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THE BRAMSON CORRECTION FOR INTEGRO-DIFFERENTIAL FISHER–KPP EQUATIONS*

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Abstract. We consider integro-differential Fisher–KPP equations with nonlocal diffusion. For typical equations, we establish the logarithmic Bramson delay for solutions with step-like initial data. That is, these solutions resemble a front at position $c_*t - \frac{3}{2\lambda_*}\log t + \mathcal{O}(1)$ for explicit constants c_* and λ_*. Certain strongly asymmetric diffusions exhibit more exotic behaviour.

Keywords. Reaction-diffusion; integro-differential equation; Fisher–KPP; asymptotics.

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1. Introduction

We study integro-differential Fisher–KPP equations:

$$\partial_t u = \mu\,(J * u - u) + f(u) \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{1.1}$$

Here $\mu > 0$ is a constant, J is a Borel probability measure on $\mathbb{R}$, and f is a KPP nonlinearity as defined below. We supplement (1.1) with the initial data $u(0,\,\cdot\,) = \mathbf{1}_{\mathbb{R}_-}$. Then a standard contraction argument yields a unique mild solution in L^∞. Moreover, (1.1) satisfies the comparison principle, so $0 \le u \le 1$.

We assume that the measure J has the following moments:

(J1) $\displaystyle\int_{\mathbb{R}} |x|\, J(\mathrm{d}x) < \infty$ and there exists $\overline{\lambda} > 0$ such that $\displaystyle\int_{\mathbb{R}} \mathrm{e}^{\overline{\lambda}x}\, J(\mathrm{d}x) < \infty.$

The latter is sometimes known as Mollison's condition [35]. Next, if J has an atom at the origin, it can be absorbed by changing μ. We are thus free to assume that

(J2) $J\big(\{0\}\big) = 0.$

Here, we deploy standard measure-theoretic notation, so that $J\big(\{0\}\big)$ is the measure of the (singleton) Borel set $\{0\}$ under J. Finally, we often assume a weak form of exponential monotonicity for J:

(J3) There exist $\Lambda, M > 0$ such that

$$\Lambda J\big([x, x+M]\big) \ge J\big((x+M,\infty)\big)$$

for all $x \ge 0$. The same holds for the spatial reverse J^* given by $J^*(A) = J(-A)$ for all Borel sets $A \subset \mathbb{R}$.

Informally, (J3) says that that the mass of J near $x \ge 0$ is not much smaller than the mass far to the right. If J has a density $\mathcal{J}$ that is eventually bounded between two positive multiples of an exponential, it satisfies (J3). Alternatively, if $\mathrm{e}^{\lambda|x|}\mathcal{J}(x)$ is eventually decreasing for each $\lambda > 0$, then J satisfies (J3). This describes Gaussian and compactly-supported kernels, for example.

The condition (J3) plays an important role in our comparison arguments. We seek comparison principles on the half-line, but nonlocal interactions with the complementary

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half-line complicate matters. Assumption (J3) limits the influence of the complement. This hypothesis is likely technical: it enables our comparison strategy, but we expect our results to hold in its absence.

We emphasise that we make no symmetry assumption on J. We choose to treat the general case due to its richer behaviour. In particular, the phenomena described in Theorem 1.2 never occur in the symmetric setting.

The nonlinear "reaction" f in (1.1) is *KPP* [30], meaning:

(F1) $f \in \mathcal{C}^1([0,1])$ and $f \in \mathcal{C}^{1,\gamma}$ near 0 for some $\gamma \in (0,1)$;

(F2) $f(0) = f(1) = 0$ and $f|_{(0,1)} > 0$;

(F3) $f(u) \leq f'(0)u$ for all $u \in [0,1]$.

If we replace the nonlocal diffusion $J * u - u$ in (1.1) by the Laplacian, we obtain the classical Fisher–KPP equation [19,30]

$$\partial_t \upsilon = \mu \Delta \upsilon + f(\upsilon). \tag{1.2}$$

This equation models numerous invasion phenomena and the propagation of the solution is a subject of intense study. We expect similar propagation in (1.1), and are thus interested in the "position" of u as $t \to \infty$. Precisely, define

$$\sigma_\theta(t) \coloneqq \sup\{x \in \mathbb{R} \mid u(t,x) \geq \theta\} \tag{1.3}$$

for $\theta \in (0,1)$. Then σ_θ tracks the leading edge of u at level θ. In this work, we study the long-time asymptotics of σ_θ.

We are motivated by the delicate behaviour exhibited by solutions υ to the classical Fisher–KPP Equation (1.2). Broadly speaking, solutions to reaction-diffusion equations tend to resemble constant-speed "travelling waves". However, when the reaction f is KPP, solutions with step initial data lag behind such waves. Let $\mathring{\sigma}_\theta$ denote the leading edge of υ. The precise structure of $\mathring{\sigma}_\theta$ was first determined by Bramson [10,12], who exploited a connection with branching Brownian motion to show that

$$\mathring{\sigma}_\theta(t) = \mathring{c}_* t - \frac{3}{2\mathring{\lambda}_*} \log t + C_\theta + o(1) \quad \text{as } t \to \infty$$

for explicit constants $\mathring{c}_*, \mathring{\lambda}_* > 0$ depending only on μ and $f'(0)$. See also [39] for a shorter proof of this result to order $\mathcal{O}(1)$. We prove an analogous result for "typical" integro-differential Equations (1.1):

$$\sigma_\theta(t) = c_* t - \frac{3}{2\lambda_*} \log t + \mathcal{O}_\theta(1) \quad \text{as } t \to \infty, \tag{1.4}$$

where $c_* \in \mathbb{R}$ and $\lambda_* > 0$ depend on μ, J, and $f'(0)$. We note that u propagates quite differently if J has sufficiently fat tails; see, for instance, [8,15,20,22,34].

The leading-order speed c_* in (1.4) is related to travelling wave solutions to (1.1). These have the form $U_c(x - ct)$ for a speed $c \in \mathbb{R}$ and profile U_c satisfying

$$\begin{gathered} \mu(J * U_c - U_c) + c\,U_c' + f(U_c) = 0, \\ 0 \leq U_c \leq 1, \quad U_c(-\infty) = 1, \quad U_c(+\infty) = 0. \end{gathered} \tag{1.5}$$

Under mild conditions on J, Coville, Dávila, and Martínez [18] prove the existence of a minimal speed $c_* \in \mathbb{R}$ such that a monotone front U_c exists for each speed $c \geq c_*$. The minimal speed is given by

$$c_* = \inf_{\lambda>0} \Gamma(\lambda) \quad \text{for} \quad \Gamma(\lambda) := \frac{1}{\lambda}\left[\mu \int_{\mathbb{R}} \mathrm{e}^{\lambda x}\, J(\mathrm{d}x) - \mu + f'(0)\right]. \tag{1.6}$$

A few comments are in order.
REMARK 1.1.

(i) The fronts in [18] connect 0 to 1 rather than 1 to 0, so our spatial signs are opposite theirs.
(ii) The infimum in (1.6) is bounded due to (J1).
(iii) We make no symmetry assumption on J, so c_* can be nonpositive.
(iv) We assume a finite first-moment in (J1) to use [18]. However, in [18], J has a continuous density with respect to Lebesgue measure. We drop this assumption, so we require extensions of the results of [18]. We collect these observations in Proposition 3.1, which we prove in Appendix B.

The dynamics of (1.1) depend on whether the infimum in (1.6) is attained. We begin by classifying these cases:

PROPOSITION 1.1. *Let J satisfy (J1)–(J2) and f satisfy (F1)–(F3). If $J(\mathbb{R}_+) > 0$ or $f'(0) < \mu$, then Γ uniquely attains its infimum. Otherwise, if $J(\mathbb{R}_+) = 0$ and $f'(0) \geq \mu$, then Γ does not attain its infimum and $c_* = 0$.*

We can now present our main results. First, we show that (1.4) holds whenever the infimum in (1.6) is attained. This determines the constant λ_* in (1.4).

THEOREM 1.1. *Suppose that J satisfies (J1)–(J3) and f satisfies (F1)–(F3). Assume that $J(\mathbb{R}_+) > 0$ or $f'(0) < \mu$, so that Γ is uniquely minimised at some $\lambda_* > 0$. Then for all $\theta \in (0,1)$, there exists a constant $C(\mu, J, f, \theta) > 0$ such that*

$$\left|\sigma_\theta(t) - c_* t + \frac{3}{2\lambda_*}\log t\right| \leq C \quad \text{for all } t \geq 1.$$

This result is closely related to the work of Addario-Berry and Reed [3] on branching random walks; we describe this connection in Appendix A. Theorem 1.1 is also similar to results of Gao [21], who considers (1.1) with an additional term of the form $\varepsilon\partial_x^2 u$. Our methods, however, are quite different.

To prove Theorem 1.1, we use the framework developed by Hamel, Nolen, Roquejoffre, and Ryzhik [25] for the classical Fisher–KPP equation. We shift to a moving frame and relate the equation to a nonlocal *linear* Dirichlet problem on $\mathbb{R}_+$. We then constrain the position σ_θ using super- and subsolutions constructed from this linear problem. Ultimately, the method relies on key estimates for the long-time behaviour of the linear Dirichlet problem. We prove these estimates using probabilistic arguments via a Feynman–Kac representation.

In a certain sense, Theorem 1.1 is the generic case: "typical" nonlocal kernels J admit a minimizer in (1.6). However, Proposition 1.1 shows that the infimum in (1.6) need not be attained when J is strongly asymmetric. Then, several behaviours are possible.

THEOREM 1.2.

(i) *Suppose that J satisfies (J1)–(J2) and f satisfies (F1)–(F3). Assume that $J(\mathbb{R}_+) = 0$ and $f'(0) > \mu$. Then for all $\theta \in (0,1)$, there exists a constant $C(\mu, J, f, \theta) > 0$ such that $|\sigma_\theta(t)| \leq C$ for all $t \geq 0$.*

(ii) *Fix $\mu > 0$ and $p > 1$. Let $f(u) = \mu(u - u^p)$ and $J = \delta_{-1}$. Then for all $\theta \in (0, 1)$, there exists $C(\mu, p, \theta) > 0$ such that*

$$\left|\sigma_\theta(t) + \frac{\log\log t}{\log p}\right| \le C \quad \textit{for all } t \ge 2.$$

Suppose $J(\mathbb{R}_+) = 0$, so the nonlocal diffusion in (1.1) only involves leftward jumps. Then by Theorem 1.1, a weak reaction with $f'(0) < \mu$ leads to a leftward drift at speed $c_* < 0$ with a logarithmic delay. In contrast, Theorem 1.2(i) shows that a strong reaction $\big(f'(0) > \mu\big)$ prevents the drift: solutions remain a bounded distance from the origin. We note that this result does not require the hypothesis (J3). In the critical case $f'(0) = \mu$, irregular modes of propagation should be possible. Theorem 1.2(ii) details one such case. This example is closely related to the main result of Bramson in [11].

Using the comparison principle, we can extend our results to solutions evolving from "step-like" initial data $u(0, \cdot\,) = u_0$. Indeed, if $0 \le u_0 \le 1$ and $u_0 - \mathbf{1}_{\mathbb{R}_-}$ is compactly supported, then we can sandwich u between translations of the special step solution considered above. Thus Theorems 1.1 and 1.2 both apply to u, with constants C depending also on the initial data u_0.

Finally, we note that our model (1.1) is distinct from the well-studied "nonlocal Fisher–KPP equation", which involves a nonlocal *nonlinearity* rather than nonlocal diffusion. The nonlocal Fisher–KPP equation has garnered much attention in the last decade. See [6] for travelling waves, [9, 37] for the Bramson correction to propagation, and [1] for a recent probabilistic interpretation. There are two principal differences between (1.1) and the nonlocal Fisher–KPP Equation: (1.1) obeys the comparison principle, but does not enjoy parabolic regularity. The technical challenges in this work are thus quite different from those overcome in [9].

In Section 2, we detail our solution theory for (1.1) and prove a version of the comparison principle. In Section 3, we prove Proposition 1.1 and our main result, Theorem 1.1. We prove Theorem 1.2 in Section 4. In Appendix A, we detail the connection between (1.1) and branching random walks. We rigorously extend the construction of travelling waves in [18] to our setting in Appendix B.

2. A mild comparison principle

In this section, we lay the technical groundwork for the remainder of the paper. We define our solution theory for (1.1) and prove a comparison principle.

For later convenience, we study a wider class of equations in this section. Consider the integro-differential equation

$$\partial_t u = \mu(J * u - u) + h(u) \tag{2.1}$$

with Lipschitz $h\colon \mathbb{R} \to \mathbb{R}$. Let $T > 0$ denote a finite time horizon. We study mild solutions of (2.1) in $L^\infty([0, T] \times \mathbb{R})$. Since T is arbitrary, our results extend to solutions in $L^\infty_{\text{loc}}\big([0, \infty); L^\infty(\mathbb{R})\big)$.

In the proof of Theorem 1.1, we will construct functions that are only super- or subsolutions of (1.1) on proper subsets of $[0, T] \times \mathbb{R}$. We therefore develop a solution theory on subsets. Let $\varsigma\colon [0, T] \to \mathbb{R}$ be $\mathcal{C}^1$ and define the domain

$$\mathcal{D} \coloneqq \big\{(t, x) \in (0, T) \times \mathbb{R} \mid x > \varsigma(t)\big\}.$$

We set $\mathcal{D}^c \coloneqq \big((0, T) \times \mathbb{R}\big) \setminus \mathcal{D}$. For each $(t, x) \in \mathcal{D}$, let

$$\rho(t, x) \coloneqq \inf\big\{s \in (0, t) \mid (s, t) \times \{x\} \subset \mathcal{D}\big\}.$$

This function is locally constant in t. We then define the incoming boundary

$$\partial_* \mathcal{D} \coloneqq \left\{ (\rho(t,x), x) \in \overline{\mathcal{D}} \mid (t,x) \in \mathcal{D} \right\}.$$

This is the subset of $\partial\mathcal{D}$ on which the vector field ∂_t points into $\mathcal{D}$. At small scales, (2.1) is dominated by its first-order part $\partial_t v = 0$, so we must impose "initial" data on the incoming boundary $\partial_* \mathcal{D}$.

In this work, we will only use convex $\mathcal{D}$. Then ρ is independent of t and

$$\partial_* \mathcal{D} = \big(\{0\} \times [\varsigma(0), \infty)\big) \cup \left\{ (t, \varsigma(t)) \mid t > 0,\, \dot{\varsigma}(t) < 0 \right\}. \tag{2.2}$$

We now define our notion of a solution of (2.1).

Definition 2.1. *Given $u_* \in L^\infty(\partial_* \mathcal{D})$ and $g \in L^\infty(\mathcal{D}^c)$, we say $u \in L^\infty([0,T] \times \mathbb{R})$ is a* solution *of* (2.1) *in $\mathcal{D}$ with* initial data u_* *and* exterior data g *if $u = g$ on $\mathcal{D}^c$ and*

$$u(t,x) = u_*(\rho(t,x), x) + \int_{\rho(t,x)}^{t} \left[\mu\big(J * u(s,x) - u(s,x)\big) + h\big(u(s,x)\big) \right] \mathrm{d}s \tag{2.3}$$

for almost every $(t,x) \in \mathcal{D}$.

Abusing terminology, we extend this definition to $\mathcal{D} = (0,T) \times \mathbb{R}$, which formally corresponds to $\varsigma = -\infty$. In this case, $\rho = 0$, $\partial_* \mathcal{D} = \{0\} \times \mathbb{R}$, and $u_* \in L^\infty(\mathbb{R})$ is the usual initial data at time zero.

Proposition 2.1. *For every $u_* \in L^\infty(\partial_* \mathcal{D})$ and $g \in L^\infty(\mathcal{D}^c)$, there exists a unique solution $u \in L^\infty([0,T] \times \mathbb{R})$ of* (2.1) *in $\mathcal{D}$ with initial data u_* and exterior data g.*

Proof. This follows from a standard contraction argument; see [5, §2.2.1] for details. ∎

When we are not concerned with uniqueness, we will often suppress reference to the exterior data, as it is incorporated in the function u.

Suppose u solves (2.1) in $\mathcal{D}$ with initial data $u_* \in L^\infty(\partial_* \mathcal{D})$. We claim that u agrees almost everywhere (a.e.) with a measurable function that is continuous in t. Indeed, because $u \in L^\infty$, there is a full-measure set $E \subset \mathbb{R}$ such that for all $x \in E$, $|u(t,x)| \leq \|u\|_\infty$ and $|J * u(t,x)| \leq \|u\|_\infty$ for a.e. $t \in \big((0,T) \times \{x\}\big) \cap \mathcal{D}$. Hence for all $x \in E$, the integral in (2.3) is continuous (in fact, Lipschitz) in t. Since ρ is locally constant in t, $u_*(\rho(t,x),x)$ is also continuous in t. Thus, the right side of (2.3) is continuous in t for all $x \in E$. By definition, u agrees with this function a.e., as claimed.

Now suppose $\varsigma' \geq \varsigma$ and define $\mathcal{D}' \subset \mathcal{D}$ with corresponding incoming boundary $\partial_* \mathcal{D}'$. If u solves (2.1) in $\mathcal{D}$, it agrees with a measurable continuous-in-t function a.e. in $\mathcal{D}$. Hence it has a well-defined trace $u|_{\partial_* \mathcal{D}'}$ in $L^\infty(\partial_* \mathcal{D}')$. We can easily check that u is a solution to (2.1) in $\mathcal{D}'$ with initial data $u|_{\partial_* \mathcal{D}'}$. Thus, our notion of solution is compatible with restriction.

Next, we define super- and subsolutions.

Definition 2.2. *We say $u^\pm \in L^\infty([0,T] \times \mathbb{R})$ is a* supersolution *(resp.* subsolution*) of* (2.1) *in $\mathcal{D}$ with initial data $u_*^\pm \in L^\infty(\partial_* \mathcal{D})$ if*

$$G^\pm(t,x) \coloneqq u^\pm(t,x) - u_*^\pm(\rho(t,x),x) - \int_{\rho(t,x)}^{t} \left[\mu\big(J * u^\pm(s,x) - u^\pm(s,x)\big) + h\big(u^\pm(s,x)\big) \right] \mathrm{d}s$$

agrees a.e. with a measurable function that is nonnegative (resp. nonpositive) and nondecreasing (resp. nonincreasing) in t.

Next, recall $\Lambda, M > 0$ from (J3) and define the strip

$$\mathcal{B} := \big\{(t,x) \in (0,T) \times \mathbb{R} \mid \varsigma(t) - M \leq x \leq \varsigma(t)\big\}.$$

We can now state our comparison principle on $\mathcal{D}$.

PROPOSITION 2.2. *Let* $u^{\pm}$ *be a supersolution (resp. subsolution) of* (2.1) *in* $\mathcal{D}$ *with initial data* $u_*^{\pm}$. *Suppose* $u_*^+ \geq u_*^-$ *a.e. on* $\partial_*\mathcal{D}$ *and* $u^+ \geq 0$ *a.e. Moreover, suppose there exists a function* $\varpi \in L^\infty([0,T])$ *such that* $\varpi \geq 0$ *a.e.,* $u^- \leq \varpi$ *a.e. in* $\mathcal{D}^c$, *and* $u^+ \geq (\Lambda + 1)\varpi$ *a.e. in* $\mathcal{B}$. *Then* $u^+ \geq u^-$ *a.e. in* $\mathcal{D}$.

REMARK 2.1. The principal novelty of this statement is the fact that u^+ and u^- need not be ordered in $\mathcal{D}^c \setminus \mathcal{B}$. Instead, u^+ is large on the strip $\mathcal{B}$. Due to the structural assumption (J3), this compensates for a lack of control elsewhere in $\mathcal{D}^c$.

REMARK 2.2. Taking $\varsigma = -\infty$, we obtain the traditional comparison principle on the whole line. Formally, $u^+ \geq u^-$ a.e. if $u^+(0,\,\cdot\,) \geq u^-(0,\,\cdot\,)$ a.e.

REMARK 2.3. We have chosen to formulate our theory in familiar L^∞ spaces, which are quotients based on the essential supremum seminorm. We could instead work in the space $\mathscr{L}^\infty$ of bounded measurable functions under the supremum *norm*. We can evaluate functions in $\mathscr{L}^\infty$ pointwise, and the proof of Proposition 2.2 goes through without the need to avoid pathological null sets. For this reason, we omit the phrase "almost everywhere" from (in)equalities in the other sections of the paper. It is implied if one works in L^∞ and unnecessary in $\mathscr{L}^\infty$. The same convention applies to suprema and infima, which are implicitly essential in the L^∞ theory.

Proof. **(Proof of Proposition 2.2.)** Let $v := u^- - u^+$ and $v_* := u_*^- - u_*^+$. Since u^+ is a supersolution of (2.1) in $\mathcal{D}$ and u^- is a subsolution,

$$\begin{aligned} G(t,x) &:= v(t,x) - v_*(\rho(t,x),x) \\ &\qquad - \int_{\rho(t,x)}^t \big[\mu\big(J * v(s,x) - v(s,x)\big) + h\big(u^-(t,x)\big) - h\big(u^+(t,x)\big)\big]\,\mathrm{d}s \end{aligned}$$

is nonpositive and nonincreasing in t a.e. in $\mathcal{D}$. Define

$$q := \begin{cases} \frac{h(u^-)-h(u^+)}{u^- - u^+} & \text{if } u^+ \neq u^-, \\ 0 & \text{if } u^+ = u^-. \end{cases}$$

Then we obtain a linear equation

$$v(t,x) = v_*(\rho(t,x),x) + G(t,x) + \int_{\rho(t,x)}^t \big[\mu\big(J * v(s,x) - v(s,x)\big) + q(s,x)v(s,x)\big]\,\mathrm{d}s$$

in $\mathcal{D}$. By hypothesis, $v_*(\rho(t,x),x) \leq 0$ a.e. in $\mathcal{D}$.

Define $\kappa := \operatorname{Lip} h + \mu$, so that $\kappa - q \geq \mu$, and let $z := \mathrm{e}^{-\kappa t} v$. Integrating by parts, we can check that z satisfies

$$\begin{aligned} z(t,x) &= z_*(\rho(t,x),x) + H(t,x) \\ &\qquad + \int_{\rho(t,x)}^t \big[\mu\big(J * z(s,x) - z(s,x)\big) - \big(\kappa - q(s,x)\big)z(s,x)\big]\,\mathrm{d}s \end{aligned} \tag{2.4}$$

a.e. in $\mathcal{D}$, where $z_* := \mathrm{e}^{-\kappa t} v_*$ on $\partial_* \mathcal{D}$ and

$$H(t,x) := \mathrm{e}^{-\kappa t} G(t,x) + \int_0^t \mathrm{e}^{-\kappa s} G(s,x)\,\mathrm{d}s.$$

Recall that G agrees almost everywhere with a measurable function that is nonincreasing in t. Replacing G by this function and integrating by parts, we see that H is nonpositive and nonincreasing in t a.e. in $\mathcal{D}$.

Now, suppose for the sake of contradiction that $A := \operatorname{ess\,sup}_{\mathcal{D}} z > 0$. We have shown that there exist a full-measure set $\mathcal{E} \subset [0,T] \times \mathbb{R}$ and bounded measurable functions $\mathbf{u}^\pm$ on $[0,T] \times \mathbb{R}$, $\mathbf{u}_*^\pm$ on $\partial_* \mathcal{D}$, $\boldsymbol{\varpi}$ on $[0,T]$, and $\mathbf{H}$ on $\mathcal{D}$ such that if $\mathbf{z} := \mathrm{e}^{-\kappa t}\big(\mathbf{u}^- - \mathbf{u}^+\big)$ and $\mathbf{z}_* := \mathrm{e}^{-\kappa t}\big(\mathbf{u}_*^- - \mathbf{u}_*^+\big)$, the following hold:

- (P1) $u^\pm = \mathbf{u}^\pm$ on $\mathcal{E}$, $H = \mathbf{H}$ on $\mathcal{E} \cap \mathcal{D}$, $\varpi = \boldsymbol{\varpi}$ a.e. in $[0,T]$, and (2.4) holds on $\mathcal{E} \cap \mathcal{D}$ with $\mathbf{z}$, $\mathbf{z}_*$, and $\mathbf{H}$ in place of z, z_*, and H;
- (P2) $\mathbf{H}$ is nonpositive and nonincreasing in t, $\mathbf{z}_* \leq 0$, and $\mathbf{u}^+ \geq 0$;
- (P3) $\mathbf{z} - \mathbf{H}$ is continuous in t;
- (P4) $\mathbf{z} \leq A$ in $\mathcal{D}$;
- (P5) $\boldsymbol{\varpi} \geq 0$, $\mathbf{u}^- \leq \boldsymbol{\varpi}$ on $\mathcal{D}^c$, and $\mathbf{u}^+ \geq (\Lambda + 1)\boldsymbol{\varpi}$ in $\mathcal{B}$.

Let $E := \big\{x \in \mathbb{R} \mid (t,x) \in \mathcal{E} \text{ for a.e. } t \in [0,T]\big\}$, which is full measure by Fubini.

Fix $\varepsilon := A/4$. Since $z = \mathbf{z}$ a.e., we have $\operatorname{ess\,sup}_{\mathcal{D}} \mathbf{z} = \operatorname{ess\,sup}_{\mathcal{D}} z = A$. Hence there exists $(t_0, x_0) \in ([0,T] \times E) \cap \mathcal{D}$ such that $\mathbf{z}(t_0, x_0) > A - \varepsilon$. Let $\rho_0 := \rho(t_0, x_0)$ and $E_0 := ([\rho_0, T] \times \{x_0\}) \cap \mathcal{E}$, which is full measure because $x_0 \in E$. By (P2) and (P3), $\mathbf{z}(\,\cdot\,, x_0)$ only has downward jump discontinuities. In particular, its left and right limits exist. Since $[\rho_0, T]$ is compact, there exists $t_* \in [\rho_0, T]$ with a left or right limit $\mathbf{z}(t_* \pm, x_0)$ that attains the supremum $\sup_{[\rho_0,T]} \mathbf{z}(\,\cdot\,, x_0) > A - \varepsilon$. Now, taking the limit $t \to 0$ within the set E_0 and using (P1) and (2.4), we find

$$\mathbf{z}(\rho_0+, x_0) = \mathbf{z}_*(\rho_0, x_0) + \mathbf{H}(\rho_0+, x_0) \leq 0.$$

Thus $t_* \neq 0$. Now, since $\mathbf{z}$ only jumps *down*, $\mathbf{z}(t_*+, x_0) = \sup_{[\rho_0,T]} \mathbf{z}(\,\cdot\,, x_0)$ would imply $\mathbf{z}(t_*-, x_0) = \sup_{[\rho_0,T]} \mathbf{z}(\,\cdot\,, x_0)$. That is, the *left* limit attains the supremum at $t_* \in (\rho_0, T]$.

In particular, there exists $\delta \in (0, t_* - \rho_0)$ such that $t_* - \delta \in E_0$ and

$$\mathbf{z}(t, x_0) > A - \varepsilon \quad \text{for all } t \in [t_* - \delta, t_*). \tag{2.5}$$

Also, because $\mathbf{z}(t_*-, x_0) = \sup_{[\rho_0,T]} \mathbf{z}(\,\cdot\,, x_0)$, there exists $t_1 \in (t_* - \delta/2, t_*) \cap E_0$ such that

$$\mathbf{z}(t_1, x_0) - \mathbf{z}(t, x_0) > -\frac{\mu \delta A}{4} \quad \text{for all } t \in [\rho_0, T]. \tag{2.6}$$

Now, (P1) and (2.4) yield

$$\begin{aligned} \mathbf{z}(t_1, x_0) &= \mathbf{z}(t_* - \delta, x_0) + \mathbf{H}(t_1, x_0) - \mathbf{H}(t_* - \delta, x_0) \\ &\quad + \int_{t_* - \delta}^{t_1} \big[\mu\big(J * \mathbf{z}(s, x_0) - \mathbf{z}(s, x_0)\big) - \big(\kappa(s, x_0) - q(s, x_0)\big)\mathbf{z}(s, x_0)\big]\,\mathrm{d}s. \end{aligned} \tag{2.7}$$

We consider this identity term by term. First, (P2) yields

$$\mathbf{H}(t_1, x_0) - \mathbf{H}(t_* - \delta, x_0) \leq 0.$$

Next, we claim that $J * \mathbf{z}(s, x_0) \leq A$ for all $s \in (\rho_0, T)$. We let $y_0 \coloneqq x_0 - \varsigma(s) > 0$ and write

$$J * \mathbf{z}(s, x_0) = \int_{(-\infty, y_0)} \mathbf{z}(s, x_0 - y)\, J(\mathrm{d}y) + \int_{[y_0, y_0+M]} \cdots + \int_{(y_0+M, \infty)} \cdots$$

and denote these integrals by I_1, I_2, and I_3. In I_1, we have $x_0 - y > \varsigma(s)$, so $(s, x_0 - y) \in \mathcal{D}$. By (P4), $\mathbf{z}(s, x_0 - y) \leq A$. Since J is a probability measure, $I_1 \leq A$. For I_2, $(s, x_0 - y) \in \mathcal{B}$, so (P5) implies that $\boldsymbol{u}^- \leq \boldsymbol{\varpi}$ and $\boldsymbol{u}^+ \geq (\Lambda + 1)\boldsymbol{\varpi} \geq 0$. Hence

$$\int_{[y_0, y_0+M]} \mathbf{z}(s, x_0 - y)\, J(\mathrm{d}y) \leq -\Lambda \mathrm{e}^{-\kappa s} \boldsymbol{\varpi}(s) J\big([y_0, y_0 + M]\big).$$

For I_3, we use $\mathbf{u}^- \leq \boldsymbol{\varpi}$ and $\mathbf{u}^+, \boldsymbol{\varpi} \geq 0$ from (P5):

$$\int_{(y_0+M, \infty)} \mathbf{z}(s, x_0 - y)\, J(\mathrm{d}y) \leq \mathrm{e}^{-\kappa s} \boldsymbol{\varpi}(s) J\big((y_0 + M, \infty)\big).$$

Combining these bounds, we find

$$J * \mathbf{z}(s, x_0) \leq A - \mathrm{e}^{-\kappa s} \boldsymbol{\varpi}(s) \left[\Lambda J\big([y_0, y_0 + M]\big) - J\big((y_0 + M, \infty)\big)\right].$$

Since $y_0 > 0$, (J3) implies that $J * \mathbf{z}(s, x_0) \leq A$ as claimed.

Finally, the definition (2.5) of δ implies $\mathbf{z}(s, x_0) > A - \varepsilon$ for all $s \in [t_* - \delta, t_1)$. Hence

$$\mu\big(J * \mathbf{z}(s, x_0) - \mathbf{z}(s, x_0)\big) < \mu\varepsilon.$$

On the other hand, $\kappa(s, x_0) - q(s, x_0) \geq \mu$, so

$$-\big(\kappa(s, x_0) - q(s, x_0)\big)\mathbf{z}(s, x_0) < -\mu(A - \varepsilon). \tag{2.8}$$

Collecting (2.7)–(2.8), $t_* - t_1 < \delta/2$ and $A - 2\varepsilon = A/2$ yield

$$\mathbf{z}(t_1, x_0) - \mathbf{z}(t_* - \delta, x_0) < -\mu(t_1 - t_* + \delta)(A - 2\varepsilon) < -\frac{\mu\delta A}{4}. \tag{2.9}$$

However, (2.9) contradicts (2.6). It follows that, in fact, $\operatorname{ess\,sup}_{\mathcal{D}} \mathbf{z} \leq 0$. Therefore $\operatorname{ess\,sup}_{\mathcal{D}}(u^- - u^+) \leq 0$, as desired. ∎

3. The generic logarithmic delay

In this section, we use the comparison principle and probabilistic arguments to prove Theorem 1.1. First, we characterise the generic equations that exhibit this logarithmic delay.

3.1. Proof of Proposition 1.1.

Proof. First, we observe that λ is a critical point of Γ precisely when

$$\mu \int_{\mathbb{R}} \mathrm{e}^{\lambda x}(\lambda x - 1)\, J(\mathrm{d}x) = f'(0) - \mu.$$

The left side is strictly increasing in λ, so Γ has at most one critical point. That is, the minimizer of Γ is unique when it exists.

To understand existence, we proceed on a case by case basis. First suppose that $J(\mathbb{R}_+) > 0$, i.e. that the measure J has positive mass on $\mathbb{R}_+$. Then the integral term in

Γ will grow exponentially as $\lambda \to \infty$, while $\Gamma(\lambda) \sim f'(0)\lambda^{-1}$ as $\lambda \to 0^+$. It follows that the infimum is attained at some intermediate $\lambda_* \in \mathbb{R}_+$.

Now suppose that $J(\mathbb{R}_+) = 0$. Then the normalisation (J2) implies that the integral term in Γ vanishes as $\lambda \to \infty$. If $f'(0) < \mu$, we have $\Gamma(\lambda) < 0$ for λ sufficiently large. Then the limits $\Gamma(0^+) = +\infty$ and $\Gamma(+\infty) = 0$ imply that Γ attains its (negative) minimum.

Finally, suppose that $J(\mathbb{R}_+) = 0$ and $f'(0) \geq \mu$. Then $\Gamma > 0$ and $\Gamma(+\infty) = 0$. Hence Γ does *not* attain its infimum and $c_* = 0$. □

3.2. Proof outline for Theorem 1.1.

Proof. In the remainder of this section, we assume that Γ is minimised at $\lambda_* > 0$ and that J satisfies (J1)–(J3). To prove Theorem 1.1, we follow the approach of Hamel, Nolen, Roquejoffre, and Ryzhik in [25].

To begin, let σ denote the expected position of the leading-edge of u:

$$\sigma(t) \coloneqq c_* t - \frac{3}{2\lambda_*}\log\frac{t+t_0}{t_0} \tag{3.1}$$

for some regularising time-shift $t_0 \geq 1$. We analyse (1.1) in a frame moving with σ. We expect, although do not show, that u eventually resembles a travelling front U_{c_*} in this moving frame. As shown in [18] under some additional assumptions,

$$U_{c_*}(x) \asymp x\mathrm{e}^{-\lambda_* x} \quad \text{when } x \geq 1.$$

We thus broadly expect u to decay like $\mathrm{e}^{-\lambda_* x}$ in the moving frame. It is convenient to preemptively remove this decay. Therefore, let

$$\overline{v}(t,x) \coloneqq \mathrm{e}^{\lambda_* x} u\big(t, x+\sigma(t)\big).$$

Next, define the tilted measure $K \in \mathcal{P}(\mathbb{R})$ by

$$K = Z_0^{-1}\mathrm{e}^{\lambda_* x} J, \quad Z_0 \coloneqq \int_{\mathbb{R}} \mathrm{e}^{\lambda_* x}\, J(\mathrm{d}x).$$

By standard manipulations, $\overline{v}$ satisfies

$$\partial_t \overline{v} = \nu K * \overline{v} + \dot{\sigma}\big(\partial_x \overline{v} - \lambda_* \overline{v}\big) + \big[f'(0) - \mu\big]\overline{v} + \mathrm{e}^{\lambda_* x} F\big(\mathrm{e}^{-\lambda_* x}\overline{v}\big), \tag{3.2}$$

where $\nu \coloneqq \mu Z_0$ and $F(u) \coloneqq f(u) - f'(0)u \leq 0$ denotes the “nonlinear part” of f.

By the definition of the shift σ,

$$\dot{\sigma}(t) = c_* - \frac{3}{2\lambda_*(t+t_0)}.$$

Extracting its leading order c_*, we define the linear operator

$$\mathcal{L}v \coloneqq \nu K * v + c_* \partial_x v + \big[f'(0) - \mu - c_*\lambda_*\big]v.$$

Then we can write (3.2) as

$$\partial_t \overline{v} = \mathcal{L}\overline{v} + \frac{3}{2(t+t_0)}\overline{v} - \frac{3}{2\lambda_*(t+t_0)}\partial_x \overline{v} + \mathrm{e}^{\lambda_* x} F\big(\mathrm{e}^{-\lambda_* x}\overline{v}\big). \tag{3.3}$$

Using the definition of c_* and λ_*, we can check that

$$\nu = c_*\lambda_* + \mu - f'(0) \quad \text{and} \quad \nu \int_{\mathbb{R}} x\, K(\mathrm{d}x) = c_*. \tag{3.4}$$

Let $m := \frac{c_*}{\nu}$ denote the mean of the probability distribution K. Then we can write

$$\mathcal{L}v = \nu\,(K * v + m\,\partial_x v - v)\,.$$

It follows that $\mathcal{L}1 = \mathcal{L}x = 0$ and $\mathcal{L}x^2 = \nu \operatorname{Var} K > 0$. Thus $\mathcal{L}$ resembles a multiple of the Laplacian to second order, and the principal part of (3.3) is a nonlocal analogue of the heat equation.

The remaining linear terms in (3.3) are due to the logarithmic delay in σ. The zeroth-order term $\frac{3}{2(t+t_0)}\overline{v}$ corresponds to multiplication by the factor $(t+t_0)^{3/2}$. It could be removed by replacing $\overline{v}$ by $(t+t_0)^{-3/2}\,\overline{v}$. The first-order term $-\frac{3}{2\lambda_*(t+t_0)}\partial_x\overline{v}$ *should* be negligible, but it is technically more difficult to handle. We therefore study the Dirichlet problem

$$\begin{cases} \partial_t z = \mathcal{L}z + \frac{D}{t+1}\partial_x z & \text{on } \mathbb{R}_+,\\ z = 0 & \text{on } (-\infty, 0],\\ z(0,x) = \mathbf{1}_{(L,2L)}(x), \end{cases} \tag{3.5}$$

for some fixed $D \in \mathbb{R}$ and $L \gg 1$ to be determined. Note that we have replaced the time-shift t_0 by 1 in (3.5). We will use the degree of freedom afforded by t_0 to relate a time-shift of (3.5) to (3.2).

To rigorously construct z, define

$$\sigma^D(t) := c_* t + D\log(t+1) \tag{3.6}$$

Then

$$Z(t,x) := \mathrm{e}^{-\lambda_*(x - c_* t)} z\big(t, x - \sigma^D(t)\big)$$

should solve the linearization

$$\partial_t Z = \mu(J * Z - Z) + f'(0)Z \tag{3.7}$$

in the domain $\mathcal{D} = \{x > \sigma^D(t)\}$ with $Z = 0$ on $(\mathbb{R}_+ \times \mathbb{R}) \setminus \mathcal{D}$. Recalling the definition of $\partial_*\mathcal{D}$ from Section 2, define $Z_*\colon \partial_*\mathcal{D} \to \mathbb{R}$ by $Z_* = \mathbf{1}_{\{0\}\times(L,2L)}(t,x)$. By Proposition 2.1, there exists a unique $Z \in L^\infty_{\text{loc}}\big([0,\infty); L^\infty(\mathbb{R})\big)$ solving (3.7) in $\mathcal{D}$ with initial data Z_* and exterior data 0. We then let

$$z(t,x) := (t+1)^{\lambda_* D}\mathrm{e}^{\lambda_* x} Z\big(t, x + \sigma^D(t)\big). \tag{3.8}$$

The Dirichlet model (3.5) is closely related to a random walk with killing. Let K^* denote the spatial reverse of the measure K, so that $K^*(A) = K(-A)$ for every Borel set $A \subset \mathbb{R}$. Then let $(X_s)_{s\geq 0}$ perform a continuous-time random walk with jump rate ν, jump law K^*, and constant drift c_* starting from 0. By (3.4), $\mathbb{E}X_s = 0$ for all $s \geq 0$. The process $(X_s)_{s\geq 0}$ is a centred walk with jump law K viewed *backwards* in time. We use it in a Feynman–Kac representation of (3.5).

To do so, we must account for the logarithmic drift caused by $\frac{D}{t+1}\partial_x$. Given $(t,x) \in [0,\infty) \times \mathbb{R}_+$, we define the log-drifting walk

$$Y_s^x := X_s + x + D\log\frac{t+1}{t-s+1} \quad \text{for } 0 \leq s \leq t.$$

Then the Itô formula for jump processes implies the following Feynman–Kac formula for (3.5):

$$z(t,x) = \mathbb{P}\Big[Y_t^x \in (L, 2L),\ Y_s^x > 0 \ \text{ for all } \ 0 \leq s \leq t\Big]. \tag{3.9}$$

To construct super- and subsolutions for (3.2), we use the behaviour of Y to control z. The following lemma is the key to our comparison arguments.

Lemma 3.1. *There exist $L > 0$ and $C_* \geq 1$ depending on $\mu, J, f'(0)$, and D such that for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$,*

$$\frac{(x - C_*)}{C_*(t+1)^{\frac{3}{2}}}\mathbf{1}_{x \leq \sqrt{t}} \leq z(t,x) \leq \frac{C_*(x+1)}{(t+1)^{\frac{3}{2}}}. \tag{3.10}$$

We recall that L is the width of the initial data $\mathbf{1}_{(L,2L)}$ in (3.5). Throughout the proof of Theorem 1.1, we let it assume the value given by Lemma 3.1.

Before proving Lemma 3.1, we use it to establish Theorem 1.1. We follow the strategy of [25] closely.

3.3. An upper bound. With Lemma 3.1, we can construct a supersolution for $\overline{v}$. Recall $\Lambda, M > 0$ from (J3). Take $D = -\frac{3}{2\lambda_*}$ in (3.5) and let C_* be the constant given by Lemma 3.1. Then the lower bound in Lemma 3.1 implies the existence of $\delta, T > 0$ such that

$$(t+1)^{\frac{3}{2}} z(t,x) \geq \delta \tag{3.11}$$

for all $(t,x) \in [T,\infty) \times [C_* + 1, C_* + M + 1]$. We define

$$\overline{w}(t,x) \coloneqq \big(\Lambda + 1\big)\delta^{-1}(t + T + 1)^{\frac{3}{2}} z(t + T, x + C_* + M + 1) \tag{3.12}$$

so that (3.11) becomes

$$\overline{w} \geq \Lambda + 1 \quad \text{on } [0,\infty) \times [-M, 0]. \tag{3.13}$$

By the upper bound in Lemma 3.1, there exists $C > 0$ such that

$$\overline{w}(t,x) \leq C(x+1) \quad \text{for all } (t,x) \in [0,\infty) \times [0,\infty). \tag{3.14}$$

Using (3.5), we can check that $\overline{w}$ satisfies

$$\partial_t \overline{w} = \mathcal{L}\overline{w} + \frac{3}{2(t+T+1)}\overline{w} - \frac{3}{2\lambda_*(t+T+1)}\partial_x \overline{w} \tag{3.15}$$

on $\mathbb{R}_+ \times (-C_* - M - 1, \infty)$. Let $t_0 = T + 1$. By (3.15), $\overline{w}$ nearly solves (3.3). It is only missing the negative nonlinearity F, so $\overline{w}$ is a supersolution of (3.3).

We will use the comparison principle to show that $\overline{w} \geq \overline{v}$ when $x > 0$. To apply Proposition 2.2, we return to the original coordinates. Define

$$\overline{W}(t,x) \coloneqq \mathrm{e}^{-\lambda_*(x - \sigma(t))}\overline{w}\big(t, x - \sigma(t)\big). \tag{3.16}$$

Since $D = -\frac{3}{2\lambda_*}$ and $t_0 = T + 1$, we can use σ^D from (3.6) to write (3.1) as

$$\sigma(t) = \sigma^D(t + T) - \sigma^D(T). \tag{3.17}$$

By (3.8) and (3.12),

$$\overline{W}(t,x) = (\Lambda+1)\delta^{-1}\mathrm{e}^{\lambda_*(C_*+M+1)}Z(t+T, x+\sigma^D(T)+C_*+M+1), \tag{3.18}$$

where Z solves (3.7) on $\{x > \sigma^D(t)\}$ with nonnegative initial data and zero exterior data. Because (3.7) is linear and translation-invariant, (3.17) and (3.18) imply that $\overline{W}$ solves (3.7) on

$$\mathcal{D} \coloneqq \{x > \sigma^D(t+T) - \sigma^D(T) - C_* - M - 1\} = \{x > \sigma(t) - C_* - M - 1\}.$$

We will apply Proposition 2.2 to $\overline{W}$ and u on the subdomain $\mathcal{D}' \coloneqq \{x > \sigma(t)\} \subset \mathcal{D}$.

To do so, we verify the hypotheses of Proposition 2.2. Restricting, $\overline{W}$ solves (3.7) in $\mathcal{D}'$. Hence (F3) implies that $\overline{W}$ is a supersolution of (1.1). Of course, u solves (1.1) in $\mathcal{D}'$, so it is also a subsolution.

Next, applying Proposition 2.2 to Z and 0 on $\mathcal{D}$, we see that $Z \geq 0$; hence $\overline{W} \geq 0$. Recall that we have well-defined restrictions $\overline{W}_* \coloneqq \overline{W}|_{\partial_*\mathcal{D}'}$ and $u_* \coloneqq u|_{\partial_*\mathcal{D}'}$. We claim that $\overline{W}_* \geq u_*$. To see this, we use (2.2) to write $\partial_*\mathcal{D}' = \partial_0\mathcal{D}' \cup \partial_+\mathcal{D}'$ for

$$\partial_0\mathcal{D}' \coloneqq \{t = 0\} \cap \partial_*\mathcal{D}' = \{0\} \times [0,\infty)$$

and

$$\partial_+\mathcal{D}' \coloneqq \{t > 0\} \cap \partial_*\mathcal{D}' = \big\{(t,\sigma(t)) \mid t > 0\, \dot{\sigma}(t) < 0\big\}.$$

Then $\overline{W}_* \geq 0 \geq u$ on $\partial_0\mathcal{D}'$ since $u(0,\,\cdot\,) = \mathbf{1}_{\mathbb{R}_-}$. Now take $(t,\sigma(t)) \in \partial_+\mathcal{D}'$, so that $\dot{\sigma}(t) < 0$. In Section 2, we showed that $\overline{W}$ is continuous in t in $\mathcal{D} \supset \partial_+\mathcal{D}'$. Thus (3.13) and (3.16) imply that

$$\overline{W}_*(t,\sigma(t)) = \lim_{s\to t^-} \overline{W}(s,\sigma(t)) \geq \limsup_{s\to t^-} \overline{w}\big(s, \sigma(t) - \sigma(s)\big) \geq \Lambda + 1$$

because $\dot{\sigma}(t) < 0$. On the other hand, $u \leq 1$ everywhere, so $\overline{W}_* \geq u$ on $\partial_+\mathcal{D}'$. We conclude that $\overline{W}_* \geq u_*$, as claimed.

Finally, (3.13) implies that $\overline{W} \geq \Lambda + 1$ in the strip $\mathcal{B} \coloneqq \{\sigma(t) - M \leq x \leq \sigma(t)\}$. Since $u \leq 1$ everywhere, we have verified the hypotheses of Proposition 2.2 with $\varpi \equiv 1$. Applying the proposition, we conclude that $\overline{W} \geq u$ in $\mathcal{D}'$. By (3.14) and (3.16), this implies

$$u\big(t, x+\sigma(t)\big) \leq \overline{W}\big(t, x+\sigma(t)\big) = \mathrm{e}^{-\lambda_* x}\overline{w}(t,x) \leq C(x+1)\mathrm{e}^{-\lambda_* x}$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$. Since the right side vanishes in the $x \to \infty$ limit,

$$\sigma_\theta(t) \leq \sigma(t) + C_\theta$$

for all $\theta \in (0,1)$ and some $C_\theta > 0$, where σ_θ is the leading edge from (1.3).

3.4. A lower bound. We now construct a subsolution to (1.1) to establish the lower bound in Theorem 1.1. For the upper bound, we studied $\mathrm{e}^{\lambda_* x}u$ in the moving frame $c_* t - \frac{3}{2\lambda_*}\log\frac{t+t_0}{t_0}$. This was chosen so that solutions to a corresponding linear Dirichlet problem remain bounded in time away from 0 and ∞ (locally in space).

We consider a similar transformation in this section, but must now contend with the nonlinear absorption. To make the nonlinearity negligible, we'd like u to be small.

Following [9], we use a different logarithmic shift to induce polynomial decay in time. Fix

$$D_\gamma > \max\left\{\frac{1}{\lambda_*}\left(\frac{1}{\gamma} - \frac{3}{2}\right), 0\right\},$$

where $\gamma \in (0,1)$ is the Hölder exponent from (F1). Then we study

$$\underline{v}(t,x) \coloneqq \mathrm{e}^{\lambda_* x} u\left(t, x + c_* t + D_\gamma \log(t+1)\right), \tag{3.19}$$

which satisfies

$$\partial_t \underline{v} = \mathcal{L}\underline{v} + \frac{D_\gamma}{t+1}\left(\partial_x \underline{v} - \lambda_* \underline{v}\right) + \mathrm{e}^{\lambda_* x} F\left(\mathrm{e}^{-\lambda_* x}\underline{v}\right). \tag{3.20}$$

Remark 3.1. We observe that this transformation breaks down as $\gamma \to 0$. In the local case, Bouin and Henderson have shown that the Bramson shift can take quite different forms when f' has a logarithmic modulus of continuity near 0. Thus, *some* additional regularity beyond $f \in \mathcal{C}^1$ is required to recover the classical Bramson shift. We expect similar behaviour in our nonlocal setting.

Now, let z solve (3.5) with $D = D_\gamma$. By Lemma 3.1,

$$z(t,x) \leq \frac{C_*(x+1)}{(t+1)^{\frac{3}{2}}} \quad \text{on } [0,\infty) \times [0,\infty).$$

Thus

$$\zeta(t,x) \coloneqq (t+1)^{-\lambda_* D_\gamma} z(t,x) \tag{3.21}$$

solves the linearisation of (3.20) and satisfies

$$\zeta(t,x) \leq \frac{C_*(x+1)}{(t+1)^{\beta}} \tag{3.22}$$

for

$$\beta \coloneqq \frac{3}{2} + \lambda_* D_\gamma > \frac{1}{\gamma}.$$

We cannot simply use ζ as a subsolution, since the nonlinearity F in (3.20) is negative. Therefore define

$$\underline{w}_+(t,x) \coloneqq a(t)\zeta(t, x + 2L) \tag{3.23}$$

for some decreasing temporal profile a to be determined. For $\underline{w}_+$ to be a subsolution to (3.20) on $\mathbb{R}_+ \times (-2L, \infty)$, we require

$$\frac{\dot{a}}{a}\,\underline{w}_+ \leq \mathrm{e}^{\lambda_* x} F\left(\mathrm{e}^{-\lambda_* x}\underline{w}_+\right). \tag{3.24}$$

By (F1), there exists $C_F > 0$ such that

$$|F(s)| \leq C_F s^{1+\gamma}.$$

By (3.22),

$$\begin{aligned}\underline{w}_+^{-1}\mathrm{e}^{\lambda_* x}\big|F\big(\mathrm{e}^{-\lambda_* x}\underline{w}_+\big)\big| &\le C_F \mathrm{e}^{-\gamma\lambda_* x}\underline{w}_+^\gamma \\ &\le C_F C_*^\gamma \mathrm{e}^{-\gamma\lambda_* x}(x+2L+1)^\gamma a(t)^\gamma (t+1)^{-\beta\gamma} \\ &\le C a(t)^\gamma (t+1)^{-\beta\gamma}\end{aligned}$$

for some $C > 0$ and all $x > -2L$. Recalling (3.24), it thus suffices to let a solve

$$\dot{a} = -Ca^{1+\gamma}(t+1)^{-\beta\gamma}.$$

Because $\beta\gamma > 1$, positive solutions will remain uniformly bounded away from 0. We choose the solution with $a(0) = \mathrm{e}^{-\lambda_* L}$, so that

$$\underline{w}_+(0, x) \le \underline{v}(0, x) \quad \text{for all } x \in \mathbb{R}. \tag{3.25}$$

Of course, we also have

$$\underline{w}_+(t, x) = 0 \le \underline{v}(t, x) \quad \text{for all } t \ge 0,\ x \le -2L. \tag{3.26}$$

To apply our comparison principle, define $\mathcal{D}_+ \coloneqq \{x > \sigma^{D_\gamma}(t) - 2L\}$ with $\sigma^{D_\gamma}(t)$ from (3.6). Inverting (3.19), we have

$$u(t, x) = \mathrm{e}^{-\lambda_*(x-\sigma^{D_\gamma}(t))}\underline{v}\big(t, x - \sigma^{D_\gamma}(t)\big).$$

We therefore define

$$\underline{W}_+(t, x) = \mathrm{e}^{-\lambda_*(x-\sigma^{D_\gamma}(t))}\underline{w}_+\big(t, x - \sigma^{D_\gamma}(t)\big).$$

Using (3.8), (3.21), and (3.23), we can compute

$$\underline{W}_+(t, x) = a(t)\mathrm{e}^{2\lambda_* L}Z(t, x + 2L).$$

By the choice of a, $\underline{W}_+$ is a subsolution of (1.1) on $\mathcal{D}_+$. Moreover, (3.25) and (3.26) imply that $\underline{W}_+ \le u$ on $\partial_*\mathcal{D}_+$ and $\underline{W}_+ = 0$ on $(\mathbb{R}_+ \times \mathbb{R}) \setminus \mathcal{D}_+$. Therefore, u and $\underline{W}_+$ satisfy the hypotheses of Proposition 2.2 on $\mathcal{D}_+$ with $\varpi \equiv 0$. It follows that $u \ge \underline{W}_+$ in $\mathcal{D}_+$.

We now use the lower bound in Lemma 3.1. By (3.10), (3.21), and (3.23), there exist $\delta, T > 0$ such that

$$u\big(t, c_* t + \sqrt{t}\big) \ge \underline{W}_+\big(t, c_* t + \sqrt{t}\big) = a(t)\mathrm{e}^{-\lambda_*\sqrt{t}}z\big(t, \sqrt{t} - D_\gamma \log(t+1) + 2L\big) \ge \frac{\delta}{t}\mathrm{e}^{-\lambda_*\sqrt{t}}$$

for all $t \ge T$. That is, we can control u at the diffusive scale. Now,

$$u(0, \,\cdot\, + h) = \mathbf{1}_{(-\infty,-h)} \le \mathbf{1}_{\mathbb{R}_-} = u(0, \,\cdot\,) \quad \text{for every } h \ge 0.$$

Applying the comparison principle, we find $u(t, \,\cdot\, + h) \le u(t, \,\cdot\,)$ for all $t, h \ge 0$. That is, u is nonincreasing in x at each fixed $t \ge 0$. So in fact

$$u\big(t, c_* t + \sqrt{t} - y\big) \ge \frac{\delta}{t}\mathrm{e}^{-\lambda_*\sqrt{t}} \tag{3.27}$$

for all $t \ge T$ and $y \ge 0$.

Before using (3.27), we need a lower bound on u on the left. This is much simpler. We extend f by zero to $[-1,1]$. Then f is a reaction of *ignition type* on this extended interval. In [17], Coville shows the existence of a non-increasing front $\underline{U}$ solving

$$\mu(J * \underline{U} - \underline{U}\,) + \underline{c}\,\underline{U}' + f(\underline{U}) = 0,$$
$$-1 \leq \underline{U} \leq 1, \quad \underline{U}(-\infty) = 1, \quad \underline{U}(+\infty) = -1$$

for a unique speed $\underline{c}$. Hence $\underline{U}(x - \underline{c}t)$ solves (1.1). If we shift $\underline{U}$ so that $u(0, \cdot\,) \geq \underline{U}$, the comparison principle implies that

$$u(t,x) \geq \underline{U}(x - \underline{c}t) \quad \text{for all } (t,x) \in [0,\infty) \times \mathbb{R}.$$

It follows that there exists $B \geq 0$ such that

$$u(T,x) \geq \frac{1}{2} \quad \text{for all } x \in \big(-\infty, c_* T + \sqrt{T} - B\big]. \tag{3.28}$$

We leverage these bounds to construct a travelling wave subsolution to (1.1). We rely on the following proposition, which is essentially due to Coville, Dávila, and Martínez [18]:

PROPOSITION 3.1. *Let J satisfy (J1) and f satisfy (F1)–(F3). Then there exists a monotone travelling wave U satisfying (1.5) with speed c_* given by (1.6). Moreover, if the rate function Γ in (1.6) attains its infimum at λ_*, then there exists $C > 0$ such that*

$$U(x) \leq C x \mathrm{e}^{-\lambda_* x} \quad \textit{for all } x \geq 1. \tag{3.29}$$

We note that U need not be continuous or unique up to translation when $c_* = 0$. Proposition 3.1 differs from results stated in [18] only in its assumptions: it applies to kernels J that are Borel measures. We defer its proof to Appendix B.

Here, we need a subsolution wave with speed c_*. Let $\tilde{f}\colon [0,1/2] \to [0,\infty)$ satisfy $\tilde{f} \leq f$, $\tilde{f}'(0) = f'(0)$, and (F1)–(F3) with $[0,1/2]$ in the place of $[0,1]$. Then $\tilde{f}$ is a KPP reaction on a restricted interval. Applying Proposition 3.1 to $\tilde{f}$, we obtain a travelling wave $\tilde{U}$ satisfying

$$\mu\big(J * \tilde{U} - \tilde{U}\big) + c_* \tilde{U}' + \tilde{f}\big(\tilde{U}\big) = 0,$$
$$0 \leq \tilde{U} \leq \frac{1}{2}, \quad \tilde{U}(-\infty) = \frac{1}{2}, \quad \tilde{U}(+\infty) = 0,$$

and

$$\tilde{U}(x) \leq C x\, \mathrm{e}^{-\lambda_* x} \quad \text{for all } x \geq 1.$$

Hence, we can translate $\tilde{U}$ so that

$$\tilde{U}\Big(\sqrt{t} + \frac{3}{2\lambda_*}\log(t+1) - B - M\Big) \leq \frac{\delta}{(\Lambda+1)t}\mathrm{e}^{-\lambda_*\sqrt{t}} \tag{3.30}$$

for all $t \geq T$, recalling $\Lambda, M > 0$ from (J3). We define

$$\underline{w}(t,x) \coloneqq \tilde{U}\Big(x - c_* t + \frac{3}{2\lambda_*}\log(t+1)\Big).$$

Let

$$\mathcal{D} \coloneqq \big\{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \mid t > T,\ \ x < c_* t + \sqrt{t} - B - M\big\},$$

and define the strip

$$\mathcal{B} := \big\{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \mid t > T, \;\; x \in [c_*t + \sqrt{t} - B - M, c_*t + \sqrt{t} - B]\big\}.$$

We wish to apply Proposition 2.2 to u and $\underline{w}$ on $\mathcal{D}$. This is not immediate, because $\mathcal{D}$ has the form $\{x < \varsigma(t)\}$ rather than $\{x > \varsigma(t)\}$. However, because (J3) extends to J^*, Proposition 2.2 applies to the spatial reverse of (1.1) as well. We can therefore reverse space, apply the comparison principle, and reverse back. It thus suffices to verify the hypotheses of Proposition 2.2 for $\mathcal{D}$.

First, $\underline{w}$ is a subsolution of (1.1) because $\tilde{f} \leq f$ and $\tilde{U}$ is decreasing in x. As a solution, u is also a supersolution. Next, (3.28) and $\tilde{U} \leq 1/2$ imply that $u \geq \underline{w}$ at the "initial" time T when $x \leq c_*T + \sqrt{T} - B$. Moreover, by (3.27) and (3.30), we have $u \geq \underline{w}$ on $\partial_*\mathcal{D}$. Finally, let

$$\varpi(t) := \frac{\delta}{(\Lambda+1)t}\mathrm{e}^{-\lambda_*\sqrt{t}}.$$

Since $\underline{w}$ is nonincreasing in space, (3.30) implies that $\underline{w} \leq \varpi$ in $((T,\infty) \times \mathbb{R}) \setminus \mathcal{D}$. Similarly, (3.27) implies that $u \geq (\Lambda+1)\varpi$ in $\mathcal{B}$. Thus u and $\underline{w}$ satisfy the hypotheses of Proposition 2.2 on $\mathcal{D}$. It follows that

$$u \geq \underline{w} \quad \text{in } \mathcal{D}. \tag{3.31}$$

Now take $\theta \in (0, 1/2)$. If we let

$$\underline{\sigma}_\theta(t) := c_*t - \frac{3}{2\lambda_*}\log(t+1) + \tilde{U}^{-1}(\theta),$$

we have $\underline{w}\big(t, \underline{\sigma}_\theta(t)\big) = \theta$. Evidently, $\big(t, \underline{\sigma}_\theta(t)\big) \in \mathcal{D}$ once t is sufficiently large. Hence (1.3) and (3.31) yield

$$\sigma_\theta(t) \geq \underline{\sigma}_\theta(t)$$

for t sufficiently large. This is a lower bound for Theorem 1.1. For $\theta \in [1/2, 1)$, we can repeat the above argument using a different $\tilde{f}$ defined instead on $[0, (1+\theta)/2]$. This completes the proof of Theorem 1.1. ◻

3.5. Proof of key estimate. To close this section, we present a proof of Lemma 3.1. By the Feynman–Kac representation (3.9), we must control the probability that the log-drifting continuous-time random walk Y_s^x moves from x to $(L, 2L)$ in time t while remaining positive. Lemma 3.1 thus belongs to a family of "ballot theorems" widely used in the theory of branching processes; see, for instance, the survey [2]. In particular, our lemma is a continuous-time version of Lemma 3.2 in [32]. Since the continuous-time theory is far less developed, it seems useful to record a proof of Lemma 3.1.

We begin by recalling the definitions of our random walks. Our central character is $(X_s)_{s\geq 0}$, a continuous-time random walk with jump rate ν, jump law K^* (the spatial reverse of K), and constant drift c_* starting from 0. Let $\mathscr{X}$ be distributed according to K^*. Then $X_s - c_*s$ is a Poissonisation of $\mathscr{X}$ with rate νs, i.e.

$$\mathrm{Law}(X_s - c_*s) = \mathrm{e}^{-\nu s}\sum_{k=0}^{\infty}\frac{(\nu s)^k}{k!}J^{*k}.$$

We use this special form to compute the characteristic function φ_{X_s} of X_s:

$$\varphi_{X_s}(\xi) \coloneqq \mathbb{E}\,\mathrm{e}^{\mathrm{i}\xi X_s} = \mathrm{e}^{-\nu s+\mathrm{i}c_* s\xi}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\nu s\mathbb{E}\mathrm{e}^{\mathrm{i}\xi\mathscr{X}}\right)^k = \mathrm{e}^{\nu s[\varphi_{\mathscr{X}}(\xi)-1+\mathrm{i}m\xi]}, \tag{3.32}$$

recalling that $m = \frac{c_*}{\nu}$ is the expectation of $-\mathscr{X}$. By (J1), $\varphi_{\mathcal{X}}$ (and hence φ_{X_s}) is analytic in an open strip containing 0. From (3.32), we can compute

$$\mathbb{E}\,X_s = 0 \quad \text{and} \quad \mathbb{E}\,X_s^2 = \nu\mathbb{E}\mathscr{X}^2 s. \tag{3.33}$$

Since X_s has mean zero, the process $(X_s)_{s\geq 0}$ is a martingale.

Next, we introduce a logarithmic drift. Given $D \in \mathbb{R}$ and $t \geq 0$, define

$$H(s) \coloneqq D\log\frac{t+1}{(t-s)_+ + 1} \quad \text{for } s \geq 0.$$

Here $(t-s)_+$ denotes the positive part of $t-s$. Of course, H also depends on t, but we suppress this dependence to minimise notation. We will also have cause to consider the random walk and drift in reverse. We therefore define

$$\overline{X}_s \coloneqq X_{t-s} - X_t \quad \text{and} \quad \overline{H}(s) \coloneqq H((t-s)_+) - H(t) = D\log\frac{1}{s\wedge t+1}.$$

Note that $\overline{X}$ is a continuous-time random walk with jump rate ν, jump law K, and constant drift $-c_*$. Finally, given $x, y > 0$, we define the log-drifting walks

$$Y_s^x \coloneqq X_s + x + H(s) \quad \text{and} \quad \overline{Y}_s^y \coloneqq \overline{X}_s + y + \overline{H}(s) \quad \text{for } s \geq 0$$

and the stopping times

$$S_x \coloneqq \inf\left\{s \geq 0 \mid Y_s^x < 0\right\} \quad \text{and} \quad \overline{S}_y \coloneqq \inf\left\{s \geq 0 \mid \overline{Y}_s^y < 0\right\}.$$

With this notation, (3.9) becomes

$$z(t,x) = \mathbb{P}\left[Y_t^x \in (L, 2L),\, S_x > t\right].$$

We adapt the proof of Theorem 1 in [2] to handle the continuous-time walk Y with logarithmic drift. To do so, we rely on two lemmas. The first controls the dispersal of X on the line.

Lemma 3.2. *For each $\mathcal{X} \in \{X, \overline{X}\}$ and $h > M$, there exists $C_S \geq 1$ depending on $h, K,$ and ν such that*

$$\mathbb{P}[x \leq \mathcal{X}_s \leq x+h] \leq \frac{C_S}{\sqrt{s}}\exp\left(-\frac{x^2}{2\nu\mathbb{E}\mathscr{X}^2\, s}\right) + o_h\left(s^{-\frac{1}{2}}\right) \tag{3.34}$$

and

$$\mathbb{P}[x \leq \mathcal{X}_s \leq x+h] \geq \frac{1}{C_S\sqrt{s}}\exp\left(-\frac{x^2}{2\nu\mathbb{E}\mathscr{X}^2\, s}\right) + o_h\left(s^{-\frac{1}{2}}\right) \tag{3.35}$$

for all $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$. The error terms satisfy $s^{\frac{1}{2}} o_h\left(s^{-\frac{1}{2}}\right) \to 0$ as $s \to \infty$ uniformly over $x \in \mathbb{R}$ and h in a compact subset of (M, ∞).

Moreover, for all $a > 0$, $s \geq 1$, and $x \in [0, a\sqrt{s}]$,

$$\mathbb{P}\left[\inf_{r\in[0,s]} \mathcal{X}_r \leq -x\right] \leq \left[1 + \mathcal{O}_a\left(s^{-\frac{1}{2}}\right)\right] \exp\left(-\frac{x^2}{2\nu\mathbb{E}\mathscr{X}^2\, s}\right). \tag{3.36}$$

Proof. The bounds (3.34) and (3.35) are forms of Stone's local limit theorem. This classical result is typically proved by Fourier analytic methods. Because X_s has a simple characteristic function (3.32), the original proof extends to continuous time; we direct the reader to [41] for details. The condition $h > M$ ensures that (3.35) remains true even if $\mathscr{X}$ is supported on a lattice. Indeed, hypothesis (J3) implies that the cell size of such a lattice can be no larger than M.

To prove (3.36), we note that Jensen's inequality implies that $\exp(\lambda\mathcal{X}_s)$ is a submartingale for all $|\lambda| < \overline{\lambda}$. Thus Doob's inequality for submartingales yields

$$\mathbb{P}\left[\inf_{r\in[0,s]} \mathcal{X}_r \leq -x\right] = \mathbb{P}\left[\sup_{r\in[0,s]} \exp(-\lambda\mathcal{X}_r) \geq \mathrm{e}^{\lambda x}\right] \leq \mathbb{E}\exp\left[-\lambda(\mathcal{X}_s + x)\right].$$

Let $M_Z(\lambda) := \mathbb{E}\mathrm{e}^{\lambda Z}$ denote the moment generating function of a random variable Z, noting that $M_Z(\lambda) = \varphi_Z(-\mathrm{i}\lambda)$. Evaluating (3.32) at complex ξ, we find

$$\begin{aligned}\mathbb{P}\left[\inf_{r\in[0,s]} \mathcal{X}_r \leq -x\right] &\leq \mathrm{e}^{-\lambda x} M_{\mathcal{X}_s}(-\lambda)\\ &\leq \exp\left\{-\lambda x + \nu s\left[M_{\pm\mathscr{X}}(-\lambda) - 1 \pm \lambda m\right]\right\},\end{aligned} \tag{3.37}$$

recalling that Law $\mathscr{X} = K^*$. (The sign before $\mathscr{X}$ depends on the identity of $\mathcal{X}$.) Because $\mathbb{E}\mathscr{X} = -m$,

$$M_{\pm\mathscr{X}}(-\lambda) - 1 \pm \lambda m = \frac{1}{2}\mathbb{E}\mathscr{X}^2\,\lambda^2 + \mathcal{O}\left(\lambda^3\right) \quad \text{as } \lambda \to 0. \tag{3.38}$$

We choose

$$\lambda := \frac{x}{\nu\mathbb{E}\mathscr{X}^2\, s} \leq \frac{a}{\nu\mathbb{E}\mathscr{X}^2} s^{-\frac{1}{2}}$$

when $\lambda < \overline{\lambda}$. Then (3.37) and (3.38) imply

$$\mathbb{P}\left[\inf_{r\in[0,s]} \mathcal{X}_r \leq -x\right] \leq \exp\left[-\frac{x^2}{2\nu\mathbb{E}\mathscr{X}^2\, s} + \mathcal{O}_a\left(s^{-\frac{1}{2}}\right)\right],$$

and (3.36) follows. If $\lambda \geq \overline{\lambda}$, $s = \mathcal{O}(a^2)$, and we make the implied constant in (3.36) large enough that the right side exceeds 1. Then (3.36) holds vacuously. ∎

Our second lemma extends several bounds of Pemantle and Peres [36] to continuous time with log-drift. It controls the stopping times S and $\overline{S}$, and shows that Y and $\overline{Y}$ disperse diffusively.

LEMMA 3.3. *There exist $C_1, c_2 > 0$ and $L \geq 1$ depending on K, ν, and D such that for each $\mathcal{S} \in \{S, \overline{S}\}$ and all $s \geq 0$:*

(i) $\mathbb{P}[\mathcal{S}_x > s] \leq C_1 \max\{x, 1\}\, s^{-\frac{1}{2}}$ *for all $x \geq 0$;*

(ii) $\mathbb{P}[\mathcal{S}_x > s] \geq c_2 \min\left\{x s^{-\frac{1}{2}}, 1\right\}$ *for all $x \geq L$.*

Moreover, there exist $C_3, \beta > 0$ and $s_0 \geq 1$ depending on K, ν, and D such that for each $(\mathcal{Y}, \mathcal{S}) \in \left\{(Y, S), \left(\overline{Y}, \overline{S}\right)\right\}$ and all $s \geq s_0$:

(iii) $\mathbb{P}\big[\mathcal{Y}_s^x \geq \beta\sqrt{s}\big] \geq \frac{1}{3}$ *for all* $x \geq 0$;

(iv) $\mathbb{E}\big[\big(\mathcal{Y}_s^{x'}\big)^2 \mid \mathcal{S}_x > s,\, \mathcal{Y}_s^{x'} \geq \beta\sqrt{s}\big] \leq C_3 s$ *for all* $x \in \big[L, 2\sqrt{s}\big]$, $x' \in \big[0, 2\sqrt{s}\big]$.

We show Lemma 3.3 using discrete-time arguments of Pemantle and Peres [36]. Before presenting its proof, we use Lemmas 3.2 and 3.3 to establish Lemma 3.1. As mentioned above, we follow the proof of Theorem 1 in [2].

Proof. **(Proof of Lemma 3.1.)** Take L from Lemma 3.3. If necessary, increase it so that $L > M$, and let $I := (L, 2L)$.

We wish to control $\mathbb{P}[Y_t^x \in I, S_x > t]$ as a function of x and t. We begin with the upper bound. We condition on the final value $y := Y_t^x = Y_{t/3}^x + \big(Y_{2t/3}^x - Y_{t/3}^x\big) - \overline{Y}_{t/3}^0$. If $Y_t^x \in I$ and $S_x > t$, the following three events must occur:

$$\{S_x \geq t/3\}, \quad \{\overline{S}_y \geq t/3\}, \quad \text{and} \quad \{Y_{2t/3}^x - Y_{t/3}^x \in \overline{Y}_{t/3}^0 - Y_{t/3}^x + I\}.$$

Using the independence of disjoint increments of X, we have

$$\mathbb{P}\big[Y_t^x \in I, S_x > t\big] \leq \mathbb{P}\Big[S_x \geq \frac{t}{3}\Big] \sup_{y \in I} \mathbb{P}\Big[\overline{S}_y \geq \frac{t}{3}\Big] \sup_{z \in \mathbb{R}} \mathbb{P}\big[Y_{2t/3}^x - Y_{t/3}^x \in z + I\big]. \tag{3.39}$$

Here y and z represent Y_t^x and $\overline{Y}_{t/3}^0 - Y_{t/3}^x$, respectively. We bound the first two terms using Lemma 3.3(i):

$$\mathbb{P}\Big[S_x \geq \frac{t}{3}\Big] \leq C \max\{x, 1\} t^{-1/2} \quad \text{and} \quad \sup_{y \in I} \mathbb{P}\Big[\overline{S}_y \geq \frac{t}{3}\Big] \leq C t^{-1/2}. \tag{3.40}$$

For the third term in (3.39), we use an elementary consequence of Lemma 3.2: the probability that X_s lands in *any* interval of bounded width is at most $Cs^{-\frac{1}{2}}$. Hence the stationarity of increments of X implies that

$$\mathbb{P}\big[Y_{2t/3}^x - Y_{t/3}^x \in z + I\big] = \mathbb{P}\big[X_{t/3} \in z + H(t/3) - H(2t/3) + I\big] \leq C t^{-1/2}.$$

Here C is a large constant that may change from expression to expression. Combining this estimate with (3.40), (3.39) yields $\mathbb{P}\big[Y_t^x \in I,\, S_x > t\big] \leq C(x+1)t^{-3/2}$. This implies the upper bound in (3.10) because probabilities are bounded by 1.

We employ a similar strategy for the lower bound, but the argument is significantly more technical. We may assume that $L \leq x \leq \sqrt{t}$, since the lower bound in (3.10) is vacuous otherwise (taking $C_* \geq L$). Also, it suffices to prove the lower bound after some fixed time $\underline{t}$ depending on K, ν, and D. After all, by increasing C_*, we can ensure that the left side of (3.10) is nonpositive when $t < \underline{t}$. We begin with $\underline{t} = 4s_0$ for s_0 as in Lemma 3.3, but we will steadily increase $\underline{t}$ over the course of the proof.

Recalling the constants C_S and β from Lemmas 3.2 and 3.3, we define α by

$$2^3 C_S^2 \exp\left(-\frac{\beta^2}{2^5 \nu \mathbb{E}\mathscr{X}^2 (1 - 2\alpha)}\right) = \frac{1}{4} \tag{3.41}$$

or $\alpha = 1/4$, whichever is greater. Thus $\alpha \in [1/4, 1/2)$. Recall also C_3 from Lemma 3.3. If all of the following events occur, we will have $Y_t^x \in I$ and $S_x > t$:

$$E_1 := \{S_x > \alpha t\}$$
$$E_2 := \{\overline{S}_y > \alpha t\}$$
$$E_3 := \Big\{\beta\sqrt{\alpha t} \leq Y_{\alpha t}^x \leq \sqrt{C_3 t}\Big\}$$

$$E_4 := \left\{\beta\sqrt{\alpha t} \le \overline{Y}^0_{\alpha t} \le \sqrt{C_3 t}\right\}$$
$$E_5 := \left\{Y^x_{(1-\alpha)t} - Y^x_{\alpha t} \in \overline{Y}^0_{\alpha t} - Y^x_{\alpha t} + I\right\}$$
$$E_6 := \left\{\inf_{\alpha t \le s \le (1-\alpha)t} \left(Y^x_s - Y^x_{\alpha t}\right) \ge -Y^x_{\alpha t}\right\}$$

By the independence of disjoint increments of X,

$$\mathbb{P}\big[Y^x_t \in I,\, S_x > t\big] \ge \mathbb{P}[E_1, E_3] \inf_{y\in I} \mathbb{P}[E_2, E_4]\, \mathbb{P}[E_5, E_6 \mid E_3, E_4]. \tag{3.42}$$

Lemma 3.3(ii) implies that $\mathbb{P}[E_1] \ge cxt^{-1/2}$ for some $c > 0$ that may change from line to line. Combining this with $\alpha t \ge \underline{t}/4 \ge s_0$, Lemma 3.3(iii), and the FKG inequality (see, for instance, [24, §2.2]), we have

$$\mathbb{P}\left[E_1,\, Y^x_{\alpha t} \ge \beta\sqrt{\alpha t}\right] \ge cxt^{-1/2}. \tag{3.43}$$

Also, $L \le x \le \sqrt{t} \le 2\sqrt{\alpha t}$, Lemma 3.3(iv), and Chebyshev's inequality imply

$$\mathbb{P}\left[Y^x_{\alpha t} > \sqrt{C_3 t} \;\middle|\; E_1,\, Y^x_{\alpha t} \ge \beta\sqrt{\alpha t}\right] \le \frac{\mathbb{E}\left[(Y^x_{\alpha t})^2 \mid E_1,\, Y^x_{\alpha t} \ge \beta\sqrt{\alpha t}\right]}{C_3 t} \le \frac{C_3 \alpha t}{C_3 t} < \frac{1}{2}.$$

By (3.43), we obtain

$$\mathbb{P}[E_1, E_3] \ge cxt^{-1/2}. \tag{3.44}$$

By identical reasoning,

$$\inf_{y\in I} \mathbb{P}[E_2, E_4] \ge ct^{-1/2}. \tag{3.45}$$

In light of (3.42), it thus suffices to show that $\mathbb{P}[E_5, E_6 \mid E_3, E_4] \ge ct^{-1/2}$.

Let $m := (1-2\alpha)t$ denote the length of the middle period $[\alpha t, (1-\alpha)t]$. For $s \in [0, m]$, we define

$$R_s := Y^x_{\alpha t + s} - Y^x_{\alpha t} \quad \text{and} \quad \overline{R}_s := Y^x_{(1-\alpha)t - s} - Y^x_{(1-\alpha)t}.$$

Then $Y^x_t \in I$ is equivalent to

$$R_m \in \overline{Y}^0_{\alpha t} - Y^x_{\alpha t} + I.$$

By the independence of disjoint increments of X,

$$\mathbb{P}[E_5, E_6 \mid E_3, E_4] \ge \inf_{p,q\in\left[\beta\sqrt{\alpha t},\, \sqrt{C_3 t}\right]} \mathbb{P}\left[R_m \in q - p + I,\, \inf_{s\in[0,\,m]} R_s \ge -p\right].$$

Here p and q represent $Y^x_{\alpha t}$ and $\overline{Y}^0_{\alpha t}$, respectively. It is convenient to separate the cases $p \ge q$ and $q > p$. We first assume that $p \ge q$ and define the events

$$A_{p,q} := \left\{R_m \in q - p + I\right\},$$

$$B_p \coloneqq \left\{ \inf_{s\in[0,\,m]} R_s \geq -p \right\}.$$

We will exploit the identity

$$\mathbb{P}[A_{p,q}, B_p] = \mathbb{P}[A_{p,q}] - \mathbb{P}\big[A_{p,q}, B_p^c\big].$$

We control the first term with Stone's local limit theorem, i.e. Lemma 3.2:

$$\mathbb{P}[A_{p,q}] \geq \frac{1}{C_S\sqrt{m}} \exp\left\{ -\frac{[q-p+L-H((1-\alpha)t)+H(\alpha t)]^2}{2\nu\mathbb{E}\mathscr{X}^2 m} \right\} + o\big(m^{-1/2}\big). \tag{3.46}$$

Recall that $H(t) = o\big(\sqrt{t}\big)$. Perhaps after increasing $\underline{t}$, we can thus assume that

$$\Sigma \coloneqq |H(t)| + L \leq \frac{\beta\sqrt{\alpha t}}{2} \leq \frac{1}{2}\min\{p, q\}. \tag{3.47}$$

Also, because $|H|$ is monotone increasing and $p \geq q$,

$$|q-p+L-H((1-\alpha)t)+H(\alpha t)| \leq |q-p| + L + |H(t)| = p - q + \Sigma.$$

Now $\Sigma \lesssim \sqrt{t}$, while $p, q \asymp \sqrt{t}$ and $m \asymp t$. Hence the argument of the exponential in (3.46) is bounded. It follows that we can absorb the $o\big(m^{-1/2}\big)$ error into the main term, perhaps after increasing $\underline{t}$. Then

$$\mathbb{P}[A_{p,q}] \geq \frac{1}{2C_S\sqrt{m}} \exp\left[-\frac{(p-q+\Sigma)^2}{2\nu\mathbb{E}\mathscr{X}^2 m} \right]. \tag{3.48}$$

We will argue that $A_{p,q} \cap B_p^c$ is significantly more unlikely than $A_{p,q}$, for then R is forced to make a large excursion. We write

$$A_{p,q} \cap B_p^c \;\subset\; A_{p,q} \cap \big(U_p \cup \overline{U}_q\big) \tag{3.49}$$

for

$$U_p \coloneqq \left\{ \inf_{s\in[0,\,m/2]} R_s < -p \right\}, \quad \overline{U}_q \coloneqq \left\{ \inf_{s\in[0,\,m/2]} \overline{R}_s < -q \right\}.$$

If the event U_p occurs, X drops by at least $p - \Sigma$ over a period of length at most $m/2$. By (3.36) in Lemma 3.2,

$$\mathbb{P}[U_p] \leq \Big[1 + \mathcal{O}\big(s^{-\frac{1}{2}}\big)\Big] \exp\left[-\frac{(p-\Sigma)^2}{\nu\mathbb{E}\mathscr{X}^2\, m} \right] \leq 2\exp\left[-\frac{(p-\Sigma)^2}{\nu\mathbb{E}\mathscr{X}^2\, m} \right], \tag{3.50}$$

provided $\underline{t}$ is sufficiently large. Now, if $A_{p,q}$ also occurs, our walk X must later climb by at least $q - \Sigma$, which is positive by (3.47). Using the independence of disjoint increments of X and Lemma 3.2,

$$\mathbb{P}[A_{p,q} \mid U_p] \leq \sup_{r\in[0,\,m/2]} \left\{ \frac{C_S}{\sqrt{m-r}} + o\big((m-r)^{-1/2}\big) \right\} \leq \frac{2C_S}{\sqrt{m}}, \tag{3.51}$$

again provided $\underline{t}$ is sufficiently large. Here, we have implicitly conditioned on the time r of the infimum of X_s in $[0, m/2]$.

Combining (3.50) and (3.51), we find

$$\mathbb{P}[A_{p,q} \cap U_p] \leq \frac{2^2 C_S}{\sqrt{m}} \exp\left[-\frac{(p-\Sigma)^2}{\nu \mathbb{E}\mathscr{X}^2\, m}\right].$$

Now (3.47), $p \geq \beta\sqrt{\alpha t}$, and $\alpha \geq 1/4$ imply that

$$(p-\Sigma)^2 - \frac{1}{2}(p-q+\Sigma)^2 \geq \frac{1}{2}(p-\Sigma)^2 \geq \frac{\beta^2 t}{2^5}.$$

By (3.48), $m = (1-2\alpha)t$, and our choice (3.41) of α,

$$\mathbb{P}[A_{p,q} \cap U_p] \leq 2^3 C_S^2 \exp\left(-\frac{\beta^2}{2^5 \nu \mathbb{E}\mathscr{X}^2 (1-2\alpha)}\right) \mathbb{P}[A_{p,q}] \leq \frac{1}{4}\mathbb{P}[A_{p,q}]. \tag{3.52}$$

Next, consider $A_{p,q} \cap \overline{U}_q$. In this case, $\overline{X}$ drops by at least $q - \Sigma$ in $[0, m/2]$, and eventually rises by at least $p - \Sigma$. Arguing from Lemma 3.2 as before,

$$\mathbb{P}\big[\overline{U}_q\big] \leq 2\exp\left[-\frac{(q-\Sigma)^2}{\nu\mathbb{E}\mathscr{X}^2\, m}\right],$$

provided $\underline{t}$ is sufficiently large. Next, we retain more information from Lemma 3.2:

$$\begin{aligned}\mathbb{P}\big[A_{p,q} \mid \overline{U}_q\big] &\leq \sup_{r \in [0,\, m/2]} \left\{\frac{C_S}{\sqrt{m-r}} \exp\left[-\frac{(p-\Sigma)^2}{2\nu\mathbb{E}\mathscr{X}^2(m-r)}\right] + o\big((m-r)^{-1/2}\big)\right\} \\ &\leq \frac{2C_S}{\sqrt{m}} \exp\left[-\frac{(p-\Sigma)^2}{2\nu\mathbb{E}\mathscr{X}^2 m}\right],\end{aligned}$$

again provided $\underline{t}$ is sufficiently large. Thus

$$\mathbb{P}\big[A_{p,q} \cap \overline{U}_q\big] \leq \frac{2^2 C_S}{\sqrt{m}} \exp\left[-\frac{(q-\Sigma)^2}{\nu\mathbb{E}\mathscr{X}^2\, m}\right] \exp\left[-\frac{(p-\Sigma)^2}{2\nu\mathbb{E}\mathscr{X}^2 m}\right].$$

Using (3.41), (3.47), (3.48), and $p \geq q \geq \beta\sqrt{\alpha t}$, we find

$$\mathbb{P}\big[A_{p,q} \cap \overline{U}_q\big] \leq 2^3 C_S^2 \exp\left(-\frac{\beta^2}{2^5 \nu \mathbb{E}\mathscr{X}^2 (1-2\alpha)}\right) \mathbb{P}[A_{p,q}] \leq \frac{1}{4}\mathbb{P}[A_{p,q}]. \tag{3.53}$$

In light of (3.49), we see that (3.48), (3.52), and (3.53) yield

$$\mathbb{P}[A_{p,q}, B_p] \geq \frac{1}{2}\mathbb{P}[A_{p,q}] \geq \frac{1}{2^2 C_S \sqrt{m}} \exp\left[-\frac{(p-q+\Sigma)^2}{2\nu\mathbb{E}\mathscr{X}^2 m}\right].$$

So far, we have assumed that $p \geq q$. The case $q > p$ can be handled similarly, so in fact

$$\mathbb{P}[A_{p,q}, B_p] \geq \frac{1}{2^2 C_S \sqrt{m}} \exp\left[-\frac{(p+q+\Sigma)^2}{2\nu\mathbb{E}\mathscr{X}^2 m}\right]$$

for any $p, q \in \big[\beta\sqrt{\alpha t},\, \sqrt{C_3 t}\big]$. The argument of the exponential is uniformly bounded and $m \asymp t$, so

$$\mathbb{P}[E_3, E_4 \mid E_1, E_2] \geq \inf_{p,q \in [\beta\sqrt{\alpha t},\, \sqrt{C_3 t}]} \mathbb{P}[A_{p,q}, B_p] \geq \frac{c}{\sqrt{t}}$$

for some $c > 0$ depending on K, ν, and D. In combination with (3.44) and (3.45), the lower bound in (3.10) follows. □

REMARK 3.2. Our use of Doob's inequality (3.36) patches a small gap in the proof of Theorem 1 in [2]. In particular, (13) in [2] is false if, say, $p \geq 3q$. In this case, the bound (3.36) gives sufficient smallness in (3.50).

We must now prove Lemma 3.3. We rely on two further lemmas that control various stopping times for X and Y. To state the first result, define the stopping times

$$T_x := \inf\{s \geq 0 \mid X_s + x < 0\} \quad \text{and} \quad \overline{T}_x := \inf\{s \geq 0 \mid \overline{X}_s + x < 0\}$$

parameterized by $x \geq 0$.

LEMMA 3.4. *There exist $C_1', c_2' > 0$ depending on K and ν such that for each $(\mathcal{T}, \mathcal{X}) \in \{(T, X), (\overline{T}, \overline{X})\}$ and all $s > 0$:*

(i) $\mathbb{P}[\mathcal{T}_x > s] \leq C_1' \max\{x, 1\} s^{-\frac{1}{2}}$ *for all $x \geq 0$;*

(ii) $\mathbb{P}[\mathcal{T}_x > s] \geq c_2' \min\left\{x s^{-\frac{1}{2}}, 1\right\}$ *for all $x \geq 1$.*

Proof. We claim that these bounds are straightforward when $x \geq C\sqrt{s}$ for $C := \sqrt{2\nu\mathbb{E}\mathscr{X}^2}$. Indeed, (i) is vacuous in this regime if we take $C_1' = C^{-1}$. Moreover, $\mathbb{E}\mathcal{X}_s^2 = \mathbb{E}X_s^2$, (3.33), and Kolmogorov's inequality imply

$$\mathbb{P}[\mathcal{T}_x > s] \geq \mathbb{P}\left[\max_{s' \in [0,\, s]} |\mathcal{X}_{s'}| \leq x\right] \geq 1 - \frac{\mathbb{E}\mathcal{X}_s^2}{x^2} \geq \frac{1}{2}$$

if $x \geq C\sqrt{s}$. This yields (ii) for such x with $c_2' = \min\{C, 1\}/2$.

We may therefore assume that $x \leq C\sqrt{s}$. In this regime, the "gambler's ruin" bounds (i) and (ii) are well known in discrete time. For instance, they follow from Theorem 5.1.7 in [31]. That proof extends to continuous time, so we do not repeat it. There is only one wrinkle worth mentioning: we forbid x to approach 0 in (ii), to prevent the drift from immediately sweeping $\mathcal{X}$ into $\mathbb{R}_-$. □

To adapt these arguments to prove Lemma 3.3, we need estimates on the hitting times S and $\overline{S}$. This is the content of Theorem 3.2(ii) in [36]; in continuous time, it reads in part:

LEMMA 3.5. *Let $\mathcal{H}\colon [0, \infty) \to [0, \infty)$ be increasing and satisfy*

$$\int_0^\infty \frac{\mathcal{H}(s)}{(s+1)^{3/2}}\, \mathrm{d}s < \infty. \tag{3.54}$$

Then there exists $C_4 > 0$ depending on $\mathcal{H}$, K, and ν such that for each $\mathcal{X} \in \left\{X, \overline{X}\right\}$ and all $s > 0$,

$$\mathbb{P}[\mathcal{X}_r \geq -\mathcal{H}(r) \ \text{ for all } 0 \leq r \leq s] \leq \frac{C_4}{\sqrt{s}}. \tag{3.55}$$

The discrete-time proof of Theorem 3.2 in [36] relies on Lemma 3.3 in [36]. Our Lemma 3.4 is an analogue of that lemma in continuous time, so the proof of Theorem 3.2 in [36] extends to continuous time.

To close this section, we prove Lemma 3.3.

Proof. **(Proof of Lemma 3.3.)** Let $(\mathcal{X}, \mathcal{T}, \mathcal{H}, \mathcal{Y}, \mathcal{S})$ be either the "forward" case (X, T, H, Y, S) or the "backward" case $\left(\overline{X}, \overline{T}, \overline{H}, \overline{Y}, \overline{S}\right)$.

Part (i). Fix $s > 0$. Taking $C_1 \geq 1$, it suffices to consider $0 \leq x \leq \sqrt{s}$. Also, if $\mathcal{H} \leq 0$, we have $\mathbb{P}[\mathcal{S}_x > s] \leq \mathbb{P}[\mathcal{T}_x > s]$, so (i) follows from Lemma 3.4(i). We can therefore assume that $\mathcal{H} \geq 0$. We follow the proof of Lemma 3.3(i) in [36].

By the central limit theorem, there exists $C \geq 1$ such that

$$\mathbb{P}\big[\mathcal{X}_{C(x^2+1)} > x\big] \geq \frac{1}{3}.$$

Thus by Lemma 3.4(ii) and the FKG inequality,

$$\mathbb{P}\big[\mathcal{X}_{C(x^2+1)} > x,\, \mathcal{T}_1 > C(x^2+1)\big] \geq \frac{1}{3}\mathbb{P}\big[\mathcal{T}_1 > C(x^2+1)\big] \geq c\min\big\{x^{-1}, 1\big\} \tag{3.56}$$

for some $c > 0$ that may change from line to line. We now form a new drift

$$\tilde{\mathcal{H}}(s) \coloneqq \begin{cases} 0 & \text{for } s < C(x^2+1), \\ \mathcal{H}\big(s - C(x^2+1)\big) & \text{for } s \geq C(x^2+1). \end{cases}$$

Let $\tilde{\mathcal{S}}_1 \coloneqq \inf\{s \geq 0 \mid \mathcal{X}_s + 1 + \tilde{\mathcal{H}}(s) < 0\}$ and define the events

$$\begin{aligned} A &\coloneqq \big\{\mathcal{X}_{C(x^2+1)} > x,\, \mathcal{T}_1 > C(x^2+1)\big\}, \\ B &\coloneqq \big\{\mathcal{X}_{C(x^2+1)+r} - \mathcal{X}_{C(x^2+1)} \geq -x - \mathcal{H}(r) \ \text{ for all } 0 \leq r \leq s\big\}, \end{aligned}$$

noting that (3.56) is a lower bound on $\mathbb{P}[A]$. Now, disjoint increments of $\mathcal{X}$ are independent and the increments are stationary. It follows that

$$\mathbb{P}[A]\mathbb{P}\big[\mathcal{S}_x > s\big] = \mathbb{P}[A]\mathbb{P}[B] = \mathbb{P}[A, B] \leq \mathbb{P}\big[\tilde{\mathcal{T}}_1 > C(x^2+1) + s\big]. \tag{3.57}$$

Lemma 3.5 applies to $\tilde{\mathcal{H}} + 1$, so (3.55) implies that

$$\mathbb{P}\big[\tilde{\mathcal{T}}_1 > C(x^2+1) + s\big] \leq \frac{C_4}{\sqrt{C(x^2+1)+s}} \leq C_4 s^{-\frac{1}{2}}.$$

Combining this with (3.56) and (3.57), we obtain

$$\mathbb{P}[\mathcal{S}_x > s] \leq C_1' \max\{x, 1\} s^{-\frac{1}{2}}$$

for some $C_1' > 0$ depending on K, ν, and D.

Part (ii). Fix $s > 0$. The lower bound for $\mathcal{S}$ follows from Lemma 3.4(ii) if $\mathcal{H} \geq 0$, so we may assume that $\mathcal{H} \leq 0$. Also, the continuous-time random walk $\mathcal{X}$ obeys the invariance principle; see, for instance, [28, Theorem 19.25]. Therefore, since $\mathcal{H}(s) = o\big(\sqrt{s}\big)$,

$$\mathbb{P}\big[\mathcal{S}_x > s\big] \geq \frac{1}{2} \quad \text{if } x \geq \underline{C}\sqrt{s} \tag{3.58}$$

for some $\underline{C} > 0$. It thus suffices to consider $L \leq x \leq \underline{C}\sqrt{s}$. We adapt the proof of Theorem 3.2(i) in [36] to handle such x.

For $m \in \mathbb{N}$, we define the event

$$V_m \coloneqq \big\{\mathcal{X}_r + x \geq |\mathcal{H}(r)| \ \text{ for all } r \in (2^{m-1}, 2^m]\big\}.$$

We claim that there exists $\tilde{C} > 0$ such that

$$\mathbb{P}\big[V_m^c \mid \mathcal{T}_x > 4N\big] \leq \tilde{C}\,|\mathcal{H}(2^m)|\, 2^{-m/2} \tag{3.59}$$

for any $N \geq 2^{m-1}$. To see this, we condition on the time of the first violation:

$$r_* := \inf\left\{r \in (2^{m-1}, 2^m] \mid \mathcal{X}_r + x < |\mathcal{H}(r)|\right\}.$$

Since $|\mathcal{H}|$ is increasing and disjoint increments of $\mathcal{X}$ are independent,

$$\begin{aligned}\mathbb{P}\big[V_m^c, \mathcal{T}_x > 4N\big] &\leq \sup_{r\in(2^{m-1},2^m]} \mathbb{P}\big[V_m^c, \mathcal{T}_x > 4N \mid r_* = r\big]\\ &\leq \mathbb{P}\big[\mathcal{T}_x > 2^{m-1}\big] \sup_r \mathbb{P}\big[\mathcal{X}_u - \mathcal{X}_r \geq -|\mathcal{H}(2^m)| \;\text{ for } u \in [r, 4N]\big]\\ &\leq \mathbb{P}\big[\mathcal{T}_x > 2^{m-1}\big]\mathbb{P}\big[T_{|\mathcal{H}(2^m)|} > 2N\big].\end{aligned}$$

By Lemma 3.4(i),

$$\mathbb{P}\big[V_m^c, \mathcal{T}_x > 4N\big] \leq (C_1')^2 x\, |\mathcal{H}(2^m)|\, \big(2^m N\big)^{-\frac{1}{2}}.$$

Dividing by $\mathbb{P}\big[\mathcal{T}_x > 4N\big]$ and using Lemma 3.4(ii), we obtain (3.59) with the constant $\tilde{C} = 2(C_1')^2(c_2')^{-1}$.

Since $|\mathcal{H}|$ satisfies (3.54) and is increasing, there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{m=m_0}^{\infty} |\mathcal{H}(2^m)|\, 2^{-m/2} \leq \frac{1}{2\tilde{C}}.$$

By (3.59),

$$\sum_{m=m_0}^{\overline{m}} \mathbb{P}\big[V_m^c \mid \mathcal{T}_x > 2^{\overline{m}+1}\big] \leq \frac{1}{2}$$

for any $\overline{m} \geq m_0$. Therefore

$$\mathbb{P}\left[\bigcap_{m=m_0}^{\overline{m}} V_m \,\middle|\, \mathcal{T}_x > 2^{\overline{m}+1}\right] \geq \frac{1}{2}.$$

Multiplying by $\mathbb{P}\big[\mathcal{T}_x > 2^{\overline{m}+1}\big]$ and using Lemma 3.4(ii), we find

$$\mathbb{P}\left[\bigcap_{m=m_0}^{\overline{m}} V_m\right] \geq cx2^{-\frac{\overline{m}}{2}}$$

for some $c > 0$ that may change from line to line. Now, the FKG inequality implies

$$\mathbb{P}\big[\mathcal{S}_x > 2^{\overline{m}}\big] = \mathbb{P}\left[\bigcap_{m=m_0}^{\overline{m}} V_m,\, \mathcal{S}_x > 2^{m_0}\right] \geq cx2^{-\frac{\overline{m}}{2}}\mathbb{P}\big[\mathcal{S}_x > 2^{m_0}\big]. \tag{3.60}$$

Finally, we choose $L := \underline{C}\, 2^{\frac{m_0}{2}}$. Then $x \geq L$ and (3.58) imply that $\mathbb{P}\big[\mathcal{S}_x > 2^{m_0}\big] \geq \frac{1}{2}$. Taking $\overline{m} = \lceil \log_2 s \rceil$, (3.60) yields (ii).

Part (iii). By the central limit theorem, there exist $\beta > 0$ and $s_0 \geq 1$ such that

$$\mathbb{P}\Big[\mathcal{X}_s \geq 2\beta\sqrt{s}\Big] \geq \frac{1}{3} \quad \text{for all } s \geq s_0.$$

Since $\mathcal{H}(s) = o\big(\sqrt{s}\big)$, we can increase s_0 to ensure that $\mathcal{H}(s) \geq -\beta\sqrt{s}$ for all $s \geq s_0$. Then

$$\mathbb{P}\Big[\mathcal{Y}_s^x \geq \beta\sqrt{s}\Big] \geq \mathbb{P}\Big[\mathcal{X}_s \geq 2\beta\sqrt{s}\Big] \geq \frac{1}{3} \quad \text{for all } s \geq s_0,\, x \geq 0.$$

Part (iv). Fix $s \geq s_0$, $x \in \big[L, 2\sqrt{s}\big]$, and $x' \in \big[0, 2\sqrt{s}\big]$. Since $x' \leq 2\sqrt{s}$ and $\mathcal{H}(s) = o\big(\sqrt{s}\big)$, Young's inequality shows that it suffices to bound

$$\mathbb{E}\Big[\mathcal{X}_s^2 \mid \mathcal{S}_x > s,\, \mathcal{Y}_s^{x'} \geq \beta\sqrt{s}\Big].$$

We then follow the proofs of Lemma 3.3(ii) in [36] and Corollary 4 in [2]. First, we claim that

$$\mathbb{E}\left[\mathcal{X}_s^2 \mid \mathcal{S}_x > s\right] \leq Cs \tag{3.61}$$

for some $C > 0$. Indeed, by (i),

$$\mathbb{E}[\mathcal{S}_x \wedge s] = \int_0^s \mathbb{P}[\mathcal{S}_x \geq r]\,\mathrm{d}r \leq 2C_1 x\sqrt{s}.$$

Thus by Wald's identity and (3.33),

$$\mathbb{E}\big[\mathcal{X}_s^2 \mathbf{1}_{\mathcal{S}_x > s}\big] \leq \mathbb{E}X_{\mathcal{S}_x \wedge s}^2 = \nu\mathbb{E}\mathscr{X}^2\,\mathbb{E}\big[\mathcal{S}_x \wedge s\big] \leq 2C_1\nu\mathbb{E}\mathscr{X}^2 x\sqrt{s}.$$

Dividing by $\mathbb{P}[\mathcal{S}_x > s]$, (ii) implies (3.61).

Next, the FKG inequality and (iii) imply

$$\mathbb{P}\big[\mathcal{Y}_s^{x'} \geq \beta\sqrt{s} \mid \mathcal{S}_x > s\big] \geq \frac{1}{3}.$$

Combining this with (3.61), we have

$$\begin{aligned}\mathbb{E}\Big[\mathcal{X}_s^2 \mid \mathcal{S}_x > s,\, \mathcal{Y}_s^{x'} \geq \beta\sqrt{s}\Big] &= \frac{\mathbb{E}\big[\mathcal{X}_s^2 \mathbf{1}_{\{\mathcal{Y}_s^{x'} \geq \beta\sqrt{s}\}} \mid \mathcal{S}_x > s\big]}{\mathbb{P}\big[\mathcal{Y}_s^{x'} \geq \beta\sqrt{s} \mid \mathcal{S}_x > s\big]} \\ &\leq 3\mathbb{E}\big[\mathcal{X}_s^2 \mid \mathcal{S}_x > s\big] \leq 3Cs.\end{aligned}$$

This completes the proof of Lemma 3.3. ∎

4. Proof of Theorem 1.2

We now turn to Theorem 1.2. By Proposition 1.1, $c_* = 0$, so we are interested in the behaviour of stationary waves. It is important to note that the continuity and uniqueness of stationary waves is a delicate issue [18]. As we shall see, both these pleasant properties can fail in this setting.

Proof. **(Proof of Theorem 1.2(i).)** We assume that $J(\mathbb{R}_+) = 0$ and $f'(0) > \mu$. By Proposition 1.1, we necessarily have $c_* = 0$. Our minimal-speed wave U thus satisfies

$$\mu(J * U - U) + f(U) = 0.$$

Such waves need not be unique up to translation, but by Proposition 3.1, they do exist. Define

$$\mathcal{F} \coloneqq \{s \in [0,1] \mid f(s) > \mu s\}.$$

Since $f'(0) > \mu$, $\mathcal{F}$ contains a nontrivial interval of the form $(0, \theta_0)$. But

$$\mu U - f(U) = \mu\, J * U \geq 0,$$

so U cannot assume any value in $\mathcal{F}$. Since $U(+\infty) = 0$, the profile U must jump *discontinuously* down to 0 at a finite position. By shifting U, we may assume that $U(x) = 0$ for all $x \geq 0$.

Now consider the evolution of u from $\mathbf{1}_{\mathbb{R}_-}$. The initial condition $\mathbf{1}_{\mathbb{R}_-}$ is a supersolution of (1.1). On the other hand, the stationary front U is a solution to (1.1), and $U \leq \mathbf{1}_{\mathbb{R}_-} = u(0, \cdot)$. By the comparison principle,

$$U(x) \leq u(t, x) \leq \mathbf{1}_{\mathbb{R}_-}(x) \quad \text{for all } (t, x) \in [0, \infty) \times \mathbb{R}.$$

So $U^{-1}(\theta) \leq \sigma_\theta(t) \leq 0$ for all $\theta \in (0, 1)$ and $t \geq 0$. Here U^{-1} denotes the left-continuous pseudo-inverse of U, i.e. $U^{-1}(\theta) \coloneqq \sup\{x \in \mathbb{R} \mid U(x) \geq \theta\}$. □

We now study a special family of equations satisfying $f'(0) = \mu$.

Proof. **(Proof of Theorem 1.2(ii).)** By rescaling time, we can reduce to the case $\mu = 1$. Then $J = \delta_{-1}$ and $f(u) = u - u^p$ for $p > 1$ fixed. A stationary front U satisfies

$$U(x + 1) = U(x)^p \quad \text{for all } x \in \mathbb{R}.$$

In this case, we can explicitly construct a monotone front:

$$U(x) \coloneqq \exp\big(-p^x\big).$$

In fact, there is a discontinuous alternative:

$$\tilde{U}(x) \coloneqq U\big(\lfloor x \rfloor\big).$$

This equation thus admits nonunique monotone stationary fronts. For analytic convenience, we work with U.

As in previous proof, $\mathbf{1}_{\mathbb{R}_-}$ is a supersolution of (1.1), so $u = 0$ on $[0, \infty) \times [0, \infty)$. Furthermore, we can explicitly compute the solution on $[0, \infty) \times [-1, 0)$. Indeed,

$$J * u(t, x) = u(t, x + 1) = 0$$

for all $x \in [-1, 0)$, so

$$\partial_t u = -u^p, \quad u(0, x) = 1.$$

Solving this Riccati-type equation, we obtain

$$u(t, x) = \big[(p - 1)t + 1\big]^{-\frac{1}{p-1}} \quad \text{for all } (t, x) \in [0, \infty) \times [-1, 0). \tag{4.1}$$

In principle, u can be found by iteratively solving ODEs for its values on $[-n, 1 - n)$ with $n \in \mathbb{N}$, but we need not perform such calculations.

Instead, we construct super- and subsolutions to (1.1) on $\mathbb{R}_-$. Combining these with the explicit solution (4.1) on the “buffer zone” $[-1, 0)$, we can control u via the comparison principle. We begin with the subsolution. Define the decreasing shift

$$\sigma_-(t) \coloneqq -U^{-1}(u(t, -1)) - 1$$

and

$$\underline{w}(t,x) \coloneqq U\big(x - \sigma_-(t)\big).$$

Note that

$$\partial_t \underline{w} - J * \underline{w} + \underline{w}^p = \partial_t \underline{w} = -\dot{\sigma}_- U' < 0,$$

so $\underline{w}$ is a subsolution to (1.1) on $\mathbb{R}_-$. By construction,

$$\underline{w}(0,\,\cdot\,) = 1 = u(0,\,\cdot\,) \quad \text{on } \mathbb{R}_-$$

and

$$\underline{w}(t,x) \leq \underline{w}(t,-1) = u(t,-1) = u(t,x) \quad \text{on } [0,\infty) \times [-1,0).$$

Thus by the comparison principle on $(-\infty,-1)$, we have

$$u(t,x) \geq \underline{w}(t,x) \quad \text{for all } (t,x) \in [0,\infty) \times \mathbb{R}_-. \tag{4.2}$$

We now construct a supersolution. Define

$$\sigma_+(t) \coloneqq -\frac{\log\log(t+1)}{\log p} + 1$$

and

$$\overline{w}(t,x) \coloneqq \Omega\, U(x - \sigma_+(t)) = \Omega(t+1)^{-p^{x-1}}$$

for some $\Omega = \Omega(p) > 1$ to be determined. Then

$$\partial_t \overline{w} = -\frac{p^{x-1}}{t+1}\overline{w}$$

and

$$J * \overline{w} - \overline{w}^p = -(1-\Omega^{1-p})\overline{w}^p = -(\Omega^{p-1}-1)(t+1)^{-(p-1)p^{x-1}}\overline{w}.$$

So

$$\partial_t \overline{w} - J * \overline{w} + \overline{w}^p = \left[(\Omega^{p-1}-1)(t+1)^{-(p-1)p^{x-1}} - p^{x-1}(t+1)^{-1}\right]\overline{w}.$$

Suppose $x \leq 0$. Then

$$(\Omega^{p-1}-1)(t+1)^{-(p-1)p^{x-1}} - p^{x-1}(t+1)^{-1} \geq \left(\Omega^{p-1} - 1 - p^{-1}\right)(t+1)^{-1}.$$

We thus choose

$$\Omega = \left(\frac{p+1}{p}\right)^{\frac{1}{p-1}},$$

so that $\overline{w}$ is a supersolution to (1.1). Also,

$$\overline{w}(0,\,\cdot\,) = \Omega > 1 \geq u(0,\,\cdot\,).$$

When $p \geq 2$,

$$\overline{w}(t,x) \geq \overline{w}(t,0) = \Omega(t+1)^{-\frac{1}{p}} \geq \left[(p-1)t+1\right]^{-\frac{1}{p}} > u(t,x)$$

for $x \in [-1,0)$. When $p \in (1,2)$, the function $(t+1)^{p-1}$ is concave in t, so

$$(t+1)^{p-1} \leq (p-1)t+1.$$

It follows that

$$\overline{w}(t,x) > (t+1)^{-1} \geq [(p-1)t+1]^{-\frac{1}{p-1}} = u(t,x)$$

for $x \in [-1,0)$. Since this holds for all $p > 1$, the comparison principle implies

$$\overline{w}(t,x) \geq u(t,x) \quad \text{for all } (t,x) \in [0,\infty) \times \mathbb{R}_-. \tag{4.3}$$

By construction, $\underline{w}$ and $\overline{w}$ are fixed profiles drifting by σ_- and σ_+, respectively. Furthermore, there exists $C(p) > 0$ such that

$$\left|\sigma_\pm(t) + \frac{\log\log t}{\log p}\right| \leq C \quad \text{for all } t \geq 2.$$

Thus (4.2) and (4.3) imply Theorem 1.2(ii). □

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Appendix A. The probabilistic connection. In this appendix, we examine the close relationship between integro-differential Fisher–KPP equations and branching random walks (BRWs).

A.1. Continuous time. A continuous-time BRW is a growing collection of particles on $\mathbb{R}$, each jumping and reproducing independently with exponential rates μ and r, respectively. When particles jump, we assume that they obey a law $J \in \mathcal{P}(\mathbb{R})$ satisfying (J1) and (J2). When they reproduce, the particles have a random number of offspring distributed according to a law $\kappa \in \mathcal{P}(\mathbb{N}_{\geq 2})$. For a detailed description and construction of branching random walks, we refer the reader to Harris [27]. Throughout, $\mathscr{X}$ and $\mathscr{Z}$ will denote random variables with laws J and κ, respectively. We will assume

$$\mathbb{E}\mathscr{Z}^{1+\gamma} < \infty \tag{A.1}$$

for some $\gamma > 0$. This condition is nearly sharp, as BRWs behave quite differently when $\mathbb{E}\mathscr{Z} = \infty$; see, for instance, [23, 38].

To understand the spreading of the population in a BRW, we study the particle with *maximal* position. That is, if $X_t^1, \ldots, X_t^{Z_t}$ denote the particle positions at time t, we study the (reversed) cumulative distribution of the maximal particle:

$$u(t,x) \coloneqq \mathbb{P}\left[\max_{1 \leq j \leq Z_t} X_t^j > x\right] \quad \text{for } (t,x) \in [0,\infty) \times \mathbb{R}.$$

Note that the population size Z_t is itself random.

Let g denote the probability generating function of κ, i.e. $g(s) = \mathbb{E}s^{\mathscr{Z}}$. Then if

$$f(s) \coloneqq r\big[1 - s - g(1 - s)\big], \tag{A.2}$$

a renewal argument along the lines of McKean [33] shows that u satisfies (1.1). Assuming the process begins from a single individual at the origin, $u(0, \cdot\,) = \mathbf{1}_{\mathbb{R}_-}$.

Using properties of the moment generating function g and (A.1), one can check that f satisfies (F1)–(F3), and is thus a KPP reaction. In fact, it is a rather special KPP reaction; for instance, it is analytic and concave. "Generic" KPP reactions are neither, and thus do not correspond to continuous-time BRWs.

A.2. Discrete sampling. While continuous-time BRWs are of analytic interest, the majority of the BRW literature concerns *discrete* time. In this setting, we replace each particle by an independent copy of a fixed point process Π when we step forward in time. For instance, Π might be $\mathscr{Z}$ particles independently sampled from J.

We can shift from continuous to discrete time by sampling a continuous BRW at evenly-spaced times. The point process Π is the set of particles in the continuous-time BRW after the first time interval. The position of the maximal particle in discrete-time BRWs is well-understood; see, for instance, [3, 4, 7, 13, 14, 26, 29]. Thus when f is of the form (A.2), our main results follow from prior work on discrete-time BRWs. In the remainder of this appendix, we describe this correspondence.

We fix a continuous-time BRW with kernels J and κ, and sample it at the discrete times $\mathbb{Z}_{\geq 0}$. Let Π denote the point process at time $t = 1$. Each particle in Π is individually distributed according to the law J_1 of a continuous-time random walk at time 1. Thus, J_1 is a Poissonisation of J:

$$J_1 = \mathrm{e}^{-\mu} \sum_{k=0}^{\infty} \frac{\mu^k}{k!} J^{*k}. \tag{A.3}$$

The total number of particles $|\Pi|$ is the population size Z_1 of the continuous-time BRW at time 1. Its law is not as easily described as J_1, but we can use a renewal argument to compute its moments. In particular,

$$\mathbb{E}Z_t = \exp\big[r(\mathbb{E}\mathscr{Z} - 1)t\big] \quad \text{for all } t \geq 0. \tag{A.4}$$

We note that (A.2) implies

$$f'(0) = r(\mathbb{E}\mathscr{Z} - 1), \tag{A.5}$$

so $f'(0)$ is the mean rate of particle production in the continuous-time BRW.

We now compute the asymptotic speed of the maximal particle in the discrete-time sampled BRW. This speed is related to the logarithmic moment generating function of Π:

$$R(\lambda) \coloneqq \log \mathbb{E}\left[\sum_{p \in \Pi} \mathrm{e}^{\lambda X(p)}\right], \tag{A.6}$$

where p denotes a point in Π with position $X(p)$. Using (A.3) and (A.4), a standard calculation yields:

$$R(\lambda) = r(\mathbb{E}\mathscr{Z} - 1) + \mu\left(\mathbb{E}\mathrm{e}^{\lambda \mathscr{X}} - 1\right).$$

As shown in [40], for instance, the asymptotic speed of the maximal particle in the sampled BRW is

$$c_* = \inf_{\lambda>0} \frac{R(\lambda)}{\lambda} = \inf_{\lambda>0} \frac{1}{\lambda}\left[r(\mathbb{E}\mathscr{Z}-1) + \mu\left(\mathbb{E}\mathrm{e}^{\lambda\mathscr{X}}-1\right)\right]. \tag{A.7}$$

In light of (A.5), (A.7) agrees with (1.6).

A.3. Sublinear behaviour. Sublinear corrections to the position of the maximal particle depend on the structure of the point process Π.

Suppose Π is unbounded from above. Then (A.6) implies that R grows superlinearly as $\lambda \to \infty$, so $\frac{R(\lambda)}{\lambda}$ attains its minimum. By Theorem 3 of [3], the maximal particle has position

$$c_* t - \frac{3}{2\lambda_*}\log t + \mathcal{O}_{\mathbb{P}}(1) \quad \text{as } t \to \infty \text{ in } \mathbb{N}, \tag{A.8}$$

as in our Theorem 1.1. Here $\mathcal{O}_{\mathbb{P}}(1)$ denotes a tight sequence of random variables.

Now suppose Π is bounded from above. Each particle in Π is distributed according to J_1, the Poissonisation of J. The Poissonisation (A.3) includes arbitrarily high convolutional powers of J, so it is only bounded from above if $J(\mathbb{R}_+) = 0$. In this case, the sublinear discrete-time behaviour hinges on the value of

$$\Xi \coloneqq \mathbb{P}[X_1 = 0]\,\mathbb{E}Z_1 = \exp\left[-\mu + r(\mathbb{E}\mathscr{Z}-1)\right] = \exp\left[f'(0) - \mu\right].$$

If $f'(0) < \mu$, then $\Xi < 1$, and Corollary 2 in [3] implies that (A.7) has a minimizer; *cf.* our Proposition 1.1. Theorem 3 of [3] then implies the log-delay (A.8), as in our Theorem 1.1.

When $\Xi > 1$, Theorem 4 in [3] states that the maximal particle remains a bounded distance from the origin. This corresponds to our Theorem 1.2(i), for then $f'(0) > \mu$. The borderline case $\Xi = 1$ can yield unusual results. Bramson neatly examined such BRWs in discrete time [11]. Due to the variety of possible behaviours, we do not comprehensively study the analogous case for (1.1), when $J(\mathbb{R}_+) = 0$ and $f'(0) = \mu$. Instead, our Theorem 1.2(ii) exhibits one family of "borderline" shifts.

In summary, our main findings parallel well-known results in discrete time. Indeed, they follow from prior results when f can be written in the form (A.2). However, generic KPP reactions do not assume this special form, and Theorems 1.1 and 1.2 are new in this setting.

Appendix B. Travelling waves for irregular kernels. In this appendix, we draw on [17] and [18] to prove Proposition 3.1.

Proof. **(Proof of Proposition 3.1.)** We first extend the existence result from [18] to our setting using an approximation argument of Coville in [17].

Let $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ be an even nonnegative bump function with $\int_{\mathbb{R}} \rho = 1$. Define $\rho^{(n)}(x) = n\rho(nx)$ and $J^{(n)} \coloneqq J * \rho^{(n)}$ for $n \in \mathbb{N}$. Let $\Gamma^{(n)}$ denote the speed functional corresponding to $J^{(n)}$ as in (1.6), and define $c_*^{(n)} \coloneqq \inf \Gamma^{(n)}$.

For the moment, fix $n \in \mathbb{N}$. For each $\varepsilon > 0$, pick $\lambda_\varepsilon > 0$ such that

$$\Gamma^{(n)}(\lambda_\varepsilon) < c_*^{(n)} + \varepsilon.$$

Then (F3) implies that $\mathrm{e}^{-\lambda_\varepsilon x}$ is a supersolution of speed $c_*^{(n)} + \varepsilon$ in the sense of Theorem 1.3 in [18]. Taking $\varepsilon \to 0$, the theorem implies the existence of a monotone travelling wave $U^{(n)}$ for $J^{(n)}$ of speed $c_*^{(n)}$. Since $U^{(n)}$ connects 1 to 0, we may shift it so that

$$U^{(n)}(0^-) \geq \frac{1}{2} \quad \text{and} \quad U^{(n)}(0^+) \leq \frac{1}{2}. \tag{B.1}$$

Note that we do not assume that $U^{(n)}$ is continuous.

Next, we take the limit $n \to \infty$. Using the evenness of ρ, standard convolution identities for the Laplace transform imply that $\Gamma^{(n)} \geq \Gamma$ and $\Gamma^{(n)} \to \Gamma$ pointwise. Hence

$$c_*^{(n)} = \inf \Gamma^{(n)} \to \inf \Gamma = c_* \quad \text{as } n \to \infty.$$

We claim that there exists a monotone $U\colon \mathbb{R} \to [0,1]$ such that

$$\mu(J * U - U) + c_* U' + f(U) = 0. \tag{B.2}$$

Our argument splits into two cases.

First, suppose that $c_* \neq 0$. Then we can assume that n is sufficiently large that $\left|c_*^{(n)}\right| \geq |c_*|/2$. Using $f \in \mathcal{C}^1$ from (F1), we can bootstrap

$$-c_*^{(n)}\big(U^{(n)}\big)' = \mu\big(J^{(n)} * U^{(n)} - U^{(n)}\big) + f\big(U^{(n)}\big) \tag{B.3}$$

to conclude that $U^{(n)}$ is uniformly bounded in $\mathcal{C}^2$. Then Arzelà–Ascoli and diagonalisation yield a $\mathcal{C}^1_{\text{loc}}$-convergent subsequence with limit U satisfying (B.2).

Next, suppose that $c = 0$. We don't have *a priori* regularity, but Helly's selection theorem allows us to extract a subsequence along which $U^{(n)}$ and $\rho^{(n)} * U^{(n)}$ both converge pointwise to limits U and $\tilde{U}$, respectively. For the remainder of the proof, we constrain n to lie in this subsequence. Dominated convergence implies that

$$J^{(n)} * U^{(n)} = J * \left(\rho^{(n)} * U^{(n)}\right) \to J * \tilde{U}.$$

Integrating (B.3) in space and taking $n \to \infty$, we see that

$$\mu(J * \tilde{U} - U) + f(U) = 0 \quad \text{a.e.}$$

Because U is monotone, we can easily check that $\rho^{(n)} * U^{(n)} \to U$ wherever U is continuous. That is, $\tilde{U} = U$ off the countable set of discontinuities of U. Thus

$$\mu(J * U - U) + f(U) = 0 \quad \text{a.e.}$$

Following [16], we define the right-continuous profile $U^+(x) \coloneqq U(x^+)$. Then dominated convergence yields

$$\mu(J * U^+ - U^+) + f(U^+) = 0 \quad \text{on } \mathbb{R}.$$

Thus U^+ satisfies (B.2). We henceforth refer to U^+ as U.

Now, we must verify the limiting behaviour in (1.5). Because U is monotone, the limits $U(\pm\infty)$ exist, lie between 0 and 1, and satisfy $f\big(U(\pm\infty)\big) = 0$. Moreover, (B.1) implies that $U(-\infty) \geq \frac{1}{2}$ and $U(+\infty) \leq \frac{1}{2}$. By (F2), we must have $U(-\infty) = 1$ and $U(+\infty) = 0$, so U is a monotone travelling wave for (1.1) of speed c_*, as desired.

Finally, suppose that Γ attains its infimum at $\lambda_* > 0$. The decay in (3.29) follows from Theorem 1.6 in [18] under additional regularity conditions on f and J. However, a careful examination of the proof reveals that (3.29) holds for *monotone* travelling waves without additional assumptions. Indeed, Lemma 5.2 in [18] shows that the integral $V(x) := \int_x^\infty U(y)\,\mathrm{d}y$ decays exponentially as $x \to \infty$ (recalling that [18] uses opposite spatial signs). The proof of Theorem 1.6 in [18] then uses Ikehara's theorem to conclude that V satisfies

$$V(x) \leq C_V x \mathrm{e}^{-\lambda_* x} \quad \text{for all } x \geq 1. \tag{B.4}$$

Neither argument makes use of any hypotheses on J or f beyond (J1) and (F1)–(F3). Now, suppose for the sake of contradiction that there exists a sequence $x_n \to \infty$ such that $U(x_n) \geq n x_n \mathrm{e}^{-\lambda_* x_n}$. Then the monotonicity of U implies:

$$V(x_n - 1) \geq \int_{x_n-1}^{x_n} U(x)\,\mathrm{d}x \geq U(x_n) \geq n x_n \mathrm{e}^{-\lambda_* x_n} \geq \mathrm{e}^{-\lambda_*} n (x_n - 1) \mathrm{e}^{-\lambda_*(x_n - 1)}.$$

This contradicts (B.4) once $n > \mathrm{e}^{\lambda_*} C_V$, so in fact U satisfies (3.29) for some $C > 0$. This completes the proof of Proposition 3.1. ∎

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