

## VANISHING VISCOSITY LIMIT FOR VISCOUS BURGERS-VLASOV EQUATIONS\*

WENTAO CAO<sup>†</sup> AND TENG WANG<sup>‡</sup>

**Abstract.** We establish the vanishing viscosity limit of viscous Burgers-Vlasov equations for one dimensional kinetic model about interactions between a viscous fluid and dispersed particles by using compensated compactness technique and the evolution of level sets arguments. The limit we obtained is exactly a finite-energy weak solution to the inviscid equations.

**Keywords.** vanishing viscosity limit; two phase flow; Vlasov equation; Burgers equation; finite-energy weak solution.

**AMS subject classifications.** 76T10; 35F20; 35Q35; 35Q72; 45K05; 82D05.

### 1. Introduction

In this note we consider the vanishing viscosity limit of the following viscous Burgers-Vlasov equations:

$$\begin{cases} u_t + uu_x = \varepsilon u_{xx} + \int_{\mathbb{R}} fvdv - u \int_{\mathbb{R}} fdv, \\ f_t + vf_x + (f(u-v))_v = 0, \end{cases} \quad (1.1)$$

with the initial data

$$u(x,0) = u_0(x), \quad f(x,v,0) = f_0(x,v), \quad (1.2)$$

such that

$$\lim_{x \rightarrow \pm\infty} u_0(x) = u^\pm, \quad \lim_{x,v \rightarrow \pm\infty} f_0(x,v) = 0, \quad (1.3)$$

with  $u^\pm$  being constant states and allowed to be different, here  $u(x,t)$  is the bulk velocity of the viscous gas at position  $x \in \mathbb{R}$  and time  $t \geq 0$ .  $f(x,v,t)$  is the distribution function of the particles occupying at time  $t$ , the position  $x$  with velocity  $v \in \mathbb{R}$ .  $\varepsilon \in (0, \varepsilon_0)$  with some  $0 < \varepsilon_0 < 1$  is the viscosity of the gas.

The system (1.1) is related to a kinetic model of a two-phase flow in which a dispersed phase interacts with a kind of viscous gas. Such model arises in the description of various combustion phenomena, e.g. diesel engines. The model reads

$$\begin{cases} \rho_g(u_t + uu_x - \varepsilon u_{xx}) = E_d, \\ f_t + vf_x + (F_d f)_v = 0, \end{cases} \quad (1.4)$$

here  $\rho_g$  is the density of the gas. The force term  $E_d$  describing the exchange of impulse between the gas and the particles has a close relation with the drag force  $F_d$  describing

---

\*Received: April 18, 2019; Accepted (in revised form): January 21, 2020. Communicated by Grigorios A. Pavliotis.

<sup>†</sup>Institute für Mathematik, Universität Leipzig, D-04109, Leipzig, Germany ([wentao.cao@math.uni-leipzig.de](mailto:wentao.cao@math.uni-leipzig.de)).

<sup>‡</sup>Corresponding author. Faculty of Science, School of Mathematics, Beijing University of Technology, Beijing 100124, China ([tengwang@amss.ac.cn](mailto:tengwang@amss.ac.cn)).

the friction of the viscous fluid in the droplets. The relation can be seen from the following formulas:

$$\begin{aligned}
 E_d &= \mathcal{C}(r)\rho_p(u_p - u), \quad F_d = \mathcal{C}(r)(u(x, t) - v), \\
 \rho_p &= \frac{4\pi}{3}\rho_l r^3 \int_{\mathbb{R}} f(x, v, t)dv, \quad \rho_p u_p = \frac{4\pi}{3}\rho_l r^3 \int_{\mathbb{R}} f(x, v, t)v dv.
 \end{aligned}
 \tag{1.5}$$

In (1.5),  $\rho_l$  is the density of liquid.  $\mathcal{C}(r)$  is a constant depending on the radius  $r$  of the droplets. In (1.4), the viscous Burgers' equation, i.e. the first equation models the evolution of viscous gas, while the Vlasov-like equation, i.e. the second equation describes the evolution of the dispersed phase. Derivation of the model can be found in [21]. Further information about our assumptions on (1.4) can also be found in [5, 7]. We remark that when all the constants in  $E_d/\rho_g$  and  $F_d$  are all assumed to be 1, then (1.4) becomes (1.1).

As for the well-posedness results of (1.1) or (1.4), global existence and uniqueness of classical solutions to the Cauchy problem with regular initial data have been considered in [5]. Global existence of weak solutions with finite energy is studied in [7]. Other complicated models of interactions between fluid-kinetic models, such as incompressible/compressible Euler/Navier-Stokes equations coupled with Vlasov/Vlasov-Fokker-Planck equation are studied in [15, 17, 18, 23] and references therein. Asymptotic problems like hydrodynamic limit and stratified limit of (1.4) are also considered in [7], and one can see [2, 8–11, 14, 16] for asymptotic problems of other models related to Vlasov equations. For the vanishing viscosity limit of Navier-Stokes equations,  $L^p$  compensated compactness framework of  $2 \times 2$  system of conservation laws is applied to yield the result in [4], which is also used in [6].

Our goal is to show that when  $\varepsilon \rightarrow 0$ , smooth solutions to (1.1)-(1.2) converge to a finite-energy weak solution to the following zero-viscosity equations:

$$\begin{cases} u_t + uu_x = \int_{\mathbb{R}} fvdv - u \int_{\mathbb{R}} f dv, \\ f_t + vf_x + (f(u - v))_v = 0. \end{cases}
 \tag{1.6}$$

The relative total energy for (1.6) is denoted as

$$E[u, f] := \frac{1}{2} \int_{\mathbb{R}} (u - \bar{u})^2 dx + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} f(1 + v^2)dv dx,$$

where a smooth monotone function  $\bar{u}(x)$  is constructed as

$$\bar{u} = \begin{cases} u^+, & x \geq L_0; \\ \text{monotone}, & -L_0 < x < L_0; \\ u^-, & x \leq -L_0 \end{cases}
 \tag{1.7}$$

with  $L_0 > 0$  large. As we can see from the formula, the relative total energy is the sum of the kinetic energy of the fluid and the particle (in the statistical sense). In the paper we denote  $[0, \infty)$  as  $\mathbb{R}_+$ ,  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ . Finite-energy weak solutions to (1.6) are defined as follows.

**DEFINITION 1.1.** *Let  $(u_0, f_0)$  be given initial data with relative finite energy with respect to the end-states  $(u^\pm, 0)$  at infinity, i.e.  $E[u_0, f_0] \leq E_0 < \infty$ . For any  $T \in \mathbb{R}_+$ , a pair of functions  $u: \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ ,  $f: \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}_+$  is called a finite-energy weak solution of Cauchy problem (1.6) and (1.2)-(1.3) if the following holds:*

(1) *There is a bounded function  $C(E_0, t)$  defined on  $\mathbb{R}_+ \times [0, T]$ , which is continuous in  $t$  for each  $E \in \mathbb{R}_+$  such that for a.e.  $t > 0$ ,*

$$E[u, f](t) + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f(u-v)^2 dv dx ds \leq C(E_0, t). \tag{1.8}$$

(2) *For any  $\phi \in C_c^1(\mathbb{R} \times [0, T])$ ,*

$$\int_{\mathbb{R}} \phi(x, 0) u_0(x) dx + \int_0^T \int_{\mathbb{R}} \left( u \phi_t + \frac{1}{2} u^2 \phi_x + \phi \int_{\mathbb{R}} f(v-u) dv \right) dx dt = 0, \tag{1.9}$$

*and for any  $\varphi \in C_c^1(\mathbb{R}^2 \times [0, T])$ ,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x, v, 0) f_0(x, v) dx dv + \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} f \varphi_t + f v \varphi_x + f(u-v) \varphi_v dv dx dt = 0. \tag{1.10}$$

(3) *The initial data is achieved in the sense of distributions.*

In fact, our general idea on the definition of finite-energy weak solution is that under the condition that the initial total energy is finite the desired solution to (1.6) shall also enjoy the finite-energy property, i.e. (1.8) and satisfies the Equations (1.6) in weak sense, i.e. (1.9) and (1.10).

Now we are ready to state our main result.

**THEOREM 1.1.** *Let the initial smooth functions  $(u_0^\varepsilon, f_0^\varepsilon)$  satisfy the following conditions:*

- (i) *There exists  $E_0 > 0$  such that  $E[u_0^\varepsilon, f_0^\varepsilon] \leq E_0 < \infty$ .*
- (ii) *It holds that*

$$(u_0^\varepsilon(x), f_0^\varepsilon(x, v)) \rightarrow (u_0(x), f_0(x, v)) \text{ as } \varepsilon \rightarrow 0$$

*in the sense of distributions with  $f_0 \geq 0$ .*

*Let  $(u^\varepsilon, f^\varepsilon)$  be the solution to the Cauchy problem (1.1) with initial data  $(u_0^\varepsilon, f_0^\varepsilon)$  for any fixed  $\varepsilon > 0$ . Then when  $\varepsilon \rightarrow 0$ , there exists  $(u, f)$  with  $u(x, t) \in L^4_{loc}(\mathbb{R} \times [0, T])$  and  $f(x, v, t) \in L^\infty([0, T], (1+v^2)L^1(\mathbb{R}^2))$ , which is a finite-energy weak solution to the Cauchy problem (1.6) and (1.2)-(1.3) in the sense of Definition 1.1, along with corresponding subsequences of  $u^\varepsilon$  and  $f^\varepsilon$  (still denoting as  $(u^\varepsilon, f^\varepsilon)$ ) such that*

$$\begin{aligned} u^\varepsilon &\rightarrow u \text{ strongly in } L^r_{loc}(\mathbb{R} \times [0, T]), \text{ for } 1 \leq r \leq 4, \text{ as } \varepsilon \rightarrow 0, \\ f^\varepsilon &\rightharpoonup f \text{ weakly in } L^\infty([0, T], L^1(\mathbb{R}^2)), \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Our strategy of proving Theorem 1.1 is to apply  $L^p$  compactness framework for scalar conservation laws and study the evolution of level sets after obtaining uniform basic energy estimate and uniform  $L^4_{loc}$  estimate. Regarding the  $L^p$  compactness framework for scalar conservation laws, it is first used in [19] and then in [12] with some improvement. The compactness framework is also generalized for more models in [22]. In the present paper, our key difficulty is the estimate of  $L^4_{loc}$  boundedness of  $u^\varepsilon$ , which is obtained by making full use of the flux term of Burgers' equation. To show the  $L^1$  weak convergence of  $f^\varepsilon$ , our technique is studying the evolution of level sets, which is also utilized to handle the convergence of approximate solutions to Vlasov-Poisson equations in [1]. For our case the novel idea is estimating the level sets through characteristic map and our key observation is that the Jacobian of the characteristic map is uniformly bounded as time grows.

In the present paper, we denote  $f = f_{\mathbb{R}}$ .  $C$  is a constant independent of  $\varepsilon$  but may vary from line to line,  $C(\cdot)$  denotes a constant depending on the parameters in the bracket. The rest of the paper is organised as follows. Section 2 is devoted to show the uniform estimates and the proof of Theorem 1.1 is provided in Section 3.

**2. Uniform estimates**

Consider the Cauchy problem (1.1) with initial conditions

$$u^\varepsilon(x, 0) = u_0^\varepsilon(x), \quad f^\varepsilon(x, v, 0) = f_0^\varepsilon(x, v) \geq 0$$

satisfying (i) and (ii) in Theorem 1.1. When the viscosity  $\varepsilon$  is fixed,  $u_0^\varepsilon \in C^1(\mathbb{R})$ ,  $f_0^\varepsilon \in C_c^1(\mathbb{R}^2)$ , according to Theorem 2.1 in [5], one is able to obtain the global existence and uniqueness of a smooth solution  $(u^\varepsilon, f^\varepsilon)$  with  $f^\varepsilon \geq 0$ . On the other hand, in Section 4 of [7], the author also gained a global weak solution to (1.1) when initial data  $(u_0^\varepsilon, f_0^\varepsilon)$  only enjoys finite-energy property and  $f_0^\varepsilon \in (L^1 \cap L^2)(\mathbb{R}^2)$ . Here, the smooth functions  $(u_0^\varepsilon, f_0^\varepsilon)$  in Theorem 1.1 are regular, and we can also cut off  $f_0^\varepsilon$  with a smooth function supported in  $\{(x, v) \mid |x| + |v| \leq \frac{1}{\varepsilon}\}$  (still denoted  $f_0^\varepsilon$ ) to make  $f_0^\varepsilon$  compactly supported and it still satisfies (i) and (ii). Additionally, the initial data satisfies (1.3), thus it is not hard to derive that

$$\lim_{x \rightarrow \pm\infty} u^\varepsilon(x, t) = u^\pm, \quad \lim_{x, v \rightarrow \pm\infty} f^\varepsilon(x, v, t) = 0. \tag{2.1}$$

Therefore, for smooth functions  $(u_0^\varepsilon, f_0^\varepsilon)$  given in Theorem 1.1 as initial data, for any fixed  $\varepsilon > 0$ , there always exists a unique smooth solution to Cauchy problem (1.1) and (1.2)-(1.3) satisfying (2.1).

We now establish two uniform estimates for solutions  $(u^\varepsilon, f^\varepsilon)$  with respect to the viscosity coefficient  $\varepsilon > 0$ , which plays a key role in our proof. For simplicity, we drop the superscript  $\varepsilon$  in this section.

**2.1. Energy estimate.** With the help of the partial dissipative effect of the source terms in (1.1), for the relative total energy  $E[u, f]$ , we have the following lemma.

LEMMA 2.1. *Let  $E[u_0, f_0] \leq E_0 < \infty$  with positive constant  $E_0$  independent of  $\varepsilon$ . Then there exists a constant  $C = C(E_0, t, \bar{u})$  such that*

$$\sup_{\tau \in [0, t]} E[u, f](\tau) + \int_0^t \int \int f(v - u)^2 dv dx d\tau + \int_0^t \int \varepsilon |u_x|^2 dx d\tau \leq C.$$

*Proof.* A direct calculation gives

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \int \frac{1}{2} (u - \bar{u})^2 dx + \frac{d}{dt} \int \int \frac{1}{2} f(1 + v^2) dx dv \\ &= \int (u - \bar{u}) u_t dx + \int \int \frac{1}{2} (1 + v^2) f_t dx dv. \end{aligned}$$

Due to

$$u_t = \varepsilon u_{xx} + \int f(v - u) dv - uu_x,$$

using (2.1) and integration by parts, we have

$$\int (u - \bar{u}) u_t dx = \int (u - \bar{u}) \varepsilon u_{xx} dx - \int (u - \bar{u}) uu_x dx + \int \int f(u - \bar{u})(v - u) dv dx$$

$$= I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= - \int \varepsilon(u_x - \bar{u}_x)u_x dx, \\ I_2 &= - \int (u - \bar{u})uu_x, \\ I_3 &= \int \int f(u - \bar{u})(v - u)dv dx. \end{aligned}$$

We then bound the three terms  $I_i, i = 1, 2, 3$  one by one. Note that from (1.7), it is easy to see that  $\bar{u}_x$  is bounded and compactly supported in  $[-L_0, L_0]$ , thus we have

$$I_1 \leq - \int \varepsilon|u_x|^2 dx + \varepsilon \int_{-L_0}^{L_0} |\bar{u}_x||u_x| dx \leq - \frac{1}{2} \int \varepsilon|u_x|^2 dx + C. \tag{2.2}$$

Similarly, for  $I_2$ , we have

$$\begin{aligned} I_2 &= - \int [(u - \bar{u})^2(u - \bar{u})_x + \bar{u}_x(u - \bar{u})^2 + (u - \bar{u})\bar{u}(u - \bar{u})_x + (u - \bar{u})\bar{u}\bar{u}_x] dx \\ &= - \int [(u - \bar{u})^2(u - \bar{u})_x + \frac{1}{2}\bar{u}_x(u - \bar{u})^2 + (u - \bar{u})\bar{u}\bar{u}_x] dx \\ &\leq C \int |u - \bar{u}|^2 dx + C \\ &\leq CE + C, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} I_3 &= \int \int f u(v - u)dv dx - \int \int \bar{u} f(v - u)dv dx \\ &\leq \int \int f u(v - u)dv dx + \frac{1}{2} \int \int f(v - u)^2 dv dx + C \int \int f dv dx. \end{aligned} \tag{2.4}$$

Furthermore, using

$$f_t = - (fv)_x - (f(u - v))_v$$

and (2.1) we gain

$$\begin{aligned} \int \int \frac{1}{2}(1 + v^2)f_t dx dv &= - \int \int \frac{1}{2}(1 + v^2)[(fv)_x + (f(u - v))_v] dx dv \\ &= \int \int f v(u - v) dv dx. \end{aligned} \tag{2.5}$$

Putting (2.2), (2.3), (2.4) and (2.5) together gives

$$\frac{dE}{dt} \leq CE + C - \frac{1}{2} \int \varepsilon|u_x|^2 dx - \frac{1}{2} \int \int f(u - v)^2 dv dx.$$

Then, direct application of Grönwall's inequality contributes to the lemma. □

**2.2. Higher integrability of velocity.** Although by Lemma 2.1, one has  $u \in L^2$ , we require much higher regularity of  $u$ . Utilizing the flux term in Burgers' equation, we improve the regularity of  $u$  to be  $L^4_{loc}$ .

LEMMA 2.2. *Let  $E[u_0, f_0] \leq E_0 < \infty$  with positive constant  $E_0$  independent of  $\varepsilon$ . Then for any compact set  $K \subset \mathbb{R}$  and all  $t > 0$ , there exists a constant  $C = C(E_0, K, \bar{u}, t)$ , independent of  $\varepsilon$  such that*

$$\int_0^t \int_K u^4 dx d\tau \leq C.$$

*Proof.* Let  $\varphi$  be an arbitrary smooth compactly supported function such that  $\varphi|_K = 1$  and  $0 \leq \varphi \leq 1$ . Motivated by the proof of Lemma 3.3 in [4], multiplying the viscous Burgers' equation by  $\varphi$  and then integrating with respect to the space variable over  $(-\infty, x)$ , we gain

$$\begin{aligned} \frac{1}{2}u^2\varphi &= \varepsilon u_x\varphi - \left(\int_{-\infty}^x u\varphi dy\right)_t + \int_{-\infty}^x \left(\frac{1}{2}u^2\varphi_x - \varepsilon u_x\varphi_x\right) dy \\ &\quad + \int_{-\infty}^x \varphi \int f(v-u) dv dy. \end{aligned}$$

Multiply the above equation by  $u^2\varphi$  and using the viscous Burgers' equation to get

$$\begin{aligned} \frac{1}{2}u^4\varphi^2 &= \varepsilon u^2 u_x \varphi^2 - \left(u^2\varphi \int_{-\infty}^x u\varphi dy\right)_t + 2\varepsilon u u_{xx} \varphi \int_{-\infty}^x u\varphi dy \\ &\quad - 2u^2 u_x \varphi \int_{-\infty}^x u\varphi dy + 2u\varphi \int f(v-u) dv \int_{-\infty}^x u\varphi dy \\ &\quad + u^2\varphi \int_{-\infty}^x \left(\frac{1}{2}u^2\varphi_x - \varepsilon u_x\varphi_x\right) dy + u^2\varphi \int_{-\infty}^x \varphi \int f(v-u) dv dy. \end{aligned}$$

Integrating over  $\mathbb{R} \times (0, t)$  gives

$$\frac{1}{2} \int_0^t \int u^4 \varphi^2 dx d\tau = \sum_{i=1}^5 J_i,$$

with

$$\begin{aligned} J_1 &= \int_0^t \int \left[ \varepsilon u^2 u_x \varphi^2 + 2\varepsilon u u_{xx} \varphi \int_{-\infty}^x u\varphi dy - u^2\varphi \int_{-\infty}^x \varepsilon u_x \varphi_x dy \right] dx d\tau, \\ J_2 &= \int \left( u_0^2 \varphi \int_{-\infty}^x u_0 \varphi dy \right) dx - \int \left( u^2 \varphi \int_{-\infty}^x u\varphi dy \right) dx, \\ J_3 &= \int_0^t \int \left[ \frac{1}{2} u^2 \varphi \int_{-\infty}^x u^2 \varphi_x dy - 2u^2 u_x \varphi \int_{-\infty}^x u\varphi dy \right] dx d\tau, \\ J_4 &= \int_0^t \int \left[ 2u\varphi \int f(v-u) dv \int_{-\infty}^x u\varphi dy \right] dx d\tau, \\ J_5 &= \int_0^t \int \left( u^2 \varphi \int_{-\infty}^x \varphi \int f(v-u) dv dy \right) dx d\tau. \end{aligned}$$

In the following, we will estimate  $J_i, i=1, \dots, 5$  one by one. For  $J_1$ , an application of integration by parts yields

$$\begin{aligned} J_1 &= \int_0^t \int \varepsilon u^2 u_x \varphi^2 dx d\tau - \int_0^t \int \left( u^2 \varphi \int_{-\infty}^x \varepsilon u_x \varphi_x dy \right) dx d\tau \\ &\quad - \int_0^t \int \left( 2\varepsilon u_x^2 \varphi \int_{-\infty}^x u \varphi dy + 2\varepsilon u^2 u_x \varphi^2 + 2\varepsilon u u_x \varphi_x \int_{-\infty}^x u \varphi dy \right) dx d\tau, \\ &\leq \delta \int_0^t \int u^4 \varphi^2 dx d\tau + C(\delta, E_0, K, \bar{u}, t) \int_0^t \int \varepsilon |u_x|^2 dx d\tau \\ &\leq \delta \int_0^t \int u^4 \varphi^2 dx d\tau + C(\delta, E_0, K, \bar{u}, t) \end{aligned}$$

with  $\delta$  to be determined, where we have used Hölder’s inequality, Lemma 2.1 and

$$\left| \int_{-\infty}^x u \varphi dy \right| \leq \int (u - \bar{u})^2 dx + C(K, \bar{u}) \leq C(E_0, K, \bar{u}).$$

Again applying basic energy estimate and Hölder’s inequality to  $J_2$  and  $J_3$ , we have

$$J_2 + J_3 \leq \delta \int_0^t \int u^4 \varphi^2 dx d\tau + C(\delta, E_0, K, \bar{u}, t).$$

For  $J_4$ , the following inequality

$$\int f u^2 dv \leq 2 \int f(v - u)^2 dv + 2 \int f v^2 dv$$

implies

$$\int_0^t \int \int f u^2 dv dx d\tau \leq C(E_0, \bar{u}, t).$$

Thus we have

$$\begin{aligned} \int_0^t \int \left| u \int f(v - u) dv \right| dx d\tau &\leq \int_0^t \int \left( \int f u^2 dv \right)^{\frac{1}{2}} \left( \int f(v - u)^2 dv \right)^{\frac{1}{2}} dx d\tau \\ &\leq C(E_0, \bar{u}, t), \end{aligned}$$

which then gives  $J_4 \leq C(E_0, K, \bar{u}, t)$ . Similarly, we can derive

$$\left| \int f(v - u) dv \right| \leq \left( \int f dv \right)^{\frac{1}{2}} \left( \int f(v - u)^2 dv \right)^{\frac{1}{2}}.$$

Hence we gain  $J_5 \leq C(E_0, K, \bar{u}, t)$ . Collecting all the estimates of  $J_i, i=1, \dots, 5$  and taking  $\delta \leq \frac{1}{16}$  yields the lemma.  $\square$

### 3. Vanishing viscosity limit

In this section, we will use the estimates in Section 2 to establish the convergence of  $(u^\varepsilon, f^\varepsilon)$ , whose limit is just a finite-energy weak solution to Cauchy problem (1.6) and (1.2)-(1.3). Based on the uniform estimates Lemma 2.1 and Lemma 2.2 in Section 2, we get the following:

$$\sup_{\tau \in [0, t]} E[u^\varepsilon, f^\varepsilon](\tau) \leq C, \tag{3.1}$$

$$\int_0^t \iint f^\varepsilon (v - u^\varepsilon)^2 dv dx d\tau, \tag{3.2}$$

$$\int_0^t \int \varepsilon |u_x^\varepsilon|^2 dx d\tau \leq C, \tag{3.3}$$

$$\int_0^t \int_K (u^\varepsilon)^4 dx d\tau \leq C, \text{ for any compact set } K \subset \mathbb{R}. \tag{3.4}$$

We then divide the proof into three subsections. In Section 3.1 and Section 3.2, we will apply the uniform estimates (3.3), (3.4) and (3.1) to show the convergence of  $f^\varepsilon$  and  $u^\varepsilon$  respectively. In Section 3.3 we will prove that the obtained limit is our desired solution.

**3.1. Limit of distribution function.** To show  $f^\varepsilon$  is weakly compact in  $L^1(\mathbb{R}^2)$ , a.e.  $t \in [0, T]$ , one can study the evolution of level sets of  $f^\varepsilon(x, v, t)$  and  $f_0^\varepsilon(x, v)$ , which is motivated by Steps 1-3 in the proof of Theorem 2.7 in [1]. For our case, the estimate on the level sets is done through the characteristic map. Our key observation is that the Jacobian of the characteristic map remains uniformly bounded as time grows.

Assume  $|\{f_0 = k\}| = 0$  for every  $k \in \mathbb{N}$ . (Otherwise one can consider  $\tau + k$  in place of  $k$  for some  $\tau \in (0, 1)$ .) From the strong convergence of  $f_0^\varepsilon$ , one could deduce that when  $\varepsilon \rightarrow 0$ ,

$$f_0^{\varepsilon, k} := \mathbf{1}_{\{k \leq f_0^\varepsilon < k+1\}} f_0^\varepsilon \rightarrow f_0^k := \mathbf{1}_{\{k \leq f_0 < k+1\}} f_0 \text{ in } L^1(\mathbb{R}^2) \text{ for any } k \in \mathbb{N},$$

where  $\mathbf{1}_A$  is the characteristic function of set  $A$ . We shall also analyze the evolution of corresponding level sets of  $f^\varepsilon(x, v, t)$ . In fact, for Vlasov equation

$$f_t^\varepsilon + (f^\varepsilon v)_x + (f^\varepsilon (u^\varepsilon - v))_v = 0$$

with  $u^\varepsilon$  being a smooth function, the equation can be rewritten as

$$f_t^\varepsilon + v f_x^\varepsilon + (u^\varepsilon - v) f_v^\varepsilon = f^\varepsilon, \quad f^\varepsilon(x, v, 0) = f_0^\varepsilon(x, v),$$

which has a unique smooth solution

$$f^\varepsilon(x, v, t) = f_0^\varepsilon(X^\varepsilon(0; x, v, t), V^\varepsilon(0; x, v, t)) e^t, \tag{3.5}$$

where  $X(s; x, v, t), V(t; x, v, t)$  are backward characteristic curves satisfying

$$\frac{dX^\varepsilon(s; x, v, t)}{ds} = V^\varepsilon(s; x, v, t), \quad X^\varepsilon(t; x, v, t) = x; \tag{3.6}$$

$$\frac{dV^\varepsilon(s; x, v, t)}{ds} = u^\varepsilon(s, X^\varepsilon(s; x, v, t)) - V^\varepsilon(s; x, v, t), \quad V^\varepsilon(t; x, v, t) = v. \tag{3.7}$$

It is not hard to see that  $X^\varepsilon, V^\varepsilon$  are well-defined from the theory of ordinary differential equation (ODE). From (3.6) and (3.7), one is able to show that the Jacobian  $J(t) = \det \nabla_{x,v}(X^\varepsilon, V^\varepsilon)$  of the map  $\mathcal{J}(s) : (x, v) \mapsto (X^\varepsilon, V^\varepsilon)$  is nonnegative and satisfies the following ODE

$$\begin{cases} \frac{dJ(t)}{dt} = J(t) \operatorname{div}_{x,v}(X^\varepsilon, V^\varepsilon) = -J(t), \\ J(t) = 1, \end{cases}$$

so  $J(\tau) = e^{-\tau}$  for any  $\tau \in [0, t]$ . For any  $t > 0$ , noting (3.5), set

$$f^{\varepsilon, k}(x, v, t) = e^t \mathbf{1}_{\{k \leq f_0^\varepsilon \circ \mathcal{J}(0) < k+1\}} f_0^\varepsilon \circ \mathcal{J}(0).$$



Then we have that  $f^{\varepsilon,k}$  is a weak solution to the Vlasov equation and

$$\begin{aligned} \iint f^{\varepsilon,k} dv dx &= \iint e^t \mathbf{1}_{\{k \leq f_0^\varepsilon \circ \mathcal{J}(0) < k+1\}} f_0^\varepsilon \circ \mathcal{J}(0) dv dx \\ &= \iint e^t \mathbf{1}_{\{k \leq f_0^\varepsilon < k+1\}} f_0^\varepsilon J(0)^{-1} dV^\varepsilon dX^\varepsilon \\ &= \iint f_0^{\varepsilon,k} dV^\varepsilon dX^\varepsilon, \end{aligned}$$

hence for any  $t > 0$ ,

$$\|f^{\varepsilon,k}(x, v, t)\|_{L^1(\mathbb{R}^2)} = \|f_0^{\varepsilon,k}(x, v)\|_{L^1(\mathbb{R}^2)}.$$

It is easy to find that  $0 \leq f^{\varepsilon,k} \leq (k+1)e^T$ , thus up to subsequences, for any  $k \in \mathbb{N}$ .

$$f^{\varepsilon,k} \rightharpoonup f^k \text{ weakly* in } L^\infty(\mathbb{R}^2 \times [0, T]), \text{ as } \varepsilon \rightarrow 0.$$

Similar to [1], one can use the test function  $\phi(t)\mathbf{1}_K \text{sign}(f^k)(x, v, t)$  for any compact subset  $K \subset \mathbb{R}^2$  and any  $\phi \in C_c^\infty(\mathbb{R}_+)$  in the above weak convergence to show

$$\|f^k(x, v, t)\|_{L^1(\mathbb{R}^2)} \leq \|f_0^k(x, v)\|_{L^1(\mathbb{R}^2)}$$

for almost all  $t$ . We define

$$f(x, v, t) := \sum_{k=0}^\infty f^k(x, v, t), \text{ for } (x, v, t) \in \mathbb{R}^2 \times [0, T],$$

then it is easy to derive

$$\|f\|_{L^1(\mathbb{R}^2)} \leq \sum_{k=0}^\infty \|f^k(x, v, t)\|_{L^1(\mathbb{R}^2)} \leq \sum_{k=0}^\infty \|f_0^k(x, v)\|_{L^1(\mathbb{R}^2)} = \|f_0\|_{L^1(\mathbb{R}^2)}.$$

We then prove

$$f^\varepsilon \rightharpoonup f \text{ weakly in } L^\infty([0, T], L^1(\mathbb{R}^2)), \text{ as } \varepsilon \rightarrow 0. \tag{3.8}$$

In fact, for any  $\varphi \in L^\infty(\mathbb{R}^2)$ , we derive

$$\begin{aligned} &\left| \iint \varphi(f^\varepsilon - f) dx dv \right| = \left| \sum_{k=0}^\infty \iint \varphi(f^{\varepsilon,k} - f^k) dx dv \right| \\ &\leq \left| \sum_{k=0}^{k_0-1} \iint \varphi(f^{\varepsilon,k} - f^k) dx dv \right| + \sum_{k=k_0}^\infty \iint |\varphi| |f^{\varepsilon,k}| dx dv + \sum_{k=k_0}^\infty \iint |\varphi| |f^k| dx dv. \end{aligned}$$

The first term converges to zero as  $\varepsilon \rightarrow 0$  due to the weak convergence of  $f^{\varepsilon,k}$  for any finite  $k_0$ . The last two terms can be estimated by

$$\begin{aligned} &\sum_{k=k_0}^\infty \iint |\varphi| |f^{\varepsilon,k}| dx dv + \sum_{k=k_0}^\infty \iint |\varphi| |f^k| dx dv \\ &\leq \|\varphi\|_{L^\infty(\mathbb{R}^2)} (\|f_0^\varepsilon \mathbf{1}_{\{f_0^\varepsilon \geq k_0\}}\|_{L^1(\mathbb{R}^2)} + \|f_0 \mathbf{1}_{\{f_0 \geq k_0\}}\|_{L^1(\mathbb{R}^2)}), \end{aligned}$$

which converges to zero as  $k_0 \rightarrow \infty$  thanks to the fact that  $f_0^\varepsilon$  and  $f_0$  are bounded in  $L^1(\mathbb{R}^2)$ . Finally, we obtain (3.8).

**3.2. Limit of velocity.** To show the convergence of  $u^\varepsilon$ , we will utilize the  $L^p$  compactness framework for Burgers' equation. Thus we first recall a proposition on such a framework, which results from [12]. The framework is shown mainly by div-curl lemma and compactness of some entropies for Burgers' equation.

PROPOSITION 3.1. *Let  $u^\varepsilon(x, t)$  satisfy the following two conditions:*

- (C1)  $u^\varepsilon(x, t)$  is uniformly bounded in  $L^p_{loc}(\mathbb{R} \times [0, T])$  for some  $p > 2$ ;
- (C2) Both  $\partial_t I_n(u^\varepsilon(x, t)) + \partial_x F_n(u^\varepsilon(x, t))$  and  $\partial_t F_n(u^\varepsilon(x, t)) + \partial_x \Phi_n(u^\varepsilon(x, t))$  lie in a compact set of  $H^{-1}_{loc}(\mathbb{R} \times [0, T])$  with respect to  $\varepsilon$  for any  $n \in \mathbb{N}$ , where

$$I_n(u) = \begin{cases} u, & \text{when } |u| \leq n, \\ 0, & \text{when } |u| \geq 2n, \end{cases} \text{ and } I_n \in C^2(\mathbb{R}), \quad |I_n(u)| \leq |u|, \quad |I'_n(u)| \leq 2,$$

$$F_n(u) = \int_0^u I'_n(s) ds, \quad \Phi_n(u) = \int_0^u F'_n(s) ds.$$

Then there exists a subsequence (still denoted  $u^\varepsilon$ ) such that  $u^\varepsilon \rightarrow u$  almost everywhere and strongly in  $L^r_{loc}(\mathbb{R} \times [0, T])$  for all  $1 \leq r \leq p$ .

With such  $L^p$  framework, we only need to verify (C1)-(C2) to show the convergence of  $u^\varepsilon$ . It is easy to see from (3.4) that we have

$$u^\varepsilon \in L^4_{loc}(\mathbb{R} \times [0, T]),$$

thus (C1) is satisfied by  $u^\varepsilon(x, t)$  for  $p = 4$ .

To verify (C2), we also require an important lemma: Murat's lemma, which is useful in proving compactness of some sequences.

LEMMA 3.1 (Murat's Lemma [3, 13, 20]). *Let  $\Omega \subset \mathbb{R}^n$  be an open bounded subset,  $D_1$  be a compact set in  $W^{-1, a}_{loc}(\Omega)$ ,  $D_2$  be a bounded set in  $W^{-1, b}_{loc}(\Omega)$  for some constants  $a, b$  satisfying  $1 < a \leq 2 < b$ . Furthermore, let  $D_0 \subset \mathcal{D}(\Omega)$  such that  $D_0 \subset D_1 \cap D_2$ . Then there exists  $D_*$ , a compact set in  $H^{-1}_{loc}(\Omega)$  such that  $D_0 \subset D_*$ .*

Then we turn to verify (C2). Noting that both  $I_n(u)$  and  $F_n(u)$  are  $C^2$  compactly supported functions of  $u$ , one gets

$$\left. \begin{aligned} &\partial_t I_n(u^\varepsilon(x, t)) + \partial_x F_n(u^\varepsilon(x, t)), \\ &\partial_t F_n(u^\varepsilon(x, t)) + \partial_x \Phi_n(u^\varepsilon(x, t)) \end{aligned} \right\} \text{ are bounded in } W^{-1, \infty}_{loc}(\mathbb{R} \times [0, T]). \quad (3.9)$$

Furthermore, from

$$\begin{aligned} \partial_t F_n(u^\varepsilon) + \partial_x \Phi_n(u^\varepsilon) &= \varepsilon u^\varepsilon_{xx} F'_n(u^\varepsilon) + F'_n(u^\varepsilon) \int f^\varepsilon(v - u^\varepsilon) dv \\ &= (\varepsilon u^\varepsilon_x F'_n(u^\varepsilon))_x - \varepsilon (u^\varepsilon_x)^2 F''_n(u^\varepsilon) + F'_n(u^\varepsilon) \int f^\varepsilon(v - u^\varepsilon) dv, \end{aligned}$$

along with  $|F'_n(u^\varepsilon)| \leq |I'_n(u^\varepsilon)u^\varepsilon| \leq C(n)$ , and  $|F''_n(u^\varepsilon)| \leq C(n)$ , one can derive

$$\partial_t F_n(u^\varepsilon) + \partial_x \Phi_n(u^\varepsilon) \text{ is compact in } W^{-1, \alpha}_{loc}(\mathbb{R} \times [0, T]) \text{ for some } \alpha \in (1, 2). \quad (3.10)$$

Indeed, using  $\sqrt{\varepsilon}u^\varepsilon_x \in L^2(\mathbb{R} \times [0, T])$  and  $|F'_n(u^\varepsilon)| \leq C(n)$ , we get that  $(\varepsilon u^\varepsilon_x F'_n(u^\varepsilon))_x$  is compact in  $H^{-1}_{loc}(\mathbb{R} \times [0, T])$ . Employing Lemma 2.1 we also have

$$-\varepsilon (u^\varepsilon_x)^2 F''_n(u^\varepsilon) + F'_n(u^\varepsilon) \int f^\varepsilon(v - u^\varepsilon) dv \in L^1(\mathbb{R} \times [0, T]),$$

which also implies that it is compact in  $W_{loc}^{-1,\alpha}(\mathbb{R} \times [0, T])$  for some  $\alpha \in (1, 2)$  by embedding theorem and Schauder's theorem, so we obtain (3.10). Similarly, one can also gain

$$\partial_t I_n(u^\varepsilon) + \partial_x F_n(u^\varepsilon) \text{ is compact in } W_{loc}^{-1,\alpha}(\mathbb{R} \times [0, T]) \text{ for some } \alpha \in (1, 2). \tag{3.11}$$

Combining with (3.9), (3.10) and (3.11), applying Murat's Lemma (see Lemma 3.1), one gets (C2). Therefore, applying Proposition 3.1 to  $u^\varepsilon$ , one can seek a  $u(x, t) \in L^4_{loc}(\mathbb{R} \times [0, T])$  and a subsequence of  $u^\varepsilon$  (still denoted as  $u^\varepsilon$ ) such that

$$\begin{aligned} u^\varepsilon &\rightarrow u \text{ a.e. } \mathbb{R} \times [0, T], \\ u^\varepsilon &\rightarrow u \text{ strongly in } L^r_{loc}(\mathbb{R} \times [0, T]), \text{ as } \varepsilon \rightarrow 0, \text{ for } 1 \leq r \leq 4. \end{aligned}$$

**3.3. Limit of equations and conclusions.** Before taking the limit of equations and proving that  $(u, f)$  is a weak solution of the Cauchy problem (1.6) and (1.2)-(1.3), we shall first get the convergences of nonlinear terms in the equations:  $\int f^\varepsilon dv$ ,  $\int f^\varepsilon v dv$  and  $u^\varepsilon \int f^\varepsilon dv$ .

(1) *Convergence of  $\int f^\varepsilon v dv \rightarrow \int f v dv$ .* From energy estimate and the lower semi-continuity of kinetic energy, we deduce that

$$\iint v^2 f dx dv \leq \liminf_{\varepsilon \rightarrow 0} \iint v^2 f^\varepsilon dx dv \leq C. \tag{3.12}$$

Let  $\chi(s) = \mathbf{1}_{[-1, 1]}(s)$ . For any  $\varphi \in C_c^\infty(\mathbb{R})$ , for any  $L > 0$ , inspired by [1], we observe that

$$\begin{aligned} &\int \left( \int f^\varepsilon v dv - \int f v dv \right) \varphi dx \\ &= \iint (f^\varepsilon - f) v \chi\left(\frac{v}{L}\right) \varphi dv dx + \iint (f^\varepsilon v (1 - \chi\left(\frac{v}{L}\right))) \varphi dv dx - \iint f v (1 - \chi\left(\frac{v}{L}\right)) \varphi dv dx. \end{aligned}$$

The first term in the right-hand side converges to 0 due to the weak convergence of  $f^\varepsilon$  to  $f$  in  $L^1(\mathbb{R}^2)$ . Thanks to (3.12), the remaining two terms can be estimated as

$$\begin{aligned} \left| \iint (f^\varepsilon v (1 - \chi\left(\frac{v}{L}\right))) \varphi dv dx \right| &\leq \frac{\|\varphi\|_{L^\infty}}{L} \iint f^\varepsilon v^2 dv dx \leq \frac{C \|\varphi\|_{L^\infty}}{L}, \\ \left| \iint (f v (1 - \chi\left(\frac{v}{L}\right))) \varphi dv dx \right| &\leq \frac{\|\varphi\|_{L^\infty}}{L} \iint f v^2 dv dx \leq \frac{C \|\varphi\|_{L^\infty}}{L}. \end{aligned}$$

Letting  $L$  go to infinity, we can get

$$\int f^\varepsilon v dv \rightharpoonup \int f v dv \text{ weakly in } L^\infty([0, T], L^1(\mathbb{R})).$$

(2) *Convergence of  $\int f^\varepsilon dv \rightarrow \int f dv$ .* Similar to Step 1, we can get

$$\int f^\varepsilon dv \rightharpoonup \int f dv \text{ weakly in } L^\infty([0, T], L^1(\mathbb{R})).$$

(3) *Convergence of  $u^\varepsilon \int f^\varepsilon dv \rightarrow u \int f dv$ .* We easily get from (3.1) and (3.2) that for any  $T \in \mathbb{R}_+$ ,

$$\int_0^T \int (u^\varepsilon)^2 \int f^\varepsilon dx dv \leq 2 \int_0^T \iint v^2 f^\varepsilon dx dv + 2 \int_0^T \iint (u^\varepsilon - v)^2 f^\varepsilon dx dv \leq C.$$

Due to the semicontinuity of the integral functional and the strong compactness of  $u^\varepsilon$  and weak compactness of  $\int f^\varepsilon dv$ , one also gets

$$\int_0^T \int u^2 \int f dx dv \leq \liminf_{\varepsilon \rightarrow 0} \int_0^T \int (u^\varepsilon)^2 \int f^\varepsilon dx dv \leq C. \tag{3.13}$$

Following the same strategy of proving the convergence of  $\int v f^\varepsilon dv$ , denoting  $\chi(s) = \mathbf{1}_{[-1,1]}(s)$ , we gain for any  $\varphi \in C_c^\infty(\mathbb{R} \times (0, T))$ , for arbitrary  $L > 0$ ,

$$\begin{aligned} & \int_0^T \int \left( u^\varepsilon \int f^\varepsilon dv - u \int f dv \right) \varphi dx dt \\ &= \int_0^T \int \int \left( u^\varepsilon \chi\left(\frac{u^\varepsilon}{L}\right) \int f^\varepsilon dv - u \chi\left(\frac{u}{L}\right) \int f dv \right) \varphi dv dx \\ & \quad + \int_0^T \int \int u^\varepsilon f^\varepsilon \left(1 - \chi\left(\frac{u^\varepsilon}{L}\right)\right) \varphi dv dx - \int_0^T \int \int f u \left(1 - \chi\left(\frac{u}{L}\right)\right) \varphi dv dx. \end{aligned}$$

For the first term in the right-hand side of the above integrals, since we have uniform bounds on the integrands, by dominated convergence theorem it converges to 0 as  $\varepsilon \rightarrow 0$ . The last two terms can be estimated through (3.13) as before,

$$\begin{aligned} \left| \int_0^T \int \int f^\varepsilon u^\varepsilon \left(1 - \chi\left(\frac{u^\varepsilon}{L}\right)\right) \varphi dv dx \right| &\leq \frac{\|\varphi\|_{L^\infty}}{L} \int_0^T \int \int f^\varepsilon (u^\varepsilon)^2 dv dx \leq \frac{C\|\varphi\|_{L^\infty}}{L}, \\ \left| \int_0^T \int \int f u \left(1 - \chi\left(\frac{u}{L}\right)\right) \varphi dv dx \right| &\leq \frac{\|\varphi\|_{L^\infty}}{L} \int_0^T \int \int f u^2 dv dx \leq \frac{C\|\varphi\|_{L^\infty}}{L}, \end{aligned}$$

both of which go to 0 upon letting  $L \rightarrow \infty$ .

Now we are in a position to show that  $(u, f)$  is a finite-energy weak solution to Cauchy problem (1.6) and (1.2)-(1.3).

(1) *Weak solutions.* It suffices to show that (1.9) and (1.10) hold for  $(u, f)$ . Here we only show (1.9) by the uniform estimate (3.3), since (1.10) can be verified similarly. Multiplying the first equation in (1.1) by  $\phi \in C_c^\infty(\mathbb{R} \times [0, T])$ , integrating over  $\mathbb{R} \times [0, T]$ , and employing integration by parts we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \phi(x, 0) u_0^\varepsilon(x) dx + \int_0^T \int_{\mathbb{R}} \left( u^\varepsilon \phi_t + \frac{1}{2} (u^\varepsilon)^2 \phi_x + \phi \int_{\mathbb{R}} f^\varepsilon (v - u^\varepsilon) dv \right) dx \\ & \quad - \varepsilon \int_0^T \int u_x^\varepsilon \phi_x dx dt = 0, \end{aligned}$$

For the last term in the left-hand side, it follows from (3.3) that

$$\left| \varepsilon \int_0^T \int u_x^\varepsilon \phi_x dx dt \right| \leq \sqrt{\varepsilon} \left( \int_0^T \int \varepsilon (u_x^\varepsilon)^2 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int \phi_x^2 dx dt \right)^{\frac{1}{2}} \leq C\sqrt{\varepsilon},$$

which goes to 0 by letting  $\varepsilon \rightarrow 0$ . The convergence of the other terms have been already obtained, thus we have (1.9). Hence, the obtained limit  $(u, f)$  is a weak solution to (1.6).

(2) *Finite energy.* We also show that the obtained limit functions also enjoy the finite-energy property. Obviously, from almost everywhere convergence of  $u^\varepsilon$  to  $u$  and Lemma 2.1, we have

$$u^\varepsilon - \bar{u} \rightarrow u - \bar{u} \text{ strongly in } L^2(\mathbb{R}).$$

Hence by the convexity of energy we gain

$$\int (u - \bar{u})^2 dx \leq \liminf_{\varepsilon \rightarrow 0} \int (u^\varepsilon - \bar{u})^2 dx \leq C.$$

Besides,

$$\int_0^T \iint \int f(u-v)^2 dv dx dt \leq 2 \int_0^T \iint \int f u^2 dv dx dt + 2 \int_0^T \iint \int f v^2 dv dx dt \leq C.$$

Combining with (3.12), we gain (1.8).

Therefore,  $(u, f)$  is a finite-energy weak solution to Cauchy problem (1.6) and (1.2)-(1.3). Besides, it is easy to verify (3) in Definition 1.1 for  $(u, f)$  by the smoothness and compactness condition (ii) of  $(u_0^\varepsilon, f_0^\varepsilon)$  in Theorem 1.1. The proof of Theorem 1.1 is then completed.

**Acknowledgments.** The research of the W. Cao is supported by ERC Grant Agreement No. 724298 and he also thanks for the hospitality of Max-Planck Institute for Mathematics in the Sciences. The work of T. Wang is partially supported by the NNSFC grant No. 11971044 and BJNSF grant No. 1202002.

#### REFERENCES

- [1] L. Ambrosio, M. Colombo, and A. Figalli, *On the Lagrangian structure of transport equations: The Vlasov-Poisson system*, Duke Math. J., **166:3505–3568**, 2017. [1](#), [3.3](#)
- [2] F. Berthelin, N.J. Mauser, and F. Poupaud, *High-field limit from a kinetic equation to multidimensional scalar conservation laws*, J. Hyperbolic Diff. Eqs., **4:123–145**, 2007. [1](#)
- [3] G.Q. Chen, *Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (III)*, Acta Math. Sci., **6:75–120**, 1986. [3.1](#)
- [4] G.Q. Chen and M. Perelitsa, *Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow*, Comm. Pure Appl. Math., **63:1469–1504**, 2010. [1](#), [2.2](#)
- [5] K. Domelevo and J.M. Roquejoffre, *Existence and stability of travelling wave solutions in a kinetic model of two-phase flows*, Comm. Part. Diff. Eqs., **24:61–108**, 1999. [1](#), [2](#)
- [6] P. Germain and P. LeFloch, *Finite energy method for compressible fluids: The Navier-Stokes-Korteweg model*, Comm. Pure Appl. Math., **69:3–61**, 2016. [1](#)
- [7] T. Goudon, *Asymptotic problems for a kinetic model of two-phase flow*, Proc. Roy. Soc. Edinb. Sect. A, **131:1371–1384**, 2001. [1](#), [2](#)
- [8] T. Goudon, P.E. Jabin, and A. Vasseur, *Hydrodynamic limit for the Vlasov-Navier-Stokes equations. I. Light particles regime*, Indiana Univ. Math. J., **53:1495–1515**, 2004. [1](#)
- [9] T. Goudon, P.E. Jabin, and A. Vasseur, *Hydrodynamic limit for the Vlasov-Navier-Stokes equations. II. Fine particles regime*, Indiana Univ. Math. J., **53:1517–1536**, 2004. [1](#)
- [10] P.E. Jabin, *Macroscopic limit of Vlasov type equations with friction*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **17:651–672**, 2000. [1](#)
- [11] P.E. Jabin, *Large time concentrations for solutions to kinetic equations with energy dissipation*, Comm. Part. Diff. Eqs., **25:541–557**, 2000. [1](#)
- [12] Y. Lu, *Cauchy problem for an extended model of combustion*, Proc. Roy. Soc. Edinb. Sect. A, **120:349–360**, 1992. [1](#), [3.2](#)
- [13] Y. Lu, *Hyperbolic Conservation Laws and the Compensated Compactness Method*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 128. Chapman & Hall/CRC, Boca Raton, FL, 2003. [3.1](#)
- [14] A. Mellet and A. Vasseur, *Asymptotic analysis for a Vlasov-Fokker-Planck/compressible Navier-Stokes system of equations*, Comm. Math. Phys., **281:573–596**, 2008. [1](#)

- [15] A. Mellet and A. Vasseur, *Global weak solutions for a Vlasov-Fokker-Planck/Navier-Stokes system of equations*, Math. Models Meth. Appl. Sci., **17:1039–1063**, 2007. [1](#)
- [16] J. Nieto, F. Poupaud, and J. Soler, *High-field limit for the Vlasov-Poisson-Fokker-Planck system*, Arch. Ration. Mech. Anal., **158:29–59**, 2001. [1](#)
- [17] G. Russo and P. Smereka, *Kinetic theory for bubbly flow. I. Collisionless case*, SIAM J. Appl. Math., **56:327–357**, 1996. [1](#)
- [18] G. Russo and P. Smereka, *Kinetic theory for bubbly flow. II. Fluid dynamic limit*, SIAM J. Appl. Math., **56:358–371**, 1996. [1](#)
- [19] M.E. Schonbek, *Convergence of solutions to nonlinear dispersive equations*, Comm. Part. Diff. Eqs., **7:959–1000**, 1982. [1](#)
- [20] L. Tartar, *Compensated compactness and applications to partial differential equations*, in R.J. Knops (ed.), Nonlinear Analysis and Mechanics, Heriot-Watt Symposium IV, Res. Notes in Math., Pitman, Boston, Mass.-London, **39:136–212**, 1979. [3.1](#)
- [21] F.A. Williams, *Combustion Theory*, Second Edition, New York, Benjamin, 1985. [1](#)
- [22] T. Yang, C. Zhu, and H. Zhao, *Compactness framework of  $L^p$  approximate solutions for scalar conservation laws*, J. Math. Anal. Appl., **220:164–186**, 1998. [1](#)
- [23] C. Yu, *Global weak solutions to the incompressible Navier-Stokes-Vlasov equations*, J. Math. Pures Appl., **100:275–293**, 2013. [1](#)