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RADON MEASURE SOLUTIONS FOR STEADY HYPERSONIC-LIMIT EULER FLOWS PASSING TWO-DIMENSIONAL FINITE NON-SYMMETRIC OBSTACLES AND INTERACTIONS OF FREE CONCENTRATION LAYERS*

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Abstract. By proposing a notion of Radon measure solutions of the compressible Euler equations, we consider in the paper uniform stationary hypersonic-limit flows passing a two-dimensional finite non-symmetric obstacle with static gas downstream behind the obstacle, and construct solutions with mass concentrated on the boundary of the obstacle and then on free layers beyond it. The Newton-Busemann pressure law on lifts/drags of the obstacle in hypersonic flow is rigorously derived. The pressure of the static gas influences the structure of the solution. Both terminations and interactions of the free concentration layers may be possible. We give some criterions about it and also present some numerical examples to demonstrate these possibilities.

Keywords. Radon measure solution; compressible Euler equations; pressureless gas; hypersonic flow; Newton-Busemann pressure law; initial-boundary value problem.

AMS subject classifications. 35L65; 35L67.

1. Introduction

The problem of supersonic flow passing bodies is both physically significant and mathematically challenging. It serves as a fundamental model problem for the studies of mathematical gas dynamics and hyperbolic systems of conservation laws. There are prominent progresses in the past decades. (See [1–6] and references therein). Motivated by the previous research works, we are interested in the case of stationary hypersonic flows passing bodies. To our knowledge, contrary to the studies of supersonic flows, there is little work on the mathematical theory of hypersonic flows [7]. There are presently two directions for the mathematical investigations of hypersonic flows. One is to consider the case that the Mach number of the flow is quite large, but not infinite. Hu [8], Hu and Zhang [9] studied the cases of hypersonic potential flow passing a curved wedge and symmetric cone. Kuang, Xiang, Zhang [10] proved rigorously the hypersonic similarity law. The other direction is to study the case that the Mach number of the upstream flow is infinite, *i.e.*, the hypersonic-limit flow (or limiting hypersonic flow).

Physicists have already noticed that hypersonic flows share some spectacular properties, such as the "Mach number independence law" [11, pp.24-26], which indicates that the flow field has a "limit" when the Mach number goes to infinity. In [12], it has been clarified mathematically that the limiting hypersonic flow is actually the pressureless Euler flows when it is away from the solid body. Moreover, it is well-known that to solve certain problems related to pressureless Euler flows and some linearly degenerate hyperbolic equations, singular measure solutions, called delta shocks, for which some physical quantity, such as density, concentrated on a curve, are necessary (see, for example, [13–22] and references therein).

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It is the first and also fundamental difficulty of studying hypersonic-limit flows that lies in the proper understanding of initial-boundary value problems of compressible Euler equations. Previous studies on delta shocks and singular measure solutions of conservation laws are about the initial value problems. Although inspirational, they cannot be applied to our case when solid boundary appears in the flow field due to the possible delta shocks on the boundary. The fact that there is no satisfactory concept of measure solutions for the general compressible Euler equations leads to many confusions: It happens for some cases that the delta shock solution cannot be uniquely solved unless some artificial values are assigned for certain components of the unknowns on the delta shock-front; How to understand the seemingly unavoidable "products or power of Dirac measures"? Do delta shocks really represent some significant physical phenomena, or are they just mathematical imagination?

We checked these issues carefully, and proposed a mathematically rigorous definition of Radon measure solutions, for initial-boundary value problems of multidimensional stationary or time-dependant compressible Euler equations, compatible with general state functions, such as polytropic gas, pressureless gas, Chaplygin gas. Equipped with it, we have successfully studied several typical problems, such as the hypersonic-limit flow past wedges [12, 26], limiting hypersonic conical flows [23], and high-Mach number limit of piston problem [24, 25]. It is demonstrated that this definition has the following merits.

Firstly, it is compatible with the standard integral weak solutions. We could easily associate to a standard weak solution a Radon measure solution of the Euler equations, which is absolute continuous with respect to the canonical measure on the physical space (the domain of independent variables) — such as the Lebesgue measure on the Euclidean spaces and Hausdorff surface measure on the sphere. For the hypersonic limit of uniform supersonic flow passing straight wedges, and the high Mach number limit for piston problems of polytropic gases, we proved that the Radon measure solutions obtained from the piecewise-discontinuous weak solutions with shocks converge vaguely as measures to a singular Radon measure, which still fulfills the definition.

Secondly, without any artificial requirements, it totally determines the singular Radon measure solution with a discontinuity curve on which the density may concentrate. The momentums etc. are considered as measures that are absolute continuous with respect to the density measure. If the curve is unknown, both the curve and various weights could also be uniquely solved from certain ordinary differential equations derived from the definition of Radon measure solutions, which could be considered as generalized Rankine-Hugoniot conditions (*cf.* Remark 3.3), noticing that they are nothing but the Rankine-Hugoniot conditions when there is no concentration on the curve, *i.e.*, the weight of the density is zero.

Thirdly, the issue of "products of Dirac measures" or "products of a discontinuous function with a Dirac measure" do not appear at all. Contrary to using multiplication, we refer to the "division of measures" provided by the Radon-Nikodym theorem. The Euler equations were relaxed to a linear differential system of Radon measures, and the nonlinearity of the Euler equations were then exhibited by some nontrivial relations of the Radon-Nikodym derivatives.

Moreover, we only require that the state function, for example, $p=\kappa\rho^\gamma$ for polytropic gases, holds *out* of the set of concentration of density measure ρ. So it is not necessary to worry about the meaning of "the power of a Dirac measure". One may doubt that the loss of the state function may lead to the non-uniqueness of solutions. However, this is not the case as we demonstrated in these works.

Fourthly, and most importantly, by the definition of Radon measure solutions, we could deduce rigorously the celebrated Newton sine-squared law and Newton-Busemann pressure law for lifts/drags of bodies in hypersonic flows (*cf.* Remark 3.2 and [12,23,26]). These laws are fundamental for hypersonic aerodynamics, and were derived from some (mathematically non-rigorous) complicated physical arguments [11, p.132 and p.137]. This evidence strongly supports the appropriateness of the definition of Radon measure solutions.

We also found that the "delta shock" could be used as a mathematical model for the "infinite-thin shock layers" proposed by engineers to describe the narrow zone between the shock front and the body in hypersonic flow [11, p.129 and p.264]. It means that studying Radon measure solutions of Euler equations is not just a mathematical curiosity, but also physically real and necessary.

Lastly, the definition may extend the research of singular measure solutions of hyperbolic conservation laws with discontinuous fluxes or multi-phase flows.

The definition of Radon measure solutions also raises the basic problem of interactions of elementary waves with delta shocks as well as that of two delta waves and the possibility of constructing a well-posedness theory of Euler equations with Radon measures as initial data. In this paper, by considering the hypersonic-limit flow passing a two-dimensional finite body, we investigate the interactions of delta shocks (also called free concentration layers following terminologies used in [11]), behind the body. It is a necessary step to construct a global solution with general initial data in future.

It should be pointed out that our definition of Radon measure solutions utilized special structure of the compressible Euler equations. It could be adapted to study systems in divergence form with similar structure, such as the potential flow equation (isentropic compressible Euler equations without rotation) [8], the minimal surface equation in Minkowskii space [22].

The related entropy conditions for the Radon measure solutions are still open. Comparing to the well-posedness theory established by Bressan *et.al.* (see [27]) for classical weak entropy solutions, and Huang and Wang [19] for one-space-dimensional zero pressure gas, the regularity of the Radon measure solution with respect to time, that we require, is quite weak. The study of this paper also implies that some improvement is necessary to define the weighted Dirac measure supported on a curve, for better understanding and presentation of the relations of weights of interacting delta shocks (*cf.* Lemma 3.3).

As already mentioned, for a general theory of Radon measure solutions of the compressible Euler equations, it is interesting and also necessary to study the interactions of the delta shocks. To this end, we study in this paper Radon measure solutions for the limiting hypersonic flows past a finite obstacle, which may generate two free concentration layers (delta shocks) beyond the obstacle. Whether these two layers intersect depend on the pressure of the static gas lying in a "stagnation zone" behind the finite obstacle. The problem is formulated in Section 2, and the definition of Radon measure solution is also presented. The concentration layer on the boundary of the body is calculated in Section 3, following the ideas introduced in [12] and [26], where the general Newton-Busemann pressure law is derived as a by-product. To show the global existence of a Radon measure solution, the interaction of these concentration layers, which leave the finite body and become free layers, should be carefully investigated. As firstly discovered in [26], for pressure of the static gas lying behind the obstacle that is quite large, the free layer might terminate and there is no global solution. For the pressure to be small, we construct global solutions assuming the two free concentration layers intersect or always separated by the static gas. In Section 4, we discuss the conditions under which the two free concentration layers intersect or not. Since the resultant explicit expressions are quite complicated to study analytically, we instead present some numerical examples, demonstrating under certain values of given data, whether the layers do meet or not. The results presented in the paper are summarized as Theorem 4.1 at the end of Section 4.

2. Mathematical formulation of the problem

We consider the following two-dimensional steady non-isentropic compressible Euler system for polytropic gases:

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho uv)_x + (\rho v^2 + p)_y = 0, \\ (\rho uE)_x + (\rho vE)_y = 0, \end{cases} \tag{2.1}$$

with the state equation taking the form

$$p = (E - \frac{1}{2}(u^2 + v^2))\frac{\gamma - 1}{\gamma}\rho, \tag{2.2}$$

where ρ, (u,v), p represent respectively the density of mass, velocity and pressure of the flow, $\gamma > 1$ is the adiabatic exponent, and E is the total enthalpy per unit mass. The local sound speed for polytropic gas is given by $c = \sqrt{\frac{\gamma p}{\rho}}$, and Mach number is defined by $M = \frac{\sqrt{u^2+v^2}}{c}$. In [12] the authors have shown that for given uniform upstream supersonic flow with fixed total enthalpy, the hypersonic limit $M \to \infty$ means $\gamma \to 1$ after suitable scalings. So from (2.2) one infers that for upstream limiting hypersonic flow, its pressure is zero. So we actually study pressureless Euler flows passing a finite obstacle in this paper. However, it should be noted that if there is a solid body in the flow field, the body will feel “pressure” due to impact (momentum) of the particles in the gas.

We now specify the finite obstacle in the hypersonic-limit flow. It is bounded by the y-axis and the line $W_3 = \{x = x_* > 0\}$, and the upper boundary $W_1 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq x_*,\ y = f_1(x)\}$, lower boundary $W_2 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq x_*,\ y = f_2(x)\}$ (see Figure 2.1 or Figure 2.2 below), where $y = f_1(x)$ and $y = f_2(x)$ are given smooth functions with $f_2'(x) > f_1'(x) \geq 0,\ f_2(x) \leq f_1(x), f_1(0) = 0$, and $y = f_2(x)$ intersects the y-axis at a point $(0, y_*)$. Behind the obstacle, we assume that there are static gases with constant pressure $\underline{p}$ and zero velocity. Thus the domain we consider, which occupied by gases, is

$$\Omega = \{(x,y) : 0 < x < x_*,\ y > f_1(x) \text{ or } y < f_2(x)\} \cup \{(x,y) : x \geq x_*\}.$$

On the solid boundary of the obstacle, the flow satisfies the slip conditions:

$$v = f_1'(x)u, \ \text{on}\ W_1, \quad v = f_2'(x)u, \ \text{on}\ W_2, \quad \text{and} \quad u = 0, \ \text{on}\ W_3. \tag{2.3}$$

The initial conditions on the y-axis and the static gas behind the obstacle are

$$\begin{aligned} &U = (\rho_1, u_1, 0, E_1)^\top, \quad \text{on}\ I_1 = \{(x,y) \in \mathbb{R}^2 : x = 0,\ y > 0\}, \\ &U = (\rho_2, u_2, v_2, E_2)^\top, \quad \text{on}\ I_2 = \{(x,y) \in \mathbb{R}^2 : x = 0,\ y < y_*\}, \\ &\textit{and} \\ &\underline{U} = (\underline{\rho}, 0, 0, \underline{E})^\top. \end{aligned} \tag{2.4}$$

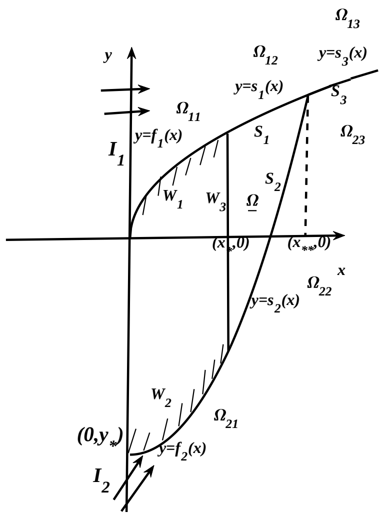

FIG. 2.1. *Two free layers intersect.*

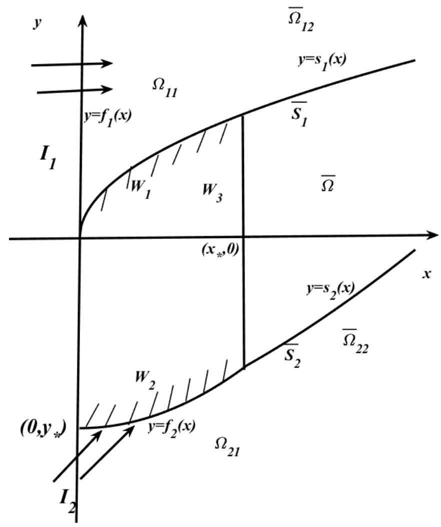

FIG. 2.2. *Two free layers never intersect.*

Notice that the domain occupied by the static gas is unknown and shall be solved. Here $(\rho_1,u_1,0,E_1)^\top$, $(\rho_2,u_2,v_2,E_2)^\top$, $\underline{\rho}$ and $\underline{E}$ are constants, and to guarantee that the gas impinges onto the obstacle, we require that

$$\frac{v_2}{u_2} \geq \max_{0\leq x\leq x_*} f_2'(x). \tag{2.5}$$

By physical observations, one may guess that particles stick to the boundary once they reach the obstacle and then move forward and form a concentration layer. Then the concentration layer will fly away from the finite obstacle and become a free layer in the space, separating the limiting hypersonic flow and the static gas behind the obstacle. For the ideal case we assume that the static gas is not affected by the free layer and the upstream limiting hypersonic flow, and remains to be $\underline{U}$. We will discuss the conditions under which the two free concentration layers intersect (or not).

For later convenience, we introduce here some notations (*cf.* Figures 2.1 and 2.2). Let $\Omega_{11}=\{(x,y)\in\mathbb{R}^2: 0\leq x<x_*, y>f_1(x)\}$ and $\Omega_{21}=\{(x,y)\in\mathbb{R}^2: 0\leq x<x_*, y<f_2(x)\}$. When the concentration layers leave from the upper and lower boundaries respectively, the two free layers in the space are denoted by $y=s_1(x)$ and $y=s_2(x)$. In case they intersect (Figure 2.1), let $\Omega_{12}=\{(x,y)\in\mathbb{R}^2: x_*\leq x<x_{**}, y>s_1(x)\}$ and $\Omega_{22}=\{(x,y)\in\mathbb{R}^2: x_*\leq x<x_{**}, y<s_2(x)\}$. They bound the domain with static gas: $\underline{\Omega}=\{(x,y)\in\mathbb{R}^2: x_*\leq x\leq x_{**}, s_2(x)<y<s_1(x)\}$. Set $\Omega_{13}=\{(x,y)\in\mathbb{R}^2: x_{**}\leq x<+\infty, y>s_3(x)\}$ and $\Omega_{23}=\{(x,y)\in\mathbb{R}^2: x_{**}\leq x<+\infty, y<s_3(x)\}$, where $y=s_3(x)$ is also a free concentration layer, resulting from the interaction of the free layers s_1 and s_2. Moreover, since the concentration layer is also continuously differentiable even if it departs from the wedge, which corresponds to the case that the pressure on the layers coming from the wedge vanishes, it is natural to assume that at $x=x_*$, $s_1'(x_*)=f_1'(x_*)$ and $s_2'(x_*)=f_2'(x_*)$ hold.

In the case that the two free layers do not intersect (Figure 2.2), we set $\overline{\Omega}_{12}=\{(x,y)\in\mathbb{R}^2: x_*\leq x<+\infty, y>s_1(x)\}$, $\overline{\Omega}_{22}=\{(x,y)\in\mathbb{R}^2: x_*\leq x<+\infty, y<s_2(x)\}$, and $\overline{\Omega}=\{(x,y)\in\mathbb{R}^2: x_*\leq x<+\infty, s_2(x)<y<s_1(x)\}$.

We wish to construct solutions with the structure shown in Figure 2.1 and Figure 2.2. To this end, we firstly need a rigorous definition of Radon measure solutions of the compressible Euler equations, to describe precisely the concentration layers.

Let $\mathcal{B}$ be the Borel $\sigma-$algebra of the Euclidean plane $\mathbb{R}^2$. In this paper we always consider Radon measures on $(\mathbb{R}^2,\mathcal{B})$, and write

$$<m,\phi>=\int_{\mathbb{R}^2}\phi(x,y)m(\mathrm{d}x\mathrm{d}y)$$

for the pairing between a Radon measure m and a compact-support test function $\phi\in C_0(\mathbb{R}^2)$. The standard Lebesgue measure of $\mathbb{R}^2$ is denoted by $\mathcal{L}^2$. A measure λ is absolutely continuous with respect to a nonnegative measure μ and is denoted by $\lambda\ll\mu$. The Dirac measure supported on a curve, which is singular to $\mathcal{L}^2$, is defined as below (*cf.* [20]).

Definition 2.1 (Weighted Dirac measure supported on a curve). *Let L be a Lipschitz curve given by $x=x(t),y=y(t)$ for $t\in[0,T)$, and $\omega_L(t)\in L^1_{\mathrm{loc}}(0,T)$. The Dirac measure supported on $L\subset\mathbb{R}^2$ with weight ω_L is defined by*

$$<\omega_L\delta_L,\phi>=\int_0^T\omega_L(t)\phi(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\mathrm{d}t,\qquad\forall\phi\in C_0(\mathbb{R}^2).\tag{2.6}$$

We now give the definition of Radon measure solutions to problem (2.1)-(2.4). (In other words, it is the precise mathematical formulation of our problem.)

Definition 2.2. *For fixed adiabatic exponent $\gamma\geq 1$, let $m^0,m^1,m^2,m^3,n^0,n^1,n^2,n^3,\wp$ be Radon measures on Ω, and ω_p^{11}, ω_p^{12}, ω_p^{21}, ω_p^{22} be nonnegative locally integrable functions on $\mathbb{R}^+\cup\{0\}$. Then (ρ,u,v,E) is called a Radon measure solution of problem (2.1)-(2.4), if the following hold:*

(1) For $\boldsymbol{n}_1=\frac{(-f_1'(x),1)}{\sqrt{1+f_1'(x)^2}}$ being the inward unit normal vector of Ω_{11} on W_1, one has

$$(\omega_p^{11},\omega_p^{21})\parallel\boldsymbol{n}_1\quad\textit{or}\quad\omega_p^{11}+f_1'(x)\omega_p^{21}=0;\tag{2.7}$$

(2) For $\boldsymbol{n}_2=\frac{(f_2'(x),-1)}{\sqrt{1+f_2'(x)^2}}$ being the inward unit normal vector of Ω_{21} on W_2, one has

$$(\omega_p^{12},\omega_p^{22})\parallel\boldsymbol{n}_2\quad\textit{or}\quad\omega_p^{12}+f_2'(x)\omega_p^{22}=0;\tag{2.8}$$

(3) For any $\phi\in C_0^1(\mathbb{R}^2)$, there hold

$$<m^0,\partial_x\phi>+<n^0,\partial_y\phi>+\int_0^{+\infty}\rho_1u_1\phi(0,y)\mathrm{d}y+\int_{-\infty}^{y_*}\rho_2u_2\phi(0,y)\mathrm{d}y=0;\tag{2.9}$$

$$\begin{aligned}&<m^1,\partial_x\phi>+<n^1,\partial_y\phi>+<\wp,\partial_x\phi>+<\omega_p^{11}\delta_{W_1},\phi>+<\omega_p^{12}\delta_{W_2},\phi>\\&\quad+\int_0^{+\infty}\rho_1u_1^2\phi(0,y)\mathrm{d}y+\int_{-\infty}^{y_*}\rho_2u_2^2\phi(0,y)\mathrm{d}y+\int_{f_2(x_*)}^{f_1(x_*)}\underline{p}\phi(x_*,y)\mathrm{d}y=0;\end{aligned}\tag{2.10}$$

$$\begin{aligned}&<m^2,\partial_x\phi>+<n^2,\partial_y\phi>+<\wp,\partial_y\phi>+<\omega_p^{21}\delta_{W_1},\phi>+<\omega_p^{22}\delta_{W_2},\phi>\\&\quad+\int_{-\infty}^{y_*}\rho_2u_2v_2\phi(0,y)\mathrm{d}y=0;\end{aligned}\tag{2.11}$$

$$<m^3,\partial_x\phi>+<n^3,\partial_y\phi>+\int_0^{+\infty}\rho_1u_1E_1\phi(0,y)\mathrm{d}y+\int_{-\infty}^{y_*}\rho_2u_2E_2\phi(0,y)\mathrm{d}y=0; \quad (2.12)$$

(4) ϱ is a nonnegative Radon measure so that $\wp\ll\varrho$, $(m^k,n^k)\ll\varrho$, $(k=0,1,2,3)$, with the Radon-Nikodym derivatives

$$u=\frac{m^0(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)} \quad \text{and} \quad v=\frac{n^0(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)} \tag{2.13}$$

satisfy that

$$u=\frac{\frac{m^1(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}{\frac{m^0(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}=\frac{\frac{n^1(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}{\frac{n^0(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}, \tag{2.14}$$

$$v=\frac{\frac{m^2(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}{\frac{m^0(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}=\frac{\frac{n^2(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}{\frac{n^0(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}; \tag{2.15}$$

and there is a $\varrho-$a.e. function E so that

$$E=\frac{\frac{m^3(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}{\frac{m^0(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}=\frac{\frac{n^3(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}{\frac{n^0(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}}; \tag{2.16}$$

(5) If $\varrho\ll\mathcal{L}^2$ with Radon-Nikodym derivative $\rho(x,y)$, and $\wp\ll\mathcal{L}^2$ with Radon-Nikodym derivative $p(x,y)$, in a neighborhood $\mathcal{N}$ of $(x,y)\in\Omega$, then in $\mathcal{N}$, $\mathcal{L}^2-$ a.e. there holds the state function

$$p=(E-\frac{1}{2}(u^2+v^2))\frac{\gamma-1}{\gamma}\rho. \tag{2.17}$$

Remark 2.1. The vector-valued functions $(\omega_p^{11},\omega_p^{21})$ and $(\omega_p^{12},\omega_p^{22})$ represent the force per unit area on the obstacle by the limiting hypersonic gas. This is why we could derive the Newton-Busemann pressure law as a by-product from the definition. For motivations of the definition, *cf.* [12, 24]. Since we only study hypersonic-limit flow, as explained before, in the rest of the paper we take $\gamma=1$ in (2.17).

3. Construction of measure solutions with concentration layers

Let I_A be the characteristic function of a set A (i.e. $I_A(x,y)=1$ for $(x,y)\in A$ and $I_A(x,y)=0$ otherwise). According to our conjecture on the structure of the Radon measure solutions as shown in Figure 2.1 and Figure 2.2, we set $\wp=\underline{p}I_{\underline{\Omega}}\mathcal{L}^2$ (the case of Figure 2.1) or $\wp=\underline{p}I_{\bar{\Omega}}\mathcal{L}^2$ (the case of Figure 2.2), with $\underline{p}$ the pressure of the static gas behind the obstacle, and furthermore,

$$\begin{aligned} m^0 =&\rho_1u_1I_{\Omega_{11}\cup\Omega_{12}\cup\Omega_{13}}\mathcal{L}^2+\rho_2u_2I_{\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}}\mathcal{L}^2+\omega_m^{01}(x)\delta_{W_1}+\omega_m^{02}(x)\delta_{W_2}\\ &+\widetilde{\omega_m^{01}}(x)\delta_{S_1}+\widetilde{\omega_m^{02}}(x)\delta_{S_2}+\widetilde{\omega_m^{03}}(x)\delta_{S_3}, \end{aligned}$$

$$
\begin{aligned}
n^0 =& \rho_1 v_1 I_{\Omega_{11}\cup\Omega_{12}\cup\Omega_{13}}\mathcal{L}^2 + \rho_2 v_2 I_{\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}}\mathcal{L}^2 + \omega_n^{01}(x)\delta_{W_1} + \omega_n^{02}(x)\delta_{W_2}\\
&+ \widetilde{\omega_n^{01}(x)}\delta_{S_1} + \widetilde{\omega_n^{02}(x)}\delta_{S_2} + \widetilde{\omega_n^{03}(x)}\delta_{S_3},
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
m^1 =& \rho_1 u_1^2 I_{\Omega_{11}\cup\Omega_{12}\cup\Omega_{13}}\mathcal{L}^2 + \rho_2 u_2^2 I_{\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}}\mathcal{L}^2 + \omega_m^{11}(x)\delta_{W_1} + \omega_m^{12}(x)\delta_{W_2}\\
&+ \widetilde{\omega_m^{11}(x)}\delta_{S_1} + \widetilde{\omega_m^{12}(x)}\delta_{S_2} + \widetilde{\omega_m^{13}(x)}\delta_{S_3},\\
n^1 =& \rho_1 u_1 v_1 I_{\Omega_{11}\cup\Omega_{12}\cup\Omega_{13}}\mathcal{L}^2 + \rho_2 u_2 v_2 I_{\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}}\mathcal{L}^2 + \omega_n^{11}(x)\delta_{W_1} + \omega_n^{12}(x)\delta_{W_2}\\
&+ \widetilde{\omega_n^{11}(x)}\delta_{S_1} + \widetilde{\omega_n^{12}(x)}\delta_{S_2} + \widetilde{\omega_n^{13}(x)}\delta_{S_3},
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
m^2 =& \rho_1 u_1 v_1 I_{\Omega_{11}\cup\Omega_{12}\cup\Omega_{13}}\mathcal{L}^2 + \rho_2 u_2 v_2 I_{\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}}\mathcal{L}^2 + \omega_m^{21}(x)\delta_{W_1} + \omega_m^{22}(x)\delta_{W_2}\\
&+ \widetilde{\omega_m^{21}(x)}\delta_{S_1} + \widetilde{\omega_m^{22}(x)}\delta_{S_2} + \widetilde{\omega_m^{23}(x)}\delta_{S_3}\\
n^2 =& \rho_1 v_1^2 I_{\Omega_{11}\cup\Omega_{12}\cup\Omega_{13}}\mathcal{L}^2 + \rho_2 v_2^2 I_{\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}}\mathcal{L}^2 + \omega_n^{21}(x)\delta_{W_1} + \omega_n^{22}(x)\delta_{W_2}\\
&+ \widetilde{\omega_n^{21}(x)}\delta_{S_1} + \widetilde{\omega_n^{22}(x)}\delta_{S_2} + \widetilde{\omega_n^{23}(x)}\delta_{S_3},
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
m^3 =& \rho_1 u_1 E_1 I_{\Omega_{11}\cup\Omega_{12}\cup\Omega_{13}}\mathcal{L}^2 + \rho_2 u_2 E_2 I_{\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}}\mathcal{L}^2 + \omega_m^{31}(x)\delta_{W_1} + \omega_m^{32}(x)\delta_{W_2}\\
&+ \widetilde{\omega_m^{31}(x)}\delta_{S_1} + \widetilde{\omega_m^{32}(x)}\delta_{S_2} + \widetilde{\omega_m^{33}(x)}\delta_{S_3},\\
n^3 =& \rho_1 v_1 E_1 I_{\Omega_{11}\cup\Omega_{12}\cup\Omega_{13}}\mathcal{L}^2 + \rho_2 v_2 E_2 I_{\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}}\mathcal{L}^2 + \omega_n^{31}(x)\delta_{W_1} + \omega_n^{32}(x)\delta_{W_2}\\
&+ \widetilde{\omega_n^{31}(x)}\delta_{S_1} + \widetilde{\omega_n^{32}(x)}\delta_{S_2} + \widetilde{\omega_n^{33}(x)}\delta_{S_3},
\end{aligned} \tag{3.4}
$$

where $\omega_m^{ij}(x)$, $\omega_n^{ij}(x)$ $(i=0,1,2,3, j=1,2)$, $\widetilde{\omega_m^{kl}(x)}$, $\widetilde{\omega_n^{kl}(x)}$ $(k=0,1,2,3, l=1,2,3)$ are functions to be determined. Notice that the terms $\widetilde{\omega_m^{k3}(x)}\delta_{S_3}$ and $\widetilde{\omega_n^{k3}(x)}\delta_{S_3}$ appear only in the case that s_1 and s_2 interact (Figure 2.1), otherwise they will not be present.

3.1. The flow in Ω_{11} and Ω_{21}. Since the initial data in Ω_{21} is slightly more complicated than that in Ω_{11}, while the overall structure of the flow fields in Ω_{21} and Ω_{11} are similar, here we will show that the flow in Ω_{21}, and then the flow in Ω_{11} can be obtained immediately. The computations are similar to those in [12] and [26], but are more complicated since we do not use the special scalings employed before. The formulas obtained here are more apparent and complete for various applications.

Now substituting (3.1) into (2.9), we have for any $\phi \in C_0^1(\Omega_{21})$ that

$$
\begin{aligned}
&\int_{\Omega_{21}} \rho_2 u_2 \partial_x \phi \mathrm{d}x\mathrm{d}y + \int_{\Omega_{21}} \rho_2 v_2 \partial_y \phi \mathrm{d}x\mathrm{d}y + \int_0^{x_*} \omega_m^{02}(x)\partial_x\phi(x,f_2(x))\sqrt{1+f_2'(x)^2}\mathrm{d}x\\
&+ \int_0^{x_*} \omega_n^{02}(x)\partial_y\phi(x,f_2(x))\sqrt{1+f_2'(x)^2}\mathrm{d}x + \int_{-\infty}^{y_*} \rho_2 u_2 \phi(0,y)\mathrm{d}y = 0.
\end{aligned} \tag{3.5}
$$

Since $\partial_x\phi(x,f_2(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\phi(x,f_2(x)) - f_2'(x)\partial_y\phi(x,f_2(x))$, using Green theorem and the integration by parts, we have

$$
\begin{aligned}
&-\omega_m^{02}(0)\sqrt{1+f_2'(0)^2}\phi(0,y_*) + \int_0^{x_*}\Big(\rho_2 v_2 - \frac{\mathrm{d}(\omega_m^{02}(x)\sqrt{1+f_2'(x)^2})}{\mathrm{d}x} - \rho_2 u_2 f_2'(x)\Big)\\
&\cdot\phi(x,f_2(x))\mathrm{d}x + \int_0^{x_*}\sqrt{1+f_2'(x)^2}(\omega_n^{02}(x) - f_2'(x)\omega_m^{02}(x))\partial_y\phi(x,f_2(x))\mathrm{d}x = 0.
\end{aligned} \tag{3.6}
$$

By the arbitrariness of ϕ, this means

$$\frac{\mathrm{d}(\omega_m^{02}(x)\sqrt{1+f_2'(x)^2})}{\mathrm{d}x}=\rho_2v_2-\rho_2u_2f_2'(x),\quad \omega_m^{02}(0)=0,\quad \omega_n^{02}(x)=f_2'(x)\omega_m^{02}(x). \tag{3.7}$$

Then we conclude that

$$\omega_m^{02}(x)=\frac{\rho_2u_2(y_*-f_2(x))+\rho_2v_2x}{\sqrt{1+f_2'(x)^2}},\quad \omega_n^{02}(x)=f_2'(x)\frac{\rho_2u_2(y_*-f_2(x))+\rho_2v_2x}{\sqrt{1+f_2'(x)^2}}. \tag{3.8}$$

Similarly, substituting (3.4) into (2.12), we have

$$\omega_m^{32}(x)=E_2\frac{\rho_2u_2(y_*-f_2(x))+\rho_2v_2x}{\sqrt{1+f_2'(x)^2}},\quad \omega_n^{32}(x)=E_2f_2'(x)\frac{\rho_2u_2(y_*-f_2(x))+\rho_2v_2x}{\sqrt{1+f_2'(x)^2}}. \tag{3.9}$$

REMARK 3.1. Actually there is some freedom to choose the total enthalpy E in the concentration layer attached to the boundary of the obstacle. We take here $E=E_2$ following the study of hypersonic limits in [12], since it is well-known that for steady flow, the total enthalpy is constant in the whole field (even across shock-fronts), if it is constant upstream.

Next substituting (3.2) into (2.10), by the assumption that $\wp=0$, we get for any $\phi\in C_0^1(\Omega_{21})$ that

$$\begin{aligned}&\int_0^{x_*}(\rho_2u_2v_2-\frac{\mathrm{d}(\omega_m^{12}(x)\sqrt{1+f_2'(x)^2})}{\mathrm{d}x}-\rho_2u_2^2f_2'(x)+\omega_p^{12}(x)\sqrt{1+f_2'(x)^2})\phi(x,f_2(x))\mathrm{d}x\\&+\int_0^{x_*}\sqrt{1+f_2'(x)^2}(\omega_n^{12}(x)-f_2'(x)\omega_m^{12}(x))\partial_y\phi(x,f_2(x))\mathrm{d}x\\&-\omega_m^{12}(0)\sqrt{1+f_2'(0)^2}\phi(0,y_*)=0.\end{aligned} \tag{3.10}$$

By arbitrariness of ϕ, which implies that

$$\begin{aligned}&\frac{\mathrm{d}(\omega_m^{12}(x)\sqrt{1+f_2'(x)^2})}{\mathrm{d}x}=\rho_2u_2v_2-\rho_2u_2^2f_2'(x)+\omega_p^{12}(x)\sqrt{1+f_2'(x)^2},\\&\omega_m^{12}(0)=0,\qquad \omega_n^{12}(x)=f_2'(x)\omega_m^{12}(x).\end{aligned} \tag{3.11}$$

While from (3.3) and (2.11), using $\wp=0$, we can get for any $\phi\in C_0^1(\Omega_{21})$ that

$$\begin{aligned}&\int_0^{x_*}(\rho_2v_2^2-\frac{\mathrm{d}(\omega_m^{22}(x)\sqrt{1+f_2'(x)^2})}{\mathrm{d}x}-\rho_2u_2v_2f_2'(x)+\omega_p^{22}(x)\sqrt{1+f_2'(x)^2})\phi(x,f_2(x))\mathrm{d}x\\&+\int_0^{x_*}\sqrt{1+f_2'(x)^2}(\omega_n^{22}(x)-f_2'(x)\omega_m^{22}(x))\partial_y\phi(x,f_2(x))\mathrm{d}x\\&-\omega_m^{22}(0)\sqrt{1+f_2'(0)^2}\phi(0,y_*)=0.\end{aligned} \tag{3.12}$$

It follows that

$$\begin{aligned}&\frac{\mathrm{d}(\omega_m^{22}(x)\sqrt{1+f_2'(x)^2})}{\mathrm{d}x}=\rho_2v_2^2-\rho_2u_2v_2f_2'(x)+\omega_p^{22}(x)\sqrt{1+f_2'(x)^2},\\&\omega_m^{22}(0)=0,\qquad \omega_n^{22}(x)=f_2'(x)\omega_m^{22}(x).\end{aligned} \tag{3.13}$$

From (2.14), (2.15), (3.2), (3.3), (2.3) and (2.8), we have $\omega_m^{22}(x)=f_2'(x)\omega_m^{12}(x)$ and $\omega_p^{22}(x)=-\frac{1}{f_2'(x)}\omega_p^{12}(x)$. Let

$$y=\omega_m^{12}(x)\sqrt{1+f_2'(x)^2}.$$

Thanks to (3.11) and (3.13), we have a linear ODE:

$$\begin{cases}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{f_2'(x)f_2''(x)}{1+f_2'(x)^2}y=\frac{-\rho_2u_2v_2f_2'(x)^2+\rho_2(v_2^2-u_2^2)f_2'(x)+\rho_2u_2v_2}{1+f_2'(x)^2},\\ y(0)=0.\end{cases}\tag{3.14}$$

It yields the solution

$$y=\frac{F_2(x)}{\sqrt{1+f_2'(x)^2}},\tag{3.15}$$

where

$$F_2(x)=\int_0^x\frac{-\rho_2u_2v_2f_2'(t)^2+\rho_2(v_2^2-u_2^2)f_2'(t)+\rho_2u_2v_2}{\sqrt{1+f_2'(t)^2}}\mathrm{d}t.\tag{3.16}$$

Moreover, we can get

$$\begin{aligned}\omega_m^{12}(x)=\frac{F_2(x)}{1+f_2'(x)^2},\quad \omega_n^{12}(x)=\frac{f_2'(x)F_2(x)}{1+f_2'(x)^2},\\ \omega_m^{22}(x)=\frac{f_2'(x)F_2(x)}{1+f_2'(x)^2},\quad \omega_n^{22}(x)=\frac{f_2'(x)^2F_2(x)}{1+f_2'(x)^2}.\end{aligned}\tag{3.17}$$

By virtue of (2.13)-(2.15),(3.1)-(3.3), (3.8) and (3.17), one has

$$\begin{aligned}u|_{W_2}&=\frac{F_2(x)}{\sqrt{1+f_2'(x)^2}(\rho_2u_2(y_*-f_2(x))+\rho_2v_2x)},\\ v|_{W_2}&=\frac{F_2(x)f_2'(x)}{\sqrt{1+f_2'(x)^2}(\rho_2u_2(y_*-f_2(x))+\rho_2v_2x)},\end{aligned}\tag{3.18}$$

$$\rho|_{W_2}=\frac{\varrho(\mathrm{d}x\mathrm{d}y)}{\delta(\mathrm{d}x\mathrm{d}y)}|_{W_2}=\frac{[\rho_2u_2(y_*-f_2(x))+\rho_2v_2x]^2}{F_2(x)}.\tag{3.19}$$

Note that $p|_{W_2}\doteq\sqrt{\omega_p^{12}(x)^2+\omega_p^{22}(x)^2}=\frac{\sqrt{1+f_2'(x)^2}}{f_2'(x)}\omega_p^{12}(x)$, we have

$$\begin{aligned}p|_{W_2}&=-\frac{f_2''(x)F_2(x)}{(1+f_2'(x)^2)^{\frac{3}{2}}}+\frac{F_2'(x)}{f_2'(x)\sqrt{1+f_2'(x)^2}}-\frac{\rho_2u_2v_2}{f_2'(x)}+\rho_2u_2^2\\ &=-\frac{f_2''(x)F_2(x)}{(1+f_2'(x)^2)^{\frac{3}{2}}}+\frac{-2\rho_2u_2v_2f_2'(x)+\rho_2v_2^2+\rho_2u_2^2f_2'(x)^2}{1+f_2'(x)^2}.\end{aligned}\tag{3.20}$$

This is the celebrated Newton-Busemann pressure law of hypersonic flow passing bodies (*cf.* [11, (3.2.7) in p.137], with scalings and different symbols, as explained in [26]).

Remark 3.2. Since only the weights on the concentration layer are unknown and interesting to us, we usually just record them, rather than present their complete expressions as measures. For example, by (3.19), we mean the density measure on Ω_{21} is

$$\varrho=\rho_2I_{\Omega_{21}}\mathcal{L}^2+\rho|_{W_2}\delta_{W_2},\tag{3.21}$$

while (3.18) means the function $v=v_2I_{\Omega_{21}}\mathcal{L}^2+v|_{W_2}I_{W_2}$ which is $\varrho-$ measurable in Ω_{21}.

By a similar manner we calculate the flow in Ω_{11} and obtain the weights

$$\omega_m^{01}(x)=\frac{\rho_1u_1f_1(x)}{\sqrt{1+f_1'(x)^2}},\quad \omega_n^{01}(x)=\frac{\rho_1u_1f_1(x)f_1'(x)}{\sqrt{1+f_1'(x)^2}},\quad \omega_m^{11}(x)=\frac{F_1(x)}{1+f_1'(x)^2},$$
$$\omega_n^{11}(x)=\frac{F_1(x)f_1'(x)}{1+f_1'(x)^2},\quad \omega_m^{21}(x)=\frac{F_1(x)f_1'(x)}{1+f_1'(x)^2},\quad \omega_n^{21}(x)=\frac{F_1(x)f_1'(x)^2}{1+f_1'(x)^2},\tag{3.22}$$

while

$$\omega_m^{31}(x)=\frac{\rho_1u_1E_1f_1(x)}{\sqrt{1+f_1'(x)^2}},\quad \omega_n^{31}(x)=\frac{\rho_1u_1E_1f_1(x)f_1'(x)}{\sqrt{1+f_1'(x)^2}},\tag{3.23}$$

where

$$F_1(x)=\int_0^x\frac{\rho_1u_1^2f_1'(t)}{\sqrt{1+f_1'(t)^2}}\mathrm{d}t.$$

From (2.13)-(2.15), (3.1)-(3.4), (3.22)-(3.23), we may get that

$$u|_{W_1}=\frac{F_1(x)}{\rho_1u_1f_1(x)\sqrt{1+f_1'(x)^2}},\quad v|_{W_1}=\frac{F_1(x)f_1'(x)}{\rho_1u_1f_1(x)\sqrt{1+f_1'(x)^2}},$$
$$p|_{W_1}=\frac{f_1''(x)F_1(x)+\rho_1u_1^2f_1'(x)^2\sqrt{1+f_2'(x)^2}}{(1+f_2'(x)^2)^{\frac{3}{2}}},\quad \rho|_{W_1}=\frac{\varrho(\mathrm{d}x\mathrm{d}y)}{\delta(\mathrm{d}x\mathrm{d}y)}|_{W_1}=\frac{(\rho_1u_1f_1(x))^2}{F_1(x)},\tag{3.24}$$

which means that the Radon measure solution in Ω_{11} is given by

$$u=u_1I_{\Omega_{11}}\mathcal{L}^2+u|_{W_1}I_{W_1},\quad v=v|_{W_1}I_{W_1},\quad E=E_1,\quad \varrho=\rho_1I_{\Omega_{11}}\mathcal{L}^2+\rho|_{W_1}\delta_{W_1}.\tag{3.25}$$

Then we arrive at the following conclusion.

Lemma 3.1. *If the surfaces of the finite obstacles $y=f_i(x)$ $(i=1,2)$, $0\le x\le x_*$ satisfy $f_2'(x)>f_1'(x)\ge 0$, $f_2(x)\le f_1(x)$, $f_1(0)=0$, $y_*=f_2(0)$ and the upcoming flow (2.4) satisfies (2.5), then there exists a Radon measure solution to the problem (2.1)-(2.4), which admits concentration layers on the surfaces $y=f_i(x)$ for $i=1,2$, $0\le x\le x_*$. Namely, they are given by (3.18), (3.19), (3.20), (3.21), (3.24) and (3.25).*

Remark 3.3. We note particularly that once concentration appears, from one single conservation law, we have two equations (which could be considered as the generalized Rankine-Hugoniot (R-H) jump conditions)(*cf.* (3.7) and (3.8)), compared to the case of standard shock-fronts, for which there is only one R-H condition from one conservation law.

Note that the concentration layers become free and should be determined together with the flow beyond $x=x_*$. As they flow downstream, the layers may extend to infinity separately, one of them terminates somewhere and then the solution could not be constructed anymore, or they interact somewhere. In the next two subsections, we will discuss the existence of global solutions for both the cases that the two free concentration layers intersect and do not, respectively.

3.2. On global solutions when the two free concentration layers intersect. In this subsection, we study the case in which the two free concentration layers intersect at a point with abscissa $x=x_{**}$ (see Figure 2.1). Before doing that we should characterize the condition under which the free layers interact as well as determine the solution from $x=x_*$ to $x=x_{**}$.

3.2.1. The flow in Ω_{12} and Ω_{22}. As said at the beginning of Section 3.1, we first analyze the flow in Ω_{22} in detail. Now $y=s_2(x)$ is free and should be determined. We start the calculation from the definition of Radon measure solution, *i.e.*, Definition 2.2.

Substituting (3.1) into (2.9), one has for any $\phi\in C_0^1(\Omega_{21}\cup\Omega_{22}\cup\underline{\Omega})$ that

$$\begin{aligned}&\int_{\Omega_{21}\cup\Omega_{22}}\rho_2u_2\partial_x\phi\mathrm{d}x\mathrm{d}y+\int_{\Omega_{21}\cup\Omega_{22}}\rho_2v_2\partial_y\phi\mathrm{d}x\mathrm{d}y+\int_0^{x_*}\omega_m^{02}(x)\partial_x\phi(x,f_2(x))\sqrt{1+f_2'(x)^2}\mathrm{d}x\\&+\int_0^{x_*}\omega_n^{02}(x)\partial_y\phi(x,f_2(x))\sqrt{1+f_2'(x)^2}\mathrm{d}x+\int_{x_*}^{+\infty}\widetilde{\omega_m^{02}(x)}\partial_x\phi(x,s_2(x))\sqrt{1+s_2'(x)^2}\mathrm{d}x\\&+\int_{x_*}^{+\infty}\widetilde{\omega_n^{02}(x)}\partial_y\phi(x,s_2(x))\sqrt{1+s_2'(x)^2}\mathrm{d}x+\int_{-\infty}^{y_*}\rho_2u_2\phi(0,y)\mathrm{d}y=0.\end{aligned}\tag{3.26}$$

Using again $\partial_x\phi(x,f(x))=\frac{\mathrm{d}}{\mathrm{d}x}\phi(x,f(x))-f'(x)\partial_y\phi(x,f(x))$, and Green theorem, by an integration-by-parts, one has

$$\begin{aligned}&\omega_m^{02}(x_*)\sqrt{1+f_2'(x_*)^2}\phi(x_*,f_2(x_*))-\widetilde{\omega_m^{02}(x_*)}\sqrt{1+s_2'(x_*)^2}\phi(x_*,s_2(x_*))\\&+\int_{x_*}^{+\infty}\left(\rho_2v_2-\frac{d(\widetilde{\omega_m^{02}(x)}\sqrt{1+s_2'(x)^2})}{\mathrm{d}x}-\rho_2u_2s_2'(x))\phi(x,s_2(x)\right)\mathrm{d}x\\&+\int_{x_*}^{+\infty}\sqrt{1+s_2'(x)^2}(\widetilde{\omega_n^{02}(x)}-s_2'(x)\widetilde{\omega_m^{02}(x)})\partial_y\phi(x,s_2(x))\mathrm{d}x=0.\end{aligned}\tag{3.27}$$

By arbitrariness of test functions ϕ, this implies that

$$\frac{\mathrm{d}(\widetilde{\omega_m^{02}(x)}\sqrt{1+s_2'(x)^2})}{\mathrm{d}x}=\rho_2v_2-\rho_2u_2s_2'(x),\quad \widetilde{\omega_m^{02}(x_*)}=\omega_m^{02}(x_*),\quad \widetilde{\omega_n^{02}(x)}=s_2'(x)\widetilde{\omega_m^{02}(x)}.\tag{3.28}$$

Then it follows that

$$\widetilde{\omega_m^{02}(x)}=\frac{\rho_2u_2(y_*-s_2(x))+\rho_2v_2x}{\sqrt{1+s_2'(x)^2}},\quad \widetilde{\omega_n^{02}(x)}=s_2'(x)\frac{\rho_2u_2(y_*-s_2(x))+\rho_2v_2x}{\sqrt{1+s_2'(x)^2}}.\tag{3.29}$$

Similarly, from (2.12) and (3.4), we can get for any $\phi\in C_0^1(\Omega_{21}\cup\Omega_{22}\cup\underline{\Omega})$ that

$$\widetilde{\omega_m^{32}(x)}=E_2\frac{\rho_2u_2(y_*-s_2(x))+\rho_2v_2x}{\sqrt{1+s_2'(x)^2}},\quad \widetilde{\omega_n^{32}(x)}=E_2s_2'(x)\frac{\rho_2u_2(y_*-s_2(x))+\rho_2v_2x}{\sqrt{1+s_2'(x)^2}}.\tag{3.30}$$

By virtue of (2.10), (3.2) and $\wp=\underline{p}I_{\underline{\Omega}}\mathcal{L}^2$, we deduce for any $\phi\in C_0^1(\Omega_{21}\cup\Omega_{22}\cup\underline{\Omega})$ that

$$\begin{aligned}&\omega_m^{12}(x_*)\sqrt{1+f_2'(x_*)^2}\phi(x_*,f_2(x_*))-\widetilde{\omega_m^{12}(x_*)}\sqrt{1+s_2'(x_*)^2}\phi(x_*,s_2(x_*))\\&+\int_{x_*}^{+\infty}(\rho_2u_2v_2-\frac{d(\widetilde{\omega_m^{12}(x)}\sqrt{1+s_2'(x)^2})}{\mathrm{d}x}-\rho_2u_2^2s_2'(x)+\underline{p}s_2'(x))\phi(x,s_2(x))\mathrm{d}x\\&+\int_{x_*}^{+\infty}\sqrt{1+s_2'(x)^2}(\widetilde{\omega_n^{12}(x)}-s_2'(x)\widetilde{\omega_m^{12}(x)})\partial_y\phi(x,s_2(x))\mathrm{d}x=0.\end{aligned}\tag{3.31}$$

Arbitrariness of ϕ implies that

$$\frac{\mathrm{d}(\widetilde{\omega_m^{12}(x)}\sqrt{1+s_2'(x)^2})}{\mathrm{d}x}=\rho_2u_2v_2+(\underline{p}-\rho_2u_2^2)s_2'(x),$$

$$\widetilde{\omega_m^{12}(x_*)}=\omega_m^{12}(x_*),\qquad \widetilde{\omega_n^{12}(x)}=s_2'(x)\widetilde{\omega_m^{12}(x)}. \tag{3.32}$$

Solving (3.32) yields

$$\begin{aligned}\widetilde{\omega_m^{12}(x)}&=\frac{(\underline{p}-\rho_2u_2^2)(s_2(x)-s_2(x_*))+\rho_2u_2v_2(x-x_*)+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}}{\sqrt{1+s_2'(x)^2}},\\ \widetilde{\omega_n^{12}(x)}&=s_2'(x)\frac{(\underline{p}-\rho_2u_2^2)(s_2(x)-s_2(x_*))+\rho_2u_2v_2(x-x_*)+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}}{\sqrt{1+s_2'(x)^2}}.\end{aligned} \tag{3.33}$$

Then using (2.11) and (3.3), one infers that for any $\phi\in C_0^1(\Omega_{21}\cup\Omega_{22}\cup\underline{\Omega})$,

$$\begin{aligned}&\omega_m^{22}(x_*)\sqrt{1+f_2'(x_*)^2}\phi(x_*,f_2(x_*))-\widetilde{\omega_m^{22}(x_*)}\sqrt{1+s_2'(x_*)^2}\phi(x_*,s_2(x_*))\\ &+\int_{x_*}^{+\infty}(\rho_2v_2^2-\frac{\mathrm{d}(\widetilde{\omega_m^{22}(x)}\sqrt{1+s_2'(x)^2})}{\mathrm{d}x}-\rho_2u_2v_2s_2'(x)-\underline{p})\phi(x,s_2(x))\mathrm{d}x\\ &+\int_{x_*}^{+\infty}\sqrt{1+s_2'(x)^2}(\widetilde{\omega_n^{22}(x)}-s_2'(x)\widetilde{\omega_m^{22}(x)})\partial_y\phi(x,s_2(x))\mathrm{d}x=0,\end{aligned} \tag{3.34}$$

which implies that

$$\begin{aligned}&\frac{\mathrm{d}(\widetilde{\omega_m^{22}(x)}\sqrt{1+s_2'(x)^2})}{\mathrm{d}x}=\rho_2v_2^2-\underline{p}-\rho_2u_2v_2s_2'(x),\\ &\widetilde{\omega_m^{22}(x_*)}=\omega_m^{22}(x_*),\qquad \widetilde{\omega_n^{22}(x)}=s_2'(x)\widetilde{\omega_m^{22}(x)}.\end{aligned} \tag{3.35}$$

We solve that

$$\begin{aligned}\widetilde{\omega_m^{22}(x)}&=\frac{-\rho_2u_2v_2(s_2(x)-s_2(x_*))+(\rho_2v_2^2-\underline{p})(x-x_*)+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}}{\sqrt{1+s_2'(x)^2}},\\ \widetilde{\omega_n^{22}(x)}&=s_2'(x)\frac{-\rho_2u_2v_2(s_2(x)-s_2(x_*))+(\rho_2v_2^2-\underline{p})(x-x_*)+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}}{\sqrt{1+s_2'(x)^2}}.\end{aligned} \tag{3.36}$$

Thanks to (2.14)-(2.15), (3.2)-(3.3), we also have

$$s_2'(x)=\frac{v|_{S_2}}{u|_{S_2}}=\frac{\widetilde{\omega_m^{22}(x)}}{\widetilde{\omega_m^{12}(x)}}, \tag{3.37}$$

where $S_2=\{(x,y)\in R^2: x_*\le x\le x_{**}, y=s_2(x)\}$.

Now from (3.33) and (3.36)-(3.37), we have

$$\begin{cases} \qquad -\rho_2u_2v_2(s_2(x)-s_2(x_*))+(\rho_2v_2^2-\underline{p})(x-x_*)+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}\\ =s_2'(x)[(\underline{p}-\rho_2u_2^2)(s_2(x)-s_2(x_*))+\rho_2u_2v_2(x-x_*)+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}],\\ s_2(x_*)=f_2(x_*).\end{cases} \tag{3.38}$$

Let $y_2(x)=s_2(x)-s_2(x_*)$. Then (3.38) becomes the following first-order separable differential equation.

$$\begin{cases} [\rho_2u_2v_2y_2(x)-(\rho_2v_2^2-\underline{p})(x-x_*)-\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}]\mathrm{d}x\\ \quad +[(\underline{p}-\rho_2u_2^2)y_2(x)+\rho_2u_2v_2(x-x_*)+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}]\mathrm{d}y_2=0,\\ y_2(x_*)=0.\end{cases} \tag{3.39}$$

Its solution is determined by

$$\begin{aligned}&\frac{1}{2}(\underline{p}-\rho_2u_2^2)y_2(x)^2+[\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\rho_2u_2v_2(x-x_*)]y_2(x)\\&-\frac{1}{2}(\rho_2v_2^2-\underline{p})(x-x_*)^2-\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}(x-x_*)=0.\end{aligned}\tag{3.40}$$

Obviously, the solvability of $y=y_2(x)$ may depend on the value of $\underline{p}$ and is somewhat complicated. We leave the discussion to Section 4, assisted with some numerical analysis. We point out here that for $\underline{p}$ large, $s_2(x)$ could only exist for a short distance, with $u=0$ at some point and then stops there. It is not clear how to prolong the solution further for this case.

Finally, we get $s_2(x)=y_2(x)+f_2(x_*)$. Moreover, from (2.14), (2.15), (3.1)-(3.3), (3.29), (3.33) and (3.36), we obtain

$$\begin{aligned}u|_{S_2}&=\frac{(\underline{p}-\rho_2u_2^2)(s_2(x)-s_2(x_*))+\rho_2u_2v_2(x-x_*)+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}}{\rho_2u_2(y_*-s_2(x))+\rho_2v_2x},\\v|_{S_2}&=\frac{-\rho_2u_2v_2(s_2(x)-s_2(x_*))+(\rho_2v_2^2-\underline{p})(x-x_*)+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}}{\rho_2u_2(y_*-s_2(x))+\rho_2v_2x},\\\varrho|_{S_2}&=\frac{(\rho_2u_2(y_*-s_2(x))+\rho_2v_2x)^2}{\sqrt{1+s_2'(x)^2}((\underline{p}-\rho_2u_2^2)(s_2(x)-s_2(x_*))+\rho_2u_2v_2(x-x_*)+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}})}.\end{aligned}\tag{3.41}$$

For the flow states in Ω_{12}, similar analysis as above yields that

$$\begin{aligned}&\widetilde{\omega_m^{01}}(x)=\frac{\rho_1u_1s_1(x)}{\sqrt{1+s_1'(x)^2}},\quad\widetilde{\omega_n^{01}}(x)=\frac{\rho_1u_1s_1(x)s_1'(x)}{\sqrt{1+s_1'(x)^2}},\quad\widetilde{\omega_m^{31}}(x)=\frac{\rho_1u_1E_1s_1(x)}{\sqrt{1+s_1'(x)^2}},\\&\widetilde{\omega_m^{11}}(x)=\frac{(\rho_1u_1^2-\underline{p})(s_1(x)-s_1(x_*))+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\sqrt{1+s_1'(x)^2}},\quad\widetilde{\omega_n^{31}}(x)=\frac{\rho_1u_1E_1s_1(x)s_1'(x)}{\sqrt{1+s_1'(x)^2}},\\&\widetilde{\omega_n^{11}}(x)=s_1'(x)\frac{(\rho_1u_1^2-\underline{p})(s_1(x)-s_1(x_*))+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\sqrt{1+s_1'(x)^2}},\quad\widetilde{\omega_m^{21}}(x)=\frac{\underline{p}(x-x_*)+\frac{f_1'(x_*)F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\sqrt{1+s_1'(x)^2}},\\&\widetilde{\omega_n^{21}}(x)=s_1'(x)\frac{\underline{p}(x-x_*)+\frac{f_1'(x_*)F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\sqrt{1+s_1'(x)^2}}.\end{aligned}\tag{3.42}$$

By (2.14)-(2.15), (3.2)-(3.3), we have

$$s_1'(x)=\frac{v|_{S_1}}{u|_{S_1}}=\frac{\widetilde{\omega_m^{21}}(x)}{\widetilde{\omega_m^{11}}(x)},\tag{3.43}$$

where $S_1=\{(x,y)\in\mathbb{R}^2:x_*\le x\le x_{**},y=s_1(x)\}$.

Using (3.42)-(3.43), we have

$$\begin{cases}\underline{p}(x-x_*)+\frac{f_1'(x_*)F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}=s_1'(x)[(\rho_1u_1^2-\underline{p})(s_1(x)-s_1(x_*))+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}],\\s_1(x_*)=f_1(x_*).\end{cases}\tag{3.44}$$

Let $y_1(x)=s_1(x)-s_1(x_*)$. From (3.44) we find that

$$\frac{1}{2}(\rho_1u_1^2-\underline{p})y_1(x)^2+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}y_1(x)-\frac{f_1'(x_*)F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}(x-x_*)-\frac{1}{2}\underline{p}(x-x_*)^2=0. \tag{3.45}$$

Finally, we take $s_1(x)=y_1(x)+f_1(x_*)$. Thanks to (2.14), (2.15), (3.1)-(3.4), (3.42), we have

$$\begin{aligned} u|_{S_1}&=\frac{(\rho_1u_1^2-\underline{p})(s_1(x)-s_1(x_*))+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\rho_1u_1s_1(x)}, \\ v|_{S_1}&=\frac{\underline{p}(x-x_*)+\frac{f_1'(x_*)F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\rho_1u_1s_1(x)}, \\ \varrho|_{S_1}&=\frac{(\rho_1u_1s_1(x))^2}{\sqrt{1+s_1'(x_*)^2}((\rho_1u_1^2-\underline{p})(s_1(x)-s_1(x_*))+\frac{F_1(x_*)}{\sqrt{1+s_1'(x_*)^2}})}. \end{aligned} \tag{3.46}$$

LEMMA 3.2. *If the problem (2.1)–(2.4) admits a Radon measure solution with two free layers for $x>x_*$, then the states of the flow on the layers should satisfy (3.41) and (3.46), and the free layers satisfy (3.40) and (3.45).*

3.2.2. The flow in $\Omega_{13}\cup\Omega_{23}$. Next we study the flow in $\Omega_{13}\cup\Omega_{23}$ by assuming that the two free layers S_1 and S_2 intersect and the sufficient condition for this will be given in Section 4. Since we should first determine those weights $\widetilde{\omega_r^{t3}(x_{**})}$ with $r=m,n;t=0,1,2,3$ at the intersection point of $y=s_1(x)$ and $y=s_2(x)$, we should also start from the definition of Radon measure solution. Similar to the previous analysis, substituting (3.1) into (2.9), one has for any $\phi\in C_0^1(\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}\cup\underline{\Omega})$ that

$$\begin{aligned} &\int_{\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}}\rho_2u_2\partial_x\phi\mathrm{d}x\mathrm{d}y \\ &\quad+\int_{\Omega_{21}\cup\Omega_{22}\cup\Omega_{23}}\rho_2v_2\partial_y\phi\mathrm{d}x\mathrm{d}y+\int_0^{x_*}\omega_m^{02}(x)\partial_x\phi(x,f_2(x))\sqrt{1+f_2'(x)^2}\mathrm{d}x \\ &\quad+\int_0^{x_*}\omega_n^{02}(x)\partial_y\phi(x,f_2(x))\sqrt{1+f_2'(x)^2}\mathrm{d}x+\int_{x_*}^{x_{**}}\widetilde{\omega_m^{02}(x)}\partial_x\phi(x,s_2(x))\sqrt{1+s_2'(x)^2}\mathrm{d}x \\ &\quad+\int_{x_*}^{x_{**}}\widetilde{\omega_n^{02}(x)}\partial_y\phi(x,s_2(x))\sqrt{1+s_2'(x)^2}\mathrm{d}x+\int_{x_{**}}^{+\infty}\widetilde{\omega_m^{03}(x)}\partial_x\phi(x,s_3(x))\sqrt{1+s_3'(x)^2}\mathrm{d}x \\ &\quad+\int_{x_{**}}^{+\infty}\widetilde{\omega_n^{03}(x)}\partial_y\phi(x,s_3(x))\sqrt{1+s_3'(x)^2}\mathrm{d}x+\int_{-\infty}^{y_*}\rho_2u_2\phi(0,y)\mathrm{d}y=0. \end{aligned} \tag{3.47}$$

It follows that

$$\begin{aligned} &\widetilde{\omega_m^{01}(x_{**})}\sqrt{1+s_1'(x_{**})^2}\phi(x_{**},s_1(x_{**}))+\widetilde{\omega_m^{02}(x_{**})}\sqrt{1+s_2'(x_{**})^2}\phi(x_{**},s_2(x_{**})) \\ &\quad-\widetilde{\omega_m^{03}(x_{**})}\sqrt{1+s_3'(x_{**})^2}\phi(x_{**},s_3(x_{**})) \\ &\quad+\int_{x_{**}}^{+\infty}\left((\rho_2v_2-\frac{d(\widetilde{\omega_m^{03}(x)}\sqrt{1+s_3'(x)^2})}{\mathrm{d}x}+(\rho_1u_1-\rho_2u_2)s_3'(x))\phi(x,s_3(x))\right)\mathrm{d}x \\ &\quad+\int_{x_{**}}^{+\infty}\sqrt{1+s_3'(x)^2}(\widetilde{\omega_n^{03}(x)}-s_3'(x)\widetilde{\omega_m^{03}(x)})\partial_y\phi(x,s_3(x))\mathrm{d}x=0. \end{aligned} \tag{3.48}$$

By the arbitrariness of ϕ and $s_1(x_{**})=s_2(x_{**})=s_3(x_{**})$, this implies that

$$\begin{aligned}&\frac{\mathrm{d}(\widetilde{\omega_m^{03}(x)}\sqrt{1+s_3'(x)^2})}{\mathrm{d}x}=\rho_2v_2+(\rho_1u_1-\rho_2u_2)s_3'(x),\quad \widetilde{\omega_n^{03}(x)}=s_3'(x)\widetilde{\omega_m^{03}(x)},\\&\widetilde{\omega_m^{03}(x_{**})}\sqrt{1+s_3'(x_{**})^2}=\widetilde{\omega_m^{01}(x_{**})}\sqrt{1+s_1'(x_{**})^2}+\widetilde{\omega_m^{02}(x_{**})}\sqrt{1+s_2'(x_{**})^2}.\end{aligned}\tag{3.49}$$

Using (3.29) and (3.42), it follows that

$$\widetilde{\omega_m^{03}(x)}=\frac{\rho_2u_2(y_*-s_3(x))+\rho_2v_2x+\rho_1u_1s_3(x)}{\sqrt{1+s_3'(x)^2}},\quad \widetilde{\omega_n^{03}(x)}=s_3'(x)\widetilde{\omega_m^{03}(x)}.\tag{3.50}$$

From (3.28), (3.42) and (3.49), we have

$$\widetilde{\omega_n^{0i}(x)}=s_i'(x)\widetilde{\omega_m^{0i}(x)}\quad (i=1,2,3),\tag{3.51}$$

while (3.49) implies

$$\widetilde{\omega_n^{03}(x_{**})}\frac{\sqrt{1+s_3'(x_{**})^2}}{s_3'(x_{**})}=\widetilde{\omega_n^{01}(x_{**})}\frac{\sqrt{1+s_1'(x_{**})^2}}{s_1'(x_{**})}+\widetilde{\omega_n^{02}(x_{**})}\frac{\sqrt{1+s_2'(x_{**})^2}}{s_2'(x_{**})}.\tag{3.52}$$

Similarly, using (2.7)-(2.12), (3.1)-(3.4) and $\wp=\underline{p}I_{\underline{\Omega}}\mathcal{L}^2$, we have

$$\begin{aligned}\widetilde{\omega_m^{13}(x)}=&\frac{\rho_1u_1^2(s_3(x)-s_1(x_*))-\rho_2u_2^2(s_3(x)-s_2(x_*))+\rho_2u_2v_2(x-x_*)}{\sqrt{1+s_3'(x)^2}}\\&+\frac{\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\underline{p}(s_1(x_*)-s_2(x_*))+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\sqrt{1+s_3'(x)^2}},\quad \widetilde{\omega_n^{13}(x)}=s_3'(x)\widetilde{\omega_m^{13}(x)},\end{aligned}\tag{3.53}$$

$$\begin{aligned}&\widetilde{\omega_m^{23}(x)}=\frac{\rho_2v_2^2(x-x_*)-\rho_2u_2v_2(s_3(x)-s_2(x_*))+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\frac{F_1(x_*)f_1'(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\sqrt{1+s_3'(x)^2}},\\&\widetilde{\omega_n^{23}(x)}=s_3'(x)\widetilde{\omega_m^{23}(x)},\end{aligned}\tag{3.54}$$

$$\widetilde{\omega_m^{33}(x)}=\frac{\rho_2u_2E_2(y_*-s_3(x))+\rho_2v_2E_2x+\rho_1u_1E_1s_3(x)}{\sqrt{1+s_3'(x)^2}},\quad \widetilde{\omega_n^{33}(x)}=s_3'(x)\widetilde{\omega_m^{33}(x)}.\tag{3.55}$$

By the above analysis, we arrive at

LEMMA 3.3. *At the interaction point $x=x_{**}$, the relation between the weight $\widetilde{\omega_r^{t3}(x)}$ on $y=s_3(x)$ and the weights $\widetilde{\omega_r^{t1}(x)}$ and $\widetilde{\omega_r^{t2}(x)}$ on $y=s_1(x)$ and $y=s_2(x)$, respectively $(r=m,n;t=0,1,2,3)$ are*

$$\widetilde{\omega_m^{t3}(x_{**})}\sqrt{1+s_3'(x_{**})^2}=\widetilde{\omega_m^{t1}(x_{**})}\sqrt{1+s_1'(x_{**})^2}+\widetilde{\omega_m^{t2}(x_{**})}\sqrt{1+s_2'(x_{**})^2}\tag{3.56}$$

and

$$\widetilde{\omega_n^{t3}(x_{**})}\frac{\sqrt{1+s_3'(x_{**})^2}}{s_3'(x_{**})}=\widetilde{\omega_n^{t1}(x_{**})}\frac{\sqrt{1+s_1'(x_{**})^2}}{s_1'(x_{**})}+\widetilde{\omega_n^{t2}(x_{**})}\frac{\sqrt{1+s_2'(x_{**})^2}}{s_2'(x_{**})}.\tag{3.57}$$

(cf. (3.49) and (3.52).)

Particularly, (3.56) indicates that for the weighted Dirac measures given by Definition 2.1, it seems more reasonable to consider the density of the concentration as product like $\widetilde{\omega_m^{t3}(x_{**})\sqrt{1+s_3'(x_{**})^2}}$, rather than the weight itself.

We continue to use (2.14)-(2.15), (3.2)-(3.3) to get

$$s_3'(x) = \frac{v|_{S_3}}{u|_{S_3}} = \frac{\widetilde{\omega_m^{23}(x)}}{\widetilde{\omega_m^{13}(x)}}, \tag{3.58}$$

where $S_3 = \{(x,y)\in\mathbb{R}^2 : x_{**}\le x<+\infty, y=s_3(x)\}$. From (3.53), (3.54) and (3.58), one has

$$\begin{aligned}
&\rho_2v_2^2(x-x_*)-\rho_2u_2v_2(s_3(x)-s_2(x_*))+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\frac{F_1(x_*)f_1'(x_*)}{\sqrt{1+f_1'(x_*)^2}}\\
=&s_3'(x)[\rho_1u_1^2(s_3(x)-s_1(x_*))-\rho_2u_2^2(s_3(x)-s_2(x_*))+\rho_2u_2v_2(x-x_*)\\
&+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\underline{p}(s_1(x_*)-s_2(x_*))+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}].
\end{aligned} \tag{3.59}$$

Let $y_3(x)=s_3(x)-s_2(x_*)$. Then (3.59) reads

$$\begin{cases}
[\rho_2v_2^2(x-x_*)-\rho_2u_2v_2y_3(x)+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\frac{F_1(x_*)f_1'(x_*)}{\sqrt{1+f_1'(x_*)^2}}]\mathrm{d}x\\
-[\rho_1u_1^2(y_3(x)+s_2(x_*)-s_1(x_*))-\rho_2u_2^2y_3(x)+\rho_2u_2v_2(x-x_*)+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}\\
+\underline{p}(s_1(x_*)-s_2(x_*))+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}]\mathrm{d}y_3(x)=0,\\
y_3(x_{**})=s_2(x_{**})-s_2(x_*).
\end{cases} \tag{3.60}$$

Its solution is given by

$$Ay_3(x)^2+By_3(x)+C=0, \tag{3.61}$$

where

$$\begin{aligned}
A=&\frac{1}{2}(\rho_2u_2^2-\rho_1u_1^2) \qquad (3.62)\\
B=&[\rho_2u_2v_2(x_*-x)-\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}-\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}-\underline{p}(s_1(x_*)-s_2(x_*))\\
&-\rho_1u_1^2(s_2(x_*)-s_1(x_*))]\\
C=&\frac{1}{2}\rho_2v_2^2(x-x_*)^2+(\frac{F_1(x_*)f_1'(x_*)}{\sqrt{1+f_1'(x_*)^2}}+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}})(x-x_{**})-\frac{1}{2}\rho_2v_2^2(x_{**}-x_*)^2\\
&+\rho_2u_2v_2(s_2(x_{**})-s_2(x_*))(x_{**}-x_*)-\frac{1}{2}(\rho_2u_2^2-\rho_1u_1^2)(s_2(x_{**})-s_2(x_*))^2\\
&+[\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}+\underline{p}(s_1(x_*)-s_2(x_*))\\
&+\rho_1u_1^2(s_2(x_*)-s_1(x_*))](s_2(x_{**})-s_2(x_*))=0.
\end{aligned}$$

The solvability of (3.61) will be discussed in Section 4.

Finally, we take $s_3(x)=y_3(x)+s_2(x_*)$. Moreover,

$$u|_{S_3}=\frac{\rho_1u_1^2(s_3(x)-s_1(x_*))-\rho_2u_2^2(s_3(x)-s_2(x_*))+\rho_2u_2v_2(x-x_*)}{\rho_2u_2(y_*-s_3(x))+\rho_2v_2x+\rho_1u_1s_3(x)} + \frac{\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\underline{p}(s_1(x_*)-s_2(x_*))+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\rho_2u_2(y_*-s_3(x))+\rho_2v_2x+\rho_1u_1s_3(x)}, \tag{3.63}$$

$$v|_{S_3}=\frac{\rho_2v_2^2(x-x_*)-\rho_2u_2v_2(s_3(x)-s_2(x_*))+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\frac{F_1(x_*)f_1'(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\rho_2u_2(y_*-s_3(x))+\rho_2v_2x+\rho_1u_1s_3(x)}, \tag{3.64}$$

$$\varrho|_{S_3}=(\rho_2u_2(y_*-s_3(x))+\rho_2v_2x+\rho_1u_1s_3(x))^2/\{\sqrt{1+s_3'(x)^2}(\rho_1u_1^2(s_3(x)-s_1(x_*)) -\rho_2u_2^2(s_3(x)-s_2(x_*))+\rho_2u_2v_2(x-x_*)+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\frac{F_1(x_*)f_1'(x_*)}{\sqrt{1+f_1'(x_*)^2}})\}. \tag{3.65}$$

We have the following conclusion.

LEMMA 3.4. *If the Radon solution exists for $x>x_{**}$, with x_{**} being the intersection point of the two free layers $y=s_i(x)$ $(i=1,2)$, then the states on the free layers satisfy (3.63)-(3.65), while the free layer $y=s_3(x)=y_3(x)+s_2(x_*)$ originating from $x=x_{**}$ satisfies (3.61).*

One aspect to ensure the global solution is that $y=s_1(x)$ and $y=s_2(x)$ intersect at some point before one of them terminates (if possible) at some point and $y=y_3(x)$ is well-defined for all $x>x_{**}$.

LEMMA 3.5. *If $y=s_1(x)$ and $y=s_2(x)$ intersect at point $x=x_{**}$ and $\frac{v_2}{u_2}>s_3'(x_{**})$, then $y=s_3(x)$ exists for all $x>x_{**}$.*

Proof. As shown in [26] (Theorem 1.3.), there is a global Radon measure solution to problem (2.1)-(2.4) after the intersection of $s_1(x)$ and $s_2(x)$ if

$$\frac{v_2}{u_2}>s_3'(x_{**}). \tag{3.66}$$

In virtue of (3.53), (3.54) and (3.58), (3.66) is equivalent to

$$\frac{v_2}{u_2}>(\rho_2v_2^2(x_{**}-x_*)-\rho_2u_2v_2(s_3(x_{**})-s_2(x_*)) +\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\frac{F_1(x_*)f_1'(x_*)}{\sqrt{1+f_1'(x_*)^2}})\backslash\{\rho_1u_1^2(s_3(x_{**}) -s_1(x_*))-\rho_2u_2^2(s_3(x_{**})-s_2(x_*))+\rho_2u_2v_2(x_{**}-x_*)+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}} +\underline{p}(s_1(x_*)-s_2(x_*))+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}\}, \tag{3.67}$$

which is ensured after some routine calculations by the facts that $s_1(x_*)>s_2(x_*)$, $s_3(x_{**})>s_1(x_*)$ and $\frac{v_2}{u_2}>f_2'(x_*)>f_1'(x_*)$. ∎

3.3. On global solution when the two free concentration layers do not intersect. In the case that the two free concentration layers do not intersect (see Figure 2.2), if they do not terminate in a finite distance from the obstacle (some sufficient conditions will be presented in Section 4), we could have a global solution, with the flow states in $\bar{\Omega}$ given by

$$
\begin{aligned}
u=&u_1 I_{\Omega_{11}\cup\bar{\Omega}_{12}}\mathcal{L}^2+\frac{F_1(x)}{\rho_1 u_1 f_1(x)\sqrt{1+f_1'(x)^2}}I_{W_1}+\frac{(\rho_1 u_1^2-\underline{p})(s_1(x)-s_1(x_*))+\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\rho_1 u_1 s_1(x)}I_{\overline{S}_1}\\
&+u_2 I_{\Omega_{21}\cup\bar{\Omega}_{22}}\mathcal{L}^2+\frac{F_2(x)}{\sqrt{1+f_2'(x)^2}(\rho_2 u_2(y_*-f_2(x))+\rho_2 v_2 x)}I_{W_2}\\
&+\frac{(\underline{p}-\rho_2 u_2^2)(s_2(x)-s_2(x_*))+\rho_2 u_2 v_2(x-x_*)+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}}{\rho_2 u_2(y_*-s_2(x))+\rho_2 v_2 x}I_{\overline{S}_2},\\
v=&\frac{F_1(x)f_1'(x)}{\rho_1 u_1 f_1(x)\sqrt{1+f_1'(x)^2}}I_{W_1}+\frac{\underline{p}(x-x_*)+\frac{f_1'(x_*)F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}}{\rho_1 u_1 s_1(x)}I_{\overline{S}_1}\\
&+v_2 I_{\Omega_{21}\cup\bar{\Omega}_{22}}\mathcal{L}^2+\frac{F_2(x)f_2'(x)}{\sqrt{1+f_2'(x)^2}(\rho_2 u_2(y_*-f_2(x))+\rho_2 v_2 x)}I_{W_2}\\
&+\frac{-\rho_2 u_2 v_2(s_2(x)-s_2(x_*))+(\rho_2 v_2^2-\underline{p})(x-x_*)+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}}{\rho_2 u_2(y_*-s_2(x))+\rho_2 v_2 x}I_{\overline{S}_2},\\
\varrho=&\rho_1 I_{\Omega_{11}\cup\bar{\Omega}_{12}}\mathcal{L}^2+\frac{(\rho_1 u_1 f_1(x))^2}{F_1(x)}\delta_{W_1}+\frac{(\rho_1 u_1 s_1(x))^2}{\sqrt{1+f_1'(x_*)^2}(\rho_1 u_1^2-\underline{p})(s_1(x)-s_1(x_*))+\frac{F_1(x_*)}{\sqrt{1+s_1'(x_*)^2}}}\delta_{\overline{S}_1}\\
&+\rho_2 I_{\Omega_{21}\cup\bar{\Omega}_{22}}\mathcal{L}^2+\frac{[\rho_2 u_2(y_*-f_2(x))+\rho_2 v_2 x]^2}{F_2(x)}\delta_{W_2}\\
&+\frac{(\rho_2 u_2(y_*-s_2(x))+\rho_2 v_2 x)^2}{\sqrt{1+s_2'(x)^2}(\underline{p}-\rho_2 u_2^2)(s_2(x)-s_2(x_*))+\rho_2 u_2 v_2(x-x_*)+\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}}\delta_{\overline{S}_2},
\end{aligned}
\tag{3.68}
$$

where $\overline{S}_1=\{(x,y)\in\mathbb{R}^2: x_*\le x<+\infty, y=s_1(x)\}$ and $\overline{S}_2=\{(x,y)\in\mathbb{R}^2: x_*\le x<+\infty, y=s_2(x)\}$. We conclude it as follows.

LEMMA 3.6. *Under the assumptions of Lemma 3.1 and for proper $\underline{p}$, there exists a solution to the problem (2.1)–(2.4), given by (3.68). The two free layers $y=s_i(x), i=1,2$ satisfying respectively (3.40) and (3.45) do not intersect for $x>x_*$.*

4. Some criteria for termination or intersection of free concentration layers and numerical results

To get the solvability of $y=s_1(x)$ and $y=s_2(x)$, we have the following conclusion.

LEMMA 4.1. *For $\underline{p}\le\rho_i(u_i^2+v_i^2)$, there exists a unique solution satisfying $s_i(x_*)=f_i(x_*)$. For $\underline{p}>\rho_i(u_i^2+v_i^2)$, the solution only exists locally. Here $i=1,2$, $v_1=0$.*

Proof. As before, we first consider the free layer $y=s_2(x)$. Recall in Section 3.2.1 that $y=y_2(x)$ is given by (3.40) and $s_2(x)=y_2(x)+f_2(x_*)$. We consider the following several cases.

Case 1: For $\underline{p}=\rho_2 u_2^2$, from (3.40) we get

$$
s_2(x)=\frac{\frac{1}{2}(\rho_2 v_2^2-\underline{p})(x-x_*)^2+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}(x-x_*)}{\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}+\rho_2 u_2 v_2(x-x_*)}+f_2(x_*),\quad \forall x\ge x_*. \tag{4.1}
$$

Case 2: For $\underline{p}\neq\rho_2u_2^2$, set

$$\Delta=(\rho_2v_2^2+\rho_2u_2^2-\underline{p})\underline{p}(x-x_*)^2+\frac{2F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}[(\underline{p}-\rho_2u_2^2)f_2'(x_*)+\rho_2u_2v_2](x-x_*)+(\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}})^2.$$

(i) If $\underline{p}\leq\rho_2v_2^2+\rho_2u_2^2$, then $\Delta>0$ is always true. So for all $x\geq x_*$, we have

$$s_2(x)=\frac{-\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}-\rho_2u_2v_2(x-x_*)}{\underline{p}-\rho_2u_2^2}+f_2(x_*)+\frac{\sqrt{(\rho_2v_2^2+\rho_2u_2^2-\underline{p})\underline{p}(x-x_*)^2+\frac{2F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}[(\underline{p}-\rho_2u_2^2)f_2'(x_*)+\rho_2u_2v_2](x-x_*)+(\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}})^2}}{\underline{p}-\rho_2u_2^2}. \tag{4.2}$$

(ii) If $\underline{p}>\rho_2v_2^2+\rho_2u_2^2$, to guarantee $\Delta\geq 0$, we need that

$$x_*\leq x\leq x^{**}\triangleq x_*+\frac{F_2(x_*)}{\underline{p}\sqrt{1+f_2'(x_*)^2}}(\sqrt{\frac{\underline{p}}{\underline{p}-\rho_2u_2^2-\rho_2v_2^2}+[\frac{(\underline{p}-\rho_2u_2^2)f_2'(x_*)+\rho_2u_2v_2}{\rho_2v_2^2+\rho_2u_2^2-\underline{p}}]^2}-\frac{(\underline{p}-\rho_2u_2^2)f_2'(x_*)+\rho_2u_2v_2}{\rho_2v_2^2+\rho_2u_2^2-\underline{p}}), \tag{4.3}$$

and then

$$s_2(x)=\frac{-\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}-\rho_2u_2v_2(x-x_*)}{\underline{p}-\rho_2u_2^2}+f_2(x_*)+\frac{\sqrt{(\rho_2v_2^2+\rho_2u_2^2-\underline{p})\underline{p}(x-x_*)^2+\frac{2F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}[(\underline{p}-\rho_2u_2^2)f_2'(x_*)+\rho_2u_2v_2](x-x_*)+(\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}})^2}}{\underline{p}-\rho_2u_2^2}, \tag{4.4}$$

$(x_*\leq x\leq x^{**})$.

Meanwhile, from (3.41), we get

$$u(x^{**},s_2(x^{**}))=0, \tag{4.5}$$
$$v(x^{**},s_2(x^{**}))=\frac{-\rho_2u_2v_2(s_2(x^{**})-s_2(x_*))+(\rho_2v_2^2-\underline{p})(x^{**}-x_*)+\frac{F_2(x_*)f_2'(x_*)}{\sqrt{1+f_2'(x_*)^2}}}{\rho_2u_2(y_*-s_2(x^{**}))+\rho_2v_2x^{**}}.$$

One may check that the line $x=x^{**}$ could not be a concentration layer, while hyperbolicity (law of causality) forbids the layer to turn backwards to $x<x^{**}$. This means that the free concentration layer $y=s_2(x)$ should stop at the point $x=x^{**}$. In other words, the Radon measure solution terminates at a finite distance from the obstacle.

Similarly, we recall that $y=y_1(x)$ is given by (3.45) and $s_1(x)=y_1(x)+f_1(x_*)$.

Case 1: For $\underline{p}=\rho_1u_1^2$, solving (3.45) yields

$$s_1(x)=\frac{\underline{p}\sqrt{1+f_1'(x_*)^2}}{2F_1(x_*)}(x-x_*)^2+f_1'(x_*)(x-x_*)+f_1(x_*),\quad\forall x\geq x_*. \tag{4.6}$$

Case 2: For $\underline{p}\neq\rho_1u_1^2$, set

$$\Delta=(\rho_1u_1^2-\underline{p})\underline{p}(x-x_*)^2+\frac{2(\rho_1u_1^2-\underline{p})f_1'(x_*)F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}(x-x_*)+(\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}})^2.$$

(i) If $\underline{p}<\rho_1u_1^2$, then $\Delta>0$. Hence

$$s_1(x)=\frac{-\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}+\sqrt{(\rho_1u_1^2-\underline{p})\underline{p}(x-x_*)^2+\frac{2(\rho_1u_1^2-\underline{p})f_1'(x_*)F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}(x-x_*)+(\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}})^2}}{\rho_1u_1^2-\underline{p}} \tag{4.7}$$
$$+f_1(x_*),\qquad \forall x\geq x_*.$$

(ii) For $\underline{p}>\rho_1u_1^2$, $\Delta\geq 0$ holds if and only if

$$x_*\leq x\leq x^*=x_*+\frac{F_1(x_*)}{\underline{p}\sqrt{1+f_1'(x_*)^2}}(\sqrt{\frac{\underline{p}}{\underline{p}-\rho_1u_1^2}+f_1'(x_*)^2}-f_1'(x_*)), \tag{4.8}$$

and we get

$$s_1(x)=\frac{-\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}+\sqrt{(\rho_1u_1^2-\underline{p})\underline{p}(x-x_*)^2+\frac{2(\rho_1u_1^2-\underline{p})f_1'(x_*)F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}}(x-x_*)+(\frac{F_1(x_*)}{\sqrt{1+f_1'(x_*)^2}})^2}}{\rho_1u_1^2-\underline{p}} \tag{4.9}$$
$$+f_1(x_*),(x_*\leq x\leq x^*).$$

Notice (3.46), we get

$$u(x^*,s_1(x^*))=0,\quad v(x^*,s_1(x^*))=\frac{(\underline{p}-\rho_1u_1^2)F_1(x_*)\sqrt{\frac{\underline{p}}{\underline{p}-\rho_1u_1^2}+f_1'(x_*)^2}}{\rho_1u_1((\underline{p}-\rho_1u_1^2)f_1(x_*)\sqrt{1+f_1'(x_*)^2}+F_1(x_*))}. \tag{4.10}$$

Hence as above, when the curve $y=s_1(x)$ arrives at the point $x=x^*$, it will stay there and the measure solution does not exist beyond the point. □

For item (ii) in Case 2, as pointed in [26], to our knowledge, this was the first time people discovered such phenomena of Radon measure solutions. We do not know how to prolong the solution downstream further in a reasonable way. Perhaps the Euler equations finally fail to model such physical process.

In Section 3.2.2, $y=y_3(x)$ is given by (3.61) and then $s_3(x)=y_3(x)+s_2(x_*)$. We have the following conclusion about the free concentration layer $y=s_3(x)$ resulting from interactions of $y=s_1(x)$ and $y=s_2(x)$.

LEMMA 4.2. *When $\rho_1u_1^2=\rho_2u_2^2$, the two layers $y=s_i(x)$, $i=1,2$ intersect at a point $x=x_{**}$. A new free concentration layer $y=s_3(x)$ resulting from their interaction is a linear function.*

Proof. When $\rho_1u_1^2=\rho_2u_2^2$, (3.61) is an algebraic equation of first order of one variable, we can easily get the expression of $y=s_3(x)=y_3(x)+s_2(x_*)=-\frac{C}{B}+s_2(x_*)$, with B,C given by (3.62). □

When $\rho_1 u_1^2 \neq \rho_2 u_2^2$, (3.61) is a quadratic equation of one variable. If $\Delta \triangleq \sqrt{B^2-4AC} > 0$, then $y=y_3(x)$ is defined for all $x > x_{**}$.

LEMMA 4.3. *Under the assumptions of Lemma (3.1) and suppose that the pressure of the static gas behind the obstacle $\underline{p}$ is suitably small, then there exists a global Radon measure solution to the problem (2.1)– (2.4).*

Proof. For $\underline{p}=0$, recall that $y=s_1(x)$ and $y=s_2(x)$ were defined by (4.7) and (4.2) respectively. Meanwhile, the order of the growth rate of functions $y=s_1(x)$ and $y=s_2(x)$ are the same as that of functions $y=\sqrt{x}$ and $y=x$ respectively. So they will intersect at some point, see Figure 4.1. Due to continuity, we conclude that when $\underline{p}$ is small, $y=s_1(x)$ and $y=s_2(x)$ will intersect. Then the conclusion follows by Lemma 3.5. ∎

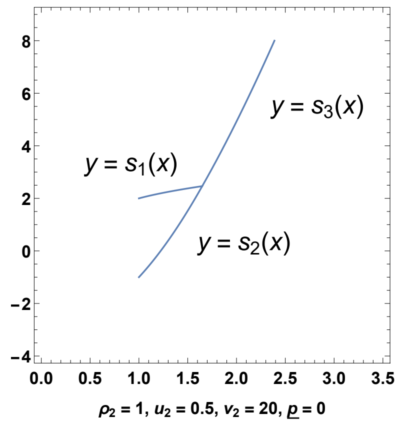

FIG. 4.1. *Two free layers intersect if there is vacuum or zero pressure gas between them.*

4.1. The influence of pressure of static gas on flow patterns. In the following, we will discuss the influence of general rather than sufficiently small pressure $\underline{p}$ in $\underline{\Omega}$ (or $\overline{\Omega}$) on the flow field behind the obstacle, and present some numerical examples. Without loss of generality, we suppose that $\rho_1=1, u_1=1, x_*=1, f_1(x)=2\sqrt{x}, f_2(x)=2x^2-3$. Hence $F_1(1)=\int_0^1 \frac{\frac{1}{\sqrt{t}}}{\sqrt{1+\frac{1}{t}}}\mathrm{d}t=2\sqrt{2}-2$, and $F_2(1)=\int_0^1 \frac{-16\rho_2u_2v_2t^2+4\rho_2(v_2^2-u_2^2)t+\rho_2u_2v_2}{\sqrt{1+16t^2}}\mathrm{d}t=-\frac{1}{8}[(-2+2\sqrt{17})u_2^2+(4\sqrt{17}-3\ln(4+\sqrt{17}))u_2v_2-2(-1+\sqrt{17})v_2^2]\rho_2$. By virtue of (4.1), (4.2), (4.4), (4.6), (4.7), (4.9) and (3.61), we analyse the relations between $y=s_1(x)$, $y=s_2(x)$ and $y=s_3(x)$. We will discuss the details about the intersection of $s_1(x)$ and $s_2(x)$ case by case.

4.1.1. The cases without termination of free layers. As indicated in the paragraph after the proof of Lemma 4.3, to avoid termination of free concentration layer and get a global solution, we consider some cases under the restriction that $0<\underline{p}\leq \min\{\rho_1u_1^2, \rho_2u_2^2+\rho_2v_2^2\}$. In this situation, there are four cases depending on the expressions for $s_1(x)$ and $s_2(x)$, which are defined by (4.6) (or (4.7)) and (4.1) (or (4.2)) respectively.

Case 1: If $\underline{p}=\rho_2u_2^2$ and $\frac{1}{2}(\frac{v_2}{u_2}-\frac{u_2}{v_2})>\sqrt{\frac{\rho_2u_2^2}{\rho_1u_1^2-\rho_2u_2^2}}$, then $y=s_1(x)$ and $y=s_2(x)$ defined by (4.7) and (4.1) respectively will intersect. In fact, the order of the growth rate of functions $y=s_1(x)$ and $y=s_2(x)$ are the same as that of functions $y=\sqrt{\frac{\rho_2u_2^2}{\rho_1u_1^2-\rho_2u_2^2}}x$ and $y=\frac{1}{2}(\frac{v_2}{u_2}-\frac{u_2}{v_2})x$ respectively. Since $\frac{1}{2}(\frac{v_2}{u_2}-\frac{u_2}{v_2})>\sqrt{\frac{\rho_2u_2^2}{\rho_1u_1^2-\rho_2u_2^2}}$, then they will intersect at some point. Here we will give an example of this case as in Figure 4.2.

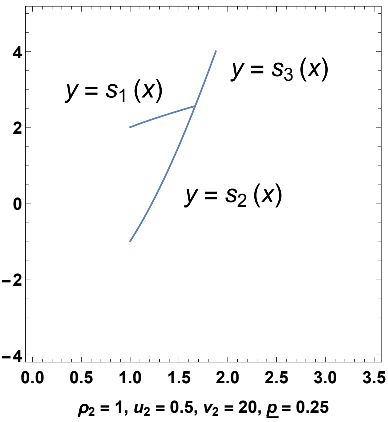

FIG. 4.2.

Case 2: If $\underline{p}\neq\rho_2u_2^2$ and $\frac{\sqrt{(\rho_2u_2^2+\rho_2v_2^2-\underline{p})\underline{p}}-\rho_2u_2v_2}{\underline{p}-\rho_2u_2^2}>\sqrt{\frac{\underline{p}}{\rho_1u_1^2-\underline{p}}}$, then $y=s_1(x)$ and $y=s_2(x)$ defined by (4.7) and (4.2) respectively will intersect. In fact, note that the order of the growth rate of functions $y=s_1(x)$ and $y=s_2(x)$ are the same as that of $y=\sqrt{\frac{\underline{p}}{\rho_1u_1^2-\underline{p}}}x$ and $y=\frac{\sqrt{(\rho_2u_2^2+\rho_2v_2^2-\underline{p})\underline{p}}-\rho_2u_2v_2}{\underline{p}-\rho_2u_2^2}x$. In view of $\frac{\sqrt{(\rho_2u_2^2+\rho_2v_2^2-\underline{p})\underline{p}}-\rho_2u_2v_2}{\underline{p}-\rho_2u_2^2}>\sqrt{\frac{\underline{p}}{\rho_1u_1^2-\underline{p}}}$, they will intersect at some point. See Figure 4.3 below for an example.

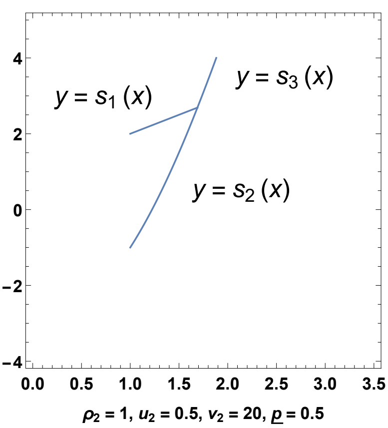

FIG. 4.3.

Case 3: If $\underline{p}=\rho_1u_1^2=\rho_2u_2^2$, then $y=s_1(x)$ and $y=s_2(x)$ are defined by (4.6) and (4.1) respectively. Meanwhile, the orders of the growth rate of functions $y=s_1(x)$ and $y=s_2(x)$ are the same as that of functions $y=\frac{\underline{p}\sqrt{1+f_1'(x_*)^2}}{2F_1(x_*)}x^2$ and $y=\frac{1}{2}(\frac{v_2}{u_2}-\frac{u_2}{v_2})x$ respectively. Figure 4.4 shows an example of this case.

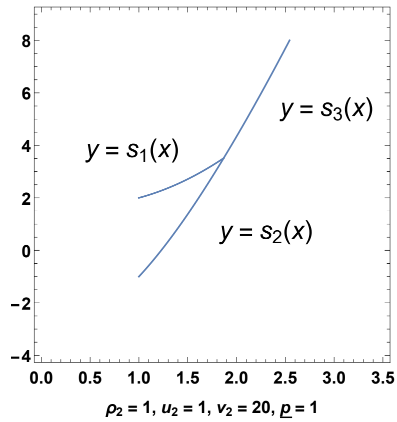

FIG. 4.4.

Case 4: If $\underline{p}=\rho_1u_1^2\neq\rho_2u_2^2$, then $y=s_1(x)$ and $y=s_2(x)$ are defined by (4.6) and (4.2). Meanwhile, the order of the growth rate of functions $y=s_1(x)$ and $y=s_2(x)$ are the same as that of functions $y=\frac{\underline{p}\sqrt{1+f_1'(x_*)^2}}{2F_1(x_*)}x^2$ and $y=\frac{\sqrt{(\rho_2u_2^2+\rho_2v_2^2-\underline{p})\underline{p}}-\rho_2u_2v_2}{\underline{p}-\rho_2u_2^2}x$ respectively. Here is an example of this case, see Figure 4.5.

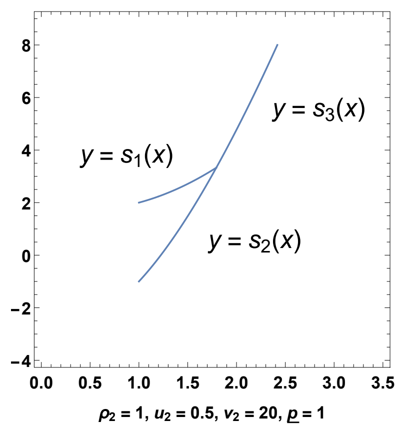

FIG. 4.5.

4.1.2. The cases for possible terminations of free layers. To get a global solution, the free layers shall interact before one of them terminates.

Case 1: $\underline{p}>\rho_1u_1^2$ and $\underline{p}\leq\rho_2u_2^2+\rho_2v_2^2$. In this case, $y=s_1(x)$ is defined by (4.9) and it terminates at $x=x^*$ and $y=s_2(x)$ is defined by (4.1) (or (4.2)). If $y=s_2(x)$ intersect $y=s_1(x)$ at some point to the left of $x=x^*$, there exists a global solution. Here we will give two examples.

(1) If $\underline{p}=\rho_2u_2^2$, $y=s_1(x)$ and $y=s_2(x)$ are defined by (4.9) and (4.1) respectively, see Figure 4.6.

(2) If $\underline{p}\neq\rho_2u_2^2$, $y=s_1(x)$ and $y=s_2(x)$ are defined by (4.9) and (4.2) respectively, see Figure 4.7.

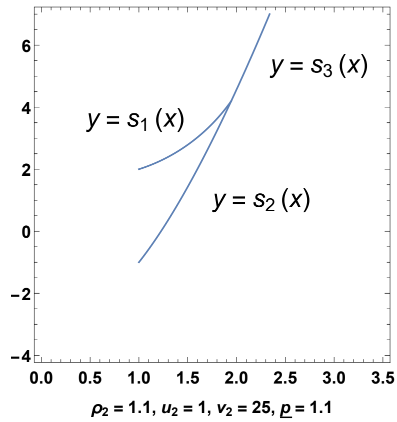

FIG. 4.6.

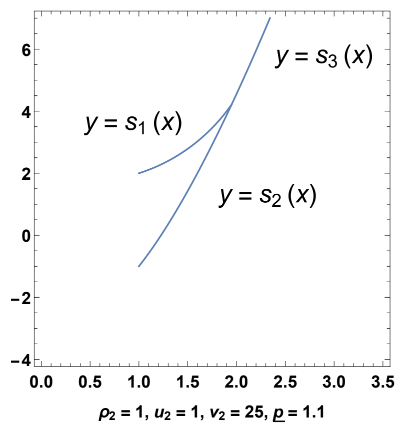

FIG. 4.7.

Case 2: $\underline{p} > \rho_2 u_2^2 + \rho_2 v_2^2$. For this case, $y = s_2(x)$ is defined by (4.4) and terminates at $x = x^{**}$. If $x_* > -\frac{F_2(x_*)((\underline{p}-\rho_2u_2^2)f_2'(x_*)+\rho_2u_2v_2)}{(\rho_2v_2^2+\rho_2u_2^2-\underline{p})\underline{p}\sqrt{1+f_2'(x_*)^2}}$, then $(\rho_2 v_2^2 + \rho_2 u_2^2 - \underline{p})\underline{p}(x - x_*)^2 + \frac{2F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}}[(\underline{p} - \rho_2 u_2^2) f_2'(x_*) + \rho_2 u_2 v_2](x - x_*) + (\frac{F_2(x_*)}{\sqrt{1+f_2'(x_*)^2}})^2$ is a decreasing function when $x > x_*$. Thus $y = s_2(x)$ is a decreasing function and $y = s_1(x)$ is an increasing function when $x > x_*$. So they will never intersect and there is no global solution.

4.2. The main theorem. Finally, by Lemmas 3.1, 3.2, 3.4, 3.5, 4.3, 4.1, we could summarize the main results of the paper in the following theorem:

THEOREM 4.1. *For the planar hypersonic-limit flow past a finite obstacle with surfaces $y = f_i(x)$ $(i = 1,2)$, $0 \le x \le x_*$ satisfying $f_2'(x) > f_1'(x) \ge 0$, $f_2(x) \le f_1(x)$, $f_1(0) = 0$, $y_* = f_2(0)$ and the upcoming flow (2.4) satisfying (2.5), there exists a Radon measure solution to the problem (2.1)– (2.4), which contains weighted Dirac measures supported on the boundary of the obstacle and beyond the obstacle on two free layers. Whether either of the two free layers terminate in a finite distance to the obstacle, and the two intersect or not, depends on the pressure $\underline{p}$ of static gas behind the obstacle. Particularly for*

$\underline{p}\geq 0$ *small, the two free layers* $y=s_1(x)$ *and* $y=s_2(x)$ *will intersect and there exists a global solution. But when* $\underline{p}$ *becomes larger and larger, they will terminate rather than intersect. Moreover, we get the equations of* $y=s_i(x)$ $(i=1,2,3)$ *and at point* $x=x_{**}$*, the relations between the weight* $\widetilde{\omega_r^{t3}(x)}$ *of* $y=s_3(x)$ *and the weights* $\widetilde{\omega_r^{t1}(x)}$*,* $\widetilde{\omega_r^{t2}(x)}$ *of* $y=s_1(x)$ *and* $y=s_2(x)$*, respectively* $(r=m,n,t=0,1,2,3)$ *are as described in Lemma 3.3.*

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