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GLOBAL EXISTENCE OF SOLUTION IN THE BESOV SPACE TO THE NONLINEAR WAVE EQUATIONS IN $\mathbb{R}^D$*

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Abstract. In [T.C. Sideris, Comm. Part. Diff. Eqs., 8:1291–1323, 1983], the author proves the solution of the nonlinear wave equations breaks down in finite time, if the initial data is radially symmetric and arbitrarily small. The present article is devoted to the study of the lower bound of blow-up rate of blow-up solution and the global solution to a class of nonlinear wave equations in $\mathbb{R}^d, d>3$. We first recall some useful lemmas in Besov spaces. Next, the local well-posedness of Equation (1.1) is obtained in $\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}$, and a lower bound of blow-up rate of blow-up solution in the space is established. Finally, by construction of the space $\mathscr{X}_R(M)$, thanks to the contraction mapping argument, we derive the global solution for the Cauchy problem of Equation (1.1) if the initial datum is sufficiently small.

Keywords. The nonlinear wave equations; well-posedness; Besov spaces; the Bony decomposition; lower bound of blow-up rate; global solution.

Subject classifications. 35G60; 35L05.

1. Introduction

In this article, we study the Cauchy problem of the following nonlinear wave equations given by

$$\begin{cases} \partial_t^2 u-\Delta u=\Phi(t,u,\nabla u)(\nabla u,\nabla u), & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ u(0,x)=u_0,\partial_t u(0,x)=u_1, & x\in\mathbb{R}^d,\end{cases}\tag{1.1}$$

which is considered as the toy model for the Einstein equations in relativity theory, where Φ denotes a bounded smooth function from $\mathbb{R}^3$ to the space of symmetric matrices on $\mathbb{R}^{d+1}$, and all of its derivatives are bounded, i.e. $\Phi(t,u,\nabla u)=(a_{ij})_{d+1\times d+1}\neq Id, a_{ij}=a_{ji}$, a_{ij} is a function of u and ∇u. $\Phi(t,u,\nabla u)(\nabla u,\nabla u)=(\nabla u)^\top\Phi(t,u,\nabla u)\nabla u$, and $\nabla=(\partial_t,\nabla_x)=(\partial_t,\partial_{x_1},\cdots,\partial_{x_d})$.

There is a long history to study the nonlinear wave equations, in the 70's, the first major breakthrough was the discovery of Stricharz estimates. The first Strichartz estimates for the wave equation with variable coefficients were obtained by Kapitanskii [6] and Mockenhaupt, Seeger and Sogge [17], if the coefficients are smooth. For the case where the coefficients are rough, the result is due to Smith [20], who showed that the Strichartz estimates hold under the condition $g\in C^2$ by wave packet techniques, for dimensions $n=2$ and $n=3$. At the same time, H.F. Smith and C.D. Sogge constructed counterexamples in [21], they showed that for all $\alpha<2$ there existed $g\in C^\alpha$ such that the Strichartz estimates failed. Furthermore, the Strichartz estimates have eventually generated some progress for all such problems (see, for example, [2,16] and the references cited therein). However, the Strichartz estimates do not provide sharp results for most nonlinear wave equations where the nonlinearity involves ∇u. The second turning point was the idea of obtaining convolution estimates in the Fourier space [8–10], By virtue of it, the researchers made great progress in the nonlinear wave equations, see for instance [4,7,12,14,15,22,23].

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The nonlinear wave equations with special nonlinearity have been paid considerable attention in recent years. In 1993, if $\Phi(t,u,\nabla u)=u^k$, k is a nonnegative integer, G. Ponce and T. Sideris [18] obtained the local well-posedness of the class of nonlinear wave equations in space $H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$, $s\in(2,5/2]$. Assume

$$\Phi(t,u,\nabla u)(\nabla u,\nabla u)=\Gamma(u)\left[-(\partial_t u)^2+\sum_{i=1}^{d}(\partial_{x_i}u)^2\right],$$

i.e., it satisfies the null condition, S. Klainerman and S. Selberg [11,13] proved the local well-posedness of the wave equations in all dimensions, in particular to the harder case of space dimension 2. It is well known that the solutions in one dimension of Equation (1.1) tend to develop singularities after a finite time, no matter how smooth and small of the initial data [3,5]. Recently, as $\Phi(t,u,\nabla u)=1$, the local well-posedness of Equation (1.1)in $H^s(\mathbb{R}^d)\times H^{s-1}(\mathbb{R}^d), s>\frac{d}{2}$ was established by D. Tataru [23] for the case $d\geq 5$. He also showed the local and global solutions to the following equation [22,24]

$$\begin{cases}\partial_t^2\phi^\alpha-\Delta\phi^\alpha=\Gamma_{jk}^\alpha\Phi_0(\phi_j,\phi_k),\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ u(0,x)=u_0,\partial_t u(0,x)=u_1,\qquad x\in\mathbb{R}^d,\end{cases}$$

for $d\geq 2$, where Γ_{jk}^α are the Riemann–Christoffel symbols, $\Phi_0(u,v)$ satisfies the null-form

$$\Phi_0(u,v)=-u_t v_t+u_x v_x.$$

The special case $d=1$ was also considered in [7]. As $\Phi(t,u,\nabla u)(\nabla u,\nabla u)=a^2u_r^2+b^2u_t^2$, $(a^2+b^2)>0$, there exists arbitrarily small, radially symmetric initial data such that the solution blows up in finite time [19].

In this paper, using the theory of Littlewood–Paley decomposition, the local well-posedness of Equation (1.1) is established in space $\dot B_{2,1}^{\frac{d}{2}-1}\cap\dot B_{2,1}^{\frac{d}{2}}$, provided $\Phi(t,u,\nabla u)=Q(t,u)+\mathcal{N}(t,\nabla u)$, where $Q(t,u),\mathcal{N}(t,\nabla u)$ are symmetric matrices. Moreover, if the nonlinear term satisfies $|Q(t,u)|,|\mathcal{N}(t,\nabla u)|\leq C|(u,\nabla u)|^\alpha$, $\alpha\geq 1$, we also derive a lower bound of the blow-up rate of blow-up solution satisfying

$$(\|\nabla u(t)\|_{\dot B_{2,1}^{\frac{d}{2}-1}}+\|\nabla u(t)\|_{\dot B_{2,1}^{\frac{d}{2}}})>C\frac{1}{(T^*-t)^{1/(1+\alpha)}}.$$

Furthermore, by construction of the space $\mathscr{X}_R(M)$ (See Section 3), we prove the existence and uniqueness of global solutions to Equation (1.1) in the space $\dot B_{2,1}^{\frac{d}{2}-\frac{1}{2}}\cap\dot B_{2,1}^{\frac{d}{2}}$ (note that $\dot B_{2,1}^{\frac{d}{2}-1}\cap\dot B_{2,1}^{\frac{d}{2}}\hookrightarrow\dot B_{2,1}^{\frac{d}{2}-\frac{1}{2}}\cap\dot B_{2,1}^{\frac{d}{2}}$), if $Q(t,u)=0$ and the initial data (u_0,u_1) is suitably chosen such that the norm $\|(u_1,\nabla_x u_0)\|_{\dot B_{2,1}^{\frac{d}{2}-\frac{1}{2}}\cap\dot B_{2,1}^{\frac{d}{2}}}$ is sufficiently small.

The remainder of this paper is organized as follows. In Section 2, we recall the Littlewood–Paley decomposition, the Bony decomposition, the definition and properties of Besov spaces. In Section 3, we first establish the local well-posedness of Equation (1.1) in space $\dot B_{2,1}^{\frac{d}{2}-1}\cap\dot B_{2,1}^{\frac{d}{2}}$, and derive a lower bound for the rate of blow-up solution in the space. Next, by detailed computation, we obtain a priori estimates in homogeneous Besov spaces to Equation (1.1). Thanks to these lemmas, the global solution for the Cauchy problem of Equation (1.1) in Besov spaces $\dot B_{2,1}^{\frac{d}{2}-\frac{1}{2}}\cap\dot B_{2,1}^{\frac{d}{2}}$ is established by Theorem 3.2.

Notation: Let $\widetilde{L}_T^\rho(\dot B_{p,r}^\sigma)$ denote the set of functions u such that

$$\|u\|_{\widetilde{L}_T^\rho(\dot B_{p,r}^\sigma)}:=\left\|\left(2^{k\sigma}\|\dot\Delta_k u\|_{L_T^\rho(L^p)}\right)_{k\in\mathbb{Z}}\right\|_{l^r},$$

for $\sigma\in\mathbb{R}, T>0$, and $(p,r,\rho)\in[1,\infty]^3$. Then via the Minkowski inequality, we have

$$\|u\|_{\widetilde{L}^\rho_T(\dot{B}^\sigma_{p,r})}\le C\|u\|_{L^\rho_T(\dot{B}^\sigma_{p,r})} \qquad \text{if} \quad 1\le\rho\le r,$$

otherwise,

$$\|u\|_{L^\rho_T(\dot{B}^\sigma_{p,r})}\le C\|u\|_{\widetilde{L}^\rho_T(\dot{B}^\sigma_{p,r})} \qquad \text{if} \quad 1\le r\le\rho.$$

If for any $T>0$, $u\in\widetilde{L}^\rho_T(\dot{B}^\sigma_{p,r})$, then we have $u\in\widetilde{L}^\rho(\dot{B}^\sigma_{p,r})$, where

$$\|u\|_{\widetilde{L}^\rho(\dot{B}^\sigma_{p,r})}:=\left(\sum_{k\in\mathbb{Z}}(2^{k\sigma}\|\dot{\Delta}_k u\|_{L^\rho(\mathbb{R}^+;L^p)})^r\right)^{1/r}.$$

2. Preliminaries

In this subsection, in order to establish local well-posedness and global solution of the Cauchy problem for Equation (1.1) in Besov spaces. First, for the convenience of the readers, we recall some facts on the Littlewood–Paley decomposition, the Bony decomposition and some useful lemmas.

PROPOSITION 2.1 ([1]). *There exist a couple of C^∞ functions (χ,φ) valued in [0, 1], such that χ is supported in the ball $\mathcal{B}=\{\xi\in\mathbb{R}^N;|\xi|\le\frac{4}{3}\}$, and φ is supported in the annulus $\mathcal{C}=\{\xi\in\mathbb{R}^N;\frac{3}{4}\le|\xi|\le\frac{8}{3}\}$. Moreover,*

$$\chi(\xi)+\sum_{q\in\mathbb{N}}\varphi(2^{-q}\xi)=1, \qquad \forall\xi\in\mathbb{R}^N,$$

$$q\ge1\Longrightarrow \mathit{supp}\,\chi(\cdot)\cap \mathit{supp}\,\varphi(2^{-q}\cdot)=\emptyset,$$

$$\mathit{supp}\,\varphi(2^{-q}\cdot)\cap \mathit{supp}\,\varphi(2^{-p}\cdot)=\emptyset,\ \mathit{if}|p-q|\ge2$$

and

$$\frac{1}{3}\le\chi^2(\xi)+\sum_{q\in\mathbb{N}}\varphi^2(2^{-q}\xi)\le1, \forall\xi\in\mathbb{R}^N.$$

Let $\tilde{h}=\mathcal{F}^{-1}\chi$ and $h=\mathcal{F}^{-1}\varphi$. Then the nonhomogeneous dyadic blocks Δ_q and the nonhomogeneous low-off operators S_q can be defined as follows:

$$\Delta_{-1}u=S_0u \quad \text{and} \quad \Delta_q u=0, \quad \text{if} \quad q\le-2,$$

$$\Delta_q u=\varphi(2^{-q}D)u=2^{qN}\int_{\mathbb{R}^N}h(2^qy)u(x-y)dy, \text{if}\, q\ge0,$$

$$S_qu=\sum_{p\ge-1}^{q-1}\Delta_pu=\chi(2^{-q}D)u=2^{qN}\int_{\mathbb{R}^N}\tilde{h}(2^qy)u(x-y)dy.$$

Moreover, if $u,v\in\mathcal{S}'(\mathbb{R}^N)$, then we have

$$\Delta_p\Delta_q=0 \quad \text{if} \quad |p-q|\ge2,$$

$$\Delta_q(S_{p-1}u\Delta_p v)=0 \quad \text{if} \quad |p-q|\geq 5.$$

Furthermore, for all $u\in\mathcal{S}'(\mathbb{R}^N)$, one can easily check that

$$u=\sum_{q\in\mathbb{Z}}\Delta_q u \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^N).$$

The homogeneous dyadic blocks $\dot{\Delta}_q$ and the homogeneous low-off operators $\dot{S}_q$ are defined for all $q\in\mathbb{Z}$ by

$$\dot{\Delta}_q u=\varphi(2^{-q}D)u=2^{qN}\int_{\mathbb{R}^N}h(2^q y)u(x-y)dy,$$

$$\dot{S}_q u=\sum_{p\leq q-1}\dot{\Delta}_p u=\chi(2^{-q}D)u=2^{qN}\int_{\mathbb{R}^N}\tilde{h}(2^q y)u(x-y)dy.$$

Then the nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^N)$ and the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^N)$ are defined as follows:

Definition 2.1. *Let $s\in\mathbb{R}, p,r\in[1,\infty]$, we set*

$$B^s_{p,r}=\left\{u\in\mathcal{S}^{'}(\mathbb{R}^N);\|u\|_{B^s_{p,r}}=\left(\sum_{k=-1}^{\infty}2^{ksr}\|\Delta_k u\|^r_{L^p}\right)^{1/r}<\infty\right\},$$

where Δ_k are the nonhomogeneous dyadic blocks. If $s=\infty$, $B^\infty_{p,r}=\cap_{\sigma\in\mathbb{R}}B^\sigma_{p,r}$.

$$\dot{B}^s_{p,r}=\left\{u\in\mathcal{S}^{'}(\mathbb{R}^N);\|u\|_{\dot{B}^s_{p,r}}=\left\|(\|2^{ks}\cdot\dot{\Delta}_k u\|_{L^p})_{k\in\mathbb{Z}}\right\|_{l^r(\mathbb{Z})}<\infty\right\},$$

where $\dot{\Delta}_k$ are the homogeneous dyadic blocks. If $s=\infty$, $\dot{B}^\infty_{p,r}=\cap_{\sigma\in\mathbb{R}}\dot{B}^\sigma_{p,r}$.

For $u,v\in\mathcal{S}'(\mathbb{R}^N)$, we have the Bony decomposition as follows:

Definition 2.2. *Let $u,v\in\mathcal{S}'(\mathbb{R}^N)$. Denote*

$$T_u v=\sum_{q\geq -1}\sum_{p\geq -1}^{q-2}\Delta_p u\Delta_q v=\sum_{q\geq -1}S_{q-1}u\Delta_q v$$

and

$$R(u,v)=\sum_{q\geq -1}\Delta_q u\widetilde{\Delta_q}v \text{ with } \widetilde{\Delta_q}=\Delta_{q-1}+\Delta_q+\Delta_{q+1}.$$

Then formally, we have the nonhomogeneous Bony decomposition

$$uv=T_u v+T_v u+R(u,v).$$

Similarly, the homogeneous Bony decomposition is given by

$$uv=\dot{T}_u v+\dot{T}_v u+\dot{R}(u,v),$$

where $\dot{T}_u v = \sum_{q\in\mathbb{Z}}\sum_{p\le q-2}\dot{\Delta}_p u\dot{\Delta}_q v = \sum_{q\in\mathbb{Z}}\dot{S}_{q-1}u\dot{\Delta}_q v$ *and*

$$\dot{R}(u,v) = \sum_{q\in\mathbb{Z}}\dot{\Delta}_q u\widetilde{\dot{\Delta}_q}v \text{ with } \widetilde{\dot{\Delta}_q} = \dot{\Delta}_{q-1}+\dot{\Delta}_q+\dot{\Delta}_{q+1}.$$

We now state the result concerning continuity of the inhomogeneous paraproduct operator T and the remainder operator R.

LEMMA 2.1 ([1]). *There exists a constant* C *such that for any couple of real numbers* (s,t) *with* t *negative and any* $(p,r,r_1,r_2)\in[1,\infty]^4$, *we have, for any* $(u,v)\in L^\infty\times B^s_{p,r}$,

$$\|T_u v\|_{B^s_{p,r}} \le C^{1+s}\|u\|_{L^\infty}\|v\|_{B^s_{p,r}},$$

for any $(u,v)\in B^t_{\infty,r_1}\times B^s_{p,r_2}$ *and* $\frac{1}{r}:=\min\{1,\frac{1}{r_1}+\frac{1}{r_2}\}$

$$\|T_u v\|_{B^{s+t}_{p,r}} \le \frac{C^{1+|s+t|}}{-t}\|u\|_{B^t_{\infty,r_1}}\|v\|_{B^s_{p,r_2}}.$$

Moreover, assume (s_1,s_2) *be real, and* $(p_1,p_2,r_1,r_2)\in[1,\infty]^4$ *such that*

$$\frac{1}{p}:=+\frac{1}{p_1}+\frac{1}{p_2}\le 1 \quad \text{and} \frac{1}{r}:=\frac{1}{r_1}+\frac{1}{r_2}\le 1.$$

If $s_1+s_2>0$, *then we imply, for any* $(u,v)\in B^{s_1}_{p_1,r_1}\times B^{s_2}_{p_2,r_2}$,

$$\|R(u,v)\|_{B^{s_1+s_2}_{p,r}} \le \frac{C^{1+|s_1+s_2|}}{s_1+s_2}\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}.$$

If $r=1$ *and* $s_1+s_2\ge 0$, *we have, for any* $(u,v)\in B^{s_1}_{p_1,r_1}\times B^{s_2}_{p_2,r_2}$,

$$\|R(u,v)\|_{B^{s_1+s_2}_{p,\infty}} \le C^{1+|s_1+s_2|}\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}.$$

REMARK 2.1. Similar to the case of inhomogeneous paraproduct operator, for the homogeneous paraproduct operator $\dot{T}$ and the remainder operator $\dot{R}$, we can get the same result.

LEMMA 2.2 ([1]). *The following properties hold:*

(i) Density: if $p,r<\infty$, *then* $\mathcal{S}(\mathbb{R}^N)$ *is dense in* $B^s_{p,r}$. *Let the space* $\mathcal{S}_0(\mathbb{R}^N)$ *denotes the function in* $\mathcal{S}(\mathbb{R}^N)$ *whose Fourier transforms are supported away from* 0. *Then the space* $\mathcal{S}_0(\mathbb{R}^N)$ *is dense in* $\dot{B}^s_{p,r}$.

(ii) Generalized derivatives: let $f\in C^\infty(\mathbb{R}^N)$ *be a homogeneous function of degree* $m\in\mathbb{R}$ *away from a neighborhood of the origin. There exists a constant* C *depending only on* f *and such that* $\|f(D)u\|_{B^s_{p,r}}\le C\|u\|_{B^{s+m}_{p,r}}$.

(iii) Sobolev embeddings: if $p_1\le p_2$ *and* $r_1\le r_2$, *then* $B^s_{p_1,r_1}\hookrightarrow B^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$. *If* $s_1<s_2, 1\le p\le\infty$ *and* $1\le r_1,r_2\le\infty$, *then the embedding* $B^{s_2}_{p,r_2}\hookrightarrow B^{s_1}_{p,r_1}$ *is locally compact.*

(iv) Algebraic properties: for $s>0$, $B^s_{p,r}\cap L^\infty$ *is an algebra. Moreover* ($B^s_{p,r}$ *is an algebra*) $\Leftrightarrow$ ($B^s_{p,r}\hookrightarrow L^\infty$) $\Leftrightarrow$ ($s>N/p$ *or* ($s\ge N/p$ *and* $r=1$)).

(v) Fatou property: if $(u_n)_{n\in\mathbb{N}}$ *is a bounded sequence of* $B^s_{p,r}$ *which tends to* u *in* $\mathcal{S}'$, *then* $u\in B^s_{p,r}$ *and* $\|u\|_{B^s_{p,r}}\le\lim_{n\to\infty}\inf\|u_n\|_{B^s_{p,r}}$.

(vi) Complex interpolation: if $u\in B^s_{p,r}\cap B^{s_1}_{p,r}$ *and* $\theta\in[0,1], p,r\in[1,\infty]$, *then* $u\in B^{\theta s+(1-\theta)s_1}_{p,r}$ *and*

$$\|u\|_{B^{\theta s+(1-\theta)s_1}_{p,r}} \le \|u\|^{\theta}_{B^s_{p,r}}\|u\|^{1-\theta}_{B^{s_1}_{p,r}}.$$

(vii) Let $m\in\mathbb{R}$ *and* f *be a* S^m*-multiplier, i.e.* $f:\mathbb{R}^n\mapsto\mathbb{R}$ *is smooth and satisfies that for all multi-index* α, *there exists a constant* C_α *such that* $\forall\xi\in\mathbb{R}^n, |\partial^\alpha f(\xi)|\le C_\alpha(1+|\xi|)^{m-|\alpha|}$. *Then for all* $s\in\mathbb{R}$ *and* $p,r\in[1,\infty]$, *the operator* $f(D)$ *is continuous from* $B^s_{p,r}$ *to* $B^{s-m}_{p,r}$.

REMARK 2.2. Properties (ii), (v), (vi), (vii) hold for the homogeneous spaces $\dot{B}^s_{p,r}$, and the following properties hold for the homogeneous spaces; Sobolev embeddings: if $p_1\le p_2$ and $r_1\le r_2$, then $\dot{B}^s_{p_1,r_1}\hookrightarrow \dot{B}^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$. Moreover, for $2\le p<\infty$, $\dot{B}^0_{p,2}\hookrightarrow L^p$. Algebraic properties: for $s>0$, $\dot{B}^s_{p,r}\cap L^\infty$ is an algebra. Moreover ($\dot{B}^s_{p,r}$ is an algebra) $\Leftrightarrow$ ($\dot{B}^s_{p,r}\hookrightarrow L^\infty$) $\Leftrightarrow$ ($s=N/p$ and $r=1$)).

By the Bony decomposition, we can infer the following estimate.

LEMMA 2.3 ([1]). *For any positive real number* s *and any* $p,r\in[1,\infty]$, *the space* $L^\infty\cap\dot{B}^s_{p,r}$ *is an algebra, and there exists a constant* C *such that*

$$\|uv\|_{\dot{B}^s_{p,r}} \le \frac{C^{s+1}}{s}(\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}}+\|v\|_{L^\infty}\|u\|_{\dot{B}^s_{p,r}}).$$

LEMMA 2.4 ([1]). *Assume* f *to be a smooth function such that* $f(0)=0$, $s>0$ *and* $p,r\in[1,\infty]$. *If* u *belongs to the space* $L^\infty\cap\dot{B}^s_{p,r}$, *then we have*

$$\|f(u)\|_{\dot{B}^s_{p,r}} \le C(s,f',\|u\|_{L^\infty})\|u\|_{\dot{B}^s_{p,r}}.$$

Moreover, if f *belongs to* $C^\infty_b(\mathbb{R})$ *and* u *belongs to* $\dot{B}^{-1}_{\infty,\infty}$, *then we also obtain*

$$\|f(u)\|_{\dot{B}^s_{p,r}} \le C(s,f,\|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}})\|u\|_{\dot{B}^s_{p,r}}.$$

Finally, taking advantage of

$$f(u)-f(v)=(u-v)\int_0^1 f'(v+\theta(u-v))d\theta,$$

we can get the following lemma.

LEMMA 2.5 ([1]). *Let* f *be a smooth function such that* $f(0)=0$, $s>0$ *and* $p,r\in[1,\infty]$. *For any couple* (u,v) *belonging to the space* $L^\infty\cap\dot{B}^s_{p,r}$, *we have the function* $f(u)-f(v)$ *belonging to* $L^\infty\cap\dot{B}^s_{p,r}$ *and*

$$\begin{aligned}\|f(u)-f(v)\|_{\dot{B}^s_{p,r}} \le& C(s,f'',\|u\|_{L^\infty},\|v\|_{L^\infty})(\sup_{\theta\in[0,1]}\|u+\theta(v-u)\|_{L^\infty}\\ &\|u-v\|_{\dot{B}^s_{p,r}}+\|u-v\|_{L^\infty}\sup_{\theta\in[0,1]}\|u+\theta(v-u)\|_{\dot{B}^s_{p,r}}).\end{aligned}$$

DEFINITION 2.3. *The pair* (q,r) *in* $[2,\infty]^2$ *is a wave-admissible if there exists some* $\widetilde{r}\in[2,r]$ *such that*

$$\frac{1}{q}+\frac{d-1}{2\widetilde{r}}=\frac{d-1}{4} \quad \textit{with} \quad (q,\widetilde{r},d)\neq(2,\infty,3).$$

Then we have the following lemma of Strichartz estimates for the wave equation.

LEMMA 2.6 ([1]). *Assume that the space dimension d is greater than or equal to 2. For any wave-admissible pairs* (q_1,r_1) *and* (q_2,r_2)*, there exists a constant C such that for any j in* $\mathbb{Z}$,

$$\|\nabla\dot{\Delta}_j u\|_{L_T^{q_1}(L^{r_1})}\le C2^{j\mu_1}\|\dot{\Delta}_j\nabla u(0)\|_{L^2}+C2^{j\mu_{12}}\|\dot{\Delta}_j\Box u\|_{L_T^{q_2'}(L^{r_2'})} \tag{2.1}$$

with

$$\mu_1:=d\left(\frac{1}{2}-\frac{1}{r_1}\right)-\frac{1}{q_1} \quad \text{and} \quad \mu_{12}:=d\left(1-\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{1}{q_1}-\frac{1}{q_2},$$

where $\Box u:=(\partial_{tt}-\Delta)u$, $\frac{1}{q_2}+\frac{1}{q_2'}=1$ *and* $\frac{1}{r_2}+\frac{1}{r_2'}=1$.

3. The existence and uniqueness of solution

3.1. Local well-posedness of solution. In this subsection, we shall establish the local well-posedness of Equation (1.1) in space $\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}$, for the convenience of presentation, we first give the following useful lemmas.

LEMMA 3.1. *Assume that* $Q\neq Id$ *is a smooth function from* $\mathbb{R}^2$ *to the space of symmetric matrices on* $\mathbb{R}^{d+1}$*, which is bounded, as are all of its derivatives. Then the following estimates hold:*

$$\|Q_1(u,v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\le C(\|(u,v)\|_{L^\infty})\|\nabla u\|^2_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla u\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}},$$

$$\begin{aligned}\|Q_2(u,v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\le& C(\|v\|_{L^\infty})\Big(\|v\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla(u+v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla(u-v)\|_{L^\infty}\\&+\|v\|_{L^\infty}\|\nabla(u+v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\Big).\end{aligned}$$

Moreover,

$$\begin{aligned}\|Q_1(u,v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\le& C(\|(u,v)\|_{L^\infty})\Big(\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla u\|^2_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\&+\|\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla u\|^2_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big),\end{aligned}$$

$$\begin{aligned}\|Q_2(u,v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\le& C(\|(u,v)\|_{L^\infty})\Big(\|v\|_{L^\infty}\|\nabla(u+v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\\&+\|\nabla(u-v)\|_{L^\infty}\Big(\|v\|_{L^\infty}\|\nabla(u+v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|\nabla v\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla(u+v)\|_{L^\infty}\Big)\Big),\end{aligned}$$

where $Q_1(u,v)=(Q(t,u)-Q(t,v))(\nabla u,\nabla u)$ *and* $Q_2(u,v)=Q(t,v)(\nabla(u+v),\nabla(u-v))$.

Proof. Denote $Q(t,u)=Q(u)$, applying Lemma 2.3 and Lemma 2.5, it is easy to check that

$$\begin{aligned}\|Q_1(u,v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}=&\|(Q(u)-Q(v))(\nabla u,\nabla u)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\\le&\|(Q(u)-Q(v))\nabla u\|_{L^\infty}\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\nabla u\|_{L^\infty}\|(Q(u)-Q(v))\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\end{aligned}$$

$$\begin{aligned}&\le C(\|(u,v)\|_{L^\infty})\|u\|_{L^\infty}\|u-v\|_{L^\infty}\|\nabla u\|_{L^\infty}\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\&\quad+\|\nabla u\|_{L^\infty}\|(Q(u)-Q(v))\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}},\end{aligned}\tag{3.1}$$

where we used $\frac{d}{2}-1>0$.

Set $f=(Q(u)-Q(v))$, by the Bony decomposition

$$(Q(u)-Q(v))\nabla u=\dot{T}_f\nabla u+\dot{T}_{\nabla u}f+\dot{R}(f,\nabla u).$$

Therefore, using Lemma 2.1 to paraproduct operator $\dot{T}$ and the remainder operator $\dot{R}$. Note that $\dot{B}^{\frac{d}{2}}_{2,1}\hookrightarrow L^\infty$ is an algebra, in view of Lemma 2.5, it follows that

$$\begin{aligned}\left\|\dot{T}_f\nabla u\right\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}&\le\|f\|_{L^\infty}\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\&\le C(\|(u,v)\|_{L^\infty})\|u\|_{L^\infty}\|u-v\|_{L^\infty}\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\&\le C(\|(u,v)\|_{L^\infty})\|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|u-v\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\&\le C(\|(u,v)\|_{L^\infty})\|\nabla u\|^2_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}},\end{aligned}\tag{3.2}$$

$$\begin{aligned}\left\|\dot{T}_{\nabla u}f\right\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}&\le\|\nabla u\|_{\dot{B}^{-1}_{\infty,1}}\|Q(u)-Q(v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\\&\le C(\|(u,v)\|_{L^\infty})\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|u-v\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\\&\le C(\|(u,v)\|_{L^\infty})\|\nabla u\|^2_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}},\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}\left\|\dot{R}(f,\nabla u)\right\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\le\left\|\dot{R}(f,\nabla u)\right\|_{\dot{B}^{d-1}_{1,1}}&\le\|f\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\&\le C(\|(u,v)\|_{L^\infty})\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla u\|^2_{\dot{B}^{\frac{d}{2}-1}_{2,1}},\end{aligned}\tag{3.4}$$

where we have used $d-1>0$ in order to estimate (3.4). Combining (3.2), (3.3) with (3.4) one has that

$$\|(Q(u)-Q(v))\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\le C(\|(u,v)\|_{L^\infty})\|\nabla u\|^2_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}},$$

substituting it into (3.1). Thanks to $\dot{B}^{\frac{d}{2}}_{2,1}\hookrightarrow L^\infty$, we imply that

$$\|Q_1(u,v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\le C(\|(u,v)\|_{L^\infty})\|\nabla u\|^2_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla u\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}.\tag{3.5}$$

We now estimate $Q_2(u,v)$. Let $\widetilde{f}=Q(v)\nabla(u+v)$. By virtue of the Bony decomposition to yield

$$\begin{aligned}&\|Q_2(u,v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}=\|Q(v)(\nabla(u+v),\nabla(u-v))\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\&\le\left\|\dot{T}_{\widetilde{f}}\nabla(u-v)\right\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\left\|\dot{T}_{\nabla(u-v)}\widetilde{f}\right\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\left\|\dot{R}(\widetilde{f},\nabla(u-v))\right\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}.\end{aligned}\tag{3.6}$$

Taking advantage of Lemma 2.1 and Lemma 2.3, the three terms in RHS of (3.6) gives, respectively

$$\begin{aligned}\left\|\dot{T}_{\widetilde{f}}\nabla(u-v)\right\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\leq&\|Q(v)\nabla(u+v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\ \leq& C(\|v\|_{L^\infty})\|v\|_{L^\infty}\|\nabla(u+v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}},\end{aligned}\tag{3.7}$$

$$\begin{aligned}\left\|\dot{R}(\widetilde{f},\nabla(u-v))\right\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\leq&\|\widetilde{f}\|_{\dot{B}^0_{\infty,\infty}}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\ \leq&\|Q(v)\nabla(u+v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\ \leq& C(\|v\|_{L^\infty})\|v\|_{L^\infty}\|\nabla(u+v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}},\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}&\left\|\dot{T}_{\nabla(u-v)}\widetilde{f}\right\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\leq\|\nabla(u-v)\|_{L^\infty}\|Q(v)\nabla(u+v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\ &\leq\|\nabla(u-v)\|_{L^\infty}\|\nabla(u+v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\left(\|Q(v)\|_{L^\infty}+\|Q(v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\right)\\ &\leq C(\|v\|_{L^\infty})\|v\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla(u+v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla(u-v)\|_{L^\infty},\end{aligned}\tag{3.9}$$

where we have used the following inequality

$$\|Q(v)\nabla(u+v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\leq\|Q(v)\|_{L^\infty}\|\nabla(u+v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\nabla(u+v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|Q(v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}},$$

which is guaranteed by the Bony decomposition.

Inserting (3.7), (3.8) and (3.9) into (3.6), we deduce that

$$\begin{aligned}\|Q_2(u,v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\leq& C(\|v\|_{L^\infty})\Big(\|v\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla(u+v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla(u-v)\|_{L^\infty}\\ &+\|v\|_{L^\infty}\|\nabla(u+v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\Big).\end{aligned}$$

Next, we deal with the second section of Lemma 3.1. Using the Bony decomposition again to yield

$$\begin{aligned}\|(Q(u)-Q(v))\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\leq&\|(Q(u)-Q(v))\|_{L^\infty}\|\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|(Q(u)-Q(v))\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\\ \leq& C(\|(u,v)\|_{L^\infty})\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\Big(\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\ &\times\|\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big).\end{aligned}\tag{3.10}$$

Therefore, in view of (3.10), we have

$$\begin{aligned}\|Q_1(u,v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\leq&\|(Q(u)-Q(v))\nabla u\|_{L^\infty}\|\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|\nabla u\|_{L^\infty}\|(Q(u)-Q(v))\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\\ \leq& C(\|(u,v)\|_{L^\infty})\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|u-v\|_{L^\infty}\|\nabla u\|^2_{\dot{B}^{\frac{d}{2}}_{2,1}}\\ &+\|\nabla u\|_{L^\infty}\|(Q(u)-Q(v))\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\end{aligned}$$

$$\le C(\|(u,v)\|_{L^\infty})\Big(\|\nabla u\|_{\dot B^{\frac d2-1}_{2,1}}\|\nabla u\|^2_{\dot B^{\frac d2}_{2,1}}\|\nabla(u-v)\|_{\dot B^{\frac d2-1}_{2,1}}$$
$$+\|\nabla u\|_{\dot B^{\frac d2}_{2,1}}\|\nabla u\|^2_{\dot B^{\frac d2-1}_{2,1}}\|\nabla(u-v)\|_{\dot B^{\frac d2}_{2,1}}\Big).$$

Similarly, one can easily check that

$$\begin{aligned}\|Q_2(u,v)\|_{\dot B^{\frac d2}_{2,1}}&=\|Q(v)(\nabla(u+v),\nabla(u-v))\|_{\dot B^{\frac d2}_{2,1}}\\&\le C(\|(u,v)\|_{L^\infty})\Big(\|v\|_{L^\infty}\|\nabla(u+v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot B^{\frac d2}_{2,1}}\\&\quad+\|\nabla(u-v)\|_{L^\infty}\Big(\|v\|_{L^\infty}\|\nabla(u+v)\|_{\dot B^{\frac d2}_{2,1}}+\|\nabla v\|_{\dot B^{\frac d2}_{2,1}}\|\nabla(u+v)\|_{L^\infty}\Big)\Big).\end{aligned}$$

This completes the proof of Lemma 3.1. □

For the symmetric matrices $\mathcal N(t,\nabla u)$, dealing with it in a manner similar to the one used in the estimate of $Q(t,u)$, we have

LEMMA 3.2. *Assume $\mathcal N\neq Id$ to be a smooth function from $\mathbb R^2$ to the space of symmetric matrices on $\mathbb R^{d+1}$, which is bounded, as are all of its derivatives. Then the following estimates hold:*

$$\|\mathcal N_1(\nabla u,\nabla v)\|_{\dot B^{\frac d2-1}_{2,1}}\le C(\|\nabla(u,v)\|_{L^\infty})\|\nabla u\|^2_{L^\infty}\Big(\|\nabla u\|_{\dot B^{\frac d2-1}_{2,1}}\|\nabla(u-v)\|_{L^\infty}$$
$$+\|\nabla u\|_{L^\infty}\|\nabla(u-v)\|_{\dot B^{\frac d2-1}_{2,1}}\Big),$$

$$\|\mathcal N_2(\nabla u,\nabla v)\|_{\dot B^{\frac d2-1}_{2,1}}\le C(\|\nabla v\|_{L^\infty})\|\nabla(u,v)\|_{L^\infty}\Big(\|\nabla(u,v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot B^{\frac d2-1}_{2,1}}$$
$$+\|\nabla(u,v)\|_{\dot B^{\frac d2-1}_{2,1}}\|\nabla(u-v)\|_{L^\infty}\Big).$$

Moreover,

$$\|\mathcal N_1(\nabla u,\nabla v)+\mathcal N_2(\nabla u,\nabla v)\|_{\dot B^{\frac d2}_{2,1}}\le C(\|\nabla(u,v)\|_{L^\infty})\|\nabla(u,v)\|_{L^\infty}\|\nabla(u,v)\|_{\dot B^{\frac d2}_{2,1}}$$
$$\times\Big(\|\nabla(u,v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot B^{\frac d2}_{2,1}}+\|\nabla(u,v)\|_{\dot B^{\frac d2}_{2,1}}\|\nabla(u-v)\|_{L^\infty}\Big),$$

$$\|\mathcal N_1(\nabla u,\nabla v)\|_{\dot B^{\frac d2-\frac12}_{2,1}}\le C(\|\nabla(u,v)\|_{L^\infty})\|\nabla v\|_{L^\infty}\|\nabla u\|_{\dot B^{\frac d2-\frac12}_{2,1}}$$
$$\times\Big(\|\nabla u\|_{L^\infty}\|\nabla(u-v)\|_{\dot B^{\frac d2}_{2,1}}+\|\nabla u\|_{\dot B^{\frac d2}_{2,1}}\|\nabla(u-v)\|_{L^\infty}\Big),$$

$$\|\mathcal N_2(\nabla u,\nabla v)\|_{\dot B^{\frac d2-\frac12}_{2,1}}\le C(\|\nabla v\|_{L^\infty})\|\nabla v\|_{L^\infty}\|\nabla v\|_{\dot B^{\frac d2}_{2,1}}$$
$$\times\Big(\|\nabla(u+v)\|_{L^\infty}\|\nabla(u-v)\|_{\dot B^{\frac d2-\frac12}_{2,1}}+\|\nabla(u-v)\|_{L^\infty}\|\nabla(u+v)\|_{\dot B^{\frac d2-\frac12}_{2,1}}\Big),$$

where the terms $\mathcal N_1(\nabla u,\nabla v)$ and $\mathcal N_2(\nabla u,\nabla v)$ are given by

$$\mathcal N_1(\nabla u,\nabla v)=(\mathcal N(t,\nabla u)-\mathcal N(t,\nabla v))(\nabla u,\nabla u),$$

$$\mathcal{N}_2(\nabla u,\nabla v)=\mathcal{N}(t,\nabla v)(\nabla(u+v),\nabla(u-v)).$$

Proof. We only prove the last two inequalities. The method of estimating the other three inequalities is similar to that used in Lemma 3.1.

$$\begin{aligned}
\|\mathcal{N}_1(\nabla u,\nabla v)\|_{\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}} &\le \|\dot{T}_{(\mathcal{N}(\nabla u)-\mathcal{N}(\nabla v))}(\nabla u\otimes\nabla u)\|_{\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}}\\
&\quad+\|\dot{T}_{\nabla u\otimes\nabla u}(\mathcal{N}(\nabla u)-\mathcal{N}(\nabla v))+\dot{R}(\nabla u\otimes\nabla u,\mathcal{N}(\nabla u)-\mathcal{N}(\nabla v))\|_{\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}}\\
&\le\|(\mathcal{N}(t,\nabla u)-\mathcal{N}(t,\nabla v))\|_{L^\infty}\|\nabla u\otimes\nabla u\|_{\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}}\\
&\quad+\|\nabla u\otimes\nabla u\|_{\dot{B}^{-\frac{1}{2}}_{\infty,1}}\|(\mathcal{N}(t,\nabla u)-\mathcal{N}(t,\nabla v))\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\\
&\le C(\|\nabla(u,v)\|_{L^\infty})\|\nabla(u-v)\|_{L^\infty}\|\nabla u\|_{L^\infty}\|\nabla u\|_{\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}}+C(\|\nabla(u,v)\|_{L^\infty})\|\nabla u\|_{L^\infty}\\
&\quad\times\|\nabla u\|_{\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}}\Big(\|\nabla u\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla(u-v)\|_{L^\infty}\Big)\\
&\le C(\|\nabla(u,v)\|_{L^\infty})\|\nabla v\|_{L^\infty}\|\nabla u\|_{\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}}\\
&\quad\times\Big(\|\nabla u\|_{L^\infty}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla(u-v)\|_{L^\infty}\Big).
\end{aligned}$$

Similarly, we can deal with $\mathcal{N}_2(\nabla u,\nabla v)$ to obtain the last inequality. □

Now, we present the main result in this subsection.

THEOREM 3.1. *Given $(u_0,u_1)=(u_0,\partial_t u_0)$ be the initial datum. Assume the space dimension $d>3$, and $\varpi:=(u_1,\nabla_x u_0)\in\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}$. Then there exists a time $T>0$ depending on u_0 such that Equation (1.1) has a unique solution $u(t,x)$ on $[0,T[$ with the initial datum (u_0,u_1). Moreover, the solution u satisfies*

$$\nabla u\in\mathcal{C}([0,T[;\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}).$$

Furthermore, if the solution u blows up in a finite time T^* in space $\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}$, then we have

$$\limsup_{t\uparrow T^*}(\|(u(t),\nabla u(t))\|_{L^\infty})=\infty.$$

Assume $|Q(u)|,|\mathcal{N}(\nabla u)|\le C|(u,\nabla u)|^\alpha$, $\alpha\ge1$. Then the lower bound for the blow-up rate of blow-up solution satisfies

$$(\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}})>C\frac{1}{(T^*-t)^{1/(1+\alpha)}}.$$

Proof. We first introduce the solution u_F of the following linear equation

$$\begin{cases}\partial_t^2u_F-\Delta u_F=0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ (u_F,\partial_t u_F)|_{t=0}=(u_0,\partial_t u_0), & x\in\mathbb{R}^d.\end{cases}\tag{3.11}$$

By virtue of Duhamel's formula, u is a solution of Equation (1.1) if and only if $u=u_F+F(u)$ with

$$\begin{cases}(\partial_t^2-\Delta)F(u)=[Q(t,u)+\mathcal{N}(t,\nabla u)](\nabla u,\nabla u), & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ (F(u),\partial_t F(u))|_{t=0}=(0,0), & x\in\mathbb{R}^d.\end{cases}\tag{3.12}$$

Since u,v are two solutions of Equation (3.12), it is obvious that $F(u)-F(v)$ solves the following equation

$$\begin{cases} (\partial_t^2-\Delta)(F(u)-F(v)) = Q_1(u,v)+Q_2(u,v) \\ \qquad\qquad\qquad\qquad +\mathcal{N}_1(\nabla u,\nabla v)+\mathcal{N}_2(\nabla u,\nabla v), \\ (F(u)-F(v),\partial_t(F(u)-F(v))|_{t=0} = (0,0), \end{cases} \tag{3.13}$$

with

$$Q_1(u,v) = (Q(t,u)-Q(t,v))(\nabla u,\nabla u),$$

$$Q_2(u,v) = Q(t,v)(\nabla(u+v),\nabla(u-v)),$$

$$\mathcal{N}_1(\nabla u,\nabla v) = (\mathcal{N}(t,\nabla u)-\mathcal{N}(t,\nabla v))(\nabla u,\nabla u),$$

$$\mathcal{N}_2(\nabla u,\nabla v) = \mathcal{N}(t,\nabla v)(\nabla(u+v),\nabla(u-v)).$$

Applying the dyadic blocks $\dot{\Delta}_k$ on both sides of Equation (3.13), after multiplying the result by $\partial_t\dot{\Delta}_k(F(u)-F(v))$, integrating by parts. Then by standard energy arguments, we end up with

$$\begin{aligned} \|\nabla\dot{\Delta}_k(F(u)-F(v))\|_{L^\infty_T(L^2)} \le& \|\dot{\Delta}_k(Q_1(u,v)+Q_2(u,v))\|_{L^1_T(L^2)} \\ &+\|\dot{\Delta}_k(\mathcal{N}_1(\nabla u,\nabla v)+\mathcal{N}_2(\nabla u,\nabla v))\|_{L^1_T(L^2)}. \end{aligned} \tag{3.14}$$

Multiplying $2^{k\frac{d}{2}-k}$ or $2^{k\frac{d}{2}}$ on both sides of inequality (3.14), after taking $l^1(\mathbb{Z})$ norm, it is easy to check that

$$\begin{aligned} \|\nabla(F(u)-F(v))\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \le& \|Q_1(u,v)+Q_2(u,v)\|_{\widetilde{L}^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\ &+\|(\mathcal{N}_1(\nabla u,\nabla v)+\mathcal{N}_2(\nabla u,\nabla v)\|_{\widetilde{L}^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \|\nabla(F(u)-F(v))\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})} \le& \|Q_1(u,v)+Q_2(u,v)\|_{\widetilde{L}^1_T(\dot{B}^{\frac{d}{2}}_{2,1})} \\ &+\|(\mathcal{N}_1(\nabla u,\nabla v)+\mathcal{N}_2(\nabla u,\nabla v)\|_{\widetilde{L}^1_T(\dot{B}^{\frac{d}{2}}_{2,1})}. \end{aligned} \tag{3.16}$$

Define the Banach space

$$\mathcal{B}(M) = \left\{ u\in \dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}; \|u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}} \le M \right\}.$$

By virtue of Lemmas 3.1, 3.2 to inequality (3.15), it follows that

$$\begin{aligned} \|\nabla(F(u)-F(v))\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \le& CT(\|\nabla(u,v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}})\|\nabla u\|^2_{\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla(u-v)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}} \\ \le& C(M)M^2T\|\nabla(u-v)\|_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1})}. \end{aligned} \tag{3.17}$$

Similarly, taking advantage of Lemmas 3.1, 3.2 to inequality (3.16), we also imply that

$$\|\nabla(F(u)-F(v))\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})} \le C(M)M^2T\|\nabla(u-v)\|_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1})}. \tag{3.18}$$

If T is sufficiently small such that $C(M)M^2T<1$, without loss of generality, letting $C(M)M^2T=\frac{1}{2}$. Combining (3.17) with (3.18), then we derive

$$\|\nabla(F(u)-F(v))\|_{L^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1})}\le\frac{1}{2}\|\nabla(u-v)\|_{L^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1})}.$$

By the contraction mapping argument we obtain the existence and uniqueness of the local solution.

Next, in order to derive the blow-up scenario, using the energy estimates to Equation (1.1), we end up with

$$\|\nabla u\|_{L^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1})}\le\|\nabla u_0\|_{\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1}}+\|[Q(t,u)+\mathcal{N}(t,\nabla u)](\nabla u,\nabla u)\|_{L^1_T(\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1})}. \tag{3.19}$$

Note that for $\nu=0,1$

$$\|\mathcal{N}(t,\nabla u)(\nabla u,\nabla u)\|_{\dot B^{\frac{d}{2}-\nu}_{2,1}}\le C(\|\nabla u\|_{L^\infty})\|\nabla u\|_{L^\infty}\|\nabla u\|_{\dot B^{\frac{d}{2}-\nu}_{2,1}}. \tag{3.20}$$

Since the term $Q(t,u)(\nabla u,\nabla u)$ is a linear combination of terms of type $\tilde Q'_{ij}(t,u)\partial_i u\partial_j u$, $i,j\in[0,d]$, where the smooth function $\tilde Q_{ij}(t,u)$ satisfies $\partial_i(\tilde Q_{ij})(t,u)=Q_{ij}(t,u)\partial_i u$. Consequently

$$\begin{aligned}\|\tilde Q'_{ij}(t,u)\partial_i u\partial_j u\|_{\dot B^{\frac{d}{2}-\nu}_{2,1}}&\le\|\tilde Q'_{ij}(t,u)\partial_i u\|_{L^\infty}\|\partial_j u\|_{\dot B^{\frac{d}{2}-\nu}_{2,1}}+\|\partial_j u\|_{L^\infty}\|\partial_i\tilde Q_{ij}(t,u)\|_{\dot B^{\frac{d}{2}-\nu}_{2,1}}\\&\le C(\|u\|_{L^\infty})\|\nabla u\|_{L^\infty}\|\nabla u\|_{\dot B^{\frac{d}{2}-\nu}_{2,1}}.\end{aligned} \tag{3.21}$$

Substituting (3.20), (3.21) into (3.19) one has that

$$\begin{aligned}\|\nabla u\|_{\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1}}\le&\|\nabla u_0\|_{\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1}}+C(\|(u,\nabla u)\|_{L^\infty_T(L^\infty)})\\&\times\int_0^t\|\nabla u(\tau)\|_{L^\infty}\|\nabla u(\tau)\|_{\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1}}\,d\tau.\end{aligned} \tag{3.22}$$

In view of the Grönwall inequality, if the solution u blows up in a finite time T^*, then

$$\limsup_{t\uparrow T^*}(\|(u(t),\nabla u(t))\|_{L^\infty})=\infty.$$

Finally, we shall derive a lower bound on the blow-up rate of blow-up solution. Thanks to the method used in the estimate of (3.16) and (3.17). Assume $|Q(u)|,|\mathcal{N}(\nabla u)|\le C|(u,\nabla u)|^\alpha$, $\alpha\ge1$, it is easy to check that

$$\|\nabla u\|_{L^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1})}\le\|\nabla u_0\|_{\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1}}+CT\|\nabla u\|^{2+\alpha}_{L^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1}\cap\dot B^{\frac{d}{2}}_{2,1})}. \tag{3.23}$$

Denote by T^* the supremum of all $T>0$ for which there exists a solution u of the nonlinear wave equation satisfying

$$\|\nabla u(t)\|_{L^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1})}+\|\nabla u(t)\|_{L^\infty_T(\dot B^{\frac{d}{2}}_{2,1})}<\infty.$$

The local well-posedness theory derives $T^*>0$ such that for all $t\in[0,T^*[$

$$\|\nabla u(t)\|_{L^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1})}+\|\nabla u(t)\|_{L^\infty_T(\dot B^{\frac{d}{2}}_{2,1})}<\infty.$$

By the maximality of T^*, it follows that

$$\|\nabla u(t)\|_{L^\infty_{T^*}(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|\nabla u(t)\|_{L^\infty_{T^*}(\dot{B}^{\frac{d}{2}}_{2,1})}=\infty.$$

Otherwise, the Cauchy problem of Equation (1.1) at time T^* with initial datum $u(T^*,\cdot)$ would be well-defined and the local existence theory would extend the solution u beyond T^*. Thus, if $T^*<\infty$, the solution blows up and

$$\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\longrightarrow\infty \qquad t\to T^*.$$

Consider the solution u posed at some time $t\in[0,T^*[$. Assume for some M

$$\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+C(T-t)M^{2+\alpha}\le M.$$

Then $T<T^*$. Consequently, $\forall M>0$

$$\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+C(T^*-t)M^{2+\alpha}>M.$$

Choosing $M=2(\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}})$, one has that

$$C(T^*-t)M^{2+\alpha}>M,$$

which is equivalent to

$$(\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\nabla u(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}})>C\frac{1}{(T^*-t)^{1/(1+\alpha)}}.$$

This completes the proof of Theorem 3.1. ∎

REMARK 3.1. The well-posedness of Equation (1.1) in space $\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}$ is inspired by

$$\|u\|_{L^\infty}\le\|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\le\|\nabla u\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}},\quad\text{and}\quad\|\nabla u\|_{L^\infty}\le\|\nabla u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}.$$

If $\mathcal{N}(t,\nabla u)=0$, then the blow-up scenario of solution in space $\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}$ can be improved to

$$\limsup_{t\uparrow T^*}\left(\|u(t)\|_{L^\infty}+\int_0^t\|\nabla u(\tau)\|d\tau\right)=\infty.$$

3.2. Global existence of solution. In [19], the author has proved that the solutions blow up in finite time, if the initial datum is arbitrarily small, radially symmetric. Whether there is the global existence of solution is a very important and difficult problem. In this subsection, letting $Q(t,u)=0$, by a large computation and narrow analysis, we obtain the existence of global solution with small initial datum, which is stated in the following theorem.

THEOREM 3.2. *Let (u_0,u_1) be the initial datum and consider the space dimension $d>3$. Assume $\varpi:=(u_1,\nabla_x u_0)\in\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}$ and $Q(t,u)=0$. Then there exists a constant M such that if*

$$\|\varpi\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}}\le M,$$

Equation (1.1) *has a unique global solution* u *such that* $\nabla u \in L^2_T(L^\infty) \cap \widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}) \cap \widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})$ *with the initial datum* (u_0,u_1)*, which satisfies*

$$\|\nabla u\|_{L^2_T(L^\infty)\cap \widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1})\cap \widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})} \le 2M.$$

Moreover, the solution u *satisfies*

$$\nabla u \in \mathcal{C}(\mathbb{R}^+; \dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1} \cap \dot{B}^{\frac{d}{2}}_{2,1}).$$

REMARK 3.2. In order to obtain the above result in Theorem 3.2, we first define the space $\mathscr{X}_R(M)$ (see Lemma 3.3) with the norm of $L^2_T(L^\infty) \cap \widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}) \cap \widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})$. On the one hand, the space $L^2_T(L^\infty)$ can absorb the time on the RHS of inequality (3.27), (3.28) and (3.30). If only considering the space $\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}) \cap \widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})$, then we only establish the local well-posedness, therefore, $L^2_T(L^\infty)$ should be added, which is guaranteed by the Strichartz estimates of Lemma 2.6. The space $\dot{B}^{\frac{d}{2}}_{2,1}$ is an algebra, and $\dot{B}^{\frac{d}{2}}_{2,1} \hookrightarrow L^\infty$, it is necessary to estimate the nonlinear term $\mathcal{N}(t,\nabla u)(\nabla u, \nabla u)$. On the other hand, note that $\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}$ is not an algebra, and

$$\dot{B}^{\frac{d}{2}}_{2,1} \nsubseteq \dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}, \qquad \dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1} \nsubseteq L^\infty,$$

however, it is necessary to introduce the space $\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1})$ to deal with the nonlinear term. Combining the three spaces, by the small initial datum, the global solution can be derived.

REMARK 3.3. If we choose a compact domain K of $\mathbb{R}^d$. Let $\dot{B}^\sigma_{p,r}(K)$ (resp., $B^\sigma_{p,r}(K)$) denote the set of distributions f in $\dot{B}^\sigma_{p,r}(\mathbb{R}^d)$ (resp., $B^\sigma_{p,r}(K)$), the support of which is included in K. By Proposition 2.93 on page 108 in [1], if $\sigma > 0$, then the spaces $\dot{B}^\sigma_{p,r}(K)$ and $B^\sigma_{p,r}(K)$ coincide. Consequently, $\dot{B}^{\frac{d}{2}}_{p,r} \hookrightarrow \dot{B}^{\frac{d}{2}-1}_{p,r}$, $d > 2$. Therefore, if $d > 3$, we can get the same result of Theorem 3.2 to Equation (1.1) in the spaces

$$\widetilde{L}^\infty(\dot{B}^{\frac{d}{2}}_{2,1}(K)) \bigcap L^2_T(L^\infty(K)).$$

Moreover, note that $\dot{B}^{\frac{d}{2}}_{2,2}(K) = B^{\frac{d}{2}}_{2,2}(K) = H^{\frac{d}{2}}(K)$ and the embeddings of Lemma 2.2, then the solution u of Equation (1.1) is global and satisfies for all $\epsilon > 0$

$$\nabla u \in \mathcal{C}(\mathbb{R}^+; H^{\frac{d}{2}+\epsilon}).$$

The global existence of solution to Equation (1.1) in Theorem 3.2 is a straightforward corollary of the following result.

LEMMA 3.3. *Let* (u_0,u_1) *be the initial datum and* $d > 3$*. For positive constants* M, R*, define the set* $\mathscr{X}_R(M)$ *of the functions* u *such that for any* $T > 0$*,*

$$\nabla u \in L^2_T(L^\infty) \bigcap \widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}) \bigcap \widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1}), \quad \textit{and}$$

$$\|\nabla u\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}) \cap \widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})} \le M, \quad \|\nabla u\|_{L^2_T(L^\infty)} \le R,$$

if the initial datum u_0 satisfies $\|\nabla u_0\|_{\dot B^{\frac{d}{2}-\frac12}_{2,1}\cap \dot B^{\frac{d}{2}}_{2,1}}\le \frac{M}{2}$, for some sufficiently small M. Then the mapping $u\longmapsto u_F+F(u)$ maps $\mathscr{X}_R(M)$ into $\mathscr{X}_R(M)$, if the initial data $u_0\in \mathscr{X}_R(M)$. More precisely, for any solutions u,v in $\mathscr{X}_R(M)$, the following result holds

$$\|\nabla((F(u)-F(v))\|_{\mathscr{X}}\le \frac{1}{2}\|\nabla(u-v)\|_{\mathscr{X}}, \tag{3.24}$$

where $\mathscr{X}=L^2_T(L^\infty)\bigcap \widetilde{L}^\infty_T(\dot B^{\frac{d}{2}-\frac12}_{2,1})\bigcap \widetilde{L}^\infty_T(\dot B^{\frac{d}{2}}_{2,1})$.

Proof. Assume u,v be two solutions of Equation (3.2). It is easy to check that $F(u)-F(v)$ solves the following equation

$$\begin{cases}(\partial_t^2-\Delta)(F(u)-F(v))=\mathcal{N}_1(\nabla u,\nabla v)+\mathcal{N}_2(\nabla u,\nabla v), & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ (F(u)-F(v),\partial_t(F(u)-F(v))|_{t=0}=(0,0), & x\in\mathbb{R}^d,\end{cases} \tag{3.25}$$

where $\mathcal{N}_1(\nabla u,\nabla v)=(\mathcal{N}(t,\nabla u)-\mathcal{N}(t,\nabla v))(\nabla u,\nabla u)$ and $\mathcal{N}_2(\nabla u,\nabla v)=(t,\nabla v)(\nabla(u+v),\nabla(u-v))$.

Applying the dyadic blocks $\dot\Delta_k$ on both sides of Equation (3.25), after multiplying $\partial_t\dot\Delta_k(F(u)-F(v))$, by standard energy estimate, multiplying the result by $2^{k\frac{d}{2}}$ and $2^{k\frac{d}{2}-\frac{k}{2}}$, respectively, after taking $l^1(\mathbb{Z})$ norm, we end up with

$$\begin{cases}\|\nabla(F(u)-F(v))\|_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}-\frac12}_{2,1})}\le\|\mathcal{N}_1(\nabla u,\nabla v)+\mathcal{N}_2(\nabla u,\nabla v)\|_{\widetilde{L}^1_T(\dot B^{\frac{d}{2}-\frac12}_{2,1})},\\ \|\nabla(F(u)-F(v))\|_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}}_{2,1})}\le\|\mathcal{N}_1(\nabla u,\nabla v)+\mathcal{N}_2(\nabla u,\nabla v)\|_{\widetilde{L}^1_T(\dot B^{\frac{d}{2}}_{2,1})}.\end{cases} \tag{3.26}$$

In view of Lemma 3.2 to inequality $(3.26)_1$, one has that

$$\begin{aligned}\|\nabla(F(u)-F(v))\|_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}-\frac12}_{2,1})}\le& C(M)MR^2\|\nabla(u-v)\|_{L^\infty_T(\dot B^{\frac{d}{2}-\frac12}_{2,1})}\\ &+C(M)MR\Big(M\|\nabla(u-v)\|_{L^2_T(L^\infty)}+R\|\nabla(u-v)\|_{L^\infty_T(\dot B^{\frac{d}{2}}_{2,1})}\Big).\end{aligned} \tag{3.27}$$

Taking advantage of Lemma 3.2 to inequality $(3.26)_2$, it follows that

$$\begin{aligned}&\|\nabla(F(u)-F(v))\|_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}}_{2,1})}\\ \le& C(M)MR\Big(M\|\nabla(u-v)\|_{L^2_T(L^\infty)}+R\|\nabla(u-v)\|_{L^\infty_T(\dot B^{\frac{d}{2}}_{2,1})}\Big).\end{aligned} \tag{3.28}$$

On the other hand, thanks to the Strichartz estimates of Lemma 2.6 with $(q_1,r_1)=(2,\infty)$ and $(q_2',r_2')=(1,2)$, we have

$$\|\nabla(F(u)-F(v))\|_{L^2_T(\dot B^0_{\infty,1})}\le\|\mathcal{N}_1(\nabla u,\nabla v)+\mathcal{N}_2(\nabla u,\nabla v)\|_{L^1_T(\dot B^{\frac{d}{2}-\frac12}_{2,1})}. \tag{3.29}$$

In view of Lemma 3.2, due to $\dot B^0_{\infty,1}\hookrightarrow L^\infty$, thus

$$\begin{aligned}&\|\nabla(F(u)-F(v))\|_{L^2_T(L^\infty)}\le\|\nabla(F(u)-F(v))\|_{L^2_T(\dot B^0_{\infty,1})}\\ \le& C(M)MR^2\|\nabla(u-v)\|_{L^\infty_T(\dot B^{\frac{d}{2}-\frac12}_{2,1})}+C(M)MR\\ &\times\Big(M\|\nabla(u-v)\|_{L^2_T(L^\infty)}+R\|\nabla(u-v)\|_{L^\infty_T(\dot B^{\frac{d}{2}}_{2,1})}\Big).\end{aligned} \tag{3.30}$$

Note that $\widetilde{L}_T^\infty(\dot{B}^s_{2,1})\hookrightarrow L_T^\infty(\dot{B}^s_{2,1})$, $s=\frac{d}{2},\frac{d}{2}-\frac{1}{2}$. Combining (3.27), (3.28) with (3.30), we have

$$\|\nabla(F(u)-F(v))\|_{\mathscr{X}}\le C(M)MR(M+R)\|\nabla(u-v)\|_{\mathscr{X}},$$

where $\mathscr{X}=L_T^2(L^\infty)\cap\widetilde{L}_T^\infty(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1})\cap\widetilde{L}_T^\infty(\dot{B}^{\frac{d}{2}}_{2,1})$. If $C(M)MR(M+R)<1$, without loss of generality, letting $C(M)MR(M+R)=\frac{1}{2}$, then we imply that

$$\|\nabla(F(u)-F(v))\|_{\mathscr{X}}\le\frac{1}{2}\|\nabla(u-v)\|_{\mathscr{X}}.$$

Next, we shall prove the existence of solution to Equation (1.1), i.e. if $u\in\mathscr{X}_R(M)$, so does $u_F+F(u)\in\mathscr{X}_R(M)$. On the one hand, applying the dyadic blocks $\dot{\Delta}_k$ on both sides of Equation (3.11), by standard energy arguments yields that

$$\begin{cases}\|\nabla u_F\|_{\widetilde{L}_T^\infty(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1})}\le\|\varpi\|_{\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}},\\ \|\nabla u_F\|_{\widetilde{L}_T^\infty(\dot{B}^{\frac{d}{2}}_{2,1})}\le\|\varpi\|_{\dot{B}^{\frac{d}{2}}_{2,1}},\end{cases}\tag{3.31}$$

where $\varpi:=(u_1,\nabla_x u_0)$.

Taking advantage of Lemma 2.6 with $(q_1,r_1)=(2,\infty)$ and $(q_2',r_2')=(1,2)$ to Equation (3.11), we have

$$|\nabla u_F\|_{L_T^2(L^\infty)}\le\|\nabla u_F\|_{L_T^2(\dot{B}^0_{\infty,1})}\le C\|\varpi\|_{\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}}.\tag{3.32}$$

On the other hand, add (3.28), (3.29) to (3.30), letting $v=0$, it follows that

$$\|\nabla(F(u))\|_{\mathscr{X}}\le C(M)MR(M+R)\|\nabla(u)\|_{\mathscr{X}}.\tag{3.33}$$

Combining (3.31), (3.32) with (3.33), one can easily check that

$$\begin{aligned}\|\nabla(u_F+F(u))\|_{\mathscr{X}}&\le C(M)MR(M+R)\|\nabla u\|_{\mathscr{X}}+M\\&\le C(M)MR(M+R)^2+M\\&\le M+R,\end{aligned}\tag{3.34}$$

where we let M small enough, and R is a fixed real number, which concludes the proof of Lemma 3.3. ∎

Proof. **(Proof of Theorem 3.2.)** Thanks to Lemma 3.3, we can get the existence of solution to Equation (1.1) by the contraction mapping argument, we will derive the result of Theorem 3.2, if we show the uniqueness of solution. Indeed, let u,v be two solutions to Equation (1.1) with initial u_0,v_0, respectively. Then, similar to the process of estimate (3.34), we obtain $u-v=(u_F-v_F)+(F(u)-F(v))$ satisfying the following inequality

$$\begin{aligned}\|\nabla(u-v)\|_{\mathscr{X}}\le C\|\varpi-\pi\|_{\mathscr{X}}+\|\nabla(F(u)-F(v))\|_{\mathscr{X}}\\ \le C\|\varpi-\pi\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}}+\frac{1}{2}\|\nabla(u-v)\|_{\mathscr{X}},\end{aligned}\tag{3.35}$$

where $\varpi:=(u_1,\nabla_x u_0)$, $\pi:=(v_1,\nabla_x v_0)$ and $\mathscr{X}=L_T^2(L^\infty)\bigcap\widetilde{L}_T^\infty(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1})$ $\bigcap\widetilde{L}_T^\infty(\dot{B}^{\frac{d}{2}}_{2,1})$. It is easy to check that the uniqueness comes from the term (3.35), which derives the result of Theorem 3.2. ∎

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