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GLOBAL SOLUTIONS OF 3D TROPICAL CLIMATE MODEL WITH FINITE ENERGY*

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Abstract. In this paper, we construct a family of finite energy classical solutions to the 3D tropical climate model equations. The approach is by studying the steady-state Beltrami flows and using the standard cut-off technique to establish the global regularity of the systems.

Keywords. Tropical climate model; Beltrami flows; Global well-posedness.

AMS subject classifications. 35D35; 76D03; 86A10.

1. Introduction

In this paper, we consider the following tropical climate model:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u + \nabla p + \nabla\cdot(v\otimes v) = 0, \\ \nabla\cdot u = 0, \\ \partial_t v + u\cdot\nabla v - \nu\Delta v + \nabla\theta + v\cdot\nabla u = 0, \\ \partial_t\theta + u\cdot\nabla\theta - \eta\Delta\theta + \nabla\cdot v = 0, \end{cases} \tag{1.1}$$

in $\mathbb{R}^3$, where the unknowns are the vector fields $u=(u^1,u^2,u^3)$, $v=(v^1,v^2,v^3)$ and the scalar functions θ and p. Here, u and v are the barotropic mode and the first baroclinic mode of the velocity, respectively, while θ and p denote the temperature and the pressure, respectively, $\mu,\nu,\eta>0$ are real parameters. For more details of the related background, see, for instance, [6, 10, 11] and the references therein.

Before going further, let us first view some results of the tropical climate model in $\mathbb{R}^2$. The inviscid version of (1.1), namely (1.1) with $\mu=\nu=\eta=0$ was first derived by Frierson, Majda and Pauluis [4] as a model for tropical geophysical flows. Fundamental issues concerning (1.1) such as the global existence and regularity of solutions have attracted considerable attention. Important results have been obtained. Li and Titi [8] proved the global well-posedness for the case $\eta=0$ by introducing a combined quantity called pseudo-baroclinic velocity. The work of Li and Titi inspired several subsequent studies, see, for instance [3, 14, 16].

From the mathematical point of view, the tropical climate model (1.1) is significantly related to Navier-Stokes equations ($v=\theta=0$). The question of whether a solution of the three-dimensional incompressible Navier-Stokes equations can develop a finite time singularity from smooth initial data with finite energy is one of the Millennium Prize problems. See [1, 9]. For general initial data, it is very difficult to obtain global solutions, hence, in recent years, people tried to construct some family of large solutions. Readers may find the following articles to be informative and relevant: [2, 7]. The work of [7] inspired several subsequent studies, see, for instance [5, 12, 13, 15].

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Lei et al. [7] constructed global smooth solutions to the three-dimensional incompressible Navier-Stokes equations with finite energy for a family of initial data. One of their starting points is to make use of the convection term to explore the cancellation structure of nonlinearities. The important feature of the initial data is that the velocity fields are almost parallel to the corresponding vorticity fields in a very large portion of spatial domain, which are perturbations of steady-state Beltrami flows. To be more precise, they defined $\widetilde{\chi}(\kappa,A)$ to be a set of functions by

$$\widetilde{\chi}(\kappa,A)=\{g\in L^1_{loc}(\mathbb{R}^3)|\nabla\cdot g=0,\nabla\times g=\kappa g,|g(x)|\leq\frac{A}{1+\kappa|x|}\}. \tag{1.2}$$

More details about its properties can be found in Section 2.

Let $\chi\in C_0^\infty(\mathbb{R}^3)$ be a cut-off function such that

$$\chi\equiv 1 \text{ for } |x|\leq 1,\quad \chi\equiv 0 \text{ for } |x|\geq 2,\quad |\nabla^k\chi|\leq 2\ (0\leq k\leq 2) \tag{1.3}$$

and

$$\chi_M(x)=\chi(\frac{x}{M}),\quad M>0. \tag{1.4}$$

Inspired by the work in [7], the aim of the paper is to construct a family of finite energy smooth large solutions for the 3D tropical climate model. We use suitably perturbed steady-state Beltrami flow, so that the velocity fields are almost parallel to the corresponding vorticity fields in a very large portion of spatial domain. We use a decay estimate developed in [7] of such family of initial data which allows us to do certain truncations over the initial data and solutions to obtain a class of finite energy solutions. By making use of the cancellation structure of nonlinearities in the convection term, we are able to prove that this class of data can be evolved globally in time.

Our main result can be stated as follows:

THEOREM 1.1. *Let χ be any standard cut-off function satisfying (1.3). Suppose that $V_0(x)\in\widetilde{\chi}(\kappa,A)$ and $u(t,x),v(t,x),\theta(t,x)$ is the unique smooth solution of 3D tropical climate model. Then there exists a positive constant $M_0\geq 1$, such that the 3D tropical climate model with the initial data $u_0=h_0(x)+\chi_M V_0(x),v_0(x),\theta_0(x)$ are globally well-posed, provided $\|h_0(x),v_0(x),\theta_0(x)\|_{H^1}\leq M^{-\frac{1}{2}}$ and $M\geq M_0$.*

REMARK 1.1. We emphasize that for barotropic velocity u, any norm of the initial data under consideration can be arbitrarily large, although there is a smallness condition on $\|h_0\|_{H^1}$. In fact, noticing the construction of $\widetilde{\chi}(\kappa,A)$: if $v_0\in\widetilde{\chi}(\kappa,A)$, then $\lambda v_0\in\widetilde{\chi}(\kappa,A)$ for all $\lambda>0$, then the term $\chi_M(\lambda v_0)$ can be arbitrarily large.

This paper is organized as follows: In Section 2, the decay properties of the background solution $V(t,x)$ with initial data $V_0(x)$ in $\widetilde{\chi}(\kappa,A)$ are given. In Section 3, we prove Theorem 1.1. Throughout the rest of this paper the letter C denotes various positive and finite constants whose exact values are unimportant and may vary from line to line.

2. Decay properties of Data

First of all, let $V(t,x)$ be the solution of the heat equation

$$V_t=\nu\Delta V,\quad V(0,x)=V_0, \tag{2.1}$$

Suppose $V_0\in\widetilde{\chi}(\kappa,A)$, without loss of generality, let $\kappa=1$. Then

$$\nabla\cdot V_0=0,\quad \nabla\times V_0=V_0.$$

It is preserved by the heat flow in (2.1), hence, one has

$$\nabla\cdot V=0,\quad \nabla\times V=V.$$

Furthermore, from the work of Lei-Lin-Zhou [7], one has the following estimate:

$$|V(x)|+|\nabla V(x)|\leq C\frac{Ae^{-\nu t/2}}{1+|x|}. \tag{2.2}$$

3. Proof of Theorem 1.1.

In this section we construct the global smooth solutions to the 3D tropical climate model equations with finite energy using the standard cut-off and perturbation arguments. Suppose that u,v,θ is the unique local smooth solution of 3D tropical climate model equations with initial data $u(0,x)=h_0(x)+\chi_M V_0(x), v(0,x)=v_0(x), \theta(0,x)=\theta_0(x)$. Here $\|h_0\|_{H^1},\|v_0\|_{H^1},\|\theta_0\|_{H^1}\leq M^{-\frac{1}{2}}$. The associated pressure is $p=-\Delta^{-1}\nabla\cdot[\nabla\cdot(u\otimes u)]-\Delta^{-1}\nabla\cdot[\nabla\cdot(v\otimes v)]$. To show that $u(t,x),v(t,x),\theta(t,x)$ is a global smooth solution, it is sufficient to prove an *a priori* estimate for $\|u(t,\cdot)\|_{H^1},\|v(t,\cdot)\|_{H^1},\|\theta(t,\cdot)\|_{H^1}$ for all $t>0$. Define

$$h=u-\chi_M V.$$

We then have

$$\begin{cases} h_t+h\cdot\nabla h+\nabla p-\Delta h+\nabla\cdot(v\otimes v)=f_1,\\ \partial_t v+h\cdot\nabla v-\Delta v+\nabla\theta+v\cdot\nabla h=f_2,\\ \partial_t\theta+h\cdot\nabla\theta-\Delta\theta+\nabla\cdot v=f_3,\\ \nabla\cdot h=-V\cdot\nabla\chi_M, \end{cases} \tag{3.1}$$

where f_1,f_2,f_3 are given by

$$\begin{aligned} f_1&=V\Delta\chi_M+2(\nabla\chi_M\cdot\nabla)V-\chi_M(V\cdot\nabla\chi_M)V-\chi_M^2 V\cdot\nabla V\\ &\quad-\ h\cdot\nabla(\chi_M V)-\chi_M V\cdot\nabla h,\\ f_2&=-\chi_M V\cdot\nabla v-v\cdot\nabla(\chi_M V),\\ f_3&=-\chi_M V\cdot\nabla\theta. \end{aligned} \tag{3.2}$$

Here and in what follows we will set $\mu=\nu=\eta=1$.

Taking the L^2 inner product of Equations $(3.1)_1$–$(3.1)_3$ with h, v and θ, respectively, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|h\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2)+\|\nabla h\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\\ =&-\int(\nabla\cdot(v\otimes v)\cdot h+v\cdot\nabla h\cdot v+\nabla\theta\cdot v+\nabla\cdot v\theta)dx\\ &-\int(h\cdot\nabla h\cdot h+h\cdot\nabla v\cdot v+h\cdot\nabla\theta\cdot\theta)dx+\int p\nabla\cdot h+f_1 h dx+\int f_2 v+f_3\theta dx=\sum_{i=1}^4 I_i. \end{aligned} \tag{3.3}$$

We need to estimate the above terms one by one. First, we have

$$\begin{aligned} I_1&=-\int(\nabla\cdot(v\otimes v)\cdot h+v\cdot\nabla h\cdot v+\nabla\theta\cdot v+\nabla\cdot v\theta)dx\\ &=-\int\nabla\cdot(v\otimes v\cdot h+v\theta)dx=0. \end{aligned} \tag{3.4}$$

Integrating by parts, noticing that $\nabla\cdot h=-V\cdot\nabla\chi_M$, by (2.2), one has

$$\begin{aligned}I_2=&-\int(h\cdot\nabla h\cdot h+h\cdot\nabla v\cdot v+h\cdot\nabla\theta\cdot\theta)dx\\ =&\frac{1}{2}\int(|h|^2+|v|^2+|\theta|^2)\nabla\cdot hdx\\ \leq& C(\|h\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2)|V\cdot\nabla\chi_M|_{L^\infty}\\ \leq& CM^{-1}e^{-\frac{t}{2}}(\|h\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2).\end{aligned}\tag{3.5}$$

Next let us use the expressions for p and f_1 to rewrite that

$$\begin{aligned}I_3=&\int p\nabla\cdot h+f_1hdx\\ =&\int(-\Delta^{-1}\nabla\cdot[\nabla\cdot(u\otimes u)]-\Delta^{-1}\nabla\cdot[\nabla\cdot(v\otimes v)])\nabla\cdot h+f_1hdx\\ =&-\int\Delta^{-1}\nabla\cdot(h\cdot\nabla h+h\cdot\nabla(\chi_MV)+\chi_MV\cdot\nabla h+\chi_MV\cdot\nabla\chi_MV)\nabla\cdot hdx\\ &-\int\Delta^{-1}\nabla\cdot(\chi_M^2V\cdot\nabla V)\nabla\cdot hdx-\int\Delta^{-1}\nabla\cdot[\nabla\cdot(v\otimes v)]\nabla\cdot hdx\\ &+\int(V\Delta\chi_M+2(\nabla\chi_M\cdot\nabla)V-\chi_M(V\cdot\nabla\chi_M)V-\chi_M^2V\cdot\nabla V\\ &-h\cdot\nabla(\chi_MV)-\chi_MV\cdot\nabla h)hdx\\ =&-\int\Delta^{-1}\nabla\cdot[\nabla\cdot(v\otimes v)]\nabla\cdot hdx+\int h\cdot\nabla h\Delta^{-1}\nabla\nabla\cdot hdx\\ &+\int(h\cdot\nabla(\chi_MV)+\chi_MV\cdot\nabla h+\chi_MV\cdot\nabla\chi_MV)(\Delta^{-1}\nabla\nabla\cdot h-h)dx\\ &+\int\chi_M^2V\cdot\nabla V(\Delta^{-1}\nabla\nabla\cdot h-h)dx+\int(V\Delta\chi_M+2(\nabla\chi_M\cdot\nabla)V)hdx\\ =&\sum_{i=1}^5 I_{3i}.\end{aligned}\tag{3.6}$$

Now, we need to estimate the above terms one by one. First, by Sobolev imbedding inequality, one immediately gets

$$\begin{aligned}I_{31}=&-\int\Delta^{-1}\nabla\cdot[\nabla\cdot(v\otimes v)]\nabla\cdot hdx\\ =&\int[\nabla\cdot(v\otimes v)]\Delta^{-1}\nabla\nabla\cdot hdx\\ \leq& C\|v\|_{L^6}\|\nabla v\|_{L^2}\|\Delta^{-1}\nabla\nabla\cdot h\|_{L^3}\\ \leq& C\|h\|_{L^3}\|\nabla v\|_{L^2}^2,\end{aligned}\tag{3.7}$$

here we used the Sobolev imbedding $\|g\|_{L^6}\leq C\|\nabla g\|_{L^2}$ and the standard Calderon-Zygmund theory $\|Zg\|_{L^p}\leq C\|g\|_{L^p}$ for Riesz operator Z and $1<p<\infty$.

Similarly, we obtain

$$\begin{aligned}I_{32}=&\int h\cdot\nabla h\Delta^{-1}\nabla\nabla\cdot hdx\\ \leq& C\|h\|_{L^6}\|\nabla h\|_{L^2}\|\Delta^{-1}\nabla\nabla\cdot h\|_{L^3}\leq C\|h\|_{L^3}\|\nabla h\|_{L^2}^2.\end{aligned}\tag{3.8}$$

For the third term of I_3, we have

$$\begin{aligned}I_{33}=&\int(h\cdot\nabla(\chi_M V)+\chi_M V\cdot\nabla h+\chi_M V\cdot\nabla\chi_M V)(\Delta^{-1}\nabla\nabla\cdot h-h)dx\\ \le& C\big(\|\nabla(\chi_M V)\|_{L^\infty}\|h\|_{L^2}+\|\chi_M V\|_{L^\infty}\|\nabla h\|_{L^2}\big)\|\Delta^{-1}\nabla\times\nabla\times h\|_{L^2}\\ &+M^{-1}\|V\|^2_{L^{\frac{12}{5}}(|x|\le 2M)}\|\Delta^{-1}\nabla\times\nabla\times h\|_{L^6}\\ \le& CM^{-1}e^{-2t}+Ce^{-t/2}\|h\|^2_{L^2}+\frac{1}{16}\|\nabla h\|^2_{L^2}.\end{aligned}\tag{3.9}$$

In order to estimate I_{34}, we first write that

$$V\cdot\nabla V=-V\times(\nabla\times V)+\frac{1}{2}\nabla|V|^2=\frac{1}{2}\nabla|V|^2,$$

then, one has

$$\begin{aligned}|I_{34}|=&\Big|\int(\chi_M^2 V\cdot\nabla V)(\Delta^{-1}\nabla\nabla\cdot h-h)dx\Big|\\ =&\Big|\int\big[\nabla(\frac{1}{2}\chi_M^2|V|^2)-\chi_M|V|^2\nabla\chi_M\big]\Delta^{-1}\nabla\times\nabla\times hdx\Big|\\ =&\Big|\int\chi_M|V|^2\nabla\chi_M\Delta^{-1}\nabla\times\nabla\times hdx\Big|\\ \le& CM^{-1}\|V\|^2_{L^{12/5}(|x|\le M)}\|\Delta^{-1}\nabla\times\nabla\times h\|_{L^6}\\ \le& CM^{-1}e^{-2t}+\frac{1}{16}\|\nabla h\|^2_{L^2}.\end{aligned}\tag{3.10}$$

For the last term of I_3, integrating by parts, we have

$$\begin{aligned}I_{35}=&\int(V\Delta\chi_M+2(\nabla\chi_M\cdot\nabla)V)hdx\\ \le& CM^{-2}\|V\|_{L^{\frac{6}{5}}(|x|\le 2M)}\|h\|_{L^6}+M^{-1}\|V\|_{L^2(|x|\le 2M)}\|\nabla h\|_{L^2}\\ \le& CM^{-1}e^{-t}+\frac{1}{16}\|\nabla h\|^2_{L^2}.\end{aligned}\tag{3.11}$$

Similarly, we obtain

$$\begin{aligned}I_4=&\int f_2 v+f_3\theta dx\\ =&-\int(\chi_M V\cdot\nabla v+v\cdot\nabla(\chi_M V))vdx-\int\chi_M V\cdot\nabla\theta\theta dx\\ \le& C\|\chi_M V\|_{L^\infty}\|\nabla v\|_{L^2}\|v\|_{L^2}+\|\nabla(\chi_M V)\|_{L^\infty}\|v\|^2_{L^2}+\|\chi_M V\|_{L^\infty}\|\nabla\theta\|_{L^2}\|\theta\|_{L^2}\\ \le& Ce^{-t/2}(\|v\|^2_{L^2}+\|\theta\|^2_{L^2})+\frac{1}{16}(\|\nabla v\|^2_{L^2}+\|\nabla\theta\|^2_{L^2}).\end{aligned}\tag{3.12}$$

Combining the above estimates, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|h\|^2_{L^2}+\|v\|^2_{L^2}+\|\theta\|^2_{L^2})+(\frac{3}{4}-C\|h\|_{L^3})(\|\nabla h\|^2_{L^2}+\|\nabla v\|^2_{L^2}+\|\nabla\theta\|^2_{L^2})\\ \le& Ce^{-t/2}(\|h\|^2_{L^2}+\|v\|^2_{L^2}+\|\theta\|^2_{L^2})+M^{-1}e^{-t}.\end{aligned}\tag{3.13}$$

Applying the curl operator to $(3.1)_1$ and then taking the L^2 inner product of the resulting equation with $\nabla\times h$, taking the L^2 inner product of Equations $(3.1)_2$–$(3.1)_3$ with $-\Delta v$ and $-\Delta\theta$, respectively, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\times h\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\|\nabla\nabla\times h\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2\\&=-\int\nabla\times(h\cdot\nabla h)\nabla\times hdx-\int\nabla\times(\nabla\cdot(v\otimes v))\nabla\times hdx\\&\quad+\int\nabla\times f_1\nabla\times hdx-\int(h\cdot\nabla v+v\cdot\nabla h)\Delta v+h\cdot\nabla\theta\Delta\theta dx+\int f_2\Delta v+f_3\Delta\theta dx,\end{aligned}\tag{3.14}$$

we have used the following fact

$$\int\nabla\theta\cdot\Delta vdx+\int\nabla\cdot v\Delta\theta dx=0.\tag{3.15}$$

We first deal with the first term on the right-hand side of (3.14). Using integration by parts and Hodge decomposition, we estimate that

$$\begin{aligned}\Big|\int\nabla\times(h\cdot\nabla h)\nabla\times hdx\Big|&\le C\|h\|_{L^3}\|\nabla h\|_{L^6}\|\nabla\times\nabla\times h\|_{L^2}\\&\le C\|h\|_{L^3}\|\nabla\nabla\times h\|_{L^2}^2+C\|h\|_{L^3}\|\nabla\cdot h\|_{L^6}\|\nabla\times\nabla\times h\|_{L^2}.\end{aligned}\tag{3.16}$$

Recalling the last equation in (3.1), one has

$$\|\nabla\cdot h\|_{L^6}=\|V\cdot\nabla\chi_M\|_{L^6}\le CM^{-1}e^{-t/2}.$$

Using interpolation $\|h\|_{L^3}\le C\|h\|_{L^2}^{\frac{1}{2}}\|\nabla h\|_{L^2}^{\frac{1}{2}}$, one finally has

$$\begin{aligned}&\Big|\int\nabla\times(h\cdot\nabla h)\nabla\times hdx\Big|\\&\le C\|h\|_{L^3}\|\nabla\nabla\times h\|_{L^2}^2+\frac{1}{16}\big(\|\nabla h\|_{L^2}^2+\|\nabla\nabla\times h\|_{L^2}^2\big)+M^{-4}e^{-2t}\|h\|_{L^2}^2.\end{aligned}\tag{3.17}$$

For the second term on the right-hand side of (3.14),

$$\begin{aligned}-\int\nabla\times(\nabla\cdot(v\otimes v))\nabla\times hdx&\le\|v\|_{L^3}\|\nabla v\|_{L^6}\|\nabla\nabla\times h\|_{L^2}\\&\le\|v\|_{L^3}\|\Delta v\|_{L^2}\|\nabla\nabla\times h\|_{L^2}.\end{aligned}\tag{3.18}$$

For the third term on the right-hand side of (3.14), we first write it as follows:

$$\begin{aligned}&\int(\nabla\times f_1)\cdot(\nabla\times h)dx\\&=\int\nabla\times\big(V\Delta\chi_M+2\nabla\chi_M\cdot\nabla V-\chi_M(V\cdot\nabla\chi_M)V\big)\nabla\times hdx\\&\quad-\int\nabla\times(\frac{1}{2}\chi_M^2\nabla|V|^2)\nabla\times hdx-\int\nabla\times\big(h\cdot\nabla(\chi_MV)+\chi_MV\cdot\nabla h\big)\nabla\times hdx.\end{aligned}\tag{3.19}$$

The first line on the right-hand side of (3.19) is treated as follows:

$$\Big|\int\nabla\times\big(V\Delta\chi_M+2\nabla\chi_M\nabla V-\chi_M(V\cdot\nabla\chi_M)V\big)\nabla\times hdx\Big|$$

$$\le C\big(M^{-2}\|V\|_{L^2(|x|\le 2M)}+M^{-1}\|\nabla V\|_{L^2(|x|\le 2M)}+M^{-1}\|V\|_{L^4}^2\big)\|\nabla\nabla\times h\|_{L^2}$$
$$\le CM^{-1}e^{-t}+\frac{1}{16}\|\nabla\nabla\times h\|_{L^2}^2. \tag{3.20}$$

Similarly, for the second term on the right-hand side of (3.19), we have

$$\Big|\int \nabla\times(\frac{1}{2}\chi_M^2\nabla|V|^2)\nabla\times h dx\Big|=\Big|\int\nabla\times(\frac{1}{2}|V|^2\nabla\chi_M^2)\nabla\times h dx\Big|$$
$$\le CM^{-2}e^{-2t}+\frac{1}{16}\|\nabla\nabla\times h\|_{L^2}^2. \tag{3.21}$$

We estimate the last term on the right-hand side of (3.19) as follows:

$$\begin{aligned}&\Big|\int\nabla\times\big(-h\cdot\nabla(\chi_M V)-\chi_M V\cdot\nabla h\big)\nabla\times h dx\Big|\\&\le\big(\|h\|_{L^2}\|\nabla(\chi_M V)\|_{L^\infty}+\|\chi_M V\|_{L^\infty}\|\nabla h\|_{L^2}\big)\|\nabla\nabla\times h\|_{L^2}\\&\le Ce^{-t/2}\big(\|h\|_{L^2}+\|\nabla h\|_{L^2}\big)\|\nabla\nabla\times h\|_{L^2}\\&\le Ce^{-t}\big(\|h\|_{L^2}^2+\|\nabla\times h\|_{L^2}^2+\|V\cdot\nabla\chi_M\|_{L^2}^2\big)+\frac{1}{16}\|\nabla\nabla\times h\|_{L^2}^2\\&\le Ce^{-t}\big(\|h\|_{L^2}^2+\|\nabla\times h\|_{L^2}^2\big)+M^{-1}e^{-2t}+\frac{1}{16}\|\nabla\nabla\times h\|_{L^2}^2.\end{aligned}$$

For the fourth term on the right-hand side of (3.14),

$$\begin{aligned}&-\int(h\cdot\nabla v+v\cdot\nabla h)\Delta v+h\cdot\nabla\theta\Delta\theta dx\\&\le C(\|h\|_{L^3}+\|v\|_{L^3})(\|\nabla v\|_{L^6}+\|\nabla h\|_{L^6}+\|\nabla\theta\|_{L^6})(\|\Delta v\|_{L^2}+\|\Delta\theta\|_{L^2})\\&\le C(\|h\|_{L^3}+\|v\|_{L^3})(\|\Delta h\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2).\end{aligned} \tag{3.22}$$

For the last term on the right-hand side of (3.14),

$$\begin{aligned}&\int f_2\Delta v+f_3\Delta\theta dx\le C\big(\|\chi_M V\|_{L^\infty}\|\nabla v\|_{L^2}\\&\quad+\|\nabla(\chi_M V)\|_{L^\infty}\|v\|_{L^2}+\|\chi_M V\|_{L^\infty}\|\nabla\theta\|_{L^2}\big)(\|\Delta v\|_{L^2}+\|\Delta\theta\|_{L^2})\\&\le Ce^{-t/2}(\|v\|_{L^2}+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\frac{1}{16}(\|\Delta v\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2).\end{aligned} \tag{3.23}$$

We finally arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\times h\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)\\&\quad+\big(\frac{11}{16}-C\|h\|_{L^3}-C\|v\|_{L^3}\big)(\|\nabla\nabla\times h\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2)\\&\le Ce^{-t}\big(\|h\|_{L^2}^2+\|\nabla\times h\|_{L^2}^2\big)+\frac{1}{16}\|\nabla h\|_{L^2}^2+M^{-1}e^{-t}.\end{aligned} \tag{3.24}$$

Now let us add up (3.13) and (3.24) to yield that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|h\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2+\|\nabla\times h\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\big)\\&\quad+\big(\frac{5}{8}-C\|h\|_{L^3}-C\|v\|_{L^3}\big)\cdot\big(\|\nabla h\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\end{aligned}$$

$$
\begin{aligned}
&+\|\nabla\nabla\times h\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2\big)\\
\le & Ce^{-t/2}\big(\|h\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2+\|\nabla\times h\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\big)+M^{-1}e^{-t}. \quad (3.25)
\end{aligned}
$$

If there holds

$$
C\|h\|_{L^3}+C\|v\|_{L^3}\le\frac{5}{8}, \tag{3.26}
$$

on some time interval $0\le t\le T$, then (3.25) implies

$$
\begin{aligned}
&\|h\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2+\|\nabla\times h\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\\
&\quad+\int_0^T\big(\|\nabla h\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla\nabla\times h\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2\big)ds\\
\le & CM^{-1}, \quad (3.27)
\end{aligned}
$$

on the same time interval. Since

$$
\begin{aligned}
&C\|h\|_{L^3}\le C\big(\|h\|_{L^2}+\|\nabla\times h\|_{L^2}+\|\nabla\cdot h\|_{L^2}\big)\le CM^{-\frac{1}{2}},\quad 0\le t\le T,\\
&C\|v\|_{L^3}\le C\big(\|v\|_{L^2}+\|\nabla v\|_{L^2}\big)\le CM^{-\frac{1}{2}},\quad 0\le t\le T,
\end{aligned} \tag{3.28}
$$

one sees that the second inequality in (3.26) is verified provided that M is sufficiently large. A standard continuation argument simply implies that (3.27) holds for all time $t\ge 0$. Then one has

$$
\|(h,v,\theta)\|_{H^1}\le CM^{-\frac{1}{2}}, \tag{3.29}
$$

for all time $t\ge 0$. Since $u=h+\chi_M V$, one has $u\in L^\infty(0,T;H^1), v\in L^\infty(0,T;H^1), \theta\in L^\infty(0,T;H^1)$, which is sufficient for the global regularity of u,v and θ.

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