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GLOBAL STRONG SOLUTIONS TO COMPRESSIBLE NAVIER-STOKES SYSTEM WITH DEGENERATE HEAT CONDUCTIVITY AND DENSITY-DEPENDING VISCOSITY*

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Abstract. We consider the compressible Navier-Stokes system where the viscosity depends on density and the heat conductivity is proportional to a positive power of the temperature under stress-free and thermally insulated boundary conditions. Under the same conditions on the initial data as those of the constant viscosity and heat conductivity case [Kazhikhov-Shelukhin. J. Appl. Math. Mech. 41, 1977], we obtain the existence and uniqueness of global strong solutions. Our result can be regarded as a natural generalization of Kazhikhov's theory for the constant heat conductivity case to the degenerate and nonlinear case under stress-free and thermally insulated boundary conditions.

Keywords. Compressible Navier-Stokes system; Density-depending viscosity; Degenerate heat conductivity; Stress-free.

AMS subject classifications. 60F10; 60J75; 62P10; 92C37.

1. Introduction

The compressible Navier-Stokes system which describes the one-dimensional motion of a viscous heat-conducting perfect polytropic gas is written in the Lagrange variables in the following form (see [4, 20]):

$$v_t = u_x, \tag{1.1}$$

$$u_t + P_x = \left(\mu \frac{u_x}{v}\right)_x, \tag{1.2}$$

$$\left(e + \frac{1}{2}u^2\right)_t + (Pu)_x = \left(\frac{\kappa\theta_x + \mu u u_x}{v}\right)_x, \tag{1.3}$$

where $t > 0$ is time, $x \in \Omega = (0,1)$ denotes the Lagrange mass coordinate, and the unknown functions $v > 0, u$ and P are, respectively, the specific volume of the gas, fluid velocity, and pressure. In this paper, we concentrate on ideal polytropic gas, that is, P and e satisfy

$$P = R\theta/v, \quad e = c_v\theta + \text{const.}, \tag{1.4}$$

where both specific gas constant R and heat capacity at constant volume c_v are positive constants. We also assume that μ and κ satisfy

$$\mu = \tilde{\mu}\left(1 + v^{-\alpha}\right), \quad \kappa = \tilde{\kappa}\theta^{\beta}, \tag{1.5}$$

with constants $\tilde{\mu}, \tilde{\kappa} > 0$ and $\alpha, \beta \geq 0$. The system (1.1)-(1.5) is supplemented with the initial conditions

$$(v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x), \quad x \in (0,1), \tag{1.6}$$

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under stress-free and thermally insulated boundary conditions

$$\left(\frac{\mu}{v}u_x - P\right)(0,t) = \left(\frac{\mu}{v}u_x - P\right)(1,t) = 0, \quad \theta_x(0,t) = \theta_x(1,t) = 0, \tag{1.7}$$

and the initial data (1.6) should be compatible with the boundary conditions (1.7).

The boundary conditions (1.7) describe the expansion of a finite mass of gas into vacuum. One also considers other kind of boundary conditions

$$u(0,t) = u(1,t) = 0, \quad \theta_x(0,t) = \theta_x(1,t) = 0, \tag{1.8}$$

which mean that the gas is confined to a fixed tube with impermeable gas.

For constant coefficients ($\alpha = \beta = 0$) with large initial data, Kazhikhov and Shelukhin [14] first obtained the global existence of solutions under boundary conditions (1.8). From then on, significant progress has been made on the mathematical aspect of the initial boundary value problems, see [1–3, 16–18] and the references therein. Moreover, much effort has been made to generalize this approach to other cases. Motivated by the fact that in the case of isentropic flow a temperature dependence on the viscosity translates into a density dependence, there is a body of literature (see [3, 6–8, 13, 19] and the references therein) studying the case that μ is independent of θ, and heat conductivity is allowed to depend on temperature in a special way with a positive lower bound and balanced with corresponding constitution relations.

Kawohl [13], Jiang [10, 11] and Wang [22] established the global existence of smooth solutions for (1.1)–(1.3), (1.6) with boundary condition of either (1.7) or (1.8) under the assumptions that $\mu(v) \geq \mu_0 > 0$ for any $v > 0$ and κ depending on both density and temperature. However, it should be mentioned here that the methods used there rely heavily on the non-degeneracy of both the viscosity μ and the heat conductivity κ and cannot be applied directly to the degenerate and nonlinear case ($\alpha \geq 0, \beta > 0$). Under the assumption that $\alpha = 0$ and $\beta \in (0, 3/2)$, Jenssen-Karper [9] proved the global existence of a weak solution to (1.1)–(1.7). Later, for $\alpha = 0$ and $\beta \in (0, \infty)$, Pan-Zhang [19] obtain the global strong solutions. In [9, 19], they only consider the case of non-slip and heat insulated boundary conditions. Recently, for the case of stress-free and heat insulated boundary condition, Duan-Guo-Zhu [5] obtain the global strong solutions of (1.1)-(1.7) under the condition that

$$(v_0, u_0, \theta_0) \in H^1 \times H^2 \times H^2. \tag{1.9}$$

In fact, one of the main aims of this paper is to prove the existence and uniqueness of global strong solutions to (1.1)-(1.7) for $\alpha \geq 0$ and $\beta > 0$ with the conditions on the initial data:

$$(v_0, u_0, \theta_0) \in H^1,$$

which are similar as those of [14]. Then we state our main result as follows.

THEOREM 1.1. *Suppose that*

$$\alpha \geq 0, \quad \beta > 0, \tag{1.10}$$

and that the initial data (v_0, u_0, θ_0) *satisfies*

$$(v_0, u_0, \theta_0) \in H^1(0,1), \tag{1.11}$$

and

$$\inf_{x\in(0,1)} v_0(x)>0, \qquad \inf_{x\in(0,1)} \theta_0(x)>0. \tag{1.12}$$

Then, the initial-boundary-value problem (1.1)-(1.7) *has a unique strong solution* (v,u,θ) *such that for each fixed* $T>0$,

$$\begin{cases} v,u,\theta\in L^\infty(0,T;H^1(0,1)),\\ v_t\in L^\infty(0,T;L^2(0,1))\cap L^2(0,T;H^1(0,1)),\\ u_t,\theta_t,u_{xx},\theta_{xx}\in L^2((0,1)\times(0,T)), \end{cases} \tag{1.13}$$

and

$$\inf_{(x,t)\in(0,1)\times(0,T)} v(x,t)\ge C^{-1}, \qquad \inf_{(x,t)\in(0,1)\times(0,T)} \theta(x,t)\ge C^{-1}, \tag{1.14}$$

where C *is a positive constant depending only on the data and* T.

A few remarks are in order.

REMARK 1.1. Our Theorem 1.1 can be regarded as a natural generalization of the Kazhikhov-Shelukhin's result ([14]) for the constant heat conductivity case to the degenerate and nonlinear case under stress-free and thermally insulated boundary conditions.

REMARK 1.2. Our result improves Duan-Guo-Zhu's result [5] where they need that the initial data satisfy (1.9) which is stronger than (1.11).

The outline of this paper is organized as follows. Section 2 is devoted to Theorem 1.1. We first present the local existence for system (1.1)-(1.7) in Lemma 2.1. Then we obtain a series of the global a priori estimates of solutions in Lemma 2.2-Lemma 2.7. Finally, Theorem 1.1 can be proved by extending the local solutions globally in time based on Lemmas 2.1-2.7.

We now comment on the analysis of this paper. After modifying slightly the method due to Kazhikhov-Shelukhin [14], we obtain a key representation of v (see (2.1)) which can be used to obtain directly not only the lower bound of v (see (2.12)) but also a pointwise estimate between v and θ (see (2.16)). A direct consequence of this pointwise estimate between v and θ (see (2.16)) is the bound on $L^\infty(0,T;L^1(0,1))$-norm of v (see (2.17)) which plays an important role in getting the upper bound of v but cannot be obtained directly from (1.1) due to the stress-free boundary condition (1.7). Next, we multiply the momentum Equation (1.2) by $(\frac{\mu}{v}u_x-P)_x$ and make full use of the stress-free boundary condition to find that the $L^2((0,1)\times(0,T))$-norm of u_{xx} can be bounded by the $L^2((0,1)\times(0,T))$-norm of $\theta^\beta\theta_x$ (see (2.28)) which indeed can be obtained by multiplying the equation of θ (see (2.13)) by $\theta^{1+\beta}$ and using Grönwall's inequality (see (2.32)). Once we get the bounds on the $L^2((0,1)\times(0,T))$-norm of both u_{xx} and u_t (see (2.27)), the desired estimates on θ_t and θ_{xx} can be obtained by standard method (see (2.33)). The whole procedure will be carried out in the next section.

2. Proof of Theorem 1.1

We first state the following local existence result which can be proved by using the principle of compressed mappings (c.f. [12,15,21]).

LEMMA 2.1. *Let* (1.10)-(1.12) *hold. Then there exists some* $T>0$ *such that the initial-*

boundary-value problem (1.1)-(1.7) *has a unique strong solution* (v,u,θ) *satisfying*

$$\begin{cases} v,\, u,\, \theta \in L^\infty(0,T;H^1(0,1)), \\ v_t \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^1(0,1)), \\ u_t,\, \theta_t,\, v_{xt},\, u_{xx},\, \theta_{xx} \in L^2((0,1)\times(0,T)). \end{cases}$$

Then, the proof of Theorem 1.1 is based on the use of a priori estimates (see (2.15), (2.21), (2.27), and (2.33) below) in which the constants depend only on the data of the problem. The estimates make it possible to continue the local solution to the whole interval $[0,\infty)$. Without loss of generality, we assume that $\tilde\mu = \tilde\kappa = R = c_v = 1$.

Next, we derive the following representation of v which is essential in obtaining the time-depending upper and lower bounds of v.

Lemma 2.2. *We have the following expression of* v

$$v(x,t) = B_0(x) D_1(x,t) D_2(x,t) \left\{ 1 + \frac{k(\alpha)}{B_0(x)} \int_0^t \frac{\theta(x,\tau)}{D_1(x,\tau) D_2(x,\tau)} d\tau \right\}, \tag{2.1}$$

where

$$B_0(x) = \begin{cases} \exp\left(\ln v_0(x) - \frac{1}{\alpha v_0(x)^\alpha}\right), & \text{if } \alpha > 0, \\ v_0(x), & \text{if } \alpha = 0, \end{cases} \tag{2.2}$$

$$D_1(x,t) = \exp\left\{ k(\alpha) \int_0^x (u(y,t) - u_0(y))\, dy \right\}, \tag{2.3}$$

$$D_2(x,t) = \begin{cases} \exp\left\{\frac{1}{\alpha v(x,t)^\alpha}\right\}, & \text{if } \alpha > 0, \\ 1, & \text{if } \alpha = 0, \end{cases} \tag{2.4}$$

and

$$k(\alpha) = \begin{cases} 1, & \text{if } \alpha > 0, \\ 1/2, & \text{if } \alpha = 0. \end{cases} \tag{2.5}$$

Proof. First, it follows from (1.2) that

$$u_t = \left(\frac{\mu}{v} u_x - P \right)_x,$$

Integrating this over $(0,x)$ and using (1.7) gives

$$\left(\int_0^x u dy \right)_t = \frac{\mu}{v} u_x - P. \tag{2.6}$$

Then, on the one hand, if $\alpha > 0$, since $u_x = v_t$, we have

$$\left(\int_0^x u dy \right)_t = \left(\ln v - \frac{1}{\alpha v^\alpha} \right)_t - \frac{\theta}{v}. \tag{2.7}$$

Integrating (2.7) over $(0,t)$ yields

$$\ln v-\frac{1}{\alpha v^{\alpha}}-\ln v_0+\frac{1}{\alpha v_0^{\alpha}}-\int_0^t\frac{\theta}{v}d\tau=\int_0^x(u-u_0)dy,$$

which implies

$$v(x,t)=B_0(x)D_1(x,t)D_2(x,t)\exp\left\{k(\alpha)\int_0^t\frac{\theta}{v}(x,\tau)d\tau\right\},\tag{2.8}$$

with $D_1(x,t)$, $D_2(x,t)$ and $B_0(x)$ as in (2.2)-(2.4) respectively. On the other hand, if $\alpha=0$, it follows from (2.6) that

$$\left(\int_0^x udy\right)_t=2(\ln v)_t-\frac{\theta}{v}.$$

Integrating this over $(0,t)$ leads to

$$2\ln v-2\ln v_0=\int_0^x(u-u_0)dy+\int_0^t\frac{\theta}{v}d\tau,$$

which shows (2.8) still holds for $\alpha=0$. Finally, denoting

$$g(x,t)=k(\alpha)\int_0^t\frac{\theta}{v}(x,\tau)d\tau,$$

we have by (2.8)

$$g_t=\frac{k(\alpha)\theta(x,t)}{v(x,t)}=\frac{k(\alpha)\theta(x,t)}{B_0(x)D_1(x,t)D_2(x,t)\exp\{g\}},$$

which gives

$$\exp\{g\}=1+\frac{k(\alpha)}{B_0(x)}\int_0^t\frac{\theta(x,\tau)}{D_1(x,\tau)D_2(x,\tau)}d\tau.$$

Putting this into (2.8) yields (2.1) and finishes the proof of Lemma 2.2. □

With Lemma 2.2 at hand, we are in a position to prove the lower bounds of v and θ.

LEMMA 2.3. *It holds*

$$\min_{(x,t)\in[0,1]\times[0,T]}v(x,t)\ge C^{-1},\qquad \min_{(x,t)\in[0,1]\times[0,T]}\theta(x,t)\ge C^{-1},\tag{2.9}$$

where (and in what follows) C denotes generic positive constant depending only on $\beta,\alpha,T,\|(v_0,u_0,\theta_0)\|_{H^1(0,1)}$, $\inf_{x\in(0,1)}v_0(x)$, and $\inf_{x\in(0,1)}\theta_0(x)$.

Proof. First, integrating (1.3) over $(0,1)$ and using (1.7) immediately leads to

$$\int_0^1\left(\theta+\frac{1}{2}u^2\right)(x,t)dx=\int_0^1\left(\theta+\frac{1}{2}u^2\right)(x,0)dx,\tag{2.10}$$

which in particular gives

$$\left|\int_0^x udy\right|\le\int_0^1|u|dy\le\left(\int_0^1u^2dy\right)^{\frac{1}{2}}\le C.$$

Combining this with (2.3) implies

$$C^{-1}\le D_1(x,t)\le C, \tag{2.11}$$

which together with (2.1) yields that for any $(x,t)\in[0,1]\times[0,T]$,

$$v(x,t)\ge B_0(x)D_1(x,t)D_2(x,t)\ge C^{-1}, \tag{2.12}$$

due to

$$D_2(x,t)\ge 1,\quad C^{-1}\le B_0(x)\le C.$$

Finally, we rewrite (1.3) as

$$\theta_t+\frac{\theta}{v}u_x=\left(\frac{\theta^\beta\theta_x}{v}\right)_x+\frac{\mu u_x^2}{v}. \tag{2.13}$$

For $r>2$, multiplying the above equality by θ^{-r} and integrating the resultant equality over $(0,1)$ yields that

$$\begin{aligned}
&\frac{1}{r-1}\frac{d}{dt}\int_0^1\left(\theta^{-1}\right)^{r-1}dx+\int_0^1\frac{\mu u_x^2}{v\theta^r}dx+r\int_0^1\frac{\theta^\beta\theta_x^2}{v\theta^{r+1}}dx\\
&=\int_0^1\frac{u_x}{v\theta^{r-1}}dx\\
&\le\frac{1}{2}\int_0^1\frac{\mu u_x^2}{v\theta^r}dx+\frac{1}{2}\int_0^1\frac{1}{\mu v\theta^{r-2}}dx\\
&\le\frac{1}{2}\int_0^1\frac{\mu u_x^2}{v\theta^r}dx+C\left\|\theta^{-1}\right\|_{L^{r-1}}^{r-2},
\end{aligned} \tag{2.14}$$

where in the second inequality we have used $\mu v=v+v^{1-\alpha}>v\ge C^{-1}$. Combining (2.14) with Grönwall's inequality yields

$$\sup_{0\le t\le T}\left\|\theta^{-1}(\cdot,t)\right\|_{L^{r-1}}\le C,$$

with C independent of r. Letting $r\to\infty$ proves the second inequality of (2.9). Thus, the proof of Lemma 2.3 is finished. □

Lemma 2.4. *There exists a positive constant C such that for each $(x,t)\in[0,1]\times[0,T]$,*

$$C^{-1}\le v(x,t)\le C. \tag{2.15}$$

Proof. First, it follows from (2.9) and (2.4) that for any $(x,t)\in[0,1]\times[0,T]$,

$$1\le D_2(x,t)\le C,$$

which together with (2.1) and (2.11) yields that for any $(x,t)\in[0,1]\times[0,T]$,

$$C^{-1}\le v(x,t)\le C+C\int_0^t\theta(x,\tau)d\tau. \tag{2.16}$$

Integrating this with respect to x over $(0,1)$ and using (2.10) leads to

$$\sup_{0\le t\le T}\int_0^1 v(x,t)dx\le C. \tag{2.17}$$

Next, for $\eta\in(0,\frac{1}{2})$ and $\varepsilon\in(0,1)$, integrating (2.13) multiplied by $\theta^{-\eta}$ over $(0,1)\times(0,T)$, we get by (2.10) and (2.9)

$$\begin{aligned}
&\int_0^T\int_0^1 \frac{\eta\theta^\beta\theta_x^2}{v\theta^{\eta+1}}dxdt+\int_0^T\int_0^1\frac{\mu u_x^2}{v\theta^\eta}dxdt\\
&=\frac{1}{1-\eta}\int_0^1\theta^{1-\eta}dx-\frac{1}{1-\eta}\int_0^1\theta_0^{1-\eta}dx+\int_0^T\int_0^1\frac{u_x}{\theta^{\eta-1}v}dxdt\\
&\le C(\eta)+\frac{1}{2}\int_0^T\int_0^1\frac{\mu u_x^2}{v\theta^\eta}dxdt+C\int_0^T\int_0^1\frac{1}{\mu v\theta^{\eta-2}}dxdt\\
&\le C(\eta)+\frac{1}{2}\int_0^T\int_0^1\frac{\mu u_x^2}{v\theta^\eta}dxdt+C\int_0^T\max_{x\in[0,1]}\theta^{1-\eta}dt\\
&\le C(\eta,\varepsilon)+\frac{1}{2}\int_0^T\int_0^1\frac{\mu u_x^2}{v\theta^\eta}dxdt+\varepsilon\int_0^T\max_{x\in[0,1]}\theta dt,
\end{aligned} \tag{2.18}$$

where in the first inequality we have used

$$\int_0^1\theta^{1-\eta}dx\le C.$$

Finally, using (2.17), we obtain that for $\eta=\min\{1,\beta\}/2$,

$$\begin{aligned}
\int_0^T\max_{x\in[0,1]}\theta dt&\le C+C\int_0^T\int_0^1|\theta_x|dxdt\\
&\le C+C\int_0^T\int_0^1\frac{\theta^\beta\theta_x^2}{v\theta^{1+\eta}}dxdt+C\int_0^T\int_0^1\frac{v\theta^{1+\eta}}{\theta^\beta}dxdt\\
&\le C+C\int_0^T\int_0^1\frac{\theta^\beta\theta_x^2}{v\theta^{1+\eta}}dxdt+C\int_0^T\max_{x\in[0,1]}\theta^{1+\eta-\beta}dt\\
&\le C+C\int_0^T\int_0^1\frac{\theta^\beta\theta_x^2}{v\theta^{1+\eta}}dxdt+\frac{1}{2}\int_0^T\max_{x\in[0,1]}\theta dt,
\end{aligned}$$

which together with (2.18) yields that

$$\int_0^T\max_{x\in[0,1]}\theta dt\le C, \tag{2.19}$$

and that for $\eta\in(0,1)$

$$\int_0^T\int_0^1\theta^{\beta-1-\eta}\theta_x^2dxdt\le C(\eta). \tag{2.20}$$

Combining (2.16) with (2.19) finishes the proof of Lemma 2.4. □

LEMMA 2.5. *There exists a positive constant C such that*

$$\sup_{0\le t\le T}\int_0^1 v_x^2dx\le C. \tag{2.21}$$

Proof. First, we rewrite (1.2) as

$$\left(\frac{\mu v_x}{v}-u\right)_t=\left(\frac{\theta}{v}\right)_x.$$

Multiplying the above equality by $\frac{\mu v_x}{v}-u$ and integrating it over $(0,1)\times(0,T)$ gives

$$\begin{aligned}&\frac{1}{2}\int_0^1\left(\frac{\mu v_x}{v}-u\right)^2dx-\frac{1}{2}\int_0^1\left(\frac{\mu v_x}{v}-u\right)^2(x,0)dx+\int_0^T\int_0^1\frac{\mu\theta v_x^2}{v^3}dxdt\\&=\int_0^T\int_0^1\frac{\theta v_x u}{v^2}dxdt+\int_0^T\int_0^1\frac{\theta_x}{v}\left(\frac{\mu v_x}{v}-u\right)dxdt\triangleq I_1+I_2.\end{aligned}\tag{2.22}$$

Then, on the one hand, Cauchy's inequality, (2.10), (2.15), and (2.19) lead to

$$\begin{aligned}|I_1|&\le\frac{1}{2}\int_0^T\int_0^1\frac{\theta u^2}{v\mu}dxdt+\frac{1}{2}\int_0^T\int_0^1\frac{\mu\theta v_x^2}{v^3}dxdt\\&\le C\int_0^T\max_{x\in[0,1]}\theta dt+\frac{1}{2}\int_0^T\int_0^1\frac{\mu\theta v_x^2}{v^3}dxdt\\&\le C+\frac{1}{2}\int_0^T\int_0^1\frac{\mu\theta v_x^2}{v^3}dxdt.\end{aligned}\tag{2.23}$$

On the other hand, we deduce from (2.15), (2.20), and (2.9) that for $\eta=\min\{1,\beta\}/2$,

$$\begin{aligned}|I_2|&\le C\int_0^T\int_0^1\theta^{\beta-1-\eta}\theta_x^2dxdt+C\int_0^T\int_0^1\theta^{1+\eta-\beta}\left(\frac{\mu v_x}{v}-u\right)^2dxdt\\&\le C+C\int_0^T\max_{x\in[0,1]}\theta^2\int_0^1\left(\frac{\mu v_x}{v}-u\right)^2dxdt.\end{aligned}\tag{2.24}$$

Next, it follows from (2.10) and (2.15) that for $\eta=\min\{1,\beta\}/2$,

$$\begin{aligned}\int_0^T\max_{x\in[0,1]}(\theta^{1+\beta}+\theta^2)dt&\le C\int_0^T\max_{x\in[0,1]}\theta^{2+\beta-\eta}dt\\&\le C+C\int_0^T\left(\max_{x\in[0,1]}\left|\theta^{\frac{2+\beta-\eta}{2}}(x,t)-(\int_0^1\theta dx)^{\frac{2+\beta-\eta}{2}}\right|\right)^2dt\\&\le C+C\int_0^T\left(\int_0^1\theta^{\frac{\beta-\eta}{2}}|\theta_x|dx\right)^2dt\\&\le C+C\int_0^T\left(\int_0^1\frac{\theta^\beta\theta_x^2}{v\theta^{\eta+1}}dx\right)\left(\int_0^1 v\theta dx\right)dt\\&\le C+C\int_0^T\int_0^1\frac{\theta^\beta\theta_x^2}{v\theta^{\eta+1}}dxdt\\&\le C,\end{aligned}\tag{2.25}$$

where in the last inequality we have used (2.20). Finally, adding (2.23) and (2.24) to (2.22), we obtain after using Grönwall's inequality and (2.25) that

$$\sup_{0\le t\le T}\int_0^1\left(\frac{\mu v_x}{v}-u\right)^2dx+\int_0^T\int_0^1\frac{\mu\theta v_x^2}{v^3}dxdt\le C,$$

which together with (2.10) and (2.15) gives

$$\sup_{0\le t\le T}\int_0^1 v_x^2dx \le C+C\sup_{0\le t\le T}\int_0^1\left(\frac{\mu v_x}{v}-u\right)^2dx \\ \le C. \tag{2.26}$$

The proof of Lemma 2.5 is finished. □

LEMMA 2.6. *There exists a positive constant C such that*

$$\sup_{0\le t\le T}\int_0^1 u_x^2dx+\int_0^T\int_0^1(u_t^2+u_{xx}^2)dxdt\le C. \tag{2.27}$$

Proof. First, integrating (1.2) multiplied by $(\frac{\mu}{v}u_x-P)_x$ over $(0,1)$, we obtain by integration by parts and (2.13) that

$$\begin{aligned}
&\int_0^1\left(\frac{\mu}{v}u_x-P\right)_x^2dx\\
&=-\int_0^1\left(\frac{\mu}{v}u_x-P\right)u_{tx}dx\\
&=-\frac12\int_0^1\frac{\mu}{v}\left(u_x^2\right)_tdx+\left(\int_0^1Pu_xdx\right)_t-\int_0^1P_tu_xdx\\
&=-\frac12\left(\int_0^1\frac{\mu}{v}u_x^2dx\right)_t+\frac12\int_0^1\left(\frac{\mu}{v}\right)_v'u_x^3dx+\left(\int_0^1Pu_xdx\right)_t\\
&\quad+2\int_0^1\frac{\theta}{v^2}u_x^2dx+\int_0^1\frac{\theta^\beta\theta_x}{v}\left(\frac{u_x}{v}\right)_xdx-\int_0^1\frac{\mu u_x^3}{v^2}dx,
\end{aligned}$$

which in particular gives

$$\begin{aligned}
&\left(\int_0^1\left(\frac{\mu}{2v}u_x^2-Pu_x\right)dx\right)_t+\int_0^1\left(\frac{\mu^2}{v^2}u_{xx}^2+\frac{\theta_x^2}{v^2}\right)dx\\
&\le C\int_0^1|u_{xx}|\left(|v_x||u_x|+|v_x|\theta+\theta^\beta|\theta_x|\right)dx+C\int_0^1|\theta_x||v_x|\theta dx\\
&\quad+C\int_0^1|\theta_x|\theta^\beta|u_x||v_x|dx+C\int_0^1v_x^2\left(u_x^2+\theta^2\right)dx+C\int_0^1\left(|u_x|^3+u_x^2\theta\right)dx\\
&\le\frac14\int_0^1\frac{\mu^2}{v^2}u_{xx}^2dx+C\int_0^1\theta^{2\beta}\theta_x^2dx+C\left(\int_0^1u_x^2dx\right)^2\\
&\quad+C\max_{x\in[0,1]}\left(u_x^2+\theta^2\right)\left(1+\int_0^1v_x^2dx+\int_0^1\theta dx\right)\\
&\le\frac12\int_0^1\frac{\mu^2}{v^2}u_{xx}^2dx+C_1\int_0^1\frac{\theta^{2\beta}\theta_x^2}{v}dx+C\max_{x\in[0,1]}\theta^2+C\left(\int_0^1u_x^2dx\right)^2+C,
\end{aligned} \tag{2.28}$$

where in the last inequality we have used (2.26) and

$$\max_{x\in[0,1]}u_x^2\le C(\varepsilon)\int_0^1u_x^2dx+\varepsilon\int_0^1u_{xx}^2dx, \tag{2.29}$$

for $\varepsilon>0$ small enough. Then, integrating (2.13) over $(0,1)\times(0,T)$ yields that

$$\begin{aligned}\int_0^T\int_0^1\frac{\mu u_x^2}{v}dxdt&=\int_0^1\theta dx-\int_0^1\theta_0dx+\int_0^T\int_0^1\frac{\theta}{v}u_xdxdt\\&\le C+\frac{1}{2}\int_0^T\int_0^1\frac{\mu u_x^2}{v}dxdt+C\int_0^T\int_0^1\frac{\theta^2}{\mu v}dxdt\\&\le C+\frac{1}{2}\int_0^T\int_0^1\frac{\mu u_x^2}{v}dxdt+C\int_0^T\max_{x\in[0,1]}\theta(x,t)dt\end{aligned}$$

which together with (2.19) and (2.15) gives

$$\int_0^T\int_0^1 u_x^2dxdt\le C. \tag{2.30}$$

Next, integrating (2.13) multiplied by $\theta^{1+\beta}$ over $(0,1)$ leads to

$$\begin{aligned}&\frac{1}{2+\beta}(\int_0^1\theta^{2+\beta}dx)_t+(1+\beta)\int_0^1\frac{\theta^{2\beta}\theta_x^2}{v}dx\\&=-\int_0^1\frac{\theta^{2+\beta}u_x}{v}dx+\int_0^1\frac{\mu\theta^{1+\beta}u_x^2}{v}dx\\&\le C\int_0^1\theta^{3+\beta}dx+C\int_0^1\theta^{1+\beta}u_x^2dx\\&\le C\max_{x\in[0,1]}\theta\int_0^1\theta^{2+\beta}dx+C\max_{x\in[0,1]}\theta^{1+\beta}\int_0^1u_x^2dx.\end{aligned} \tag{2.31}$$

Choosing $C_2\ge C_1+1$ suitably large such that

$$C_2\theta^{2+\beta}+\mu v^{-1}u_x^2\ge 4(2+\beta)\theta v^{-1}|u_x|,$$

adding (2.31) multiplied by C_2 to (2.28), and choosing ε sufficiently small, we obtain from Grönwall's inequality, (2.30), and (2.25) that

$$\sup_{0\le t\le T}\int_0^1\left(\theta^{2+\beta}+u_x^2\right)dx+\int_0^T\int_0^1u_{xx}^2dxdt+\int_0^T\int_0^1\theta^{2\beta}\theta_x^2dxdt\le C. \tag{2.32}$$

Finally, rewriting (1.2) as

$$u_t=\frac{\mu u_{xx}}{v}+\left(\frac{\mu}{v}\right)_v'v_xu_x-\frac{\theta_x}{v}+\frac{\theta v_x}{v^2},$$

we deduce from (2.15), (2.32), (2.26), (2.30), (2.29), and (2.25) that

$$\begin{aligned}\int_0^T\int_0^1u_t^2dxdt&\le C\int_0^T\int_0^1\left(u_{xx}^2+u_x^2v_x^2+\theta_x^2+\theta^2v_x^2\right)dxdt\\&\le C+C\int_0^T\max_{x\in[0,1]}\theta^2dt\\&\le C,\end{aligned}$$

which together with (2.32) finishes the proof of Lemma 2.6. □

LEMMA 2.7. *There exists a positive constant C such that*

$$\sup_{0\le t\le T}\int_0^1 \theta_x^2dx+\int_0^T\int_0^1\left(\theta_t^2+\theta_{xx}^2\right)dxdt\le C. \tag{2.33}$$

Proof. First, multiplying (2.13) by $\theta^\beta\theta_t$ and integrating the resultant equality over (0,1) yields

$$\begin{aligned}&\int_0^1\theta^\beta\theta_t^2dx+\int_0^1\frac{\theta^{\beta+1}u_x\theta_t}{v}dx\\ &=-\int_0^1\left(\frac{\theta^\beta\theta_x}{v}\right)\left(\theta^\beta\theta_x\right)_tdx+\int_0^1\frac{\mu u_x^2}{v}\theta^\beta\theta_tdx\\ &=-\frac{1}{2}\frac{d}{dt}\int_0^1\frac{(\theta^\beta\theta_x)^2}{v}dx+\frac{1}{2}\int_0^1\left(\theta^\beta\theta_x\right)^2\left(\frac{1}{v}\right)_tdx+\int_0^1\frac{\mu u_x^2\theta^\beta\theta_t}{v}dx\\ &=-\frac{1}{2}\frac{d}{dt}\int_0^1\frac{(\theta^\beta\theta_x)^2}{v}dx-\frac{1}{2}\int_0^1\frac{(\theta^\beta\theta_x)^2u_x}{v^2}dx+\int_0^1\frac{\mu u_x^2\theta^\beta\theta_t}{v}dx,\end{aligned}$$

which, combined with the Hölder inequality, (2.21), and (2.32), leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\frac{\left(\theta^\beta\theta_x\right)^2}{v}dx+\int_0^1\theta^\beta\theta_t^2dx\\ &=-\frac{1}{2}\int_0^1\frac{\left(\theta^\beta\theta_x\right)^2u_x}{v^2}dx+\int_0^1\frac{\mu u_x^2\theta^\beta\theta_t}{v}dx-\int_0^1\frac{\theta^{\beta+1}u_x\theta_t}{v}dx\\ &\le C\int_0^1\theta^{2\beta}\theta_x^2|u_x|dx+\frac{1}{2}\int_0^1\theta^\beta\theta_t^2dx+C\int_0^1u_x^4\theta^\beta dx+C\int_0^1\theta^{\beta+2}u_x^2dx\\ &\le C\max_{x\in[0,1]}|u_x|\int_0^1\theta^{2\beta}\theta_x^2dx+\frac{1}{2}\int_0^1\theta^\beta\theta_t^2dx+C\max_{x\in[0,1]}\left(u_x^2\theta^\beta+\theta^{\beta+2}\right)\\ &\le\frac{1}{2}\int_0^1\theta^\beta\theta_t^2dx+C\Big(\int_0^1\theta^{2\beta}\theta_x^2dx\Big)^2+C\max_{x\in[0,1]}\left(u_x^4+\theta^{2\beta+2}\right)+C.\end{aligned} \tag{2.34}$$

It follows from (2.21), (2.29), (2.27), and the Hölder inequality that

$$\begin{aligned}\int_0^T\max_{x\in[0,1]}u_x^4dt&\le C\int_0^T\int_0^1u_x^4dxdt+C\int_0^T\int_0^1|u_x^3u_{xx}|dxdt\\ &\le C\int_0^T\max_{x\in[0,1]}u_x^2\int_0^1u_x^2dxdt\\ &\quad+C\int_0^T\max_{x\in[0,1]}u_x^2\left(\int_0^1u_x^2dx\right)^{\frac{1}{2}}\left(\int_0^1u_{xx}^2dx\right)^{\frac{1}{2}}dt\\ &\le C\int_0^T\int_0^1\left(u_x^2+u_{xx}^2\right)dxdt+\frac{1}{2}\int_0^T\max_{x\in[0,1]}u_x^4dt\\ &\le C+\frac{1}{2}\int_0^T\max_{x\in[0,1]}u_x^4dt\end{aligned} \tag{2.35}$$

which, combined with (2.34), (2.32), and the Grönwall inequality, yields

$$\sup_{0\le t\le T}\int_0^1\left(\theta^\beta\theta_x\right)^2dx+\int_0^T\int_0^1\theta^\beta\theta_t^2dxdt\le C, \tag{2.36}$$

where we have used

$$\begin{aligned}\max_{x\in[0,1]}\theta^{2\beta+2}&\le C+C\left(\max_{x\in[0,1]}|\theta^{\beta+1}(x,t)-(\int_0^1\theta dx)^{\beta+1}|\right)^2\\&\le C+C\left(\int_0^1|\theta^\beta\theta_x|dx\right)^2\\&\le C+C\int_0^1(\theta^\beta\theta_x)^2dx.\end{aligned}\tag{2.37}$$

Combining (2.36) with (2.37) implies for all $(x,t)\in(0,1)\times(0,T)$

$$\theta(x,t)\le C.\tag{2.38}$$

Meanwhile, both (2.36) and (2.9) lead to

$$\sup_{0\le t\le T}\int_0^1\theta_x^2dx+\int_0^T\int_0^1\theta_t^2dxdt\le C.\tag{2.39}$$

Finally, it follows from (2.13) that

$$\frac{\theta^\beta\theta_{xx}}{v}=-\frac{\beta\theta^{\beta-1}\theta_x^2}{v}+\frac{\theta^\beta\theta_xv_x}{v^2}-\frac{\mu u_x^2}{v}+\frac{\theta u_x}{v}+\theta_t,$$

which together with (2.39), (2.9), (2.38), (2.35), and (2.21) gives

$$\begin{aligned}\int_0^T\int_0^1\theta_{xx}^2dxdt&\le C\int_0^T\int_0^1\left(\theta_x^4+v_x^2\theta_x^2+u_x^4+\theta^2u_x^2+\theta_t^2\right)dxdt\\&\le C+C\int_0^T\max_{x\in[0,1]}\theta_x^2dt\\&\le C+C\int_0^T\int_0^1\theta_x^2dxdt+\frac{1}{2}\int_0^T\int_0^1\theta_{xx}^2dxdt.\end{aligned}$$

Combining this with (2.39) gives (2.33) and finishes the proof of Lemma 2.7. □

REFERENCES

[1] A.A. Amosov and A.A. Zlotnik, *Global generalized solutions of the equations of the one-dimensional motion of a viscous heat-conducting gas*, Soviet Math. Dokl., 38:1–5, 1989. 1

[2] A.A. Amosov and A.A. Zlotnik, *Solvability "in the large" of a system of equations of the one-dimensional motion of an inhomogeneous viscous heat-conducting gas*, Math. Notes, 52:753–763, 1992. 1

[3] S.N. Antontsev, A.V. Kazhikhov, and V.N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, North-Holland, Amsterdam, 1990. 1

[4] G.K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, London, 1967. 1

[5] R. Duan, A. Guo, and C.J. Zhu, *Global strong solution to compressible Navier-Stokes equations with density dependent viscosity and temperature dependent heat conductivity*, J. Diff. Eqs., 262:4314–4335, 2017. 1, 1.2

[6] L. Hsiao, *Quasilinear Hyperbolic Systems and Dissipative Mechanisms*, World Scientific Publishing Co., 1998. 1

[7] L. Hsiao and S. Jiang, *Nonlinear hyperbolic-parabolic coupled systems*, in C. Dafermos and E. Feireisl (eds.), Handbook of Differential Equations: Evolutionary Equations, Elsevier, 1:287–384, 2004. 1

[8] B. Huang and X.D. Shi, *Nonlinearly exponential stability of compressible Navier-Stokes system with degenerate heat-conductivity*, J. Diff. Eqs., 268(5):2464-2490. 1

[9] H.K. Jenssen and T.K. Karper, *One-dimensional compressible flow with temperature dependent transport coefficients*, SIAM J. Math. Anal., 42:904–930, 2010. 1

[10] S. Jiang, *On initial-boundary value problems for a viscous, heat-conducting, one-dimensional real gas*, J. Diff. Eqs., 110:157–181, 1994. 1

[11] S. Jiang, *On the asymptotic behavior of the motion of a viscous, heat-conducting, one-dimensional real gas*, Math. Z., 216:317–336, 1994. 1

[12] S. Kawashima and T. Nishida, *Global solutions to the initial value problem for the equations of one dimensional motion of viscous polytropic gases*, J. Math. Kyoto Univ., 21:825–837, 1981. 2

[13] B. Kawohl, *Global existence of large solutions to initial-boundary value problems for a viscous, heat-conducting, one-dimensional real gas*, J. Diff. Eqs., 58:76–103, 1985. 1

[14] A.V. Kazhikhov and V.V. Shelukhin, *Unique global solution with respect to time of initial boundary value problems for one-dimensional equations of a viscous gas*, J. Appl. Math. Mech., 41:273–282, 1977. 1, 1, 1.1, 1

[15] J. Nash, *Le problème de Cauchy pour les équations différentielles d'un fluide général*, Bull. Soc. Math. France, 90:487–497, 1962. 2

[16] T. Nagasawa, *On the asymptotic behavior of the one-dimensional motion of the polytropic ideal gas with stress-free condition*, Quart. Appl. Math., 46:665–679, 1988. 1

[17] T. Nagasawa, *On the one-dimensional motion of the polytropic ideal gas non-fixed on the boundary*, J. Diff. Eqs., 65:49–67, 1986. 1

[18] T. Nishida, *Equations of motion of compressible viscous fluids*, in T. Nishida, M. Mimura, and H. Fujii, (eds.), Patterns and Waves, Kinokuniya/North-Holland, Amsterdam, 97–128, 1986. 1

[19] R.H. Pan and W.Z. Zhang, *Compressible Navier-Stokes equations with temperature dependent heat conductivities*, Commun. Math. Sci., 13:401–425, 2015. 1

[20] J. Serrin, *Mathematical principles of classical fluid mechanics*, in C. Truesdell (Eds.), Fluid Dynamics I/Strömungsmechanik I, Encyclopedia of Physics/Handbuch der Physik, Springer, Berlin, 125–263, 1972. 1

[21] A. Tani, *On the first initial-boundary value problem of compressible viscous fluid motion*, Publ. Res. Inst. Math. Sci., 13:193–253, 1977. 2

[22] D.H. Wang, *Global solutions of the Navier-Stokes equations for viscous compressible flows*, Nonlinear Anal., 52:1867–1890, 2003. 1