2\right)dt, \qquad \forall \eta > 0. \tag{2.15}$$

This concept of well-posedness, introduced by Kreiss [11], is based on the Laplace transform in time, where we define $\mathbf{u}(\mathbf{x},t) = 0$ for $t < 0$. One therefore needs $\mathbf{u}(\mathbf{x},0) = \mathbf{h}(\mathbf{x}) \equiv 0$ for the solution $\mathbf{u}$ to the half-space problem to be continuous. A major importance of Definition 2.5 is that it is not limited to symmetric hyperbolic systems with maximally dissipative boundary conditions. By applying the mode analysis (Laplace-Fourier technique), constructing Kreiss smooth symmetrizers, and utilizing the theory of pseudo-differential operators, one can obtain necessary and sufficient conditions for the well-posedness of more general linear hyperbolic IBVPs with variable coefficients. The theory can aslo be applied to quasi-linear IBVPs; see e.g. [30]. It is to be noted that the assumption $\mathbf{h} \equiv 0$ is not restrictive. One may think that since the transformation $\mathbf{v} = \mathbf{u} - e^{-t}\mathbf{h}(\mathbf{x})$ will introduce derivatives of $\mathbf{h}$ in the source term $\mathbf{f}$, such derivatives would appear in the estimate (2.15), and we would "lose" a derivative over $\mathbf{h}$. However, as shown in [27–29], this will not cause the loss of derivatives: the IBVPs for strictly hyperbolic and symmetric hyperbolic systems that are strongly well-posed in the generalized sense are also strongly well-posed in the sense of Definition 2.4.

IV. Well-posedness in the generalized sense. Kreiss has further proposed a weaker definition of well-posedness, without presenting any analysis, by assuming that both initial and boundary data vanish [12, 15]. As we will discuss in Section 3, this type of well-posedness is particularly suitable for the study of boundary-type wave phenomena, such as surface waves and glancing waves, for which the boundary estimates in Definition 2.4 and Definition 2.5 do not exist.

DEFINITION 2.6. *Let $\mathbf{h} \equiv \mathbf{g} \equiv \mathbf{0}$. We call the half-space problem* (2.10)-(2.12) *well-posed in the generalized sense if there is a constant $K > 0$ such that for all smooth, compactly supported $\mathbf{f}$ a unique solution $\mathbf{u}$ exists that satisfies*

$$\int_0^\infty e^{-2\eta t}||\mathbf{u}(.,t)||_{R_0}^2 dt \leq \frac{K}{\eta^2}\int_0^\infty e^{-2\eta t}||\mathbf{f}(.,t)||_{R_0}^2 dt, \qquad \forall \eta > 0. \tag{2.16}$$

As noted by Kreiss and Lorenz in [15], at present there is no general theory for well-posed problems in the generalized sense. They further anticipated that any such theory would be extremely complicated; see Section 7.3 in [15]. In Section 3 we will indeed develop a theory for such problems utilizing the mode analysis and Kreiss smooth symmetrizers. In fact we will show that the Kreiss theory can be extended to study this type of well-posedness.

In the remainder of Section 2, we focus on the strong well-posedness in the generalized sense (Definition 2.5) and review the Kreiss theory.

2.2.2. A necessary condition for strong well-posedness in the generalized sense. Consider the half-space problem (2.10)-(2.12) with the homogeneous initial condition $\mathbf{u}(\mathbf{x},0)=\mathbf{h}\equiv 0$. We further set $\mathbf{u}(\mathbf{x},t)\equiv 0$ for $t<0$, and define a new variable

$$\mathbf{u}_\eta(\mathbf{x},t)=e^{-\eta t}\mathbf{u}(\mathbf{x},t)=\begin{cases} e^{-\eta t}\mathbf{u}(\mathbf{x},t) & t>0,\ \mathbf{x}\in R_0 \\ 0 & t\le 0,\ \mathbf{x}\in R_0 \end{cases}, \qquad \eta>0.$$

Let $\tilde{\mathbf{u}}$ denote the Fourier transform of the new variable $\mathbf{u}_\eta(\mathbf{x},t)$ with respect to t and $\mathbf{x}_-$, with real duals ξ and $\boldsymbol{\omega}_-=(\omega_2,\dots,\omega_m)$, respectively:

$$\tilde{\mathbf{u}}=\tilde{\mathbf{u}}(x_1,\boldsymbol{\omega}_-,s)=\mathcal{F}[\mathbf{u}_\eta(x_1,\mathbf{x}_-,t)]=(2\pi)^{-m/2}\int_0^\infty\int_{R_-} e^{-st-i\langle\boldsymbol{\omega}_-,\mathbf{x}_-\rangle}\,\mathbf{u}(x_1,\mathbf{x}_-,t)\,d\mathbf{x}_-\,dt,$$

where $s=\eta+i\xi$ and $\langle\boldsymbol{\omega}_-,\mathbf{x}_-\rangle:=\sum_{j=2}^m\omega_j x_j$. With this setup, $\tilde{\mathbf{u}}$ can be thought of as the Laplace transform in t and Fourier transform in $\mathbf{x}_-$ of the solution $\mathbf{u}$ to the half-space problem (2.10)-(2.12), differing only up to a factor of $(2\pi)^{1/2}$. Similarly, we set $\mathbf{f}(\mathbf{x},t)\equiv\mathbf{g}(\mathbf{x}_-,t)\equiv 0$ for $t\le 0$, and define $\mathbf{f}_\eta(\mathbf{x},t)=e^{-\eta t}\mathbf{f}(\mathbf{x},t)$ and $\mathbf{g}_\eta(\mathbf{x}_-,t)=e^{-\eta t}\mathbf{g}(\mathbf{x}_-,t)$, for all $t\in\mathbb{R}$, $\mathbf{x}\in R_0$, and $\mathbf{x}_-\in R_-$, and let $\tilde{\mathbf{f}}(x_1,\boldsymbol{\omega}_-,s)=\mathcal{F}[\mathbf{f}_\eta(x_1,\mathbf{x}_-,t)]$ and $\tilde{\mathbf{g}}(\boldsymbol{\omega}_-,s)=\mathcal{F}[\mathbf{g}_\eta(\mathbf{x}_-,t)]$. Introducing the new variables $\mathbf{u}_\eta$, $\mathbf{f}_\eta$, and $\mathbf{g}_\eta$ into (2.10) and (2.12) and Laplace-Fourier transforming the resulting equations, we obtain the resolvent system

$$s\tilde{\mathbf{u}}=A\frac{d\tilde{\mathbf{u}}}{dx_1}+iB(\boldsymbol{\omega}_-)\tilde{\mathbf{u}}+\tilde{\mathbf{f}},\qquad \text{for } x_1\ge 0, \tag{2.17}$$

$$\tilde{\mathbf{u}}_I=S\,\tilde{\mathbf{u}}_{II}+\tilde{\mathbf{g}},\qquad \text{at } x_1=0. \tag{2.18}$$

In order to derive a necessary condition for well-posedness, we consider a homogeneous eigenvalue problem connected with (2.17)-(2.18) as follows. Let $L^2([0,\infty))$ denote the space of all quadratically integrable complex-valued n-dimensional vector functions $\boldsymbol{\phi}$ on $x_1\in[0,\infty)$, and define

$$||\boldsymbol{\phi}||_0^2:=\int_0^\infty|\boldsymbol{\phi}|^2\,dx_1,\qquad |\boldsymbol{\phi}|^2:=\boldsymbol{\phi}^*\boldsymbol{\phi},\qquad \boldsymbol{\phi}\in\mathbb{C}^n.$$

Then $\boldsymbol{\phi}\in L^2([0,\infty))$ is called an eigenfunction of (2.17)-(2.18) corresponding to an eigenvalue $s\in\mathbb{C}$, if $\boldsymbol{\phi}$ is the nontrivial solution of the homogeneous eigenvalue problem

$$s\boldsymbol{\phi}=A\frac{d\boldsymbol{\phi}}{dx_1}+iB(\boldsymbol{\omega}_-)\,\boldsymbol{\phi},\qquad \text{for } x_1\ge 0, \tag{2.19}$$

$$\boldsymbol{\phi}_I=S\,\boldsymbol{\phi}_{II},\qquad \text{at } x_1=0. \tag{2.20}$$

Definition 2.7. *Let $\boldsymbol{\omega}_-\in\mathbb{R}^{m-1}$ be fixed. A number $s\in\mathbb{C}$ is called an eigenvalue of the eigenvalue problem (2.19)-(2.20) for the parameter $\boldsymbol{\omega}_-$ if there is a nontrivial solution $\boldsymbol{\phi}$ to the eigenvalue problem.*

We have the following result due to Agmon [1].

Theorem 2.2 (Agmon eigenvalue condition). *A necessary condition for the half-space problem to be well-posed is that the eigenvalue problem (2.19)-(2.20) has no eigenvalue s with $\Re s>0$.*

Proof. Suppose that for some $\boldsymbol{\omega}_-\in\mathbb{R}^{m-1}$ there is an eigenvalue s with $\Re s>0$, and let $\boldsymbol{\phi}$ be the corresponding eigenfunction. Then it is easy to see that simple wave functions of the form

$$\mathbf{u}(\mathbf{x},t)=e^{st+i\langle\boldsymbol{\omega}_-,\mathbf{x}_-\rangle}\,\boldsymbol{\phi}(x_1), \tag{2.21}$$

solve (2.10) and (2.12) with $\mathbf{f} \equiv \mathbf{g} \equiv \mathbf{0}$. The homogeneity of the problem implies that for every $\alpha > 0$,

$$\mathbf{u}_\alpha(\mathbf{x},t) = e^{\alpha s t + i\alpha\langle \boldsymbol{\omega}_-, \mathbf{x}_-\rangle} \boldsymbol{\phi}(\alpha x_1),$$

is also a solution to (2.10) and (2.12) with homogeneous data. Since $\Re s > 0$, we can choose α arbitrarily large and have an exponentially arbitrarily growing solution, indicating that the problem is not well-posed. □

From the proof it is obvious that the non-existence of simple solutions of type (2.21) is an equivalent necessary condition for well-posedness (equivalent to the Agmon eigenvalue condition). This condition is referred to as the *Lopatinsky condition.*

Before further discussing sufficient conditions for well-posedness, it will be useful and important to derive algebraic conditions that determine whether s with $\Re s > 0$ is an eigenvalue. Let κ be the solution of the following characteristic equation corresponding to (2.19),

$$\text{Det}|A\kappa - (sI - iB(\boldsymbol{\omega}_-))| = 0. \tag{2.22}$$

It is to be noted that the solutions of (2.22) are the eigenvalues of the matrix $M = A^{-1}(sI - iB(\boldsymbol{\omega}_-))$. We have the following lemma [9].

LEMMA 2.1. *For solutions κ of the characteristic Equation (2.22), we have:*

(1) For $\Re s > 0$, there are no κ with $\Re \kappa = 0$.

(2) There are precisely r solutions with $\Re\kappa < 0$ and $n - r$ solutions with $\Re\kappa > 0$.

(3) There exists a constant $\delta > 0$ such that $|\Re\kappa| > \delta\eta$ for all $s = i\xi + \eta$, with $\xi \in \mathbb{R}$ and $\eta > 0$, and for all $\boldsymbol{\omega}_- \in \mathbb{R}^{m-1}$.

Proof. For the proof see Lemma 2.1 in [11]. □

Assuming all eigenvalues $\kappa_j = \kappa_j(\boldsymbol{\omega}_-, s)$, with $j = 1, \ldots, n$, are distinct, we can write the solution of (2.19)-(2.20) as

$$\boldsymbol{\phi} = \sum_{\Re\kappa_j < 0} \sigma_j e^{\kappa_j x_1} \mathbf{v}_j + \sum_{\Re\kappa_j > 0} \sigma_j e^{\kappa_j x_1} \mathbf{v}_j, \tag{2.23}$$

where $\mathbf{v}_j = \mathbf{v}_j(\boldsymbol{\omega}_-, s)$, with $j = 1, \ldots, n$, are the eigenvectors of M corresponding to the eigenvalues κ_j. Note that if the eigenvalues κ_j are not distinct, the usual d'Alembert-type modifications apply. Since we are only interested in bounded solutions $\boldsymbol{\phi} \in L^2([0,\infty))$, we set $\sigma_j = 0$ corresponding to $\Re\kappa_j > 0$ in the second sum in (2.23). The solution $\boldsymbol{\phi}$ is then given by the first sum, consisting of r terms with $\Re\kappa_j < 0$, $j = 1, \ldots, r$. We partition the eigenvectors $\mathbf{v}_j = (v_{j,1}, \ldots, v_{j,n})^\top$ to $\mathbf{v}_j^I = (v_{j,1}, \ldots, v_{j,r})^\top$ and $\mathbf{v}_j^{II} = (v_{j,r+1}, \ldots, v_{j,n})^\top$, for every $j = 1, \ldots, r$. Introducing $\boldsymbol{\phi}$ into the boundary condition (2.20) at $x_1 = 0$, we get a linear system of r equations for r unknowns $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r)^\top$,

$$\tilde{S}(s, \boldsymbol{\omega}_-)\boldsymbol{\sigma} = 0, \qquad \tilde{S} = V_I - S V_{II} \in \mathbb{C}^{r\times r}, \tag{2.24}$$

where

$$V_I = [\mathbf{v}_1^I \; \ldots \; \mathbf{v}_r^I] \in \mathbb{C}^{r\times r}, \qquad V_{II} = [\mathbf{v}_1^{II} \; \ldots \; \mathbf{v}_r^{II}] \in \mathbb{C}^{(n-r)\times r}.$$

Therefore s with $\Re s > 0$ is an eigenvalue if and only if $\text{Det}|\tilde{S}(s, \boldsymbol{\omega}_-)| = 0$. This suggests that the Agmon eigenvalue condition in Theorem 2.2 is equivalent to the algebraic condition

$$\text{Det}|\tilde{S}(s, \boldsymbol{\omega}_-)| \neq 0, \qquad \text{for} \quad \Re s > 0. \tag{2.25}$$

The Agmon eigenvalue condition, or equivalently the algebraic condition (2.25), is a necessary condition for the IBVP to be well-posed. Hersch [9] has shown that this condition is also sufficient for the problem to be weakly well-posed in the sense of Hadamard. The IBVP can indeed be solved by the Laplace-Fourier transform as follows. If $\mathbf{u}$ is the solution of the IBVP with $\mathbf{h}\equiv\mathbf{0}$, then its Laplace-Fourier transform $\tilde{\mathbf{u}}$ satisfies the resolvent problem (2.17)-(2.18). If (2.25) holds for every s with $\Re s>0$, we know that (2.17)-(2.18) has a unique solution. Inverting the Laplace-Fourier transform we obtain the solution to the IBVP. In general, however, the eigenvalue condition does not give stability against lower order terms and hence is not sufficient for strong well-posedness in the generalized sense.

2.2.3. A sufficient condition for strong well-posedness in the generalized sense. We now introduce the concept of generalized eigenvalues and present a sufficient condition for strong well-posedness in the generalized sense.

Consider the following normalized variables

$$s' := \frac{s}{\sqrt{|s|^2+|\boldsymbol{\omega}_-|^2}} = \frac{i\xi+\eta}{\sqrt{|s|^2+|\boldsymbol{\omega}_-|^2}} =: i\xi'+\eta', \qquad \boldsymbol{\omega}'_- := \frac{\boldsymbol{\omega}_-}{\sqrt{|s|^2+|\boldsymbol{\omega}_-|^2}}, \tag{2.26}$$

and write the resolvent problem (2.17)-(2.18) as

$$-A\frac{d\tilde{\mathbf{u}}}{dx_1} + \sqrt{|s|^2+|\boldsymbol{\omega}_-|^2}\,\big(s'I - iB(\boldsymbol{\omega}'_-)\big)\,\tilde{\mathbf{u}} = \tilde{\mathbf{f}}, \qquad \text{for } x_1 \geq 0, \tag{2.27}$$

$$\tilde{\mathbf{u}}_I - S\,\tilde{\mathbf{u}}_{II} = \tilde{\mathbf{g}}, \qquad \text{at } x_1 = 0. \tag{2.28}$$

The corresponding normalized homogeneous eigenvalue problem then reads

$$-A\frac{d\boldsymbol{\phi}}{dx_1} + \sqrt{|s|^2+|\boldsymbol{\omega}_-|^2}\,\big(s'I - iB(\boldsymbol{\omega}_-)\big)\,\boldsymbol{\phi} = \mathbf{0}, \qquad \text{for } x_1 \geq 0, \tag{2.29}$$

$$\boldsymbol{\phi}_I - S\,\boldsymbol{\phi}_{II} = \mathbf{0}, \qquad \text{at } x_1 = 0. \tag{2.30}$$

One can show that the normalized determinant $\mathrm{Det}|\tilde{S}(s',\boldsymbol{\omega}'_-)|$ is a continuous function of its arguments for all $\Re s' \geq 0$; see [11]. This leads to the following definition of generalized eigenvalues.

DEFINITION 2.8. *Let $(i\xi'_0,\boldsymbol{\omega}'_0)$, with $\xi'_0\in\mathbb{R}$ and $\boldsymbol{\omega}'_0\in\mathbb{R}^{m-1}$, be fixed and consider the eigenvalue problem (2.29)-(2.30) for $s'=i\xi'_0+\eta'$ and $\boldsymbol{\omega}'_-=\boldsymbol{\omega}'_0$, with $\eta'>0$. Then $(i\xi'_0,\boldsymbol{\omega}'_0)$ is called a generalized eigenvalue corresponding to a given boundary condition if in the limit $\eta'\to 0^+$ the boundary condition (2.30) is satisfied, or equivalently if*

$$\lim_{\eta'\to 0^+} Det|\tilde{S}(i\xi'_0+\eta',\boldsymbol{\omega}'_0)| = 0,$$

where $\tilde{S}$ is given by (2.24).

We can now formulate the Kreiss Eigenvalue Theorem (Main Theorem 1 in [11]) that gives a necessary and sufficient condition for strong well-posedness in the generalized sense.

THEOREM 2.3 (Kreiss eigenvalue condition). *Suppose that (2.10) is strictly hyperbolic with a nonsingular boundary matrix A of the form (2.9). The half-space problem (2.10)-(2.12), with $\mathbf{h}\equiv\mathbf{0}$, is strongly well-posed in the generalized sense, that is, an estimate of type (2.15) holds, if and only if the Kreiss eigenvalue condition is fulfilled, that is,* *the eigenvalue problem (2.29)-(2.30) has no eigenvalue or generalized eigenvalue for* $\Re s' \geq 0$.

One can also phrase the Kreiss eigenvalue condition as a strong boundary stability condition [18].

DEFINITION 2.9. *Let* $\mathbf{f} \equiv \mathbf{h} \equiv 0$. *We call the IBVP (2.10)-(2.12) strongly boundary stable if there is a constant* $K > 0$ *such that for all smooth boundary data* $\mathbf{g}$ *a unique solution* $\mathbf{u}$ *exists that satisfies*

$$\int_0^\infty e^{-2\eta t} ||\mathbf{u}(0,.,t)||^2_{L^2(R_-)} dt \leq K \int_0^\infty e^{-2\eta t} ||\mathbf{g}(.,t)||^2_{L^2(R_-)} dt, \qquad \forall \eta > 0. \tag{2.31}$$

The Kreiss Eigenvalue Theorem can then be formulated in terms of Definition 2.9 as follows.

THEOREM 2.4 (Kreiss boundary stability condition). *Suppose that* (2.10) *is strictly hyperbolic with a nonsingular boundary matrix* A *of the form* (2.9). *The half-space problem* (2.10)-(2.12), *with* $\mathbf{h} \equiv \mathbf{0}$, *is strongly well-posed in the generalized sense, that is, an estimate of type* (2.15) *holds, if and only if the problem is strongly boundary stable, that is, an estimate of type* (2.31) *holds.*

The proof of the above equivalent theorems, i.e. Theorem 2.3 and Theorem 2.4, is based on the construction of Kreiss symmetrizers [11], formulated in Theorem 2.5. A major advantage of strong boundary stability is that its corresponding estimate (2.31) is easier to verify compared to the equivalent estimate (2.15) corresponding to strong well-posedness in the generalized sense.

THEOREM 2.5. *Suppose that* (2.10) *is strictly hyperbolic with a nonsingular boundary matrix* A *of the form* (2.9). *Assume further that the half-space problem* (2.10)-(2.12), *with* $\mathbf{h} \equiv \mathbf{0}$, *is strongly boundary stable, or equivalently the Kreiss eigenvalue condition is satisfied. Then there exists a complex symmetrizer matrix* $\hat{R} = \hat{R}(s', \boldsymbol{\omega}'_-)$ *that has the following properties for all* $\boldsymbol{\omega}'_- \in \mathbb{R}^{m-1}$, $\xi' \in \mathbb{R}$, *and* $\eta' \in \mathbb{R}$ *with* $\eta > 0$:

i) $\hat{R}$ *is uniformly bounded and a smooth function of* s' *and* $\boldsymbol{\omega}'_-$ *and of the coefficient matrices* A, B_j, *with* $j = 2, \ldots, m$, *and* S.

ii) $\hat{R}A$ *is Hermitian.*

iii) For all vectors $\tilde{\mathbf{u}}_0$ *satisfying the boundary condition, we have* $\tilde{\mathbf{u}}_0^* \hat{R} A \tilde{\mathbf{u}}_0 \geq \delta |\tilde{\mathbf{u}}_0|^2 - C |\tilde{\mathbf{g}}|^2$, *where* δ *and* C *are constants independent of* $\boldsymbol{\omega}'_-$, ξ', *and* η'. *Here,* $|\mathbf{v}|^2 := \mathbf{v}^* \mathbf{v}$ *for every* $\mathbf{v} \in \mathbb{C}^n$.

iv) $\sqrt{|s|^2 + |\boldsymbol{\omega}_-|^2}\, \Re\{\hat{R}\big(s' I - i B(\boldsymbol{\omega}'_-)\big)\} \geq \eta I$.

Proof. For the proof see Section 4 in [11]. ∎

Proof. (**Proof of Theorem 2.3 and Theorem 2.4.**) Clearly, with $\mathbf{f} \equiv \mathbf{0}$, the estimate (2.15) implies the estimate (2.31). It remains to show that the converse is also true. If (2.31) holds, then by Theorem 2.5, we know that there exists a symmetrizer $\hat{R}$ with the properties (i)-(iv) above. Let $(\mathbf{u}, \mathbf{v})_0 := \int_0^\infty \mathbf{u}^* \mathbf{v}\, dx_1$ and $||\mathbf{u}||_0^2 = (\mathbf{u}, \mathbf{u})_0$. We multiply (2.27)-(2.28) by $\hat{R}$ and obtain

$$\begin{aligned}\Re(\tilde{\mathbf{u}}, \hat{R}\tilde{\mathbf{f}})_0 &= -\Re\big(\tilde{\mathbf{u}}, \hat{R}A \frac{d\tilde{\mathbf{u}}}{dx_1}\big)_0 + \Re\big(\tilde{\mathbf{u}}, \sqrt{|s|^2 + |\boldsymbol{\omega}_-|^2}\, \hat{R}\big(s' I - i B(\boldsymbol{\omega}'_-)\big)\tilde{\mathbf{u}}\big)_0 \\ &= -\frac{1}{2}\Re\big[\tilde{\mathbf{u}}^* \hat{R} A \tilde{\mathbf{u}}\big]_0^\infty + \Re\big(\tilde{\mathbf{u}}, \sqrt{|s|^2 + |\boldsymbol{\omega}_-|^2}\, \hat{R}\big(s' I - i B(\boldsymbol{\omega}'_-)\big)\tilde{\mathbf{u}}\big)_0\end{aligned}$$

$$\geq \frac{1}{2}\delta|\tilde{\mathbf{u}}(0,\boldsymbol{\omega}_-,s)|^2 - \frac{1}{2}C|\tilde{\mathbf{g}}|^2 + \eta||\tilde{\mathbf{u}}(.,\boldsymbol{\omega}_-,s)||_0^2.$$

The second equality above follows by integration by parts and property (ii) in Theorem 2.5. The last inequality follows from properties (iii)-(iv) in Theorem 2.5. We then use the Cauchy-Schwarz inequality and the Young's inequality with ε,

$$2\Re(\mathbf{a}^*\mathbf{b}) = \mathbf{a}^*\mathbf{b} + \mathbf{b}^*\mathbf{a} \leq 2|\mathbf{a}|\ |\mathbf{b}| \leq \varepsilon|\mathbf{a}|^2 + \varepsilon^{-1}|\mathbf{b}|^2, \qquad \forall \mathbf{a},\mathbf{b} \in \mathbb{C}^n, \quad \varepsilon > 0,$$

and set $\mathbf{a} = \tilde{\mathbf{u}}$, $\mathbf{b} = \hat{R}\tilde{\mathbf{f}}$, and $\varepsilon = \eta$. After integrating with respect to x_1, we get

$$\Re(\tilde{\mathbf{u}},\hat{R}\tilde{\mathbf{f}})_0 \leq \frac{\eta}{2}||\tilde{\mathbf{u}}(.,\boldsymbol{\omega}_-,s)||_0^2 + \frac{c}{2\eta}||\tilde{\mathbf{f}}||_0^2,$$

where the constant c is related to the upper bound of $||\hat{R}||$, due to property (i) in Theorem 2.5. Combining the above two inequalities for $\Re(\tilde{\mathbf{u}},\hat{R}\tilde{\mathbf{f}})_0$, we obtain

$$\eta||\tilde{\mathbf{u}}(.,\boldsymbol{\omega}_-,s)||_0^2 + \delta|\tilde{\mathbf{u}}(0,\boldsymbol{\omega}_-,s)|^2 \leq \frac{c}{\eta}||\tilde{\mathbf{f}}||_0^2 + C|\tilde{\mathbf{g}}|^2.$$

Inverting the Fourier and Laplace transforms, the estimate (2.15) follows by Parseval's relation. ∎

It is to be noted that strong well-posedness in the generalized sense has several good properties. First, the estimate (2.31), and consequently the estimate (2.15), can easily be derived by direct algebraic manipulations, as we only need to solve the system of ordinary differential Equations (2.27)-(2.28) with constant coefficients and homogeneous data ($\tilde{\mathbf{g}} \equiv \mathbf{0}$). Secondly, Definition 2.5 (or equivalently Definition 2.9) implies robustness, i.e. stability against both lower order terms and boundary perturbations [2, 11, 20]. If we add a zeroth-order term to the PDE or if we add a low-order term to the boundary condition (here since we have a Dirichlet condition a low-order term means a term independent of $\mathbf{u}$), then the perturbed problem remains well-posed if the original unperturbed problem is well-posed in the sense of Definition 2.5. Moreover, if the problem is strongly boundary stable, one can derive similar estimates for higher-order derivatives of the solution in terms of derivatives of the data. Finally, since the derivation of the estimates is through the construction of smooth symmetrizers, one can use the theory of pseudo-differential operators and treat problems with variable coefficients in general smooth domains. In this case the principle of frozen coefficients holds: the variable-coefficient problem is strongly well-posed in the generalized sense if all corresponding frozen-coefficient problems are strongly well-posed in the generalized sense. We can formulate this result as follows [11].

THEOREM 2.6. *Consider the half-space problem* (2.10)-(2.12) *with variable and sufficiently smooth coefficients. Then an estimate of type* (2.15) *holds if we can construct a smooth symmetrizer* $\hat{R}(\mathbf{x},t,\boldsymbol{\omega}_-,s)$ *with the properties stated in Theorem 2.5 for every fixed boundary point* $(0,\mathbf{x}_-,t)$.

Proof. For the proof see the proof of Theorem 8.4.9 in [15]. ∎

3. Extension of Kreiss theory

The Kreiss theory gives necessary and sufficient conditions for the well-posedness and robustness of strongly boundary stable problems. There are, however, problems that are not strongly boundary stable and yet are well-posed and robust. Examples include surface waves and glancing waves which are boundary-type wave phenomena that often occur in elastodynamics and electromagnetism.

In this section we will consider hyperbolic problems that describe boundary-type wave phenomena and characterize them by boundary-type eigenmodes. Using the Laplace-Fourier technique and through the construction of smooth symmetrizers, we will develop a theory for the well-posedness and robustness of such class of problems in the sense of Definition 2.6. After formulating the main theorem and related definitions in Section 3.1, we will present a two-dimensional model problem in a half-plane subject to different types of boundary conditions resulting in different types of well-posedness in Section 3.2. This simple model problem will serve as an illustrative example to better understand various concepts of well-posedness and to shed light on the construction of more general symmetrizers, which is known to be a technical task [10]. The proof of the main theorem and the construction of symmetrizers for general systems will be presented in Section 4.

3.1. Well-posedness in the generalized sense. The main goal is to relax the Kreiss eigenvalue condition, stated in Theorem 2.3, and obtain a sufficient condition for the half-space problem (2.10)-(2.12) to be well-posed in the sense of Definition 2.6. Clearly, the Agmon eigenvalue condition, stated in Theorem 2.2, is a necessary condition for well-posedness in any sense, and we cannot have eigenvalues $s' \in \mathbb{C}$ with $\Re s' > 0$. We will hence keep the Agmon eigenvalue condition and relax the requirement on the generalized eigenvalues with $\Re s' = 0$. We recognize a particular class of generalized eigenvalues, referred to as *boundary-type* generalized eigenvalues, defined below.

Definition 3.1. *Let $(i\xi_0', \boldsymbol{\omega}_0')$, with $\xi_0' \in \mathbb{R}$ and $\boldsymbol{\omega}_0' \in \mathbb{R}^{m-1}$, be fixed and consider the eigenvalue problem (2.29)-(2.30) for $s' = i\xi_0' + \eta'$ and $\boldsymbol{\omega}_-' = \boldsymbol{\omega}_0'$, with $\eta' > 0$. Then $(i\xi_0', \boldsymbol{\omega}_0')$ is called a boundary-type generalized eigenvalue corresponding to a boundary condition if the following conditions hold:*

(1) In the limit $\eta' \to 0^+$ the boundary condition (2.30) is satisfied, that is, $(i\xi_0', \boldsymbol{\omega}_0')$ is a generalized eigenvalue according to Definition 2.8.

(2) The eigenfunction corresponding to the generalized eigenvalue is of either surface mode, decaying exponentially in the direction normal to the boundary, or glancing mode, being independent of the variable normal to the boundary. This can be characterized by solutions κ of the characteristic Equation (2.22):
- *For surface eigenmodes: $\lim_{\eta \to 0^+} \kappa = -c|\boldsymbol{\omega}_0|$, with $c > 0$ and $\forall \kappa$ with $\Re\kappa < 0$.*
- *For glancing eigenmodes: $\lim_{\eta \to 0^+} \kappa = 0$.*

We can now formulated the main theorem of the present work as follows.

Theorem 3.1 (**Main Theorem**). *Consider the half-space problem (2.10)-(2.12) with $\mathbf{h} \equiv \mathbf{0}$ and constant, complex-valued coefficient matrices $A, B_2, \ldots, B_m \in \mathbb{C}^{n \times n}$ and $S \in \mathbb{C}^{r \times (n-r)}$, where A is nonsingular and has the form (2.9). Suppose that the system (2.10) is strictly hyperbolic, and assume further that there is no eigenvalue s with $\Re s > 0$ to the corresponding homogeneous eigenvalue problem (2.29)-(2.30). Then we have the following results:*

1) The problem is strongly well-posed in the generalized sense if and only if there is no generalized eigenvalue s with $\Re s = 0$.

2) The problem is well-posed in the generalized sense if the generalized eigenvalue s with $\Re s = 0$ is of either glancing or surface type.

The first part of the Main Theorem is the Kreiss eigenvalue theorem (Theorem 2.3). The second part of the theorem is the new contribution of the present work. We will present the proof in Section 4.

It is to be noted that well-posedness in the generalized sense (Definition 2.6) has several good properties, similar to the case of strong well-posedness in the generalized sense. First, it can be verified by algebraic manipulations. Secondly, the definition implies robustness, i.e. stability against both low-order terms and boundary perturbations. For instance, if we assume that the problem $\partial_t \mathbf{u} = P(\partial_{\mathbf{x}})\mathbf{u} + \mathbf{f}$ is well-posed in the generalized sense, then the perturbed problem $\partial_t \mathbf{v} = P(\partial_{\mathbf{x}})\mathbf{v} + D\mathbf{v} + \mathbf{f}$ is also well-posed in the generalized sense, provided D is a bounded matrix. Indeed, we can think of $\mathbf{q} := D\mathbf{v} + \mathbf{f}$ as a forcing function and consider the system $\partial_t \mathbf{v} = P(\partial_{\mathbf{x}})\mathbf{v} + \mathbf{q}$. Then the available estimate for $\mathbf{v}$ in terms of $\mathbf{q}$, thanks to the well-posedness of the unperturbed problem, can be utilized to derive the desired estimate for $\mathbf{v}$ in terms of $\mathbf{f}$; see Theorem 2.4.1 in [13] for the proof. We conjecture that well-posedness in the generalized sense is the weakest definition of well-posedness that has this stability property, i.e., if the problem is not well-posed in the generalized sense, then we can make a perturbation such that there are solutions which grow exponentially arbitrarily fast. Finally, as we will show in the following sections, one can extend the construction of Kreiss-type symmetrizers to this case. One is therefore able to treat systems with variable coefficients in general smooth domains. In particular, the same result as the result in the Main Theorem holds for hyperbolic systems with variable coefficients.

REMARK 3.1. Similar to the original work of Kreiss [11], discussed in Section 2, we assume strict hyperbolicity and consider a nonsingular boundary matrix A. The former assumption seems to be restrictive for many applications. However, we conjecture that, similar to the work of Agranovich [2] on the original theory of Kreiss, it is possible to show that the Main Theorem remains valid for strongly hyperbolic systems with variable coefficients if there exists smooth transformations $S(\mathbf{x},t,\boldsymbol{\omega})$ that diagonalize the symbol $P(\mathbf{x},t,\boldsymbol{\omega})$ of the differential operator. For singular boundary matrices, similar modifications to [21] need to be applied.

3.2. A 2D model problem. In this section we consider a model problem in a 2D half-plane $R_0 = \{(x,y) \,|\, x \geq 0, -\infty < y < \infty\}$. We discuss in details different types of well-posedness, emphasizing on the well-posedness in the generalized sense (Definition 2.6). We obtain desired interior estimates (2.16) via two approaches: 1) employing Laplace-Fourier technique and directly solving the resulting family of ordinary boundary value problems, and 2) constructing Kreiss-type symmetrizers. The first approach will help understand the behavior of the solution in the presence of boundary-type eigenmodes. The second approach will facilitate the construction of symmetrizers for more general problems, needed to extend the theory to problems with variable coefficients.

Consider the following model problem,

$$\partial_t \mathbf{u} = A\partial_x \mathbf{u} + B\partial_y \mathbf{u} + \mathbf{f}(x,y,t), \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (x,y) \in R_0, \quad t \geq 0, \tag{3.1}$$

with the initial condition

$$\mathbf{u}(x,y,0) = \mathbf{0}, \tag{3.2}$$

and the boundary condition at $x = 0$,

$$u_1(0,y,t) = a u_2(0,y,t) + g(y,t), \qquad a \in \mathbb{C}. \tag{3.3}$$

Here $\mathbf{u} = (u_1, u_2)^\top$ is a complex-valued two-dimensional vector function. The data are assumed to be smooth, compactly supported, and compatible at $t = 0$ and $x = 0$. Moreover, we are only interested in solutions with bounded L^2-norm and therefore assume that $||\mathbf{u}(.,t)||_{L^2(R_0)} < \infty$ for every fixed t. It is to be noted that the general theory of hyperbolic systems suggests that we impose only one boundary condition at $x=0$ to express the in-going characteristic field in terms of the out-going one.

We Fourier-Laplace transform the problem in y-t, respectively, and obtain the ordinary boundary problem

$$\frac{d\tilde{\mathbf{u}}}{dx} = M\,\tilde{\mathbf{u}} - A^{-1}\tilde{\mathbf{f}}, \qquad M = \begin{pmatrix} -s & i\omega \\ -i\omega & s \end{pmatrix}, \qquad s\in\mathbb{C}, \quad \omega\in\mathbb{R}, \tag{3.4}$$

with the boundary conditions

$$\tilde{u}_1(0,\omega,s) = a\,\tilde{u}_2(0,\omega,s) + \tilde{g}(\omega,s), \qquad ||\tilde{\mathbf{u}}||_0^2 = \int_0^\infty |\tilde{\mathbf{u}}(x)|^2 dx < \infty. \tag{3.5}$$

We have indeed considered the boundedness requirement $||\tilde{\mathbf{u}}||_0 < \infty$ as the boundary condition at infinity.

A family of eigenvalue problems. We let $\mathbf{f}\equiv g\equiv 0$ and construct simple wave solutions of type

$$\mathbf{u}(x,y,t) = e^{st+i\omega y}\,\boldsymbol{\phi}(x), \qquad \Re s > 0, \tag{3.6}$$

where $\boldsymbol{\phi} = (\phi_1,\phi_2)^\top$ and $||\boldsymbol{\phi}||_0^2 < \infty$. Substituting (3.6) into (3.1) and (3.3), we obtain the following family of eigenvalue problems

$$s\,\boldsymbol{\phi} = A\frac{d\boldsymbol{\phi}}{dx} + i\omega B\,\boldsymbol{\phi}, \qquad x \geq 0, \tag{3.7}$$

$$\phi_1(0) = a\,\phi_2(0), \qquad ||\boldsymbol{\phi}||_0^2 < \infty. \tag{3.8}$$

Agmon and Kreiss eigenvalue conditions. We now discuss the requirements on the boundary parameter $a\in\mathbb{C}$ so that the necessary and sufficient conditions for well-posedness in the Kreiss sense are fulfilled.

Lemma 3.1. *The Agmon eigenvalue condition is satisfied, that is, there is no eigenvalue of (3.7)-(3.8) with $\Re s > 0$, only in two cases: when $a\in\mathbb{R}$, or when $a\in\mathbb{C}$ with $|a|\leq 1$.*

Proof. For the proof see Lemma 8.4.2 in [15]. □

This lemma gives the Agmon eigenvalue condition that is necessary for well-posedness: if $|a|>1$ and $a\notin\mathbb{R}$, there exist eigenvalues of (3.7)-(3.8) with $\Re s > 0$ and therefore the problem will be ill-posed. Now assume that $a\in\mathbb{R}$ or $|a|\leq 1$. We can then solve (3.4)-(3.5) with $\mathbf{f}, g \not\equiv 0$. Inverting the Laplace-Fourier transforms, we obtain the solution of the IBVP (3.1)-(3.3). Obtaining the solution will however not imply well-posedness. The problem will be well-posed only if we can derive proper solution estimates in terms of the data. To derive a sufficient condition for well-posedness, we first check if there is any generalized eigenvalue.

Lemma 3.2. *The Kreiss eigenvalue condition is satisfied, that is, there is no eigenvalue and generalized eigenvalue of (3.7)-(3.8) with $\Re s \geq 0$, only in the case $|a|<1$.*

Proof. For the proof see Section 8.4.3 in [15]. □

This lemma shows that the condition $|a|<1$ is both necessary and sufficient for the problem to be strongly boundary stable in the sense of Kreiss.

Discussion of generalized eigenvalues. We now focus on the case $\Re s = 0$ where a generalized eigenvalue exists. In this case the Agmon necessary condition for well-posedness is still fulfilled, but the Kreiss necessary and sufficient condition is not satisfied. Hence the problem is neither ill-posed nor strongly boundary stable. By Lemma 3.1 and Lemma 3.2, there are two cases for which there exists a generalized eigenvalue:

(1) $|a| > 1, a \in \mathbb{R}$,

(2) $|a| = 1, a \in \mathbb{C}$.

These two cases are fundamentally different and correspond to different types of generalized eigenvalues. As we will show, in the first case the problem is not well-posed, while in the second case the problem is well-posed in the generalized sense. In fact, only the second case corresponds to boundary-type eigenmodes.

Before getting into the discussion, let us first write the Equation (3.7) in the form

$$\frac{d\boldsymbol{\phi}}{dx} = M\boldsymbol{\phi}, \qquad M = \begin{pmatrix} -s & i\omega \\ -i\omega & s \end{pmatrix}, \qquad s \in \mathbb{C}, \quad \omega \in \mathbb{R}. \tag{3.9}$$

The matrix M has two eigenvalues

$$\kappa_1 = -\kappa, \qquad \kappa_2 = \kappa, \qquad \kappa = \sqrt{s^2 + \omega^2}, \tag{3.10}$$

with the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} s+\kappa \\ i\omega \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} s-\kappa \\ i\omega \end{pmatrix}. \tag{3.11}$$

Since for $\Re s > 0$, we have $\Re\kappa_1 < 0 < \Re\kappa_2$, then the general L_2-solution of (3.7) is given by

$$\boldsymbol{\phi} = \sigma e^{-\kappa x}\begin{pmatrix} s+\kappa \\ i\omega \end{pmatrix}, \qquad \sigma \in \mathbb{C}. \tag{3.12}$$

Inserting this expression into the boundary condition (3.8), we get

$$\tilde{S}\sigma = 0, \qquad \tilde{S} = s + \kappa - ia\omega.$$

Hence, by definition 2.8, $(i\xi_0, \omega_0)$ is a generalized eigenvalue if

$$\lim_{\eta\to 0^+} \tilde{S}(\eta;\xi_0,\omega_0,a) = 0, \qquad \tilde{S}(\eta;\xi,\omega,a) = (\eta + i\xi + \sqrt{(\eta+i\xi)^2+\omega^2} - ia\omega). \tag{3.13}$$

We notice that for a complex number $z \in \mathbb{C}$, the argument of $\sqrt{z}$ is defined by

$$\arg\sqrt{z} = \frac{1}{2}\arg z, \qquad -\pi < \arg z \le \pi. \tag{3.14}$$

In what follows, we will need the following two transformations of the matrix M in (3.9). First, if the two eigenvalues in (3.10) are distinct ($\kappa \neq 0$), we can diagonalize the matrix M,

$$\Lambda_D = T^{-1}MT = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \qquad T = (\mathbf{v}_1 \ \ \mathbf{v}_2) = \begin{pmatrix} s+\kappa & s-\kappa \\ i\omega & i\omega \end{pmatrix}, \tag{3.15}$$

and write the ordinary boundary problem (3.4)-(3.5) as

$$\frac{d\tilde{\mathbf{v}}}{dx} = \Lambda_D\tilde{\mathbf{v}} + F, \qquad \tilde{\mathbf{v}} = T^{-1}\tilde{\mathbf{u}}, \qquad F = -T^{-1}A^{-1}\tilde{\mathbf{f}}, \tag{3.16}$$

$$S_1\tilde{v}_1(0,\omega,s)+S_2\tilde{v}_2(0,\omega,s)=\tilde{g}(\omega,s), \quad S_1=s+\kappa-ia\omega, \quad S_2=s-\kappa-ia\omega. \tag{3.17}$$

Secondly, if the two eigenvalues in (3.10) are multiple ($\kappa=0$), we use the Schur decomposition and by a unitary transformation matrix we transform M into an upper triangular form

$$\Lambda_T=T^*MT=\begin{pmatrix}\kappa_1 & p\\ 0 & \kappa_2\end{pmatrix}, \qquad T=\left((s+\kappa)(\bar{s}+\bar{\kappa})+\omega^2\right)^{-\frac{1}{2}}\begin{pmatrix}s+\kappa & i\omega\\ i\omega & \bar{s}+\bar{\kappa}\end{pmatrix}, \tag{3.18}$$

where

$$p=-i\omega\frac{(2s-\bar{s}-\bar{\kappa})(\bar{s}+\bar{\kappa})+\omega^2}{(s+\kappa)(\bar{s}+\bar{\kappa})+\omega^2}.$$

The ordinary boundary problem (3.4)-(3.5) is then written as

$$\frac{d\tilde{\mathbf{v}}}{dx}=\Lambda_T\tilde{\mathbf{v}}+F, \qquad \tilde{\mathbf{v}}=T^*\tilde{\mathbf{u}}, \qquad F=-T^*A^{-1}\tilde{\mathbf{f}}, \tag{3.19}$$

$$S_1\tilde{v}_1(0,\omega,s)+S_2\tilde{v}_2(0,\omega,s)=\tilde{g}(\omega,s), \quad S_1=s+\kappa-ia\omega, \quad S_2=-a(\bar{s}+\bar{\kappa})+i\omega. \tag{3.20}$$

We note that in both cases (distinct κ's and multiple κ's) we have $\tilde{S}=S_1$. This will be used later to find the generalized eigenvalues. We are now ready to study the two cases of a that lead to generalized eigenvalues. We treat each case separately.

3.2.1. The case $|a|>1, a\in\mathbb{R}$. As discussed above, the problem is not strongly boundary stable in this case. Here, we will show that the problem is also not well-posed in the generalized sense. We only consider the case when $a\in\mathbb{R}$ and $a>1$. The other case when $a<-1$ can be treated similarly. We first find the generalized eigenvalue.

THEOREM 3.2. *The generalized eigenvalue of (3.7)-(3.8) with $a>1, a\in\mathbb{R}$ is*

$$s_0=i\xi_0, \qquad \xi_0=\frac{1+a^2}{2a}\omega_0. \tag{3.21}$$

and the corresponding eigen-solution reads

$$\mathbf{u}=\exp\left(i\omega_0(-\frac{a^2-1}{2a}x+y+\frac{1+a^2}{2a}t)\right)\mathbf{v}_1. \tag{3.22}$$

Furthermore, the generalized eigenvalue (3.21) is not of boundary-type.

Proof. Let $s=i\frac{1+a^2}{2a}\omega+\eta$, where $0<\eta\ll|\omega|$. Then

$$\begin{aligned}\kappa=\sqrt{s^2+\omega^2}&\approx\sqrt{\frac{-(a^2-1)^2}{4a^2}\omega^2+2i\eta\frac{1+a^2}{2a}\omega}\\ &=i\frac{a^2-1}{2a}|\omega|\sqrt{1-2i\eta\frac{2a(1+a^2)}{(a^2-1)^2}\frac{1}{\omega}}\\ &\approx i\frac{a^2-1}{2a}|\omega|\left(1-i\eta\frac{2a(1+a^2)}{(a^2-1)^2}\frac{1}{\omega}\right).\end{aligned}$$

Since, by Lemma 5.3 of the Appendix, $\Re\kappa>0$ for $\Re s>0$, we only consider $\omega>0$ and therefore

$$\kappa\approx i\frac{a^2-1}{2a}\omega+\frac{1+a^2}{a^2-1}\eta, \qquad \omega>0. \tag{3.23}$$

This gives

$$\begin{aligned}\tilde{S}(\eta;\xi,\omega,a) &= (\eta+i\xi+\sqrt{(\eta+i\xi)^2+\omega^2}-ia\omega)\\ &\approx \eta(1+\frac{1+a^2}{a^2-1})+i\omega(\frac{1+a^2}{2a}+\frac{a^2-1}{2a}-a)=\frac{2a^2}{a^2-1}\eta,\end{aligned}$$

and hence $\lim_{\eta\to 0^+}\tilde{S}(\eta;\xi,\omega,a)=0$, meaning that (3.21) is the generalized eigenvalue according to (3.13). The eigen-solution (3.22) is easily obtained by plugging (3.12) into (3.6) at the generalized eigenvalue (3.21). Clearly, κ in (3.23) is purely imaginary in the limit $\eta\to 0^+$, and hence by Definition 3.1 the generalized eigenvalue (3.21) is not of boundary-type. ∎

We next obtain the solution estimates. We need to derive the estimates only in a neighborhood of the generalized eigenvalue (3.21). We set

$$s=i\frac{1+a^2}{2a}\omega+\eta, \qquad \omega=\omega_0, \qquad 0<\eta\ll|\omega_0|. \tag{3.24}$$

Since $\kappa\neq 0$, we can diagonalize the system and employ (3.16)-(3.17) with

$$\begin{aligned}S_1 &= s+\kappa-ia\omega\approx\eta(1+\frac{1+a^2}{a^2-1})+i\omega(\frac{1+a^2}{2a}+\frac{a^2-1}{2a}-a)=\frac{2a^2}{a^2-1}\eta,\\ S_2 &= s-\kappa-ia\omega\approx\frac{-2}{a^2-1}\eta-i\frac{a^2-1}{a}\omega.\end{aligned}$$

We hence have

$$|S_1|\approx\frac{2a^2}{a^2-1}\eta, \qquad |S_2|\approx\frac{a^2-1}{a}\omega. \tag{3.25}$$

In order to obtain estimates, we split the solution of (3.16)-(3.17) into two parts; one part solving the homogeneous Equation (3.16) with an inhomogeneous boundary condition (3.17), and the other part satisfying the full inhomogeneous equation but with a homogeneous boundary condition.

We first assume $F\equiv 0$ and $\tilde{g}\not\equiv 0$. From the second equation of (3.16), we get

$$\tilde{v}_2(x,\omega,s)=0, \tag{3.26}$$

because otherwise, the solution would not be bounded (since $\Re\kappa>0$). The boundary condition (3.17) and the relation (3.25) give us

$$|\tilde{v}_1(0,\omega,s)|^2\approx\left(\frac{a^2-1}{2a^2}\right)^2\frac{1}{\eta^2}|\tilde{g}|^2. \tag{3.27}$$

To obtain interior estimates, we use the first equation of (3.16), and by (3.27) we write

$$||\tilde{v}_1(x,\omega,s)||_0^2=\int_0^\infty|\tilde{v}_1(0,\omega,s)e^{-\Re\kappa x}|^2\,dx=\frac{1}{2\Re\kappa}|\tilde{v}_1(0,\omega,s)|^2\approx\frac{(a^2-1)^3}{8a^4(1+a^2)}\frac{1}{\eta^3}|\tilde{g}|^2. \tag{3.28}$$

We next let $F=(F_1,F_2)^\top\not\equiv 0$ and $\tilde{g}\equiv 0$. By Lemma 5.1 of the Appendix for the second equation of (3.16), we obtain

$$|\tilde{v}_2(0,\omega,s)|^2\leq\frac{2}{\Re\kappa}||F_2||_0^2\approx\frac{2(a^2-1)}{\eta(1+a^2)}||F_2||_0^2, \tag{3.29}$$

$$||\tilde{v}_2(.,\omega,s)||_0^2 \le \frac{1}{(\Re\kappa)^2}||F_2||_0^2 \approx \frac{(a^2-1)^2}{\eta^2(1+a^2)^2}||F_2||_0^2. \tag{3.30}$$

For the first equation of (3.16) we use Lemma 5.2 and write

$$||\tilde{v}_1(.,\omega,s)||_0^2 \le \frac{1}{(\Re\kappa)^2}||F_1||_0^2 + \frac{1}{2\Re\kappa}|\tilde{v}_1(0,\omega,s)|^2.$$

Moreover, from the boundary condition (3.17) and the relation (3.25) we get

$$|\tilde{v}_1(0,\omega,s)| \approx \frac{(a^2-1)^2}{2a^3}\frac{\omega}{\eta}|\tilde{v}_2(0,\omega,s)|.$$

We therefore obtain

$$|\tilde{v}_1(0,\omega,s)|^2 \le \frac{(a^2-1)^5}{2a^6(1+a^2)}\frac{\omega^2}{\eta^3}||F_2||_0^2, \tag{3.31}$$

$$||\tilde{v}_1(.,\omega,s)||_0^2 \le \left(\frac{a^2-1}{1+a^2}\right)^2\frac{1}{\eta^2}||F_1||_0^2 + \frac{(a^2-1)^6}{4a^6(1+a^2)^2}\frac{\omega^2}{\eta^4}||F_2||_0^2. \tag{3.32}$$

We collect the estimates (3.26)-(3.32) in the following lemma.

LEMMA 3.3. *For the solution* $\tilde{\mathbf{v}}$ *of the transformed resolvent system (3.16)-(3.17) with* $a>1, a\in\mathbb{R}$ *the following estimates near the generalized eigenvalue (3.21) hold*

$$\begin{aligned}
|\tilde{v}_1(0,\omega,s)|^2 &\le C_1\left(\frac{\omega^2}{\eta^3}||F_2||_0^2 + \frac{1}{\eta^2}|\tilde{g}|^2\right),\\
|\tilde{v}_2(0,\omega,s)|^2 &\le C_2\frac{1}{\eta}||F_2||_0^2,\\
||\tilde{v}_1(.,\omega,s)||_0^2 &\le C_3\left(\frac{1}{\eta^2}||F_1||_0^2 + \frac{\omega^2}{\eta^4}||F_2||_0^2 + \frac{1}{\eta^3}|\tilde{g}|^2\right),\\
||\tilde{v}_2(.,\omega,s)||_0^2 &\le C_4\frac{1}{\eta^2}||F_2||_0^2,
\end{aligned}$$

where the constants C_1, C_2, C_3, C_4 *are independent of* ω*,* η*, and the data. The same estimates hold for the solution* $\tilde{\mathbf{u}}$ *of the resolvent system* (3.4)-(3.5)*, with* F *replaced by* $\tilde{\mathbf{f}}$ *in the right-hand side of the estimates.*

THEOREM 3.3. *The IBVP (3.1)-(3.3) with* $a>1, a\in\mathbb{R}$ *is not well-posed in the generalized sense.*

Proof. Let $g\equiv 0$. Then by Lemma 3.3, we have the interior estimate

$$||\tilde{\mathbf{u}}(.,\omega,s)||_0 \le K_\eta(1+\omega)||\tilde{\mathbf{f}}||_0, \tag{3.33}$$

where K_η is a constant of the form $1/\eta^\alpha$, independent of ω and $\tilde{\mathbf{f}}$. This estimate is equivalent with

$$\int_0^\infty e^{-2\eta t}||\mathbf{u}(.,t)||_{R_0}^2 dt \le K_\eta^2\int_0^\infty e^{-2\eta t}||\mathbf{f}(.,t)||_{H^1(R_0)}^2 dt, \tag{3.34}$$

where

$$||\mathbf{f}||^2_{H^1(R_0)} := \sum_{|\boldsymbol{\nu}|\leq 1} ||\frac{\partial^{|\boldsymbol{\nu}|}\mathbf{f}}{\partial_x^{\nu_1}\partial_y^{\nu_2}}||^2_{R_0}, \qquad \boldsymbol{\nu}=(\nu_1,\nu_2)\in\mathbb{Z}^2_+, \qquad |\boldsymbol{\nu}|=\nu_1+\nu_2.$$

This simply follows by inverting the Fourier and Laplace transforms and employing Parseval's relation:

$$\int_0^\infty e^{-2\eta t}||\mathbf{u}(.,t)||^2_{R_0}dt = \int_0^\infty ||e^{-\eta t}\mathbf{u}(.,t)||^2_{R_0}dt = \int_{-\infty}^\infty\int_{-\infty}^\infty ||\tilde{\mathbf{u}}(.,\omega,\eta+i\xi)||^2_0 d\omega\, d\xi.$$

Obviously, due to the proportionalities in (3.25), the interior solution in Fourier space cannot be bounded independent of ω, and we cannot get a sharper estimate than (3.33) independent of ω. Hence, the interior solution in the physical space cannot be bounded by the data. It is rather bounded by the first derivative of the data. The estimate (3.34) is weaker than the desired interior estimate of form (2.16), and hence by Definition 2.6 the problem is not well-posed in the generalized sense. ◻

Remark 3.2. The weak estimate (3.34) in the proof of Theorem 3.3 implies that the solution in the interior loses one derivative over the data at each reflection from the boundary. Therefore, if we consider the problem in the strip $0\leq x\leq 1$, $-\infty<y<\infty$ and add another boundary condition

$$u_1(1,y,t)=b u_2(1,y,t), \qquad |b|>1, b\in\mathbb{R},$$

the solution will lose many derivatives over the data as the time goes by. We refer to such problems as *ill-posed in the asymptotic sense.*

Remark 3.3. The generalized eigenvalue (3.21) is not of boundary-type. This suggests that the existence of boundary-type generalized eigenvalues may also be a necessary condition for well-posedness in the generalized sense, in addition to being a sufficient condition.

3.2.2. The case $|a|=1, a\in\mathbb{C}$. We now show that in this case the problem is well-posed in the generalized sense. We will indeed obtain desired interior solution estimates of type (2.16). We consider two approaches. First, we employ Fourier and Laplace transforms and directly solve the resulting family of ordinary boundary value problems. Secondly, we construct Kreiss-type symmetrizers to obtain interior estimates. The latter approach is intended to show that the Kreiss theory can indeed be extended to treat well-posed problems in the generalized sense. We distinguish between the following two different cases, as depicted in Figure 3.1, and study each case separately:

I. $|a|=1, \Im a\neq 0$.

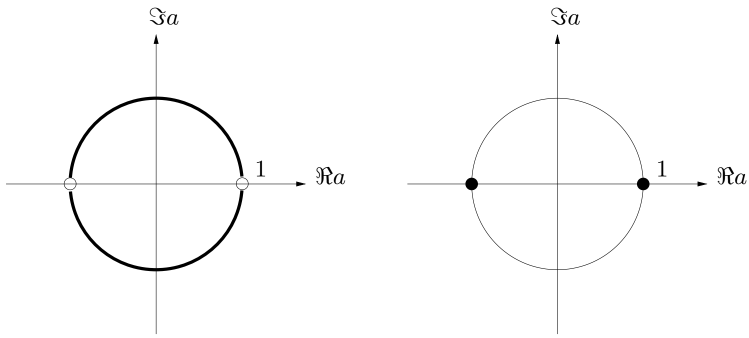
II. $a=\pm 1$.

As we will show, these two different cases correspond to two different boundary-type eigenmodes: I) surface modes, and II) glancing modes.

I. Surface eigenmodes ($|a|=1, \Im a\neq 0$). Let $a=e^{i\theta}$ with $\theta\neq k\pi$, $k=\mathbb{Z}$. We first find the generalized eigenvalue.

Theorem 3.4. *The generalized eigenvalue of (3.7)-(3.8) with $|a|=1, \Im a\neq 0$ is of boundary-type, given by*

$$s_0=i\xi_0, \qquad \xi_0=\cos\theta\,\omega_0, \qquad \sin\theta\,\omega_0<0, \tag{3.35}$$



(a) $|a| = 1$ and $\Im a \neq 0$

(b) $a = \mp 1$

FIG. 3.1. *A schematic representation of the two different cases of* $|a| = 1$.

and the corresponding eigen-solution reads

$$\mathbf{u} = \exp\Big(-|\sin\theta\,\omega_0|\,x + i\omega_0(\cos\theta\, t + y)\Big)\,\mathbf{v}_1, \tag{3.36}$$

which represents a surface wave decaying exponentially in the direction normal to the boundary.

Proof. At the point (3.35) we have

$$\tilde{S}(s_0,\omega_0) = s_0 + \sqrt{s_0^2 + \omega_0^2} - ia\omega_0 = i\cos\theta\,\omega_0 + \sqrt{\sin^2\theta\,\omega_0^2} - i(cos\theta + i\sin\theta)\omega_0 = 0,$$

where $\sin\theta\,\omega_0 < 0$. Hence this point corresponds to a generalized eigenvalue. The eigen-solution (3.36) is easily obtained by plugging (3.12) into (3.6) at the generalized eigenvalue (3.35), noting that at this point we have $\kappa = -\sin\theta\,\omega_0 = |\sin\theta\,\omega_0| > 0$. Moreover, since the first eigenvalue κ_1 of M in (3.10) with $\Re\kappa_1 < 0$ is negative and proportional to $|\omega_0|$ in the limit $\eta \to 0^+$, by Definition 3.1 the generalized eigenvalue (3.35) is of boundary-type. ∎

We consider a neighborhood of the generalized eigenvalue (3.35), and set

$$s = i\cos\theta\,\omega + \eta, \qquad \sin\theta\,\omega < 0, \qquad \omega = \omega_0, \qquad 0 < \eta \ll |\omega_0|.$$

Then

$$\kappa = \sqrt{s^2 + \omega^2} = \sqrt{\sin^2\theta\,\omega^2 + 2i\cos\theta\,\eta\,\omega + \eta^2} \approx |\sin\theta\,\omega| \left(1 + i\eta\frac{\cos\theta}{\sin^2\theta}\frac{1}{\omega}\right).$$

Since $\sin\theta\,\omega < 0$, we have

$$\kappa \approx -\sin\theta\,\omega - i\frac{\cos\theta}{\sin\theta}\eta, \qquad \sin\theta\,\omega < 0. \tag{3.37}$$

Hence $\kappa \neq 0$, and we can employ (3.16)-(3.17) with

$$S_1 = s + \kappa - ia\omega \approx (1 - i\frac{\cos\theta}{\sin\theta})\eta,$$
$$S_2 = s - \kappa - ia\omega \approx (1 + i\frac{\cos\theta}{\sin\theta})\eta + 2\sin\theta\,\omega.$$

We have

$$|S_1| \approx \frac{1}{|\sin\theta|}\eta, \qquad |S_2| \approx 2|\sin\theta|\,|\omega|. \tag{3.38}$$

We are now ready to obtain solution estimates. We will employ two approaches: first by directly obtaining estimates for the solution of (3.16)-(3.17), and next by constructing Kreiss-type symmetrizers.

Direct calculation of estimates. We first assume that $F \equiv 0$ and $\tilde{g} \not\equiv 0$ in (3.16)-(3.17). From the second equation of (3.16), since $\Re\kappa > 0$ in (3.37), we get

$$\tilde{v}_2(x,\omega,s) = 0, \tag{3.39}$$

because otherwise, the solution would not be bounded. The boundary condition (3.17) and the relation (3.38) give

$$|\tilde{v}_1(0,\omega,s)|^2 \approx \frac{|\sin\theta|^2}{\eta^2}|\tilde{g}|^2. \tag{3.40}$$

We obtain the following interior estimate from the first equation of (3.16) and using (3.40),

$$||\tilde{v}_1(.,\omega,s)||_0^2 = \int_0^\infty |\tilde{v}_1(0,\omega,s)e^{-\Re\kappa x}|^2\,dx = \frac{1}{2\Re\kappa}|\tilde{v}_1(0,\omega,s)|^2 \approx \frac{|\sin\theta|}{2\eta^2|\omega|}|\tilde{g}|^2. \tag{3.41}$$

We next let $F = (F_1, F_2)^\top \not\equiv 0$ and $\tilde{g} \equiv 0$. By Lemma 5.1 of the Appendix from the second equation of (3.16), we obtain

$$|\tilde{v}_2(0,\omega,s)|^2 \le \frac{2}{\Re\kappa}||F_2||_0^2 \approx \frac{C_1}{|\omega|}||F_2||_0^2, \tag{3.42}$$

$$||\tilde{v}_2(.,\omega,s)||_0^2 \le \frac{1}{(\Re\kappa)^2}||F_2||_0^2 \approx \frac{C_2}{|\omega|^2}||F_2||_0^2. \tag{3.43}$$

From the first equation of (3.16) and using Lemma 5.2, we get

$$||\tilde{v}_1(.,\omega,s)||_0^2 \le \frac{1}{(\Re\kappa)^2}||F_1||_0^2 + \frac{1}{2\Re\kappa}|\tilde{v}_1(0,\omega,s)|^2.$$

Moreover, from the boundary condition (3.17), and the relations (3.38) and (3.42), we have

$$|\tilde{v}_1(0,\omega,s)| \approx \frac{|\omega|}{\eta}|\tilde{v}_2(0,\omega,s)| \le \frac{C|\omega|^{1/2}}{\eta}||F_2||_0. \tag{3.44}$$

We therefore obtain

$$||\tilde{v}_1(.,\omega,s)||_0^2 \le \frac{C_3}{|\omega|^2}||F_1||_0^2 + \frac{C_4}{\eta^2}||F_2||_0^2. \tag{3.45}$$

We collect the estimates (3.39)-(3.45) in the following lemma.

LEMMA 3.4. *For the solution $\tilde{\mathbf{v}}$ of the transformed resolvent system (3.16)-(3.17) with $|a|=1, \Im a \neq 0$ the following estimates near the generalized eigenvalue (3.35) hold*

$$|\tilde{v}_1(0,\omega,s)|^2 \leq C_1 \left(\frac{|\omega|}{\eta^2} ||F_2||_0^2 + \frac{1}{\eta^2} |\tilde{g}|^2 \right),$$
$$|\tilde{v}_2(0,\omega,s)|^2 \leq C_2 \frac{1}{|\omega|} ||F_2||_0^2,$$
$$||\tilde{v}_1(.,\omega,s)||_0^2 \leq C_3 \left(\frac{1}{|\omega|^2} ||F_1||_0^2 + \frac{1}{\eta^2} ||F_2||_0^2 + \frac{1}{\eta^2 |\omega|} |\tilde{g}|^2 \right),$$
$$||\tilde{v}_2(.,\omega,s)||_0^2 \leq C_4 \frac{1}{|\omega|^2} ||F_2||_0^2,$$

where the constants C_1, C_2, C_3, C_4 are independent of ω, η, and the data. The same estimates hold for the solution $\tilde{\mathbf{u}}$ of the resolvent system (3.4)-(3.5).

THEOREM 3.5. *The IBVP (3.1)-(3.3) with $|a|=1, \Im a \neq 0$ is well-posed in the generalized sense.*

Proof. Let $g \equiv 0$. Inverting the Fourier and Laplace transforms and employing Parseval's relation, from Lemma 3.4 we obtain the interior estimate of form (2.16), and hence by Definition 2.6 the problem is well-posed in the generalized sense. ∎

REMARK 3.4. It is to be noted that the first estimate in Lemma 3.4 indicates that the solution at the boundary loses half a derivative over the force data. Indeed, at each reflection at the boundary, the amplitude $|\tilde{\mathbf{v}}|$ jumps by a factor of $|\omega|^{1/2}$. This will cause the problem not to be strongly boundary stable. However, the jump in amplitude concentrates only in a boundary layer and decays exponentially off the boundary, thanks to the eigen-solution that corresponds to a surface eigenmode. Hence, in the interior we obtain the desired estimate of the solution in terms of data without any loss of derivatives.

REMARK 3.5 (In-going and out-going characteristics at the boundary). The jump in the amplitude at each reflection can also be explained in terms of characteristics. Consider the boundary condition (3.17) with homogeneous data ($\tilde{g} \equiv 0$). By (3.38) we have

$$|\tilde{v}_1| \approx c \frac{|\omega|}{\eta} |\tilde{v}_2|, \qquad \text{at } x=0.$$

In other words, when the out-going characteristic, corresponding to $|\tilde{v}_2|$, reaches the boundary, it gets reflected and turns into an in-going characteristic, corresponding to $|\tilde{v}_1|$, amplified by a large number proportional to $|\omega|/\eta$. In fact, $|\hat{S}| := |S_2|/|S_1| \propto |\omega|/\eta$ is typical to the generalized eigenvalues of surface type. Fortunately, as noted in Remark 3.4, the in-going characteristic field decays exponentially as it propagates off the boundary and in the interior.

Construction of Kreiss-type symmetrizers. We now let $\tilde{g} \equiv 0$. Since $\kappa \neq 0$ at the generalized eigenvalue (3.35), we consider the diagonal system (3.16) augmented with a homogeneous boundary condition (3.17). We will show how to obtain the desired interior estimate by constructing a symmetrizer in a neighborhood of the generalized eigenvalue. It is to be noted that away from the generalized eigenvalue the general theory of Kreiss works, and hence we do not need to consider it here. We use the normalized variables (2.26) and consider a neighborhood of the normalized generalized eigenvalue,

$$s' = i\cos\theta\,\omega_0' + \eta', \qquad \sin\theta\,\omega_0' < 0, \qquad 0 < \eta' \ll 1. \tag{3.46}$$

We use the transformation matrix T in (3.15) in the neighborhood of the generalized eigenvalue, i.e. for the variables in (3.46),

$$T = \begin{pmatrix} i\cos\theta\,\omega_0' - \sin\theta\,\omega_0' & i\cos\theta\,\omega_0' + \sin\theta\,\omega_0' \\ i\omega_0' & i\omega_0' \end{pmatrix} + \eta' \begin{pmatrix} 1 - i\cot\theta & 1 + i\cot\theta \\ 0 & 0 \end{pmatrix},$$

and use the system (3.16) with $\Lambda_D = \sqrt{|s|^2 + |\omega|^2}\,\Lambda_D'$, where

$$\Lambda_D' = T^{-1} M' T = \begin{pmatrix} -\kappa' & 0 \\ 0 & \kappa' \end{pmatrix} = \begin{pmatrix} \sin\theta\,\omega_0' + i\cot\theta\,\eta' & 0 \\ 0 & -\sin\theta\,\omega_0' - i\cot\theta\,\eta' \end{pmatrix}. \tag{3.47}$$

Following [11], we consider a symmetrizer of the form

$$\tilde{R} = \begin{pmatrix} -b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \qquad b_1, b_2 > 0. \tag{3.48}$$

Clearly, $\tilde{R}$ is Hermitian. We note that although the matrix (3.48) is a Kreiss-type symmetrizer, the Kreiss approach in [11] cannot be applied here. The Kreiss approach is suitable for obtaining boundary estimates in the case of inhomogeneous boundary conditions and in the absence of generalized eigenvalues. We take a different approach that consists of: 1) splitting the symmetrized system into two scalar equations, one corresponding to the in-going characteristic and the other corresponding to the out-going characteristic; 2) deriving estimates for each component of the solution; and 3) connecting the two components through the homogeneous boundary condition.

Let $\mathbf{F} = (F_1, F_2)^\top$. Since $\tilde{R}$ and Λ_D' are both diagonal, we have

$$\begin{aligned}
\Re(\tilde{\mathbf{v}}, -\tilde{R}\mathbf{F})_0 &= \Re(\tilde{v}_1, b_1 F_1)_0 + \Re(\tilde{v}_2, -b_2 F_2)_0 \\
&= \Re(\tilde{v}_1, \sqrt{|s|^2 + |\omega_0|^2}\, b_1 \kappa \tilde{v}_1)_0 - \frac{b_1}{2}|\tilde{v}_1(0)|^2 + \Re(\tilde{v}_2, \sqrt{|s|^2 + |\omega_0|^2}\, b_2 \kappa \tilde{v}_2)_0 + \frac{b_2}{2}|\tilde{v}_2(0)|^2 \\
&= b_1 \Re\kappa ||\tilde{v}_1||_0^2 - \frac{b_1}{2}|\tilde{v}_1(0)|^2 + b_2 \Re\kappa ||\tilde{v}_2||_0^2 + \frac{b_2}{2}|\tilde{v}_2(0)|^2.
\end{aligned}$$

We now use the Cauchy-Schwarz inequality and Young's inequality with $\varepsilon_1 = b_1 \Re\kappa > 0$ and $\varepsilon_2 = b_2 \Re\kappa > 0$ for $\tilde{v}_1$ and $\tilde{v}_2$, respectively, and write

$$\Re(\tilde{v}_1, b_1 F_1)_0 \le \frac{1}{2} b_1 \Re\kappa ||\tilde{v}_1||_0^2 + \frac{b_1}{2\Re\kappa}||F_1||_0^2, \qquad \Re(\tilde{v}_2, -b_2 F_2)_0 \le \frac{1}{2} b_2 \Re\kappa ||\tilde{v}_2||_0^2 + \frac{b_2}{2\Re\kappa}||F_2||_0^2.$$

Hence, for $\tilde{v}_1$ we obtain

$$||\tilde{v}_1||_0^2 \le \frac{1}{\Re\kappa}|\tilde{v}_1(0)|^2 + \frac{1}{(\Re\kappa)^2}||F_1||_0^2, \tag{3.49}$$

and for $\tilde{v}_2$ we get

$$||\tilde{v}_2||_0^2 \le \frac{1}{(\Re\kappa)^2}||F_2||_0^2, \qquad |\tilde{v}_2(0)|^2 \le \frac{1}{\Re\kappa}||F_2||_0^2. \tag{3.50}$$

We also recall that from the boundary condition we have

$$|\tilde{v}_1|^2 = |\hat{S}|^2 |\tilde{v}_2(0)|^2, \qquad |\hat{S}| = \frac{|S_2|}{|S_1|} \approx \frac{|\omega|}{\eta}. \tag{3.51}$$

Noting that $\Re\kappa \approx |\sin\theta\omega|$, we obtain the desired interior estimate in a neighborhood of the generalized eigenvalue by (3.49), (3.50), and (3.51),

$$||\tilde{\mathbf{v}}||_0^2 = ||\tilde{v}_1||_0^2 + ||\tilde{v}_2||_0^2 \leq \frac{c}{\eta^2}(||F_1||_0^2 + ||F_2||_0^2) = \frac{c}{\eta^2}||\mathbf{F}||_0^2.$$

The problem is therefore well-posed in the generalized sense. We notice that the values of b_1 and b_2 do not matter as long as they are both positive. Generally speaking, the desired interior estimate is obtained thanks to the fact that $\Re\kappa \propto |S_2| \propto |\omega|$, a typical characteristic of the generalized eigenvalues of surface type.

II. Glancing eigenmodes ($a = \pm 1$). We only consider the case $a = 1$. The other case $a = -1$ can be treated similarly. We first find the generalized eigenvalue.

THEOREM 3.6. *The generalized eigenvalue of (3.7)-(3.8) with $a = 1$ is of boundary-type, given by*

$$s_0 = i\xi_0, \qquad \xi_0 = \omega_0, \qquad \omega_0 > 0, \tag{3.52}$$

and the corresponding eigen-solution reads

$$\mathbf{u} = \exp\Big(i\omega_0(t+y)\Big)\mathbf{v}_1, \tag{3.53}$$

which represents a glancing wave being independent of the variable x.

Proof. At the point (3.52) we have

$$\tilde{S}(s_0,\omega_0) = s_0 + \sqrt{s_0^2 + \omega_0^2} - ia\omega_0 = i\omega_0 + \sqrt{-\omega_0^2 + \omega_0^2} - i\omega_0 = 0,$$

Hence this point corresponds to a generalized eigenvalue. The eigen-solution (3.53) is easily obtained by plugging (3.12) into (3.6) at the generalized eigenvalue (3.52), where $\kappa = 0$. Clearly, by Definition 3.1 the generalized eigenvalue (3.52) is of boundary-type. ∎

We consider a neighborhood of the generalized eigenvalue (3.52), and set

$$s = i\omega_0 + \eta, \qquad \omega = \omega_0 > 0, \qquad 0 < \eta \ll |\omega_0|,$$

Then

$$\kappa \approx (2i\eta\omega)^{1/2} = (1+i)|\eta\omega|^{1/2}, \qquad \omega > 0. \tag{3.54}$$

Note that since $\Re\kappa > 0$, we only consider $\omega > 0$. Since at $\eta = 0$, we have $\kappa = 0$, the eigenvalues are multiple ($\kappa_1 = \kappa_2 = 0$), and hence we employ the system (3.19)-(3.20) with

$$S_1 = s + \kappa - i\omega = \eta + \kappa,$$
$$S_2 = -\bar{s} - \bar{\kappa} + i\omega = -\eta + 2i\omega - \bar{\kappa}.$$

We hence have

$$|S_1|^2 \approx 2\eta\omega, \qquad |S_2|^2 \approx 4\omega^2. \tag{3.55}$$

We are now ready to obtain interior estimates of the solution. Again, we will employ two approaches: first by directly obtaining estimates for the solution of (3.19)-(3.20), and next by constructing Kreiss-type symetrizers.

Direct calculation of estimates. We first assume that $F \equiv 0$ and $\tilde{g} \not\equiv 0$. From the second equation of (3.19), since $\Re\kappa > 0$ in (3.54), we get

$$\tilde{v}_2(x,\omega,s) = 0, \tag{3.56}$$

because otherwise, the solution would not be bounded. The boundary condition (3.20) and the first relation in (3.55) give

$$|\tilde{v}_1(0,\omega,s)|^2 \approx \frac{1}{2\eta\omega}|\tilde{g}|^2. \tag{3.57}$$

To obtain interior estimates, we use the first equation of (3.19) together with (3.54) and (3.57) and get

$$||\tilde{v}_1(.,\omega,s)||_0^2 = \int_0^\infty |\tilde{v}_1(0,\omega,s)e^{-\Re\kappa x}|^2\,dx = \frac{1}{2\Re\kappa}|\tilde{v}_1(0,\omega,s)|^2 \approx \frac{1}{4(\eta\omega)^{3/2}}|\tilde{g}|^2. \tag{3.58}$$

We next let $F = (F_1, F_2)^\top \not\equiv 0$ and $\tilde{g} \equiv 0$. By Lemma 5.1 of the Appendix, from the second equation of (3.19) we obtain

$$|\tilde{v}_2(0,\omega,s)|^2 \le \frac{2}{\Re\kappa}||F_2||_0^2 \approx \frac{C_1}{(\eta\omega)^{1/2}}||F_2||_0^2, \tag{3.59}$$

$$||\tilde{v}_2(.,\omega,s)||_0^2 \le \frac{1}{(\Re\kappa)^2}||F_2||_0^2 \approx \frac{C_2}{\eta\omega}||F_2||_0^2. \tag{3.60}$$

From the first equation of (3.19) and using Lemma 5.2, we have

$$||\tilde{v}_1(.,\omega,s)||_0^2 \le \frac{1}{(\Re\kappa)^2}||F_1||_0^2 + \frac{(\Re p)^2}{(\Re\kappa)^2}||\tilde{v}_2||_0^2 + \frac{1}{2\Re\kappa}|\tilde{v}_1(0,\omega,s)|^2.$$

By simple algebraic manipulations, from (3.18) we obtain $\Re p \approx 2(\eta\omega)^{1/2}$ in a neighborhood of the generalized eigenvalue. Moreover, from the boundary condition (3.20), and the relations (3.55) and (3.59), we have

$$|\tilde{v}_1(0,\omega,s)|^2 \approx \frac{2\omega^2}{\eta\omega}|\tilde{v}_2(0,\omega,s)|^2 \le \frac{2C_1\omega^{1/2}}{\eta^{3/2}}||F_2||_0^2. \tag{3.61}$$

We therefore obtain

$$||\tilde{v}_1(.,\omega,s)||_0^2 \le \frac{C_3}{\eta\omega}||F_1||_0^2 + C_4\left(\frac{1}{\eta^2} + \frac{1}{\eta\omega}\right)||F_2||_0^2. \tag{3.62}$$

We collect the estimates (3.56)-(3.62) in the following lemma.

LEMMA 3.5. *For the solution* $\tilde{\mathbf{v}}$ *of the transformed resolvent system (3.19)-(3.20) with* $a = 1$ *the following estimates near the generalized eigenvalue (3.35) hold*

$$|\tilde{v}_1(0,\omega,s)|^2 \le C_1\left(\frac{\omega^{1/2}}{\eta^{3/2}}||F_2||_0^2 + \frac{1}{\eta\omega}|\tilde{g}|^2\right),$$

$$|\tilde{v}_2(0,\omega,s)|^2 \le C_2\frac{1}{(\eta\omega)^{1/2}}||F_2||_0^2,$$

$$||\tilde{v}_1(.,\omega,s)||_0^2 \le C_3\left(\frac{1}{\eta\omega}||F_1||_0^2 + \left(\frac{1}{\eta^2} + \frac{1}{\eta\omega}\right)||F_2||_0^2 + \frac{1}{(\eta\omega)^{3/2}}|\tilde{g}|^2\right),$$

$$||\tilde{v}_2(.,\omega,s)||_0^2 \le C_4\frac{1}{\eta\omega}||F_2||_0^2,$$

where the constants C_1, C_2, C_3, C_4 are independent of ω, η, and the data. The same estimates hold for the solution $\tilde{\mathbf{u}}$ of the resolvent system (3.4)-(3.5).

THEOREM 3.7. *The IBVP (3.1)-(3.3) with $a=1$ is well-posed in the generalized sense.*

Proof. Let $g \equiv 0$. Inverting the Fourier and Laplace transforms and employing Parseval's relation, from Lemma 3.5 we obtain the interior estimate of form (2.16), and hence by Definition 2.6 the problem is well-posed in the generalized sense. ∎

REMARK 3.6. It is to be noted that the first estimate in Lemma 3.5 indicates that the solution at the boundary loses a quarter of derivative over the force data. This is indeed why the problem is not strongly boundary stable. However, in the interior we obtain the desired estimate of the solution in terms of data without any loss of derivatives. Physically, this is due to the fact that the eigen-solution corresponds to a glancing wave that propagates only in the boundary layer and will not get reflected back into the interior of domain.

Construction of Kreiss-type symmetrizers. We now let $\tilde{g} \equiv 0$ and consider (3.19) augmented with a homogeneous boundary condition (3.20). We show how to obtain the desired interior estimate by constructing the symmetrizer in a neighborhood of the generalized eigenvalue (3.52). We use the normalized variables (2.26) and consider a neighborhood of the normalized generalized eigenvalue,

$$s' = i\omega_0' + \eta', \qquad \omega_0' > 0, \qquad 0 < \eta' \ll 1.$$

In general, we can use the unitary transformation matrix T in (3.18) in the neighborhood of the generalized eigenvalue and obtain the desired interior estimate with a proper choice of symmetrizer. However, as we will show here, it turns out that the derivations will be easier if we consider the unitary transformation matrix T in (3.18) precisely at the generalized eigenvalue (with $\eta' = 0$),

$$T_0 = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

By this choice we can easily cancel out the boundary terms. We then employ the system (3.19) with $\Lambda_T = \sqrt{|s|^2 + |\omega|^2}\,\Lambda_T'$, where

$$\Lambda_T' = T_0^* M' T_0 = \begin{pmatrix} 0 & -2i\omega_0' - \eta' \\ -\eta' & 0 \end{pmatrix}. \tag{3.63}$$

With the transformation T_0, we let $\tilde{\mathbf{v}} = T_0^* \tilde{\mathbf{u}}$, and hence from the boundary condition (3.20) with $\tilde{g} = 0$ we obtain

$$\tilde{v}_2(0, \omega, s) = 0, \tag{3.64}$$

because otherwise, since $S_1 = 0$ and $S_2 \neq 0$, there will be no solution $\tilde{\mathbf{v}}$ that would satisfy the boundary condition. Following [11], we consider a symmetrizer of the form,

$$\tilde{R} = \begin{pmatrix} 0 & d_1 \\ d_1 & d_2 \end{pmatrix} - i\eta' \begin{pmatrix} 0 & -f \\ f & 0 \end{pmatrix}. \tag{3.65}$$

We then obtain

$$2\Re(\tilde{R}\Lambda_T') = \begin{pmatrix} -2d_1\eta' & -d_2\eta' \\ -d_2\eta' & -2d_1\eta' - 4f\omega_0'\eta' \end{pmatrix} + \mathcal{O}(\eta'^2).$$

By choosing $d_2=0$, $d_1=-1$, and $f\le 0$, we get

$$\Re(\tilde{R}\Lambda_T')\ge \eta' I. \tag{3.66}$$

For the boundary terms, by (3.64) we get

$$\langle \tilde{\mathbf{v}},\tilde{R}\tilde{\mathbf{v}}\rangle_{x=0}=(d_1-i f\eta')\tilde{v}_1\tilde{v}_2^*+(d_1+i f\eta')\tilde{v}_1^*\tilde{v}_2=0. \tag{3.67}$$

We finally obtain the desired interior estimate,

$$\begin{aligned}\frac{\eta}{2}||\tilde{\mathbf{v}}||_0^2+\frac{c}{2\eta}||F||_0^2\ge \Re(\tilde{\mathbf{v}},-\tilde{R}F)_0 &= -\Re(\tilde{\mathbf{v}},\tilde{R}\frac{d\tilde{\mathbf{v}}}{dx})_0+\Re\Big(\tilde{\mathbf{v}},\sqrt{|s|^2+|\omega|^2}\tilde{R}\Lambda_T'\tilde{\mathbf{v}}\Big)_0\\ &=\frac{1}{2}\Re\langle\tilde{\mathbf{v}},\tilde{R}\tilde{\mathbf{v}}\rangle_{x=0}+\Re\Big(\tilde{\mathbf{v}},\sqrt{|s|^2+|\omega|^2}\tilde{R}\Lambda_T'\tilde{\mathbf{v}}\Big)_0\\ &\ge \eta'\sqrt{|s|^2+|\omega|^2}\,||\tilde{\mathbf{v}}(x,\omega,s)||_0^2.\end{aligned}$$

The first inequality is a direct application of the Cauchy-Schwarz inequality and Young's inequality with $\varepsilon=\eta$, where the constant c is related to the upper bound of $||\tilde{R}||$. The second inequality is a consequence of (3.66)-(3.67). We finally obtain the desired interior estimate

$$||\tilde{\mathbf{v}}||_0^2\le \frac{c}{\eta^2}||F||_0^2.$$

The problem is therefore well-posed in the generalized sense.

Summary of well-posedness for the model problem. We summarize the results for the well-posedness of the IBVP (3.1)-(3.3):

1) if $|a|<1$, then the problem is strongly well-posed in the generalized sense.
2) if $|a|=1$, then the problem is well-posed in the generalized sense.
3) if $|a|>1, a\in\mathbb{R}$, then the problem is ill-posed in the asymptotic sense, i.e. the solution loses one derivative over the data at each reflection from the boundary.
4) if $|a|>1, a\notin\mathbb{R}$, then the problem is ill-posed in the sense that there are solutions which grow exponentially, arbitrarily fast.

It is to be noted that Kreiss theory can address only the first and fourth cases. With the generalization presented here, we can further study the second and third cases, and hence making the Kreiss theory a far more comprehensive theory applicable to all types of well-posedness.

4. Proof of the main theorem (Theorem 3.1)

In this section we present the proof of Theorem 3.1. We consider the half-space problem (2.10)-(2.12) with homogeneous initial and boundary conditions ($\mathbf{h}\equiv\mathbf{g}\equiv\mathbf{0}$). We Fourier and Laplace transform the problem with respect to the tangential variables $\mathbf{x}_-$ and t, respectively, and obtain the resolvent problem

$$\frac{d\tilde{\mathbf{u}}}{dx_1}=M\tilde{\mathbf{u}}-A^{-1}\tilde{\mathbf{f}},\qquad M=A^{-1}(sI-iB(\boldsymbol{\omega}_-)),\qquad \text{for } x_1\ge 0, \tag{4.1}$$

$$\tilde{\mathbf{u}}^I=S\tilde{\mathbf{u}}^{II},\qquad \text{at } x_1=0. \tag{4.2}$$

We assume that there are no eigenvalues s with $\Re s>0$ to the corresponding homogeneous eigenvalue problem. Because otherwise, by Theorem 2.2, the problem would be ill-posed.

The first part of the Main Theorem is indeed the Kreiss eigenvalue theorem (Theorem 2.3). We therefore concentrate on the second part of the theorem and assume that there exists a boundary-type generalized eigenvalue $(i\xi_0', \boldsymbol{\omega}_0')$. Using the normalized variables (2.26), we consider a neighborhood of this generalized eigenvalue

$$s' = i\xi_0' + \eta', \qquad \boldsymbol{\omega}_-' = \boldsymbol{\omega}_0', \qquad 0 < \eta' \ll 1, \tag{4.3}$$

and write the matrix M in (4.1) as

$$M(s,\boldsymbol{\omega}_-) = \sqrt{|s|^2 + |\boldsymbol{\omega}_-|^2}\, M(s',\boldsymbol{\omega}_-'), \qquad M(s',\boldsymbol{\omega}_-') = A^{-1}(s'I - iB(\boldsymbol{\omega}_-')). \tag{4.4}$$

The eigenvalues κ' of $M(s',\boldsymbol{\omega}_-')$ are the solutions of the characteristic equation

$$\mathrm{Det}|\kappa' I - A^{-1}(s'I - iB(\boldsymbol{\omega}_-'))| = 0. \tag{4.5}$$

Depending on the type of the generalized eigenvalue (either surface or glancing eigenmodes), the eigenvalues κ' of $M(s',\boldsymbol{\omega}_-')$ will have different properties. Accordingly, we will use different types of matrix transformations for M. We treat each case separately and derive desired interior estimates of type (2.16) by constructing appropriate Kreiss-type symmetrizers.

4.1. Surface eigenmodes. We will take an approach similar to the approach presented for the model problem in Section 3.2.2 and decompose the resolvent system (4.1) into two systems, one with in-going characteristics and one with out-going characteristics. We first formulate the following important lemma.

LEMMA 4.1. *Let $(s',\boldsymbol{\omega}_-')$ be given by (4.3), where $(i\xi_0',\boldsymbol{\omega}_0')$ is a generalized eigenvalue of surface type. For $M(s',\boldsymbol{\omega}_-')$ in (4.4), there exists a smooth transformation $T = T(s',\boldsymbol{\omega}_-')$ such that $\Lambda(s',\boldsymbol{\omega}_-') = T^{-1}MT$ has block diagonal form*

$$\Lambda(s',\boldsymbol{\omega}_-') = \begin{pmatrix} \Lambda_I & 0 \\ 0 & \Lambda_{II} \end{pmatrix}, \qquad \Lambda_I \in \mathbb{C}^{r\times r}, \qquad \Lambda_{II} \in \mathbb{C}^{(n-r)\times(n-r)}, \tag{4.6}$$

with

$$\Lambda_I + \Lambda_I^* \le -\delta|\boldsymbol{\omega}_0'|I, \qquad \Lambda_{II} + \Lambda_{II}^* \ge \delta|\boldsymbol{\omega}_0'|I, \qquad \delta > 0. \tag{4.7}$$

Proof. By definition 3.1, in a neighborhood (4.3) of a generalized eigenvalue of surface type, there is a constant $c > 0$ such that $|\Re\kappa'| \ge c|\boldsymbol{\omega}_0'| > 0$. Then similar to Lemma 2.1, there are exactly r eigenvalues κ' with $\Re\kappa' > 0$ and $n-r$ eigenvalues with $\Re\kappa' < 0$. Finally, the smoothness of the transformation T is due to the strict hyperbolicity assumption on the system (2.1). ∎

Corresponding to the block form (4.6) we have a partition of vectors $\tilde{\mathbf{v}} = T^{-1}\tilde{\mathbf{u}} = (\tilde{\mathbf{v}}_I^\top, \tilde{\mathbf{v}}_{II}^\top)^\top$. We also partition the force term $\mathbf{F} = -T^{-1}A^{-1}\tilde{\mathbf{f}} = (\mathbf{F}_I^\top, \mathbf{F}_{II}^\top)^\top$. The resolvent problem (4.1)-(4.2) is then transformed into two separate systems

$$\frac{d\tilde{\mathbf{v}}_I}{dx_1} = \sqrt{|s|^2 + |\boldsymbol{\omega}_-|^2}\,\Lambda_I\tilde{\mathbf{v}}_I + \mathbf{F}_I, \qquad \frac{d\tilde{\mathbf{v}}_{II}}{dx_1} = \sqrt{|s|^2 + |\boldsymbol{\omega}_-|^2}\,\Lambda_{II}\tilde{\mathbf{v}}_{II} + \mathbf{F}_{II}, \qquad \text{for } x_1 \ge 0, \tag{4.8}$$

coupled through the boundary condition

$$S_I\tilde{\mathbf{v}}_I = S_{II}\tilde{\mathbf{v}}_{II}, \qquad \text{at } x_1 = 0, \tag{4.9}$$

We note that in a neighborhood of the generalized eigenvalue, S_I is non-singular. We then write

$$|\tilde{\mathbf{v}}_I| \leq ||\hat{S}||\,|\tilde{\mathbf{v}}_{II}|, \qquad \hat{S} = S_I^{-1} S_{II}, \qquad \text{at } x_1 \geq 0. \tag{4.10}$$

According to the structure of Λ in (4.6), we consider a Hermitian symmetrizer of the form

$$\tilde{R} = \begin{pmatrix} -b_1 I & \\ & b_2 I \end{pmatrix}, \qquad b_1, b_2 > 0.$$

With this choice, similar to the proof for the model problem, we can write

$$\begin{aligned}
\Re(\tilde{\mathbf{v}}, -\tilde{R}\mathbf{F})_0 &= \Re(\tilde{\mathbf{v}}_I, b_1 \mathbf{F}_I)_0 + \Re(\tilde{\mathbf{v}}_{II}, -b_2 \mathbf{F}_{II})_0 \\
&= \Re(\tilde{\mathbf{v}}_I, -\sqrt{|s|^2 + |\omega_0|^2}\, b_1 \Lambda_I \tilde{\mathbf{v}}_I)_0 - \frac{b_1}{2} |\tilde{\mathbf{v}}_I(0)|^2 + \\
&\quad + \Re(\tilde{\mathbf{v}}_{II}, \sqrt{|s|^2 + |\omega_0|^2}\, b_2 \Lambda_{II} \tilde{\mathbf{v}}_{II})_0 + \frac{b_2}{2} |\tilde{\mathbf{v}}_{II}(0)|^2 \\
&\geq \frac{b_1}{2} \delta |\boldsymbol{\omega}_0|\; ||\tilde{\mathbf{v}}_I||_0^2 - \frac{b_1}{2} |\tilde{\mathbf{v}}_I(0)|^2 + \frac{b_2}{2} \delta |\boldsymbol{\omega}_0|\; ||\tilde{\mathbf{v}}_{II}||_0^2 + \frac{b_2}{2} |\tilde{\mathbf{v}}_{II}(0)|^2,
\end{aligned}$$

where we have used the first system in (4.8) and the first inequality in (4.7). We now use the Cauchy-Schwarz inequality and Young's inequality with $\varepsilon_1 = b_1 \delta |\boldsymbol{\omega}_0|/2 > 0$ and $\varepsilon_2 = b_2 \delta |\boldsymbol{\omega}_0|/2 > 0$ for $\tilde{\mathbf{v}}_I$ and $\tilde{\mathbf{v}}_{II}$, respectively, and write

$$\begin{aligned}
\Re(\tilde{\mathbf{v}}_I, b_1 \mathbf{F}_I)_0 &\leq \frac{b_1}{4} \delta |\boldsymbol{\omega}_0|\; ||\tilde{\mathbf{v}}_I||_0^2 + \frac{b_1}{\delta |\boldsymbol{\omega}_0|} ||\mathbf{F}_I||_0^2, \\
\Re(\tilde{\mathbf{v}}_{II}, -b_2 \mathbf{F}_{II})_0 &\leq \frac{b_2}{4} \delta |\boldsymbol{\omega}_0|\; ||\tilde{\mathbf{v}}_{II}||_0^2 + \frac{b_2}{\delta |\boldsymbol{\omega}_0|} ||\mathbf{F}_{II}||_0^2.
\end{aligned}$$

Hence, for $\tilde{\mathbf{v}}_I$ we obtain

$$||\tilde{\mathbf{v}}_I||_0^2 \leq \frac{2}{\delta |\boldsymbol{\omega}_0|} |\tilde{\mathbf{v}}_I(0)|^2 + \frac{4}{\delta^2 |\boldsymbol{\omega}_0|^2} ||\mathbf{F}_I||_0^2, \tag{4.11}$$

and for $\tilde{\mathbf{v}}_{II}$ we get

$$||\tilde{\mathbf{v}}_{II}||_0^2 \leq \frac{2}{\delta^2 |\boldsymbol{\omega}_0|^2} ||\mathbf{F}_{II}||_0^2, \qquad |\tilde{\mathbf{v}}_{II}(0)|^2 \leq \frac{2}{\delta |\boldsymbol{\omega}_0|} ||\mathbf{F}_{II}||_0^2. \tag{4.12}$$

Noting that $||\hat{S}|| \propto |\boldsymbol{\omega}_0|/\eta$, from the boundary condition we will have

$$|\tilde{\mathbf{v}}^I(0)| \leq c \frac{|\boldsymbol{\omega}_0|}{\eta} |\tilde{\mathbf{v}}^{II}(0)|.$$

Hence by (4.11)-(4.12), we obtain the desired interior estimate in a neighborhood of the generalized eigenvalue

$$||\tilde{\mathbf{v}}||_0^2 = ||\tilde{\mathbf{v}}_I||_0^2 + ||\tilde{\mathbf{v}}_{II}||_0^2 \leq \frac{c}{\eta^2} (||\mathbf{F}_I||_0^2 + ||\mathbf{F}_{II}||_0^2) = \frac{c}{\eta^2} ||\mathbf{F}||_0^2.$$

Inverting the Fourier and Laplace transforms, the estimate (2.16) follows by Parseval's relation. The problem is therefore well-posed in the generalized sense.

4.2. Glancing eigenmodes. Similar to the approach we took for the model problem in Section 3.2.2, it is possible to use a unitary transformation matrix precisely at the glancing generalized eigenvalue and cancel the boundary terms. Nevertheless, here we closely follow [11] and transform the resolvent system (4.1) into a Jordan normal form in a neighborhood of the generalized eigenvalue. This can be done thanks to the fact that in the case of glancing eigenmodes, the eigenvalues κ of matrix M vanish as $\eta \to 0^+$. We first formulate the following important lemma.

LEMMA 4.2. *Let* $(s', \boldsymbol{\omega}'_-)$ *be given by* (4.3), *where* $(i\xi'_0, \boldsymbol{\omega}'_0)$ *is a generalized eigenvalue of glancing type. For* $M(s', \boldsymbol{\omega}'_-)$ *in* (4.4), *there exists a smooth transformation* $T = T(s', \boldsymbol{\omega}'_-)$ *such that* $\Lambda(s', \boldsymbol{\omega}'_-) = T^{-1} M T$ *has the block diagonal form*

$$\Lambda(s', \boldsymbol{\omega}'_-) = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_l \end{pmatrix}, \tag{4.13}$$

where

$$M_j(s', \boldsymbol{\omega}'_-) = M_j(i\xi'_0, \boldsymbol{\omega}'_-) + \eta' N_j(i\xi'_0, \boldsymbol{\omega}'_-) + \mathcal{O}(\eta'^2) \in \mathbb{C}^{m_j \times m_j}, \qquad 1 \le j \le l, \tag{4.14}$$

with

$$M_j(i\xi'_0, \boldsymbol{\omega}'_-) = \begin{pmatrix} \lambda_j & i & & \\ & \lambda_j & i & \\ & & \ddots & i \\ & & & \lambda_j \end{pmatrix}, \qquad \Re\lambda_j = 0, \qquad \lambda_p \neq \lambda_q \ \textit{ for } \ p \neq q, \ \ 1 \le p, q \le l.$$

Furthermore, the real part of the left lower corner element of N_j *is non-zero, i.e.* $\Re(N_j)_{m_j,1} \neq 0$, *and* $M_j(s', \boldsymbol{\omega}'_-)$ *has precisely* ρ_j *eigenvalues with negative real parts standing in the first rows, where*

$$\rho_j = \begin{cases} m_j/2, & m_j \equiv 0 \pmod 2, \\ (m_j - 1)/2, & m_j \equiv 1 \pmod 2 \ \textit{and} \ \Re(N_j)_{m_j,1} > 0, \\ (m_j + 1)/2, & m_j \equiv 1 \pmod 2 \ \textit{and} \ \Re(N_j)_{m_j,1} < 0. \end{cases}$$

Proof. The proof follows by Lemmas 2.3-2.7 in [11] for the case when A and B_j are real-valued matrices. We also refer to [26] where the proof is extended to the case of complex-valued matrices. The present lemma is a particular case of the lemmas in [11, 26] with $\lim_{\eta \to 0^+} \kappa = 0$. ∎

Corresponding to the block form (4.13) we have a partition of vectors $\tilde{\mathbf{v}} = T^{-1}\tilde{\mathbf{u}} = (\tilde{\mathbf{v}}_1^\top, \dots, \tilde{\mathbf{v}}_l^\top)^\top$. Now assume that we can find a set of smooth Hermitian matrices $\tilde{R}_j$, with $1 \le j \le l$, such that for each block M_j,

$$\Re\{\tilde{R}_j M_j\} \ge \eta' I, \tag{4.15}$$

and for any vector $\tilde{\mathbf{v}}_j$,

$$\tilde{\mathbf{v}}_j^* \tilde{R}_j \tilde{\mathbf{v}}_j \ge 0. \tag{4.16}$$

Then with a symmetrizer of the form

$$\tilde{R} = (T^{-1})^* \begin{pmatrix} \tilde{R}_1 & & & \\ & \tilde{R}_2 & & \\ & & \ddots & \\ & & & \tilde{R}_l \end{pmatrix} (T^{-1}),$$

we will have by (4.13) and (4.15)

$$\Re\{\tilde{R}M\} = (T^{-1})^* \begin{pmatrix} \Re\{\tilde{R}_1 M_1\} & & & \\ & \Re\{\tilde{R}_2 M_2\} & & \\ & & \ddots & \\ & & & \Re\{\tilde{R}_l M_l\} \end{pmatrix} (T^{-1}) \geq \delta\eta' I, \qquad \delta > 0. \tag{4.17}$$

Furthermore, for all vectors $\tilde{\mathbf{u}}$ satisfying the boundary condition, by (4.16) we have at the boundary

$$\tilde{\mathbf{u}}^* \tilde{R} \tilde{\mathbf{u}}\big|_{x_1=0} = \tilde{\mathbf{v}}^* \begin{pmatrix} \tilde{R}_1 & & & \\ & \tilde{R}_2 & & \\ & & \ddots & \\ & & & \tilde{R}_l \end{pmatrix} \tilde{\mathbf{v}} = \sum_{j=1}^{l} \tilde{\mathbf{v}}_j^* \tilde{R}_j \tilde{\mathbf{v}}_j \big|_{x_1=0} \geq 0. \tag{4.18}$$

Now if we multiply (4.1)-(4.2) by $\tilde{R}$, thanks to (4.17)-(4.18), we obtain

$$\begin{aligned} \Re(\tilde{\mathbf{u}}, \tilde{R} A^{-1} \tilde{\mathbf{f}})_0 &= -\Re\big(\tilde{\mathbf{u}}, \tilde{R} \frac{d\tilde{\mathbf{u}}}{dx_1}\big)_0 + \Re\big(\tilde{\mathbf{u}}, \sqrt{|s|^2 + |\boldsymbol{\omega}_-|^2}\, \tilde{R} M \tilde{\mathbf{u}}\big)_0 \\ &= -\frac{1}{2} \Re\big[\tilde{\mathbf{u}}^* \tilde{R} \tilde{\mathbf{u}}\big]_0^\infty + \Re\big(\tilde{\mathbf{u}}, \sqrt{|s|^2 + |\boldsymbol{\omega}_-|^2}\, \tilde{R} M \tilde{\mathbf{u}}\big)_0 \geq \delta\eta ||\tilde{\mathbf{u}}||_0^2. \end{aligned}$$

We then use the Cauchy-Schwarz inequality and Young's inequality with $\varepsilon = \delta\eta$ and write

$$\Re(\tilde{\mathbf{u}}, \tilde{R} A^{-1} \tilde{\mathbf{f}})_0 \leq \frac{\delta\eta}{2} ||\tilde{\mathbf{u}}||_0^2 + \frac{c}{2\delta\eta} ||\tilde{\mathbf{f}}||_0^2,$$

where the constant c is related to the upper bound of $||\tilde{R} A^{-1}||$. Combining the above two inequalities, we obtain

$$||\tilde{\mathbf{u}}||_0^2 \leq \frac{c}{\eta^2} ||\tilde{\mathbf{f}}||_0^2.$$

Inverting the Fourier and Laplace transforms, the estimate (2.16) follows by Parseval's relation.

It now remains to construct $\tilde{R}_j$ such that (4.15) and (4.16) hold. We set

$$\tilde{R}_j = D - i\eta' F, \quad D = \begin{pmatrix} & & & d_1 \\ & & d_1 & d_2 \\ & & & \vdots \\ d_1 & d_2 & \dots & d_{m_j} \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & -f_1 & & & \\ f_1 & 0 & -f_2 & & \\ & f_2 & \ddots & \ddots & \\ & & \ddots & 0 & -f_{m_j-1} \\ & & & f_{m_j-1} & 0 \end{pmatrix},$$

where the matrices D and F are real symmetric ($D = D^*$) and real anti-symmetric ($F = -F^*$), respectively. It is easy to see that

$$2\Re\{\tilde{R}_j M_j\} = \tilde{R}_j M_j + M_j^* \tilde{R}_j = \eta' (D N_j + N_j^* D + F C + C^* F^*) + \mathcal{O}(\eta'^2),$$

where C is the nilpotent matrix with 1's along the superdiagonal and zeros everywhere else. The left upper corner element of $D N_j + N_j^* D$ is $2 d_1 \Re(N_j)_{m_j,1}$. By Lemma 4.2, we know that $\Re(N_j)_{m_j,1} \neq 0$. Therefore, if we choose d_1 so that $2 d_1 \Re(N_j)_{m_j,1} \geq 3$, then there is a constant $\beta = \beta(||N_j||, ||D||) > 0$ such that

$$D N_j + N_j^* D \geq \begin{pmatrix} 3 & & & \\ & -\beta & & \\ & & \ddots & \\ & & & -\beta \end{pmatrix}.$$

We further choose the elements of F to be $f_k = d^{k^2}$, with $k = 1, \dots, m_j - 1$, for some $d > 0$ to be determined. Then by Lemma 4.4 in [11], we have

$$F C + C^* F^* \geq \begin{pmatrix} -1 & & & \\ & d/2 & & \\ & & \ddots & \\ & & & d/2 \end{pmatrix}, \qquad d > 0.$$

We then have

$$D N_j + N_j^* D + F C + C^* F^* \geq \begin{pmatrix} 2 & \\ & (d/2 - \beta) I \end{pmatrix}.$$

Then (4.15) will follow if we choose $d \geq 4 + 2\beta$. The inequality (4.16) follows easily noting that for any vector $\tilde{\mathbf{v}}_j$, we have

$$\mathbf{v}_i^* \tilde{R}_j \mathbf{v}_j = \mathbf{v}_j^* D \mathbf{v}_j + \mathcal{O}(\eta') |\mathbf{v}_j|^2.$$

We therefore need to choose $d_2, \dots, d_{m_j}$ as a sufficiently fast increasing positive sequence. This completes the proof.

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Appendix. In this Appendix we collect a number of auxiliary lemmas.

LEMMA 5.1. *Consider the ordinary differential equation $u_x = \lambda u + F$ with $\Re\lambda > 0$, $0 \leq x < \infty$. Then if the solution u vanishes at infinity, it satisfies the estimate*

$$|u(0)|^2 \leq \frac{2}{\Re\lambda} ||F||_0^2, \qquad ||u||_0^2 \leq \frac{1}{(\Re\lambda)^2} ||F||_0^2.$$

Proof. Integration by parts gives us

$$(u, u_x) = -|u(0)|^2 - (u_x, u),$$

i.e.,

$$2\Re(u,u_x) = -|u(0)|^2.$$

Therefore

$$\frac{1}{2}|u(0)|^2 + \Re\lambda ||u||^2 \leq ||u|| \, ||F||,$$

and the lemma follows, considering each term in the left-hand side of the above inequality separately. □

LEMMA 5.2. *Consider* $u_x = -\lambda u + F$ *with* $\Re\lambda > 0$, $0 \leq x < \infty$. *Then if the solution* u *vanishes at infinity, it satisfies the estimate*

$$||u||_0^2 \leq \frac{1}{(\Re\lambda)^2}||F||_0^2 + \frac{1}{2\Re\lambda}|u(0)|^2.$$

Proof. For $u(0)=0$, we use integration by parts, and for $F \equiv 0$, we can explicitly calculate the solution,

$$u(x) = u(0)\, e^{-\lambda x}.$$

This gives

$$||u||_0^2 = |u(0)|^2 \int_0^\infty e^{-\Re\lambda x} dx.$$

The lemma follows after simple manipulations. □

LEMMA 5.3. *There is a constant* $\delta > 0$ *such that for all* $\omega \in \mathbb{R}$,

$$\Re\kappa = \Re\sqrt{s^2+\omega^2} \geq \delta\eta, \qquad \eta = \Re s > 0.$$

Proof. For the proof see Lemma 2 in [18]. □

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