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ELECTROMAGNETO-ELASTICITY SYSTEM WITH MOVING BOUNDARY*

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Abstract. Nonlinear electromagneto-elasticity with moving boundary is presented in this work. We introduce a diffeomorphism operator that transforms the moving boundary system into an equivalent one with fixed boundary and we apply Galerkin's method and results of compactness to obtain the existence and uniqueness of solution.

Keywords. Electromagneto-elasticity; Galerkin method; existence and uniqueness; moving boundary.

AMS subject classifications. 35L20; 34K10; 35R37.

1. Introduction

The determination of deformation, stress, motion, and electromagnetic field in a medium on the applications of external loads are studied in the electromagnetic continuum theory. These external loads may be of mechanical origin (e.g., constraints placed on the surface of the body, and initial and boundary conditions arising from thermal and other changes, forces, couples) and of electromagnetic origin (e.g., magnetic, electric, and current fields).

If an electro-conductive medium is subjected to an electromagnetic field, the propagation of elastic and electromagnetic waves will influence each other through the coupling effects of mechanical stresses in the medium and electromagnetic oscillations.

The foundation of magneto-elasticity was first presented by Knopoff [6] in 1955. This pioneering work was followed by Dunkin et al. [3], in which the authors investigated the coupling of electromagnetic and elastic waves from the standpoint of linear elasticity and a linearized electromagnetic theory. An investigation of the same problem for a uniform electrostatic field showed that the usual plane waves propagate without any change in their phase velocities but the mechanical waves are accompanied by small fluctuating electromagnetic fields. The problem of vibration of a free infinite elastic plate in a large magnetostatic field were examined under the assumption that the resulting electromagnetic fields are quasi-stationary.

Mathematical problems of interaction between the propagations of elastic and electromagnetic waves were treated by Avdeev et al. [1]. We also refer the readers to the following related works: Imomnazarov [5], Merazhov [11], Yakhno [8,9], Priimenko et al. [15,16] Romanov et al. [17,18] and Yakhno et al. [19,20].

In recent years, the interaction of electromagnetic fields in deformable media has been a subject of many theoretical and experimental investigations in continuum mechanics and geophysics, such as, among others, [2,14]. A special attention is being given to the interaction between the magnetic and the deformation fields in a thermoelastic

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solid due to its relevance in the fields of geophysics, plasma and nuclear physics, for which we can mention the works of [10,12].

For the theory of electromagneto-elasticity, we can mention, among others, the works [2,4,13].

2. Mathematical model

Consider a mathematical model of an isotropic electromagneto-elastic medium describing the coupling mechanism of electromagnetic and elastic effects due to small deformation of the medium and the variation of electromagnetic field. The system of governing equations are given by the following electromagnetic dynamics:

$$rot\vec{H} = \sigma\vec{E} + \sigma\mu_e\frac{\partial\vec{U}}{\partial t}\times\vec{H} + \vec{J}, \tag{2.1}$$

$$rot\vec{E} = -\mu_e\frac{\partial\vec{H}}{\partial t}, \quad div(\mu_e\vec{H}) = 0, \tag{2.2}$$

and the equation of motion of the elastic medium with the Lorentz force:

$$\rho\,\frac{\partial^2\vec{U}}{\partial t^2} = div\;T(\vec{U}) + \mu_e\;rot\vec{H}\times\vec{H} + \vec{F}, \tag{2.3}$$

where the stress tensor $T(\vec{U})$ is given by

$$T_{ij} = \lambda(\nabla\cdot\vec{U})\delta_{ij} + \mu(U_{i,j}+U_{j,i}),\;\; 1\le i,j\le 3. \tag{2.4}$$

The vectors $\vec{E}=(E_1,E_2,E_3)$ and $\vec{H}=(H_1,H_2,H_3)$ are the components of electric and magnetic fields respectively; $\vec{U}=(U_1,U_2,U_3)$ is the displacement vector of the medium; σ, μ_e, ρ, λ and μ are electric conductivity, magnetic permeability, density and the Lamé coefficients of the medium, respectively; $\vec{J}$ is the source of electromagnetic field; $\vec{F}$ is the external force field and δ_{ij} is the Kronecker symbol.

We shall consider the case where all functions in Equations (2.1)-(2.4) depend on variables (z,t) and $\vec{U}$, $\vec{E}$, $\vec{H}$, $\vec{J}$ and $\vec{F}$ are vectorial functions defined by $\vec{E} = e(z,t)(0,1,0)$, $\vec{H}=h(z,t)(1,0,0)$, $\vec{U}=u(z,t)(0,0,1)$, $\vec{J}=j(z,t)(0,1,0)$, $\vec{F}=f(z,t)(0,0,1)$ where e,h,u,j and f are scalar functions and z represents the variable x_3.

For the particular case of $\rho=$ constant and $\mu_e=$ constant, the system can be written in the form:

$$\begin{aligned} &h_t = (rh_z)_z - (hu_t)z - (rj)_z,\\ &u_{tt} = (\nu^2u_z)_z - phh_z + f,\\ &e = rh_z - hu_t, \end{aligned} \tag{2.5}$$

where, $v_p = ((\lambda+2\mu)/\rho)^{1/2}$ is the velocity of longitudinal elastic wave; $r^{-1}=\mu_eLV_0\sigma$ is the magnetic Reynolds number, $p=\mu_eH_0^2\rho^{-1}V_0^{-2}$, $\nu=v_p/V_0$ are the dimensionless speeds of propagation of elastic field and L,V_0 and H_0 are the characteristic values of length, velocity and magnetic field, respectively.

In this paper we shall formulate the problem based on the coupled system (2.5) in a domain with moving boundary.

2.1. Formulation of the problem. Let $\hat{Q}=\{(x,t)\in\mathbb{R}^2;\,\alpha(t)<x<\beta(t),\;0<t<T\}$ be a noncylindrical domain, with its lateral boundary denoted by $\hat{\Sigma}=$

$\cup_{\{0<t<T\}}\{\alpha(t),\beta(t)\}\times\{t\}$. We shall consider the following coupled system of nonlinear electromagneto-elasticity with moving ends:

$$\begin{cases} h_t-(rh_x)_x+(hu_t)_x+(rj)_x=0, & \text{in } \hat{Q} \\ u_{tt}-(\nu^2u_x)_x+phh_x+\delta u_t-f=0, & \text{in } \hat{Q} \\ u=h=0, & \forall (x,t)\in\hat{\Sigma} \\ h(x,0)=h_0(x), & \alpha(0)<x<\beta(0) \\ u(x,0)=u_0(x),\, u_t(x,0)=u_1(x), & \alpha(0)<x<\beta(0) \end{cases} \tag{2.6}$$

where we have added a damping term δu_t with damping coefficient $\delta>0$.

The existence and uniqueness of weak solutions for the fixed domain of the model (2.6) was proved in [14]. In this work, we will investigate the existence and uniqueness of weak solutions of electromagnetic and elastic fields for the model (2.6) with moving boundary.

2.2. Notations and hypotheses. Let the space $V=H_0^1(\Omega)\cap H^2(\Omega)$ be equipped with the scalar product and norm given by

$$(u,v)_V=\int_{\alpha_0}^{\beta_0}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}\,dx;\quad |u|_V^2=\int_{\alpha_0}^{\beta_0}\Big|\frac{\partial^2 u}{\partial x^2}\Big|_{\mathbb{R}}^2 dx,\quad \forall u\in V.$$

The scalar product and norm in $L^2(\Omega)$ are represented by

$$(u,v)=\int_{\alpha_0}^{\beta_0}u(x)v(x)\,dx;\quad |u|^2=\int_{\alpha_0}^{\beta_0}|u(x)|_{\mathbb{R}}^2\,dx,\quad \forall u,v\in L^2(\Omega).$$

and by $L^\infty(Q)$ we represent the Banach space of bounded measurable functions equipped with the norm

$$\|u\|_\infty=\sup_{(x,t)\in Q}\text{ess}\,|u(x,t)|_{\mathbb{R}}.$$

Consider the following hypotheses for the boundary $\{\alpha(t),\beta(t)\}$:

H1: $\alpha,\beta\in C^2([0,T];\mathbb{R})$, where $0<\gamma_0=\min_{0\le t\le T}\gamma(t)\le\gamma_1$;

H2: $\beta'(t)>0$ and $\alpha'(t)<0$, $\forall t\in[0,T]$.

In order to solve problem (2.6), we shall now consider a change of variables to transform the domain Q_t to a cylindrical domain Q using the following transformation:

$$\begin{aligned} \mathcal{T}:Q_t&\to Q=(0,1)\times(0,T)\\ (x,t)&\mapsto(y,t)=\Big(\frac{x-\alpha(t)}{\gamma(t)},t\Big), \end{aligned} \tag{2.7}$$

which is C^2. The inverse $\mathcal{T}^{-1}$ is also C^2. Using the application (2.7) and notations

$$\phi(y,t)=h(\alpha(t)+\gamma(t)y,t),\qquad \gamma(t)=\beta(t)-\alpha(t)$$
$$u(x,t)=u\big(\mathcal{T}^{-1}(y,t)\big)=v(y,t).$$

With the transformation $\mathcal{T}$, the coupled system (2.6) is transformed into the following equivalent coupled system with the cylindrical domain $Q=(0,1)\times(0,T)$

$$\begin{cases} \phi' - \dfrac{1}{\gamma^2(t)}\Big(((r_1(y)\phi_y)_y - (\alpha'(t)+\gamma'(t)y))\phi v_y\Big)_y + \dfrac{1}{\gamma(t)}(\phi v')_y \\ \qquad + \dfrac{1}{\gamma(t)}\Big((r_1(y)j_1(y,t))_y - (\alpha'(t)+\gamma'(t)y)\Big)\phi_y = 0, & \text{in } Q \\ v'' + \dfrac{1}{\gamma^2(t)}\Big((\alpha'(t)+\gamma'(t)y)^2 - (\nu_1^2(y)v_y)_y\Big) + \dfrac{p}{\gamma(t)}(\phi\phi_y) - \dfrac{2}{\gamma^2(t)}(\alpha'(t)+\gamma'(t)y)^2 v'_y \\ -\dfrac{1}{\gamma(t)}\Big((\alpha''(t)+\gamma''(t)y) + \delta\big(\alpha'(t)+\gamma'(t)y\big)\Big)v_y + \delta v' - f_1(y,t) = 0, & \text{in } Q, \\ v=\phi=0, \qquad \forall (y,t)\in\Sigma = \displaystyle\bigcup_{0<t<1}\{0,1\}\times\{t\}, \\ \phi(y,0)=\phi_0(y), \quad v(y,0)=v_0(y), \quad v'(y,0)=v_1(y), \qquad 0<y<1, \end{cases} \tag{2.8}$$

where the prime $\{'\}$ denotes the partial derivative with respect to t and

$$\begin{aligned} &\nu_1(y)=\nu(\alpha(t)+\gamma(t)y), && j_1(y,t)=j(\alpha(t)+\gamma(t)y,t), && f_1(y,t)=f(\alpha(t)+\gamma(t)y,t) \\ &\nu_1^2(y)=\nu^2(\alpha(t)+\gamma(t)y), && \phi_0(y)=h_0(\alpha(0)+\gamma(0)y), && v_0(y)=u_0(\alpha(0)+\gamma(0)y) \\ &v_1(y)=u_1(\alpha(0)+\gamma(0)y)+\gamma(0)u_{0x}(\alpha(0)+\gamma(0)y). \end{aligned}$$

With this change of variables, $u(x,t)$ and $h(x,t)$ are solutions of the coupled problem (2.6) if and only if $v(y,t)$ and $\phi(y,t)$ are solutions of the coupled problem (2.8).

To establish the existence and uniqueness of the solution to the coupled problem (2.8), we shall require the following additional hypotheses:

H3: $M=\max\limits_{0\le t\le T}\{|\alpha'(t)|+|\beta'(t)|,\ |\alpha''(t)|+|\beta''(t)|\}$, $M_1=\max\limits_{0\le t\le T}\{|\alpha'(t)|+|\beta'(t)|\}$

$$4c_7^2M^2+2c_5^2M\gamma_1+2\delta c_7^2M\gamma_1<\nu_2^2/4.$$

The $\alpha(t)$ and $\beta(t)$ functions, which define boundary conditions, will be conveniently chosen to satisfy the following assumptions,

$$\frac{4\gamma_0^{-1}M^3}{\delta}+\Big(\frac{4}{\delta}+1+\frac{8(1+\delta)^2}{\delta^2}+2(c_0+1)^2\Big)M^2+\Big(\frac{4\nu_3^2\gamma_0^{-1}}{\delta}+(1+\delta)c_0\gamma_1\Big)M<\nu_2^2/2,$$

$$M\gamma_1c_0^2<r_2,\ M/\gamma_0<\delta/8,\ M_1^2<\nu_2^2/2 \text{ and } 4c_5^2M^2+2c_5^2M\gamma_1+2\delta c_5^2M\gamma_1<\nu_2^2/4,$$

where ν_2 and ν_3 are defined by the hypothesis **(H4)** below and c_5 is a constant due to immersion $H^1(0,1)\subset C^0([0,1])$.

H4: $r\in W^{1,\infty}(a,b)$, $\nu\in W^{2,\infty}(a,b)$, $j\in W^{1,\infty}(\hat{Q})$, $\nu_x(\alpha(t))=\nu_x(\beta(t))=0$,
where $(\alpha(t),\beta(t))\subset(a,b)$, and there are positive constants r_2, ν_2, r_3 and ν_3 such that $r_2\le r(x)\le r_3$ and $\nu_2\le\nu(x)\le\nu_3$, $\forall x\in(a,b)$ with p and δ being positive constants.

H5: $\{h_0,u_1\}\in H_0^1(\Omega_0)$, $u_0\in H_0^1(\Omega_0)\cap H^2(\Omega_0)$, $\Omega_0=(\alpha(0),\beta(0))$, $f\in L^2(0,T;H_0^1(\Omega_t))$, where $\Omega_t=(\alpha(t),\beta(t))$

The functions $\{u_0,\ u_1,\ f\}$ are chosen small enough, so that, after changing variables, the functions $\{v_0,\ v_1\}$ satisfy the conditions:

$$\textbf{H6:}\quad \begin{aligned} &\Big(4c_1\gamma_1+\frac{1}{\nu_2}(8pMc_1\gamma_1)\Big(K_0+p\int_0^T|\sqrt{r_1(y)}j_1(y,t)|^2dt \\ &\quad+\frac{1}{8\delta\nu_2^2}(\delta^2\gamma_1^2c_0^2+16\nu_2^2)\int_0^T|f_1(t)|^2dt\Big)^{1/2}<r_2, \end{aligned}$$

where $K_0=\bar{c}\ (|\phi_0|^2+|v_1|^2+|v_0|^2+\|v_0\|^2)$ and $\bar{c}$ a positive constant.

REMARK 2.1.

(i) From hypothesis **(H4)** and using the change of variables $\mathcal{T}$, we obtain

$$r_1 \in W^{1,\infty}(0,1), \nu_1 \in W^{2,\infty}(0,1), \; j_1 \in W^{1,\infty}(Q), \quad \nu_{1y}(0) = \nu_{1y}(1) = 0.$$

Besides that, there are positive constants $\{r_2,\, r_3,\, \nu_2,\, \nu_3,\, \nu_4,\, \nu_5\}$ such that

$$r_2 \le r_1(y) \le r_3,\; \nu_2 \le \nu_1(y) \le \nu_3,\; \nu_{1y}(y) \le \nu_4,\; \nu_{1yy}(y) \le \nu_5,\; \forall y \in (0,1).$$

(ii) Similarly, from hypothesis **(H5)**, we obtain

$$\{\phi_0, v_1\} \in H_0^1(0,1), \, v_0 \in H_0^1(0,1) \cap H^2(0,1), \, f_1 \in L^2(0,T;H_0^1(0,1)).$$

3. Existence of solution

To show the existence of a solution to the coupled system (2.6), we will use the Galerkin method. First, let us define the concept of a solution for (2.6):

DEFINITION 3.1. *Suppose that h_0, u_1, u_0, satisfying* **(H5)**. *A global solution for the coupled system (2.6), is a pair of functions $\{h,\, u\}$ belonging to the class*

$$h \in L^\infty(0,T;H_0^1(\Omega_t)) \cap L^2(0,T;H_0^1(\Omega_t) \cap H^2(\Omega_t)),\; h_t \in L^2(0,T;L^2(\Omega_t)),$$
$$u \in L^\infty(0,T;H_0^1(\Omega_t) \cap H^2(\Omega_t)),\; u_t \in L^\infty(0,T;H_0^1(\Omega_t)),\; u_{tt} \in L^2(0,T;L^2(\Omega_t))$$

satisfying the following integral identities,

$$\begin{aligned}
&\int_{\hat{Q}} h_t(x,t)\varphi(x,t)dxdt + \int_{\hat{Q}} r(x)h_x(x,t)\varphi_x(x,t)dxdt \\
&\qquad - \int_{\hat{Q}} h(x,t)u_t(x,t)\varphi_x(x,t)dxdt - \int_{\hat{Q}} r(x)j(x,t)\varphi_x(x,t)dxdt = 0 \\
&\int_{\hat{Q}} u_{tt}(x,t)\varphi(x,t)dxdt + \int_{\hat{Q}} \nu^2(x)u_x(x,t)\varphi_x(x,t)dxdt \\
&\qquad + \int_{\hat{Q}} ph(x,t)h_x(x,t)\varphi(x,t)dxdt - \int_{\hat{Q}} f(x,t)\varphi(x,t)dxdt = 0, \quad \forall \varphi \in L^2(0,T;H_0^1(\Omega_t)).
\end{aligned}$$

Moreover, $\{h,u\}$ satisfy the initial conditions $u(0) = u_0$, $u_t(0) = u_1$, $h(0) = h_0$.

Similarly, the concept of global solution for the coupled system (2.8) is given by:

DEFINITION 3.2. *Suppose that $\{\phi_0, v_1\} \in H_0^1(0,1)$, $v_0 \in H_0^1(0,1) \cap H^2(0,1)$, $f_1 \in L^2(0,T;H_0^1(0,1))$. A global solution for the coupled system (2.8), is a pair of functions $(\phi;v): Q = \Omega \times (0,T) \to \mathbb{R}$, belonging to the class*

$$\phi \in L^\infty(0,T;H_0^1(0,1)) \cap L^2(0,T;H_0^1(0,1) \cap H^2(0,1)),\; \phi_t \in L^2(0,T;L^2(0,1)),$$
$$v \in L^\infty(0,T;H_0^1(0,1) \cap H^2(0,1)),\; v_t \in L^\infty(0,T;H_0^1(0,1)),\; v_{tt} \in L^2(0,T;L^2(0,1))$$

satisfying the following integral identities

$$\begin{aligned}
&\int_Q \phi_t(y,t)\varphi(y,t)dQ + \int_Q \frac{1}{\gamma^2(t)}(r_1(y)\phi_y(y,t))\varphi_y(y,t)dQ \\
&\quad + \int_Q \frac{1}{\gamma^2(t)}(\alpha'(t) + \gamma'(t)y)\phi(y,t)v_y(y,t)\varphi_y(y,t)dQ - \int_Q \frac{1}{\gamma(t)}(\phi(y,t)v_t(y,t))\varphi_y(y,t)dQ \\
&\quad - \int_Q \frac{1}{\gamma(t)}(r_1(y)j_1(y,t))\varphi_y(y,t)dQ - \int_Q \frac{1}{\gamma(t)}(\alpha'(t) + \gamma'(t)y)\phi_y(y,t)\varphi_y(y,t)dQ = 0,
\end{aligned}$$

$$\int_Q v_{tt}(y,t)\varphi(y,t)dQ - \int_Q \frac{1}{\gamma^2(t)}\Big((\alpha'(t)+\gamma'(t)y)^2-\nu_1^2(y)\Big)v_y(y,t)\varphi_y(y,t)dQ$$
$$+\int_Q \frac{p}{\gamma(t)}\Big(\phi(y,t)\phi_y(y,t)\varphi(y,t)\Big)dQ - 2\int_Q \frac{1}{\gamma(t)}(\alpha'(t)+\gamma'(t)y)v_{yt}^2(y,t)\varphi(y,t)dQ$$
$$-\int_Q \frac{1}{\gamma(t)}\Big(\alpha''(t)+\gamma''(t)y+\delta(\alpha'(t)+\gamma'(t)y)\Big)v_y(y,t)\varphi(y,t)dQ + \delta\int_Q v_t(y,t)\varphi(y,t)dQ$$
$$-\int_Q f_1(y,t)\varphi(y,t)dQ = 0, \quad \forall\varphi\in L^2(0,T;H_0^1(0,1)).$$

Moreover, $\{\phi,\ v\}$ *satisfy the initial conditions* $v(0)=v_0$, $\phi(0)=\phi_0$, $v_t(0)=v_1$.

THEOREM 3.1. *From the hypotheses* **(H1) - (H6)** *and given* $\{r_1,j_1,\nu_1,f_1,\phi_0,v_0,v_1\}$, *there exists a pair of functions* $(\phi;\ v):\Omega\times(0,T)\to\mathbb{R}$ *for the electromagnetoelasticity problem* (2.8), *in the class*

$$\phi\in L^\infty(0,T;H_0^1(0,1))\cap L^2(0,T;H_0^1(0,1)\cap H^2(0,1)),\ \phi_t\in L^2(0,T;L^2(0,1)),$$
$$v\in L^\infty(0,T;H_0^1(0,1)\cap H^2(0,1)),\ v_t\in L^\infty(0,T;H_0^1(0,1)),\ v_{tt}\in L^2(0,T;L^2(0,1)).$$

Proof. Since the coupled system (2.8) has time-dependent coefficient, a natural method to show the existence of solution for system (2.6) is the classical Galerkin's method. For this, we choose a special spectral basis $(w_i)_{i\in\mathbb{N}}$ of the Laplace operator in $L^2(\Omega)$ which is an orthogonal and complete system in $H_0^1(\Omega)$. For each $m\in\mathbb{N}$ we set $V_m=\text{span}\,\{w_i,\dots,v_m\}\subset H_0^1(0,1)$ generated by the first m basis vectors.

Approximation Problem: For each $m\in\mathbb{N}$, we find a real number $t_m>0$ and real functions g_{im} and d_{im} defined on $[0,t_m]$ such that setting

$$v_m(t)=\sum_{i=1}^m g_{im}(t)w_i\in V_m \ \text{ and } \ \phi_m(t)=\sum_{i=1}^m d_{im}(t)w_i\in V_m, \tag{3.1}$$

where (ϕ_m,v_m) for $y\in\Omega$ and $t\in[0,\ t_m]$ is the solution to the approximate problem:

$$(\phi_m'(t),w)+\frac{r_1}{\gamma^2}(\phi_{my}(t),w_y)-\frac{1}{\gamma}\Big((\alpha'+\gamma'y)\phi_{my}(t),w\Big)-\frac{1}{\gamma}(r_1j_1(t),w_y)$$
$$+\frac{1}{\gamma^2}((\alpha'+\gamma'y)v_{my}(t)\phi_m(t),w_y)-\frac{1}{\gamma}(\phi_m(t)v_m'(t),w_y)=0 \tag{3.2}$$

$$(v_m''(t),w)+\frac{1}{\gamma^2(t)}\Big(\Big(\nu_1^2(y)-(\alpha'+\gamma'y)^2\Big)v_{my}(t),w_y\Big)+\frac{p}{\gamma}\Big(\phi_m(t)\phi_{my}(t),w\Big)$$
$$+\frac{-2}{\gamma}\Big((\alpha'+\gamma'y)v_{my}',w\Big)-\frac{1}{\gamma}\Big(\Big((\alpha''+\gamma''y)+\delta(\alpha'+\gamma'y)\Big)v_{my},w\Big)$$
$$+\big(\delta v_m'(t),w\big)=(f_1(t),w). \tag{3.3}$$

Associated with the initial data (v_0,v_1,ϕ_0) we will take $(v_{0m},\ v_{1m},\ \theta_{0m})_{m\in N}$ such that

$$v_m(0)=v_{0m}\longrightarrow v_0 \ \text{ in } H_0^1(0,1)\cap H^2(0,1), \tag{3.4}$$
$$\phi_m(0)=\phi_{0m}\longrightarrow \phi_0 \ \text{ in } H_0^1(0,1), \tag{3.5}$$
$$v_m'(0)=v_{1m}\longrightarrow v_1 \ \text{ in } H_0^1(0,1). \tag{3.6}$$

The existence of local approximate solution $\{\phi_m, v_m\}$ of (3.2) - (3.6) is a consequence of the standard ordinary differential equation theory. To extend the local solution to the interval $[0,T]$ independent of m, in the sense of Definition (2.3), the following a priori estimates are needed.

Estimate I: Taking $w = p\phi_m(t)$ in (3.2), $w = v_m'(t)$ in (3.3), and adding the equations, we have

$$
\begin{aligned}
&\frac{p}{2}\frac{d}{dt}|\phi_m(t)|^2 + p\int_0^1 \frac{r_1(y)}{\gamma^2(t)}\phi_{my}^2(y,t)dy + \frac{1}{2}\frac{d}{dt}|v_m'(t)|^2 \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^1 \frac{1}{\gamma^2(t)}\Big(\nu_1^2(y) - (\alpha'(t)+\gamma'(t)y)^2\Big)v_{my}^2(y,t)dy + \frac{p}{2}\frac{\gamma'(t)}{\gamma(t)}|\phi_m(t)|^2 \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + \frac{\gamma'+\delta}{\gamma}|v_m'|^2 - \Big(\nu_1^2(y) - 2(\alpha'+\gamma'y)^2\gamma\gamma'\Big)v_{my}^2(y,t) \qquad (3.7)
\end{aligned}
$$

where $\{I_1,\ I_2,\ I_3,\ I_4,\ I_5\}$ are defined as follows:

Analysis of the terms on the right side of (3.7):

$$
I_1 = p\Big(\frac{1}{\gamma(t)}r_1 j_1(t), \phi_{my}(t)\Big) \le p\Big|\sqrt{r_1}j_1(t)\Big|^2_{L^\infty(0,1)} + p\int_0^1 \frac{|r_1(y)|}{4\gamma^2(t)}|\phi_{my}(y,t)|^2 dy
$$

$$
I_2 = \frac{-p}{\gamma^2(t)}\Big(\big(\alpha'(t)+\gamma'(t)y\big)v_{my}(t)\phi_m(t), \phi_{my}(t)\Big) \le \frac{pMc_0}{\gamma^2(t)}|v_{my}(t)||\phi_{my}(t)|^2
$$

where c_0 is the immersion constant $|.|_{L^\infty(0,1)} \le c_0\|.\|_{H_0^1(0,1)}$ and $|\alpha'(t)+\gamma'(t)y| \le |\alpha'(t)| + |\beta'(t)| \le M$.

Similarly for the I_3, I_4 and I_5 term, using Cauchy-Schwarz inequality and remembering that $\gamma(t)$ is increasing, we have,

$$
\begin{aligned}
I_3 &= \frac{1}{2\gamma^4(t)}\int_0^1 \Big(-2(\alpha'(t)+\gamma'(t)y)(\alpha''(t)+\gamma''(t)y)\gamma^2(t) - (\nu_1^2(y) - (\alpha'+\gamma'y)^2)2\gamma\gamma'\Big)v_{my}^2(y,t)dy \\
&\le \Big(\frac{1}{\gamma^2}M^2\gamma^2 + (\nu_3^2+M^2)M\gamma_0^{-1}\Big)|v_{my}(t)|^2.
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \int_0^1 \frac{1}{\gamma(t)}\Big(\Big(\alpha''(t)+\gamma''(t)y\Big) + \delta\Big(\alpha'(t)+\gamma'(t)y\Big)\Big)v_{my}(y,t)v_m'(y,t)dy \\
&\le \frac{1}{\gamma^2(t)}\Big((1+\delta)M\Big)^2 \frac{2}{\delta}|v_{my}(t)|^2 + \frac{\delta}{8}|v_m'(t)|^2 \\
I_5 &= (f_1(t), v_m'(t)) \le \frac{2}{\delta}\big|f_1(t)\big|^2 + \frac{\delta}{8}\big|v_m'(t)\big|^2.
\end{aligned}
$$

Now, taking $w = v_m(t)$ in (3.2), we have

$$
\begin{aligned}
&\frac{d}{dt}(v_m'(t), v_m(t)) - |v_m'(t)|^2 + \int_0^1 \frac{1}{\gamma^2(t)}\Big(\nu_1^2(y) - (\alpha'(t)+\gamma(t)y)^2\Big)v_{my}^2(y,t) + \frac{\delta}{2}\frac{d}{dt}|v_m(t)|^2 \\
&= I_6 + I_7 + I_8 + I_9 \qquad (3.8)
\end{aligned}
$$

where $\{I_6,\ I_7,\ I_8,\ I_9\}$ are defined as follows:

Analysis of the terms (3.8):

$$I_6 = \frac{-p}{\gamma}(\phi_m(t)\phi_{my}(t), v_m(t)) \leq \frac{p}{\gamma}|\phi(t)|_{L^\infty(0,1)}||\phi_{my}(t)||v_m(t)| \leq \frac{pc_0\gamma_1}{\gamma^2}|\phi_{my}(t)|^2|v_m(t)|.$$

Integrating by parts, we obtain

$$\begin{aligned}
I_7 &= \frac{2}{\gamma}\Big((\alpha'+\gamma' y)v'_{my}(t), v_m(t)\Big) = -2\int_0^1 \frac{1}{\gamma}\Big((\alpha'+\gamma' y)v_m(y,t)\Big)_y v'_m(y,t)dy \\
&= -2\int_0^1 \Big(\gamma'\gamma v_m(y,t)v'_m(y,t) + \frac{1}{\gamma}(\alpha'+\gamma' y)v_{my}(y,t)v'_m(y,t)\Big)dy \\
&\leq \frac{2}{\gamma}(|\alpha'|+|\beta'|)(c_1+1)|v_{my}(t)||v'_m(t)| \leq \frac{1}{\gamma^2}\big(\sqrt{2}M(c_1+1)\big)^2|v_{my}(t)|^2 + \frac{1}{2}|v'_m(t)|^2
\end{aligned}$$

where c_1 is the immersion constant, i.e, $|.|_{L^2(0,1)} \leq c_1\|.\|_{H_0^1(0,1)}$.

$$\begin{aligned}
I_8 &= \frac{1}{\gamma(t)}\Big(\big((\alpha''(t)+\gamma''(t)y) + \delta(\alpha'(t)+\gamma'(t)y)\big)v_{my}(t), v_m(t)\Big) \\
&\leq \Big(|\alpha''(t)|+|\beta''(t)|+\delta(|\alpha'(t)|+|\beta'(t)|)\Big)c_1|v_{my}(t)|^2 \leq \frac{1}{\gamma^2}(1+\delta)Mc_1\gamma_1|v_{my}|^2 \qquad (3.9)
\end{aligned}$$

where, we have used Young's inequality. Similarly, we have

$$I_9 = (f_1(t), v_m(t) \leq |f_1(t)||v_m(t)| \leq \frac{1}{\gamma^2(t)}(\gamma_1^2 c_1^2 \nu_2^2|f_1(t)|^2) + \nu_2^2|v_{my}(t)|^2. \qquad (3.10)$$

Considering the inequalities $(I_1 - I_9)$, multiplying (3.8) by $\delta/4$, adding with the terms of (3.7), we obtain, after some calculus, that

$$\begin{aligned}
&\frac{p}{2}\frac{d}{dt}|\phi_m(t)|^2 + p\int_0^1 \frac{r_1(y)}{\gamma^2(t)}\phi_{my}^2(y,t)dy + \frac{1}{2}\frac{d}{dt}|v'_m(t)|^2 + I_{10} + I_{11} + I_{12} \\
&\quad + \delta|v'_m(t)|^2 + \frac{\delta}{4}\frac{d}{dt}(v'_m(t), v_m(t)) - \frac{\delta}{4}|v'_m(t)|^2 + \frac{\delta^2}{8}\frac{d}{dt}|v_m(t)|^2 \\
&\quad + \frac{\delta}{4}\int_0^1 \frac{1}{\gamma^2(t)}\Big(\nu_1^2(y) - (\alpha'(t)+\gamma'(t)y)^2\Big)v_{my}^2(y,t)dy \\
&\leq p\Big|\sqrt{r_1}j_1(t)\Big|_{L^\infty}^2 + p\int_0^1 \frac{r_1(y)}{4\gamma^2}\phi_{my}^2(y,t)dy + \frac{pMc_0}{\gamma^2(t)}|v_{my}||\phi_{my}|^2 + \frac{\delta}{8}|v'_m|^2 \\
&\quad + \frac{1}{\gamma^2(t)}\Big(M^2 + ((\nu_3^2+M^2)M)\gamma_0^{-1}\Big)|v_{my}(t)|^2 + \frac{1}{\gamma^2(t)}\Big((1+\delta)M\Big)^2\frac{2}{\delta}|v_{my}(t)|^2 \\
&\quad + \frac{2}{\delta}|f_1|^2 + \frac{\delta}{8}|v'_m|^2 + \frac{\delta pc_0\gamma_1}{4\gamma^2}|\phi_{my}|^2|v_m(t)| + \frac{\delta}{2\gamma^2}\Big(M(c_1+1)\gamma\Big)^2|v_{my}|^2 \\
&\quad + \frac{\delta}{8}|v'_m(t)|^2 + \frac{1}{\gamma^2(t)}\Big(\delta(1+\delta)Mc_1\gamma_1\Big)|v_{my}(t)|^2 + \frac{\delta\gamma_1^2c_1^2}{8\nu_2^2}|f_1|^2 + \frac{\delta\nu_2^2}{8\gamma^2}|v_{my}|^2 \qquad (3.11)
\end{aligned}$$

where $\{I_{10},\ I_{11},\ I_{12}\}$ terms, on the left side of (3.10) are defined as follows:

$$\begin{aligned}
I_{10} &= \frac{p\gamma'(t)}{2\gamma(t)}|\phi_m(t)|^2 \leq \frac{p}{2\gamma^2}(|\alpha'|+|\beta'|)|\phi_m|^2\gamma \leq \frac{pM\gamma_1c_1^2}{2\gamma^2}|\phi_{my}|^2 \\
I_{11} &= \frac{\gamma'(t)}{\gamma(t)}|v'_m(t)|^2 \leq \frac{1}{\gamma_0}M|v'_m(t)|^2
\end{aligned}$$

$$\begin{aligned} I_{12} &= \frac{\delta}{4}\int_0^1 \frac{1}{\gamma^2(t)}\left(\nu_1^2(y) - (\alpha'(t)+\gamma'(t)y)^2\right)v_{my}^2(y,t)dy \\ &= \frac{\delta}{4}\int_0^1 \frac{\nu_1^2(y)}{\gamma^2(t)}v_{my}^2(y,t)dy - \frac{\delta}{4}\int_0^1 \frac{1}{\gamma^2(t)}(\alpha'(t)+\gamma'(t)y)^2 v_{my}^2(y,t)dy \\ &\geq \frac{\delta}{4}\frac{\nu_2^2}{\gamma^2(t)}|v_{my}(t)|^2 - \frac{\delta M^2}{4\gamma^2(t)}|v_{my}(t)|^2 \end{aligned} \tag{3.12}$$

where, we use the (**H3**) hypothesis to obtain the inequalities.

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{p}{2}|\phi_m(t)|^2 + \frac{1}{2}|v_m'(t)|^2 + \frac{1}{2}\int_0^1 \frac{1}{\gamma^2(t)}\Big(\nu_1^2(y) - (\alpha'(t)+\gamma'(t)y)^2\Big)v_{my}^2(y,t)dy \\ &+ \frac{\delta}{4}(v_m', v_m) + \frac{\delta^2}{8}|v_m|^2\Big] + \frac{p}{4\gamma^2}|\phi_{my}|^2\Big(r_2 - \big(\delta c_0\gamma_1|v_m| + 4Mc_0|v_{my}|\big)\Big) \\ &+ \frac{p}{2\gamma^2}|\phi_{my}|^2(r_2 - M\gamma_1 c_1^2) + \frac{\delta\widehat{M}}{4\gamma^2}|v_{my}|^2 + \frac{\delta}{4}|v_m'|^2 + |v_m'|^2\Big(\frac{\delta}{8} - \frac{M}{\gamma_0}\Big) \\ &\leq p\Big|\sqrt{r_1}j_1(t)\Big|_{L^\infty}^2 + \frac{\delta^2\gamma_1^2 c_1^2}{8\nu_2^2}|f_1(t)|^2 + \frac{2}{\delta}|f_1(t)|^2 \end{aligned} \tag{3.13}$$

where

$$\widehat{M} = \Big[\frac{\nu_2^2}{2} - \big(\frac{4M^2}{\delta} + \frac{4}{\delta}(\nu_3^2 + M^2)M\gamma_0^{-1} + \frac{8}{\delta^2}(1+\delta)^2M^2 + 2M^2(c_1+1)^2 + (1+\delta)Mc_1\gamma_1 + M^2\big)\Big].$$

Let us denote the function $K(t)$ by

$$\begin{aligned} K(t) &:= \frac{p}{2}|\phi_m(t)|^2 + \frac{1}{2}|v_m'(t)|^2 + \frac{\delta^2}{8}|v_m(t)|^2 + \frac{\delta}{4}(v_m'(t), v_m(t)) \\ &\quad + \frac{1}{2}\int_0^1 \frac{1}{\gamma^2(t)}\Big(\nu_1^2(y) - (\alpha'(t)+\gamma'(t)y)^2\Big)v_{my}^2(y,t)dy. \end{aligned} \tag{3.14}$$

REMARK 3.1. Now, we want to show that there are constants $\overline{c}$ and $\widetilde{c}$ such that

$$\begin{aligned} K(t) &\geq \widetilde{c}\ (|\phi_m(t)|^2 + |v_m'(t)|^2 + |v_m(t)|^2 + \|v_m(t)\|^2) \\ K(t) &\leq \overline{c}\ (|\phi_m(t)|^2 + |v_m'(t)|^2 + |v_m(t)|^2 + \|v_m(t)\|^2) \end{aligned} \tag{3.15}$$

Proof. Indeed, $K(t) \geq \widetilde{c}\ (|\phi_m(t)|^2 + |v_m'(t)|^2 + |v_m(t)|^2 + \|v_m(t)\|^2$.
By using Cauchy's inequality, one gets

$$\Big|\frac{\delta}{4}(v_m'(t), v_m(t))\Big| \leq |v_m'(t)|\Big(\frac{\delta}{4}|v_m(t)|\Big) \leq \frac{1}{4}|v_m'(t)|^2 + \frac{\delta^2}{16}|v_m(t)|^2,$$

then

$$\frac{\delta}{4}(v_m'(t), v_m(t)) \geq -\frac{1}{4}|v_m'(t)|^2 - \frac{\delta^2}{16}|v_m(t)|^2.$$

Note that,

$$\begin{aligned} K(t) &= \frac{p}{2}|\phi_m(t)|^2 + \frac{1}{2}|v_m'(t)|^2 + \frac{\delta^2}{8}|v_m(t)|^2 + \frac{\delta}{4}(v_m'(t), v_m(t)) \\ &+ \frac{1}{2}\int_0^1 \frac{1}{\gamma^2}\Big(\nu_1^2(y) - (\alpha'+\gamma'y)^2\Big)v_{my}^2(y,t)dy \geq \frac{p}{2}|\phi_m(t)|^2 + \frac{1}{2}|v_m'(t)|^2 \end{aligned}$$

$$+\frac{\delta^2}{8}|v_m|^2-\frac{1}{4}|v_m'|^2-\frac{\delta^2}{16}|v_m|^2+\frac{1}{2}\int_0^1\frac{1}{\gamma^2}\Big(\nu_1^2(y)-(\alpha'+\gamma'y)^2\Big)v_{my}^2dy. \tag{3.16}$$

Therefore

$$K(t)\geq\frac{p}{2}|\phi_m(t)|^2+\frac{1}{4}|v_m'(t)|^2+\frac{\delta^2}{16}|v_m(t)|^2+\frac{1}{2}\int_0^1\frac{1}{\gamma^2}\Big(\nu_1^2(y)-(\alpha'+\gamma'y)^2\Big)v_{my}^2dy. \tag{3.17}$$

However, using the hypotheses that $M_1^2<\frac{1}{2}\nu_2^2$, it follows that

$$\frac{1}{2}\int_0^1\frac{1}{\gamma^2(t)}\Big(\nu_1^2(y)-(\alpha'(t)+\gamma'(t)y)^2\Big)v_{my}^2(y,t)dy\geq\frac{1}{4}\Big(\frac{\nu_2^2}{\gamma^2(t)}\Big)|v_{my}(t)|^2.$$

Thus,

$$K(t)\geq\frac{p}{2}|\phi_m(t)|^2+\frac{1}{4}|v_m'(t)|^2+\frac{\delta^2}{16}|v_m(t)|^2+\frac{\nu_2^2}{4\gamma^2(t)}\|v_m(t)\|^2. \tag{3.18}$$

Taking $\widetilde{c}=\min\Big\{\frac{p}{2},\ \frac{1}{4},\ \frac{\delta^2}{16},\ \frac{\nu_2^2}{4\gamma_1^2}\Big\}$ results that

$$K(t)\geq\widetilde{c}\ (|\phi_m(t)|^2+|v_m'(t)|^2+|v_m(t)|^2+\|v_m(t)\|^2). \tag{3.19}$$

Now, let us prove the second inequality of $(3.15)_2$. Indeed; from (3.16) we have

$$\begin{aligned}K(t)\leq&\frac{p}{2}|\phi_m(t)|^2+\frac{1}{2}|v_m'(t)|^2+\frac{\delta^2}{8}|v_m(t)|^2+\frac{1}{4}|v_m'(t)|^2+\frac{\delta^2}{16}|v_m(t)|^2\\&+\frac{1}{2}\int_0^1\frac{1}{\gamma^2(t)}(\nu_1^2(y)-(\alpha'(t)+\gamma'(t)y)^2)v_{my}^2(y,t)dy,\end{aligned} \tag{3.20}$$

Therefore,

$$\begin{aligned}K(t)\ &\leq\frac{p}{2}|\phi_m(t)|^2+\frac{3}{4}|v_m'|^2+\frac{3\delta^2}{16}|v_m|^2+\frac{1}{2}\int_0^1\frac{1}{\gamma^2}(\nu_1^2(y)-(\alpha'+\gamma'y)^2)v_{my}^2dy\\&\leq\frac{p}{2}|\phi_m(t)|^2+\frac{3}{4}|v_m'(t)|^2+\frac{3\delta^2}{16}|v_m(t)|^2+\frac{1}{2\gamma^2(t)}\Big(\frac{\nu_3^2+M^2}{\gamma^2(t)}\Big)|v_{my}(t)|^2.\end{aligned}$$

Thus,

$$K(t)\leq\frac{p}{2}|\phi_m(t)|^2+\frac{3}{4}|v_m'(t)|^2+\frac{3\delta^2}{16}|v_m(t)|^2+\frac{1}{2\gamma_0^2}\Big(\nu_3^2+M^2\Big)\|v_m(t)\|^2 \tag{3.21}$$

Taking

$$\bar{c}=\max\Big\{\frac{p}{2},\ \frac{3}{4},\ \frac{3\delta^2}{16},\ \frac{\nu_3^2+M^2}{2\gamma_0^2}\Big\}, \tag{3.22}$$

Therefore,

$$K(t)\leq\bar{c}\ (|\phi_m(t)|^2+|v_m'(t)|^2+|v_m(t)|^2+\|v_m(t)\|^2), \tag{3.23}$$

thus, we conclude the Remark (3.1). □

Returning to (3.13) and with the help of Remark (3.1), we can write

$$\frac{d}{dt}K(t)+\frac{p}{4\gamma^2(t)}|\phi_{my}(t)|^2\Big(r_2-\big(\delta c_0\gamma_1|v_m(t)|+4Mc_0|v_{my}(t)|\big)\Big)+\frac{\delta}{4}|v'_m(t)|^2$$
$$\leq p\Big|\sqrt{r_1}j_1(t)\Big|^2+\frac{|f_1(t)|^2}{8\delta\nu_2^2}\big(\delta^2\gamma_1^2c_1^2+16\nu_2^2\big) \tag{3.24}$$

and

$$R(t)=\delta c_0\gamma_1\big|v_m(t)\big|+4Mc_0\big|v_{my}(t)\big|\leq\Big(4c_0\gamma_1+\frac{8Mc_0\gamma_1}{\nu_2}\Big)\sqrt{K(t)}. \tag{3.25}$$

Now, suppose that $\{v_0,\ \phi_0,\ j_1,\ f_1\}$ are such that

$$\Big(4c_0\gamma_1+\frac{8Mc_0\gamma_1}{\nu_2}\Big)\Big[K(0)+p\int_0^T\Big|\sqrt{r_1}j_1(t)\Big|^2+\Big(\frac{\delta^2\gamma_1^2c_1^2+16\nu_2^2}{8\delta\nu_2^2}\Big)\int_0^T|f_1|^2\Big]^{1/2}<r_2. \tag{3.26}$$

Affirmation:
The function $R(t)$, defined in (3.25) satisfies: $R(t)<r_2, \forall t\geq 0$.

Indeed, the proof will be made by contradiction. Suppose, by contradiction, that is false. From (3.25) and (3.26), we have that $R(0)<r_2$. From the continuity of R, there exists a minimum t^* such that

$$\begin{cases} R(t)<r_2, & \forall t\in[0,\ t^*[\\ R(t^*)=r_2 \end{cases} \tag{3.27}$$

Thus, integrating (3.13) from 0 to t^*, we have

$$K(t^*)\leq K(0)+\int_0^T p\big|\sqrt{r_1}j_1(t)\big|^2dt+\frac{1}{8\nu_2^2\delta}\big(\delta^2\gamma_1^2c_1^2+16\nu_2^2\big)\int_0^T\big|f_1(t)\big|^2dt. \tag{3.28}$$

From (3.25), it follows that

$$\begin{aligned} R(t^*) \leq \Big(4c_0\gamma_1+\frac{8Mc_0\gamma_1}{\nu_2}\Big)\sqrt{K(t^*)}\leq\Big(4c_0\gamma_1+\frac{8Mc_0\gamma_1}{\nu_2}\Big) \\ \Big[K(0)+\int_0^T p\big|\sqrt{r_1}j_1(t)\big|^2dt+\Big(\frac{\delta^2\gamma_1^2c_1^2+16\nu_2^2}{8\delta\nu_2^2}\Big)\int_0^T|f_1|^2\Big]^{1/2}<r_2. \end{aligned} \tag{3.29}$$

Then,

$$R(t^*)<r_2, \qquad \text{which contradicts} \quad (3.27)_2. \tag{3.30}$$

Therefore,

$$R(t)<r_2,\ \ \forall t\geq 0. \tag{3.31}$$

Thus, integrating (3.24) over $(0,t)$, we have,

$$K(t)+\frac{\delta}{4}\int_0^t|v'_m(s)|^2ds\leq K(0)+p\int_0^T|\sqrt{r_1}j_1(t)|^2_{L^\infty(0,1)}dt$$
$$+\frac{1}{8\delta\nu_2^2}(\delta^2\gamma_1^2c_1^2+16\nu_2^2)\int_0^T|f_1(t)|^2dt\leq\overline{c}\Big(|\phi_{0m}|^2+|v_{1m}|^2+|v_{0m}|^2+\|v_{0m}\|^2$$

$$+p\int_0^T \left|\sqrt{r_1}j_1(t)\right|^2_{L^\infty(0,1)}dt+\frac{1}{8\delta\nu_2^2}\left(\delta^2\gamma_1^2c_1^2+16\nu_2^2\right)\int_0^T \left|f_1(t)\right|^2dt\Big)\le c_2. \tag{3.32}$$

As $\widetilde{c}\ (|\phi_m(t)|^2+|v_m'(t)|^2+|v_m(t)|^2+\|v_m(t)\|^2)\le K(t)$, it follows that

$$\widetilde{c}\big(|\phi_m(t)|^2+|v_m'(t)|^2+|v_m(t)|^2+\|v_m(t)\|^2\big)+\frac{\delta}{4}\int_0^t |v_m'(s)|^2ds\le c_3. \tag{3.33}$$

Thus, we have

$$|\phi_m(t)|^2+|v_m'(t)|^2+|v_m(t)|^2+\|v_m(t)\|^2+\int_0^t |v_m'(s)|^2ds\le c_4, \tag{3.34}$$

where c_4 is independent of m.

Estimate II: Taking $w=-p\phi_{myy}(t)$ in (3.2), $w=-v'_{myy}(t)$ in (3.3) and adding the equations, we have

$$\begin{aligned}&\frac{p}{2}\frac{d}{dt}\|\phi_m(t)\|^2+p\int_0^1 \frac{r_1(y)}{\gamma^2(t)}\phi^2_{myy}dy+\frac{1}{2}\frac{d}{dt}\|v_m'(t)\|^2\ dy\\ &\quad+2\int_0^1 \frac{1}{\gamma^2(t)}\big(\nu_1(y)\nu_{1y}(y)-(\alpha'(t)+\gamma'(t)y)\gamma'(t)\big)v_{my}v'_{myy}\ dy\\ &\quad+\frac{1}{2}\frac{d}{dt}\int_0^1 \frac{1}{\gamma^2(t)}\big(\nu_1^2(y)-(\alpha'(t)+\gamma'(t)y)^2\big)v^2_{myy}\ dy\\ &\quad-\frac{1}{2}\int_0^1 \frac{d}{dt}\Big(\frac{1}{\gamma^2(t)}\big(\nu_1^2(y)-(\alpha'(t)+\gamma'(t)y)^2\big)\Big)v^2_{myy}\ dy+\{I_{13}-I_{19}\}\\ &\quad+\int_0^1 \frac{1}{\gamma}\Big((\alpha''+\gamma''y)+\delta\big(\alpha'+\gamma'y\big)\Big)v_{my}(v')^2_{myy}dy+\delta\|v_m'\|^2=I_{20}.\end{aligned} \tag{3.35}$$

Analysis of the terms (3.35):

Let us first analyze the fourth and last term on the left-hand side of (3.35), then the fifth and sixth term, and finally the remainder terms $(I_{13}-I_{20})$. We denote,

$$\omega=\frac{2}{\gamma^2}\Big(\nu_1(y)\nu_{1y}(y)-(\alpha'+\gamma'y)\gamma'\Big)+\frac{1}{\gamma}\Big((\alpha''+\gamma''y)+\delta(\alpha'+\gamma'y)\Big). \tag{3.36}$$

Since that,

$$\frac{d}{dt}\int_0^1 \omega v_{my}v_{myy}dy=\int_0^1 \omega' v_{my}v_{myy}dy+\int_0^1 \omega v'_{my}v_{myy}dy+\int_0^1 \omega v_{my}v'_{myy}dy,$$

it follows that

$$\int_0^1 \omega v_{my}v'_{myy}dy=\frac{d}{dt}\int_0^1 \omega v_{my}v_{myy}dy-\int_0^1 \omega' v_{my}v_{myy}dy-\int_0^1 \omega v'_{my}v_{myy}dy.$$

Then, we have

$$\int_0^1 \omega v_{my}v_{myy}dy=\frac{1}{2}\int_0^1 \omega(v^2_{my})_y\ dy=\frac{1}{2}\omega v^2_{my}\Big|_0^1-\frac{1}{2}\int_0^1 \omega_y v^2_{my}\ dy.$$

Therefore,

$$\frac{1}{2}\omega v^2_{my}\Big|_0^1\le\Big|\frac{1}{2\gamma(t)}\Big(\frac{-2\beta'(t)\gamma'(t)}{\gamma(t)}+\beta''(t)\gamma(t)+\delta\beta'(t)\Big)v^2_{my}(1,t)\Big|$$

$$+\Big|\frac{1}{2\gamma(t)}\Big(\frac{-2\alpha'(t)\gamma'(t)}{\gamma(t)}+\alpha''(t)+\delta\alpha'(t)\Big)v_{my}^2(0,t)\Big|$$
$$\leq\frac{1}{2\gamma}\Big(\frac{2M^2}{\gamma}+(M(1+\delta))\Big)c_7^2\Big|v_{myy}\Big|^2+\frac{1}{2\gamma}\Big(\frac{2M^2}{\gamma}+M(1+\delta)\Big)c_7^2\Big|v_{myy}^2\Big|^2, \quad (3.37)$$

where $\{c_5,\ c_6,\ c_7\}$ are positive constants due to the continuous immersion of space $H^1(0,1)\subset C^0([0,1])$.

Thus, $|v_{my}(1)|\leq c_7|v_{myy}|$ and $|v_{my}(0)|\leq c_7|v_{myy}|$. Therefore,

$$\frac{1}{2}\omega v_{my}^2\Big|_0^1\leq\frac{M}{\gamma^2(t)}\Big(2M+\gamma_1(1+\delta)\Big)c_7^2|v_{myy}(t)|^2.$$

Also,

$$2\int_0^1\Big(\frac{1}{\gamma}(\alpha'+\gamma'y)\Big)v'_{my}v'_{myy}dy=\frac{1}{\gamma}(\alpha'+\gamma'y)v'^2_{my}\Big|_0^1-\int_0^1\frac{\gamma'}{\gamma}(v')^2_{my}dy$$
$$=\frac{\beta'}{\gamma(t)}(v')^2_{my}(1,t)-\frac{\alpha'}{\gamma}v'^2_{my}(0,t)-\int_0^1\frac{\gamma'}{\gamma}v'^2_{my}dy. \quad (3.38)$$

Since that, by hypothesis, $\gamma(t)$ is increasing, then the following inequality is true:

$$\frac{1}{\gamma(t)}\big(\beta'(t)v'^2_{my}(1,t)-\alpha'(t)\big)(v')^2_{my}(0,t)\geq 0.$$

Estimate of the terms $(I_{13}-I_{20})$. The procedure for estimating each $(I_{13}-I_{20})$ term is very similar to that done for the terms (I_1-I_9) and thus we will not detail all of the calculus. Indeed:

$$I_{13}=\frac{p}{\gamma^2(t)}\int_0^1 r_{1y}(y)\phi_{my}\phi_{myy}dy\leq\frac{2p}{r_2\gamma^2(t)}|r_{1y}|^2_{L^\infty(0,1)}\|\phi_m(t)\|^2+\frac{pr_2}{8\gamma^2(t)}|\phi_{myy}(t)|^2$$
$$I_{14}=p\int_0^1\frac{1}{\gamma(t)}\big(\alpha'(t)+\gamma'(t)y\big)\phi_{my}\phi_{myy}\ dy\leq\frac{2pM^2}{r_2}\|\phi_m(t)\|^2\ +\ \frac{pr_2}{8\gamma^2(t)}|\phi_{myy}(t)|^2$$
$$I_{15}=-\frac{p}{\gamma(t)}\int_0^1(r_1(y)j_1)_y\phi_{myy}dy+p\frac{\gamma'(t)}{\gamma^2(t)}\int_0^1 v_{my}\phi_m\phi_{myy}\ dy$$
$$\leq\frac{2p}{r_2}\ |(r_1j_1(t))_y|^2_{L^\infty(0,1)}+\frac{pr_2}{8\gamma^2(t)}\ |\phi_{myy}(t)|^2$$
$$I_{16}=\frac{p}{\gamma(t)}\int_0^1\frac{1}{\gamma(t)}\big(\alpha'(t)+\gamma'(t)y\big)v_{myy}\phi_m\phi_{myy}$$
$$\leq\frac{2pM^2}{r_2\gamma^2(t)}\|v_m(t)\|^2\|\phi_m(t)\|^2+\frac{pr_2}{8\gamma^2(t)}|\phi_{myy}(t)|^2$$
$$I_{17}=\frac{p}{\gamma(t)}\int_0^1\frac{1}{\gamma(t)}\big(\alpha'(t)+\gamma'(t)y\big)v_{my}\phi_{my}\phi_{myy}$$
$$\leq\frac{2pM^2}{r_2\gamma^2(t)}|v_{myy}(t)|^2|\phi_m(t)|^2+\frac{pr_2}{8\gamma^2(t)}|\phi_{myy}(t)|^2$$
$$I_{18}=2\int_0^1\frac{1}{\gamma(t)}\big(\alpha'(t)+\gamma'(t)y\big)v'_{my}v'_{myy}\ dy\leq\frac{M}{\gamma_0}\|v'_m(t)\|^2$$
$$I_{19}=-\frac{3p}{\gamma(t)}\int_0^1\phi_{my}v'_m\phi_{myy}dy\leq\frac{18p}{r_2}\|\phi_m(t)\|^2|v'_m(t)|^2+\frac{pr_2}{8\gamma^2(t)}|\phi_{myy}(t)|^2$$

$$I_{20}=\int_0^1 f_{1y}v'_{my}\ dy\le\frac{1}{2}\|f_1(t)\|^2+\frac{1}{2}\|v'_m(t)\|^2.$$

Let us now make estimates for the remaining terms of the (3.35).

$$I_{21}=\int_0^1\frac{1}{2\gamma^4(t)}\Big(-2(\alpha'(t)+\gamma'(t)y)(\alpha''(t)+\gamma''(t)y)\gamma^2(t)-$$
$$(\nu_1^2(y)-(\alpha'(t)+\gamma'(t)y)^2 2\gamma(t)\gamma'(t))\Big)v_{my}^2\le\Big(\frac{M^2}{\gamma_0^2}+\frac{(\nu_3^2+M^2)M}{\gamma_0^3}\Big)\|v_m(t)\|^2$$

$$I_{22}=\frac{p}{\gamma^2}\int_0^1\Big(\alpha'+\gamma'y\Big)v_{my}\phi_{my}\phi_{myy}\le\frac{2pM^2}{r_2\gamma^2}\|v_m(t)\|^2\|\phi_m(t)\|^2+\frac{pr_2}{8\gamma^2}|\phi_{myy}(t)|^2$$

$$I_{23}=\frac{1}{2}\int_0^1\omega'_y v_{my}v_{my}\le\frac{c_0^2}{8}\Big(\frac{2}{\gamma_0^3}(\nu_4^2+\nu_3\nu_5+2M^3)+\ (1+\delta)\frac{M}{\gamma_0}+(5+\delta)\frac{M^2}{\gamma_0^2}\Big)^2v_{myy}^2(t)$$
$$+\frac{1}{2}v_m^2(t)\le c_8|v_{myy}(t)|^2+\frac{1}{2}\|v_m(t)\|^2$$

$$I_{24}=\int_0^1|\omega_y v_{my}v'_{my}|dy\le\frac{1}{2}\Big(\frac{2}{\gamma_0^2}(\nu_4^2+\nu_3\nu_5+M^2)+(1+\delta)\frac{M}{\gamma_0}\Big)^2\|v_m(t)\|^2$$
$$+\frac{1}{2}\|v'_m(t)\|^2\le c_9\|v_m(t)\|^2+\frac{1}{2}\|v'_m(t)\|^2$$

$$I_{25}=\int_0^1|\omega' v_{my}v_{myy}|dy\le\frac{1}{2\gamma_0^3}\Big((3+\delta)\gamma_0M^2\ +\ 4(\nu_3\nu_4+M^2)M+\gamma_0^2(1+\delta)M\Big)^2|v_{myy}|^2$$
$$+\frac{1}{2}\|v_m(t)\|^2\le c_{10}|v_{myy}(t)|^2+\frac{1}{2}\|v_m(t)\|^2$$

$$I_{26}=\int_0^1|\omega v'_{my}v_{myy}|dy\le\frac{1}{\gamma_0^2}\Big(\Big(\nu_3\nu_4+M^2\Big)+\frac{M\gamma_0}{2}(1+\delta)\Big)|v_{myy}(t)|^2+\frac{1}{2}\|v'_m(t)\|^2$$
$$\le c_{11}|v_{myy}(t)|^2+\frac{1}{2}\|v'_m(t)\|^2$$

$$I_{27}=\int_0^1\frac{1}{\gamma^2}\Big(\Big(\nu_1^2(y)-(\alpha'+\gamma'y)^2\Big)v_{myy}dy-\Big(2c_7^2M^2+c_7^2M\gamma_1+\delta c_7^2M\gamma_1\Big)\Big)\int_0^1 v_{myy}^2dy$$
$$\ge\frac{1}{2}\int_0^1\frac{1}{\gamma^2}\Big(\Big(\nu_2^2-M_1^2\Big)-\Big(4c_7^2M^2+2c_7^2M\gamma_1+2\delta c_7^2M\gamma_1\Big)\Big)v_{myy}^2(y,t)dy$$
$$\ge\frac{1}{2\gamma^2(t)}\Big((\nu_2^2/2)-(4c_7^2M^2+2c_7^2M\gamma_1+2\delta c_7^2M\gamma_1)\Big)|v_{myy}(t)|^2\ge\frac{1}{8}\frac{\nu_2^2}{\gamma^2(t)}|v_{myy}(t)|^2$$

Here, we used the hypotheses: $M_1^2<\nu_2^2/2$ and $4c_7^2M^2+2c_7^2M\gamma_1+2\delta c_7^2M\gamma_1<\nu_2^2/4$. Thus from (3.35), using $(I_{13}-I_{27})$, grouping common terms and integrating from 0 to t we have

$$\frac{p}{2}\|\phi_m(t)\|^2+\frac{pr_2}{8\gamma_1}\int_0^t|\phi_{myy}^2(s)|^2ds+\frac{1}{2}\|v'_m(t)\|^2+\delta\int_0^t\|v'_m(s)\|^2ds$$
$$+\frac{\nu_2^2}{8\gamma_1}|v_{myy}(t)|^2\le\frac{p}{2}\|\phi_{0m}\|^2+\frac{1}{2}\|v_{1m}\|^2+\frac{\nu_2^2}{8\gamma_1^2}|v_{0myy}|^2$$
$$+\frac{2pT}{r_2}|(r_1j_1)_y|^2_{L^\infty(Q)}+\frac{1}{\gamma_0^3}\Big(M^2\gamma_0+(\nu_3^2M+M^3)+c_9\gamma_0^3\Big)\|v_m\|^2_{L^\infty(0,T;H_0^1)}\cdot T$$
$$+\frac{2p}{r_2\gamma_0^2}\int_0^1(M^2\gamma_0^2+|r_{1y}|^2_{L^\infty(0,1)}+2M^2\|v_m(s)\|^2+9\gamma_0^2|v'_m(s)|^2)\|\phi_m(s)\|^2ds$$

$$+\int_0^t \Big((c_8+c_{10}+c_{11})+\frac{2pM^2}{r_2\gamma_0^2}|\phi_m(s)|^2\Big)|v_{myy}(s)|^2ds+\frac{1}{2}\int_0^T \|f_1(t)\|^2dt$$
$$+\Big(\frac{3}{2}+\frac{M}{\gamma_0}\Big)\int_0^t \|v_m'(s)\|^2ds. \tag{3.39}$$

Therefore, using the estimate (3.34) and Grönwall's inequality in (3.39), we obtain

$$\frac{p}{2}\|\phi_m\|^2+\frac{pr_2}{8\gamma_1}\int_0^t |\phi_{myy}(s)|^2+\frac{1}{2}\|v_m'\|^2+\delta\int_0^t \|v_m'(s)\|^2+\frac{\nu_2^2}{8\gamma_1}|v_{myy}|^2\le c_{12}. \tag{3.40}$$

where c_{12} depends on $\{\phi_0,v_0,v_1,r_1,j_1,\nu_1,f_1,T\}$ and is independent of m.

Estimate III: Taking $w=\phi_m'(t)$ in (3.2)and $w=v_m''(t)$ in (3.3), we have

$$|\phi_m'(t)|^2=\Big(\frac{1}{\gamma^2(t)}\ r_{1y}\phi_{my}(t)+\frac{r_1}{\gamma^2(t)}\phi_{myy}(t)+\frac{1}{\gamma(t)}(\alpha'(t)+\gamma'(t)'y)\phi_{my}(t)\Big)$$
$$-\Big(\Big(\frac{r_1j_1(t)}{\gamma(t)}\Big)_y+\frac{\gamma'(t)}{\gamma^2(t)}v_{my}(t)\phi_m(t)+\frac{1}{\gamma(t)}\Big(\frac{\alpha'(t)+\gamma'(t)y}{\gamma(t)}\Big)v_{myy}(t)\phi_m(t)\Big)$$
$$+\Big(\frac{1}{\gamma}\Big(\frac{\alpha'+\gamma'y}{\gamma}\Big)v_{my}(t)\phi_{my}(t)-\frac{1}{\gamma}\phi_{my}v_m'-\frac{1}{\gamma}\phi_m(t)v_{my}'(t),\phi_m'(t)\Big) \tag{3.41}$$

$$|v_m''(t)|^2=\Big(\frac{2}{\gamma^2(t)}(\nu_1\nu_{1y}-(\alpha'(t)+\gamma'(t)y)\gamma'(t))v_{my}(t)\Big)$$
$$+\Big(\frac{1}{\gamma^2}(\nu_1^2-(\alpha'+\gamma'y)^2)v_{myy}(t)-\frac{p}{\gamma}\phi_m(t)\phi_{my}(t)+\frac{2}{\gamma}\Big((\alpha'+\gamma'y)v_{my}'(t)\Big)$$
$$+\Big(\Big(\frac{1}{\gamma(t)}(\alpha''(t)+\gamma'(t)y)+\delta(\alpha'(t)+\gamma'(t)y)\Big)v_{my}(t)-\delta v_m'(t)+f_1(t),v_m''(t)\Big). \tag{3.42}$$

The analysis to make the estimates of the terms is done in a similar way to the Estimate II, that is, we have from (3.41) and (3.42), that

$$J_1=\int_0^1 \frac{1}{\gamma^2(t)}r_{1y}(y)\phi_{my}\phi_m'dy\le\frac{5}{2\gamma_0^4}|r_{1y}|^2_{L^\infty(0,1)}\|\phi_m(t)\|^2+\frac{1}{10}|\phi_m'(t)|^2$$
$$J_2=\int_0^1 \frac{1}{\gamma^2(t)}r_1(y)\phi_{myy}\phi_m'dy\le\frac{5}{2\gamma_0^4}|r_1|^2_{L^\infty(0,1)}|\phi_{myy}(t)|^2+\frac{1}{10}|\phi_m'(t)|^2$$
$$J_3=\int_0^1 \Big(\frac{\alpha'(t)+\gamma'(t)y}{\gamma(t)}\Big)\ \phi_{my}\phi_m'dy\le\frac{5M^2}{2\gamma_0^2}\|\phi_m(t)\|^2+\frac{1}{10}|\phi_m'(t)|^2$$
$$J_4=-\int_0^1 \Big(\frac{r_1(y)j_1}{\gamma(t)}\Big)_y\phi_m'dy\le\frac{5}{2\gamma_0^2}|(r_1j_1(t))_y|^2_{L^\infty(0,1)}+\frac{1}{10}|\phi_m'(t)|^2$$
$$J_5=\int_0^1 \frac{\gamma'(t)}{\gamma^2(t)}v_{my}\phi_m\phi_m'\le\frac{5M^2}{2\gamma_0^4}\|v_m(t)\|^2|\phi_m(t)|^2+\frac{1}{10}|\phi_m'(t)|^2$$
$$J_6=\frac{1}{\gamma(t)}\int_0^1 \Big(\frac{\alpha'(t)+\gamma'(t)y}{\gamma(t)}\Big)v_{myy}\phi_m\phi_m'dy\le\frac{5M^2}{2\gamma_0^4}|v_{myy}(t)|^2|\phi_m(t)|^2+\frac{1}{10}|\phi_m'(t)|^2$$
$$J_7=\frac{1}{\gamma}\int_0^1 \Big(\frac{1}{\gamma}(\alpha'+\gamma'y)\Big)v_{my}\phi_{my}\phi_m'dy\le\frac{5M^2}{2\gamma_0^4}\|v_m(t)\|^2\|\phi_m(t)\|^2+\frac{1}{10}|\phi_m'(t)|^2$$
$$J_8=-\frac{1}{\gamma(t)}\int_0^1 \phi_{my}v_m'\phi_m'dy\le\frac{5}{2\gamma_0^2}\|\phi_m(t)\|^2|v_m'(t)|^2+\frac{1}{10}|\phi_m'(t)|^2$$

$$J_9 = -\frac{1}{\gamma(t)}\int_0^1 \phi_m(t)v'_{my}\phi'_m dy \le \frac{5}{2\gamma_0^2}|\phi_m(t)|^2\|v'_m(t)\|^2 + \frac{1}{10}|\phi'_m(t)|^2$$
$$J_{10} = 2\int_0^1 \frac{1}{\gamma^2}(\nu_1(y)\nu_{1y}(y) - (\alpha' + \gamma' y)\gamma')v_{my}v''_m \le \frac{8}{\gamma_0^4}(\nu_3\nu_4 + M^2)^2\|v_m\|^2 + \frac{1}{8}|v''_m|^2$$
$$J_{11} = \int_0^1 \Big(\frac{1}{\gamma^2}(\nu_1^2(y) - (\alpha' + \gamma' y)^2)\Big)v_{myy}v''_m dy \le \frac{2}{\gamma_0^4}(\nu_3^2 + M^2)^2|v_{myy}|^2 + \frac{1}{8}|v''_m(t)|^2$$
$$J_{12} = -\int_0^1 \frac{p}{\gamma(t)}\phi_m\phi_{my}v''_m dy \le \frac{2p^2}{\gamma_0^2}\ |\phi_m(t)|^2\ \|\phi_m(t)\|^2 + \frac{1}{8}|v''_m(t)|^2$$
$$J_{13} = 2\int_0^1 \frac{1}{\gamma(t)}(\alpha'(t) + \gamma'(t)y)v'_{my}v''_m dy \le \frac{8M^2}{\gamma_0^2}\ \|v'_m(t)\|^2 + \frac{1}{8}|v''_m(t)|^2$$
$$J_{14} = \int_0^1 \Big(\frac{1}{\gamma}(\alpha'' + \gamma'' y) + \frac{\delta}{\gamma}\Big(\alpha' + \gamma' y\Big)\Big)v_{my}v''_m \le \frac{2M^2}{\gamma_0^2}(1+\delta)^2\|v_m(t)\|^2 + \frac{1}{8}|v''_m(t)|^2$$
$$J_{15} = \int_0^1 \delta v'_m v''_m \le \delta|v'_m(t)||v''_m(t)| \le 2\delta^2|v'_m(t)|^2 + \frac{1}{8}|v''_m(t)|^2$$
$$J_{16} = \int_0^1 f_1 v''_m dy \le |f_1(t)||v''_m(t)| \le 2|f_1(t)|^2 + \frac{1}{8}|v''_m(t)|^2.$$

Substituting the inequalities ($J_1 - J_{16}$) in (3.41) and (3.42) and integrating from 0 to t, we have

$$\begin{aligned}\frac{1}{10}\int_0^t |\phi'_m|^2 &\le \frac{5}{2\gamma_0^4}|r_{1y}|^2_{L^\infty}\int_0^t \|\phi_m\|^2 + \frac{5}{2\gamma_0^4}|r_1|^2_{L^\infty}\int_0^t |\phi_{myy}|^2\\ &+ \frac{5M^2}{2\gamma_0^2}\|\phi_m\|^2_{L^\infty(0,T;H_0^1)}\cdot T + \frac{5}{2\gamma_0^2}|(r_1 j_1)_y|^2_{L^\infty(Q)}\cdot T + \frac{5M^2}{2\gamma_0^4}\|v_m\|^2_{L^\infty(0,T;H_0^1)}\int_0^t |\phi_m|^2\\ &+ \frac{5M^2}{2\gamma_0^4}|v_{myy}|^2_{L^\infty(0,T;L^2)}\int_0^t |\phi_m(s)|^2 + \frac{5M^2}{2\gamma_0^4}\|\phi_m\|^2_{L^\infty(0,T;H_0^1)}\int_0^t \|v_m(s)\|^2\\ &+ \frac{5}{2\gamma_0^2}\|\phi_m\|^2_{L^\infty(0,T;H_0^1)}\int_0^t |v'_m(s)|^2 + \frac{5}{2\gamma_0^2}\|v'_m\|^2_{L^\infty(0,T;H_0^1)}\int_0^t |\phi_m|^2 \end{aligned} \tag{3.43}$$

and

$$\begin{aligned}\frac{1}{8}\int_0^t |v''_m|^2 &\le \frac{8}{\gamma_0^4}(\nu_3\nu_4 + M^2)^2\|v_m\|^2_{L^\infty(0,T;H_0^1)}\cdot T + \frac{2}{\gamma_0^4}(\nu_3^2 + M^2)^2|v_{myy}|^2_{L^\infty(0,T;L^2)}\cdot T\\ &+ \frac{2p^2}{\gamma_0^2}\|\phi_m\|^2_{L^\infty(0,T;H_0^1(0,1))}\int_0^t |\phi_m(s)|^2 ds + \frac{8M^2}{\gamma_0^2}\int_0^t \|v'_m(s)\|^2 ds\\ &+ \frac{2M^2}{\gamma_0^2}(1+\delta)^2\|v_m\|^2_{L^\infty(0,T;H_0^1(0,1))}T + 2\delta^2\int_0^t |v'_m(s)|^2 ds + 2\int_0^T |f_1(t)|^2 dt. \end{aligned} \tag{3.44}$$

By using the estimates (3.34) and (3.40) in (3.43) and (3.44), it yields

$$\int_0^t |\phi'_m(s)|^2 ds \le c_{13}, \qquad \text{where } c_{13} \text{ is independent of } m. \tag{3.45}$$

Besides that,

$$\int_0^t |v''_m(s)|^2 ds \le c_{14}, \qquad \text{where } c_{14} \text{ is independent of } m. \tag{3.46}$$

From the estimates (3.34), (3.40), (3.45) and (3.46), using Banach-Bourbaki-Alaoglu, Kakutani and Aubin-Lions theorems, there exists a subsequence of $(u_m),(h_m)$; still denoted by $(u_m),(h_m)$ such that

$$\begin{cases} \phi_m \overset{*}{\rightharpoonup} \phi \,(weak\,star) \text{ in } L^\infty(0,T;H_0^1(0,1)) \\ v_m \overset{*}{\rightharpoonup} v\,(weak\,star) \text{ in } L^\infty(0,T;H_0^1(0,1)\cap H^2(0,1)) \\ v_{my} \to v_y\,(strong) \text{ in } L^2(0,T;L^2(0,1)) \text{ and a.e. in } \Omega\times(0,T)=Q \\ \phi_m \rightharpoonup \phi\,(weak) \text{ in } L^2(0,T;H_0^1(0,1)\cap H^2(0,1)) \\ \phi_m \to \phi\,(strong) \text{ in } L^2(0,T;H_0^1(0,1)) \text{ and a.e. in } \Omega\times(0,T)=Q \\ \phi_{my} \to \phi_y\,(strong) \text{ in } L^2(0,T;L^2(0,1)) \text{ and a.e. in } \Omega\times(0,T)=Q \\ v'_m \overset{*}{\rightharpoonup} v'\,(weak\,star) \text{ in } L^\infty(0,T;H_0^1(0,1)) \\ v'_m \to v'\,(strong) \text{ in } L^2(0,T;L^2(0,1)) \text{ and a.e. in } \Omega\times(0,T)=Q \\ \phi'_m \rightharpoonup \phi'\,(weak) \text{ in } L^2(0,T;L^2(0,1)) \\ v''_m \rightharpoonup v''\,(weak) \text{ in } L^2(0,T;L^2(0,1)). \end{cases} \tag{3.47}$$

We also have, from (3.47), that $\phi_m\phi_{my}\to\phi\phi_y$ a.e. in $Q=\Omega\times(0,T)$ and

$$\int_Q |\phi_m\phi_{my}|^2 dydt \le \Big(|\phi_m(t)|^2_{L^\infty(0,1)}\int_0^1 |\phi_m|^2 dy\Big)dt \le \int_0^T |\phi_m(t)|^2_{L^\infty(0,1)}\|\phi_m(t)\|^2 dt$$
$$\le c_0^2\int_0^T \|\phi_m(t)\|^4 dt \le c_{15}\|\phi_m\|^4_{L^\infty(0,T;H_0^1(0,1))}T \le c_{17}.$$

Therefore, from Lions [7], $\phi_m\phi_{my}\rightharpoonup\phi\phi_y$ (weak) in $L^2(Q)$.

On the other hand, as $\phi_m v'_m \to \phi v'$ a.e. in Q and

$$\int_Q |\phi_m v'_m|^2 dydt \le \int_0^T \Big(|\phi_m(t)|^2_{L^\infty(0,1)}\int_0^1 |v'_m|^2 dy\Big)dt \le \int_0^T (c_0^2\|\phi_m(t)\|^2|v'_m(t)|^2)dt$$
$$\le c_{15}\|\phi_m\|^2_{L^\infty(0,T;H_0^1(0,1))}\int_0^T |v'_m(t)|^2 dt \le c_{16}.$$

Then, from Lions [7], $\phi_m v'_m \rightharpoonup \phi v'$ (weak) in $L^2(Q)$ and similarly, $\phi_m v_{my}\rightharpoonup\phi v_y$ (weak) in $L^2(Q)$.

Uniqueness of solution.

We will prove the uniqueness of solution using the argument by contradiction.

In fact, consider two solutions $\{\phi_1,v_1\}$ and $\{\phi_2,v_2\}$ solutions of the coupled systems (2.8) in the sense defined in Theorem (3.2).

Taking $w=\phi_1-\phi_2$ and $z=v_1-v_2$, it yields

$$\begin{cases} w_t - \Big(\frac{1}{\gamma^2(t)}(r_1(y)w_y)_y - \Big(\frac{1}{\gamma^2(t)}(\alpha'(t)+\gamma'(t)y)(\phi_1v_{1y}-\phi_2v_{2y})\Big)_y \\ \quad +\Big(\frac{1}{\gamma(t)}(\phi_1v_{1t})\Big)_y - \Big(\frac{1}{\gamma(t)}(\phi_2v_{2t})\Big)_y - \Big(\frac{\alpha'(t)+\gamma'(t)y}{\gamma(t)}\Big)w_y = 0, & \text{in } Q \\ z_{tt} + \Big(\Big(\frac{1}{\gamma^2(t)}(\alpha'(t)+\gamma'(t)y)^2 - \frac{\nu_1^2(y)}{\gamma^2(t)}\Big)z_y\Big)_y + \frac{p}{\gamma(t)}(\phi_1\phi_{1y}-\phi_2\phi_{2y}) \\ \quad -\frac{2}{\gamma}\Big(\alpha'+\gamma'y\Big)z_{yt} - \Big(\Big(\frac{1}{\gamma}\Big(\alpha''+\gamma''y\Big)+\delta\Big(\alpha'+\gamma'y\Big)\Big)\Big)z_y + \delta z_t = 0 \\ w = z = 0, \qquad \forall t\ge 0 \\ w(y,0)=0, \qquad 0<y<1 \\ z(y,0)=0,\ z_t(y,0)=0,\ 0<y<1 \end{cases} \tag{3.48}$$

Taking the composition of $(3.48)_1$ with w and $(3.48)_2$ with z_t, we have

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2+\int_0^1\frac{r_1(y)}{\gamma^2(t)}w_y^2+\int_0^1\Big(\frac{1}{\gamma^2(t)}(\alpha'(t)+\gamma'(t)y)(\phi_1v_{1y}-\phi_2v_{2y})\Big)w_y\,dy$$
$$+\int_0^1\frac{1}{\gamma(t)}(\phi_2v_{2t}-\phi_1v_{1t})w_ydy-\frac{1}{2}\int_0^1\Big(\frac{1}{\gamma(t)}(\alpha'(t)+\gamma'(t)y)\Big)w_y^2\,dy=0 \tag{3.49}$$

$$\frac{1}{2}\frac{d}{dt}|z_t'|^2+\frac{1}{2}\int_0^1\Big(\Big((\frac{\nu_1^2(y)}{\gamma^2})-(\frac{\alpha'+\gamma'y}{\gamma})^2\Big)\frac{d}{dt}z_y^2\Big)dy+\frac{p}{\gamma}\int_0^1(\phi_1\phi_{1y}-\phi_2\phi_{2y})z_t$$
$$-\int_0^1\Big(\frac{1}{\gamma(t)}(\alpha'+\gamma'y)(z_t^2)_y\Big)-\int_0^1\frac{1}{\gamma}\Big((\alpha''+\gamma''y)+\delta(\alpha'+\gamma'y)\Big)z_yz_t+\delta|z_t(t)|^2=0. \tag{3.50}$$

Analysis of the terms (3.49) and (3.50).

For the third term of the left-hand side from (3.49), we have:

$$I_1=\int_0^1\Big(\Big(\frac{\alpha'(t)+\gamma'(t)y}{\gamma^2(t)}\Big)(\phi_1v_{1y}-\phi_2v_{2y})\Big)w_ydy\le\frac{M}{\gamma^2(t)}\int_0^1|\phi_1||v_{1y}-v_{2y}||w_y|$$
$$+\frac{M}{\gamma^2}\int_0^1|v_{2y}||\phi_1-\phi_2||w_y|\le\frac{Mc_{20}}{\gamma^2}|z_y(t)||w_y(t)|+\frac{M}{\gamma^2(t)}|v_{2y}(t)|_{L^\infty(0,1)}|w(t)||w_y(t)|$$
$$\le\frac{Mc_{20}}{\gamma^2}|z_y(t)||w_y(t)|+\frac{Mc_0}{\gamma^2}|v_2(t)|_{H^2}|w(t)||w_y(t)|$$
$$\le\frac{2M^2c_{20}^2}{r_2\gamma^2(t)}\|z(t)\|^2+\frac{r_2}{4\gamma^2(t)}|w_y(t)|^2+\frac{2M^2c_0^2}{r_2\gamma^2(t)}|v_2(t)|^2_{H^2(0,1)}|w(t)|^2,$$

where $|\phi_1(t)|_{L^\infty(0,1)}\le c_{20}$, since $\phi\in L^\infty(0,T;H_0^1(0,1))\subset L^\infty(0,T;L^\infty(0,1))$.

For the fourth term of the left-hand side from (3.49), we have:

$$I_2=\frac{1}{\gamma}\int_0^1(\phi_2v_{2t}-\phi_1v_{1t})w_y\le\frac{1}{\gamma}\int_0^1|\phi_2||v_{2t}-v_{1t}||w_y|+\frac{1}{\gamma}\int_0^1|\phi_2-\phi_1(t)||v_{1t}||w_y|$$
$$\le\frac{c_{20}}{\gamma}|z_t(t)||w_y(t)|+\frac{c_{20}}{\gamma}|w(t)||w_y(t)|\le\frac{2c_{20}^2}{r_2}|z_t(t)|^2+\frac{r_2}{4\gamma^2}|w_y(t)|^2+\frac{2c_{20}^2}{r_2}|w(t)|^2,$$

where $|\phi_2(t)|_{L^\infty(0,1)}\le c_{20}$ since that $\phi\in L^\infty(0,T;L^\infty(0,1))$. We also use $|v_{1t}|_{L^\infty(0,1)}\le c_{20}$ since that $v_t\in L^\infty(0,T;H_0^1)\subset L^\infty(0,T;L^\infty)$.

Similarly, for the last term from (3.49), we have:

$$I_3=-\frac{1}{2}\int_0^1\frac{1}{\gamma(t)}(\alpha'(t)+\gamma'(t)y)w_y^2dy\le\frac{M}{\gamma(t)}|w(t)|^2\le\frac{M}{\gamma_0}|w(t)|^2.$$

Consider now the terms from (3.50).

For the fourth and fifth terms of the left side from (3.50), we have:

$$I_4=-\frac{1}{2}\int_0^1\frac{1}{\gamma(t)}(\alpha'(t)+\gamma'(t)y)(z_t^2)_ydy\le\frac{M}{\gamma_0}|z_t(t)|^2$$
$$I_5=-\int_0^1\frac{1}{\gamma(t)}\Big((\alpha''(t)+\gamma''(t)y)+\delta(\alpha'(t)+\gamma'(t)y)\Big)z_yz_tdy$$

$$\leq \frac{M(1+\delta)}{\gamma(t)}\|z(t)\||z_t(t)| \leq \frac{(M(1+\delta))^2}{2\gamma^2(t)}\|z(t)\|^2 + \frac{1}{2}|z_t(t)|^2.$$

For the second term of the left-hand side from (3.50), we have:

$$\begin{aligned}
I_6 &= \frac{1}{2}\int_0^1 \Big(\Big(\frac{\nu_1^2(y)}{\gamma^2(t)} - \Big(\frac{\alpha'(t)+\gamma'(t)y}{\gamma(t)}\Big)^2\Big)\frac{d}{dt}z_y^2\Big)dy \\
&= \frac{1}{2}\frac{d}{dt}\int_0^1 \Big(\frac{\nu_1^2(y)-(\alpha'(t)+\gamma'(t)y)^2}{\gamma^2(t)}\Big)z_y^2 dy - \frac{1}{2}\int_0^1 \frac{d}{dt}\Big(\frac{\nu_1^2(y)-(\alpha'(t)+\gamma'(t)y)^2}{\gamma^2(t)}\Big)z_y^2\, dy \\
&\leq \frac{1}{2}\frac{d}{dt}\int_0^1 \Big(\frac{\nu_1^2(y)-(\alpha'(t)+\gamma'(t)y)^2}{\gamma^2(t)}\Big)z_y^2 dy + \Big(\frac{M^2}{\gamma^2(t)} + \frac{(\nu_3^2+M^2)M\gamma_0^{-1}}{\gamma^2(t)}\Big)|z_y(t)|^2.
\end{aligned}$$

For the third term of the left-hand side from (3.50), we have:

$$\begin{aligned}
I_7 &= \frac{p}{\gamma(t)}\int_0^1 (\phi_1\phi_{1y} - \phi_2\phi_{2y})z_t dy \\
&\leq \frac{p}{\gamma(t)}\int_0^1 |\phi_1||\phi_{1y}-\phi_{2y}(t)||z_t|dy + \frac{p}{\gamma(t)}\int_0^1 |\phi_{2y}||\phi_1-\phi_2||z_t|dy \\
&\leq \frac{pc_{18}}{\gamma(t)}|w_y(t)||z_t(t)| + \frac{p}{\gamma(t)}|\phi_{2y}(t)|_{L^\infty(0,1)}|w(t)||z_t(t)| \\
&\leq \frac{pc_{18}}{\gamma(t)}|w_y(t)||z_t(t)| + \frac{pc_0}{\gamma(t)}|\phi_2(t)|_{H^2(0,1)}|w(t)||z_t(t)| \\
&\leq \frac{r_2}{8\gamma^2(t)}|w_y(t)|^2 + \frac{2p^2c_{18}^2}{r_2}|z_t(t)|^2 + \frac{r_2}{8\gamma^2(t)}|\phi_2(t)|^2_{H^2(0,1)}|w(t)|^2 + \frac{2p^2c_0^2}{r_2}|z_t(t)|^2.
\end{aligned}$$

Substituting the inequations $(I_1 - I_7)$ in (3.49) and (3.50), and then integrating from 0 to t, we have

$$\begin{aligned}
\frac{1}{2}|w(t)|^2 + \frac{r_2}{2\gamma_1^2}\int_0^t |w_y(s)|^2 ds &\leq \Big(\frac{2M^2c_0^2}{r_2\gamma_0^2}\Big)\int_0^t |v_2(s)|^2_{H^2(0,1)}|w(s)|^2 ds \\
&- \Big(\frac{2M^2c_{20}^2}{r_2\gamma_0^2}\Big)\int_0^t \|z(s)\|^2 ds + \Big(\frac{2c_{20}^2}{r_2}\Big)\int_0^t |z_t(s)|^2 ds + \Big(\frac{2c_{20}^2}{r_2} + \frac{M}{\gamma_0}\Big)\int_0^t |w(s)|^2 ds \\
\frac{1}{2}|z_t(t)|^2 + \frac{\nu_2^2}{4\gamma_1^2}\|z(t)\|^2 ds &\leq \Big(\frac{M^2}{\gamma_0^2} + \frac{(\nu_3^2+M^2)M\gamma_0^{-1}}{\gamma_0^2}\Big)\int_0^t |z_y(s)|^2 ds + \frac{r_2}{8\gamma_0^2}\int_0^t |w_y(s)|^2 ds \\
&+ \Big(\frac{2p^2(c_{20}+c_0^2)}{r_2} + \frac{M}{\gamma_0} + \frac{1}{2}\Big)\int_0^t |z_t(s)|^2 ds + \frac{r_2}{8\gamma_0^2}\int_0^t |\phi_2(s)|^2_{H^2(0,1)}|w(s)|^2 ds \\
&+ \frac{(M(1+\delta))^2}{2\gamma_0^2}\int_0^t \|z(s)\|^2 ds. \qquad (3.51)
\end{aligned}$$

Thus, from (3.51), we obtain

$$\begin{aligned}
\frac{1}{2}|w(t)|^2 + \frac{r_2}{2\gamma_1^2}\int_0^t |w_y(s)|^2 ds &\leq c_{21}\int_0^t (1+|v_2|^2_{H^2(0,1)})|w(s)|^2 ds \\
&+ c_{22}\int_0^t \|z(s)\|^2 ds + c_{23}\int_0^t |z_t(s)|^2 ds \qquad (3.52)
\end{aligned}$$

$$\frac{1}{2}|z_t(t)|^2 + \frac{\nu_2^2}{4\gamma_1^2}\|z(t)\|^2 ds \leq c_{24}\int_0^t \|z(s)\|^2 ds + c_{25}\int_0^t \|w(s)\|^2 ds$$

$$+c_{26}\int_0^t |z_t(s)|^2 ds + c_{27}\int_0^t |\phi_2(s)|^2_{H^2(0,1)}|w(s)|^2 ds. \tag{3.53}$$

Therefore, summing (3.52) and (3.53), we obtain

$$\begin{aligned}&\frac{1}{2}|w(t)|^2+\frac{3r_2}{8\gamma_1^2}\int_0^t \|w(s)\|^2 ds+\frac{1}{2}|z_t(t)|^2+\frac{\nu_2^2}{4\gamma_1^2}\|z(t)\|^2\\ \leq & c_{28}\int_0^t (1+|v_2(s)|^2_{H^2(0,1)}+|\phi_2(s)|^2_{H^2(0,1)})(|w(s)|^2+\|z(s)\|^2+|z_t(s)|^2)ds.\end{aligned} \tag{3.54}$$

Applying the Grönwall inequality in (3.54), we obtain the uniqueness. ◻

THEOREM 3.2. *From the hypotheses* **(H1) - (H5)** *and given* $\{r,j,\nu,f,h_0,u_0,u_1\}$, *there exists a pair of functions* $(h;\ u):\Omega_t\times(0,T)\to\mathbb{R}$, *for the electromagnetoelasticity problem* (2.6), *in the class*

$$\begin{aligned}&h\in L^\infty(0,T;H_0^1(\Omega_t))\cap L^2(0,T;H_0^1(\Omega_t)\cap H^2(\Omega_t)),\ h_t\in L^2(0,T;L^2(\Omega_t)),\\ &u\in L^\infty(0,T;H_0^1(\Omega_t)\cap H^2(\Omega_t)),\ u_t\in L^\infty(0,T;H_0^1(\Omega_t)),\ u_{tt}\in L^2(0,T;L^2(\Omega_t)).\end{aligned}$$

Proof. The transformation $\mathcal{T}$, given in (2.7), is invertible; then the solution of the coupled system (2.8) can be transformed into the solution of the coupled system (2.6), i.e, $u(x,t)$ and $h(x,t)$ are solutions of the coupled system (2.6) if and only if $v(y,t)$ and $\phi(y,t)$ are solutions of the coupled system (2.8). ◻

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