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ASYMPTOTIC BEHAVIOR OF 3-D STOCHASTIC PRIMITIVE EQUATIONS OF LARGE-SCALE MOIST ATMOSPHERE WITH ADDITIVE NOISE*

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Abstract. The primitive equations (PEs) are a basic model in the study of large scale oceanic and atmospheric dynamics. Its high non-linearity and anisotropic structure attract much attention from mathematicians.

In the present article, we consider the corresponding stochastic model. As studies from climate sciences show that the complex multi-scale nature of the earth's climate system results in many uncertainties that should be accounted for in the basic dynamical models of atmospheric and oceanic processes. It is further pointed out by [Palmer, Q.J.R. Meteorol. Soc., 127:279–304, 2001] and [Majda, Timofeyev and Vanden-Eijinden, Commun. Pure Appl. Math., 54:891–974, 2001] that stochastic modeling for climate is important for understanding the intrinsic variability of dominant low-frequency teleconnection patterns in climate, to provide cheap low-dimensional computational models for the coupled atmosphere-ocean system and to reduce model error in standard deterministic computer models for extended-range prediction through appropriate stochastic noise.

This is the first attempt to consider stochastic moist PEs defined on manifolds. Using a new and general way, we prove the existence of random attractor (strong attractor) for the three dimensional stochastic moist primitive equations defined on a manifold in 3D space improving the existence of weak attractor for the corresponding deterministic model [Guo, Huang, J. Diff. Eqs., 251:457–491, 2011]. As an application of the result, we show the existence of the invariant measure. The technique presented in this work can be applied to common classes of dissipative stochastic partial differential equations and it has some advantages over the common method of using compact Sobolev imbedding theorem, i.e., if the absorbing set in some Sobolev space does exist in view of the common method, our method would then further imply the existence of random attractor in this space.

Keywords. Stochastic moist primitive equations; manifold; Random attractor; Invariant measure.

AMS subject classifications. 60H15; 35Q35.

1. Introduction

Given the space domain as a manifold $\mathbf{D}=\mathbf{S}^2\times(0,1)$, where $\mathbf{S}^2$ is a two-dimensional unit sphere, we consider the following three-dimensional viscous stochastic primitive equations in the pressure coordinate system (θ,φ,ξ):

$$\partial_t\mathbf{v}+\nabla_{\mathbf{v}}\mathbf{v}+w\partial_\xi\mathbf{v}+\frac{f}{R_0}\vec{k}\times\mathbf{v}+\nabla\Phi+L_1\mathbf{v}=\dot{W}_1, \tag{1.1a}$$

$$\partial_\xi\Phi+\frac{br_s}{r}(1+aq)T=0, \tag{1.1b}$$

$$\text{div }\mathbf{v}+\partial_\xi w=0, \tag{1.1c}$$

$$\partial_t T+\nabla_{\mathbf{v}}T+w\partial_\xi T-\frac{br_s}{r}(1+aq)w+L_2T=Q_T+\dot{W}_2, \tag{1.1d}$$

$$\partial_t q+\nabla_{\mathbf{v}}q+w\partial_\xi q+L_3q=Q_q+\dot{W}_3. \tag{1.1e}$$

In this geophysical system, unknown functions are $\mathbf{v},w,\Phi,T,q$ and the physical meanings are as follows: $(\mathbf{v},w)=(v_\theta,v_\varphi,w)$ is the 3-D fluid velocity field, with $\mathbf{v}=(v_\theta,v_\varphi)$ being

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the horizontal velocity and w being the vertical velocity in the pressure coordinate system; Φ is the geopotential, T is the temperature, and q is the mixing ratio of water vapor in the air.

$f=2\cos\theta$ is the Coriolis parameter, R_0 is the Rossby number, $\vec{k}$ is the vertical unit vector. r is the pressure function depending on the variable ξ: $r=(r_s-r_0)\xi+r_0$, where $0<r_0\le r\le r_s$. Q_T,Q_q are given functions on $\mathbf{S}^2\times(0,1)$, a,b are positive constants. The viscosity and the heat diffusion operators L_1,L_2,L_3 are given by

$$L_i=-\nu_i\Delta-\mu_i\partial_{\xi\xi}, \qquad i=1,2,3,$$

where the positive constants $\nu_i,\mu_i,i=1,2,3$ are the horizontal and vertical Reynolds numbers. To simplify the notations, we assume $\nu_i=\mu_i=1,\quad i=1,2,3$. The results in this paper are still valid when we consider the general cases. Note that in this paper we are discussing operators defined on manifolds, and in the pressure coordinate system. The formal definitions of $\nabla_{\mathbf{v}}\mathbf{v},\Delta\mathbf{v},\Delta T,\Delta q,\nabla_{\mathbf{v}}T,\nabla_{\mathbf{v}}q,\text{div }\mathbf{v},\nabla\Phi$ will be given in Section 2. $\dot{W}_j,j=1,2,3$ are independent Gaussian white noise processes which are formally delta correlated in time.

The boundary conditions of the system (1.1) are given by

$$\text{On }\xi=1:\ \partial_\xi\mathbf{v}=0,w=0,\partial_\xi T=-\alpha(T-T^*),\partial_\xi q=-\beta(q-q^*), \tag{1.2a}$$
$$\text{On }\xi=0:\ \partial_\xi\mathbf{v}=0,w=0,\partial_\xi T=0,\partial_\xi q=0, \tag{1.2b}$$

where α,β are positive constants, T^*,q^* are the given temperature and mixing ratio of water vapor on the surface of the earth, respectively. For simplicity, we assume that $T^*=q^*=0$. One can always homogenize boundary conditions for nonzero T^*,q^* (see [16]).

Now integrating (1.1b) and (1.1c) and applying boundary conditions (1.2), with $\Phi_s(t;\theta,\varphi)$ being a certain unknown function on the isobaric surface $\xi=1$, one can get

$$\Phi(t;\theta,\varphi,\xi)=\Phi_s(t;\theta,\varphi)+\int_\xi^1\frac{br_s}{r}(1+aq)Td\xi', \tag{1.3}$$

$$w(t;\theta,\varphi,\xi)=\int_\xi^1\text{div }\mathbf{v}(t;\theta,\varphi,\xi')d\xi', \tag{1.4}$$

$$\int_0^1\text{div }\mathbf{v}d\xi=0. \tag{1.5}$$

In addition, we supply the system with the initial conditions:

$$\mathbf{v}(t_0;\theta,\varphi,\xi)=\mathbf{v}_0(\theta,\varphi,\xi), \tag{1.6a}$$
$$T(t_0;\theta,\varphi,\xi)=T_0(\theta,\varphi,\xi), \tag{1.6b}$$
$$q(t_0;\theta,\varphi,\xi)=q_0(\theta,\varphi,\xi). \tag{1.6c}$$

With all the above discussions, we have the following equivalent formulation for the 3-D stochastic PEs:

$$\begin{aligned}\partial_t\mathbf{v}+\nabla_{\mathbf{v}}\mathbf{v}+\left(\int_\xi^1\text{div }\mathbf{v}(t;\theta,\varphi,\xi')d\xi'\right)\partial_\xi\mathbf{v}+\frac{f}{R_0}\vec{k}\times\mathbf{v}+\nabla\Phi_s\\+\int_\xi^1\frac{br_s}{r}\nabla[(1+aq)T]d\xi'-\Delta\mathbf{v}-\partial_{\xi\xi}\mathbf{v}=\dot{W}_1,\end{aligned} \tag{1.7a}$$

$$\partial_t T+\nabla_{\mathbf{v}} T+\left(\int_{\xi}^{1} \operatorname{div} \mathbf{v}(t;\theta,\varphi,\xi')d\xi'\right)\partial_{\xi} T-\frac{br_s}{r}(1+aq)\left(\int_{\xi}^{1} \operatorname{div} \mathbf{v}(t;\theta,\varphi,\xi')d\xi'\right)$$
$$-\Delta T-\partial_{\xi\xi} T=Q_T+\dot{W}_2, \tag{1.7b}$$

$$\partial_t q+\nabla_{\mathbf{v}} q+\left(\int_{\xi}^{1} \operatorname{div} \mathbf{v}(t;\theta,\varphi,\xi')d\xi'\right)\partial_{\xi} q-\Delta q-\partial_{\xi\xi} q=Q_q+\dot{W}_3, \tag{1.7c}$$

$$\int_0^1 \operatorname{div} \mathbf{v} d\xi=0, \tag{1.7d}$$

$$\text{On } \xi=1\ (r=r_s):\ \partial_{\xi}\mathbf{v}=0, w=0, \partial_{\xi} T=-\alpha T, \partial_{\xi} q=-\beta q, \tag{1.7e}$$

$$\text{On } \xi=0\ (r=r_0):\ \partial_{\xi}\mathbf{v}=0, w=0, \partial_{\xi} T=0, \partial_{\xi} q=0. \tag{1.7f}$$

The primitive equations (PEs) are the basic models to study the mechanism of long-term weather prediction and climate changes, whose mathematical study was initiated by Lions, Teman and Wang ([24]- [26]). This research field has received wide attention from mathematical community over the last two decades. Taking advantage of the fact that the pressure is essentially two-dimensional in the PEs, Cao and Titi [5] proved the global results for the existence of strong solutions of three-dimensional PEs. Subsequently, Kukavica and Ziane [22] developed a different proof which allows one to treat non-rectangular domains as well as different and physically realistic boundary conditions. Ju [21] studied the long-time behavior of the PEs. We refer the reader to the survey papers [28, 29] for further references and background about the deterministic mathematical theory for the PEs.

Although the PEs express very fundamental laws of physics, the deterministic models are numerically intractable. Studies have shown that resolved states are associated with many possible unresolved states. This calls for stochastic methods for numerical weather and climate prediction which potentially allow a proper representation of the uncertainties, a reduction of systematic biases and improved representation of long-term climate variability (see [1, 27, 30]).

Despite the developments in the deterministic case, the theory for the stochastic PEs remains relatively underdeveloped. Ewald, Petcu, Teman [11] and Glatt-Holtz, Ziane [15] considered two-dimensional stochastic PEs. Then Glatt-Holtz and Temam [13, 14] extended the case to the greater generality of physically relevant boundary conditions and nonlinear multiplicative noise. Following the methods similar to [5], Guo and Huang [16] studied the global well-posedness of the three-dimensional system with an additive noise in the horizontal momentum equations and obtained some kind of weak-type compactness properties of the solutions to the stochastic system. Using methods different from [16], Debussche, Glatt-Holtz, Temam and Ziane [9] considered a three-dimensional system with multiplicative noise. A similar result is also obtained by Gao and Sun [18]. Moreover, when the noise tends to zero, Gao and Sun [19] established the large deviation principle for this stochastic system. In [20], Gao and Sun studied the long-time behavior of stochastic PEs when the velocity is perturbed by an additive noise. Under the periodic conditions, Glatt-Holtz, Kukavica, Vicol and Ziane constructed an invariant measure for the 3D PEs in [12]. The uniqueness of the invariant measure for the 3D stochastic PEs were obtained by Dong, Zhai and Zhang in [6] under the periodic conditions. In [8], Dong, Zhai and Zhang established the large deviation principle for the 3D stochastic PEs. Some analytical properties of weak solutions of 3D stochastic PEs with periodic boundary conditions were obtained by Dong and Zhang in [7], in which the martingale problem associated to this model is shown to have a family of solutions satisfying the Markov property.

In this paper, we mainly study the existence of random attractor and invariant measure for the stochastic PEs **defined on manifolds**, which **is the first result** for this kind of stochastic model. Note that the definition of attractors in our paper is different from that in [16]. The random attractor obtained in our work is $\mathbb{P}$-a.e. ω compact in $(H^1(\mathbf{D}))^4$ and attracts any orbit starting from $-\infty$ in the strong topology of $(H^1(\mathbf{D}))^4$. While the attractor studied in [16] is not necessarily a compact subset in $(H^1(\mathbf{D}))^4$, and the attractor attracts any orbit in the weak topology of $(H^1(\mathbf{D}))^4$.

Since the uniqueness of the weak solution to the 3D stochastic PEs is still open, we have to choose $(H^1(\mathbf{D}))^4$ as the phase space to work with. After following the method in [17] to prove the global existence of the strong solutions, we show the continuity of the strong solutions to the 3D stochastic PEs in the space $(H^1(\mathbf{D}))^4$ with respect to time t as well as with respect to the initial condition $(\mathbf{v}_0, T_0, q_0)$. Notice that [17] only proved that the strong solution is Lipschitz continuous in the space $(L^2(\mathbf{D}))^4$ with respect to the initial data, but this is not enough to study the asymptotic behavior in $(H^1(\mathbf{D}))^4$ considered here. The first new difficulty arising here is to obtain the regularities of the strong solution about time t and initial condition, the key problem is that we have no valid boundedness for the derivatives of the vertical velocity. The second difficulty is that the geometric structure of the manifold is more complicated than the case of $\mathbb{R}^n$. For example, in order to obtain *a priori* estimates in $(L^4(\mathbf{D}))^4$, there is no estimate like the following:

$$|\nabla_{e_\varphi}\mathbf{v}^3| \leq C|\nabla_{e_\varphi}\mathbf{v}||\mathbf{v}|^2,$$

where $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2, \mathbf{v}^3 = (v_1^3, v_2^3) \in \mathbb{R}^2$ and C is a constant. To overcome the difficulties, we should delicately and carefully use the geometric structure of the manifold involved with the velocity to obtain the *a priori* estimates. Finally, with the above mentioned *a priori* estimates and continuity properties, we try to find a compact absorbing set in $(H^1(\mathbf{D}))^4$ to establish the existence of the random attractor which is in fact the most difficult problem for this stochastic moist PEs. For a bounded domain, the common method is to find an absorbing set for the solutions in the functional space with higher regularity than $(H^1(\mathbf{D}))^4$. However, this seems to be very complicated for the 3D stochastic PEs. To overcome the difficulties, we adopt a new method inspired by [21] to prove the existence of random attractor. We should mention that the extension of the existence of global attractor of PEs to stochastic moist PEs is not straightforward. Firstly, there is a qualitative difference between the definitions of attractor and random attractor (see [3, 4]). Secondly, since PEs is an autonomous differential system, its solution operator is a semigroup whilst stochastic moist PEs is a nonautonomous differential system which generates a cocycle but not a semigroup. Thirdly, the regularity of the solutions to the stochastic equations with respect to time is much weaker than the deterministic cases. Finally, the stochastic moist PEs defined on manifolds is more complicated than the PEs defined on Cartesian coordinate system. To overcome the difficulties, **our main idea** is that we firstly prove that $\mathbb{P}$-a.e. ω the solution operator $\mathcal{S}(t,s;\omega)_{t\geq s;\omega\in\Omega}$ of stochastic PEs is compact in the functional space $(H^1(\mathbf{D}))^4$ for any fixed time $t, s \in \mathbb{R}$. Then by virtue of the regularity of strong solutions, we use the solution operator to act on an absorbing ball to construct a compact absorbing ball, which implies the existence of random attractor and invariant measure. We would like to mention that our method provides a general way for proving the existence of random attractor for common classes of dissipative stochastic partial differential equations with Wiener noises and has some advantages over the common method of using compact Sobolev imbedding theorem, i.e., if an absorbing ball for the solutions in space

$(H^2(\mathbf{D}))^4$ does exist, our method will then further imply the existence of global random attractor in $(H^2(\mathbf{D}))^4$.

REMARK 1.1. In the Appendix, we discuss the *a priori* estimates in various spaces for the solution $(\mathbf{v},T,q)$ to (1.7). Through the discussion of this part, we see that **the stochastic moist PEs defined on manifolds are much different from the stochastic model defined on a bounded subset of $\mathbb{R}^3$ and the deterministic model defined on manifolds** (see (A.5)). More precisely, for the stochastic model defined on manifolds, the positive constants α and β in the boundary conditions of (1.7) play very important roles in ensuring the existence of random attractor, on the contrary, the conditions $\alpha=0$ and $\beta=0$ are more preferable for the other two cases (see [16, 20]).

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and then present the main results, including local and global existence of solutions, as well as the existence of random attractor and invariant measure, the proof of global existence is given in Section 3. Since in Section 2 we point out that the existence of the random attractor implies the existence of the invariant measure, in Section 4 we only study the existence of random attractor. The *a priori* estimates for the global existence of strong solutions are shown in the Appendix. As usual, constant C may change from one line to the next, unless we give a special declaration; we denote by $C(a)$ a constant which depends on some parameter a.

2. Preliminaries and main results

In this section, we will give the formal definitions for differential operators in the pressure coordinate system, and the stochastic terms, then reformulate this geophysical system into an abstract setting. We will present the main results at the end of this section.

Let $e_\theta, e_\varphi, e_\xi$ be the unit vectors in θ,φ,ξ directions of the space domain $\mathbf{D}=\mathbf{S}^2\times(0,1)$, respectively,

$$e_\theta=\partial_\theta,\ e_\varphi=\frac{1}{\sin\theta}\partial_\varphi,\ e_\xi=\partial_\xi.$$

Correspondingly, define the following spaces

$$L^p(\mathbf{D}):=\{h:\mathbf{D}\to\mathbb{R},\int_{\mathbf{D}}|h|^p d\mathbf{D}<\infty\},\ \text{for } 1\le p<\infty,$$

$$L^2(T\mathbf{D}|T\mathbf{S}^2):=\{\mathbf{v}=(v_\theta,v_\varphi):\mathbf{D}\to T\mathbf{S}^2,\int_{\mathbf{D}}(|v_\theta|^2+|v_\varphi|^2)d\mathbf{D}<\infty\},$$

$C^\infty(\mathbf{S}^2)$, $C^\infty(\mathbf{D})$ are smooth function spaces defined on $\mathbf{S}^2,\mathbf{D}$, respectively,

$$C^\infty(T\mathbf{D}|T\mathbf{S}^2):=\{\mathbf{v}=(v_\theta,v_\varphi):v_\theta,v_\varphi\in C^\infty(\mathbf{D})\}.$$

$H^m(\mathbf{D})$ is the Sobolev space of functions which are in L^2, with all covariant derivatives with respect to e_θ,e_φ,e_ξ of order $\le m$, then analogously, we define

$$H^m(T\mathbf{D}|T\mathbf{S}^2):=\{\mathbf{v}=(v_\theta,v_\varphi):v_\theta,v_\varphi\in H^m(\mathbf{D})\}.$$

In the pressure coordinate system, given $\mathbf{v}=v_\theta e_\theta+v_\varphi e_\varphi,\mathbf{u}=u_\theta e_\theta+u_\varphi e_\varphi\in C^\infty(T\mathbf{D}|T\mathbf{S}^2)$, $T,q\in C^\infty(\mathbf{D})$ and $\Phi_s\in C^\infty(\mathbf{S}^2)$, we first define the horizontal gradient ∇ for T,Φ_s on $\mathbf{S}^2$ as follows:

$$\nabla T=(\partial_\theta T)e_\theta+\Big(\frac{1}{\sin\theta}\partial_\varphi T\Big)e_\varphi, \tag{2.1a}$$

$$\nabla \Phi_s = (\partial_\theta \Phi_s) e_\theta + \Big(\frac{1}{\sin\theta}\partial_\varphi \Phi_s\Big) e_\varphi. \tag{2.1b}$$

We then define the covariant derivatives of $\mathbf{u}, T, q$ with respect to $\mathbf{v}$ as follows:

$$\nabla_{\mathbf{v}} \mathbf{u} = \Big(v_\theta \partial_\theta u_\theta + \frac{v_\varphi}{\sin\theta}\partial_\varphi u_\theta - v_\varphi u_\varphi \cot\theta\Big) e_\theta + \Big(v_\theta \partial_\theta u_\varphi + \frac{v_\varphi}{\sin\theta}\partial_\varphi u_\varphi + v_\varphi u_\theta \cot\theta\Big) e_\varphi, \tag{2.2a}$$

$$\nabla_{\mathbf{v}} T = v_\theta \partial_\theta T + \frac{v_\varphi}{\sin\theta}\partial_\varphi T, \tag{2.2b}$$

$$\nabla_{\mathbf{v}} q = v_\theta \partial_\theta q + \frac{v_\varphi}{\sin\theta}\partial_\varphi q. \tag{2.2c}$$

Finally, the divergence form of $\mathbf{v}$ and the Laplace-Beltrami operator Δ for scalar and vector functions are defined as follows:

$$\operatorname{div} \mathbf{v} = \operatorname{div}(v_\theta e_\theta + v_\varphi e_\varphi) = \frac{1}{\sin\theta}(\partial_\theta(v_\theta \sin\theta) + \partial_\varphi v_\varphi), \tag{2.3a}$$

$$\Delta T = \operatorname{div}(\nabla T) = \frac{1}{\sin\theta}[\partial_\theta(\sin\theta \partial_\theta T) + \frac{1}{\sin\theta}\partial_{\varphi\varphi} T], \tag{2.3b}$$

$$\Delta q = \operatorname{div}(\nabla q) = \frac{1}{\sin\theta}[\partial_\theta(\sin\theta \partial_\theta q) + \frac{1}{\sin\theta}\partial_{\varphi\varphi} q], \tag{2.3c}$$

$$\Delta \mathbf{v} = \Big(\Delta v_\theta - \frac{2\cos\theta}{\sin^2\theta}\partial_\varphi v_\varphi - \frac{v_\theta}{\sin^2\theta}\Big) e_\theta + \Big(\Delta v_\varphi + \frac{2\cos\theta}{\sin^2\theta}\partial_\varphi v_\theta - \frac{v_\varphi}{\sin^2\theta}\Big) e_\varphi. \tag{2.3d}$$

We then define our working spaces for the stochastic PE system as follows:

$$\begin{aligned}
\mathcal{V}_1^0 &:= \Big\{\mathbf{v} \in C^\infty(T\mathbf{D}|T\mathbf{S}^2) : \partial_\xi \mathbf{v}\big|_{\xi=0} = \partial_\xi \mathbf{v}\big|_{\xi=1} = 0, \int_0^1 \operatorname{div} \mathbf{v} d\xi = 0\Big\},\\
\mathcal{V}_2^0 &:= \Big\{T \in C^\infty(\mathbf{D}) : \partial_\xi T\big|_{\xi=0} = 0,\ \partial_\xi T\big|_{\xi=1} = -\alpha T\Big\},\\
\mathcal{V}_3^0 &:= \Big\{q \in C^\infty(\mathbf{D}) : \partial_\xi q\big|_{\xi=0} = 0,\ \partial_\xi q\big|_{\xi=1} = -\beta q\Big\}.
\end{aligned}$$

We denote by $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_3$ the closure spaces of $\mathcal{V}_1^0$ in $H^1(T\mathbf{D}|T\mathbf{S}^2)$, $\mathcal{V}_2^0$ and $\mathcal{V}_3^0$ in $H^1(\mathbf{D})$ with respect to H^1 norm. Also define $\mathcal{H}_1$ as the closure space of $\mathcal{V}_1^0$ with respect to L^2 norm in $L^2(T\mathbf{D}|T\mathbf{S}^2)$, $\mathcal{H}_2 = \mathcal{H}_3 = L^2(\mathbf{D})$. Now set

$$\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3,\ \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3.$$

Let $U := (\mathbf{v}, T, q), \widetilde{U} := (\widetilde{\mathbf{v}}, \widetilde{T}, \widetilde{q}) \in \mathcal{V}$ and we equip $\mathcal{V}$ with the inner product

$$\begin{aligned}
\langle U, \widetilde{U}\rangle_{\mathcal{V}} &:= \langle \mathbf{v}, \widetilde{\mathbf{v}}\rangle_{\mathcal{V}_1} + \langle T, \widetilde{T}\rangle_{\mathcal{V}_2} + \langle q, \widetilde{q}\rangle_{\mathcal{V}_3},\\
\langle \mathbf{v}, \widetilde{\mathbf{v}}\rangle_{\mathcal{V}_1} &:= \int_{\mathbf{D}} (\nabla_{e_\theta}\mathbf{v} \cdot \nabla_{e_\theta}\widetilde{v} + \nabla_{e_\varphi}\mathbf{v} \cdot \nabla_{e_\varphi}\widetilde{\mathbf{v}} + \partial_\xi \mathbf{v} \partial_\xi \widetilde{\mathbf{v}} + \mathbf{v} \cdot \widetilde{\mathbf{v}}) d\mathbf{D},\\
\langle T, \widetilde{T}\rangle_{\mathcal{V}_2} &:= \int_{\mathbf{D}} (\nabla T \cdot \nabla \widetilde{T} + \partial_\xi T \partial_\xi \widetilde{T}) d\mathbf{D} + \alpha \int_{\mathbf{S}^2} T|_{\xi=1} \widetilde{T}|_{\xi=1} d\mathbf{S}^2,\\
\langle q, \widetilde{q}\rangle_{\mathcal{V}_3} &:= \int_{\mathbf{D}} (\nabla q \cdot \nabla \widetilde{q} + \partial_\xi q \partial_\xi \widetilde{q}) d\mathbf{D} + \beta \int_{\mathbf{S}^2} q|_{\xi=1} \widetilde{q}|_{\xi=1} d\mathbf{S}^2.
\end{aligned}$$

Similarly, we equip the Hilbert space $\mathcal{H}$ with the inner product

$$\langle U, \widetilde{U}\rangle_{\mathcal{H}} := \langle \mathbf{v}, \widetilde{\mathbf{v}}\rangle + \langle T, \widetilde{T}\rangle + \langle q, \widetilde{q}\rangle,$$

$$\langle \mathbf{v},\widetilde{\mathbf{v}}\rangle := \int_{\mathbf{D}} \mathbf{v}\cdot\widetilde{\mathbf{v}} d\mathbf{D},$$
$$\langle T,\widetilde{T}\rangle := \int_{\mathbf{D}} T\widetilde{T} d\mathbf{D},$$
$$\langle q,\widetilde{q}\rangle := \int_{\mathbf{D}} q\widetilde{q} d\mathbf{D}.$$

We also denote different norms for $U,\mathbf{v},T,q$ by

$$\|U\|_1 = \langle U,U\rangle_{\mathcal{V}}^{\frac{1}{2}},\ \|\mathbf{v}\|_1 = \langle \mathbf{v},\mathbf{v}\rangle_{\mathcal{V}_1}^{\frac{1}{2}},\ \|T\|_1 = \langle T,T\rangle_{\mathcal{V}_2}^{\frac{1}{2}},\ \|q\|_1 = \langle q,q\rangle_{\mathcal{V}_3}^{\frac{1}{2}}.$$

$$|U|_2 = \langle U,U\rangle_{\mathcal{H}}^{\frac{1}{2}},\ |\mathbf{v}|_2 = \langle \mathbf{v},\mathbf{v}\rangle^{\frac{1}{2}},\ |T|_2 = \langle T,T\rangle^{\frac{1}{2}},\ |q|_2 = \langle q,q\rangle^{\frac{1}{2}}.$$

We here directly quote Lemma 4.1 in [17], without a proof, to obtain some integral equalities:

Lemma 2.1. *Let* $\mathbf{u} = (u_\theta,u_\varphi),\widetilde{\mathbf{u}} = (\widetilde{u}_\theta,\widetilde{u}_\varphi) \in C^\infty(T\mathbf{D}|T\mathbf{S}^2)$*, and* $h \in C^\infty(\mathbf{S}^2)$*, we have*
(i)

$$\int_{\mathbf{S}^2} h\,div\,\mathbf{u} d\mathbf{S}^2 = -\int_{\mathbf{S}^2} \nabla h\cdot \mathbf{u} d\mathbf{S}^2, \quad \textit{in particular, } \int_{\mathbf{D}} \nabla h\cdot\mathbf{v} d\mathbf{D} = 0,\ \textit{for any } \mathbf{v}\in\mathcal{V}_1.$$

(ii)

$$\int_{\mathbf{D}} (-\Delta\mathbf{u})\cdot\widetilde{\mathbf{u}} d\mathbf{D} = \int_{\mathbf{D}} (\nabla_{e_\theta}\mathbf{u}\cdot\nabla_{e_\theta}\widetilde{\mathbf{u}} + \nabla_{e_\varphi}\mathbf{u}\cdot\nabla_{e_\varphi}\widetilde{\mathbf{u}} + \mathbf{u}\cdot\widetilde{\mathbf{u}}) d\mathbf{D}.$$

In our paper, we will frequently use the following inequalities, so we state them in the following lemmas. For their proof, one can refer to [16].

Lemma 2.2. *Let* $v \in H^2(T\mathbf{D}|T\mathbf{S}^2), \mu \in H^1(T\mathbf{D}|T\mathbf{S}^2)\Big($*or* $\mu \in H^1(\mathbf{D})\Big)$ *and* $\nu \in L^2(T\mathbf{D}|T\mathbf{S}^2)\Big($*or* $\nu \in L^2(\mathbf{D})\Big)$*. Then, there exists a positive constant c independent of* v,μ *and* ν *such that*

$$\begin{aligned}
&\Big|\langle\Big(\int_\xi^1 \mathrm{div} v(t;\theta,\phi,\xi')d\xi'\Big)\mu,\nu\rangle\Big| \\
&\le c|\mathrm{div} v|_2^{\frac{1}{2}}(|\mathrm{div} v|_2^{\frac{1}{2}} + |\Delta v|_2^{\frac{1}{2}})|\mu|_2^{\frac{1}{2}}(|\nabla_{e_\theta}\mu|_2^{\frac{1}{2}} + |\nabla_{e_\varphi}\mu|_2^{\frac{1}{2}} + |\Delta\mu|_2^{\frac{1}{2}})|\nu|_2, \\
&\Big(or\ \le c|\mathrm{div} v|_2^{\frac{1}{2}}(|\mathrm{div} v|_2^{\frac{1}{2}} + |\Delta v|_2^{\frac{1}{2}})|\mu|_2^{\frac{1}{2}}(|\nabla\mu|_2^{\frac{1}{2}} + |\Delta\mu|_2^{\frac{1}{2}})|\nu|_2\Big).
\end{aligned}$$

Lemma 2.3. *Let* $v \in H^1(T\mathbf{D}|T\mathbf{S}^2), \mu \in H^1(T\mathbf{D}|T\mathbf{S}^2)\Big($*or* $\mu \in H^1(\mathbf{D})\Big)$ *and* $\nu \in H^1(T\mathbf{D}|T\mathbf{S}^2)\Big($*or* $\nu \in H^1(\mathbf{D})\Big)$*. Then, there exists a positive constant c independent of* v,μ *and* ν *such that*

$$\begin{aligned}
&\Big|\langle\Big(\int_\xi^1 \mathrm{div} v(t;\theta,\phi,\xi')d\xi'\Big)\mu,\nu\rangle\Big| \\
&\le c|\mathrm{div} v|_2|\mu|_2^{\frac{1}{2}}(|\mu|_2^{\frac{1}{2}} + |\nabla_{e_\theta}\mu|_2^{\frac{1}{2}} + |\nabla_{e_\varphi}\mu|_2^{\frac{1}{2}})|\nu|_2^{\frac{1}{2}}(|\nu|_2^{\frac{1}{2}} + |\nabla_{e_\theta}\nu|_2^{\frac{1}{2}} + |\nabla_{e_\varphi}\nu|_2^{\frac{1}{2}}),
\end{aligned}$$

$\Big(or\ \le c|\text{div}v|_2|\mu|_2^{\frac12}(|\mu|_2^{\frac12}+|\nabla\mu|_2^{\frac12})|\nu|_2^{\frac12}(|\nu|_2^{\frac12}+|\nabla\nu|_2^{\frac12})\Big)$.

Now define linear operators $A_i:\mathcal{V}_i\mapsto\mathcal{V}_i', i=1,2,3:$

$$\langle A_1\mathbf{v},\widetilde{\mathbf{v}}\rangle=\langle\mathbf{v},\widetilde{\mathbf{v}}\rangle_{\mathcal{V}_1},\ \text{for any }\mathbf{v},\widetilde{\mathbf{v}}\in\mathcal{V}_1,$$
$$\langle A_2T,\widetilde{T}\rangle=\langle T,\widetilde{T}\rangle_{\mathcal{V}_2},\ \text{for any }T,\widetilde{T}\in\mathcal{V}_2,$$
$$\langle A_3q,\widetilde{q}\rangle=\langle q,\widetilde{q}\rangle_{\mathcal{V}_3},\ \text{for any }q,\widetilde{q}\in\mathcal{V}_3.$$

Denote by $D(A_i)=\{\eta\in\mathcal{V}_i,A_i\eta\in\mathcal{H}_i\}$. From (ii) in Lemma 2.1, we see that A_i's are positive definite, self-adjoint operators, according to the classic spectral theory we can define the power A_i^s for any $s\in\mathbb{R}$. Then we have $D(A_i^{\frac12})=\mathcal{V}_i$ and $D(A_i^{-\frac12})=\mathcal{V}_i'$. Moreover,

$$D(A_i)\subset\mathcal{V}_i\subset\mathcal{H}_i\subset\mathcal{V}_i'\subset D(A_i)',$$

where $D(A_i)'$ is the dual space of $D(A_i)$, and the embeddings above are all compact. Now we denote by P_H the Leray-type projection operator from $L^2(T\mathbf{D}|T\mathbf{S}^2)\times(L^2(\mathbf{D}))^2$ onto $\mathcal{H}$, and take $\mathcal{V}^{(2)}$ as the closure of $\mathcal{V}$ in the $H^2(T\mathbf{D}|T\mathbf{S}^2)\times(H^2(\mathbf{D}))^2$ norm, then we define the principal linear portion of the system:

$$AU=P_H\begin{pmatrix}-\Delta\mathbf{v}-\partial_{\xi\xi}\mathbf{v}\\-\Delta T-\partial_{\xi\xi}T\\-\Delta q-\partial_{\xi\xi}q\end{pmatrix},\ \text{for any }U\in D(A),$$

where

$$\begin{aligned}D(A):=\{U=(\mathbf{v},T,q)\in\mathcal{V}^{(2)};\ &\text{On }\xi=1:\ \partial_\xi\mathbf{v}=0,w=0,\partial_\xi T=-\alpha T,\partial_\xi q=-\beta q;\\&\text{On }\xi=0:\ \partial_\xi\mathbf{v}=0,w=0,\partial_\xi T=0,\partial_\xi q=0\},\end{aligned}$$

and $\langle AU,\widetilde{U}\rangle=\langle U,\widetilde{U}\rangle_{\mathcal{V}}$ for all $U,\widetilde{U}\in D(A)$. In particular, let $\mathbf{u}=(u_\theta,u_\varphi)\in C^\infty(T\mathbf{D}|T\mathbf{S}^2),\widetilde{\mathbf{u}}=(-\Delta-\partial_{\xi\xi})\mathbf{u}$. Applying Lemma 2.1 and boundary conditions, we have

$$\begin{aligned}&\langle(-\Delta-\partial_{\xi\xi})\mathbf{u},(-\Delta-\partial_{\xi\xi})\mathbf{u}\rangle\\=&\langle(-\Delta-\partial_{\xi\xi})\mathbf{u},\widetilde{\mathbf{u}}\rangle\\=&\langle-\Delta\mathbf{u},\widetilde{\mathbf{u}}\rangle+\langle-\partial_{\xi\xi}\mathbf{u},\widetilde{\mathbf{u}}\rangle\\=&\int_{\mathbf{D}}(\nabla_{e_\theta}\mathbf{u}\cdot\nabla_{e_\theta}\widetilde{\mathbf{u}}+\nabla_{e_\varphi}\mathbf{u}\cdot\nabla_{e_\varphi}\widetilde{\mathbf{u}}+\mathbf{u}\cdot\widetilde{\mathbf{u}})d\mathbf{D}+\int_{\mathbf{D}}\partial_\xi\mathbf{u}\cdot\partial_\xi\widetilde{\mathbf{u}}d\mathbf{D}.\end{aligned}$$

Next, we define the diagnostic function:

$$w(\mathbf{v}):=\int_\xi^1\text{div }\mathbf{v}(t;\theta,\varphi,\xi')d\xi',\qquad\mathbf{v}\in\mathcal{V}_1.\tag{2.4}$$

Now take $U,\widetilde{U}\in D(A)$ and define the nonlinear operator as

$$B(U,\widetilde{U}):=P_H\begin{pmatrix}\nabla_{\mathbf{v}}\widetilde{\mathbf{v}}\\\nabla_{\mathbf{v}}\widetilde{T}\\\nabla_{\mathbf{v}}\widetilde{q}\end{pmatrix}+P_H\begin{pmatrix}w(\mathbf{v})\partial_\xi\widetilde{\mathbf{v}}\\w(\mathbf{v})\partial_\xi\widetilde{T}\\w(\mathbf{v})\partial_\xi\widetilde{q}\end{pmatrix}.\tag{2.5}$$

Also, we define the pressure operator, Coriolis operator and external operator as

$$A_pU:=P_H\begin{pmatrix}\int_\xi^1\frac{br_s}{r}\nabla[(1+aq)T]d\xi'\\-\frac{br_s}{r}(1+aq)\Big(\int_\xi^1\text{div }\mathbf{v}(t;\theta,\varphi,\xi')d\xi'\Big)\\0\end{pmatrix},\ U\in\mathcal{V},\tag{2.6}$$

$$EU := P_H \begin{pmatrix} \frac{f}{R_0}\vec{k}\times\mathbf{v} \\ 0 \\ 0 \end{pmatrix},\ U\in\mathcal{H}, \tag{2.7}$$

$$F := P_H \begin{pmatrix} 0 \\ Q_T \\ Q_q \end{pmatrix}. \tag{2.8}$$

Finally, Let $(B_i(t))_{i\in\mathbb{N}^+}$ be a sequence of one-dimensional, independent, identically distributed, two-sided Brownian motions, defined on the complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. For $j=1,2,3$, we write $(e_{i,j})_{i\in\mathbb{N}^+}$, for an orthonormal basis of $\mathcal{H}_j$, consisting of eigenfunctions of the operator A_j and $(\gamma_{i,j})_{i\in\mathbb{N}^+,}$ for the sequence of the corresponding eigenvalues. We introduce the $\mathcal{H}_j$-valued Wiener process $(W_j(.,t))_{t\in\mathbb{R}^+}$ with $j=1,2,3$ by setting

$$W_j(.,t) := \sum_{i=1}^{\infty}\lambda_{i,j}^{\frac{1}{2}}e_{i,j}(.)B_i(t), \tag{2.9}$$

where $(\lambda_{i,j})_{i\in\mathbb{N}^+}$ is a sequence of positive numbers such that the series converge a.s. in the strong topology of $\mathcal{H}_j$.

With all the above operator notations, we could reformulate (1.7) into the following abstract evolution system,

$$dU+(AU+B(U)+A_pU+EU)dt = Fdt+dW,\ U(t_0)=U_0, \tag{2.10}$$

where $U_0=(\mathbf{v}_0,T_0,q_0)(\theta,\varphi,\xi)$.

Definition 2.1. *For $\mathbb{P}$-a.e. $\omega\in\Omega$, we say a continuous $\mathcal{V}$-valued $\mathcal{F}_{t_0,t}=\sigma(W_i(s)-W_i(t_0),s\in[t_0,t],i=1,2,3)$ adapted random field $(U(.,t))_{t\in[t_0,\tau]} := (\mathbf{v}(.,t),T(.,t),q(.,t))_{t\in[t_0,\tau]}$ defined on $(\Omega,\mathcal{F},\mathbb{P})$ is a* ***strong solution*** *to problem (1.7a)-(1.7f) with $\mathbf{v}_0\in\mathcal{V}_1,T_0\in\mathcal{V}_2,q_0\in\mathcal{V}_3$ and $t_0,\tau\in\mathbb{R},\tau\geq t_0$, if $(U(.,t))_{t\in[t_0,\tau]}$ satisfies (1.7a)-(1.7f) in the weak sense such that*

$$\begin{aligned}&\mathbf{v}\in C([t_0,\tau];\mathcal{V}_1)\cap L^2([t_0,\tau];(H^2(\mathbf{D}))^2),\\&T\in C([t_0,\tau];\mathcal{V}_2)\cap L^2([t_0,\tau];H^2(\mathbf{D})),\\&q\in C([t_0,\tau];\mathcal{V}_3)\cap L^2([t_0,\tau];H^2(\mathbf{D})).\end{aligned}$$

Similarly, we can define the strong solution to (3.2a)-(3.2h). In this part, we state our results about the local well-posedness, the global well-posedness, the existence of random attractor and invariant measure.

Theorem 2.1 (Existence of local solutions). *If $Q_T,\partial_\xi Q_T,\ Q_q,\ \partial_\xi Q_q\in L^2(\mathbf{D})$, $(\mathbf{v}_0,T_0,q_0)\in\mathcal{V}$ then, for $\mathbb{P}$-a.e. $\omega\in\Omega$, there exists a stopping time $\mathcal{T}>0$ such that $(\mathbf{v},T,q)$ is a strong solution of the system* (1.7a)-(1.7f) *on the interval* $[0,\mathcal{T}]$.

To consider the local well-posedness, we separate the Equation (1.7a)-(1.7f) into a deterministic linear equation corresponding to (1.7a) and the stochastic nonlinear part with zero initial condition, i.e., (1.7a)-(1.7f) with $\mathbf{u}(t_0)=\mathbf{u}_0$ replaced by $\mathbf{u}(t_0)=0$. The global well-posedness of linear part is well known. The proof of the local well-posedness of the nonlinear part with zero initial condition is also classic. We first obtain that the sequence of solutions to the approximation of (1.7a)-(1.7f) is bounded in $L^2([t_0,\tau];(H^2(\mathbf{D}))^4)$, then we use Aubin-Lions Lemma to obtain a strongly convergent subsequence of solutions in $L^2([t_0,\tau];(H^1(\mathbf{D}))^4)$. Reasoning on weakly and strongly convergent subsequences one gets the existence of a solution with the regularity specified by Theorem 2.1. The proof is similar to [17], so we omit it here.

THEOREM 2.2 (Existence of global solutions). *Let Q_T, $\partial_\xi Q_T$, Q_q, $\partial_\xi Q_q \in L^2(\mathbf{D})$, $(\mathbf{v}_0,T_0,q_0)\in\mathcal{V}$, and $\sum_{i=1}^{\infty}\lambda_{i,j}^2\gamma_{i,j}^{2+\sigma}<\infty, j=1,2,3,$ for small positive constant σ. Then, for arbitrary $\tau>t_0$, there exists a unique strong solution $(\mathbf{v},T,q)$ to the system* (1.7a)-(1.7f) *on the interval $[t_0,\tau]$, which is Lipschitz continuous with respect to the initial data in $\mathcal{V}$.*

Now we give preliminary knowledge about random attractors. Let (X,d) be a Polish space and $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ be a probability space, where $\widetilde{\Omega}$ is the two-sided Wiener space $C_0(\mathbb{R};X)$ of continuous functions with values in X, equaling 0 at $t=0$. We consider a family of mappings $\mathcal{S}(t,s;\omega):X\to X$, $-\infty<s\le t<\infty$, parametrized by $\omega\in\widetilde{\Omega}$, satisfying for $\widetilde{\mathbb{P}}$-*a.s.* ω, the following properties (i)-(iv):

(i) $\mathcal{S}(t,r;\omega)\mathcal{S}(r,s;\omega)x=\mathcal{S}(t,s;\omega)x$ for all $s\le r\le t$ and $x\in X$,

(ii) $\mathcal{S}(t,s;\omega)$ is continuous in X, for all $s\le t$,

(iii) for all $s<t$ and $x\in X$, the mapping

$$\omega\mapsto\mathcal{S}(t,s;\omega)x$$

is measurable from $(\widetilde{\Omega},\widetilde{\mathcal{F}})$ to $(X,\mathcal{B}(X))$ where $\mathcal{B}(X)$ is the Borel-σ- algebra of X,

(iv) for all $t,x\in X$, the mapping $s\mapsto\mathcal{S}(t,s;\omega)$ is right-continuous at any point.

We define for $A,B\in 2^X$ with $A,B\neq\emptyset$, $d(A,B)=\sup\{\inf\{d(x,y):y\in B\}:x\in A\}$, and it follows that $d(x,B)=d(\{x\},B)$. We now give the following definitions.

DEFINITION 2.2. *A set-valued map $K:\widetilde{\Omega}\to 2^X$ taking values in the closed subsets of X is said to be* ***measurable****, if for each $x\in X$, the map $\omega\mapsto d(x,K(\omega))$ is measurable. A closed set-valued measurable map $K:\widetilde{\Omega}\to 2^X$ is called a* ***random closed set****.*

DEFINITION 2.3. *Given $t\in\mathbb{R}$ and $\omega\in\widetilde{\Omega}, K(t,\omega)\subset X$ is called an* ***attracting set*** *at time t if, for all bounded sets $B\subset X$,*

$$d(\mathcal{S}(t,s;\omega)B,K(t,\omega))\to 0,\quad \textit{provided}\quad s\to-\infty.$$

Moreover, if for all bounded sets $B\subset X$, there exists $t_B(\omega)$ such that for all $s\le t_B(\omega)$,

$$\mathcal{S}(t,s;\omega)B\subset K(t,\omega),$$

we say $K(t,\omega)$ is an ***absorbing set*** *at time t.*

Let $\{\vartheta_t:\widetilde{\Omega}\to\widetilde{\Omega}\}_{t\in\mathbb{R}}$ be a family of measure-preserving transformations of the probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ such that for all $s<t$ and $\omega\in\widetilde{\Omega}$,

(a) $(t,\omega)\to\vartheta_t\omega$ is measurable,

(b) $\vartheta_t(\omega)(s)=\omega(t+s)-\omega(t)$,

(c) $\mathcal{S}(t,s;\omega)x=\mathcal{S}(t-s,0;\vartheta_s\omega)x$.

We defined $(\vartheta_t)_{t\in T}$ as a flow, and $((\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}}),(\vartheta_t)_{t\in\mathbb{R}})$ is a measurable dynamical system.

DEFINITION 2.4. *Given a bounded set $B\subset X$, the set*

$$\mathcal{A}(B,t,\omega)=\bigcap_{T\le t}\overline{\bigcup_{s\le T}\mathcal{S}(t,s,\omega)B}$$

is said to be the Ω***-limit set of*** B *at time* t*. Obviously, if we denote* $\mathcal{A}(B,0,\omega)=\mathcal{A}(B,\omega)$*, we have* $\mathcal{A}(B,t,\omega)=\mathcal{A}(B,\vartheta_t\omega)$*.*

We may identify

$$\mathcal{A}(B,t,\omega)=\{x\in X:\ \text{there exist } s_n\to-\infty,\ x_n\in B \text{ such that } \lim_{n\to\infty}\mathcal{S}(t,s_n,\omega)x_n=x\}. \tag{2.11}$$

Furthermore, if there exists a compact attracting set $K(t,\omega)$ at time t, it is not difficult to check that $\mathcal{A}(B,t,\omega)$ is a nonempty compact subset of X and $\mathcal{A}(B,t,\omega)\subset K(t,\omega)$. Now we are ready to give the definition of random attractors as follows:

Definition 2.5. *If, for any* $t\in\mathbb{R}$ *and* $\omega\in\widetilde{\Omega}$*, the random closed set* $\omega\to\mathcal{A}(t,\omega)$ *satisfies the following properties:*

(1) $\mathcal{A}(t,\omega)$ *is a nonempty compact subset of* X*,*

(2) $\mathcal{A}(t,\omega)$ *is the minimal closed attracting set, i.e., if* $\widetilde{\mathcal{A}}(t,\omega)$ *is another closed attracting set, then* $\mathcal{A}(t,\omega)\subset\widetilde{\mathcal{A}}(t,\omega)$*,*

(3) it is invariant, in the sense that, for all $s\le t$*,*

$$\mathcal{S}(t,s;\omega)\mathcal{A}(s,\omega)=\mathcal{A}(t,\omega).$$

$\mathcal{A}(t,\omega)$ *is called the* ***random attractor****.*

We finish this section with our main result, the existence of random attractor and invariant measure for (1.7a)-(1.7f).

Theorem 2.3 (Existence of random attractor). *In addition to the conditions in Theorem 2.2, we assume* $|\frac{br_s}{r}|\le\min\{\frac{1}{2},\alpha,\beta\}$*. Then the solution operator* $(\mathcal{S}(t,s;\omega))_{t\ge s,\omega\in\tilde{\Omega}}$ *of 3D stochastic PEs* (1.7a)-(1.7f)*:* $\mathcal{S}(t,s;\omega)(\mathbf{v}_s,T_s,q_s)=(\mathbf{v}(t),T(t),q(t))$ *has properties* (i)$-$(iv) *and possesses a compact absorbing ball* $\mathcal{B}(0,\omega)$ *in* $\mathcal{V}$ *at time* 0*. Furthermore, for* $\tilde{\mathbb{P}}$*-a.e.* ω*, the set*

$$\mathcal{A}(\omega)=\overline{\bigcup_{B\subset\mathcal{V}}\mathcal{A}(B,\omega)}$$

is the random attractor of stochastic PEs, where the union is taken over all the bounded subsets of $\mathcal{V}$*.*

With the above conclusions, we can prove the existence of invariant measures for the system (1.7).

Let $U_0:=(\mathbf{v}_0,T_0,q_0)\in\mathcal{V}$. In the following, we denote by

$$U(t,\omega;U_0):=(\mathbf{v}(t,\omega;t_0,\mathbf{v}_0),T(t,\omega;t_0,T_0),q(t,\omega;t_0,q_0))$$

the solution to (1.7) with $(\mathbf{v}(t_0)=\mathbf{v}_0,T(t_0)=T_0,q(t_0)=q_0)$. Following the standard argument, we can show that $U(t,\omega;U_0),t\in[t_0,\mathcal{T}],t_0\le\mathcal{T}$ is Markov in the following sense:

For every bounded, $\mathcal{B}(\mathcal{V})$-measurable $F:\mathcal{V}\to\mathbb{R}$, and all $s,t\in[t_0,\mathcal{T}]$, $t_0\le s\le t\le\mathcal{T}$

$$\mathbb{E}[F(U(t,\omega;U_0))|\mathcal{F}_s](\omega)=\mathbb{E}[F(U(t,s,U(s)))]\ \text{ for } \tilde{\mathbb{P}}-a.e.\ \omega\in\Omega.$$

where $\mathcal{F}_s=\mathcal{F}_{t_0,s}$ (see Definition 2.1) and $U(t,s,U(s))$ is the solution to (1.7) at time t with initial data $U(s)$.

For $B\in\mathcal{B}(\mathcal{V})$ the collection of Borel-measurable subset on $\mathcal{V}$, we define

$$\tilde{\mathbb{P}}_t(U_0,B)=\tilde{\mathbb{P}}((U(t,\omega;U_0)\in B).$$

For any probability measure ν defined on $\mathcal{B}(\mathcal{V})$, we denote by $(\nu\tilde{\mathbb{P}}_t)(\cdot)=\int_{\mathcal{V}}\tilde{\mathbb{P}}_t(x,\cdot)\nu(dx)$ the distribution at time t of the solution to (1.7) with initial condition having the distribution ν.

For $t\geq t_0$ and any function $f\in C_b(\mathcal{V};\mathbb{R})$, which is the set of continuous and bounded functions from $\mathcal{V}$ into $\mathbb{R}$, denoted by

$$\tilde{\mathbb{P}}_t f(U_0)=\mathbb{E}[f(U(t,\omega;U_0))]=\int_V f(x)\tilde{\mathbb{P}}_t(U_0,dx).$$

Definition 2.6. *Let ρ be a probability measure on $\mathcal{B}(\mathcal{V})$. We say that ρ is* **an invariant measure** *for $\tilde{\mathbb{P}}_t$ if we have*

$$\int_{\mathcal{V}} f(x)\rho(dx)=\int_{\mathcal{V}}\tilde{\mathbb{P}}_t f(x)\rho(dx)$$

for all $f\in C_b(\mathcal{V};\mathbb{R})$ and $t\geq 0$.

Let $\mu.$ be a transition probability from $\tilde{\Omega}$ to $\mathcal{V}$, i.e., $\mu.$ is a Borel probability measure on $\mathcal{V}$ and $\omega\to\mu.(B)$ is measurable for every Borel set $B\subset\mathcal{V}$. Denote by $\mathcal{P}_{\tilde{\Omega}}(\mathcal{V})$ the set of transition probabilities with $\mu.$ and $\nu.$ identified if $\tilde{\mathbb{P}}\{\omega:\mu_\omega\neq\nu_\omega\}=0$.

In view of Proposition 4.5 in [4], the existence of random attractor obtained in Theorem 2.3 implies the existence of an invariant Markov measure $\mu.\in\mathcal{P}_{\tilde{\Omega}}(V)$ for $\mathcal{S}$ such that $\mu_\omega(\mathcal{A}(\omega))=1$ $\tilde{\mathbb{P}}$-a.e.. Therefore, by [2] there exists an invariant measure for the Markov semigroup $\tilde{\mathbb{P}}_t$ and it is given by

$$\rho(B)=\int_{\tilde{\Omega}}\mu_\omega(B)\tilde{\mathbb{P}}(d\omega),$$

where $B\subseteq\mathcal{V}$ is a Borel set and $f\in C_b(\mathcal{V};\mathbb{R})$. If the invariant measure ρ for $\tilde{\mathbb{P}}$ is unique, the invariant Markov measure $\mu.$ for $\mathcal{S}$ is unique and given by

$$\mu_\omega=\lim_{t\to\infty}\mathcal{S}(0,-t,\omega)\rho.$$

Summarizing the above argument, we arrive at the following result:

Theorem 2.4. *The Markov semigroup $(\tilde{\mathbb{P}}_t)_{t\geq 0}$ induced by the solution $(U(t,\omega;U_0))_{t\geq 0}$ to (1.7) has an invariant measure ρ with $\rho(\mathcal{A}(\omega))=1$ $\tilde{\mathbb{P}}$-a.e..*

3. Global well-posedness of strong solutions

We need the regularity of the strong solution and *a priori* estimates to prove the compact property of the solution operator, which is the key to prove Theorem 2.3.

The following lemma, a special case of a general result of Lions and Magenes [23], will help us to show the continuity of the solution to stochastic PEs with respect to time in $(H^1(\mathbf{D}))^4$.

Lemma 3.1. *Let V,H,V' be three Hilbert spaces such that $V\subset H=H'\subset V'$, where H' and V' are the dual spaces of H and V respectively. Suppose $u\in L^2(0,T;V)$ and $u'\in L^2(0,T;V')$. Then u is almost everywhere equal to a function continuous from $[0,T]$ into H.*

Before giving our proof, we first introduce a modified stochastic convolution, which is an Ornstein-Uhlenbeck process satisfying

$$dZ+(AZ+\gamma Z)dt=dW, \qquad \text{with } \gamma>0. \tag{3.1}$$

Denote $Z=(Z_1,Z_2,Z_3)$, and define $\widehat{U}=U-Z=(\mathbf{u},S,p)$ which satisfies

$$\frac{\partial \widehat{U}}{\partial t}+A\widehat{U}+B(Z+\widehat{U})+E(Z+\widehat{U})+A_p(Z+\widehat{U})=F+\gamma W,$$
$$\widehat{U}(t_0)=U_0-Z(t_0)=(\mathbf{v}_0,T_0,q_0).$$

Using Kolmogorov test and Lemma 2.1, we can get the regularity of $Z_i, i=1,2,3$. One can also see the standard arguments in [10], so we omit the proof of the following lemma.

LEMMA 3.2. *Assume $\tau\geq t_0$ and $\sum_{i=1}^{\infty}\lambda_{i,j}\gamma_{i,j}^{2+\sigma}<\infty, j=1,2,3$, for small positive constant σ. Then*

$$Z_1\in C([t_0,\tau];H^3(T\mathbf{D}|T\mathbf{S}^2)) \text{ and } Z_2,Z_3\in C([t_0,\tau];H^3(\mathbf{D})).$$

In details, the components of $\widehat{U}$, $\mathbf{u},S,p$, satisfy

$$\partial_t\mathbf{u}+\nabla_{Z_1+\mathbf{u}}(Z_1+\mathbf{u})+w(Z_1+\mathbf{u})\partial_\xi(Z_1+\mathbf{u})+\frac{f}{R_0}\vec{k}\times(Z_1+\mathbf{u})+\nabla\Phi_s$$
$$=\Delta\mathbf{u}+\partial_{\xi\xi}\mathbf{u}-\int_\xi^1\frac{br_s}{r}\nabla[(1+a(Z_3+p))(Z_2+S)]d\xi'+\gamma Z_1, \tag{3.2a}$$
$$\partial_t S+\nabla_{Z_1+\mathbf{u}}(Z_2+S)+w(Z_1+\mathbf{u})\partial_\xi(Z_2+S)$$
$$=\Delta S+\partial_{\xi\xi}S+\frac{br_s}{r}(1+a(Z_3+p))w(Z_1+\mathbf{u})+Q_T+\gamma Z_2, \tag{3.2b}$$
$$\partial_t p+\nabla_{Z_1+\mathbf{u}}(Z_3+p)+w(Z_1+\mathbf{u})\partial_\xi(Z_3+p)=\Delta p+\partial_{\xi\xi}p+Q_q+\gamma Z_3, \tag{3.2c}$$
$$w(Z_1+\mathbf{u})=\int_\xi^1 \text{div }(Z_1+\mathbf{u})d\xi', \tag{3.2d}$$
$$\int_0^1 \text{div }\mathbf{u}d\xi=0, \tag{3.2e}$$
$$\text{On } \xi=1\ (r=r_s):\ \partial_\xi\mathbf{u}=0, w=0, \partial_\xi S=-\alpha S, \partial_\xi p=-\beta p, \tag{3.2f}$$
$$\text{On } \xi=0\ (r=r_0):\ \partial_\xi\mathbf{u}=0, w=0, \partial_\xi S=0, \partial_\xi p=0, \tag{3.2g}$$
$$(\mathbf{u}(t_0),S(t_0),p(t_0))=(\mathbf{u}_0,S_0,p_0). \tag{3.2h}$$

We should notify that the global well-posedness of (1.7a)-(1.7f) with initial condition (1.6a)-(1.6c) is equivalent to the system (3.2a)-(3.2h).

Proof. **(Proof of Theorem 2.2.)** In the following, we will complete our proof of the global well-posedness of stochastic PEs by three steps. Firstly, we will prove the global existence of strong solution. Then, we will show that the solution is continuous in the space $\mathcal{V}$ with respect to t. Finally, we will obtain the continuity in $\mathcal{V}$ with respect to the initial data.

Step 1: We prove the global existence of the strong solutions.

We denote by $[t_0,\tau_*)$ the maximal interval of existence of the solution of (3.2a)-(3.2h), we infer that $\tau_*=\infty$, a.s.. Otherwise, if there exists $A\in\mathcal{F}$ such that $\mathbb{P}(A)>0$ and for fixed $\omega\in A, \tau_*(\omega)<\infty$, it is clear that

$$\limsup_{t\to\tau_*(\omega)-}(\|\mathbf{u}(t)\|_1+\|S(t)\|_1+\|p\|_1)=\infty, \quad \text{for any } \omega\in A,$$

which contradicts with the estimates (A.40), (A.43), (A.46) and (A.52) given in the Appendix. Therefore, $\tau_*=\infty$, a.s., and the strong solution $(\mathbf{u},S,p)$ exists globally in time a.s..

Step 2: We show the continuity of strong solutions with respect to t.

Taking inner product between $\partial_t A_1^{\frac{1}{2}}\mathbf{u}$ and η, by (3.2a), one can get

$$\begin{aligned}\langle\partial_t A_1^{\frac{1}{2}}\mathbf{u},\eta\rangle=&\langle\partial_t\mathbf{u},A_1^{\frac{1}{2}}\eta\rangle=-\langle A_1\mathbf{u},A_1^{\frac{1}{2}}\eta\rangle-\langle\nabla_{\mathbf{u}+Z_1}(\mathbf{u}+Z_1),A_1^{\frac{1}{2}}\eta\rangle\\&-\langle w(\mathbf{u}+Z_1)\partial_\xi(\mathbf{u}+Z_1),A_1^{\frac{1}{2}}\eta\rangle-\frac{f}{R_0}\langle(\mathbf{u}+Z_1)^{\perp},A_1^{\frac{1}{2}}\eta\rangle\\&-\langle\int_\xi^1\frac{br_s}{r}\nabla[(1+a(Z_3+p))(Z_2+S)]d\xi',A_1^{\frac{1}{2}}\eta\rangle+\gamma\langle Z_1,A_1^{\frac{1}{2}}\eta\rangle,\end{aligned}$$

where we have used $\langle\nabla\Phi_s,A_1^{\frac{1}{2}}\eta\rangle=0$ which follows by integration by parts formula. By the Hölder inequality and the interpolation inequality, we get

$$\langle w(\mathbf{u}+Z_1)\partial_\xi(\mathbf{u}+Z_1),A_1^{\frac{1}{2}}\eta\rangle\le C\|\mathbf{u}+Z_1\|_1\|\mathbf{u}+Z_1\|_2|A_1^{\frac{1}{2}}\eta|_2.$$

By the Hölder inequality and the Sobolev imbedding theorem, we have

$$\begin{aligned}&-\langle\int_\xi^1\frac{br_s}{r}\nabla[(1+a(Z_3+p))(Z_2+S)]d\xi',A_1^{\frac{1}{2}}\eta\rangle\\\le& C|A_1^{\frac{1}{2}}\eta|_2(\|Z_3+p\|_2\|Z_2+S\|_1+\|Z_3+p\|_1\|Z_2+S\|_2).\end{aligned}$$

Similarly,

$$-\langle\nabla_{\mathbf{u}+Z_1}(\mathbf{u}+Z_1),A_1^{\frac{1}{2}}\eta\rangle\le C\|\mathbf{u}+Z_1\|_1\|\mathbf{u}+Z_1\|_2|A_1^{\frac{1}{2}}\eta|_2.$$

Therefore, combining the above estimates yields

$$\begin{aligned}\|\partial_t(A_1^{\frac{1}{2}}\mathbf{u})\|_{\mathcal{V}_1'}\le C(&\|\mathbf{u}\|_2+\|\mathbf{u}+Z_1\|_1\|\mathbf{u}+Z_1\|_2+|\mathbf{u}|_2+|Z_1|_2\\&+\|Z_3+p\|_2\|Z_2+S\|_1+\|Z_3+p\|_1\|Z_2+S\|_2).\end{aligned}$$

By *a priori* estimates of $\mathbf{u}$ in the Appendix,

$$\mathbf{u}\in L^\infty([t_0,\tau];\mathcal{V}_1)\cap L^2([t_0,\tau];H^2(T\mathbf{D}|T\mathbf{S}^2)),Z_1\in C([t_0,\tau];H^3(T\mathbf{D}|T\mathbf{S}^2)),$$

and $Z_2,Z_3\in C([t_0,\tau];H^3(\mathbf{D}))$ for all $\tau>t_0$, we obtain

$$A_1^{\frac{1}{2}}\mathbf{u}\in L^2([t_0,\tau];\mathcal{V}_1),\quad \partial_t(A_1^{\frac{1}{2}}\mathbf{u})\in L^2([t_0,\tau];\mathcal{V}_1'),$$

which together with Lemma 3.1 implies

$$A_1^{\frac{1}{2}}\mathbf{u}\in C([t_0,\tau];\mathcal{H}_1)\text{ or }\mathbf{u}\in C([t_0,\tau];\mathcal{V}_1)\text{ a.s..}$$

Similarly, we can prove

$$S\in C([t_0,\tau];\mathcal{V}_2)\text{ and }p\in C([t_0,\tau];\mathcal{V}_3).$$

Step 3: We obtain the continuity of strong solutions in $\mathcal{V}$ with respect to the initial data.

Let $(\mathbf{v}_1,T_1,q_1)$ and $(\mathbf{v}_2,T_2,q_2)$ be two solutions of the system (1.7a)-(1.7f) with corresponding pressure $\Phi_s{}'$ and $\Phi_s{}''$, and initial data $(\mathbf{v}_{t_0}^1,T_{t_0}^1,q_{t_0}^1)$ and $(\mathbf{v}_{t_0}^2,T_{t_0}^2,q_{t_0}^2)$

respectively. Denote by $\mathbf{v}=\mathbf{v}_1-\mathbf{v}_2, T=T_1-T_2, q=q_1-q_2$ and $\Phi_s={\Phi_s}'-{\Phi_s}''$. Then we derive from (1.7a)-(1.7f) that

$$\begin{aligned}
&\partial_t\mathbf{v}+L_1\mathbf{v}+\nabla_{\mathbf{v}_1}\mathbf{v}+\nabla_{\mathbf{v}}\mathbf{v}_2+\Big(\int_\xi^1 \mathrm{div}\mathbf{v}_1(x,y,\xi',t)d\xi'\Big)\partial_\xi\mathbf{v}\\
&\quad+\Big(\int_\xi^1 \mathrm{div}\mathbf{v}(x,y,\xi',t)d\xi'\Big)\partial_\xi\mathbf{v}_2+\frac{f}{R_0}\mathbf{v}^\perp+\nabla\Phi_s\\
&\quad+\int_\xi^1\frac{bP}{p}\nabla T d\xi'+\int_\xi^1\frac{abP}{p}\nabla(q_1T)d\xi'+\int_\xi^1\frac{abP}{p}\nabla(qT_2)d\xi'=0, \qquad (3.3)\\
&\partial_tT+L_2T+\nabla_{\mathbf{v}_1}T+\nabla_{\mathbf{v}}T_2+\Big(\int_\xi^1 \mathrm{div}\mathbf{v}_1(x,y,\xi',t)d\xi'\Big)\partial_\xi T\\
&\quad+\Big(\int_\xi^1 \mathrm{div}\mathbf{v}(x,y,\xi',t)d\xi'\Big)\partial_\xi T_2-\frac{bP}{p}\Big(\int_\xi^1 \mathrm{div}\mathbf{v}(x,y,\xi',t)d\xi'\Big)\\
&\quad-\frac{abP}{p}q_1\Big(\int_\xi^1 \mathrm{div}\mathbf{v}(x,y,\xi',t)d\xi'\Big)-\frac{abP}{p}q\Big(\int_\xi^1 \mathrm{div}\mathbf{v}_2(x,y,\xi',t)d\xi'\Big)=0, \qquad (3.4)\\
&\partial_tq+L_3q+\nabla_{\mathbf{v}_1}q+\nabla_{\mathbf{v}}q_2+\Big(\int_\xi^1 \mathrm{div}\mathbf{v}_1(x,y,\xi',t)d\xi'\Big)\partial_\xi q\\
&\quad+\Big(\int_\xi^1 \mathrm{div}\mathbf{v}(x,y,\xi',t)d\xi'\Big)\partial_\xi q_2=0, \qquad (3.5)\\
&\mathbf{v}|_{t_0}=\mathbf{v}^1_{t_0}-\mathbf{v}^2_{t_0},\quad T|_{t_0}=T^1_{t_0}-T^2_{t_0},\quad q|_{t_0}=q^1_{t_0}-q^2_{t_0}, \qquad (3.6)\\
&\xi=1:\quad \partial_\xi\mathbf{v}=0,\quad \partial_\xi T=-\alpha T,\quad \partial_\xi q=-\beta q, \qquad (3.7)\\
&\xi=0:\quad \partial_\xi u=0,\quad \partial_\xi T=0,\quad \partial_\xi q=0. \qquad (3.8)
\end{aligned}$$

Taking inner product of (3.3) with $A_1\mathbf{v}$ in $L^2(T\mathbf{D}|T\mathbf{S}^2)$ we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|_1^2+|A_1\mathbf{v}|_2^2=&-\langle\nabla_{\mathbf{v}_1}\mathbf{v},A_1\mathbf{v}\rangle-\langle\nabla_{\mathbf{v}}\mathbf{v}_2,A_1\mathbf{v}\rangle-\langle\Big(\int_\xi^1 \mathrm{div}\mathbf{v}(x,y,\xi',t)d\xi'\Big)\partial_\xi\mathbf{v}_2,A_1\mathbf{v}\rangle\\
&-\langle\Big(\int_\xi^1 \mathrm{div}\mathbf{v}_1(x,y,\xi',t)d\xi'\Big)\partial_\xi\mathbf{v},A_1\mathbf{v}\rangle-\langle\int_\xi^1\frac{bP}{p}\nabla Td\xi',A_1\mathbf{v}\rangle\\
&-\langle(\frac{f}{R_0}\mathbf{v}^\perp+\nabla\Phi_s),A_1\mathbf{v}\rangle-\langle\int_\xi^1\frac{abP}{p}\nabla(q_1T)d\xi',A_1\mathbf{v}\rangle\\
&-\langle\int_\xi^1\frac{abP}{p}\nabla(qT_2)d\xi',A_1\mathbf{v}\rangle\\
=:&\sum_{i=1}^8 k_i. \qquad (3.9)
\end{aligned}$$

By the Hölder inequality, the Agmon inequality and Young's inequality, we have

$$\begin{aligned}
k_1\le&|\mathbf{v}_1|_\infty(|\nabla_{e_\theta}\mathbf{v}|_2+|\nabla_{e_\varphi}\mathbf{v}|_2)|A_1\mathbf{v}|_2\\
\le&c\|\mathbf{v}_1\|_1^{\frac12}|A_1\mathbf{v}_1|_2^{\frac12}\|\mathbf{v}\|_1|A_1\mathbf{v}|_2\\
\le&\varepsilon|A_1\mathbf{v}|_2^2+c\|\mathbf{v}_1\|_1|A_1\mathbf{v}_1|_2\|\mathbf{v}\|_1^2.
\end{aligned}$$

Similarly, we obtain

$$k_2\le|\mathbf{v}|_\infty\|\mathbf{v}_2\|_1|A_1\mathbf{v}|_2\le\varepsilon|A_1\mathbf{v}|_2^2+c\|\mathbf{v}\|_1^2\|\mathbf{v}_2\|_1^4.$$

By the Hölder inequality and the Sobolev imbedding theorem, we get

$$k_3+k_4\le \varepsilon|A_1\mathbf{v}|_2^2+c\|\mathbf{v}\|_1^2\|\mathbf{v}_2\|_1^2\|\mathbf{v}_2\|_2^2+c\|\mathbf{v}\|_1^2\|\mathbf{v}_1\|_1^2\|\mathbf{v}_1\|_2^2.$$

In view of Lemma 2.1 and the Hölder inequality, we obtain

$$k_5+k_6\le \varepsilon|A_1\mathbf{v}|_2^2+c\|T\|_1^2+c|\mathbf{v}|_2^2.$$

To estimate k_7, by the Hölder inequality, the Minkowski inequality and the interpolation inequality we have

$$\begin{aligned}
k_7\le& c\int_{\mathbf{S}^2}\Big(\int_0^1|\nabla q_1||T|d\xi\int_0^1|A_1\mathbf{v}|d\xi\Big)d\mathbf{S}^2+c\int_{\mathbf{S}^2}\Big(\int_0^1|q_1||\nabla T|d\xi\int_0^1|A_1\mathbf{v}|d\xi\Big)d\mathbf{S}^2\\
\le& c|A_1\mathbf{v}|_2\Big(\int_{\mathbf{S}^2}\big(\int_0^1|\nabla q_1|^2d\xi\big)^2d\mathbf{S}^2\Big)^{\frac14}\Big(\int_{\mathbf{S}^2}\big(\int_0^1|T|^2d\xi\big)^2d\mathbf{S}^2\Big)^{\frac14}\\
&+c|A_1\mathbf{v}|_2\Big(\int_{\mathbf{S}^2}\big(\int_0^1|q_1|^2d\xi\big)^2d\mathbf{S}^2\Big)^{\frac14}\Big(\int_{\mathbf{S}^2}\big(\int_0^1|\nabla T|^2d\xi\big)^2d\mathbf{S}^2\Big)^{\frac14}\\
\le& \varepsilon|A_1\mathbf{v}|_2^2+c|\nabla q_1|_2(|\nabla q_1|_2+|\Delta q_1|_2)(|T|_2^2+|T|_2|\nabla T|_2)\\
&+c|\nabla T|_2(|\nabla T|_2+|\Delta T|_2)|q_1|_4^2\\
\le& \varepsilon|A_1\mathbf{v}|_2^2+\varepsilon|\Delta T|_2^2+c\|T\|_1^2(\|q_1\|_1^4+\|q_1\|_2^2).
\end{aligned}$$

Similarly, we have

$$k_8\le \varepsilon|A_1\mathbf{v}|_2^2+\varepsilon|\Delta q|_2^2+c\|q\|_1^2(\|T_2\|_1^4+\|T_2\|_2^2).$$

By (3.9) and estimates of k_i, $i=1,\dots,8$, we get

$$\begin{aligned}
&\frac12\frac{d\|\mathbf{v}\|_1^2}{dt}+|A_1\mathbf{v}|_2^2\\
&\le \varepsilon|A_1\mathbf{v}|_2^2+\varepsilon|A_2T|_2^2+\varepsilon|A_3q|_2^2+c\|\mathbf{v}\|_1^2(\|\mathbf{v}_1\|_1^2\|\mathbf{v}_1\|_2^2+\|\mathbf{v}_2\|_1^2\|\mathbf{v}_2\|_2^2+\|\mathbf{v}_2\|_1^4+1)\\
&\quad+c\|T\|_1^2(\|q_1\|_1^4+\|q_1\|_2^2)+c\|q\|_1^2(\|T_2\|_1^4+\|T_2\|_2^2).
\end{aligned}\tag{3.10}$$

Taking an analogous argument as above, from (3.4) and (3.5) we have

$$\begin{aligned}
\frac12\frac{d\|T\|_1^2}{dt}+|A_2T|_2^2\le& \varepsilon|A_2T|_2^2+\varepsilon|A_1\mathbf{v}|_2^2+c\|q\|_1^2\|\mathbf{v}_2\|_1\|\mathbf{v}_2\|_2+c\|T\|_1^2(1+\|\mathbf{v}_1\|_1^2\|\mathbf{v}_1\|_2^2)\\
&+c\|\mathbf{v}\|_1^2(1+\|q_1\|_1^4+\|T_2\|_2^2+\|T_2\|_1^2\|T_2\|_2^2),
\end{aligned}\tag{3.11}$$

and

$$\frac12\frac{d\|q\|_1^2}{dt}+|A_3q|_2^2\le \varepsilon|A_3q|_2^2+c\|q\|_1^2(1+\|\mathbf{v}_1\|_1^2\|\mathbf{v}_1\|_2^2)+c\|\mathbf{v}\|_1^2(\|q_2\|_2^2+\|q_2\|_1^2\|q_2\|_2^2).\tag{3.12}$$

Let

$$\begin{aligned}
g_1:=&1+\|\mathbf{v}_1\|_1^2\|\mathbf{v}_1\|_2^2+\|\mathbf{v}_2\|_1^4+\|\mathbf{v}_2\|_1^2\|\mathbf{v}_2\|_2^2+\|T_2\|_2^2\\
&+\|T_2\|_1^2\|T_2\|_2^2+\|q_1\|_1^4+\|q_2\|_2^2+\|q_2\|_1^2\|q_2\|_2^2,\\
g_2:=&1+\|q_1\|_2^2+\|q_1\|_1^4+\|\mathbf{v}_1\|_1^2\|\mathbf{v}_1\|_2^2,
\end{aligned}$$

and

$$g_3:=1+\|T_2\|_2^2+\|T_2\|_1^4+\|\mathbf{v}_2\|_1\|\mathbf{v}_2\|_2+\|\mathbf{v}_1\|_1^2\|\mathbf{v}_1\|_2^2.$$

It is obvious that for arbitrary $0 \le a < b < \infty$,

$$\int_a^b (g_1(t)+g_2(t)+g_3(t))dt < \infty.$$

Therefore, we get

$$\frac{d(\|\mathbf{v}\|_1^2+\|T\|_1^2+\|q\|_1^2)}{dt} \le c(g_1(t)+g_2(t)+g_3(t))(\|\mathbf{v}\|_1^2+\|T\|_1^2+\|q\|_1^2),$$

applying Grönwall inequality implies

$$\begin{aligned}&\|\mathbf{v}(t)\|_1^2+\|T(t)\|_1^2+\|q(t)\|_1^2\\ \le& c(\|\mathbf{v}_{t_0}^1-\mathbf{v}_{t_0}^2\|_1^2+\|T_{t_0}^1-T_{t_0}^2\|_1^2+\|q_{t_0}^1-q_{t_0}^2\|_1^2)e^{\int_0^t (g_1(s)+g_2(s)+g_3(s))ds}.\end{aligned}$$

So far, we have shown that for $t > t_0$, the strong solution $(\mathbf{v}(t), T(t), q(t))$ to (1.1) is Lipschitz continuous in $\mathcal{V}$ with respect to the initial data $(\mathbf{v}_0, T_0, q_0)$. □

4. Random attractors

We denote by $W = (W_1, W_2, W_3)$ the $\mathcal{V}$-valued Wiener process, which has a version ω in $C_0(\mathbb{R},\mathcal{V}) := \widetilde{\Omega}$, the space of continuous functions which are zero at zero. In what follows we consider a canonical version of W given by the probability space $(C_0(\mathbb{R},\mathcal{V}), B(C_0(\mathbb{R},\mathcal{V})), \widetilde{\mathbb{P}})$ where $\widetilde{\mathbb{P}}$ is the Wiener-measure generated by W. On this probability space we also introduce the shift

$$\vartheta_s \omega(t) = \omega(t+s) - \omega(s), \qquad s,t \in \mathbb{R}.$$

Going back to the abstract evolution system defined in (2.10),

$$dU + (AU + B(U) + A_p U + EU)dt = Fdt + dW,\ U(t_0) = U_0, \tag{4.1}$$

we define an Ornstein-Uhlenbeck process by

$$Z(t) = \int_{-\infty}^t e^{-(A+\gamma)(t-s)} dW(s), \qquad \widehat{U} = U - Z.$$

Z is a stationary process and its trajectories are $\widetilde{\mathbb{P}}$-a.s. continuous. $\widehat{U}$ satisfies another evolution system:

$$\frac{d\widehat{U}}{dt} + A\widehat{U} + B(\widehat{U}+Z) + A_p(\widehat{U}+Z) + E(\widehat{U}+Z) = F + \gamma Z - AZ. \tag{4.2}$$

Again, using Galerkin approximating method, we have for any $\omega \in \widetilde{\Omega}$, and any fixed $s \in \mathbb{R}$, and $\widehat{U}_s \in \mathcal{V}$, a.s., there exists a unique solution, $\widehat{U}(t,\omega)$, defined on $[s,\infty)$, satisfying the above equation and

$$\widehat{U}(s,\omega) = \widehat{U}_s(\omega),\ \widetilde{\mathbb{P}}\text{-a.s.} \tag{4.3}$$

We now define the stochastic dynamical system $(\mathcal{S}(t,s;\omega))_{t \ge s, \omega \in \widetilde{\Omega}}$ by

$$\mathcal{S}(t,s;\omega)U_s = \widehat{U}(t,\omega) + Z(t,\omega),$$

with $\widehat{U}(s,\omega)=\widehat{U}_s(\omega)=U_s-Z_s(\omega)$. It's obvious that $\mathcal{S}(t,s;\omega)$ satisfies (i)-(iv) (see Section 2), and also satisfies for any $s<t$ and $h\in\mathcal{V}$,

$$\mathcal{S}(t,s;\omega)h=\mathcal{S}(t-s,0;\vartheta_s\omega)h,\ \widetilde{\mathbb{P}}\text{-a.s.}$$

To prove the compact property of solution operator, we need Aubin's Lemma which is cited below.

LEMMA 4.1 (Aubin's Lemma). *Let B_0,B,B_1 be Banach spaces such that B_0,B_1 are reflexive and $B_0\overset{c}{\subset}B\subset B_1$. Define for $0<K<\infty$,*

$$X:=\{h|h\in L^2([0,K],B_0),h'(t)\in L^2([0,K];B_1)\}.$$

Then X is a Banach space equipped with the norm $|h|_{L^2([0,K];B_0)}+|h|_{L^2([0,K];B_1)}$. Moreover,

$$X\overset{c}{\subset}L^2([0,K];B_1).$$

Finally, we restate our main result of the existence of the random attractor for stochastic PEs.

THEOREM 4.1. *Let Q_T, $\partial_\xi Q_T$, Q_q, $\partial_\xi Q_q\in L^2(\mathbf{D})$, $(\mathbf{v}_0,T_0,q_0)\in\mathcal{V}$, and $\sum_{i=1}^{\infty}\lambda_{i,j}^2\gamma_{i,j}^{2+\sigma}<\infty,j=1,2,3$, for small positive constant σ. Furthermore we assume $|\frac{br_s}{r}|\le\min\{\frac{1}{2},\alpha,\beta\}$. Then the solution operator $(\mathcal{S}(t,s;\omega))_{t\ge s,\omega\in\widetilde{\Omega}}$ of the 3D stochastic PEs (1.1), defined as $\mathcal{S}(t,s;\omega)(\mathbf{v}_s,T_s,q_s)=(\mathbf{v}_t,T_t,q_t)$ has properties* (i)$-$(iv) *(see Section 2) and possesses a compact absorbing set $\mathcal{B}(0,\omega)$ at time t. Moreover, for $\widetilde{\mathbb{P}}$-a.s. ω, the set $\mathcal{A}(\omega)=\overline{\bigcup_{B\subset\mathcal{V}}\mathcal{A}(B,\omega)}$ is the random attractor of stochastic PEs, where the union is taken over all the bounded subsets of $\mathcal{V}$.*

Proof. Following the classic arguments (see [10]), we can prove that for arbitrarily small $\varepsilon>0$, we can choose γ big enough such that $E\|Z_i(0)\|_3^2\le\varepsilon,i=1,2,3$ and $\|Z_i(t)\|_3$ has polynomial growth when $t\to-\infty$. Furthermore, the process $Z(t)$ is stationary and ergodic, thus, we know from the ergodic theorem that

$$-\frac{1}{s}\int_s^0(\|Z_1\|_3^2+\|Z_2\|_3^2+\|Z_3\|_3^2)dr\to\mathbb{E}[\|Z_1(0)\|_3^2+\|Z_2(0)\|_3^2+\|Z_3(0)\|_3^2]\text{ as }s\to-\infty. \tag{4.4}$$

Since we can choose γ big enough such that

$$\mathbb{E}[\|Z_1(0)\|_3^2+\|Z_2(0)\|_3^2+\|Z_3(0)\|_3^2]\le\frac{\gamma_1}{4},$$

there exists $s_0(\omega)$ such that for any $s<s_0(\omega)$,

$$-\frac{1}{s}\int_s^0(\|Z_1\|_3^2+\|Z_2\|_3^2+\|Z_3\|_3^2)dr\le\frac{\gamma_1}{4}.$$

Using similar discussion with respect to negative time t, and referring to the result (A.6), we have

$$\begin{aligned}&|\mathbf{u}(-4)|_2^2+|S(-4)|_2^2+|p(-4)|_2^2\\ \le&(|\mathbf{v}(t_0)|_2^2+|T(t_0)|_2^2+|q(t_0)|_2^2)\exp\left[C\int_{t_0}^{-4}(-\gamma_1+\|Z_1(s)\|_3^2+\|Z_2(s)\|_3^2+\|Z_3(s)\|_3^2)ds\right]\end{aligned}$$

$$+\int_{t_0}^{-4}(|Q_T|_2^2+|Q_q|_2^2+|Z|_2^2)\exp\left[C\int_s^{-4}(-\gamma_1+\|Z_1(r)\|_3^2+\|Z_2(r)\|_3^2+\|Z_3(r)\|_3^2)dr\right]ds. \tag{4.5}$$

Applying (A.6) again, we have for $t\in[-4,0]$,

$$\begin{aligned}&|\mathbf{u}(t)|_2^2+|S(t)|_2^2+|p(t)|_2^2\\ \le&(|\mathbf{u}(-4)|_2^2+|S(-4)|_2^2+|p(-4)|_2^2)\exp\left[C\int_{-4}^{t}(-\gamma_1+\|Z_1(s)\|_3^2+\|Z_2(s)\|_3^2+\|Z_3(s)\|_3^2)ds\right]\\ &+\int_{-4}^{t}(|Q_T|_2^2+|Q_q|_2^2+|Z|_2^2)\exp\left[C\int_s^{-4}(-\gamma_1+\|Z_1(r)\|_3^2+\|Z_2(r)\|_3^2+\|Z_3(r)\|_3^2)dr\right]ds. \end{aligned} \tag{4.6}$$

Now we denote by $(\mathbf{u}(t,\omega;t_0,\mathbf{u}_*),S(t,\omega;t_0,S_*),p(t,\omega;t_0,p_*))$ the solution to the system (3.2) with $(\mathbf{u}(t_0),S(t_0),p(t_0))=(\mathbf{u}_*,S_*,p_*)$. Then, by (4.5) and (4.6), there exists $r_1(\omega)$ depending on γ_1,Z_1,Z_2 and Z_3, such that for arbitrarily fixed $\rho>0$ there exists $t(\omega)\le -4$, $\mathbb{P}$-a.s. for all $t_0\le t(\omega)$ and $(\mathbf{u}_*,S_*,p_*)\in\mathcal{V}$ with $\|\mathbf{u}_*\|_1+\|S_*\|_1+\|p_*\|_1<\rho$, the solution $(\mathbf{u}(t,\omega;t_0,\mathbf{u}_*),S(t,\omega;t_0,S_*),p(t,\omega;t_0,p_*))$ on $[t_0,\infty)$ satisfies

$$|\mathbf{u}(t,\omega;t_0,\mathbf{u}_*)|_2^2+|S(t,\omega;t_0,S_*)|_2^2+|p(t,\omega;t_0,p_*)|_2^2\le r_1(\omega) \text{ for } t\in[-4,0]. \tag{4.7}$$

Moreover, integrating (A.5), we have

$$\begin{aligned}&\int_{-4}^{0}(\|\mathbf{u}\|_1^2+\|S\|_1^2+\|p\|_1^2)ds\\ \le&|\mathbf{u}(-4)|_2^2+|S(-4)|_2^2+|p(-4)|_2^2\\ &+C\int_{-4}^{0}(|\mathbf{u}|_2^2+|S|_2^2+|p|_2^2)(\|Z_1(s)\|_3^2+\|Z_2(s)\|_3^2+\|Z_3(s)\|_3^2))ds\\ &+C\int_{-4}^{0}(|Q_T|_2^2+|Q_q|_2^2+|Z|_2^2)ds,\end{aligned} \tag{4.8}$$

thus, there exists $c_1(\omega)$ depending on γ_1,Z_1,Z_2 and Z_3,

$$\int_{-4}^{0}(\|\mathbf{u}(t,\omega;t_0,\mathbf{u}_*)\|_1^2+\|S(t,\omega;t_0,S_*)\|_1^2+\|p(t,\omega;t_0,p_*)\|_1^2)ds\le c_1(\omega). \tag{4.9}$$

We continue the discussion of L^4 norms. For $t<-3$, by (A.13) we have

$$\begin{aligned}|p(-3,\omega;t_0,p_*)|_4^2\le&|p(t,\omega;t_0,p_*)|_4^2e^{-C(-3-t)}\\ &+C\int_t^{-3}e^{-C(-3-s)}(|Q_q|_2^2+\|Z_3\|_3^2+\|Z_1\|_1^2+\|Z_3\|_3^2\|\mathbf{u}(s,\omega;t_0,\mathbf{u}_*)\|_1^2)ds.\end{aligned} \tag{4.10}$$

We now integrate both sides over $[-4,-3]$,

$$\begin{aligned}&|p(-3,\omega;t_0,p_*)|_4^2\\ \le&\int_{-4}^{-3}|p(t,\omega;t_0,p_*)|_4^2e^{-C(-3-t)}dt\\ &+C\int_{-4}^{-3}\int_t^{-3}e^{-C(-3-s)}(|Q_q|_2^2+\|Z_3\|_3^2+\|Z_1\|_1^2+\|Z_3\|_3^2\|\mathbf{u}(s,\omega;t_0,\mathbf{u}_*)\|_1^2)dsdt\end{aligned}$$

$$\leq C\int_{-4}^{-3}(\|p(t,\omega;t_0,p_*)\|_1^2+\|\mathbf{u}(t,\omega;t_0,\mathbf{u}_*)\|_1^2)dt$$
$$+C\int_{-4}^{-3}e^{-C(-3-s)}(|Q_q|_2^2+\|Z_3\|_3^2+\|Z_1\|_1^2)dt. \tag{4.11}$$

By (4.9), there exists $c_2(\omega)$, depending only on γ_1, Z_1, Z_2, Z_3 such that for arbitrarily fixed $\rho>0$, there exists $t(\omega)\leq -3$, $\widetilde{\mathbb{P}}$-a.s. for all $t_0\leq t(\omega)$ and $(\mathbf{u}_*,S_*,p_*)\in\mathcal{V}$ with $\|\mathbf{u}_*\|_1+\|S_*\|_1+\|p_*\|_1<\rho$, and the solution $p(t,\omega;t_0,p_*)$ satisfies

$$|p(-3,\omega;t_0,p_*|_4\leq c_2(\omega). \tag{4.12}$$

Similar as in (4.7), there exists $c_3(\omega)$ depending only on γ_1, Z_1, Z_2, Z_3 such that for arbitrarily fixed $\rho>0$ there exists $t(\omega)\leq -3$, $\widetilde{\mathbb{P}}$-a.s. for all $t_0\leq t(\omega)$ and $(\mathbf{u}_*,S_*,p_*)\in\mathcal{V}$ with $\|\mathbf{u}_*\|_1+\|S_*\|_1+\|p_*\|_1<\rho$, and the solution $p(t,\omega;t_0,p_*)$ satisfies

$$|p(t,\omega;t_0,p_*)|_4^2+\int_{-3}^{0}|p|_{\xi=1}|_4^4 ds\leq c_3(\omega), \text{ for any } t\in[-3,0]. \tag{4.13}$$

Analogously, by (A.20) and (A.21), we conclude that there exists $c_4(\omega)$ depending on γ_1, Z_1, Z_2, Z_3 such that for arbitrarily fixed $\rho>0$ there exists $t(\omega)\leq -3$, $\widetilde{\mathbb{P}}$-a.s. for all $t_0\leq t(\omega)$ and $(\mathbf{u}_*,S_*,p_*)\in\mathcal{V}$ with $\|\mathbf{u}_*\|_1+\|S_*\|_1+\|p_*\|_1<\rho$, and the solution $S(t,\omega;t_0,S_*)$ satisfies

$$|S(t,\omega;t_0,S_*)|_4^2+\int_{-3}^{0}|S|_{\xi=1}|_4^4\leq c_4(\omega), \text{ for any } t\in[-3,0]. \tag{4.14}$$

By (A.31), (A.32), and with similar discussion as the above, there exist constants $c_5(\omega)$, $c_6(\omega)$ depending on γ_1, Z_1, Z_2, Z_3 such that for arbitrarily fixed $\rho>0$ there exists $t(\omega)\leq -3$, $\widetilde{\mathbb{P}}$-a.s. for all $t_0\leq t(\omega)$ and $(\mathbf{u}_*,S_*,p_*)\in\mathcal{V}$ with $\|\mathbf{u}_*\|_1+\|S_*\|_1+\|p_*\|_1<\rho$, and the solution $\widetilde{\mathbf{u}}(t,\omega;t_0,\widetilde{\mathbf{u}}_*)$ satisfies

$$|\widetilde{\mathbf{u}}(t,\omega;t_0,\widetilde{\mathbf{u}}_*)|_4^2\leq c_5(\omega), \text{ for any } t\in[-3,0], \tag{4.15}$$
$$\int_{-3}^{0}\int_{\mathbf{D}}|\widetilde{\mathbf{u}}|^2\Big(|\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2+|\nabla_{e_\varphi}\widetilde{\mathbf{u}}|^2+|\partial_\xi\widetilde{\mathbf{u}}|^2\Big)d\mathbf{D}ds\leq c_6(\omega). \tag{4.16}$$

By (A.35), we proceed to have the existence of a constant $c_7(\omega)$ depending on γ_1, Z_1, Z_2, Z_3 such that for arbitrarily fixed $\rho>0$ there exists $t(\omega)\leq -2$, $\widetilde{\mathbb{P}}$-a.s. for all $t_0\leq t(\omega)$ and $(\mathbf{u}_*,S_*,p_*)\in\mathcal{V}$ with $\|\mathbf{u}_*\|_1+\|S_*\|_1+\|p_*\|_1<\rho$, and the solution $\bar{\mathbf{u}}(t,\omega;t_0,\bar{\mathbf{u}}_*)$ satisfies

$$\int_{\mathbf{S}^2}|\nabla_{e_\theta}\bar{\mathbf{u}}|^2+|\nabla_{e_\varphi}\bar{\mathbf{u}}|^2 d\mathbf{S}^2\leq c_7(\omega), \text{ for any } t\in[-2,0]. \tag{4.17}$$

By (A.39), and together with results in (4.9) and (4.15), (4.16), (4.17), we conclude that there exist constants $r_2(\omega)$ and $c_8(\omega)$ depending on γ_1, Z_1, Z_2, Z_3 such that for arbitrarily fixed $\rho>0$ there exists $t(\omega)\leq -1$, $\widetilde{\mathbb{P}}$-a.s. for all $t_0\leq t(\omega)$ and $(\mathbf{u}_*,S_*,p_*)\in\mathcal{V}$ with $\|\mathbf{u}_*\|_1+\|S_*\|_1+\|p_*\|_1<\rho$, $\mathbf{u}_\xi(t,\omega;t_0,\mathbf{u}_*)$ satisfies

$$|\mathbf{u}_\xi(t,\omega;t_0,\mathbf{u}_*)|_2^2\leq r_2(\omega), \text{ for any } t\in[-1,0], \tag{4.18}$$
$$\int_{-1}^{0}\int_{\mathbf{S}^2}|\nabla_{e_\theta}\mathbf{u}_\xi(s,\omega;t_0,\mathbf{u}_*)|^2+|\nabla_{e_\varphi}\mathbf{u}_\xi(s,\omega;t_0,\mathbf{u}_*)|^2+|\mathbf{u}_{\xi\xi}(s,\omega;t_0,\mathbf{u}_*)|^2 d\mathbf{S}^2 ds\leq c_8(\omega). \tag{4.19}$$

By (A.42), and together with results in (4.9) and (4.15)–(4.19), we conclude that there exist constants $r_3(\omega)$ and $c_9(\omega)$ depending on γ_1, Z_1, Z_2, Z_3 such that for arbitrarily fixed $\rho > 0$ there exists $t(\omega) \leq -1$, $\widetilde{\mathbb{P}}$-as for all $t_0 \leq t(\omega)$ and $(\mathbf{u}_*, S_*, p_*) \in \mathcal{V}$ with $\|\mathbf{u}_*\|_1 + \|S_*\|_1 + \|p_*\|_1 < \rho$, $p_\xi(t,\omega;t_0,p_*)$ satisfies

$$|p_\xi(t,\omega;t_0,p_*)|_2^2 \leq r_3(\omega), \text{ for any } t \in [-1,0], \tag{4.20}$$

$$\int_{-1}^{0} (|\nabla p_\xi|_2^2 + |p_{\xi\xi}|_2^2) ds \leq c_9(\omega). \tag{4.21}$$

By (A.45), and together with results in (4.9) and (4.15))–(4.21), we conclude that there exist constants $r_4(\omega)$ and $c_{10}(\omega)$ depending on γ_1, Z_1, Z_2, Z_3 such that for arbitrarily fixed $\rho > 0$ there exists $t(\omega) \leq -1$, $\widetilde{\mathbb{P}}$-a.s. for all $t_0 \leq t(\omega)$ and $(\mathbf{u}_*, S_*, p_*) \in \mathcal{V}$ with $\|\mathbf{u}_*\|_1 + \|S_*\|_1 + \|p_*\|_1 < \rho$, $S_\xi(t,\omega;t_0,p_*)$ satisfies

$$|S_\xi(t,\omega;t_0,S_*)|_2^2 \leq r_4(\omega), \text{ for any } t \in [-1,0], \tag{4.22}$$

$$\int_{-1}^{0} (|\nabla S_\xi|_2^2 + |S_{\xi\xi}|_2^2) ds \leq c_{10}(\omega). \tag{4.23}$$

In view of (A.51), there exists a constant $r_5(\omega)$ such that for arbitrarily fixed $\rho > 0$, there exists $t(\omega) \leq -1$, $\widetilde{\mathbb{P}}$-a.s. for all $t_0 \leq t(\omega)$ and $(\mathbf{u}_*, S_*, p_*) \in \mathcal{V}$ with $\|\mathbf{u}_*\|_1 + \|S_*\|_1 + \|p_*\|_1 < \rho$, $\mathbf{u}(t,\omega;t_0,\mathbf{u}_*), S$ and p satisfy

$$\begin{aligned} &|\nabla_{e_\theta} \mathbf{u}(t,\omega;t_0,\mathbf{u}_*)|_2^2 + |\nabla_{e_\varphi} \mathbf{u}(t,\omega;t_0,\mathbf{u}_*)|_2^2 \\ &\quad + |\nabla S(t,\omega;t_0,S_*)|_2^2 + |\nabla p(t,\omega;t_0,p_*)|_2^2 \leq r_5(\omega), \text{ for any } t \in [-1,0], \end{aligned} \tag{4.24}$$

which together with (4.7), (4.18), (4.20) and (4.22) implies that there exists a constant $r_6(\omega)$ such that for arbitrarily fixed $\rho > 0$, there exists $t(\omega) \leq -1$, $\widetilde{\mathbb{P}}$-a.s. for all $t_0 \leq t(\omega)$ and $(\mathbf{u}_*, S_*, p_*) \in \mathcal{V}$ with $\|\mathbf{u}_*\|_1 + \|S_*\|_1 + \|p_*\|_1 < \rho$, $(\mathbf{u}, S, p)$ satisfy

$$\|\mathbf{u}(t,\omega;t_0,\mathbf{u}_*)\|_1^2 + \|S(t,\omega;t_0,S_*)\|_1^2 + \|p(t,\omega;t_0,p_*)\|_1^2 \leq r_6(\omega), \text{ for any } t \in [-1,0]. \tag{4.25}$$

Now we are ready to prove the desired compactness result. Let $r(\omega) = r_6(\omega) + \|Z(-1)\|_1^2$, then $B(-1, r(\omega))$, the ball of center $0 \in \mathcal{V}$ and radius $r(\omega)$, is an absorbing set at time -1 for $(\mathcal{S}(t,s;\omega))_{t \geq s, \omega \in \tilde{\Omega}}$. Therefore, in order to prove the existence of the random attractor of the stochastic dynamical system in space $\mathcal{V}$, we need to construct a compact absorbing set at time 0 in $\mathcal{V}$. Denote by $\mathcal{B}$ a bounded subset of $\mathcal{V}$ and set $\mathcal{C}_T$ as a subset of the functional space:

$$\begin{aligned} \mathcal{C}_{T,q} := \Big\{ (A_1^{1/2}\mathbf{v}, A_2^{1/2}T, A_3^{1/2}q) \Big| &(\mathbf{v}(-1), T(-1), q(-1)) \in \mathcal{B}, \\ &(\mathbf{v}(t), T(t), q(t)) = \mathcal{S}(t,-1;\omega)(\mathbf{v}(-1), T(-1), q(-1)), t \in [-1,0]. \Big\} \end{aligned} \tag{4.26}$$

Obviously the embedding $\mathcal{V} \subset \mathcal{H}$ is compact. Let $(\mathbf{v}(-1), T(-1), q(-1)) \in \mathcal{B}$, by the continuity of strong solutions with respect to time t, we know

$$\begin{aligned} &(A_1^{1/2}\mathbf{u}, A_2^{1/2}S, A_3^{1/2}p) \in L^2([-1,0]; \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3), \\ &(\partial_t A_1^{1/2}\mathbf{u}, \partial_t A_2^{1/2}S, \partial_t A_3^{1/2}p) \in L^2([-1,0]; \mathcal{V}_1' \times \mathcal{V}_2' \times \mathcal{V}_3'). \end{aligned}$$

Now we apply Aubin's Lemma with

$$B_0 = \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3, \quad B = \mathcal{H}_1 \times (L^2(\mathbf{D}))^2, \quad B_1 = \mathcal{V}_1' \times \mathcal{V}_2' \times \mathcal{V}_3',$$

$\mathcal{C}_{T,q}$ is compact in $L^2([-1,0];\mathcal{H})$.

In order to show that for any fixed $t\in(-1,0],\omega\in\tilde{\Omega},\mathcal{S}(t,-1;\omega)$ is a compact operator in $\mathcal{V}$, we take any bounded sequences $\{(\mathbf{v}_{0,n},T_{0,n},q_{0,n})\}_{n\in\mathbb{N}}\subset\mathcal{B}$ and we want to extract, for any fixed $t\in(-1,0]$ and $\omega\in\tilde{\Omega}$, a convergent subsequence from $\{\mathcal{S}(t,-1;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})\}$. Since $\{(A_1^{\frac{1}{2}}\mathbf{v},A_2^{\frac{1}{2}}T,A_3^{\frac{1}{2}}q)\}\subset\mathcal{C}_{T,q}$, by Aubin's Lemma, there is a function $(\mathbf{v}^*,T^*,q^*)$:

$$(\mathbf{v}^*,T^*,q^*)\in L^2([-1,0];\mathcal{V}),$$

and there exists a subsequence of $\{\mathcal{S}(t,-1;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})\}_{n\in\mathbb{N}}$, for simplicity, we still denote it by $\{\mathcal{S}(t,-1;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})\}_{n\in\mathbb{N}}$, and it satisfies

$$\lim_{n\to\infty}\int_{-1}^{0}\|\mathcal{S}(t,-1;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})-(\mathbf{v}^*(t),T^*(t),q^*(t))\|_1^2dt=0. \tag{4.27}$$

By elementary measure theory, there exists a subsequence of $\{\mathcal{S}(t,-1;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})\}_{n\in\mathbb{N}}$, still denoted by $\{\mathcal{S}(t,-1;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})\}_{n\in\mathbb{N}}$ for simplicity, such that

$$\lim_{n\to\infty}\|\mathcal{S}(t,-1;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})-(\mathbf{v}^*(t),T^*(t),q^*(t))\|_1=0,\quad a.e.\ t\in(-1,0]. \tag{4.28}$$

Fix any $t\in(-1,0]$, we can select a $t_0\in(-1,t)$ such that

$$\lim_{n\to\infty}\|\mathcal{S}(t_0,-1,\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})-(\mathbf{v}^*(t_0),T^*(t_0),q^*(t))\|_1=0.$$

Then by the continuity of the map $\mathcal{S}(t-t_0,t_0;\omega)$ in $\mathcal{V}$ with respect to initial value, we have

$$\begin{aligned}\mathcal{S}(t,-1;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})=&\mathcal{S}(t-t_0,t_0;\omega)\mathcal{S}(t_0,-1;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})\\ &\to\mathcal{S}(t-t_0,t_0;\omega)(\mathbf{v}^*(t_0),T^*(t_0),q^*(t_0))\quad\text{in }\mathcal{V}.\end{aligned}$$

Hence for any $t\in(-1,0],\{\mathcal{S}(t,-1;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})\}_{n\in\mathbb{N}}$ contains a subsequence which is convergent in $\mathcal{V}$, which implies that for any fixed $t\in(-1,0],\omega\in\tilde{\Omega},\mathcal{S}(t,-1;\omega)$ is a compact operator in $\mathcal{V}$. Let $\mathcal{B}(0,\omega)=\overline{\mathcal{S}(0,-1;\omega)B(-1,r(\omega))}$ be the closed set of $\mathcal{S}(0,-1;\omega)B(-1,r(\omega))$ in $\mathcal{V}$. Then, by the above arguments, we know $\mathcal{B}(0,\omega)$ is a random compact set in $\mathcal{V}$. More precisely, $\mathcal{B}(0,\omega)$ is a compact absorbing set in $\mathcal{V}$ at time 0. Indeed, for $(\mathbf{v}_{0,n},T_{0,n},q_{0,n})\in\mathcal{B}$, there exists $s(\mathcal{B})\in\mathbb{R}_-$ such that if $s\le s(\mathcal{B})$, we have

$$\begin{aligned}\mathcal{S}(0,s;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})=&\mathcal{S}(0,-1;\omega)\mathcal{S}(-1,s;\omega)(\mathbf{v}_{0,n},T_{0,n},q_{0,n})\\ &\subset\mathcal{S}(0,-1;\omega)B(-1,r(\omega))\subset\mathcal{B}(0,\omega).\end{aligned}$$

Therefore, the existence of the random attractor follows. □

Appendix. A priori estimates. REMARK A.1. In this section, we discuss the a priori estimates in $(L^2(\mathbf{D}))^4$ space and $(L^4(\mathbf{D}))^4$ space in details for the solution $(\mathbf{v},T,q)$ to (1.7). The estimates in $(H^1(\mathbf{D}))^4$ space is similar to [17], so we just state the result, for the details, one can refer to [17]. From (A.5) we see that **the stochastic moist PEs are much different from the deterministic moist PEs**. For the stochastic model, the existence of the positive constants α and β is very important, on the contrary, if $\alpha=0$ and $\beta=0$, it is much better for the deterministic case.

In the later part of this section, we improve the bounds of $(\mathbf{v},T,q)$ in L^4 space and observe that the estimates in L^3 (obtained in [17]) are not necessary.

A.1. L^2 estimates of $\mathbf{u}$,S,p. We first take the inner product of Equation (3.2c) with p, in $L^2(\mathbf{D})$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d|p|_2^2}{dt}+|\nabla p|_2^2+|\partial_\xi p|_2^2+\beta|p(\xi=1)|_2^2\\ =&\int_{\mathbf{D}}(Q_q+\gamma Z_3)pd\mathbf{D}-\int_{\mathbf{D}}\Big(\nabla_{Z_1+\mathbf{u}}(Z_3+p)+w(Z_1+\mathbf{u})\partial_\xi(Z_3+p)\Big)pd\mathbf{D}.\end{aligned}$$

By Lemma 2.1 and integration by parts, we have

$$\int_{\mathbf{D}}\Big(\nabla_{Z_1+\mathbf{u}}p+w(Z_1+\mathbf{u})\partial_\xi p\Big)pd\mathbf{D}=0.$$

Applying the Hölder inequality and the Sobolev imbedding theorem, we get

$$\begin{aligned}&\int_{\mathbf{D}}(w(Z_1+\mathbf{u})\partial_\xi Z_3+\nabla_{Z_1+\mathbf{u}}Z_3)pd\mathbf{D}\\ \le& C|\partial_\xi Z_3|_\infty|\text{div } Z_1+\text{div } \mathbf{u}|_2|p|_2+C|\nabla Z_3|_\infty|Z_1+\mathbf{u}|_2|p|_2\\ \le&\varepsilon(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|^2+|\mathbf{u}|_2^2)+C\|Z_3\|_3^2|p|_2^2+C\|Z_1\|_1^2.\end{aligned}$$

We also have

$$|p|_2^2-|p(\xi=1)|_2^2\le 2\int_{\mathbf{D}}\int_0^1|p\partial_\xi p|d\mathbf{D}\le\frac{1}{2}|p|_2^2+2|\partial_\xi p|_2^2.$$

This gives

$$|p|_2^2\le 2|p(\xi=1)|_2^2+4|\partial_\xi p|_2^2,$$

similarly we also have

$$|S|_2^2\le 2|S(\xi=1)|_2^2+4|\partial_\xi S|_2^2.$$

Therefore, we obtain that

$$\begin{aligned}\int_{\mathbf{D}}(Q_q+\gamma Z_3)pd\mathbf{D}\le& C(|Q_q|_2^2+|Z_3|_2^2)+\frac{\varepsilon}{16}|p|_2^2\\ \le& C(|Q_q|_2^2+|Z_3|_2^2)+\varepsilon|\partial_\xi p|_2^2+\varepsilon|p(\xi=1)|_2^2.\end{aligned}$$

Altogether, we have the estimate for p as

$$\begin{aligned}&\frac{1}{2}\frac{d|p|_2^2}{dt}+|\nabla p|_2^2+(1-\varepsilon)|\partial_\xi p|_2^2+(\beta-\varepsilon)|p(\xi=1)|_2^2\\ \le& C(|Q_q|_2^2+|Z_3|_2^2)+\varepsilon(|\nabla_{e_\theta}\mathbf{u}|^2+|\nabla_{e_\varphi}\mathbf{u}|^2+|\mathbf{u}|_2^2)+C\|Z_1\|_1^2+C\|Z_3\|_3^2|p|_2^2.\end{aligned}\tag{A.1}$$

Similarly, we get the estimate for S from (3.2b) as

$$\begin{aligned}&\frac{1}{2}\frac{d|S|_2^2}{dt}+|\nabla S|_2^2+|\partial_\xi S|_2^2+\alpha|S(\xi=1)|_2^2\\ =&\int_{\mathbf{D}}(Q_T+\gamma Z_2)Sd\mathbf{D}-\int_{\mathbf{D}}\Big(\nabla_{Z_1+\mathbf{u}}Z_2+w(Z_1+\mathbf{u})\partial_\xi Z_2\Big)Sd\mathbf{D}\\ &+\int_{\mathbf{D}}\frac{br_s}{r}(1+a(Z_3+p))w(Z_1+\mathbf{u})Sd\mathbf{D}=:I_1+I_2+I_3.\end{aligned}$$

With same discussion, we have

$$I_1 \le C(|Q_T|_2^2+|Z_2|_2^2)+\varepsilon|\partial_\xi S|_2^2+\varepsilon|S(\xi=1)|_2^2.$$

In view of the Hölder inequality, the Sobolev imbedding, Lemma 2.1 and Young's inequality, we arrive at

$$I_2 \le \varepsilon(|\mathbf{u}|_2^2+|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\phi}\mathbf{u}|_2^2+|S|_2^2)+C\|Z_2\|_3^2|S|_2^2+C\|Z_1\|_1^2.$$

Similarly, we deduce that

$$\begin{aligned} I_3 \le& \varepsilon(|S|_2^2+|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\phi}\mathbf{u}|_2^2+|p|_2^2) \\ &+C\|Z_1\|_1^2+C\|Z_3\|_2^2|S|_2^2+C\|Z_1\|_3^2|S|_2^2+\int_{\mathbf{D}}\frac{br_s}{r}(1+ap)w(\mathbf{u})Sd\mathbf{D}. \end{aligned}$$

Therefore, combining the above estimates about I_1–I_3, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d|S|_2^2}{dt}+(1-\varepsilon)|\nabla S|_2^2+(1-\varepsilon)|\partial_\xi S|_2^2+(\alpha-\varepsilon)|S(\xi=1)|_2^2 \\ \le& C(|Q_T|_2^2+|Z_2|_2^2)+\varepsilon(|\nabla_{e_\theta}\mathbf{u}|^2+|\nabla_{e_\varphi}\mathbf{u}|^2+|\mathbf{u}|_2^2)+C\|Z_1\|_1^2 \\ &+C(\|Z_2\|_3^2+\|Z_1\|_3^2)|S|_2^2+\varepsilon|p|_2^2+\int_{\mathbf{D}}\frac{br_s}{r}(1+ap)w(\mathbf{u})Sd\mathbf{D}. \end{aligned} \tag{A.2}$$

For $\mathbf{u}$ in (3.2a), since $(\frac{f}{R_0}\times\mathbf{u})\cdot\mathbf{u}=0$, and by Lemma 2.1 and integration by parts,

$$\begin{aligned} &\frac{1}{2}\frac{d|\mathbf{u}|_2^2}{dt}+(1-2\varepsilon)\left(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2\right)+|\partial_\xi\mathbf{u}|_2^2+|\mathbf{u}|_2^2 \\ \le& C\|Z_1\|_1^2+C\|Z_1\|_3^2|\mathbf{u}|_2^2+\varepsilon|\mathbf{u}|_2^2-\int_{\mathbf{D}}\left(\int_\xi^1\frac{br_s}{r}\nabla[(1+aq)T]d\xi'\right)\cdot\mathbf{u}d\mathbf{D}. \end{aligned} \tag{A.3}$$

By integration by parts,

$$\begin{aligned} \int_{\mathbf{D}}\frac{br_s}{r}(1+aq)Tw(\mathbf{u})d\mathbf{D} =& -\int_{\mathbf{D}}\nabla\left[\frac{br_s}{r}(1+aq)T\right]\int_\xi^1 ud\xi'd\mathbf{D} \\ =& \int_{\mathbf{D}}\left(\partial_\xi\int_\xi^1\nabla\left[\frac{br_s}{r}(1+aq)T\right]d\xi'\right)\int_\xi^1 ud\xi'd\mathbf{D} \\ =& \int_{\mathbf{D}}\left(\int_\xi^1\nabla\left[\frac{br_s}{r}(1+aq)T\right]d\xi'\right)\mathbf{u}d\mathbf{D}. \end{aligned}$$

Therefore, we estimate the sum of the last term on the right-hand side of (A.2) and (A.3) that

$$\begin{aligned} &\int_{\mathbf{D}}\frac{br_s}{r}(1+ap)w(\mathbf{u})Sd\mathbf{D}-\int_{\mathbf{D}}\left(\int_\xi^1\frac{br_s}{r}\nabla[(1+aq)T]d\xi'\right)\cdot\mathbf{u}d\mathbf{D} \\ \le& \varepsilon(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2)+\frac{br_s}{2r}(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2+|S|_2^2) \\ &+C\|Z_2\|_2^2(1+|p|_2^2)+C\|Z_2\|_2^2\|Z_3\|_2^2. \end{aligned} \tag{A.4}$$

Combining (A.1)-(A.4), we get

$$\frac{1}{2}\frac{d(|\mathbf{u}|_2^2+|S|_2^2+|p|_2^2)}{dt}+(1-\varepsilon)(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2+|\mathbf{u}|_2^2+|\nabla S|_2^2+|\nabla p|_2^2)$$

$$
\begin{aligned}
&+(1-\varepsilon)(|\partial_\xi \mathbf{u}|_2^2+|\partial_\xi S|_2^2+|\partial_\xi p|_2^2)+(\alpha-\varepsilon)|S(\xi=1)|_2^2+(\beta-\varepsilon)|p(\xi=1)|_2^2\\
\leq&\frac{br_s}{2r}(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2+|S|_2^2)+C(|\mathbf{u}|_2^2+|S|_2^2+|p|_2^2)(\|Z_1\|_3^2+\|Z_2\|_3^2+\|Z_3\|_3^2)\\
&+C(|Q_T|_2^2+|Q_q|_2^2+\|Z_1\|_1^2+|Z_2|_2^2+|Z_3|_2^2+\|Z_2\|_2^2\|Z_3\|_2^2). \qquad (A.5)
\end{aligned}
$$

With the assumption that $\frac{br_s}{2r}$ is small enough, there exists $\gamma_1>0$ such that

$$
\begin{aligned}
&(2-2\varepsilon-\frac{br_s}{r})(|\nabla_{e_\theta}\mathbf{u}|^2+|\nabla_{e_\varphi}\mathbf{u}|^2)+(2-2\varepsilon)(|\nabla S|_2^2+|\nabla p|_2^2)+(2-2\varepsilon)|\partial_\xi\mathbf{u}+\partial_\xi S+\partial_\xi p|_2^2\\
&+(2\alpha-2\varepsilon)|S(\xi=1)|_2^2+(2\beta-2\varepsilon)|p(\xi=1)|_2^2>(\gamma_1-2+2\varepsilon)(|\mathbf{u}|_2^2+|S|_2^2+|p|_2^2).
\end{aligned}
$$

Therefore, by the Grönwall inequality, we have for any t, there exists γ_2,

$$
\begin{aligned}
&|\mathbf{u}(t)|_2^2+|S(t)|_2^2+|p(t)|_2^2\\
\leq&(|\mathbf{u}_0|_2^2+|S_0|_2^2+|p_0|_2^2)\exp\left[-\gamma_1 t+\gamma_2\int_{t_0}^t(\|Z_1(s)\|_3^2+\|Z_2(s)\|_3^2+\|Z_3(s)\|_3^2)ds\right]\\
&+\int_{t_0}^t(|Q_T|_2^2+|Q_q|_2^2+\|Z_1\|_1^2+|Z_2|_2^2+|Z_3|_2^2+\|Z_2\|_2^2\|Z_3\|_2^2)\\
&\times\exp\left[-\gamma_1(t-s)+\gamma_2\int_s^t(\|Z_1(r)\|_3^2+\|Z_2(r)\|_3^2+\|Z_3(r)\|_3^2)dr\right]ds. \qquad (A.6)
\end{aligned}
$$

A.2. L^4 estimates of p,S. Taking the inner product of Equation (3.2c) with p^3 in $L^2(\mathbf{D})$, we obtain that

$$
\begin{aligned}
&\frac{1}{4}\frac{d|p|_4^4}{dt}+\frac{3}{4}|\nabla p^2|_2^2+\frac{3}{4}|\partial_\xi p^2|_2^2+\beta\int_{\mathbf{S}^2}|p(\xi=1)|^4 d\mathbf{S}^2\\
=&\int_{\mathbf{D}}(Q_q+\gamma Z_3)p^3 d\mathbf{D}-\int_{\mathbf{D}}\Big(\nabla_{Z_1+\mathbf{u}}(Z_3+p)+w(Z_1+\mathbf{u})\partial_\xi(Z_3+p)\Big)p^3 d\mathbf{D}. \qquad (A.7)
\end{aligned}
$$

Again, applying integration by parts, we have

$$
\int_{\mathbf{D}}\Big(\nabla_{Z_1+\mathbf{u}}p+w(Z_1+\mathbf{u})\partial_\xi p\Big)p^3 d\mathbf{D}=0.
$$

Applying the Hölder inequality and the Sobolev imbedding theorem, we get

$$
\begin{aligned}
&\left|\int_{\mathbf{D}}(w(Z_1+\mathbf{u})\partial_\xi Z_3+\nabla_{Z_1+\mathbf{u}}Z_3)p^3 d\mathbf{D}\right|\\
\leq&C|\partial_\xi Z_3|_\infty|\text{div }\mathbf{u}+\text{div }Z_1|_2|p^3|_2+C|\nabla Z_3|_\infty|Z_1+\mathbf{u}|_2|p^3|_2\\
\leq&C\|Z_3\|_3(|\nabla_{e_\theta}\mathbf{u}|_2+|\nabla_{e_\varphi}\mathbf{u}|_2+|u|_2)|p^2|_3^{3/2}+C\|Z_3\|_3\|Z_1\|_3|p^2|_3^{3/2}. \qquad (A.8)
\end{aligned}
$$

Applying the interpolation inequality to $|p^2|_3$, we obtain

$$
|p^2|_3\leq C|p^2|_2^{\frac{1}{2}}(|\nabla p^2|_2^{\frac{1}{2}}+|\partial_\xi p^2|_2^{\frac{1}{2}}+\beta|p^2(\xi=1)|_2^{\frac{1}{2}}).
$$

Therefore, by Young's inequality we have

$$
\begin{aligned}
&\|Z_3\|_3(|\nabla_{e_\theta}\mathbf{u}|_2+|\nabla_{e_\varphi}\mathbf{u}|_2+|u|_2)|p^2|_3^{3/2}\\
\leq&C\|Z_3\|_3(|\nabla_{e_\theta}\mathbf{u}|_2+|\nabla_{e_\varphi}\mathbf{u}|_2+|u|_2)|p^2|_2^{\frac{3}{4}}(|\nabla p^2|_2^{\frac{3}{4}}+|\partial_\xi p^2|_2^{\frac{3}{4}}+\beta|p^2(\xi=1)|_2^{\frac{3}{4}})
\end{aligned}
$$

$$\leq\varepsilon(|\nabla p^2|_2^2+|\partial_\xi p^2|_2^2+\beta|p^2(\xi=1)|_2^2)+C\|Z_3\|_3^{\frac{8}{5}}(|\nabla_{e_\theta}\mathbf{u}|_2^{\frac{8}{5}}+|\nabla_{e_\varphi}\mathbf{u}|_2^{\frac{8}{5}}+|u|_2^{\frac{8}{5}})|p|_4^{\frac{12}{5}}. \quad (A.9)$$

Similarly, we obtain

$$\|Z_3\|_3\|Z_1\|_3|p^2|_3^{3/2}\leq\varepsilon(|\nabla p^2|_2^2+|\partial_\xi p^2|_2^2+\beta|p^2(\xi=1)|_2^2)+C\|Z_3\|_3^{\frac{8}{5}}\|Z_1\|_3^{\frac{8}{5}}|p|_4^{\frac{12}{5}}, \quad (A.10)$$

and

$$\begin{aligned}&\int_{\mathbf{D}}(Q_q+\gamma Z_3)p^3d\mathbf{D}\\ \leq&(|Q_q|_2+\gamma|Z_3|_2)|p^2|_3^{3/2}\\ \leq&\varepsilon(|\nabla p^2|_2^2+|\partial_\xi p^2|_2^2+\beta|p^2(\xi=1)|_2^2)+C(|Q_q|_2^{8/5}+\gamma|Z_3|_2^{8/5})|p|_4^{12/5}. \quad (A.11)\end{aligned}$$

From (A.8)-(A.10), we conclude that

$$\begin{aligned}&\left|\int_{\mathbf{D}}(w(Z_1+\mathbf{u})\partial_\xi Z_3+\nabla_{Z_1+\mathbf{u}}Z_3)p^3d\mathbf{D}\right|\\ \leq&\varepsilon(|\nabla p^2|_2^2+|\partial_\xi p^2|_2^2+\beta|p^2(\xi=1)|_2^2)\\ &+C\|Z_3\|_3^{\frac{8}{5}}(|\nabla_{e_\theta}\mathbf{u}|_2^{\frac{8}{5}}+|\nabla_{e_\varphi}\mathbf{u}|_2^{\frac{8}{5}}+|u|_2^{\frac{8}{5}})|p|_4^{\frac{12}{5}}+C\|Z_3\|_3^{\frac{8}{5}}\|Z_1\|_3^{\frac{8}{5}}|p|_4^{\frac{12}{5}}. \quad (A.12)\end{aligned}$$

Now back to the Equation (A.7), together with estimates (A.11), (A.12), we have

$$\begin{aligned}&\frac{d|p|_4^4}{dt}+(3-2\varepsilon)(|\nabla p^2|_2^2+|\partial_\xi p^2|_2^2)+(4\beta-2\varepsilon)\int_{\mathbf{S}^2}|p(\xi=1)|^4d\mathbf{S}^2\\ \leq&C\|Z_3\|_3^{8/5}(|\nabla_{e_\theta}\mathbf{u}|_2^{8/5}+|\nabla_{e_\varphi}\mathbf{u}|_2^{8/5}+|\mathbf{u}|_2^{8/5})|p|_4^{12/5}\\ &+C(|Q_q|_2^{8/5}+|Z_3|_2^{8/5})|p|_4^{12/5}+C\|Z_1\|_3^{8/5}\|Z_3\|_3^{8/5}|p|_4^{12/5}.\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}|p|_4^4=&\int_{\mathbf{D}}p^4d\mathbf{D}=-\int_{\mathbf{S}^2}\int_0^1\int_\xi^1\partial_\xi p^4+\int_{\mathbf{S}^2}\int_0^1p^4(\xi=1)\\ \leq&2|\partial_\xi p^2|_2^2+\frac{1}{2}|p|_4^4+\int_{\mathbf{S}^2}p^4(\xi=1)d\mathbf{S}^2,\end{aligned}$$

which implies that

$$\frac{|p|_4^2}{2}\frac{d|p|_4^2}{dt}+|p|_4^4\leq C(|Q_q|_2^{8/5}+|Z_3|_2^{8/5}+\|Z_1\|_3^{8/5}\|Z_3\|_3^{8/5}+\|Z_3\|_3^{8/5}\|\mathbf{u}\|_1^{8/5})|p|_4^{12/5}.$$

Thus, we can apply Grönwall's inequality to

$$\frac{d|p|_4^2}{dt}+|p|_4^2\leq C(|Q_q|_2^{8/5}+|Z_3|_2^{8/5}+\|Z_1\|_3^{8/5}\|Z_3\|_3^{8/5}+\|Z_3\|_3^{8/5}\|\mathbf{u}\|_1^{8/5})|p|_4^{2/5},$$

and get

$$|p|_4^2\leq|q_0|_4^2e^{-Ct}+C\int_{t_0}^te^{-C(t-s)}(|Q_q|_2^{8/5}+|Z_3|_2^{8/5}+\|Z_1\|_3^{8/5}\|Z_3\|_3^{8/5}+\|Z_3\|_3^{8/5}\|\mathbf{u}\|_1^{8/5})ds. \quad (A.13)$$

Taking inner product of (3.2b) with S^3 in $L^2(\mathbf{D})$, we have

$$\begin{aligned}
&\frac{1}{4}\frac{d|S|_4^4}{dt}+\frac{3}{4}|\nabla S^2|_2^2+\frac{3}{4}|\partial_\xi S^2|_2^2+\alpha\int_{\mathbf{S}^2}|S(\xi=1)|_4 d\mathbf{S}^2\\
=&\int_{\mathbf{D}}(Q_T+\gamma Z_2)S^3 d\mathbf{D}-\int_{\mathbf{D}}\Big(\nabla_{Z_1+\mathbf{u}}(Z_2+S)+w(Z_1+\mathbf{u})\partial_\xi(Z_2+S)\Big)S^3 d\mathbf{D}\\
&+\int_{\mathbf{D}}\frac{br_s}{r}(1+a(Z_3+p))w(Z_1+\mathbf{u})S^3 d\mathbf{D}. \qquad (A.14)
\end{aligned}$$

Similarly as the evaluations in the previous subsection, by integration by parts, the Hölder inequality and the Sobolev imbedding theorem, one has that

$$\int_{\mathbf{D}}\Big(\nabla_{Z_1+\mathbf{u}}S+w(Z_1+\mathbf{u})\partial_\xi S\Big)S^3 d\mathbf{D}=0.$$

Taking an analogous argument of (A.11) and (A.12), we have

$$\begin{aligned}
&\int_{\mathbf{D}}(Q_T+\gamma Z_2)S^3 d\mathbf{D}\\
\leq&\varepsilon(|\nabla S^2|_2^2+|\partial_\xi S^2|_2^2+\beta|S^2(\xi=1)|_2^2)+C(|Q_T|_2^{8/5}+|Z_2|_2^{8/5})|S|_4^{12/5}, \qquad (A.15)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbf{D}}(w(Z_1+\mathbf{u})\partial_\xi Z_2+\nabla_{Z_1+\mathbf{u}}Z_2)S^3 d\mathbf{D}\right|\\
\leq&\varepsilon(|\nabla S^2|_2^2+|\partial_\xi S^2|_2^2+\beta|S^2(\xi=1)|_2^2)\\
&+C\|Z_2\|_3^{\frac{8}{5}}(|\nabla_{e_\theta}\mathbf{u}|_2^{\frac{8}{5}}+|\nabla_{e_\varphi}\mathbf{u}|_2^{\frac{8}{5}}+|u|_2^{\frac{8}{5}})|S|_4^{\frac{12}{5}}+C\|Z_2\|_3^{\frac{8}{5}}\|Z_1\|_3^{\frac{8}{5}}|S|_4^{\frac{12}{5}}. \qquad (A.16)
\end{aligned}$$

Repeating the argument of (5.19) we obtain the estimates of (A.14) by

$$\begin{aligned}
&\left|\int_{\mathbf{D}}\frac{br_s}{r}\Big(\int_\xi^1 \mathrm{div}\ (Z_1+\mathbf{u})d\xi'\Big)S^3 d\mathbf{D}\right|\\
\leq&\varepsilon(|\nabla S^2|_2^2+|\partial_\xi S^2|_2^2)+C(\|Z_1\|_1^{8/5}+|\nabla_{e_\theta}\mathbf{u}|_2^{8/5}+|\nabla_{e_\varphi}\mathbf{u}|_2^{8/5})|S|_4^{12/5}\\
&+C(\|Z_1\|_1+|\nabla_{e_\theta}\mathbf{u}|_2+|\nabla_{e_\varphi}\mathbf{u}|_2)|S|_4^3. \qquad (A.17)
\end{aligned}$$

By Lemma 2.3 and Young's inequality, we get

$$\left|\int_{\mathbf{D}}\frac{abr_s}{r}(p+Z_3)\Big(\int_\xi^1 \mathrm{div}\mathbf{u}d\xi'\Big)S^3\right|\leq\varepsilon|\nabla S^2|_2^2+C(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2)|p+Z_3|_4^2|S|_4^2. \qquad (A.18)$$

Now back to the Equation (A.14), together with estimates in (A.15)-(A.18), we have

$$\begin{aligned}
&\frac{|S|_4^2}{2}\frac{d|S|_4^2}{dt}+(\frac{3}{4}-4\varepsilon)|\nabla S^2|_2^2+(\frac{3}{4}-4\varepsilon)|\partial_\xi S^2|_2^2+(\alpha-2\varepsilon)\int_{\mathbf{S}^2}|S(\xi=1)|_4 d\mathbf{S}^2\\
\leq&C(|Q_T|_2^{8/5}+|Z_2|_2^{8/5}+\|Z_1\|_1^{8/5}+\|\mathbf{u}\|_1^{8/5}\\
&+\|Z_2\|_3^{8/5}\|\mathbf{u}\|_1^{8/5}+\|Z_1\|_3^{8/5}\|Z_2\|_3^{8/5})|S|_4^{12/5}\\
&+C(\|Z_1\|_1+\|\mathbf{u}\|_1)|S|_4^3+C\|\mathbf{u}\|_1^2(|p|_4^2+|Z_3|_4^2)|S|_4^2. \qquad (A.19)
\end{aligned}$$

Since $\|S^2\|_1^2$ is equivalent to $|\nabla S^2|_2^2+|\partial_\xi S^2|_2^2+\alpha|S^2(\xi=1)|_2^2$, there exists a constant C such that

$$|S|_4^4=|S^2|_2^2\le C|\nabla S^2|_2^2+|\partial_\xi S^2|_2^2+\alpha|S^2(\xi=1)|_2^2.$$

Then by (A.19) we have

$$\begin{aligned}\frac{d|S|_4^2}{dt}+C|S|_4^2\le C(|Q_T|_2^2+|Z_2|_2^2+\|Z_1\|_1^2+\|Z_1\|_3^2\|Z_2\|_3^2+\|\mathbf{u}\|_1^2\\ +\|Z_3\|_1^2\|\mathbf{u}\|_1^2+\|Z_2\|_3^2\|\mathbf{u}\|_1^2+\|\mathbf{u}\|_1^2|p|_4^2).\end{aligned}\tag{A.20}$$

Applying Grönwall inequality, for $t<\tau$,

$$\begin{aligned}|S(t)|_4^2\le|S_0|_4^2e^{-Ct}+C\int_{t_0}^t e^{-C(t-s)}(|Q_T|_2^2+|Z_2|_2^2+\|Z_1\|_1^2+\|Z_1\|_3^2\|Z_2\|_3^2)ds\\ +C\int_{t_0}^t e^{-C(t-s)}(1+\|Z_3\|_1^2+\|Z_2\|_3^2+|p|_4^2)\|\mathbf{u}\|_1^2ds.\end{aligned}\tag{A.21}$$

A.3. L^4 estimates of u. To estimate L^4 norm of $\mathbf{u}$, we denote by

$$\bar{\mathbf{u}}(\theta,\varphi)=\int_0^1\mathbf{u}(\theta,\varphi,\xi)d\xi,\ \text{for}\ (\theta,\varphi)\in\mathbf{S}^2,$$

and $\widetilde{\mathbf{u}}=\mathbf{u}-\bar{\mathbf{u}}$. Note that

$$\bar{\widetilde{\mathbf{u}}}=0\ \text{and div}\ \bar{\mathbf{u}}=0.$$

Now we take the average value of (3.2a) with respect to ξ

$$\begin{aligned}&\partial_t\bar{\mathbf{u}}+\overline{\nabla_{Z_1+\mathbf{u}}(Z_1+\mathbf{u})+w(Z_1+\mathbf{u})\partial_\xi(Z_1+\mathbf{u})}+\frac{f}{R_0}\vec{k}\times(\bar{Z}_1+\bar{\mathbf{u}})+\nabla\Phi_s\\ &=\Delta\bar{\mathbf{u}}+\gamma\bar{Z}_1-\int_0^1\int_\xi^1\frac{br_s}{r}\nabla[1+a(Z_3+p)(Z_2+S)]d\xi'd\xi.\end{aligned}$$

Now applying integration by parts and boundary conditions,

$$\int_0^1\nabla_{Z_1+\mathbf{u}}(Z_1+\mathbf{u})d\xi=\int_0^1\nabla_{\widetilde{Z}_1+\widetilde{\mathbf{u}}}(\widetilde{Z}_1+\widetilde{\mathbf{u}})d\xi+\nabla_{\bar{Z}_1+\bar{\mathbf{u}}}(\bar{Z}_1+\bar{\mathbf{u}}),$$

and

$$\int_0^1 w(Z_1+\mathbf{u})\partial_\xi(Z_1+\mathbf{u})d\xi=\int_0^1(Z_1+\mathbf{u})\text{div}\ (Z_1+\mathbf{u})d\xi=\int_0^1(\widetilde{Z}_1+\widetilde{\mathbf{u}})\text{div}\ (\widetilde{Z}_1+\widetilde{\mathbf{u}})d\xi.$$

Thus, we have $\bar{\mathbf{u}}$ satisfy the following equation and boundary conditions

$$\begin{aligned}&\partial_t\bar{\mathbf{u}}+\overline{\nabla_{\widetilde{Z}_1+\widetilde{\mathbf{u}}}(\widetilde{Z}_1+\widetilde{\mathbf{u}})+(\widetilde{Z}_1+\widetilde{\mathbf{u}})\text{div}\ (\widetilde{Z}_1+\widetilde{\mathbf{u}})}+\nabla_{\bar{Z}_1+\bar{\mathbf{u}}}(\bar{Z}_1+\bar{\mathbf{u}})\\ &\quad+\frac{f}{R_0}\vec{k}\times(\bar{Z}_1+\bar{\mathbf{u}})+\nabla\Phi_s=\Delta\bar{\mathbf{u}}+\gamma\bar{Z}_1-\int_0^1\int_\xi^1\frac{br_s}{r}\nabla[1+a(Z_3+p)(Z_2+S)]d\xi'd\xi,\end{aligned}\tag{A.22a}$$

$$\text{div}\ \bar{\mathbf{u}}=0,\ \text{on}\ \mathbf{S}^2.\tag{A.22b}$$

By subtracting (A.22) from (3.2a), we can also conclude that $\widetilde{\mathbf{u}}$ satisfies the following equation and boundary conditions

$$\begin{aligned}
&\partial_t\widetilde{\mathbf{u}}+\nabla_{\widetilde{Z}_1+\widetilde{\mathbf{u}}}(\widetilde{Z}_1+\widetilde{\mathbf{u}})+w(\widetilde{Z}_1+\widetilde{\mathbf{u}})\partial_\xi(\widetilde{Z}_1+\widetilde{\mathbf{u}})+\nabla_{\widetilde{Z}_1+\widetilde{\mathbf{u}}}(\bar{Z}_1+\bar{\mathbf{u}})+\nabla_{\bar{Z}_1+\bar{\mathbf{u}}}(\widetilde{Z}_1+\widetilde{\mathbf{u}})\\
&\quad-\overline{\nabla_{\widetilde{Z}_1+\widetilde{\mathbf{u}}}(\widetilde{Z}_1+\widetilde{\mathbf{u}})+(\widetilde{Z}_1+\widetilde{\mathbf{u}})\mathrm{div}\ (\widetilde{Z}_1+\widetilde{\mathbf{u}})}+\frac{f}{R_0}\vec{k}\times(\widetilde{Z}_1+\widetilde{\mathbf{u}})-\Delta\widetilde{\mathbf{u}}-\partial_{\xi\xi}\widetilde{\mathbf{u}}\\
&=\int_\xi^1\frac{br_s}{r}\nabla[1+a(Z_3+p)(Z_2+S)]d\xi'-\int_0^1\int_\xi^1\frac{br_s}{r}\nabla[1+a(Z_3+p)(Z_2+S)]d\xi'd\xi, \qquad \text{(A.23a)}\\
&\partial_\xi\widetilde{\mathbf{u}}=0,\ \text{when } \xi=0 \text{ and } \xi=1. \qquad \text{(A.23b)}
\end{aligned}$$

By the definition of covariant derivative, for $h\in C^\infty(\mathbf{D})$ and $u=(u_\theta,u_\varphi)\in C^\infty(T\mathbf{D}|T\mathbf{S}^2)$, we have

$$\begin{aligned}
&\nabla_{e_\theta}(h\widetilde{\mathbf{u}})=h\nabla_{e_\theta}\widetilde{\mathbf{u}}+\widetilde{\mathbf{u}}\nabla_{e_\theta}h,\\
&\nabla_{e_\varphi}(h\widetilde{\mathbf{u}})=h\nabla_{e_\varphi}\widetilde{\mathbf{u}}+\widetilde{\mathbf{u}}\nabla_{e_\varphi}h,\\
&\nabla_{e_\theta}(u\cdot\widetilde{\mathbf{u}})=u\cdot\nabla_{e_\theta}\widetilde{\mathbf{u}}+\widetilde{\mathbf{u}}\cdot\nabla_{e_\theta}u,\\
&\nabla_{e_\varphi}(u\cdot\widetilde{\mathbf{u}})=u\cdot\nabla_{e_\varphi}\widetilde{\mathbf{u}}+\widetilde{\mathbf{u}}\cdot\nabla_{e_\varphi}u,\\
&\nabla_u\widetilde{\mathbf{u}}=u_\theta\nabla_{e_\theta}\widetilde{\mathbf{u}}+u_\varphi\nabla_{e_\varphi}\widetilde{\mathbf{u}}.
\end{aligned}$$

Applying integration by parts, we have

$$\int_{\mathbf{D}}\left[\nabla_{\widetilde{\mathbf{u}}}\widetilde{\mathbf{u}}+\left(\int_\xi^1\mathrm{div}\ \widetilde{\mathbf{u}}d\xi'\right)\partial_\xi\widetilde{\mathbf{u}}\right]\cdot|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}}d\mathbf{D}=0.$$

Using integration by parts together with div $\bar{\mathbf{u}}=0$,

$$\int_{\mathbf{D}}\nabla_{\bar{Z}_1+\bar{\mathbf{u}}}\widetilde{\mathbf{u}}\cdot(|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}})d\mathbf{D}=-\frac{1}{4}\int_{\mathbf{D}}|\widetilde{\mathbf{u}}|^4\mathrm{div}\ (\bar{Z}_1+\bar{\mathbf{u}})d\mathbf{D}=0.$$

Similarly, we will also have

$$\begin{aligned}
&\int_{\mathbf{D}}\overline{\nabla_{\widetilde{Z}_1+\widetilde{\mathbf{u}}}(\widetilde{Z}_1+\widetilde{\mathbf{u}})+(\widetilde{Z}_1+\widetilde{\mathbf{u}})\mathrm{div}\ (\widetilde{Z}_1+\widetilde{\mathbf{u}})}\cdot|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}}d\mathbf{D}\\
&=-\int_{\mathbf{D}}\overline{(\widetilde{Z}_{1,\theta}+\widetilde{u}_\theta)(\widetilde{Z}_1+\widetilde{\mathbf{u}})}\cdot\nabla_{e_\theta}(|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}})d\mathbf{D}-\int_{\mathbf{D}}\overline{(\widetilde{Z}_{1,\varphi}+\widetilde{u}_\varphi)(\widetilde{Z}_1+\widetilde{\mathbf{u}})}\cdot\nabla_{e_\varphi}(|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}})d\mathbf{D}.
\end{aligned}$$

Now taking inner product of the (A.23) with $|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}}$ in $(L^2(\mathbf{D}))^2$, and using the above equalities, we get

$$\begin{aligned}
&\frac{1}{4}\frac{d|\widetilde{\mathbf{u}}|_4^4}{dt}+\frac{1}{2}\int_{\mathbf{D}}\Big(|\nabla_{e_\theta}|\widetilde{\mathbf{u}}^2||^2+|\nabla_{e_\varphi}|\widetilde{\mathbf{u}}^2||^2+|\partial_\xi|\widetilde{\mathbf{u}}^2||^2\Big)d\mathbf{D}\\
&\quad+\int_{\mathbf{D}}|\widetilde{\mathbf{u}}|^2\Big(|\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2+|\nabla_{e_\varphi}\widetilde{\mathbf{u}}|^2+|\partial_\xi\widetilde{\mathbf{u}}|^2+|\widetilde{\mathbf{u}}|^2\Big)d\mathbf{D}\\
&=-\int_{\mathbf{D}}[\nabla_{\widetilde{Z}_1}\widetilde{\mathbf{u}}+\nabla_{\widetilde{\mathbf{u}}}\widetilde{Z}_1+\nabla_{\widetilde{Z}_1}\widetilde{Z}_1]\cdot|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}}d\mathbf{D}\\
&\quad-\int_{\mathbf{D}}[w(\widetilde{Z}_1)\partial_\xi\widetilde{\mathbf{u}}+w(\widetilde{\mathbf{u}})\partial_\xi\widetilde{Z}_1+w(\widetilde{Z}_1)\partial_\xi\widetilde{Z}_1]\cdot|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}}d\mathbf{D}\\
&\quad+\int_{\mathbf{D}}\nabla_{\widetilde{Z}_1+\widetilde{\mathbf{u}}}(\bar{Z}_1+\bar{\mathbf{u}})\cdot|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}}d\mathbf{D}
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbf{D}} \nabla_{\bar{Z}_1+\bar{\mathbf{u}}} \widetilde{Z}_1 \widetilde{\mathbf{u}}^3 d\mathbf{D} - \int_{\mathbf{D}} \left(\frac{f}{R_0} \vec{k} \times \widetilde{Z}_1 \right) \cdot |\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}} d\mathbf{D} \\
&-\int_{\mathbf{D}} \overline{(\widetilde{Z}_{1,\theta} + \widetilde{u}_\theta)(\widetilde{Z}_1 + \widetilde{\mathbf{u}})} \cdot \nabla_{e_\theta} (|\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}}) d\mathbf{D} - \int_{\mathbf{D}} \overline{(\widetilde{Z}_{1,\varphi} + \widetilde{u}_\varphi)(\widetilde{Z}_1 + \widetilde{\mathbf{u}})} \cdot \nabla_{e_\varphi} (|\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}}) d\mathbf{D} \\
&+\int_{\mathbf{D}} \left(\int_\xi^1 \frac{br_s}{r} \nabla [1 + a(Z_3 + p)](Z_2 + S) d\xi' \right) \cdot |\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}} d\mathbf{D} \\
&-\int_{\mathbf{D}} \left(\int_0^1 \int_\xi^1 \frac{br_s}{r} \nabla [1 + a(Z_3 + p)](Z_2 + S) d\xi' d\xi \right) \cdot |\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}} d\mathbf{D} := \sum_{i=1}^9 I_i. \qquad \text{(A.24)}
\end{aligned}
$$

We will estimate I_i respectively for $i=1,\cdots,9$. Now applying integration by parts to first terms of I_1 and I_2,

$$
\begin{aligned}
&\int_{\mathbf{D}} \nabla_{\widetilde{Z}_1} \widetilde{\mathbf{u}} \cdot |\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}} d\mathbf{D} + \int_{\mathbf{D}} w(\widetilde{Z}_1) \partial_\xi \widetilde{\mathbf{u}} \cdot |\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}} d\mathbf{D} \\
&= -\frac{1}{4} \int_{\mathbf{D}} \nabla_{\widetilde{Z}_1} |\widetilde{\mathbf{u}}|^4 d\mathbf{D} - \frac{1}{4} \int_{\mathbf{D}} w(\widetilde{Z}_1) \partial_\xi |\widetilde{\mathbf{u}}|^4 = 0.
\end{aligned}
$$

Then applying the Hölder inequality and the interpolation inequality to the other terms in I_1, we get

$$
|I_1| \le \varepsilon \int_{\mathbf{D}} \left(|\nabla_{e_\theta} |\widetilde{\mathbf{u}}|^2|^2 + |\nabla_{e_\varphi} |\widetilde{\mathbf{u}}|^2|^2 + |\partial_\xi |\widetilde{\mathbf{u}}|^2|^2 + |\widetilde{\mathbf{u}}|^4 \right) d\mathbf{D} + C\|Z_1\|_2^2 |\widetilde{\mathbf{u}}|_4^4 + C\|Z_1\|_2^{16/5} |\widetilde{\mathbf{u}}|_4^{12/5}.
$$

Applying integration by parts and the interpolation inequality on $\mathbf{S}^2$ to the other terms of I_2,

$$
\begin{aligned}
-\int_{\mathbf{D}} w(\widetilde{\mathbf{u}}) \partial_\xi \widetilde{Z}_1 \cdot (|\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}}) d\mathbf{D} &\le |\partial_\xi Z_1|_\infty \int_{\mathbf{S}^2} \left(\int_0^1 |\mathrm{div} \widetilde{\mathbf{u}}| d\xi \int_0^1 |\widetilde{\mathbf{u}}|^3 d\xi \right) d\mathbf{S}^2 \\
&\le \varepsilon (|\nabla_{e_\theta} |\widetilde{\mathbf{u}}|^2|_2^2 + |\nabla_{e_\varphi} |\widetilde{\mathbf{u}}|^2|_2^2) \\
&\quad + C\|Z_1\|_3^{4/3} \|\mathbf{u}\|_1^{4/3} |\widetilde{\mathbf{u}}|_4^{8/3} + C\|Z_1\|_3 \|\mathbf{u}\|_1 |\widetilde{\mathbf{u}}|_4^3.
\end{aligned}
$$

Using the Hölder inequality, Sobolev imbedding theorem and Lemma 2.1 yields that

$$
-\int_{\mathbf{D}} (\nabla_{\widetilde{Z}_1} \widetilde{Z}_1) |\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}} d\mathbf{D} \le |Z_1|_\infty (|\nabla_{e_\theta} Z_1|_\infty + |\nabla_{e_\varphi} Z_1|_\infty) |\widetilde{\mathbf{u}}|_4^3 \le C\|Z_1\|_3^2 |\widetilde{\mathbf{u}}|_4^3.
$$

By the argument above, we have estimates for I_2 as

$$
|I_2| \le \varepsilon \left(|\nabla_{e_\theta} |\widetilde{\mathbf{u}}|^2|_2^2 + |\nabla_{e_\varphi} |\widetilde{\mathbf{u}}|^2|_2^2 \right) + C\|Z_1\|_3^2 |\widetilde{\mathbf{u}}|_4^3 + C\|Z_1\|_3^{4/3} \|\mathbf{u}\|_1^{4/3} |\widetilde{\mathbf{u}}|_4^{8/3} + C\|Z_1\|_3 \|\mathbf{u}\|_1 |\widetilde{\mathbf{u}}|_4^3.
$$

By the interpolation inequality, the Hölder inequality and the Minkowski inequality,

$$
\begin{aligned}
\int_{\mathbf{D}} (\nabla_{\widetilde{\mathbf{u}}} \bar{\mathbf{u}}) \cdot |\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}} d\mathbf{D} &\le \int_{\mathbf{S}^2} \left((|\nabla_{e_\theta} \bar{\mathbf{u}}| + |\nabla_{e_\varphi} \bar{\mathbf{u}}|) \int_0^1 |\widetilde{\mathbf{u}}|^4 d\xi \right) d\mathbf{S}^2 \\
&\le \varepsilon (|\nabla_{e_\theta} |\widetilde{\mathbf{u}}|^2|_2^2 + |\nabla_{e_\varphi} |\widetilde{\mathbf{u}}|^2|_2^2) + C(1 + |\nabla_{e_\theta} \bar{\mathbf{u}}|_2^2 + |\nabla_{e_\varphi} \bar{\mathbf{u}}|_2^2) |\widetilde{\mathbf{u}}|_4^4.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{\mathbf{D}} (\nabla_{\widetilde{Z}_1} \bar{Z}_1) \cdot (|\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}}) d\mathbf{D} + \int_{\mathbf{D}} (\nabla_{\widetilde{Z}_1} \bar{\mathbf{u}}) \cdot (|\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}}) d\mathbf{D} + \int_{\mathbf{D}} (\nabla_{\widetilde{\mathbf{u}}} \bar{Z}_1) \cdot (|\widetilde{\mathbf{u}}|^2 \widetilde{\mathbf{u}}) d\mathbf{D} \\
&\le \varepsilon (|\nabla_{e_\theta} |\widetilde{\mathbf{u}}|^2|_2^2 + |\nabla_{e_\varphi} |\widetilde{\mathbf{u}}|^2|_2^2 + |\partial_\xi |\widetilde{\mathbf{u}}|^2|_2^2) + C\|Z_1\|_2^{8/5} \|\mathbf{u}\|_1^{8/5} |\widetilde{\mathbf{u}}|_4^{12/5}
\end{aligned}
$$

$$+C\|Z_1\|_2\|\mathbf{u}\|_1|\widetilde{\mathbf{u}}|_4^3+C\|Z_1\|_2^2|\widetilde{\mathbf{u}}|_4^3+C\|Z_1\|_3|\widetilde{\mathbf{u}}|_4^4.$$

Therefore, combining the above estimates we obtain

$$\begin{aligned}I_3\le&\varepsilon(|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|_2^2+|\nabla_{e_\varphi}|\widetilde{\mathbf{u}}|^2|_2^2+|\partial_\xi|\widetilde{\mathbf{u}}|^2|_2^2)+C\|Z_1\|_2^{8/5}\|\mathbf{u}\|_1^{8/5}|\widetilde{\mathbf{u}}|_4^{12/5}\\&+C\|Z_1\|_2\|\mathbf{u}\|_1|\widetilde{\mathbf{u}}|_4^3+C\|Z_1\|_2^2|\widetilde{\mathbf{u}}|_4^3+C(1+\|Z_1\|_2+\|\mathbf{u}\|_1^2)|\widetilde{\mathbf{u}}|_4^4.\end{aligned}$$

Analogously, we have

$$\begin{aligned}|I_4|=&\left|\int_{\mathbf{D}}\nabla_{\bar{Z}_1+\bar{\mathbf{u}}}\widetilde{Z}_1\cdot(|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}})d\mathbf{D}\right|\\\le&\varepsilon\int_{\mathbf{D}}\Big(|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|^2+|\nabla_{e_\varphi}|\widetilde{\mathbf{u}}|^2|^2+|\partial_\xi|\widetilde{\mathbf{u}}|^2|^2\Big)d\mathbf{D}\\&+C\|\widetilde{Z}_1\|_2(|\bar{\mathbf{u}}|_4+|\bar{Z}_1|_4)|\widetilde{\mathbf{u}}|_4^3+C\|\widetilde{Z}_1\|_2^{8/5}(|\bar{\mathbf{u}}|_{L^4(\mathbf{S}^2)}^{8/5}+|\bar{Z}_1|_{L^4(\mathbf{S}^2)}^{8/5})|\widetilde{\mathbf{u}}|_4^{12/5}.\end{aligned}$$

By the Hölder inequality and the Sobolev imbedding theorem, we have

$$|I_5|\le C\|Z_1\|_1|\widetilde{\mathbf{u}}|_4^3.$$

To estimate I_5, using the similar calculations above we first obtain

$$\begin{aligned}&\int_{\mathbf{D}}\overline{\widetilde{u}_\theta\widetilde{\mathbf{u}}}\cdot(\widetilde{\mathbf{u}}\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2)\\\le&\int_{\mathbf{S}^2}\overline{|\widetilde{\mathbf{u}}|^2}\Big(\int_0^1|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|^2d\xi\Big)^{1/2}\Big(\int_0^1|\widetilde{\mathbf{u}}|^2d\xi\Big)^{1/2}d\mathbf{S}^2\\\le&\varepsilon|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|_2^2+C|\widetilde{\mathbf{u}}|_4^4\|\mathbf{u}\|_1^2.\end{aligned}\tag{A.25}$$

Then

$$\begin{aligned}&-\int_{\mathbf{D}}\overline{\widetilde{Z}_{1,\theta}\widetilde{Z}_1}\nabla_{e_\theta}(|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}})d\mathbf{D}\\=&-\int_{\mathbf{D}}\overline{\widetilde{Z}_{1,\theta}\widetilde{Z}_1}(|\widetilde{\mathbf{u}}|^2\nabla_{e_\theta}\widetilde{\mathbf{u}}+\widetilde{\mathbf{u}}\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2)d\mathbf{D}\\\le&\varepsilon\Big(\int_{\mathbf{D}}(|\widetilde{\mathbf{u}}|^2|\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2+|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|^2)d\mathbf{D}\Big)+C\|Z_1\|_2^4|\mathbf{u}|_2^2.\end{aligned}\tag{A.26}$$

Using the Hölder inequality and the Minkowski inequality, the interpolation inequality and Young's inequality we obtain

$$\begin{aligned}&-\int_{\mathbf{D}}\overline{\widetilde{Z}_{1,\theta}\widetilde{\mathbf{u}}}\nabla_{e_\theta}(|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}})d\mathbf{D}\\\le&|Z_1|_\infty\int_{\mathbf{S}^2}\Big[\int_0^1|\widetilde{\mathbf{u}}|d\xi\Big(\int_0^1|\widetilde{\mathbf{u}}|^2d\xi\Big)^{1/2}\Big(\int_0^1|\widetilde{\mathbf{u}}|^2|\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2d\xi\Big)^{1/2}\Big]d\mathbf{S}^2\\&+|Z_1|_\infty\int_{\mathbf{S}^2}\Big[\int_0^1|\widetilde{\mathbf{u}}|d\xi\Big(\int_0^1|\widetilde{\mathbf{u}}|^2d\xi\Big)^{1/2}\Big(\int_0^1|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|^2d\xi\Big)^{1/2}\Big]d\mathbf{S}^2\\\le&\varepsilon(\int_{\mathbf{D}}(|\widetilde{\mathbf{u}}|^2||\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2+|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|^2)d\mathbf{D})+C\|Z_1\|_2^2|\widetilde{\mathbf{u}}|_4^3.\end{aligned}\tag{A.27}$$

Analogously, we have

$$-\int_{\mathbf{D}}\overline{\widetilde{u}_\theta\widetilde{Z}_1}\nabla_{e_\theta}(|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}})d\mathbf{D}\le\varepsilon(\int_{\mathbf{D}}(|\widetilde{\mathbf{u}}|^2||\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2+|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|^2)d\mathbf{D})+C\|Z_1\|_2^2|\widetilde{\mathbf{u}}|_4^3.\tag{A.28}$$

Combining the estimates (A.25)-(A.28) yields that

$$\begin{aligned}I_6=&-\int_{\mathbf{D}}\overline{(\widetilde{Z}_{1,\theta}+\widetilde{u}_\theta)(\widetilde{Z}_1+\widetilde{\mathbf{u}})}\cdot\nabla_{e_\theta}(|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}})d\mathbf{D}\\ \le&\varepsilon(|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|_2^2+||\widetilde{\mathbf{u}}||\nabla_{e_\theta}\widetilde{\mathbf{u}}||_2^2)+C\|Z_1\|_2^4|\mathbf{u}|_2^2+C\|Z_1\|_2^2|\widetilde{\mathbf{u}}|_4^3+C\|\mathbf{u}\|_1^2|\widetilde{\mathbf{u}}|_4^4.\end{aligned}\tag{A.29}$$

Repeating the argument of (A.29) we get

$$\begin{aligned}I_7=&-\int_{\mathbf{D}}\overline{(\widetilde{Z}_{1,\varphi}+\widetilde{u}_\varphi)(\widetilde{Z}_1+\widetilde{\mathbf{u}})}\cdot\nabla_{e_\varphi}(|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}})d\mathbf{D}\\ \le&\varepsilon(|\nabla_{e_\varphi}|\widetilde{\mathbf{u}}|^2|_2^2+||\widetilde{\mathbf{u}}||\nabla_{e_\varphi}\widetilde{\mathbf{u}}||_2^2)+C\|Z_1\|_2^4|\mathbf{u}|_2^2+C\|Z_1\|_2^2|\widetilde{\mathbf{u}}|_4^3+C\|\mathbf{u}\|_1^2|\widetilde{\mathbf{u}}|_4^4.\end{aligned}\tag{A.30}$$

By the definition of the horizontal covariant derivative and the horizontal divergence (see (2.2a) and (2.3a)), we have

$$\mathrm{div}(|\widetilde{\mathbf{u}}|^2\widetilde{\mathbf{u}})=(\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2+\nabla_{e_\varphi}|\widetilde{\mathbf{u}}|^2)\cdot\widetilde{\mathbf{u}}+|\widetilde{\mathbf{u}}|^2\mathrm{div}\widetilde{\mathbf{u}},$$

and

$$|\mathrm{div}\widetilde{\mathbf{u}}|\le|\nabla_{e_\theta}\widetilde{\mathbf{u}}|+|\nabla_{e_\varphi}\widetilde{\mathbf{u}}|.$$

Therefore applying integration by parts to I_8 and I_9, we obtain

$$\begin{aligned}|I_8+I_9|\le&C\int_{\mathbf{S}^2}(|\overline{Z_2+S}|+|\overline{(Z_3+p)(Z_2+S)}|)\int_0^1|\widetilde{\mathbf{u}}|^2\Big(|\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2+|\nabla_{e_\varphi}\widetilde{\mathbf{u}}|^2\Big)^{1/2}d\xi d\mathbf{S}^2\\ &+C\int_{\mathbf{S}^2}(|\overline{Z_2+S}|+|\overline{(Z_3+p)(Z_2+S)}|)\int_0^1|\widetilde{\mathbf{u}}|\Big(|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|+|\nabla_{e_\varphi}|\widetilde{\mathbf{u}}|^2|\Big)d\xi d\mathbf{S}^2\\ =&J_1+J_2.\end{aligned}$$

We first estimate J_1, then the estimate of J_2 follows similarly. By the Hölder inequality, the interpolation inequality and Young's inequality we have

$$\begin{aligned}J_1\le&|\widetilde{\mathbf{u}}|\nabla_{e_\theta}\widetilde{\mathbf{u}}||_2|\widetilde{\mathbf{u}}|_4\Big(|\overline{Z_2+S}|_{L^4(\mathbf{S}^2)}+|\overline{(Z_3+p)(Z_2+S)}|_{L^4(\mathbf{S}^2)}\Big)\\ \le&\varepsilon\int_{\mathbf{D}}|\widetilde{\mathbf{u}}|^2|\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2d\mathbf{D}+C|\widetilde{\mathbf{u}}|_4^2(\|Z_2\|_2^2+\|Z_3\|_2^2+|p|_4\|p\|_1+|S|_4\|S\|_1).\end{aligned}$$

Similarly, we have the estimate of J_2,

$$J_2\le\varepsilon\int_{\mathbf{D}}|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|^2d\mathbf{D}+C|\widetilde{\mathbf{u}}|_4^2(\|Z_2\|_2^2+\|Z_3\|_2^2+|p|_4\|p\|_1+|S|_4\|S\|_1).$$

By virtue of Estimates of J_1 and J_2 we have

$$|I_8+I_9|\le\varepsilon\int_{\mathbf{D}}(|\nabla_{e_\theta}|\widetilde{\mathbf{u}}|^2|^2+|\widetilde{\mathbf{u}}|^2|\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2)d\mathbf{D}+C|\widetilde{\mathbf{u}}|_4^2(\|Z_2\|_2^2+\|Z_3\|_2^2+|p|_4\|p\|_1+|S|_4\|S\|_1).$$

Throughout the estimates I_1-I_{10}, we have

$$\begin{aligned}&\frac{d|\widetilde{\mathbf{u}}|_4^4}{dt}+\int_{\mathbf{D}}\Big(|\nabla_{e_\theta}\widetilde{\mathbf{u}}^2|^2+|\nabla_{e_\varphi}\widetilde{\mathbf{u}}^2|^2+|\partial_\xi|\widetilde{\mathbf{u}}|^2|^2\Big)d\mathbf{D}\\ &+\int_{\mathbf{D}}|\widetilde{\mathbf{u}}|^2\Big(|\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2+|\nabla_{e_\varphi}\widetilde{\mathbf{u}}|^2+|\partial_\xi\widetilde{\mathbf{u}}|^2+|\widetilde{\mathbf{u}}|^2\Big)d\mathbf{D}\end{aligned}$$

$$
\begin{aligned}
&\leq C(1+\|Z_1\|_2^4+\|\mathbf{u}\|_1^2)|\widetilde{\mathbf{u}}|_4^4+C(\|Z_1\|_3^2+\|Z_1\|_3\|\mathbf{u}\|_1+\|Z_1\|_1)|\widetilde{\mathbf{u}}|_4^3\\
&\quad+C\|Z_1\|_3^{4/3}\|\mathbf{u}\|_1^{4/3}|\widetilde{\mathbf{u}}|_4^{8/3}+C(\|Z_1\|_2^{16/5}+\|Z_1\|_2^{8/5}\|\mathbf{u}\|_1^{8/5})|\widetilde{\mathbf{u}}|_4^{12/5}\\
&\quad+C(\|Z_1\|_2^4+\|Z_2\|_2^2+\|Z_3\|_2^2+\|Z_1\|_2^2\|\mathbf{u}\|_1^2+|S|_4\|S\|_1+|p|_4\|p\|_1)|\widetilde{\mathbf{u}}|_4^2, \qquad (A.31)
\end{aligned}
$$

and we also have

$$
\begin{aligned}
&\frac{d|\widetilde{\mathbf{u}}|_4^2}{dt}+|\widetilde{\mathbf{u}}|_4^2\\
&\leq C(1+\|Z_1\|_2^4+\|\mathbf{u}\|_1^2)|\widetilde{\mathbf{u}}|_4^2+C\|Z_1\|_3^2\|\mathbf{u}\|_1^2\\
&\quad+C|S|_4\|S\|_1+C|p|_4\|p\|_1+C(\|Z_1\|_1^2+\|Z_2\|_2^2+\|Z_3\|_2^2+\|Z_1\|_3^4). \qquad (A.32)
\end{aligned}
$$

Now applying Grönwall's inequality gives that

$$
\begin{aligned}
&\sup_{t\in[0,\tau)}|\widetilde{\mathbf{u}}(t)|_4^4+\int_0^\tau\int_{\mathbf{D}}\Big(|\nabla_{e_\theta}\widetilde{\mathbf{u}}^2|^2+|\nabla_{e_\varphi}\widetilde{\mathbf{u}}^2|^2+|\partial_\xi|\widetilde{\mathbf{u}}|^2|^2\Big)d\mathbf{D}\\
&\quad+\int_0^\tau\int_{\mathbf{D}}|\widetilde{\mathbf{u}}|^2\Big(|\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2+|\nabla_{e_\varphi}\widetilde{\mathbf{u}}|^2+|\partial_\xi\widetilde{\mathbf{u}}|^2+|\widetilde{\mathbf{u}}|^2\Big)d\mathbf{D}\\
&\leq C(\tau,Z_1,Z_2,Z_3,U_0). \qquad (A.33)
\end{aligned}
$$

Taking the inner product of (A.22) with $-\Delta\bar{\mathbf{u}}$ in $L^2(\mathbf{S}^2)$, we get

$$
\begin{aligned}
&\frac{1}{2}\partial_t(|\nabla_{e_\theta}\bar{\mathbf{u}}|_2^2+|\nabla_{e_\varphi}\bar{\mathbf{u}}|_2^2+|\bar{\mathbf{u}}|_2^2)+|\Delta\bar{\mathbf{u}}|_2^2\\
=&\int_{\mathbf{S}^2}\overline{\nabla_{\widetilde{Z}_1+\widetilde{\mathbf{u}}}(\widetilde{Z}_1+\widetilde{\mathbf{u}})+(\widetilde{Z}_1+\widetilde{\mathbf{u}})\text{div }(\widetilde{Z}_1+\widetilde{\mathbf{u}})}\cdot\Delta\bar{\mathbf{u}}d\mathbf{S}^2+\int_{\mathbf{S}^2}\nabla_{\bar{Z}_1+\bar{\mathbf{u}}}(\bar{Z}_1+\bar{\mathbf{u}})\cdot\Delta\bar{\mathbf{u}}d\mathbf{S}^2\\
&+\int_{\mathbf{S}^2}\frac{f}{R_0}\vec{k}\times\bar{Z}_1\cdot\Delta\bar{\mathbf{u}}d\mathbf{S}^2, \qquad (A.34)
\end{aligned}
$$

because by integration by parts, we have

$$
\int_{\mathbf{S}^2}\frac{f}{R_0}\vec{k}\times\bar{\mathbf{u}}\cdot\Delta\bar{\mathbf{u}}d\mathbf{S}^2=0,\quad\int_{\mathbf{S}^2}\nabla\Phi_s\cdot\Delta\bar{\mathbf{u}}d\mathbf{S}^2=0,
$$

and

$$
\int_{\mathbf{S}^2}\nabla\int_0^1\int_\xi^1\frac{br_s}{r}[1+a(Z_3+p)(Z_2+S)]d\xi'd\xi\cdot\Delta\bar{\mathbf{u}}d\mathbf{S}^2=0.
$$

Applying the Hölder inequality, the Minkowski inequality and the Sobolev imbedding theorem, we first have

$$
\begin{aligned}
&\left|\int_{\mathbf{S}^2}\overline{\nabla_{\widetilde{Z}_1+\widetilde{\mathbf{u}}}(\widetilde{Z}_1+\widetilde{\mathbf{u}})+(\widetilde{Z}_1+\widetilde{\mathbf{u}})\text{div }(\widetilde{Z}_1+\widetilde{\mathbf{u}})}\cdot\Delta\bar{\mathbf{u}}d\mathbf{S}^2\right|\\
&\leq\varepsilon|\Delta\bar{\mathbf{u}}|_2^2+C\int_{\mathbf{D}}|\widetilde{\mathbf{u}}|^2(|\nabla_{e_\theta}\widetilde{\mathbf{u}}|^2+|\nabla_{e_\varphi}\widetilde{\mathbf{u}}|^2)d\mathbf{D}\\
&\qquad+C(|\widetilde{\mathbf{u}}|_4^2\|Z_1\|_2^2+|Z_1|_\infty\|\mathbf{u}\|_1^2+|Z_1|_\infty\|Z_1\|_1^2).
\end{aligned}
$$

By the Hölder inequality and Young's inequality we have

$$
\int_{\mathbf{S}^2}\bar{\mathbf{u}}_\theta\nabla_{e_\theta}\bar{\mathbf{u}}\cdot\Delta\bar{\mathbf{u}}d\mathbf{S}^2\leq\varepsilon|\Delta\bar{\mathbf{u}}|_2^2+C|\mathbf{u}|_2^2\|\mathbf{u}\|_1^2|\nabla_{e_\theta}\bar{\mathbf{u}}|_2^2.
$$

Similarly, we can get

$$\begin{aligned}&\left|\int_{\mathbf{S}^2}\nabla_{\bar{Z}_1+\bar{\mathbf{u}}}(\bar{Z}_1+\bar{\mathbf{u}})\cdot\Delta\bar{\mathbf{u}}d\mathbf{S}^2\right|\\ \leq&\varepsilon|\Delta\bar{\mathbf{u}}|_2^2+C|\mathbf{u}|_2^2\|\mathbf{u}\|_1^2(|\nabla_{e_\theta}\bar{\mathbf{u}}|_2^2+|\nabla_{e_\varphi}\bar{\mathbf{u}}|_2^2)+C\|Z_1\|_3^2|\mathbf{u}|_2^2+C\|Z_1\|_2\|\mathbf{u}\|_1^2+C\|Z_1\|_2^4.\end{aligned}$$

Therefore, combining the above arguments and (A.34) yields

$$\begin{aligned}&\frac{1}{2}\partial_t(|\nabla_{e_\theta}\bar{\mathbf{u}}|_2^2+|\nabla_{e_\varphi}\bar{\mathbf{u}}|_2^2+|\bar{\mathbf{u}}|_2^2)+|\Delta\bar{\mathbf{u}}|_2^2\\ \leq&C\int_{\mathbf{D}}|\tilde{\mathbf{u}}|^2(|\nabla_{e_\theta}\tilde{\mathbf{u}}|^2+|\nabla_{e_\varphi}\tilde{\mathbf{u}}|^2)d\mathbf{D}+C|\mathbf{u}|_2^2\|\mathbf{u}\|_1^2(|\nabla_{e_\theta}\bar{\mathbf{u}}|_2^2+|\nabla_{e_\varphi}\bar{\mathbf{u}}|_2^2)\\ &+C\|Z_1\|_2\|\mathbf{u}\|_1^2+C|\tilde{\mathbf{u}}|_4^2\|Z_1\|_2^2+C|\mathbf{u}|_2^2\|Z_1\|_3^2\\ &+C\|Z_1\|_2^4+C\|Z_1\|_2\|Z_1\|_1^2+C|Z_1|_2^2.\end{aligned}\tag{A.35}$$

Applying Grönwall's inequality and (A.33)-(A.35) we obtain

$$\sup_{t\in[t_0,\tau)}\Big(|\nabla_{e_\theta}\bar{\mathbf{u}}(t)|_2^2+|\nabla_{e_\varphi}\bar{\mathbf{u}}(t)|_2^2+|\bar{\mathbf{u}}(t)|_2^2\Big)\leq C(\tau,Z_1,Z_2,Z_3,U_0).\tag{A.36}$$

A.4. H^1 **estimates of** $\mathbf{v},T,q$**.** Taking the derivative of (3.2a) with respect to ξ, we get

$$\begin{aligned}&\partial_t\mathbf{u}_\xi-\Delta\mathbf{u}_\xi-\partial_{\xi\xi}\mathbf{u}_\xi+\nabla_{Z_1+\mathbf{u}}(\partial_\xi Z_1+\mathbf{u}_\xi)+\nabla_{\partial_\xi Z_1+\mathbf{u}_\xi}(Z_1+\mathbf{u})\\ &\quad-(\mathrm{div}\ (Z_1+\mathbf{u}))(\partial_\xi Z_1+\mathbf{u}_\xi)+w(Z_1+\mathbf{u})(\partial_{\xi\xi}Z_1+\mathbf{u}_{\xi\xi})\\ =&-\frac{f}{R_0}\vec{k}\times(\partial_\xi Z_1+\mathbf{u}_\xi)+\frac{br_s}{r}\nabla[(1+aq)T]+\gamma\partial_\xi Z_1.\end{aligned}\tag{A.37}$$

Applying integration by parts, we have

$$\int\Big(\nabla_{\mathbf{u}}\mathbf{u}_\xi+w(\mathbf{u})\mathbf{u}_{\xi\xi}\Big)\cdot\mathbf{u}_\xi d\mathbf{D}=0.$$

Since Φ_s is independent of ξ, also with $[\frac{f}{R_0}\vec{k}\times\mathbf{u}_\xi]\cdot\mathbf{u}_\xi=0$, then taking the inner product with $\mathbf{u}_\xi$ in $(L^2(\mathbf{D}))^2$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d|\mathbf{u}_\xi|_2^2}{dt}+\int_{\mathbf{D}}[|\nabla_{e_\theta}\mathbf{u}_\xi|^2+|\nabla_{e_\varphi}\mathbf{u}_\xi|^2+|\mathbf{u}_\xi|^2]d\mathbf{D}+\int_{\mathbf{D}}|\mathbf{u}_{\xi\xi}|^2\\ =&-\int_{\mathbf{D}}\nabla_{\mathbf{u}}\partial_\xi Z_1\cdot\mathbf{u}_\xi d\mathbf{D}-\int_{\mathbf{D}}\nabla_{Z_1}(\partial_\xi Z_1+\mathbf{u}_\xi)\cdot\mathbf{u}_\xi d\mathbf{D}-\int_{\mathbf{D}}\nabla_{\partial_\xi Z_1+\mathbf{u}_\xi}(Z_1+\mathbf{u})\cdot\mathbf{u}_\xi d\mathbf{D}\\ &+\int_{\mathbf{D}}\mathrm{div}\ (Z_1+\mathbf{u})(\partial_\xi Z_1+\mathbf{u}_\xi)\cdot\mathbf{u}_\xi d\mathbf{D}+\int_{\mathbf{D}}w(Z_1)\mathbf{u}_{\xi\xi}\cdot\mathbf{u}_\xi d\mathbf{D}+\int_{\mathbf{D}}w(Z_1+\mathbf{u})\partial_{\xi\xi}Z_1\cdot\mathbf{u}_\xi d\mathbf{D}\\ &+\int_{\mathbf{D}}\frac{br_s}{r}\nabla[(1+aq)T]\cdot\mathbf{u}_\xi d\mathbf{D}+\gamma\int_{\mathbf{D}}\partial_\xi Z_1\cdot\mathbf{u}_\xi d\mathbf{D}-\int_{\mathbf{D}}\frac{f}{R_0}\vec{k}\times\partial_\xi Z_1\cdot\mathbf{u}_\xi d\mathbf{D}.\end{aligned}\tag{A.38}$$

Using a similar argument as in [17] yields,

$$\begin{aligned}&\partial_t|\mathbf{u}_\xi|_2^2+|\nabla_{e_\theta}\mathbf{u}_\xi|_2^2+|\nabla_{e_\varphi}\mathbf{u}_\xi|_2^2+|\mathbf{u}_{\xi\xi}|_2^2\\ \leq&C(|\tilde{\mathbf{u}}|_4^8+\|\bar{\mathbf{u}}\|_1^8+\|Z_1\|_2^8+\|Z_1\|_3+1)|\mathbf{u}_\xi|_2^2+C\|\mathbf{u}\|_1^2\|Z_1\|_3^2+C\|Z_1\|_2^4+C|T|_2^2+C|q|_4^2|T|_4^2.\end{aligned}\tag{A.39}$$

Therefore, by the Grönwall inequality, and L^2, L^4 estimates of T and q, one can have

$$\begin{aligned}&\sup_{t\in[t_0,\tau)} |\mathbf{u}_\xi(t)|_2^2 + \int_0^\tau \Big(|\nabla_{e_\theta}\mathbf{u}_\xi(s)|_2^2 + |\nabla_{e_\varphi}\mathbf{u}_\xi(s)|_2^2 + |\mathbf{u}_{\xi\xi}(s)|_2^2\Big)ds \\ &\leq C(\tau, Q_T, Q_q, Z_1, Z_2, Z_3, U_0). \end{aligned} \tag{A.40}$$

Taking the derivative of (3.2c) with respect to ξ, we obtain

$$\begin{aligned}&\partial_t p_\xi + \nabla_{\partial_\xi Z_1 + \mathbf{u}_\xi}(Z_3 + p) + \nabla_{Z_1+\mathbf{u}}(\partial_\xi Z_3 + p_\xi) \\ &\quad - (\mathrm{div}(Z_1+\mathbf{u}))\partial_\xi(Z_3+p) + w(Z_1+\mathbf{u})\partial_{\xi\xi}(Z_3+p) \\ &= \Delta p_\xi + \partial_{\xi\xi} p_\xi + \partial_\xi Q_q + \gamma \partial_\xi Z_3. \end{aligned} \tag{A.41}$$

Taking a similar argument as in [17] we have

$$\begin{aligned}&\partial_t(|p_\xi|_2^2 + |p(\xi=1)|_2^2) + |\nabla p_\xi|_2^2 + |p_{\xi\xi}|_2^2 + |\nabla p|_{\xi=1}|_2^2 \\ &\leq C(|p|_{\xi=1}|_4^2 + |p|_{\xi=1}|_4^4) + C(|Q_q|_2^2 + |\partial_\xi Q_q|_2^2) + C(\|Z_3\|_3^2 + 1)\|\mathbf{u}\|_1^2 + C\|Z_1\|_1^2\|Z_3\|_3^2 \\ &\quad + C\|Z_1\|_2^2 + C\|\mathbf{u}_\xi\|_1^2 + C\|Z_3\|_2^2 + C(1 + \|Z_3\|_3^2 + \|Z_1\|_3^4 + \|p\|_1^2 + \|\bar{\mathbf{u}}\|_1^8 + |\widetilde{\mathbf{u}}|_4^8)|p_\xi|_2^2. \end{aligned} \tag{A.42}$$

Applying the Grönwall inequality to (A.42) yields that

$$\begin{aligned}&\sup_{t\in[t_0,\tau)} (|p_\xi|_2^2 + |p(\xi=1)|_2^2) + \int_{t_0}^\tau (|\nabla p_\xi(t)|_2^2 + |p_{\xi\xi}(t)|_2^2 + |\nabla p|_{\xi=1}(t)|_2^2)dt \\ &< C(\tau, Z_1, Z_2, Z_3, Q_q). \end{aligned} \tag{A.43}$$

Taking the derivative of (3.2b) with respect to ξ, we obtain

$$\begin{aligned}&\partial_t S_\xi + \nabla_{\partial_\xi Z_1 + \mathbf{u}_\xi}(Z_2 + S) + \nabla_{Z_1+\mathbf{u}}(\partial_\xi Z_2 + S_\xi) \\ &\quad - (\mathrm{div}(Z_1+\mathbf{u}))\partial_\xi(Z_2+S) + w(Z_1+\mathbf{u})\partial_{\xi\xi}(Z_2+S) \\ &= \Delta S_\xi + \partial_{\xi\xi} S_\xi + \partial_\xi Q_T + \gamma\partial_\xi Z_2 - \frac{br_s^2}{r^2}(1 + a(Z_3+p))w(Z_1+\mathbf{u}) \\ &\quad + \frac{br_s}{r}a(\partial_\xi Z_3 + p_\xi)w(Z_1+\mathbf{u}) - \frac{br_s}{r}(1 + a(Z_3+p))\mathrm{div}(Z_1+\mathbf{u}). \end{aligned} \tag{A.44}$$

Similarly to [17],

$$\begin{aligned}&\partial_t(|S_\xi|_2^2 + |S|_{\xi=1}|_2^2) + |\nabla S_\xi|_2^2 + |S_{\xi\xi}|_2^2 + |\nabla S|_{\xi=1}|_2^2 \\ &\leq C(|S|_{\xi=1}|_4^2 + |S|_{\xi=1}|_4^4) + C(|Q_T|_2^2 + |\partial_\xi Q_T|_2^2) \\ &\quad + C(\|Z_2\|_3^2 + \|Z_3\|_2^2 + |p|_4^2 + 1)\|\mathbf{u}\|_1^2 + C\|Z_1\|_1^2(\|Z_2\|_3^2 + \|Z_3\|_2^2) \\ &\quad + C\|Z_1\|_2^2(1 + |p|_4^2) + C\|\mathbf{u}_\xi\|_1^2 + C\|Z_2\|_2^2 + C(1 + \|Z_2\|_3^2 + \|Z_1\|_3^4 + \|Z_3\|_3^2 + |p_\xi|_2^4 \\ &\quad + |p_\xi|_2^2|\nabla p_\xi|_2^2 + \|S\|_1^2 + \|\bar{\mathbf{u}}\|_1^8 + |\widetilde{\mathbf{u}}|_4^8)|S_\xi|_2^2. \end{aligned} \tag{A.45}$$

Applying the Grönwall inequality to (A.45) yields that

$$\begin{aligned}&\sup_{t\in[t_0,\tau)} (|S_\xi|_2^2 + |S|_{\xi=1}|_2^2) + \int_0^\tau (|\nabla S_\xi(t)|_2^2 + |S_{\xi\xi}(t)|_2^2 + |\nabla S|_{\xi=1}(t)|_2^2)dt \\ &< C(\tau, Q_T, Z_1, Z_2, Z_3, U_0). \end{aligned} \tag{A.46}$$

Now we take the inner product of (3.2a) with $-\Delta\mathbf{u}$, and because $\int_{\mathbf{D}}(\frac{f}{R_0}\vec{k}\times\mathbf{u})\cdot\Delta\mathbf{u}d\mathbf{D}=0$, $\int_{\mathbf{D}}\nabla\Phi_s\cdot\Delta\mathbf{u}d\mathbf{D}=0$, we get

$$\begin{aligned}
&\partial_t(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2+|\mathbf{u}|_2^2)+|\Delta\mathbf{u}|_2^2+|\nabla_{e_\theta}\mathbf{u}_\xi|_2^2+|\nabla_{e_\varphi}\mathbf{u}_\xi|_2^2+|\mathbf{u}_\xi|_2^2\\
=&-\int_{\mathbf{D}}\nabla_{Z_1+\mathbf{u}}(Z_1+\mathbf{u})\cdot\Delta\mathbf{u}d\mathbf{D}-\int_{\mathbf{D}}w(Z_1+\mathbf{u})\partial_\xi(Z_1+\mathbf{u})\cdot\Delta\mathbf{u}d\mathbf{D}\\
&-\int_{\mathbf{D}}\frac{f}{R_0}\vec{k}\times Z_1\cdot\Delta\mathbf{u}d\mathbf{D}+\gamma\int_{\mathbf{D}}Z_1\cdot\Delta\mathbf{u}d\mathbf{D}\\
&-\int_{\mathbf{D}}\int_\xi^1\frac{br_s}{r}\nabla[(1+a(Z_3+p))(Z_2+S)]d\xi'\cdot\Delta\mathbf{u}d\mathbf{D}.
\end{aligned}\tag{A.47}$$

Following the same steps as in [16] yields,

$$\begin{aligned}
&\partial_t(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2+|\mathbf{u}|_2^2)+|\Delta\mathbf{u}|_2^2+|\nabla_{e_\theta}\mathbf{u}_\xi|_2^2+|\nabla_{e_\varphi}\mathbf{u}_\xi|_2^2+|\mathbf{u}_\xi|_2^2\\
\le&\varepsilon(|\Delta\mathbf{u}|_2^2+|\nabla_{e_\theta}\mathbf{u}_\xi|_2^2+|\nabla_{e_\varphi}\mathbf{u}_\xi|_2^2+|\Delta p|_2^2+|\Delta S|_2^2)\\
&+C(|Z_1+\mathbf{u}|_4^8+1)(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2)+C|Z_1+\mathbf{u}|_4^2\|Z_1\|_2^2\\
&+C(1+|\partial_\xi Z_1|_2^2+|\mathbf{u}_\xi|_2^2)(\|Z_1\|_2^2+|\nabla_{e_\theta}\mathbf{u}_\xi|_2^2\\
&+|\nabla_{e_\varphi}\mathbf{u}_\xi|_2^2+|\mathbf{u}_\xi|_2^2)(\|Z_1\|_2^2+|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2)+C(\|Z_2\|_2^2+\|Z_3\|_2^2)\\
&+C(|Z_2|_4^4+|S|_4^4)(|\nabla Z_3|_2^2+|\nabla p|_2^2)+C(1+|Z_3|_4^4+|p|_4^4)(|\nabla Z_2|_2^2+|\nabla S|_2^2).
\end{aligned}\tag{A.48}$$

Similarly, we have

$$\begin{aligned}
&\frac{1}{2}\partial_t|\nabla S|_2^2+|\Delta S|_2^2+|\nabla S_\xi|_2^2+\alpha|\nabla S|_{\xi=1}|_2^2\\
\le&\varepsilon(|\Delta S|_2^2+|\Delta\mathbf{u}|_2^2)+C(\|Z_1\|_1^8+|\mathbf{u}|_4^8)|\nabla S|_2^2+C\bigg((\|Z_2\|_1^2+|S_\xi|_2^2)(\|Z_2\|_2^2+|\nabla S_\xi|_2^2)\\
&+(1+\|Z_1\|_2^2+\|Z_3\|_1^4+|p|_4^4)\bigg)(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2)\\
&+C\|Z_2\|_2^2(\|Z_1\|_1^2+|\mathbf{u}|_4^2)+C\|Z_1\|_1^2(\|Z_2\|_1^2+|S_\xi|_2^2)(\|Z_2\|_2^2+|\nabla S_\xi|_2^2)\\
&+C(\|Z_1\|_2^2+1)(1+\|Z_1\|_2^2+\|Z_3\|_1^4+|p|_4^4)+C|Q_T|_2^2+C|Z_2|_2^2,
\end{aligned}\tag{A.49}$$

and

$$\begin{aligned}
&\frac{1}{2}\partial_t|\nabla p|_2^2+|\Delta p|_2^2+|\nabla p_\xi|_2^2+\beta|\nabla S|_{\xi=1}|_2^2\\
\le&\varepsilon(|\Delta p|_2^2+|\Delta\mathbf{u}|_2^2)+C(\|Z_1\|_1^8+|\mathbf{u}|_4^8)|\nabla p|_2^2\\
&+C(\|Z_3\|_1^2+|p_\xi|_2^2)(\|Z_3\|_2^2+|\nabla p_\xi|_2^2)(|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2)\\
&+C\|Z_3\|_2^2(\|Z_1\|_1^2+|\mathbf{u}|_4^2)+C\|Z_1\|_1^2(\|Z_3\|_1^2+|p_\xi|_2^2)(\|Z_3\|_2^2+|\nabla p_\xi|_2^2)\\
&+C\|Z_1\|_2^2+C|Q_q|_2^2+C|Z_3|_2^2.
\end{aligned}\tag{A.50}$$

To estimate $(\mathbf{u},S,p)$ in $\mathcal{V}$, we denote by

$$f:=|\nabla_{e_\theta}\mathbf{u}|_2^2+|\nabla_{e_\varphi}\mathbf{u}|_2^2+|\mathbf{u}|_2^2+|\nabla S|_2^2+|\nabla p|_2^2,$$

$$g:=|\Delta\mathbf{u}|_2^2+|\nabla_{e_\theta}\mathbf{u}_\xi|_2^2+|\nabla_{e_\varphi}\mathbf{u}_\xi|_2^2+|\Delta S|_2^2+|\nabla S_\xi|_2^2+|\Delta p|_2^2+|\nabla p_\xi|_2^2,$$

$$h:=1+|\mathbf{u}|_4^8+|p|_4^8+|S|_4^8+\|Z_1\|_2^8+\|Z_2\|_2^8+\|Z_3\|_2^8+(\|Z_2\|_1^2+|S_\xi|_2^2)(\|Z_2\|_2^2+|\nabla S_\xi|_2^2)$$

$$+(\|Z_3\|_1^2+|p_\xi|_2^2)(\|Z_3\|_2^2+|\nabla p_\xi|_2^2)+(1+\|Z_1\|_1^2+|\mathbf{u}_\xi|_2^2)(\|Z_1\|_2^2+\|\mathbf{u}_\xi\|_1^2),$$

and

$$\begin{aligned}k:=&(1+\|Z_2\|_1^4+|S|_4^4+\|Z_3\|_1^4+|p|_4^4)(\|Z_2\|_1^2+\|Z_3\|_1^2)+(1+\|Z_1\|_2^2)(1+\|Z_1\|_2^2+\|Z_3\|_1^4+|p|_4^4)\\&+\|Z_1\|_1^2(\|Z_3\|_1^2+|p_\xi|_2^2)(\|Z_3\|_2^2+|\nabla p_\xi|_2^2)+\|Z_1\|_1^2(\|Z_2\|_1^2+|S_\xi|_2^2)(\|Z_2\|_2^2+|\nabla S_\xi|_2^2)\\&+|Q_T|_2^2+|Q_q|_2^2+|Z_2|_2^2+|Z_3|_2^2+(\|Z_2\|_2^2+\|Z_3\|_2^2)(\|Z_1\|_1^2+|\mathbf{u}|_4^2).\end{aligned}$$

In view of (A.13), (A.21), (A.33), (A.36), (A.40), (A.43) and (A.46), we have $f,h,k\in L^2([t_0,\tau];\mathbb{R})$. Combining (A.48)-(A.50), we obtain that

$$\partial_t f(t)+g(t)\le h(t)f(t)+k(t), \tag{A.51}$$

for $t\ge t_0$. By the Grönwall inequality, we have

$$\sup_{t\in[t_0,\tau)} f(t)+\int_{t_0}^{\tau} g\,dt\le C(\tau,Q_T,Q_q,Z_1,Z_2,Z_3,U_0). \tag{A.52}$$

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