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ON UNIFORM SECOND-ORDER NONLOCAL APPROXIMATIONS TO DIFFUSION AND SUBDIFFUSION EQUATIONS WITH NONLOCAL EFFECT PARAMETER*

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Abstract. In this paper we focus on uniform convergence rates from nonlocal diffusion and subdiffusion solutions to the corresponding local limit with respect to a nonlocal effect parameter without extra assumptions on the regularity of nonlocal solutions, and present sufficient conditions to guarantee first- and second-order convergence rates, respectively. To do so, we first revisit the maximum principle for nonlocal models using the idea in [Luchko, J. Math. Anal. Appl., 2009], and present the uniqueness of the nonlocal solutions. After that, we extend the methodology developed in [Du, Zhang and Zheng, Commun. Math. Sci., 2019] to address the truncated errors on the volume constraints, and then combine the resulting errors from the boundary domain with the maximum principle to obtain the uniform convergence rates. Our analysis shows that the constant value continuation of the boundary conditions of local problems only leads to a first-order convergence rate. If one expects a second-order convergence rate, the information of first-order derivatives for local problems on the boundaries is required. One- and two-dimensional numerical examples are provided to validate our theoretical analysis.

Keywords. Subdiffusion equation; Caputo derivative; maximum principle; nonlocal model; local limit; asymptotically compatible.

AMS subject classifications. 82C21; 65R20; 65M60; 46N20; 45A05.

1. Introduction

Nonlocal models have received much attention due to their practical applications to various fields such as the peridynamical theory of continuum mechanics, nonlocal wave propagation and nonlocal diffusion process [1, 5, 19, 26, 29]. Nonlocal models associated with nonlocal diffusion and nonlocal peridynamics often involve a parameter δ to describe the range of nonlocal effect [5]. As the nonlocal effect parameter vanishes (i.e., $\delta \to 0$), solutions of nonlocal models converge to that of the corresponding local problems under suitable assumptions on the kernel functions and the given data [4, 5]. Such local limits play important roles in both modeling and the validation/verification of numerical simulations to capture the underlying physical and mathematical meanings.

Many studies on the local limits have been carried out, for instance, Taylor expansions with sufficiently smooth solutions [19, 20], functional analytical means without extra regularity assumptions [15, 16], the maximum principle of nonlocal operator demonstrated in analog of the local second-order elliptic operator [9], and asymptotically compatible schemes proposed in [24, 25] to guarantee the consistency of numerical schemes with the local limit. More investigations on local limits for bounded domain problems refer to [2, 5, 7, 9, 10, 25, 29].

Generically, in free space, the smoothness of nonlocal solutions can be increased if the smoothness of the external force is increased. In this situation, by directly using the Taylor expansion, nonlocal solutions converge to the local limit in second-order rate with respect to δ [6, 29]. This property becomes more complicated for problems with

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nonlocal volume constraints. This is to say, nonlocal solutions may possibly be discontinuous across the domain boundary even if the external force is smooth enough. The discontinuity will propagate into the interior domain, which leads to the weak regularity of the solution [8, 9, 21]. Under the assumption on this practical weak regularity, it is interesting to ask if it still remains valid that nonlocal solutions converge to their local limits in second-order rate? As mentioned above, the conventional methods that were based on the high regularity of the nonlocal solution will fail in proving the local limits. Recently, the study in [9] considers the nonlocal approximations to linear two-point boundary value problems (BVPs) with Dirichlet and mixed boundary conditions: The authors first investigate the well-posedness and regularity of the resulting nonlocal problems, then apply the maximum principle (more suitable for the practical regularity condition) to present the convergence of nonlocal solutions to the local solution as $\delta \to 0$, and finally give a sufficient condition to ensure the second-order convergence rate. The maximum principle is the key step to circumvent the use of strong regularity of nonlocal solutions for their local limits. It is worth mentioning that convergence of nonlocal solutions to the corresponding local limits for Neumann-type boundary condition are studied in [3, 23, 28].

In this paper, we will study the convergence for the solutions of the following nonlocal equations

$$(\partial_t^\alpha + \mathcal{L}_\delta) u_\delta(\mathbf{x},t) = f_\delta(\mathbf{x},t), \quad 0<\alpha\le 1,\, (\mathbf{x},t)\in\Omega\times(0,T], \tag{1.1}$$

to that of the local equation

$$(\partial_t^\alpha + \mathcal{L}_0) u_0(\mathbf{x},t) = f_0(\mathbf{x},t), \quad (\mathbf{x},t)\in\Omega\times(0,T], \tag{1.2}$$

where $f_\delta(\mathbf{x},t)$ and $f_0(\mathbf{x},t)$ are given source functions, the nonlocal operator $\mathcal{L}_\delta$ is defined in (2.2), $\mathcal{L}_0 = -\Delta$, $\partial_t^1 = \partial_t$ represents the standard first-order derivative, ∂_t^α $(0<\alpha<1)$ represents the Caputo fractional derivative, namely,

$$\partial_t^\alpha v(\mathbf{x},t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{\partial_s v(\mathbf{x},s)}{(t-s)^\alpha}\,\mathrm{d}s, & 0<\alpha<1, \\ \partial_t, & \alpha = 1. \end{cases}$$

Diffusion is one of the most prominent transport mechanisms in nature. The classical diffusion problem (1.2) describes the Brownian motion of the particle. Over the last decades, various experimental studies illustrated that the Brownian motion assumption may not be adequate for accurately describing some physical processes, which follow the subdiffusion or anomalous diffusion, also referred to as the non-Gaussian process, and can be better described by model (1.1). One can refer to the comprehensive surveys [17, 18] and the monograph [12] for several practical applications and physical modeling of the subdiffusion.

As stated earlier, we consider the convergence rate for solutions of (1.1) to that of (1.2) with respect to δ as an asymptotic parameter. To be specific, we address the convergence behaviors with respect to δ without extra assumptions on the regularity of nonlocal solutions, and present different sufficient conditions to guarantee the first- and second-order convergence rates, respectively. To this end, we first present the maximum principle for problem (1.1) based on the results in [14], and prove the uniqueness of the solution for (1.1). After that, we use the methodology developed in [9] to address the truncated errors on the volume constraints, and then combine the resulting errors from boundary domain with the maximum principle to obtain the uniform convergence rate.

We find that not all imposed boundary conditions lead to the second-order convergence with respect to δ. Specifically, if the constant value continuation of the boundary conditions of local problems is used, only first-order convergence rate could be obtained. If we expect to ensure second-order convergence rate, the information of the first-order derivative for the local solution on the boundary is required.

The rest of the paper is organized as follows. In section 2, we streamline several useful notations to introduce the definition of nonlocal operator, initial and boundary conditions. In section 3, based on the techniques in [14], we present the maximum principle and a priori estimate for the nonlocal problem (1.1) with initial and Dirichlet boundary conditions (2.3), and then prove the uniqueness of the nonlocal solution. In section 4, the convergence rates for the nonlocal solutions to their local limits are discussed for various initial and Dirichlet boundary conditions. In section 5, numerical examples are provided to validate our theoretical analysis.

2. Notation

We begin with several useful notations to introduce the definition of nonlocal operator. Let $\Omega \subset \mathbb{R}^d$ be a bounded, open and convex domain with Lipschitz continuous boundary. Denote $\overline{\Omega}$ and $\partial\Omega$ by the closure and boundary of Ω, respectively. We also introduce the following notations:

$$\Omega_\delta^c = \{\mathbf{y} \in \mathbb{R}^d \backslash \Omega : \mathrm{dist}(\mathbf{y}, \partial\Omega) < \delta\}, \quad \Omega_\delta^+ = \Omega \cup \Omega_\delta^c,$$
$$\Omega_T = \Omega \times (0,T), \quad \Omega_T^\partial = \partial\Omega \times [0,T],$$
$$\Omega_{\delta,T}^c = \Omega_\delta^c \times [0,T], \quad \Omega_{\delta,T}^+ = \Omega_\delta^+ \times (0,T).$$

Let $L((0,T))$ be the set of functions Lebesgue-integrable in $(0,T)$, $W_t^1((0,T))$ be the space of functions $f \in C^1((0,T])$ and $f' \in L((0,T))$. Define the space

$$CW_T(\Omega) = C(\overline{\Omega}_T) \cup W_t^1((0,T)) \cup C_{\mathbf{x}}^2(\Omega).$$

In this work, the non-negative symmetric kernel $\gamma_\delta(\mathbf{x},\mathbf{y})$ is assumed to satisfy the following conditions: $\forall \mathbf{x} \in \Omega$,

$$\begin{cases} \gamma_\delta(\mathbf{x},\mathbf{y}) = 0, \ \forall \mathbf{y} \in \Omega_\delta^+ \backslash B_\delta(\mathbf{x}), \\ \gamma_\delta(\mathbf{x},\mathbf{y}) = \tilde{\gamma}_\delta(|\mathbf{y}-\mathbf{x}|), \\ \frac{1}{2}\int_{B_\delta(\mathbf{0})} z_i^2 \tilde{\gamma}_\delta(|\mathbf{z}|)\,\mathrm{d}\mathbf{z} = 1, \ i = 1, \cdots, d. \end{cases} \tag{2.1}$$

Here $B_\delta(\mathbf{x}) := \{\mathbf{y} \in \Omega_\delta^+ : |\mathbf{y}-\mathbf{x}| \le \delta\}$. This implies that nonlocal interactions are radially symmetric and limited to a spherical neighborhood of radius δ.

2.1. Definition of nonlocal operator.

For $v(\mathbf{x}) : \Omega_\delta^+ \to \mathbb{R}$, the nonlocal operator $\mathcal{L}_\delta$ on $v(\mathbf{x})$ is defined as

$$\mathcal{L}_\delta v(\mathbf{x}) = \int_{\Omega_\delta^+} \big(v(\mathbf{x}) - v(\mathbf{y})\big)\gamma_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} \quad \forall \mathbf{x} \in \Omega. \tag{2.2}$$

The nonlocal operator (2.2) has an intimate connection with the local differential operator [4]. If a function $v(\mathbf{x}) \in C^4(\overline{\Omega_\delta^+})$, it holds for any $\mathbf{x} \in \Omega$ that

$$\mathcal{L}_\delta v(\mathbf{x}) = -\int_{\Omega_\delta^+} \big(v(\mathbf{y}) - v(\mathbf{x})\big)\gamma_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}$$
$$= -\int_{B_\delta(\mathbf{0})} \Big(\mathbf{z}\cdot\nabla v(\mathbf{x}) + \frac{1}{2}\sum_{i,j=1}^d z_i z_j \partial_{ij}^2 v(\mathbf{x}) + \frac{1}{6}\sum_{i,j,k=1}^d z_i z_j z_k \partial_{ijk}^3 v(\mathbf{x}) + \mathcal{O}(\delta^4)|v|_4\Big)\gamma_\delta(|\mathbf{z}|)\,\mathrm{d}\mathbf{z}$$

$$= -\frac{1}{2}\int_{B_\delta(\mathbf{0})} \Big(\sum_{i=1}^{d} z_i^2 \partial_{ii}^2 v(\mathbf{x})\Big)\gamma_\delta(|\mathbf{z}|)\,\mathrm{d}\mathbf{z} + \mathcal{O}(\delta^2) = \mathcal{L}_0 v(\mathbf{x}) + \mathcal{O}(\delta^2).$$

This implies that, acting on the smooth functions suitably away from the boundary, the nonlocal operator converges to the local differential operator in second-order rate with respect to δ. As mentioned before, the smoothness of the nonlocal solution will probably decrease while imposing volume constraints. In this article we will discuss how to circumvent the weak regularity of nonlocal solution to obtain the corresponding convergence rates.

2.2. Initial and Dirichlet boundary conditions. We consider the initial and Dirichlet boundary conditions (IDBCs) for nonlocal problem (1.1) as follows

$$\begin{cases} u_\delta(\mathbf{x},t) = g_\delta(\mathbf{x},t), \ (\mathbf{x},t)\in\Omega^c_{\delta,T}, \\ u_\delta(\mathbf{x},0) = \varphi_\delta(\mathbf{x}), \quad \mathbf{x}\in\Omega_\delta^+, \end{cases} \tag{2.3}$$

and for local problem (1.2) as follows

$$\begin{cases} u_0(\mathbf{x},t) = g_0(\mathbf{x},t), \ (\mathbf{x},t)\in\Omega^\partial_T, \\ u_0(\mathbf{x},0) = \varphi_0(\mathbf{x}), \quad \mathbf{x}\in\overline{\Omega}, \end{cases} \tag{2.4}$$

where the consistency relationship between the boundary and initial conditions should be satisfied. This means that

$$g_\delta(\mathbf{x},0) = \varphi_\delta(\mathbf{x}), \forall \mathbf{x}\in\Omega^c_\delta, \quad g_0(\mathbf{x},0) = \varphi_0(\mathbf{x}), \forall\mathbf{x}\in\partial\Omega.$$

3. A priori estimate for local and nonlocal problems

In this section, we first revisit the useful results in [14] for the extreme principle of Caputo derivative, maximum and minimum principles, and a priori estimate for the local problem (1.2) with IDBCs (2.4). After that, we present the corresponding properties of nonlocal problem (1.1) with IDBCs (2.3).

3.1. A priori estimate for local problem. It is well-known that, unlike the conventional first-order derivative, the condition that the Caputo fractional derivative of a given function $y(t)$ is positive can not guarantee the monotonicity of the function $y(t)$. But, the following lemma shows an important property that the Caputo derivative of a given function at its maximum point is non-negative.

LEMMA 3.1 (Extreme principle [14]). *Assume $y(t)\in W_t^1((0,T))\cup C([0,T])$ has its maximum over the interval $[0,T]$ at point $t_0\in(0,T]$. Then it holds that*

$$\partial_t^\alpha y(t_0) \geq 0, \quad \forall \alpha\in(0,1). \tag{3.1}$$

As in [14], by defining the bottom and back-side parts of the boundary for the domain Ω_T as $B_T := (\overline{\Omega}\times\{0\})\cup\Omega_T^\partial$, the following lemma on the maximum and minimum principles holds true.

LEMMA 3.2 (Maximum and minimum principles [14]). *Assume the function $u_0(\mathbf{x},t)\in CW_T(\Omega)$ is the solution of local problem (1.2) with IDBCs (2.4). If $f_0(\mathbf{x},t)\leq 0, \forall(\mathbf{x},t)\in\Omega_T$, then either $u_0(\mathbf{x},t)\leq 0$, $\forall(\mathbf{x},t)\in\Omega_T$ or u_0 attains its positive maximum on B_T, namely,*

$$u_0(\mathbf{x},t) \leq \max\Big\{0, \max_{(\mathbf{x},t)\in B_T} u_0(\mathbf{x},t)\Big\}, \quad \forall(\mathbf{x},t)\in\overline{\Omega}_T.$$

On the other hand, if $f_0(\mathbf{x},t)\geq 0, \forall(\mathbf{x},t)\in\Omega_T$*, then either* $u_0(\mathbf{x},t)\geq 0$*,* $\forall(\mathbf{x},t)\in\Omega_T$ *or* u_0 *attains its negative minimum on* B_T*, namely,*

$$u_0(\mathbf{x},t)\geq \min_{(\mathbf{x},t)\in B_T} u_0(\mathbf{x},t), \quad \forall(\mathbf{x},t)\in\overline{\Omega}_T.$$

The *a priori* estimate for the solution to local problem (1.2) with IDBCs (2.4) is given as follows.

THEOREM 3.1 (A priori estimate [14]). *Assume* $u_0(\mathbf{x},t)\in CW_T(\Omega)$ *is the solution of the local problem (1.2) with IDBCs* (2.4) *and* $f_0(\mathbf{x},t)\in C(\overline{\Omega}_T)$ *with the norm* $M:=\|f_0\|_{C(\overline{\Omega}_T)}$*. Then the following a priori estimate holds*

$$\|u_0\|_{C(\overline{\Omega}_T)}\leq \max\{M_0,M_1\}+\frac{T^\alpha}{\Gamma(1+\alpha)}M,$$

with $M_0=\|\varphi_0\|_{C(\overline{\Omega}_T)}$ *and* $M_1=\|g_0\|_{C(\Omega_T^\partial)}$*.*

3.2. A priori estimate for nonlocal problem. In this subsection, we present the maximum and minimum principles, a priori estimate, and the uniqueness of the solutions for nonlocal problems (1.1) with IDBCs (2.3). The technique we use to prove them is similar to that in [14], however, there are some details we need to modify since nonlocal problems have their own distinguished properties.

First, denote the bottom and back-side parts of the boundary of Ω_T by

$$B_T^c:=(\Omega_\delta^+\times\{0\})\cup\Omega_{\delta,T}^c,$$

and define the space

$$CW_{\delta,T}(\Omega):=C(\overline{\Omega}_T)\cup W_t^1((0,T]).$$

We have the following lemma which we refer to as the nonlocal maximum and minimum principles.

LEMMA 3.3 (Nonlocal maximum and minimum principles). *Assume the function* $u_\delta(\mathbf{x},t)\in CW_{\delta,T}(\Omega)$ *is the solution of nonlocal problem (1.1) with IDBCs* (2.3)*. If* $f_\delta(\mathbf{x},t)\leq 0,\forall(\mathbf{x},t)\in\Omega_T$*, then either* $u_\delta(\mathbf{x},t)\leq 0$*,* $\forall(\mathbf{x},t)\in\Omega_T$ *or* u_δ *attains its positive maximum on* B_T^c*, that is,*

$$u_\delta(\mathbf{x},t)\leq \max\left\{0,\max_{(\mathbf{x},t)\in B_T^c} u_\delta(\mathbf{x},t)\right\}, \quad \forall(\mathbf{x},t)\in\Omega_{\delta,T}^+. \tag{3.2}$$

If $f_\delta(\mathbf{x},t)\geq 0,\forall(\mathbf{x},t)\in\Omega_T$*, then either* $u_\delta(\mathbf{x},t)\geq 0$*,* $\forall(\mathbf{x},t)\in\Omega_T$ *or* u_δ *attains its negative minimum on* B_T^c*, that is,*

$$u_\delta(\mathbf{x},t)\geq \min_{(\mathbf{x},t)\in B_T^c} u_\delta(\mathbf{x},t), \quad \forall(\mathbf{x},t)\in\Omega_{\delta,T}^+. \tag{3.3}$$

Proof. We just provide the proof for (3.2). We proceed by contradiction and assume that u_δ is the solution of nonlocal problem (1.1) with IDBCs (2.3), however, there is an interior point $(\mathbf{x}_0,t_0)\in\Omega_T$ such that

$$u_\delta(\mathbf{x}_0,t_0)>M:=\max\left\{0,\max_{(\mathbf{x},t)\in B_T^c} u_\delta(\mathbf{x},t)\right\}>0.$$

Set $\varepsilon = u_\delta(\mathbf{x}_0,t_0) - M > 0$ and construct an auxiliary function

$$w(\mathbf{x},t) = u_\delta(\mathbf{x},t) + \frac{\varepsilon}{2}\cdot\frac{T-t}{T}, \quad (\mathbf{x},t)\in\Omega^+_{\delta,T}. \tag{3.4}$$

Thus, it holds for any $(\mathbf{x},t)\in B^c_T$ that

$$w(\mathbf{x}_0,t_0) \geq u_\delta(\mathbf{x}_0,t_0) = \varepsilon + M \geq \varepsilon + u_\delta(\mathbf{x},t) \geq \frac{\varepsilon}{2} + w(\mathbf{x},t),$$

which implies that the function $w(\mathbf{x},t)$ has its maximum in the interior domain of Ω_T, say $(\mathbf{x}_1,t_1)\in\Omega_T$, to satisfy

$$w(\mathbf{x}_1,t_1) \geq w(\mathbf{x}_0,t_0) \geq \varepsilon + M > \varepsilon.$$

It follows from the extreme principle of Caputo derivative in Lemma 3.1 and the definition of nonlocal operator that

$$\partial_t^\alpha w(\mathbf{x}_1,t_1) \geq 0 \text{ and } \mathcal{L}_\delta w(\mathbf{x}_1,t_1) \geq 0. \tag{3.5}$$

Using the relationship between $u_\delta(\mathbf{x},t)$ and $w(\mathbf{x},t)$ in (3.4), we have

$$\partial_t^\alpha u_\delta(\mathbf{x},t) = \partial_t^\alpha w(\mathbf{x},t) + \frac{\varepsilon}{2T}\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{3.6}$$

where we also use the well-known formula for the Caputo fractional derivative

$$\partial_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} t^{\beta-\alpha}, \beta > 0, 0 < \alpha \leq 1.$$

Combining (3.5) and (3.6) it is straightforward to verify that

$$\begin{aligned}&\partial_t^\alpha u_\delta(\mathbf{x}_1,t_1) + \mathcal{L}_\delta u_\delta(\mathbf{x}_1,t_1) - f_\delta(\mathbf{x}_1,t_1)\\ =&\partial_t^\alpha w(\mathbf{x}_1,t_1) + \frac{\varepsilon}{2T}\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \mathcal{L}_\delta w(\mathbf{x}_1,t_1) - f_\delta(\mathbf{x}_1,t_1) \geq \frac{\varepsilon}{2T}\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} > 0,\end{aligned}$$

which contradicts the fact that $u_\delta(\mathbf{x},t)$ is a solution of Equation (1.1). The proof is complete. ∎

Now we apply the nonlocal maximum and minimum principles to prove that the nonlocal problem (1.1) with IDBCs (2.3) has at most one solution and this solution (if it exists) depends continuously on the given data.

First, the *a priori* estimate for the nonlocal solution u_δ is given as follows.

THEOREM 3.2 (Nonlocal a priori estimate). *Assume $u_\delta(\mathbf{x},t)\in CW_{\delta,T}(\Omega)$ is the solution of nonlocal problem (1.1) with IDBCs (2.3), and $f_\delta(\mathbf{x},t)\in C(\overline{\Omega}_T)$ with the norm $M := \|f_\delta\|_{C(\overline{\Omega}_T)}$. Then the following a priori estimate holds*

$$\|u_\delta\|_{C(\overline{\Omega}_T)} \leq \max\{M_0,M_1\} + \frac{T^\alpha}{\Gamma(1+\alpha)}M, \tag{3.7}$$

where $M_0 = \|\varphi_\delta\|_{C(\overline{\Omega}^+_\delta)}$ and $M_1 = \|g_\delta\|_{C(\Omega^c_{\delta,T})}$.

Proof. Set an auxiliary function

$$w(\mathbf{x},t) = u_\delta(\mathbf{x},t) - \frac{Mt^\alpha}{\Gamma(1-\alpha)}, \quad (\mathbf{x},t)\in\Omega^+_{\delta,T}.$$

Then $w(\mathbf{x},t)$ is the solution of nonlocal problem (1.1) with the Dirichlet boundary condition $g_1(\mathbf{x},t)=g_\delta(\mathbf{x},t)-\frac{t^\alpha}{\Gamma(1-\alpha)}M$ instead of $g_\delta(\mathbf{x},t)$, and the source function $f_1(\mathbf{x},t)=f_\delta(\mathbf{x},t)-M$ instead of $f_\delta(\mathbf{x},t)$. Noting that $f_1(\mathbf{x},t)\le 0$, the nonlocal maximum principle (3.2) leads to

$$w(\mathbf{x},t)\le \max\{M_0,M_1\},\quad \forall(\mathbf{x},t)\in\Omega^+_{\delta,T},$$

which implies that

$$u_\delta(\mathbf{x},t)=w(\mathbf{x},t)+\frac{Mt^\alpha}{\Gamma(1-\alpha)}\le \max\{M_0,M_1\}+\frac{MT^\alpha}{\Gamma(1-\alpha)}. \tag{3.8}$$

Similarly, set another auxiliary function

$$w(\mathbf{x},t)=u_\delta(\mathbf{x},t)+\frac{Mt^\alpha}{\Gamma(1-\alpha)},\quad (\mathbf{x},t)\in\Omega^+_{\delta,T}.$$

The nonlocal minimum principle (3.3) leads to

$$u_\delta(\mathbf{x},t)\ge -\max\{M_0,M_1\}-\frac{MT^\alpha}{\Gamma(1-\alpha)},\quad \forall(\mathbf{x},t)\in\Omega^+_{\delta,T}. \tag{3.9}$$

The combination of equations (3.8) and (3.9) completes the proof. □

By the nonlocal *a priori* estimate, we immediately obtain the following result on the uniqueness of the solution for nonlocal problem (1.1) with IDBCs (2.3).

THEOREM 3.3 (Uniqueness). *The nonlocal problem (1.1) with IDBCs (2.3) has at most one solution. The solution continuously depends on the data given in the problem in the sense that if*

$$\varepsilon_0=\|f_\delta-\tilde f_\delta\|_{C(\overline{\Omega}_T)},\quad \varepsilon_1=\|\varphi_\delta-\tilde\varphi_\delta\|_{C(\overline{\Omega})},\quad \varepsilon_2=\|g_\delta-\tilde g_\delta\|_{C(\Omega^c_{\delta,T})},$$

the corresponding solutions u_δ and $\tilde u_\delta$ hold the estimate

$$\|u_\delta-\tilde u_\delta\|_{C(\overline{\Omega}_T)}\le \max\{\varepsilon_1,\varepsilon_2\}+\frac{\varepsilon_0 T^\alpha}{\Gamma(1+\alpha)}. \tag{3.10}$$

As pointed in [14], the maximum principle cannot ensure the existence of the solution. Since the nonlocal operator $\mathcal{L}_\delta$ defined in (2.2) is positive-definite and self-adjoint under the condition (2.1), the method of separation of variables might be used to construct a formal solution to (1.1).

4. Convergence rates for the approximation of nonlocal solutions to their local limits

As taking $\delta\to 0$, it is well-known that $\mathcal{L}_\delta v$ converges to $\mathcal{L}_0 v$ in a second-order rate with respect to δ when v is smooth enough. Here we investigate the asymptotic error of the nonlocal diffusion problem when nonlocal volume constraints on the boundary layers are imposed, which will lead to a weak regularity of the solution. In this situation, we expect to figure out, for what kind of nonlocal IDBCs, nonlocal solutions converge to their local limits with a second-order asymptotic rate in δ. We first consider 1D case for simplicity and then express the idea for general dimensional cases.

4.1. One dimensional case. Denote by $B_T^l=(a-\delta,a)\times(0,T]$, and $B_T^r=(b,b+\delta)\times(0,T]$.

THEOREM 4.1. *Let $\Omega=(a,b)$ be a bounded interval and $\delta<(b-a)/2$. Suppose $u_0(x,t)\in C_x^4(\Omega)\times W_t^1((0,T])$ is the unique solution to the local problem (1.2) with the following IDBCs*

$$\begin{cases} u_0(a,t)=g_0(a,t), & t\in(0,T],\\ u_0(b,t)=g_0(b,t), & t\in(0,T],\\ u_0(x,0)=\varphi_0(x), & x\in[a,b]. \end{cases}$$

If u_δ is the unique solution of nonlocal problem (1.1) with IDBCs given as

$$\begin{cases} u_\delta(x,t)=g_0(a,t),\ (x,t)\in B_T^l,\\ u_\delta(x,t)=g_0(b,t),\ (x,t)\in B_T^r,\\ u_\delta(x,0)=\varphi_\delta(x),\ \ x\in\Omega_\delta^+, \end{cases} \tag{4.1}$$

and

$$\|\varphi_\delta-\varphi_0\|_{C(\overline{\Omega})}=\mathcal{O}(\delta),\quad \|f_\delta-f_0\|_{C(\overline{\Omega}_T)}=\mathcal{O}(\delta).$$

Then it holds that

$$\|u_\delta-u_0\|_{C(\overline{\Omega}_T)}=\mathcal{O}(\delta). \tag{4.2}$$

On the other hand, if u_δ is the unique solution of nonlocal problem (1.1) with IDBCs given as

$$\begin{cases} u_\delta(x,t)=g_0(a,t)+(x-a)\partial_x u_0(a,t), & (x,t)\in B_T^l\\ u_\delta(x,t)=g_0(b,t)+(x-b)\partial_x u_0(b,t), & (x,t)\in B_T^r\\ u_\delta(x,0)=\varphi_\delta(x), & x\in\Omega_\delta^+, \end{cases} \tag{4.3}$$

and

$$\|\varphi_\delta-\varphi_0\|_{C(\overline{\Omega})}=\mathcal{O}(\delta^2),\quad \|f_\delta-f_0\|_{C(\overline{\Omega}_T)}=\mathcal{O}(\delta^2).$$

Then it holds that

$$\|u_\delta-u_0\|_{C(\overline{\Omega}_T)}=\mathcal{O}(\delta^2). \tag{4.4}$$

Proof. We first prove (4.2), to this aim we construct an auxiliary solution

$$\tilde{u}_0(x,t)=\begin{cases} \displaystyle\sum_{m=0}^{4}\frac{(x-a)^m}{m!}\partial_x^4 u_0(a,t), & (x,t)\in B_T^l\\ u_0(x,t), & (x,t)\in\Omega_T,\\ \displaystyle\sum_{m=0}^{4}\frac{(x-b)^m}{m!}\partial_x^4 u_0(b,t), & (x,t)\in B_T^r. \end{cases}$$

Since $\tilde{u}_0\in C_b^4(\Omega_\delta^+)\times W_t^1((0,T])$, the direct calculation leads to

$$\begin{aligned}(\partial_t^\alpha+\mathcal{L}_\delta)\tilde{u}_0(x,t)&=(\partial_t^\alpha+\mathcal{L}_0)u_0(x,t)+\mathcal{O}(\delta^2)\\ &=f_0(x,t)+\mathcal{O}(\delta^2)\\ &=f_\delta(x,t)+\mathcal{O}(\delta),\forall(x,t)\in\Omega_T.\end{aligned}$$

Thus, it holds that

$$\begin{cases} (\partial_t^\alpha + \mathcal{L}_\delta)(u_\delta(x,t) - \tilde{u}_0(x,t)) = \mathcal{O}(\delta), & (x,t) \in \Omega_T, \\ u_\delta(x,t) - \tilde{u}_0(x,t) = \mathcal{O}(\delta), & (x,t) \in B_T^l, \\ u_\delta(x,t) - \tilde{u}_0(x,t) = \mathcal{O}(\delta), & (x,t) \in B_T^r, \\ u_\delta(x,0) - \tilde{u}_0(x,0) = \varphi_\delta(x) - \varphi_0(x) = \mathcal{O}(\delta), & x \in \Omega_\delta^+, \end{cases}$$

where in the derivation of the last formula we use the consistency relationship between the boundary and initial conditions. The direct application of Theorem 3.3 to the above problem produces

$$\|u_\delta - \tilde{u}_0\|_{C(\overline{\Omega}_T)} = \mathcal{O}(\delta).$$

Thus we arrive at (4.2) by the definition of function $\tilde{u}_0$.

Using the same argument for IDBCs (4.3) to arrive at

$$\begin{cases} (\partial_t^\alpha + \mathcal{L}_\delta)(u_\delta(x,t) - \tilde{u}_0(x,t)) = \mathcal{O}(\delta^2), & (x,t) \in \Omega_T, \\ u_\delta(x,t) - \tilde{u}_0(x,t) = \mathcal{O}(\delta^2), & (x,t) \in B_T^l, \\ u_\delta(x,t) - \tilde{u}_0(x,t) = \mathcal{O}(\delta^2), & (x,t) \in B_T^r, \\ u_\delta(x,0) - \tilde{u}_0(x,0) = \mathcal{O}(\delta^2), & x \in \Omega_\delta^+. \end{cases}$$

We finally have $\|u_\delta - \tilde{u}_0\|_{C(\overline{\Omega}_T)} = \mathcal{O}(\delta^2)$ from Theorem 3.3 and

$$\|u_\delta - u_0\|_{C(\overline{\Omega}_T)} = \mathcal{O}(\delta^2),$$

by the definition of function $\tilde{u}_0$. The proof is complete. ∎

4.2. General dimensional case. We have derived for what kind of nonlocal IDBCs, nonlocal solutions converge to their local limits in second-order rate with respect to δ, for 1D case. In fact, the key point is to require the nonlocal IDBCs be a second-order approximation to an extension of the local IDBCs. To be specific, we construct the function $\tilde{u}_0$ as the extension of u_0 by Taylor expansion. However, for high dimensional case, this kind of extension is rather complicated. Here we describe the extension by a universal way for general dimensional case. It is worth mentioning that a C^4 nonlocal extension is given in Appendix A.2 of [28] by using Taylor expansions with respect to normal and tangential derivatives up to the fourth orders, under the condition that the boundary $\partial\Omega$ is regular enough.

DEFINITION 4.1. *Given $u_0 \in C_{\mathbf{x}}^4(\Omega) \times W_t^1((0,T])$, $u_0(\mathbf{x},t)|_{\Omega_T^\partial} = g_0(\mathbf{x},t)$. A function $\tilde{u}_0$ is called a C^4 nonlocal extension of u_0, if the following conditions hold*

$$\begin{cases} \tilde{u}_0(\mathbf{x},t) \in C_{\mathbf{x}}^4(\Omega_\delta^+) \times W_t^1((0,T]), \\ \tilde{u}_0(\mathbf{x},t)|_{\Omega_T^\partial} = g_0(\mathbf{x},t), \\ \tilde{u}_0(\mathbf{x},t)|_{\Omega\times[0,T]} = u_0(\mathbf{x},t). \end{cases}$$

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, open and convex domain with Lipschitz continuous boundary, and $\delta < diam(\Omega)/2$. Suppose $u_0(\mathbf{x},t) \in C_{\mathbf{x}}^4(\Omega) \times W_t^1((0,T])$ is the unique solution to the local problem (1.2) with IDBCs (2.4). Let $u_\delta(\mathbf{x},t)$ be the unique solution to the nonlocal problem (1.1) with IDBCs (2.3). If $\tilde{u}_0(\mathbf{x},t)$ is a C^4 nonlocal extension of $u_0(\mathbf{x},t)$ and*

$$\|\tilde{u}_0 - g_\delta\|_{C(\Omega_{\delta,T}^c)} = \mathcal{O}(\delta^k),$$

and

$$\|f_\delta - f_0\|_{C(\overline{\Omega}_T)} = \mathcal{O}(\delta^k),$$

then it holds that

$$\|u_\delta - u_0\|_{C(\overline{\Omega}_T)} = \mathcal{O}(\delta^{\min\{k,2\}}). \tag{4.5}$$

Proof. Since $\tilde{u}_0(\mathbf{x},t) \in C_b^4(\Omega_\delta^+) \times W_t^1((0,T])$, the direct calculation leads to

$$\begin{aligned}(\partial_t^\alpha + \mathcal{L}_\delta)\tilde{u}_0(\mathbf{x},t) &= (\partial_t^\alpha + \mathcal{L}_0)u_0(\mathbf{x},t) + \mathcal{O}(\delta^2)\\ &= f_\delta(\mathbf{x},t) + \mathcal{O}(\delta^{\min\{k,2\}}), \forall (\mathbf{x},t) \in \Omega_T.\end{aligned}$$

Thus, it holds that

$$\begin{cases} (\partial_t^\alpha + \mathcal{L}_\delta)(u_\delta(\mathbf{x},t) - \tilde{u}_0(\mathbf{x},t)) = \mathcal{O}(\delta^{\min\{k,2\}}), & (\mathbf{x},t) \in \Omega_T,\\ u_\delta(\mathbf{x},t) - \tilde{u}_0(\mathbf{x},t) = g_\delta(\mathbf{x},t) - \tilde{u}_0(\mathbf{x},t) = \mathcal{O}(\delta^k), & (\mathbf{x},t) \in \Omega_{\delta,T}^c,\\ u_\delta(\mathbf{x},0) - \tilde{u}_0(\mathbf{x},0) = \varphi_\delta(\mathbf{x}) - \varphi_0(\mathbf{x}) = \mathcal{O}(\delta^k), & \mathbf{x} \in \Omega_\delta^+, \end{cases}$$

where we use the consistency relationship between the boundary and initial conditions in the derivation of the last formula. The direct application of Theorem 3.3 to the above problem produces

$$\|u_\delta(\mathbf{x},t) - \tilde{u}_0(\mathbf{x},t)\|_{C(\overline{\Omega}_T)} = \mathcal{O}(\delta^{\min\{k,2\}}).$$

The proof is complete. ∎

We point out that although the constructed solution $\tilde{u}_0(\mathbf{x},t)$ is required to be C^4 with respect to $\mathbf{x}$, the nonlocal solution itself, i.e. $u_\delta(\mathbf{x},t)$, does not require this restriction. As we will see in Section 5, $u_\delta(\mathbf{x},t)$ could be discontinuous across $\partial\Omega$ with respect to $\mathbf{x}$.

5. Numerical Experiments

As the nonlocal effect parameter $\delta \to 0$, the above theoretical analysis shows that nonlocal solutions converge to the local one in different orders based on how to impose the volume constraints by using information of the local problem. We now give 1D and 2D numerical examples to verify the sharpness of our theoretical analysis by using kernel function γ_δ in 1D case given by

$$\tilde{\gamma}_\delta(r) = \begin{cases} 3/\delta^3, & r \le \delta,\\ 0, & r > \delta, \end{cases}$$

and in 2D case by

$$\tilde{\gamma}_\delta(r) = \begin{cases} 8/\pi/\delta^4, & r \le \delta,\\ 0, & r > \delta. \end{cases}$$

It is straightforward to verify that these two kernel functions satisfy the condition (2.1). In all calculations, we use the Crank-Nicolson scheme to discretize the time derivative for the diffusion equation with $\alpha = 1$, and the L^1-scheme [13,22] to discretize the Caputo fractional derivative for subdiffusion equation with $0 < \alpha < 1$ while the conforming DG method proposed in [8] is adopted to discretize the space direction for the nonlocal operator. We first introduce the two methods, L^1-scheme and the conforming DG method, respectively.

L^1-scheme and its fast algorithm. Set the kernel $\omega_\beta(t)=t^{\beta-1}/\Gamma(\beta)$, $u^k=g(t_k)$ and the difference operator $\nabla_\tau u^k=u^k-u^{k-1}$ for $k\geq 1$. The nonuniform L^1 formula of Caputo derivative is given by

$$D_N^\alpha u^n=\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\omega_{1-\alpha}(t_n-s)\nabla_\tau u^k/\tau_k\,\mathrm{d}s=\sum_{k=1}^{n}a_{n-k}^{(n)}\nabla_\tau u^k, \tag{5.1}$$

where the time-level-dependent convolution coefficient $a_{n-k}^{(n)}$ is defined by

$$a_{n-k}^{(n)}=\int_{t_{k-1}}^{t_k}\frac{\omega_{1-\alpha}(t_n-s)}{\tau_k}\,\mathrm{d}s,\quad 1\leq k\leq n.$$

It is obvious that the cost of the direct evaluation of L^1-scheme (5.1) for small time steps or long-time simulations is quite expensive since it needs all history information of solutions. So the fast algorithm [11, 27] is often used to speed up the evaluation of the Caputo derivative. A basic result of sum-of-exponentials (SOEs) approximation (see [11, Theorem 2.5] or [27, Lemma 2.2]) is given as follows:

THEOREM 5.1. *Given $\alpha\in(0,1)$, an absolute tolerance error $\varepsilon\ll 1$, a cut-off time $\Delta t>0$ and a final time T, there exists a positive integer N_q, positive quadrature nodes θ^ℓ and positive weights ϖ^ℓ $(1\leq\ell\leq N_q)$ such that*

$$\left|\omega_{1-\alpha}(t)-\sum_{\ell=1}^{N_q}\varpi^\ell e^{-\theta^\ell t}\right|\leq\varepsilon\quad\forall t\in[\Delta t,T],$$

where the number N_q of quadrature nodes satisfies

$$N_q=O\left(\log\frac{1}{\varepsilon}\left(\log\log\frac{1}{\varepsilon}+\log\frac{T}{\Delta t}\right)+\log\frac{1}{\Delta t}\left(\log\log\frac{1}{\varepsilon}+\log\frac{1}{\Delta t}\right)\right).$$

To obtain the fast L^1 algorithm, the Caputo derivative $\partial_t^\alpha u(t_n)$ is split into a sum of a local part (an integral over $[t_{n-1},t_n]$) and a history part (an integral over $[0,t_{n-1}]$), and approximate u' by a linear interpolation of the local part (same as the standard L^1 method) and use the SOE technique in Lemma 5.1 to approximate the kernel $\omega_{1-\alpha}(t-s)$ in the history part, thus

$$\begin{aligned}(\partial_t^\alpha u)(t_n)&\approx\int_{t_{n-1}}^{t_n}\omega_{1-\alpha}(t_n-s)\frac{\nabla_\tau u^n}{\tau_n}\,\mathrm{d}s+\int_0^{t_{n-1}}\sum_{\ell=1}^{N_q}\varpi^\ell e^{-\theta^\ell(t_n-s)}u'(s)\,\mathrm{d}s\\&=a_0^{(n)}\nabla_\tau u^n+\sum_{\ell=1}^{N_q}\varpi^\ell e^{-\theta^\ell\tau_n}\mathcal{H}^\ell(t_{n-1}),\quad n\geq 1,\end{aligned}$$

where $\mathcal{H}^\ell(t_k):=\int_0^{t_k}e^{-\theta^\ell(t_k-s)}u'(s)\,\mathrm{d}s$ with $\mathcal{H}^\ell(t_0)=0$ for $1\leq\ell\leq N_q$. $\mathcal{H}^\ell(t_k)$ can be efficiently computed by applying a linear interpolation on each cell $[t_{k-1},t_k]$, namely

$$\begin{aligned}\mathcal{H}^\ell(t_k)&=e^{-\theta^\ell\tau_k}\mathcal{H}^\ell(t_{k-1})+\int_{t_{k-1}}^{t_k}e^{-\theta^\ell(t_k-s)}u'(s)\,\mathrm{d}s\\&\approx e^{-\theta^\ell\tau_k}\mathcal{H}^\ell(t_{k-1})+b^{(k,\ell)}\nabla_\tau u^k,\end{aligned}$$

where

$$b^{(k,\ell)} := \frac{1}{\tau_k}\int_{t_{k-1}}^{t_k} e^{-\theta^\ell(t_k-s)}\,\mathrm{d}s, \quad k\ge 1,\, 1\le \ell\le N_q. \tag{5.2}$$

The conforming DG method. We use the conforming DG method proposed in [8] to discretize the space direction, i.e., to obtain the corresponding two stiff matrices, A and C. Here we take the two-dimensional case for illustration. For a given triangulation $\mathcal{T}_{\delta,h}$ of Ω_δ^+ that simultaneously triangulates Ω, let $\Omega_{\delta,h}=\mathcal{T}_{\delta,h}\cap\overline{\Omega}$ and $\Omega_{\delta,h}^c=\mathcal{T}_{\delta,h}\cap\overline{\Omega_\delta^c}$. The set of inner nodes of $\Omega_{\delta,h}$, i.e., all nodes in $\mathcal{T}_{\delta,h}\cap\Omega$, is denoted by $NI=\{\mathbf{x}_j : j=1,2,\cdots,m\}$, with piecewise-linear basis functions defined on $\mathcal{T}_{\delta,h}$ being $\phi_j(\mathbf{x}), j=1,2,\cdots,m$. The set of all nodes in $\mathcal{T}_{\delta,h}\cap\partial\Omega$ is denoted by $NB=\{\mathbf{x}_{m+j}: j=1,2,\cdots,n\}$ with piecewise-linear basis functions defined on $\mathcal{T}_{\delta,h}$ being $\phi_{m+j}(\mathbf{x}), j=1,2,\cdots,n$. The basis functions for the space $V_{\delta,h}^0$ are as follows: for $j=1,2,\cdots,m+n$,

$$\widetilde{\phi}_j(\mathbf{x})=\begin{cases}\phi_j(\mathbf{x})|_{\Omega_{\delta,h}}, & \mathbf{x}\in\Omega_{\delta,h},\\ 0, & \mathbf{x}\in\mathcal{T}_{\delta,h}\backslash\Omega_{\delta,h}.\end{cases}$$

The set of all nodes in $\Omega_{\delta,h}^c$ is denoted by $NC=\{\mathbf{x}_j : j=m+1,m+2,\cdots,m+n+p\}$ with piecewise-linear basis functions defined on $\mathcal{T}_{\delta,h}$ being $\phi_j(\mathbf{x}), j=m+1,m+2,\cdots,m+n+p$. Note that $NB\subset NC$. The basis functions for the space $V_{\delta,h}^g$ are as follows: for $j=1,2,\cdots,p+n$,

$$\widetilde{\phi}_j^c(\mathbf{x})=\begin{cases}\phi_{m+j}(\mathbf{x})|_{\Omega_{\delta,h}^c}, & \mathbf{x}\in\Omega_{\delta,h}^c,\\ 0, & \mathbf{x}\in\mathcal{T}_{\delta,h}\backslash\Omega_{\delta,h}^c.\end{cases}$$

Then we define the finite element space $V_{\delta,h}^+=V_{\delta,h}^0\bigoplus V_{\delta,h}^g$, or

$$V_{\delta,h}^+=\left\{\mathbf{u}\cdot\left[\widetilde{\phi}_1,\cdots,\widetilde{\phi}_{m+n}\right]^T+\mathbf{g}\cdot\left[\widetilde{\phi}_1^c,\cdots,\widetilde{\phi}_{p+n}^c\right]^T : \mathbf{u}\in\mathbb{R}^{m+n},\mathbf{g}\in\mathbb{R}^{p+n}\right\}.$$

Denote by $n_1=m+n$, and $n_2=n+p$, the two stiff matrices are given by

$$A=(a_{ij})_{n_1\times n_1},\, a_{i,j}=2\int_\Omega\widetilde{\phi}_i(\mathbf{x})\int_\Omega\widetilde{\phi}_j(\mathbf{y})\gamma_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}.$$

and

$$C=(c_{ij})_{n_1\times n_2},\, c_{i,j}=2\int_\Omega\widetilde{\phi}_i(\mathbf{x})\int_{\Omega_\delta^c}\widetilde{\phi}_j^c(\mathbf{y})\gamma_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}.$$

Here we use the finite element space $V_{\delta,h}^+$ which is continuous on $\Omega_{\delta,h}$ and $\Omega_{\delta,h}^c$, respectively. However, it is possibly discontinuous across $\partial\Omega$, thus we regard it a conforming but hybrid version of DG and continuous FEM [8].

This paper mainly focuses on the model errors with respect to δ as an asymptotic parameter, and considers the convergence rate by imposing various IDBCs. In this situation, we take the temporal and spatial mesh sizes τ and h small enough, say $\tau,h\ll\delta$, until the mesh sizes do not affect the convergence with respect to δ. On the other hand, the theory developed in [24, 25] ensures the convergence since the linear element is an asymptotically compatible scheme such that its numerical solutions can correctly converge to the corresponding local limit. So we fix $h=\delta/4$ and refine δ for 1D case, while fixing h and refining δ for the 2D case. However, both strategies will lead to the same convergence rates for 1D and 2D cases.

5.1. One dimensional case.

EXAMPLE 5.1. *Consider the convergence rate in Theorem 4.1 by constructing an exact solution to local problem (1.2),*

$$u_0(x,t) = (1+t^\sigma)(x^2+x^4+x^5+x^7),$$

with the computation domain $(a,b)=(0,1)$. Here σ is a regularity parameter to indicate the weak regularity of the solution of subdiffusion equation at the initial time, and generally taken as $\sigma \geq \alpha$. For the local problem (1.2), we have the source term

$$f_0(x,t) = -(1+t^\sigma)(42x^5+20x^3+12x^2+2) + \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-\alpha)} t^{\sigma-\alpha}(x^2+x^4+x^5+x^7),$$

and the Dirichlet boundary conditions $u_0(a,t)=0$, $u_0(b,t)=4(1+t^\sigma)$. For the nonlocal problem (1.1), we take $f_\delta(x,t)=f_0(x,t)$, $(x,t)\in\Omega_T$ and consider the following two kinds of IDBCs:

1. *(4.1) which leads to the first-order convergence w.r.t. δ by Theorem 4.1,*
2. *(4.3) which leads to the second-order convergence w.r.t. δ by Theorem 4.1.*

The first kind of IDBCs, i.e. (4.1), means the boundary condition of nonlocal problem over a layer with the width δ is extended by directly using the information of boundary values of the local problem. The second kind of IDBCs, i.e. (4.3), means the boundary condition of nonlocal problems over a layer with the width δ is extended by using the information of both boundary values and first-order derivative values of the local problem, this is, the first-order Taylor expansion.

δ	$\alpha=0.2,\sigma=0.4$		$\alpha=\sigma=0.5$		$\alpha=\sigma=0.8$		$\alpha=\sigma=1$	
	Error	Order	Error	Order	Error	Order	Error	Order
1/32	4.06e-01	–	3.95e-01	–	3.64e-01	–	3.47e-01	–
1/64	2.00e-01	1.02	1.94e-01	1.02	1.79e-01	1.04	1.71e-01	1.02
1/128	9.92e-02	1.01	9.63e-02	1.01	8.88e-02	1.01	8.45e-02	1.01
1/256	4.94e-02	1.01	4.80e-02	1.01	4.42e-02	1.01	4.73e-02	0.96

TABLE 5.1. *L^∞-errors and convergence rates between nonlocal solutions and local solutions for IDBCs (4.1) by fixing $h=\delta/4$ and refining δ.*

δ	$\alpha=0.2,\sigma=0.4$		$\alpha=\sigma=0.5$		$\alpha=\sigma=0.8$		$\alpha=\sigma=1$	
	Error	Order	Error	Order	Error	Order	Error	Order
1/8	2.77e-01	–	2.48e-01	–	1.77e-01	–	2.37e-01	–
1/16	6.98e-02	1.97	6.22e-02	1.99	4.45e-02	1.99	5.94e-02	1.99
1/32	1.75e-02	1.99	1.55e-02	2.00	1.12e-02	2.00	1.47e-02	2.01
1/64	4.39e-03	2.00	3.86e-03	2.00	2.80e-03	2.00	3.49e-03	2.07

TABLE 5.2. *L^∞-errors and convergence rates between nonlocal solutions and local solutions for IDBCs (4.3) by fixing $h=\delta/4$ and refining δ.*

To demonstrate the convergence rates for (4.1), we take the final time $T=0.5$, the time size $\tau=1.25\times10^{-4}$ for $0<\alpha<1$ and $\tau=6.25\times10^{-5}$ for $\alpha=1$, fix $h=\delta/4$ and refine $\delta=2^{-5},2^{-6},2^{-7},2^{-8}$. Table 5.1 shows the convergence rates for different α. The convergence rates for (4.3) are studied similarly where the results are shown in Table 5.2.

5.2. Two dimensional case.

EXAMPLE 5.2. *Consider the convergence rate in Theorem 4.2 by constructing an exact solution to the local problem (1.2) as*

$$u_0(\mathbf{x},t) = (1+t^\sigma)x_1x_2(1-x_1)(1-x_2),\, (\mathbf{x},t)\in\Omega_T,$$

with $\Omega = (0,1)\times(0,1)$. For the local problem (1.2), we have the source term

$$f_0(\mathbf{x},t) = 2(1+t^\sigma)(x_1(1-x_1)+x_2(1-x_2)) + \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-\alpha)}t^{\sigma-\alpha}x_1x_2(1-x_1)(1-x_2),$$

and $u_0(\mathbf{x},t)=0$ for $\mathbf{x}\in\partial\Omega$. For the nonlocal problem, we take $f_\delta(\mathbf{x},t)=f_0(\mathbf{x},t)$, $(\mathbf{x},t)\in\Omega_T$, and consider the following two kinds of IDBCs:

(1) *$g_\delta(\mathbf{x},t)=0$, $(\mathbf{x},t)\in\Omega^c_{\delta,T}$, $\varphi_\delta(\mathbf{x})$ is determined by the consistency relationship between the boundary and the initial conditions.*
(2) *$g_\delta(\mathbf{x},t)=(1+t^\sigma)x_1x_2(1-x_1)(1-x_2)$, $(\mathbf{x},t)\in\Omega^c_{\delta,T}$, $\varphi_\delta(\mathbf{x})$ is determined by the consistency relationship between the boundary and the initial conditions.*

Here we set

$$\tilde{u}_0(\mathbf{x},t) = (1+t^\sigma)x_1x_2(1-x_1)(1-x_2),\, (\mathbf{x},t)\in\Omega^+_{\delta,T}.$$

Obviously, it is a C^4 nonlocal extension of $u_0(\mathbf{x},t)$. Then for the first kind of IDBCs, it holds that

$$\|\tilde{u}_0(\mathbf{x},t)-g_\delta(\mathbf{x},t)\|_{C(\Omega^c_{\delta,T})} = \mathcal{O}(\delta),$$

so we expect by Theorem 4.2

$$\|u_\delta(\mathbf{x},t)-u_0(\mathbf{x},t)\|_{C(\overline{\Omega}_T)} = \mathcal{O}(\delta).$$

While for the second kind of IDBCs, it holds that

$$\tilde{u}_0(\mathbf{x},t) = g_\delta(\mathbf{x},t),\, (\mathbf{x},t)\in\Omega^c_{\delta,T},$$

so we expect by Theorem 4.2

$$\|u_\delta(\mathbf{x},t)-u_0(\mathbf{x},t)\|_{C(\overline{\Omega}_T)} = \mathcal{O}(\delta^2).$$

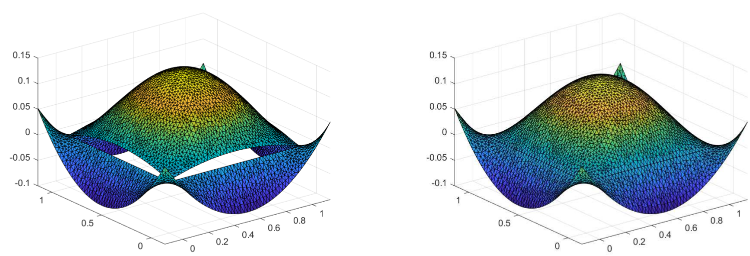
To demonstrate the convergence rates, we take the final time $T=1$, the time size $\tau = 2\times10^{-2}$ and fix $h=0.8\cdot2^{-5}$ and refine $\delta=0.8\cdot2^{-k/2}$. We also test different time step sizes, for example $\tau=1.25\times10^{-4}$, etc.. The results are similar to that for $\tau=2\times10^{-2}$, however, as τ becomes smaller, the computational time will grow. Tables 5.3 and 5.4 show the convergence rates by using the first and second kinds of IDBCs, respectively, by taking $\alpha=0.2,0.5,0.8,1$. We also plot in Figure 5.1 the nonlocal solutions at $t=T$ for the two kinds of IDBCs with $\alpha=0.2$ and 1. As we see, although $f_\delta(\mathbf{x},t)$ is a very smooth function with respect to $\mathbf{x}$, $u_\delta(\mathbf{x},t)$ is discontinuous across $\partial\Omega$ with respect to $\mathbf{x}$. Note that the discontinuity could be recognized clearly for the first kind of IDBCs. The case for the second kind of IDBCs is not clear to distinguish, at least from intuitive feeling. However, from the construction of the finite element space $V^+_{\delta,h}$, we know that the discontinuity across $\partial\Omega$ is permitted. Thus our results on uniform convergence rates from nonlocal solutions to the corresponding local limits with respect to δ do not require extra assumptions on the regularity of nonlocal solutions.

$\delta=\frac{0.8}{2^{k/2}}$	$\alpha=\sigma=0.2$		$\alpha=\sigma=0.5$		$\alpha=\sigma=0.8$		$\alpha=\sigma=1$	
	Error	Order	Error	Order	Error	Order	Error	Order
$k=0$	1.38e-01	–	1.41e-01	–	1.43e-01	–	1.44e-01	–
$k=1$	1.01e-01	0.91	1.02e-01	0.93	1.03e-01	0.94	1.04e-01	0.95
$k=2$	7.16e-02	0.99	7.21e-02	1.00	7.26e-02	1.01	7.28e-02	1.02
$k=3$	5.07e-02	1.00	5.09e-02	1.00	5.12e-02	1.01	5.12e-02	1.01
$k=4$	3.58e-02	1.01	3.59e-02	1.01	3.60e-02	1.01	3.60e-02	1.02
$k=5$	2.52e-02	1.01	2.53e-02	1.01	2.53e-02	1.01	2.53e-02	1.02

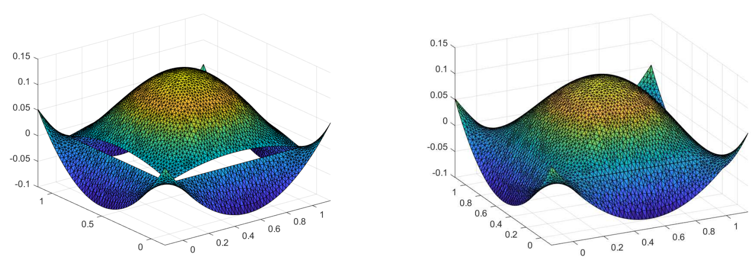
TABLE 5.3. *L^∞-errors and convergence rates between nonlocal solutions and their local limits for the first kind of IDBCs by fixing $h=0.8\cdot 2^{-5}$ and refining δ.*

$\delta=\frac{0.8}{2^{k/2}}$	$\alpha=\sigma=0.2$		$\alpha=\sigma=0.5$		$\alpha=\sigma=0.8$		$\alpha=\sigma=1$	
	Error	Order	Error	Order	Error	Order	Error	Order
$k=0$	5.65e-02	–	5.75e-02	–	5.87e-02	–	5.92e-02	–
$k=1$	2.56e-02	2.29	2.60e-02	2.29	2.65e-02	2.30	2.66e-02	2.31
$k=2$	1.10e-02	2.42	1.12e-02	2.43	1.14e-02	2.43	1.14e-02	2.44
$k=3$	5.00e-03	2.29	5.08e-03	2.28	5.20e-03	2.27	5.14e-03	2.30
$k=4$	2.34e-03	2.19	2.39e-03	2.18	2.48e-03	2.14	2.40e-03	2.20
$k=5$	1.14e-03	2.09	1.17e-03	2.06	1.24e-03	2.00	1.16e-03	2.10

TABLE 5.4. *L^∞-errors and convergence rates between nonlocal solutions and their local limits for the second kind of IDBCs by fixing $h=0.8\cdot 2^{-5}$ and refining δ.*



(a) The first kind of IDBCs, $\alpha=0.2$ (b) The second kind of IDBCs, $\alpha=0.2$



(c) The first kind of IDBCs, $\alpha=1$ (d) The second kind of IDBCs, $\alpha=1$

FIG. 5.1. *Example 5.2: nonlocal solutions at $t=T$ for different IDBCs, $k=5$*

6. Conclusion

The issue of the local limit for nonlocal problems is very important in both modeling and the validation/verification of numerical simulations to capture the underlying physical and mathematical meanings. In this paper, the local limit for diffusion and subdiffusion equations is considered with respect to the nonlocal effect parameter δ. Based on the maximum principle for nonlocal problems and methodology developed in [9], we show that the convergence rates to the local limit depends on how to impose the initial and Dirichlet boundary conditions, and it is not necessary to impose the extra assumptions on the regularity of nonlocal solutions. Specifically for the 1D case, if we simply use the constant value continuation of the Dirichlet boundary conditions of local problems, there is only first-order convergence rate. If we expect second-order convergence rate, the information of the first-order derivatives for local problems on the boundaries is required.

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