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NON-UNIQUENESS OF TRANSONIC SHOCK SOLUTIONS TO NON-ISENTROPIC EULER-POISSON SYSTEM*

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Abstract. In this paper, we study the non-isentropic Euler-Poisson system and the non-uniqueness of transonic shock solutions is obtained. More precisely, prescribing a class of physical boundary conditions on the boundary of a flat nozzle with finite length, we prove that there exist two and only two transonic shocks. This is motivated by the result of existence of multiple transonic shock solutions for isentropic Euler-Poisson system (Tao Luo, Zhouping Xin, Commun. Math. Sci., 10:419–462, 2012). Moreover, the monotonicity with a threshold between the location of the transonic shock and the density at the exit of the nozzle is established.

Keywords. Euler-Poisson system; non-isentropic; non-uniqueness; transonic shocks.

AMS subject classifications. 35A02; 35Q35; 35L67.

1. Introduction

Considering the non-isentropic Euler-Poisson system

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ (\rho u)_t+(\rho u^2+p(\rho,S))_x=\rho E, \\ (\rho\varepsilon)_t+(\rho u\mathcal{E}+up)_x=\rho uE, \\ E_x=\rho-b, \end{cases} \tag{1.1}$$

where S is the specific entropy, the pressure p and the energy density $\mathcal{E}$ are given by

$$p=e^S\rho^\gamma, \quad \mathcal{E}=\frac{|u|^2}{2}+\frac{p}{(\gamma-1)\rho}. \tag{1.2}$$

Several physical flows including the propagation of electrons in submicron semiconductor devices and plasma, and the biological transport of ions for channel proteins are modeled as one-dimensional Euler-Poisson system. For instance, in the hydrodynamical model of semiconductor devices or plasma, ρ, u and p represent the average electron density, particle velocity and pressure, respectively. E is the electric field, which is generated by the Coulomb force of particles. $b>0$ stands for the density of fixed, positively charged background ions. The biological model describes the transport of ions between the extracellular side and the cytoplasmic side of the membranes. In this case, ρ, ρu and E are the concentration, the ion's translational mass and the electric field, respectively.

This paper is concerned with the non-uniqueness of transonic shock solutions for the following one-dimensional steady Euler-Poisson system:

$$\begin{cases} (\rho u)_x=0, \\ (\rho u^2+p(\rho,S))_x=\rho E, \\ (\rho u\mathcal{E}+up)_x=\rho uE, \\ E_x=\rho-b, \end{cases} \tag{1.3}$$

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Assuming that p satisfies

$$p(0)=0, \quad p'(\rho)>0, \quad p''(\rho)>0, \text{ for } \rho>0, \quad p(+\infty)=+\infty. \tag{1.4}$$

For the fixed L, which denotes the length of the nozzle, the boundary conditions for (1.1) in $0\leq x\leq L$ are given as follows

$$(\rho,E,S)(0)=(\rho_0,E_0,S_0), \quad \rho(L)=\rho_e. \tag{1.5}$$

We call $c=\sqrt{p_\rho(\rho,S)}$ the sound speed. Correspondingly, there is a unique solution $\rho=\rho_s$, which is the sonic state, when $p'(\rho)\rho^2=J^2$. In this case, if

$$p'(\rho)\rho^2<J^2, \text{i.e.}, \rho<\rho_s, \tag{1.6}$$

we call the flow supersonic. If

$$p'(\rho)\rho^2>J^2, \text{i.e.}, \rho>\rho_s, \tag{1.7}$$

the flow is called subsonic.

As is well known, the transonic shock problem has always been a hot topic among researchers. In [7], Courant and Friedrichs described the following transonic shock phenomena in a de Laval nozzle: Given the appropriately large receiver pressure p_r, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. So it is expected that a curved transonic shock is still formed in a nozzle when the end pressure p_r varies and lies in an appropriate scope. Later, J. Li, Z. P. Xin and H. C. Yin solved this problem for the two-dimensional steady Euler system with a variable exit pressure in a nozzle whose divergent part is an angular sector in [13]. They studied the transonic shock problem in a nozzle when the given variable end pressure at the exit of the nozzle lies in an appropriate scope. They also in [12] worked on the uniqueness problem of 3-D transonic shock solution in a conic nozzle when the variable end pressure in the diverging part of the nozzle lies in an appropriate scope. For steady isentropic irrotational inviscid potential flows, the uniqueness of a solution with a transonic shock in a duct was considered by G. Q. Chen, H. R. Yuan in [5]. In particular, they investigated a finite duct with the uniform supersonic flow at one entrance of the duct and the subsonic flow at the exit of the duct.

Moreover, there are some results concerning transonic shock solutions for Euler-Poisson system. Significant progress on transonic shock solutions was made by T. Luo and Z. P. Xin, they constructed detailed structure of the solutions to the boundary value problem for Euler-Poisson system in [9]. According to the cases of different boundary conditions and physical interval length L, they established the existence, non-existence, uniqueness, and non-uniqueness of transonic shock solutions. This inspires us to pay particular attention to the non-uniqueness of the non-isentropic issue in the present paper. Furthermore, T. Luo, J. Ranch, C. J. Xie and Z. P. Xin investigated the structural and dynamical stabilities of steady shock solutions for one-dimensional Euler-Poisson systems in [11], provided that the electric field is positive at the shock location. Besides the transonic flows, the research on subsonic flows for Euler-Poisson system has also made important achievements. In [1], M. Bae, B. Duan and C. J. Xie established the existence and stability of subsonic potential flow for the steady Euler-Poisson system in a multidimensional nozzle of a finite length, when prescribing the electric potential difference on a non-insulated boundary from a fixed point at the exit and the pressure at the exit of the nozzle. Later, M. Bae and S. K. Weng addressed the structural stability of 3-D axisymmetric subsonic flows with nonzero swirl for the steady compressible Euler-Poisson system in [3], by introducing a special Helmholtz decomposition. Recently, S. K. Weng provided a deformation-curl-Poisson decomposition for the three dimensional steady Euler-Poisson system in [14]. In addition, the structural stability of 1-D subsonic background solutions with multidimensional perturbations on the entrance and exit of a rectangular cylinder was established. For more interesting results of transonic shock solutions, one may refer to [16]. It deserves to be mentioned that there are also several works focusing on the transonic shock solutions and subsonic flows for Euler system [2, 4, 6, 8, 10, 15]. Motivated by the works on the non-uniqueness of transonic shock solutions for the isentropic Euler-Poisson system in [9], we extend their results to the non-isentropic case.

This paper is organized as follows. In Section 2, we estimate the derivatives of density and entropy respectively with respect to the shock location, which are used to determine the monotonicity. In Section 3, we present a thorough proof of the non-uniqueness of transonic shock solutions for non-isentropic Euler-Poisson system (1.3) by means of analysing the different monotonicity in different intervals, the crucial ingredient of this paper.

2. Preliminary formulas of the problem

In this section, we employ the R-H conditions to derive the derivatives of density and entropy of subsonic flow with respect to the shock location, as well as the relationship between density of subsonic flow and supersonic flow, the real worth of which is later revealed. The exact structure of these formulas will be needed below, when we prove monotonicity.

To follow the subsequent discussion, it is necessary to focus on some conditions that matter in the paper. Suppose the shock location is at a point $a\in(0,L)$, we have the following Rankine-Hugoniot conditions

$$\begin{cases} \left(p(\rho,S)+\frac{J^2}{\rho}\right)(a_-)=\left(p(\rho,S)+\frac{J^2}{\rho}\right)(a_+), \\ \left(\frac{J^2}{2\rho^2}+\frac{\gamma p(\rho,S)}{(\gamma-1)\rho}\right)(a_-)=\left(\frac{J^2}{2\rho^2}+\frac{\gamma p(\rho,S)}{(\gamma-1)\rho}\right)(a_+), \\ E(a_-)=E(a_+), \end{cases} \tag{2.1}$$

and entropy condition

$$p(\rho(a_+),S_+)>p(\rho(a_-),S_-). \tag{2.2}$$

When the solutions of (1.3) have no singularity, the entropy S is a constant along the stream line. Therefore, the smooth solutions of (1.3) can be analyzed in (ρ, E)-phase plane. Any trajectory in (ρ, E)-phase plane satisfies the following equation:

$$d\left(\frac{1}{2}E^2-H(\rho,S)\right)=0, \text{ where } H_\rho^{'}(\rho,S)=\frac{\rho-b}{\rho}\left(p_\rho^{'}(\rho,S)-\frac{J^2}{\rho^2}\right). \tag{2.3}$$

The trajectory passing through the point (ρ_0,E_0) with ρ_0 is given by

$$\frac{1}{2}E^2-\int_{\rho_0}^{\rho}H_\tau^{'}(\tau,S)d\tau=\frac{1}{2}E_0^2. \tag{2.4}$$

For any $\rho\in(0,\rho_s)$, there exists one and only one $F(\rho)$ satisfying the Rankine-Hugoniot condition (2.1), i.e., the state (ρ,E) and $(F(\rho),E)$ can be connected by a transonic shock. Then we define

$$\mathscr{B}=\{(\rho,E):\rho<\rho_s,E>0,\frac{1}{2}E^2-H(\rho)>-H(F^{-1}(b))\}.$$

The definition of transonic shock solutions for the boundary value problem (1.3) and (1.5) is given as follows.

Definition 2.1. *A piecewise smooth solution (ρ,E,S) with $\rho>0$ to the boundary value problem (1.3) and (1.5) is said to be a transonic shock if it is separated by a discontinuity located at $x^*\in(0,L)$, and of the form*

$$(\rho,E,S)=\begin{cases}(\rho_-,E_-,S_-)(x), & 0<x<x^*,\\ (\rho_+,E_+,S_+)(x), & x^*<x<L,\end{cases}$$

satisfying the Rankine-Hugoniot condition

$$\begin{cases}\left(p(\rho_-,S_-)+\frac{J^2}{\rho_-}\right)(x^*)=\left(p(\rho_+,S_+)+\frac{J^2}{\rho_+}\right)(x^*),\\ \left(\frac{J^2}{2\rho_-^2}+\frac{\gamma p(\rho_-,S_-)}{(\gamma-1)\rho_-}\right)(x^*)=\left(\frac{J^2}{2\rho_+^2}+\frac{\gamma p(\rho_+,S_+)}{(\gamma-1)\rho_+}\right)(x^*),\\ E_-(x^*)=E_+(x^*).\end{cases}\tag{2.5}$$

and the Lax's entropy condition

$$\rho_+(x^*)>\rho_-(x^*).\tag{2.6}$$

Furthermore, the solution (ρ,E,S) satisfies the boundary value condition (1.5). Moreover, the shock is transonic if

$$\rho(a_+)>\rho_s>\rho(a_-).$$

For given positive constants J, L, S_0, $\gamma>1$, let $\rho_{min}=\left(\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{\gamma+1}}\rho_s<\rho_s$ satisfying

$$\frac{1}{2}E(\rho_{min})^2-\int_{\rho_s}^{\rho_{min}}H_\tau(\tau,S_0)d\tau=0.$$

Then it follows from $u_{max}=\frac{J}{\rho_{min}}$ that

$$M_{max}^2=\frac{u_{max}^2}{c^2(\rho_{min})}=\frac{J^2}{\gamma e^{S_0}\rho_{min}^{\gamma+1}}=\left(\frac{\rho_s}{\rho_{min}}\right)^{\gamma+1}=\frac{2\gamma}{\gamma-1},$$

here we introduce the Mach number $M=\frac{u}{c}$.

Next, elementary calculations give the derivative formulas of $\rho_+(x^*)$ and $S_+(x^*)$ with respect to x^* respectively, as well as the relation between $\rho_+(x^*)$ and $\rho_-(x^*)$. Indeed, it follows from the first equation in R-H conditions (2.5) that,

$$e^{S_+(x^*)}=\frac{1}{\rho_+^\gamma(x^*)}\left(e^{S_-(x^*)}\rho_-^\gamma(x^*)+\frac{J^2}{\rho_-^2(x^*)}-\frac{J^2}{\rho_+^2(x^*)}\right),$$

put it into $(2.5)_2$ gives

$$\left(\frac{J^2}{2\rho_-^2}+\frac{\gamma e^{S_-}\rho_-^\gamma}{(\gamma-1)\rho_-}\right)(x^*)=\left(\frac{J^2}{2\rho_+^2}+\frac{\gamma e^{S_-}\rho_-^\gamma}{(\gamma-1)\rho_+}+\frac{\gamma}{\gamma-1}\frac{J^2}{\rho_-\rho_+}-\frac{\gamma}{\gamma-1}\frac{J^2}{\rho_+^2}\right)(x^*).$$

Multiplying it by $\rho_+^2(x^*)$ implies

$$\frac{\gamma}{\gamma-1}e^{S_-}\rho_-^\gamma(x^*)\rho_+(x^*)+\frac{\gamma}{\gamma-1}\frac{J^2}{\rho_-(x^*)}\rho_+(x^*)-\frac{u_-^2\rho_+^2(x^*)}{2}-\frac{c_-^2\rho_+^2(x^*)}{\gamma-1}=(\frac{\gamma}{\gamma-1}-\frac{1}{2})J^2,$$

$$(\frac{1}{2}u_-^2+\frac{c_-^2}{\gamma-1})\rho_+^2(x^*)-\frac{\gamma}{\gamma-1}\left(e^{s_-}\rho_-^\gamma(x^*)+\frac{J^2}{\rho_-(x^*)}\right)\rho_+(x^*)+(\frac{\gamma}{\gamma-1}-\frac{1}{2})J^2=0.$$

Then it can be rewritten as

$$c_-^2(\frac{1}{2}M_-^2+\frac{1}{\gamma-1})\rho_+(x^*)^2-\rho_-(x^*)c_-^2(\frac{1}{\gamma-1}+\frac{\gamma}{\gamma-1}M_-^2)\rho_+(x^*)+(\frac{\gamma}{\gamma-1}-\frac{1}{2})J^2=0. \tag{2.7}$$

Hence, it holds that

$$\rho_+(x^*)=\frac{(\gamma+1)M_-^2}{(\gamma-1)M_-^2+2}\rho_-(x^*). \tag{2.8}$$

This, combined with $M_-^2(x^*)<M_{max}^2=\frac{2\gamma}{\gamma-1}$, gives

$$\frac{\rho_+(x^*)}{\rho_-(x^*)}<\frac{\gamma}{\gamma-1}, \tag{2.9}$$

a relation formula which will be used later. Differentiating $(2.5)_1$ and $(2.5)_2$ with respect to x^* yields

$$\begin{aligned}&\left(\gamma e^{S_-}\rho_-^{\gamma-1}(x^*)-\frac{J^2}{\rho_-^2(x^*)}\right)\frac{d\rho_-(x^*)}{dx^*}\\ =&\left(\gamma e^{S_+}\rho_+^{\gamma-1}(x^*)-\frac{J^2}{\rho_+^2(x^*)}\right)\frac{d\rho_+(x^*)}{dx^*}+e^{S_+}\rho_+^\gamma(x^*)\frac{dS_+(x^*)}{dx^*}\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}&\left(\gamma e^{S_-}\rho_-^{\gamma-2}(x^*)-\frac{J^2}{\rho_-^3(x^*)}\right)\frac{d\rho_-(x^*)}{dx^*}\\ =&\left(\gamma e^{S_+}\rho_+^{\gamma-2}(x^*)-\frac{J^2}{\rho_+^3(x^*)}\right)\frac{d\rho_+(x^*)}{dx^*}+\frac{\gamma e^{S_+}\rho_+^{\gamma-1}(x^*)}{\gamma-1}\frac{dS_+(x^*)}{dx^*}.\end{aligned} \tag{2.11}$$

In addition, one has

$$\frac{d\rho_-(x^*)}{dx^*}=\frac{\rho_-(x^*)E_-(x^*)}{\gamma e^{S_-(x^*)}\rho_-^{\gamma-1}(x^*)-\frac{J^2}{\rho_-^2(x^*)}}, \tag{2.12}$$

combining (2.10), (2.11) and (2.12), we derive

$$\frac{d\rho_+(x^*)}{dx^*}=\frac{\gamma\rho_-(x^*)E_-(x^*)-(\gamma-1)\rho_+(x^*)E_+(x^*)}{\gamma e^{S_+(x^*)}\rho_+^{\gamma-1}(x^*)-\frac{J^2}{\rho_+^2(x^*)}} \tag{2.13}$$

and

$$\frac{dS_+(x^*)}{dx^*}=\frac{(\gamma-1)(\rho_+(x^*)-\rho_-(x^*))E_-(x^*)}{e^{S_+(x^*)}\rho_+^\gamma(x^*)}. \tag{2.14}$$

By virtue of (2.9) and the entropy condition $\rho_+(x^*) > \rho_-(x^*)$, we obtain

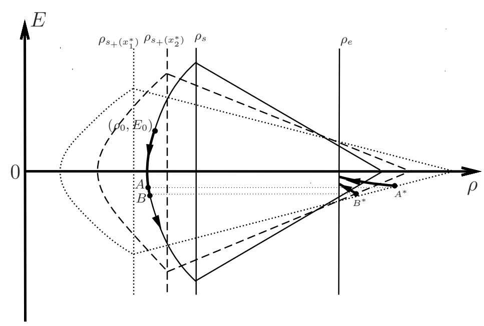
$$\frac{d\rho_+(x^*)}{dx^*} < 0 \tag{2.15}$$

and

$$\frac{dS_+(x^*)}{dx^*} < 0 \tag{2.16}$$

respectively.

The transonic shock solutions of problem (1.3) and (1.5) can be analyzed in the following (ρ, E)-phase plane. For any (ρ_0, E_o, S_0), there exist two states $(\rho_-(x_1^*), E_-(x_1^*), S_-)$ and $(\rho_-(x_2^*), E_-(x_2^*), S_-)$ at shock locations $x = x_1^*$ and $x = x_2^*$, respectively. Because of the change of entropy, the phase plane changes (See Figure 2.1). Across the shock at x_1^* and x_2^*, the trajectory separately jumps from A to A^* and B to B^*. It's worth noting that, for the two new sonic states $\rho_{s_+}(x_1^*)$ and $\rho_{s_+}(x_2^*)$, the trajectory starting from A^* and B^* respectively intersects the line $\rho = \rho_e$ at two different points.



$A: \ (\rho_-(x_1^*), E_-(x_1^*), S_-),$ $\qquad A^*: \ (\rho_+(x_1^*), E_+(x_1^*), S_+(x_1^*)).$

$B: \ (\rho_-(x_2^*), E_-(x_2^*), S_-),$ $\qquad B^*: \ (\rho_+(x_2^*), E_+(x_2^*), S_+(x_2^*)).$

FIG. 2.1. *Phase portrait of transonic shock solutions*

3. The non-uniqueness of transonic shock solutions

In this section, we present the main result in this paper. We will prove the existence of two transonic solutions for the non-isentropic Euler-Poisson system (1.3) with prescribing physical boundary conditions and the fixed length of nozzle. Let $\ell((\rho_1, E_1); (\rho_2, E_2); S)$ be the length for the trajectory of (1.3) to travel from the state $(\rho_1, E_1; S)$ to the state $(\rho_2, E_2; S)$ when $(\rho_2, E_2; S)$ and $(\rho_1, E_1; S)$ are on the same trajectory and $T(\rho_0, E_0; S)$ denotes the trajectory passing through the state $(\rho_0, E_0; S)$. An important property of these trajectories is displayed in the following lemma.

LEMMA 3.1. *Suppose that the two states $(\rho_1, E_1; S)$ and $(\rho_2, E_2; S)$ are on the same trajectory of system (1.1), i.e., $(\rho_2, E_2; S) \in T(\rho_1, E_1; S)$. Then, if on the trajectory connecting these two states, E does not change sign (then E is a function of ρ, denoted by $E(\rho)$),*

$$\ell((\rho_1, E_1); (\rho_2, E_2); S) = \int_{\rho_1}^{\rho_2} \frac{p'_\rho(\rho, S) - \frac{J^2}{\rho^2}}{\rho E(\rho)} d\rho. \tag{3.1}$$

Proof. It follows from $(1.3)_2$ that $\frac{p'_\rho(\rho,S)-\frac{J^2}{\rho^2}}{\rho E}d\rho=dx$, which yields (3.1). ∎

In the following, we consider the existence of two transonic shock solutions for the case of $b>\rho_s$ in [9], the key ingredient of which is to establish the monotonic property of the end density on the position of the shock.

LEMMA 3.2. *For given positive constants J, L, S_0, $\gamma>1$ and $(\rho_0,E_0)\in\mathscr{B}$, there exists a unique shock location $\hat{x}^*$, s.t.*

$$\lim_{x^*\to\underline{x}^*}\frac{\partial\rho_+(L)}{\partial x^*}=-\infty,\qquad \begin{cases}\frac{\partial\rho_+(L)}{\partial x^*}<0, \text{ for } \underline{x}^*<x^*<\hat{x}^*,\\ \frac{\partial\rho_+(L)}{\partial x^*}>0, \text{ for } \hat{x}^*<x^*<L.\end{cases} \tag{3.2}$$

So

$$\rho_+(L)(\hat{x}^*)=\min_{\underline{x}^*\le x^*\le L}\rho_+(L)(x^*), \tag{3.3}$$

where $\underline{x}^*$ satisfies $E_+(L,\underline{x}^*)=0$.

Proof. The proof is divided into three steps. To obtain the monotonicity of $\rho_+(L)$ with respect to x^*, it is necessary to determine the representation formula of $\frac{\partial\rho_+(L)}{\partial x^*}$ at first.

Step 1. Note that

$$x^*+\int_{\rho_+(x^*)}^{\rho_+(L)}\frac{\gamma e^{S_+}\tau^{\gamma-1}-\frac{J^2}{\tau^2}}{\tau E_+(\tau,x^*)}d\tau=L. \tag{3.4}$$

With respect to x^*, differentiating (3.4) yields

$$\begin{aligned}1&+\frac{h(\rho_+(L),S_+)}{\rho_+(L)E_+(L)}\frac{\partial\rho_+(L)}{\partial x^*}-\frac{h(\rho_+(x^*),S_+)}{\rho_+(x^*)E_+(x^*)}\frac{\partial\rho_+(x^*)}{\partial x^*}\\&+\int_{\rho_+(x^*)}^{\rho_+(L)}\frac{\gamma e^{S_+}\tau^{\gamma-1}}{\tau E_+(\tau,x^*)}\frac{\partial S_+}{\partial x^*}d\tau-\int_{\rho_+(x^*)}^{\rho_+(L)}\frac{\gamma e^{S_+}\tau^{\gamma-1}-\frac{J^2}{\tau^2}}{\tau E_+^2(\tau,x^*)}\frac{\partial E_+(\tau,x^*)}{\partial x^*}d\tau=0,\end{aligned} \tag{3.5}$$

where $h(\tau,S)=\gamma e^S\tau^{\gamma-1}-\frac{J^2}{\tau^2}$, and $E_+(\tau,x^*)$ satisfies

$$\begin{aligned}\frac{1}{2}E_+^2(\tau,x^*)&=\frac{1}{2}E_+^2(x^*)+\int_{\rho_+(x^*)}^{\tau}H_t(t,S_+)dt\\&=\frac{1}{2}E_+^2(x^*)+\int_{\rho_+(x^*)}^{\tau}\frac{t-b}{t}(\gamma e^{S_+}t^{\gamma-1}-\frac{J^2}{t^2})dt.\end{aligned} \tag{3.6}$$

Differentiating (3.6) with respect to x^* gives

$$\begin{aligned}E_+(\tau,x^*)\frac{\partial E_+(\tau,x^*)}{\partial x^*}&=E_-(x^*)\frac{dE_-(x^*)}{dx^*}-\frac{(\rho_+(x^*)-b)h(\rho_+(x^*),S_+)}{\rho_+(x^*)}\frac{d\rho_+(x^*)}{dx^*}\\&\quad+\int_{\rho_+(x^*)}^{\tau}\frac{t-b}{\tau}\gamma e^{S_+}t^{\gamma-1}dt\cdot\frac{dS_+(x^*)}{dx^*}\\&=\frac{(\rho_-(x^*)-b)h_-(\rho_-(x^*),S_-)}{\rho_-(x^*)}\frac{\rho_-(x^*)E_-(x^*)}{h_-(\rho_-(x^*),S_-)}\\&\quad-\frac{(\rho_+(x^*)-b)h_+(\rho_+(x^*),S_+)}{\rho_+(x^*)}\cdot\frac{(\gamma\rho_--(\gamma-1)\rho_+)E_-(x^*)}{h_+(\rho_+(x^*),S_+)}\\&\quad+\int_{\rho_+(x^*)}^{t}\frac{t-b}{t}\gamma e^{S_+(x^*)}t^{\gamma-1}dt\cdot\frac{dS_+(x^*)}{dx^*}.\end{aligned}$$

Applying $\rho_+(x^*)=\frac{(\gamma+1)M_-^2}{(\gamma-1)M_-^2+2}\rho_-(x^*)$ yields

$$
\begin{aligned}
&\gamma\rho_-(x^*)E_-(x^*)-(\gamma-1)\rho_+(x^*)E_-(x^*)\\
=&\gamma\rho_-(x^*)E_-(x^*)-(\gamma-1)\frac{(\gamma+1)M_-^2}{(\gamma-1)M_-^2+2}\rho_-(x^*)E_-(x^*)\\
=&\rho_-(x^*)E_-(x^*)\cdot\frac{\frac{2\gamma}{(\gamma-1)}-M_-^2}{M_-^2+\frac{2}{(\gamma-1)}}.
\end{aligned}
$$

Then it holds that

$$
\begin{aligned}
&E_+(\tau,x^*)\frac{\partial E_+(\tau,x^*)}{\partial x^*}\\
=&\left((1-\frac{b}{\rho_-})-(1-\frac{b}{\rho_+})\frac{\frac{2\gamma}{\gamma-1}-M^2}{M^2+\frac{2}{\gamma-1}}\right)\rho_-(x^*)E_-(x^*)+\int_{\rho_+(x^*)}^{\tau}\frac{t-b}{t}\gamma e^{S_+}t^{\gamma-1}dt\cdot\frac{dS_+(x^*)}{dx^*}\\
=&\left(1-\frac{\rho_-(x^*)}{\rho_+(x^*)}\right)\Big((\gamma-1)\rho_+(x^*)-\gamma b\Big)E_-(x^*)+\int_{\rho_+(x^*)}^{t}\frac{t-b}{t}\gamma e^{S_+}\tau^{\gamma-1}dt\cdot\frac{dS_+(x^*)}{dx^*}.
\end{aligned}
\tag{3.7}
$$

By virtue of $\left(1-\frac{\rho_-(x^*)}{\rho_+(x^*)}\right)((\gamma-1)\rho_+(x^*)-\gamma b)<0$, a result of (2.9) and $b>\rho_-(x^*)$, one has

$$
\frac{\partial E_+(\tau,x^*)}{\partial x^*}<0. \tag{3.8}
$$

Substituting (3.7) into (3.5), one derives

$$
\frac{\partial\rho_+(L)}{\partial x^*}=\frac{\rho_+(L)E_+(L)}{h(\rho_+(L),S_+)}Q(x^*), \tag{3.9}
$$

where

$$
\begin{aligned}
Q(x^*)=&\frac{h(\rho_+(x^*),S_+)}{\rho_+(x^*)E_+(x^*)}\frac{\partial\rho_+(x^*)}{\partial x^*}-1\\
&+\int_{\rho_+(x^*)}^{\rho_+(L)}\frac{h(\tau,S_+)E_-(x^*)}{\tau E_+^3(\tau,x^*)}\left(1-\frac{\rho_-(x^*)}{\rho_+(x^*)}\right)\Big((\gamma-1)\rho_+(x^*)-\gamma b\Big)d\tau\\
&-\int_{\rho_+(x^*)}^{\rho_+(L)}\left(\frac{\gamma e^{S_+}\tau^{\gamma-1}}{\tau E_+(\tau,x^*)}-\frac{h(\tau,S_+)}{\tau E_+^3(\tau,x^*)}\int_{\rho_+(x^*)}^{\tau}\frac{t-b}{t}\gamma e^{S_+}t^{\gamma-1}dt\right)d\tau\cdot\frac{\partial S_+(x^*)}{\partial x^*}.
\end{aligned}
\tag{3.10}
$$

Step 2. The existence of transonic shock solutions. Now we determine the sign of $\frac{\partial\rho_+(L,x^*)}{\partial x^*}$ to investigate the monotonicity between x^* and $\rho_+(L)$ at the supersonic flow and the subsonic flow, so as to derive the existence of transonic shock solutions. Since

$$
\operatorname{sgn}Q(x^*)=-\operatorname{sgn}\frac{\partial\rho_+(L,x^*)}{\partial x^*}, \tag{3.11}
$$

one may consider the sign of $Q(x^*)$ instead. Observe that when $x^*=L$,

$$
Q(x^*)=\left(\frac{h(\rho_+(x^*),S_+)}{\rho_+(x^*)E_+(x^*)}\frac{\partial\rho_+(x^*)}{\partial x^*}-1\right)
$$

$$=\gamma\left(\frac{\rho_-(x^*)}{\rho_+(x^*)}-1\right)$$
$$<0. \tag{3.12}$$

Noting that

$$-\infty<\frac{h(\rho_+(x^*),S_+)}{\rho_+(x^*)E_+(x^*)}\frac{\partial\rho_+(x^*)}{\partial x^*}-1<0, \tag{3.13}$$

and

$$-\int_{\rho_+(x^*)}^{\rho_+(L)}\left(\frac{\gamma e^{S_+}\tau^{\gamma-1}}{\tau E_+(\tau,x^*)}-\frac{h(\tau,S_+)}{\tau E_+^3(\tau,x^*)}\int_{\rho_+(x^*)}^{\tau}\frac{t-b}{t}\gamma e^{S_+}t^{\gamma-1}dt\right)d\tau\cdot\frac{\partial S_+(x^*)}{\partial x^*}>0, \tag{3.14}$$

we now claim that

$$\lim_{x\to\underline{x}^*}\int_{\rho_+(x^*)}^{\rho_+(L)}\frac{h(\tau,S_+)E_-(x^*)}{\tau E_+^3(\tau,x^*)}\left(1-\frac{\rho_-(x^*)}{\rho_+(x^*)}\right)\left((\gamma-1)\rho_+(x^*)-\gamma b\right)d\tau=+\infty. \tag{3.15}$$

In fact, for fixed $\underline{x}^*$, let

$$f(\tau)=E_+^2(\tau,\underline{x}^*),\qquad \rho_+(L)\le\tau\le\rho_+(\underline{x}^*).$$

Then

$$\frac{1}{2}f(\tau)-H(\tau,S_+)=\frac{1}{2}E_+^2(\rho_+(\underline{x}^*),S_+)-H(\rho_+(\underline{x}^*),S_+).$$

Therefore,

$$f'(\tau)=2H'_\tau(\tau,S_+)=2\left(1-\frac{b}{\tau}\right)\left(p'_\tau(\tau,S_+)-\frac{J^2}{\tau^2}\right),\qquad \rho_+(L)\le\tau\le\rho_+(\underline{x}^*).$$

Since $\rho_+(\underline{x}^*)>\rho_+(L)>b>\rho_s$, there exist positive constants C_1 and C_2 such that

$$C_1\le f'_\tau(\tau)\le C_2,\qquad \rho_+(L)\le\tau\le\rho_+(\underline{x}^*).$$

Due to $f(\rho_+(\underline{x}^*))=0$, it holds that

$$f(\tau)=o(|\tau-\rho_+(\underline{x}^*)|^{\frac{1}{2}})$$

for small $|\tau-\rho_+(\underline{x}^*)|$. Equation (3.15) now follows owing to $\rho_+(\underline{x}^*)>\rho_+(L)$ and $E_+(\underline{x}^*,S_+)<0$ for $\rho_+(L)\le\tau\le\rho_+(\underline{x}^*)$. As a consequence of (3.10), (3.13), (3.14) and (3.15), one has

$$\lim_{x^*\to\underline{x}^*}Q(x^*)=+\infty. \tag{3.16}$$

Step 3. The non-uniqueness of solutions. So far, we have shown that there indeed exist transonic solutions for boundary value problem (1.3) and (1.5) according to the different monotonicity of $\rho_+(x^*)$ at the supersonic flow and the subsonic flow with respect to x^*. In order to prove that there exist two and only two transonic solutions, we now claim that $\frac{\partial\rho_+(x^*)}{\partial x^*}$ is monotonically increasing with respect to x^*, which is needed to judge the sign of $\frac{\partial^2\rho_+(x^*)}{\partial(x^*)^2}$. It is easy to see that

$$\frac{\partial\rho_+(L,\hat{x}^*)}{\partial\hat{x}^*}=0 \tag{3.17}$$

at the location $x^*=\hat{x}^*$, i.e.

$$\gamma\left(\frac{\rho_-(\hat{x}^*)}{\rho_+(\hat{x}^*)}-1\right)+\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)}\frac{h(\tau,S_+)E_-(\hat{x}^*)}{\tau E_+^3(\tau,\hat{x}^*)}\left(1-\frac{\rho_-(x^*)}{\rho_+(x^*)}\right)\left((\gamma-1)\rho_+(x^*)-\gamma b\right)d\tau$$
$$-\int_{\rho_+(\hat{x^*})}^{\rho_+(L)}\left(\frac{\gamma e^{S_+}\tau^{\gamma-1}}{\tau E_+(\tau,\hat{x^*})}-\frac{h(\tau,S_+)}{\tau E_+^3(\tau,\hat{x^*})}\cdot\int_{\rho_+(\hat{x^*})}^{\tau}\frac{t-b}{t}\gamma e^{S_+}t^{\gamma-1}dt\right)d\tau\cdot\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}=0. \tag{3.18}$$

Let us for simplicity write $\rho_+(\hat{x}^*)$ as ρ_+, $\rho_-(\hat{x}^*)$ as ρ_-, and $S_+(\hat{x}^*)$ as S_+. Because of (3.9) and (3.17), one has

$$Q(\hat{x}^*)=\frac{\partial\rho_+(L,\hat{x}^*)}{\partial\hat{x}^*}=0. \tag{3.19}$$

Straightforward computation gives

$$\frac{dQ(\hat{x}^*)}{d\hat{x}^*}=\sum_{i=1}^{4}I_i, \tag{3.20}$$

where

$$\begin{aligned}I_1=&\frac{h(\rho_+(L),S_+)E_-(\hat{x}^*)}{\rho_+(L)E_+^3(L,\hat{x}^*)}\left(1-\frac{\rho_-}{\rho_+}\right)\left((\gamma-1)\rho_+-\gamma b\right)\frac{\partial\rho_+(L)}{\partial\hat{x}^*}\\&-\frac{\gamma e^{S_+}\rho_+^{\gamma-1}(L)}{\rho_+(L)E_+(L,\hat{x}^*)}\frac{\partial\rho_+(L)}{\partial\hat{x}^*}\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}\\&+\frac{h(\rho_+(L),S_+)}{\rho_+(L)E_+^3(L,\hat{x}^*)}\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)}\frac{t-b}{t}\gamma e^{S_+}t^{\gamma-1}dt\cdot\frac{\partial\rho_+(L)}{\partial\hat{x}^*}\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}\\&-\frac{h(\rho_+(\hat{x}^*),S_+)}{\rho_+(\hat{x}^*)E_+^3(x^*)}\int_{\rho_+(\hat{x}^*)}^{\rho_+(\hat{x}^*)}\frac{t-b}{t}\cdot\gamma e^{S_+}t^{\gamma-1}dt\cdot\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*},\end{aligned} \tag{3.21}$$

$$\begin{aligned}I_2=&-\frac{h(\rho_+(\hat{x}^*),S_+)E_-(\hat{x}^*)}{\rho_+(\hat{x}^*)E_+^3(\hat{x}^*)}\left(1-\frac{\rho_-}{\rho_+}\right)\left((\gamma-1)\rho_+-\gamma b\right)\cdot\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}\\&+\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)}\frac{\frac{\partial h(\tau,S_+)}{\partial\hat{x}^*}E_-(\hat{x}^*)}{\tau E_+^3(\tau,\hat{x}^*)}\left(1-\frac{\rho_-}{\rho_+}\right)\cdot\left((\gamma-1)\rho_+-\gamma b\right)d\tau\\&+\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)}\frac{h(\tau,S_+)E_-(\hat{x}^*)}{\tau E_+^3(\tau,\hat{x}^*)}\left((\gamma-1)-\frac{\gamma b\rho_-}{\rho_+^2}\right)\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}d\tau\\&+\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)}\frac{h(\tau,S_+)E_-(\hat{x}^*)}{\tau E_+^3(\tau,\hat{x}^*)}\cdot\left(-(\gamma-1)+\frac{\gamma b}{\rho_+}\right)\frac{\partial\rho_-(\hat{x}^*)}{\partial\hat{x}^*}d\tau\\&-\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)}\frac{3h(\tau,S_+)E_-(\hat{x}^*)}{\tau E_+^4(\tau,\hat{x}^*)}\frac{\partial E_+(\tau,\hat{x}^*)}{\partial\hat{x}^*}\cdot\left(1-\frac{\rho_-}{\rho_+}\right)\left((\gamma-1)\rho_+-\gamma b\right)d\tau\\&+\frac{\gamma e^{S_+}\rho_+^{\gamma-1}(\hat{x}^*)}{\rho_+(\hat{x}^*)E_+(\hat{x}^*)}\cdot\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}\end{aligned}$$

$$
\begin{aligned}
&-\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{\gamma e^{S_+}\tau^{\gamma-1}}{\tau E_+(\tau,\hat{x}^*)}\cdot\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}d\tau\cdot\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}\\
&+\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{\gamma e^{S_+}\tau^{\gamma-1}}{\tau E_+^2(\tau,\hat{x}^*)}\frac{\partial E_+(\tau,\hat{x}^*)}{\partial\hat{x}^*}d\tau\cdot\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}\\
&+\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \left(\frac{\frac{\partial h(\tau,S_+)}{\partial\hat{x}^*}}{\tau E_+^3(\tau,\hat{x}^*)}-\frac{3h(\tau,S_+)}{\tau E_+^4(\tau,\hat{x}^*)}\frac{\partial E_+(\tau,\hat{x}^*)}{\partial\hat{x}^*}\right)\cdot\int_{\rho_+(\hat{x}^*)}^{\tau}\frac{t-b}{t}\gamma e^{S_+}t^{\gamma-1}dtd\tau\cdot\frac{\partial S_+}{\partial\hat{x}^*}\\
&+\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{h(\tau,S_+)}{\tau E_+^3(\tau,\hat{x}^*)}\left(-\frac{\rho_+-b}{\rho_+}\gamma e^{S_+}\rho_+^{\gamma-1}\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}\right)d\tau\cdot\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}\\
&+\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{h(\tau,S_+)}{\tau E_+^3(\tau,\hat{x}^*)}\left(\int_{\rho_+(\hat{x}^*)}^{\tau}\frac{t-b}{t}\cdot\gamma e^{S_+}t^{\gamma-1}\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}dt\right)d\tau\cdot\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*},
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
I_3=&\gamma\cdot\frac{\frac{\partial\rho_-(\hat{x}^*)}{\partial\hat{x}^*}\rho_+-\rho_-\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}}{\rho_+^2(\hat{x}^*)}\\
&+\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)}\frac{h(\tau,S_+)}{\tau E_+^3(\tau,\hat{x}^*)}\frac{\partial E_-(\hat{x}^*)}{\partial\hat{x}^*}\cdot\left(1-\frac{\rho_-}{\rho_+}\right)\Big((\gamma-1)\rho_+-\gamma b\Big)d\tau,
\end{aligned}
\tag{3.23}
$$

and

$$
I_4=-\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)}\left(\frac{\gamma e^{S_+}\tau^{\gamma-1}}{\tau E_+(\tau,\hat{x}^*)}-\frac{h(\tau,S_+)}{\tau E_+^3(\tau,\hat{x}^*)}\cdot\int_{\rho_+(\hat{x}^*)}^{\tau}\frac{t-b}{t}\gamma e^{S_+}t^{\gamma-1}dt\right)d\tau\cdot\frac{\partial^2 S_+(\hat{x}^*)}{\partial(\hat{x}^*)^2}.
\tag{3.24}
$$

Note that

$$
I_2<0,
\tag{3.25}
$$

thanks to $E_+(x^*)<0, E_+(\tau,x^*)<0, \frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}<0$ and (2.9), (2.10), (2.15), (2.16), (3.8). Moreover,

$$
I_1=0
\tag{3.26}
$$

holds by $\frac{\partial\rho_+(L,\hat{x}^*)}{\partial\hat{x}^*}=0$. On the other hand, it holds that

$$
I_3>0,
\tag{3.27}
$$

owing to $\frac{dE_-(\hat{x}^*)}{d\hat{x}^*}=\rho_-(\hat{x}^*)-b<0$.

Let us take attention to the sign of I_4, which is unknown. Direct calculations yield

$$
\begin{aligned}
\frac{\partial^2 S_+(\hat{x}^*)}{\partial(\hat{x}^*)^2}=&\frac{\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}-\frac{\partial\rho_-(\hat{x}^*)}{\partial\hat{x}^*}}{\rho_+-\rho_-}\cdot\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}+\frac{\rho_--b}{E_-(\hat{x}^*)}\cdot\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}\\
&-\left(\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}+\frac{\gamma}{\rho_+}\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}\right)\cdot\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}\\
=&\left(\frac{\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}-\frac{\partial\rho_-(\hat{x}^*)}{\partial\hat{x}^*}}{\rho_+(\hat{x}^*)-\rho_-(\hat{x}^*)}+\frac{\rho_-(\hat{x}^*)-b}{E_-(\hat{x}^*)}-\left(\frac{\gamma}{\rho_+(\hat{x}^*)}\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}+\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*}\right)\right)\cdot\frac{\partial S_+}{\partial\hat{x}^*}.
\end{aligned}
\tag{3.28}
$$

Thus applying (3.17) gives

$$
\begin{aligned}
I_4 =& -\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \left(\frac{\gamma e^{S_+}\tau^{\gamma-1}}{\tau E_+(\tau,\hat{x}^*)} - \frac{h(\tau,S_+)}{\tau E_+^3(\tau,\hat{x}^*)} \cdot \int_{\rho_+(\hat{x}^*)}^{\tau} \frac{t-b}{t}\gamma e^{S_+}t^{\gamma-1}dt \right) d\tau \cdot \frac{\partial S_+(\hat{x}^*)}{\partial \hat{x}^*} \\
&\cdot \left(\frac{\frac{\partial\rho_+}{\partial\hat{x}^*} - \frac{\partial\rho_-}{\partial\hat{x}^*}}{\rho_+ - \rho_-} + \frac{\rho_- - b}{E_-(\hat{x}^*)} - \left(\frac{\gamma}{\rho_+(\hat{x}^*)} \frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*} + \frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*} \right) \right) \\
=& \left(-\gamma\Big(\frac{\rho_-}{\rho_+} - 1\Big) - \int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{h(\tau,S_+)E_-(\hat{x}^*)}{\tau E_+^3(\tau,\hat{x}^*)} \left(1 - \frac{\rho_-}{\rho_+}\right) \Big((\gamma-1)\rho_+ - \gamma b \Big) d\tau \right) \\
&\cdot \left(\frac{\frac{\partial\rho_+}{\partial\hat{x}^*} - \frac{\partial\rho_-}{\partial\hat{x}^*}}{\rho_+ - \rho_-} + \frac{\rho_- - b}{E_-(\hat{x}^*)} - \frac{\gamma}{\rho_+} \frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*} \right) \\
&+ \int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \left(\frac{\gamma e^{S_+}\tau^{\gamma-1}}{\tau E_+(\tau,\hat{x}^*)} - \frac{h(\tau,S_+)}{\tau E_+^3(\tau,\hat{x}^*)} \cdot \int_{\rho_+(\hat{x}^*)}^{\tau} \frac{t-b}{t}\gamma e^{S_+}t^{\gamma-1}dt \right) d\tau \cdot \frac{\partial S_+}{\partial\hat{x}^*} \cdot \frac{\partial S_+}{\partial\hat{x}^*} \\
=:& \sum_{j=1}^{6} J_j,
\end{aligned}
\tag{3.29}
$$

where

$$
\begin{aligned}
J_1 =& \frac{\gamma}{\rho_+} \frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*} - \frac{\gamma}{\rho_+} \frac{\partial\rho_-(\hat{x}^*)}{\partial\hat{x}^*} - \frac{\gamma^2}{\rho_+^2}(\rho_+ - \rho_-) \frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}, \\
J_2 =& \frac{\gamma}{\rho_+ E_-(\hat{x}^*)} (\rho_+ - \rho_-)(\rho_- - b), \\
J_3 =& -\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{h(\tau,S_+)E_-(\hat{x}^*)}{\tau E_+^3(\tau,\hat{x}^*)} \left(1 - \frac{\rho_-}{\rho_+}\right) \Big((\gamma-1)\rho_+ - \gamma b \Big) \left(\frac{1}{\rho_+ - \rho_-} - \frac{\gamma}{\rho_+} \right) \\
&\cdot \frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*} d\tau, \\
J_4 =& \int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{h(\tau,S_+)E_-(\hat{x}^*)}{\tau E_+^3(\tau,\hat{x}^*)} \left(1 - \frac{\rho_-}{\rho_+}\right) \Big((\gamma-1)\rho_+ - \gamma b \Big) \frac{1}{\rho_+ - \rho_-} \frac{\partial\rho_-(\hat{x}^*)}{\partial\hat{x}^*} d\tau, \\
J_5 =& -\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{h(\tau,S_+)(\rho_- - b)}{\tau E_+^3(\tau,\hat{x}^*)} \left(1 - \frac{\rho_-}{\rho_+}\right) \Big((\gamma-1)\rho_+ - \gamma b \Big) d\tau,
\end{aligned}
$$

and

$$
J_6 = \int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \left(\frac{\gamma e^{S_+}\tau^{\gamma-1}}{\tau E_+(\tau,\hat{x}^*)} - \frac{h(\tau,S_+)}{\tau E_+^3(\tau,\hat{x}^*)} \int_{\rho_+(\hat{x}^*)}^{\tau} \frac{t-b}{t}\gamma e^{S_+}t^{\gamma-1}dt \right) d\tau \cdot \left(\frac{\partial S_+(\hat{x}^*)}{\partial\hat{x}^*} \right)^2 .
\tag{3.30}
$$

First, straightforward computations give

$$
J_6 + (I_2)_7 + (I_2)_{11} = 0,
\tag{3.31}
$$

$$
\begin{aligned}
(I_3)_1 + (I_2)_6 + J_1 =& -\frac{\gamma}{\rho_+}\frac{\rho_-}{\rho_+}\frac{\partial\rho_+}{\partial\hat{x}^*} + \frac{\gamma}{\rho_+}\frac{\partial\rho_+}{\partial\hat{x}^*} + \frac{\gamma(\gamma-1)(\rho_+ - \rho_-)}{\rho_+^2}\frac{\partial\rho_+}{\partial\hat{x}^*} - \frac{\gamma^2(\rho_+ - \rho_-)}{\rho_+^2}\frac{\partial\rho_+}{\partial\hat{x}^*} \\
=& \frac{\gamma}{\rho_+}\Big(1 - \frac{\rho_-}{\rho_+}\Big)\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*} + (\gamma^2 - \gamma - \gamma^2)\cdot\frac{\rho_+ - \rho_-}{\rho_+^2}\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*} \\
=& 0,
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
(I_2)_3+(I_2)_{10}+J_3 &= \int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{h(\tau,S_+)E_-(\hat{x}^*)}{\tau E_+^3(\tau,\hat{x}^*)}\Bigg(-\frac{\rho_+-b}{\rho_+}\frac{\gamma(\gamma-1)(\rho_+-\rho_-)}{\rho_+}-\frac{\gamma b\rho_-}{\rho_+^2} \\
&\quad +\frac{\gamma b}{\rho_+}+\frac{\gamma(\gamma-1)(\rho_+-\rho_-)}{\rho_+}-\frac{\gamma^2 b(\rho_+-\rho_-)}{\rho_+^2}\Bigg)\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}d\tau \\
&= \int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{h(\tau,S_+)E_-(\hat{x}^*)}{\tau E_+^3(\tau,\hat{x}^*)}\cdot 0\cdot\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*}d\tau \\
&= 0,
\end{aligned} \tag{3.33}
$$

$$
(I_2)_4+J_4=\int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{h(\tau,S_+)E_-(\hat{x}^*)}{\tau E_+^3(\tau,\hat{x}^*)}\left(-(\gamma-1)+\frac{\gamma b}{\rho_+}+(\gamma-1)-\frac{\gamma b}{\rho_+}\right)\frac{\partial\rho_-(\hat{x}^*)}{\partial\hat{x}^*}d\tau=0, \tag{3.34}
$$

and

$$
\begin{aligned}
&(I_3)_2+J_5 \\
&= \int_{\rho_+(\hat{x}^*)}^{\rho_+(L)} \frac{h(\tau,S_+)(\rho_--b)}{\tau E_+^3(\tau,\hat{x}^*)}\left((1-\frac{\rho_-}{\rho_+})((\gamma-1)\rho_+-\gamma b)-(1-\frac{\rho_-}{\rho_+})((\gamma-1)\rho_+-\gamma b)\right)d\tau \\
&= 0
\end{aligned} \tag{3.35}
$$

by using $\frac{\partial E_-(x^*)}{\partial x^*}=\rho_-(x^*)-b$. Second, it holds that

$$
\begin{aligned}
(I_2)_1+J_2 &= \frac{\gamma(\rho_+-\rho_-)(\rho_--b)}{\rho_+E_-(\hat{x}^*)} \\
&\quad -\frac{h(\rho_+(\hat{x}^*),S_+)E_-(\hat{x}^*)}{\rho_+(\hat{x}^*)E_+^3(\hat{x}^*)}\left(1-\frac{\rho_-}{\rho_+}\right)\left((\gamma-1)\rho_+-\gamma b\right)\cdot\frac{\partial\rho_+(\hat{x}^*)}{\partial\hat{x}^*} \\
&= -\frac{(\rho_+-\rho_-)\rho_-}{\rho_+E_-(\hat{x}^*)}\left(-(\gamma-1)^2\frac{\rho_+}{\rho_-}+\gamma^2 b(\frac{1}{\rho_-}-\frac{1}{\rho_+})+\gamma(\gamma-2)\right) \\
&< -\frac{(\rho_+-\rho_-)\rho_-}{\rho_+E_-(\hat{x}^*)}\left(-(\gamma-1)^2\frac{\rho_+}{\rho_-}+\gamma^2 \rho_+(\frac{1}{\rho_-}-\frac{1}{\rho_+})+\gamma(\gamma-2)\right) \\
&= -\frac{(\rho_+-\rho_-)\rho_-}{\rho_+E_-(\hat{x}^*)}\left((2\gamma-1)\frac{\rho_+}{\rho_-}-2\gamma\right),
\end{aligned} \tag{3.36}
$$

here inequality holds according to $b<\rho_+$ and $\rho_-<\rho_+$.

It remains to claim that $-\frac{(\rho_+-\rho_-)\rho_-}{\rho_+E_-(\hat{x}^*)}\left((2\gamma-1)\frac{\rho_+}{\rho_-}-2\gamma\right)<0$, i.e., $(2\gamma-1)\frac{\rho_+}{\rho_-}-2\gamma<0$. Observe that $(2\gamma-1)\frac{\rho_+}{\rho_-}-2\gamma<0$, provided that $M_-^2<\frac{4\gamma}{3\gamma-1}$. Indeed, it follows from $M_-^2<\frac{4\gamma}{3\gamma-1}$ that $(3\gamma-1)M_-^2<4\gamma$, which implies $(\gamma+1)(2\gamma-1)M_-^2<2\gamma(\gamma-1)M_-^2+4\gamma$, then

$$
\frac{(\gamma+1)M_-^2}{(\gamma-1)M_-^2+2}<\frac{2\gamma}{2\gamma-1},
$$

namely,

$$
\frac{\rho_+}{\rho_-}<\frac{2\gamma}{2\gamma-1}. \tag{3.37}
$$

Hence,

$$(I_2)_1 + J_2 < 0 \tag{3.38}$$

is valid. Combining with (3.23)-(3.26), (3.29)-(3.35) and (3.38), (3.20) implies that

$$\frac{dQ(\hat{x}^*)}{d\hat{x}^*} < 0. \tag{3.39}$$

It follows now from (3.11), (3.15) and (3.8) that $Q(\underline{x}^*) = +\infty$, $Q(L) < 0$, $Q'(\hat{x}^*) < 0$ as $Q(\hat{x}^*) = 0$ for $\hat{x}^* \in (\underline{x}^*, L)$. Therefore, $Q(x^*)$ only change the sign once for $x^* \in (\underline{x}^*, L)$ at $x^* = \hat{x}^*$. Eventually, we conclude that

$$\lim_{x^* \to \underline{x}^*} Q(x^*) = +\infty, \begin{cases} Q(x^*) > 0, \text{ for } \underline{x}^* < x^* < \hat{x}^*, \\ Q(\hat{x}^*) = 0, \\ Q(x^*) < 0, \text{ for } \hat{x}^* < x^* < L. \end{cases} \tag{3.40}$$

This confirms (3.2) and (3.3), as asserted. □

Based on Lemma 3.2, we establish the following theorem.

THEOREM 3.1. *For given positive constants J, L, $S_0 > 0$ and $\gamma > 1$ and a non-empty parameter set $\mathscr{B}$, for any $(\rho_0, E_0; S_0) \in \mathscr{B}$, there exists an interval $I = (\underline{\rho}, \overline{\rho})$. Then for any $\rho_e \in I$, there exist two solutions $(\rho_1, E_1; S_1)$ and $(\rho_2, E_2; S_2)$ on $[0, L]$ for the boundary value problem (1.3) and (1.5) satisfying Rankine-Hugoniot condition (2.5) at $x = x_1^*$ and x_2^* and the entropy condition (2.6).*

Proof. As showed in Lemma 3.2, we have verified that $\frac{\partial \rho_+(L)}{\partial x^*}$ has the different monotonicity with respect to x^* in different intervals, that is,

$$\lim_{x^* \to \underline{x}^*} \frac{\partial \rho_+(L)}{\partial x^*} = -\infty, \quad \begin{cases} \frac{\partial \rho_+(L)}{\partial x^*} < 0, \text{ for } \underline{x}^* < x^* < \hat{x}^*, \\ \frac{\partial \rho_+(L)}{\partial x^*} > 0, \text{ for } \hat{x}^* < x^* < L, \\ \frac{\partial^2 \rho_+(L)}{\partial (\hat{x}^*)^2} > 0. \end{cases} \tag{3.41}$$

Set $\rho_{\hat{x}^*}(L) = \underline{\rho}$, $\rho_L(L) = \overline{\rho}$, then $I = (\underline{\rho}, \overline{\rho})$. For any $\rho_e \in I$, there exist two and only two shock locations x_1^* and x_2^* satisfying $\rho_{x_1^*}(L) = \rho_e, \rho_{x_2^*}(L) = \rho_e$. Therefore, there exist two and only two transonic solutions for the boundary value problem (1.3) and (1.5) as displayed in Figure 3.1. Combining the proofs of Lemma 3.2 and the computations in the Section 2, the two solutions satisfy Rankine-Hugoniot condition (2.5) at $x = x_1^*$ and

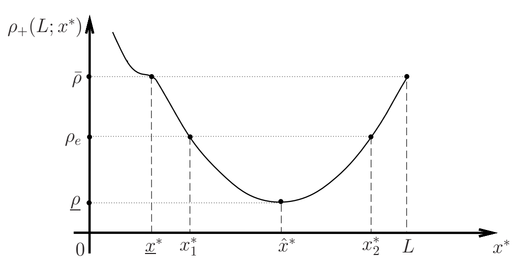


FIG. 3.1. *Analysis of non-uniqueness of solutions*

x_2^* and the entropy condition (2.6). □

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