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# OPTIMAL LARGE TIME BEHAVIOR OF THE COMPRESSIBLE BIPOLAR NAVIER–STOKES–POISSON SYSTEM*

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**Abstract.** This paper is concerned with the Cauchy problem of the 3D compressible bipolar Navier–Stokes–Poisson (BNSP) system. Our main purpose is three–fold: First, under the assumption that $H^l\cap L^1(l\geq 3)$–norm of the initial data is small, we prove the time decay rates of the solution as well as its spatial derivatives from the first–order to the highest–order. Similar to the results on the heat equation and the compressible Navier–Stokes equations, these decay rates for the BNSP system are optimal. Second, for well–chosen initial data, we also show the lower bounds on the decay rates. Third, we give the explicit influences of the electric field on the qualitative behaviors of solutions, which are totally new as compared to the results for the compressible unipolar Navier–Stokes–Poisson (UNSP) system [H.l. Li et al., Arch. Ration. Mech. Anal., 196:681–713, 2010; Y.J. Wang, J. Differ. Equ., 253:273–297, 2012]. More precisely, we show that the densities of the BNSP system converge to their corresponding equilibriums at the same $L^2$–rate $(1+t)^{-\frac{3}{4}}$ as the compressible Navier–Stokes equations, but the momentums of the BNSP system decay at the $L^2$–rate $(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}$ with $1\leq p\leq \frac{3}{2}$, which depend directly on the initial low frequency assumption of electric field, namely, the smallness of $\|\nabla\phi_0\|_{L^p}$. This phenomenon is the most important difference from the compressible Navier–Stokes equations.

**Keywords.** Bipolar compressible Navier–Stokes–Poisson system; unequal viscosities; optimal time decay rates.

**AMS subject classifications.** 35D35; 35B40.

## 1. Introduction

At high temperature and velocity, ions and electrons in a plasma tend to become two separate fluids due to their different physical properties (inertia, charge). One of the fundamental fluid models for describing plasma dynamics is the two–fluid model, in which two compressible ion and electron fluids penetrate each other through their own self–consistent electromagnetic field. In this paper, we formally take the magnetic field equal to zero and study the optimal time decay rate for the global classical solution to the bipolar Navier–Stokes–Poisson (BNSP) system in 3D:

$$\begin{cases} \partial_t\rho_1+\mathrm{div}m_1=0,\\ \partial_t m_1+\mathrm{div}\left(\frac{m_1\otimes m_1}{\rho_1}\right)+\nabla P_1=\mu_1\Delta\left(\frac{m_1}{\rho_1}\right)+\nu_1\nabla\mathrm{div}\left(\frac{m_1}{\rho_1}\right)+Z\rho_1\nabla\phi,\\ \partial_t\rho_2+\mathrm{div}m_2=0,\\ \partial_t m_2+\mathrm{div}\left(\frac{m_2\otimes m_2}{\rho_2}\right)+\nabla P_2=\mu_2\Delta\left(\frac{m_2}{\rho_2}\right)+\nu_2\nabla\mathrm{div}\left(\frac{m_2}{\rho_2}\right)-\rho_2\nabla\phi,\\ \Delta\phi=Z\rho_1-\rho_2,\qquad \lim_{|x|\to\infty}\phi(x,t)=0 \end{cases} \tag{1.1}$$

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for $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$ and with initial data

$$(\rho_1,m_1,\rho_2,m_2,\phi)(x,0)=(\rho_{10},m_{10},\rho_{20},m_{20},\phi_0)(x)\to\left(\frac{1}{Z},0,1,0,0\right)\quad as\quad |x|\to\infty. \tag{1.2}$$

The (BNSP) system describes a plasma composed of ions and electrons. The unknown functions are density $\rho_1$ and momentum $m_1$ of the ions, density $\rho_2$ and momentum $m_2$ of the electrons, and the electrostatic potential $\phi$. The positive constant $Z$ represents the charge of the ions. For $i=1,2$, the viscosity coefficients $\mu_i>0$ and $\nu_i$ satisfy $3\mu_i+2\nu_i>0$, and the pressure functions $P_i$ of the two fluids terms of $\rho_i$ are smooth functions and satisfy $P_i'(\rho_i)>0$ if $\rho_i>0$. Without loss of generality, we assume other physical parameters to be 1.

Owing to the physical importance and mathematical challenges, there is an extensive literature on the long-time behavior of global smooth solutions to the Navier–Stokes–Poisson (NSP) system. If only considering the dynamics of one fluid in plasmas, the system (1.1) would reduce to the unipolar Navier–Stokes–Poisson (UNSP) system. For the UNSP system, Li–Matsumura–Zhang [20] proved that the density of the NSP system converges to its equilibrium state at the same $L^2$–rate $(1+t)^{-\frac{3}{4}}$ as the compressible Navier–Stokes (NS) system, but the momentum of the UNSP system decays at the $L^2$–rate $(1+t)^{-\frac{1}{4}}$, which is slower than the $L^2$–rate $(1+t)^{-\frac{3}{4}}$ for the compressible NS system, due to the effect of the electric field. And then the authors extended similar result to the non–isentropic case [35]. Wu–Wang [32] investigated the pointwise estimates of the solution and showed the pointwise profile of the solution contains the D–wave but does not contain the H–wave, which is different from the compressible NS system. Wang [29] obtained the optimal asymptotic decay of solutions just by pure energy estimates, and particularly proved that the density of the UNSP system decays at $L^2$–rate $(1+t)^{-\frac{5}{4}}$, which is faster than $L^2$–rate $(1+t)^{-\frac{3}{4}}$ for the NS system due to the effect of the electric field. Hao–Li [17], Tan–Wu [26], Chikami–Danchin [4] and Bie–Wang–Yao [2] also established the unique global solvability and the optimal decay rates for small perturbations on a linearly stable constant state. We mention that there are many results on the existence and long-time behavior of the weak solutions or non–constant stationary solutions, see [1, 9, 13, 25, 36] and the references therein. We also mention that the quasi–neutral phenomenon to the UNSP system is studied in [10, 19, 27].

For the BNSP system (1.1), there are very few results due to its non–conservative structure and the interaction of two fluids through the electric field. Duan–Yang [12] showed global well–posedness and asymptotic behavior of smooth solutions for the Cauchy problem in one dimension, and Zhou–Li [37] obtained the corresponding convergence rate. Recently, Li–Yang–Zou [21], Zou [38], and Wu–Wang [32] investigated the global existence and the optimal decay rates of the classical solution around a constant state by detailed analysis of the Green's function to the related linearized equations. And Wang–Xu [28] proved the $L^2$–decay rate of the solution by using long wave and short wave decomposition method. It should be noted that in [21, 32, 38], the viscosity coefficients of two fluids are taken to be equal to each other, i.e.,

$$\mu_1=\mu_2,\quad \nu_1=\nu_2.$$

Thus by taking a linear combination of the system (1.1), the system (1.1) can be reformulated into one for the compressible NS system and another one for the UNSP system which however are coupled with each other through the nonlinear terms. Thus in order to obtain a priori estimates of the solutions, one can employ similar arguments as in [3,22,23] for the Navier–Stokes system and in [20,29] for the UNSP system. Recently, under the assumption that the initial perturbation is small in $H^l(\mathbb{R}^3)$ with $l\geq 3$, Wu–Zhang–Zhang [33] established the global existence for the BNSP system with unequal viscosities, i.e.,

$$\mu_1\neq\mu_2,\quad \nu_1\neq\nu_2. \tag{1.3}$$

Moreover, if in addition, the initial perturbation is small in $\dot{H}^{-s}(\mathbb{R}^3)$–norm with $\frac{1}{2}\leq s<\frac{3}{2}$ or $\dot{B}^{-s}_{2,\infty}$–norm with $\frac{1}{2}<s\leq\frac{3}{2}$, it is shown that the densities $\rho_i$ and the velocities $u_i=\frac{m_i}{\rho_i}$ with $i=1,2$ have the following convergent decay estimates

$$\left\|\nabla^k\left(\rho_1-\frac{1}{Z},u_1,\rho_2-1,u_2,\nabla\phi\right)(t)\right\|_{\ell-k}\lesssim(1+t)^{-\frac{k+s}{2}},$$

for $k=0,1,\cdots,l-1$, which together with the fact that for $p\in(1,3/2]$, $L^p\subset\dot{H}^{-s}$ with $s=3(\frac{1}{p}-\frac{1}{2})\in[1/2,3/2)$, and for $p\in[1,3/2)$, $L^p\subset\dot{B}^{-s}_{2,\infty}$ with $s=3(\frac{1}{p}-\frac{1}{2})\in(1/2,3/2]$ imply

$$\left\|\nabla^k\left(\rho_1-\frac{1}{Z},u_1,\rho_2-1,u_2,\nabla\phi\right)(t)\right\|_{\ell-k}\lesssim(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}, \tag{1.4}$$

for $p\in[1,3/2]$ and $k=0,1,\cdots,l-1$. For the compressible Euler–Poisson system and related models, we refer to [14–16,30,31] and the references therein.

We remark that this work is strongly motivated by [20,29,33]. Particularly, we are interested in the following three problems:

(i) Both of the results in [20,29] show that the electric field has influence on the qualitative behaviors of solutions. However, neither of them explains that how the electric field affects the qualitative behaviors of solutions. Can we give an exact influence of the electric field on the qualitative behaviors of solutions?

(ii) Noticing the decay rate (1.4) of [33], for the $l$-th–order (i.e. **the highest–order**) spatial derivative of the solution, it holds that

$$\left\|\nabla^l\left(\rho_1-\frac{1}{Z},u_1,\rho_2-1,u_2,\nabla\phi\right)(t)\right\|_{L^2}\lesssim(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{l-1}{2}}. \tag{1.5}$$

On the other hand, let us revisit the following classical result of the heat equation:

$$\begin{cases}\partial_t u-\Delta u=0, & \text{in } \mathbb{R}^3,\\ u|_{t=0}=u_0.\end{cases} \tag{1.6}$$

If $u_0\in H^l(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$ with $l\geq 0$ be an integer and $1\leq p\leq 2$, then for any $0\leq k\leq l$, the solution of the heat equation (1.6) has the following decay rate:

$$\|\nabla^k u(t)\|_{L^2}\leq C((1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}},$$

which particularly implies

$$\|\nabla^l u(t)\|_{L^2}\leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{l}{2}}. \tag{1.7}$$

Therefore, in view of (1.5) and (1.7), it is clear that the decay rate of the $l$-th–order spatial derivative of the solution in (1.5) is slower than that of the heat equation as in

(1.7). Therefore, the decay rate of the $l$-th–order spatial derivative of the solution in (1.5) is not optimal in this sense. So, can we develop a method to capture the optimal decay rate of **the highest–order** spatial derivative of the solution?

(iii) All the results in [20, 29, 33] do not give any information on the lower bounds of decay rate for the solution. Can we provide some lower bounds of decay rate for the solution as well as its spatial derivatives of all orders?

The main motivation of this article is to give a clear answer to these issues mentioned above. Before stating our results, let us introduce some notations and conventions used throughout this paper. We employ $H^\ell(\mathbb{R}^3)$ and $W^{m,p}(\mathbb{R}^3)$ to denote the usual Sobolev spaces with norm $||\cdot||_m$ and $||\cdot||_{m,p}$ for $m\geq 0$ and $1\leq p\leq +\infty$. If $m=0$, we just use $||\cdot||_{L^2}$ and $||\cdot||_{L^q}$ for convenience. Set a radial function $\varphi\in C_0^\infty(\mathbb{R}^3_\xi)$ such that $\varphi(\xi)=1$ while $|\xi|\leq 1$ and $\varphi(\xi)=0$ while $|\xi|\geq 2$. Define the low-frequency part of $f$ by

$$f^L=\mathfrak{F}^{-1}[\varphi(\xi)\widehat{f}],$$

and the high-frequency part of $f$ by

$$f^H=\mathfrak{F}^{-1}[(1-\varphi(\xi))\widehat{f}],$$

then $f=f^L+f^H$ if the Fourier transform of $f$ exists. We will employ the notation $a\lesssim b$ to mean that $a\leq Cb$ for a universal constant $C>0$ that only depends on the parameters coming from the problem. And $C_i(i=1,2,\cdots,9)$ will also denote some positive constants depending only on the parameters of the problem.

Our main results are stated in the following theorem.

THEOREM 1.1.

- ***Global existence.*** *Assume that* $\left(\rho_{10}-\frac{1}{Z},m_{10},\rho_{20}-1,m_{20},\nabla\phi_0\right)\in H^l(\mathbb{R}^3)$ *with* $l\geq 3$. *Then if there exists a sufficiently small constant* $\delta_0>0$, *such that*

$$C_0=\left\|\left(\rho_{10}-\frac{1}{Z},m_{10},\rho_{20}-1,m_{20},\nabla\phi_0\right)\right\|_l\leq\delta_0, \tag{1.8}$$

*then the Cauchy problem* (1.1)–(1.2) *with unequal viscosities* (1.3) *admits a unique globally classical solution* $(\rho_1,m_1,\rho_2,m_2,\phi)$ *satisfying that for any* $t\in[0,\infty)$,

$$\begin{aligned}&\left\|\left(\rho_1-\frac{1}{Z},m_1,\rho_2-1,m_2,\nabla\phi\right)\right\|_l^2+\int_0^t\left(\|\nabla(\rho_1,\rho_2,\nabla\phi)(\tau)\|_{l-1}^2+\|\nabla(m_1,m_2)(\tau)\|_l^2\right)d\tau\\ \leq &C\left\|\left(\rho_{10}-\frac{1}{Z},m_{10},\rho_{20}-1,m_{20},\nabla\phi_0\right)\right\|_l^2.\end{aligned}$$

- ***Upper decay rates.*** *Under the assumption in Theorem 1.1, if additionally for* $1\leq p\leq\frac{3}{2}$,

$$K_0=\left\|\left(\rho_{10}-\frac{1}{Z},m_{10},\rho_{20}-1,m_{20}\right)\right\|_{L^1}+\|\nabla\phi_0\|_{L^p}<\delta_0, \tag{1.9}$$

*then for all* $t\geq 0$, *it holds that*

$$\left\|\nabla^k\left(\rho_1-\frac{1}{Z},\rho_2-1\right)\right\|_{L^2}\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{1.10}$$

*and*

$$\|\nabla^k(m_1,m_2,\nabla\phi)\|_{L^2}\le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{k}{2}}, \tag{1.11}$$

*for* $0\le k\le l$.

• ***Lower decay rates.*** *Moreover, assume that* (1.9) *holds for* $p=1$ *and the Fourier transform functions* $\left(\rho_{10}-\frac{1}{Z},m_{10},\rho_{20}-1,m_{20}\right)$ *satisfy*

$$\mathfrak{F}\left[\rho_{10}-\frac{1}{Z}\right]=\mathfrak{F}[m_{10}]=\mathfrak{F}[\rho_{20}-1]=\mathfrak{F}[\Lambda^{-1}\mathrm{curl}m_{20}]=0,\quad \text{and}\quad |\mathfrak{F}[\Lambda^{-1}\mathrm{div}m_{20}]|\ge C\delta_0^{\frac{3}{2}}, \tag{1.12}$$

*for any* $|\xi|\le\eta$. *Then there exists a positive constant* $C_1$ *independent of time such that for any large enough* $t$,

$$\min\left\{\left\|\nabla^k\left(\rho_1-\frac{1}{Z}\right)\right\|_{L^2},\|\nabla^k m_1\|_{L^2},\|\nabla^k(\rho_2-1)\|_{L^2},\|\nabla^k m_2\|_{L^2},\|\nabla^k\nabla\phi\|_{L^2}\right\}$$
$$\ge C_1\delta_0^{\frac{3}{2}}(1+t)^{-\frac{3}{4}-\frac{k}{2}},$$

*for* $0\le k\le l$.

REMARK 1.1. The global existence of the Cauchy problem (1.1)–(1.2) with unequal viscosities (1.3) under the small initial perturbation assumption has been proved in [33] by the standard continuity argument. In this paper, we focus on the upper–lower time decay rates of the solution as well as spatial derivatives of all orders.

REMARK 1.2. Both of the results in [20, 29] imply that the electric field affects the decay rates of the solutions. However, neither of them clarifies that how the electric field affects the decay rates of the solutions. We gives a clear answer to this issue. More precisely, the decay rate in (1.11) shows the explicit influences of the electric field on the decay rates of solutions. In particular, by employing our method to the UNSP system, one can easily find that the electric field plays the same role for decay rate to the two systems. Therefore, this phenomenon is the most important difference between the NS system and the NSP system. In addition, by noticing the fact that $Z\rho_1-\rho_2\sim\Delta\phi$ and using (1.11), the difference $Z\rho_1-\rho_2$ between two densities has the following decay rate

$$\|\nabla^k(Z\rho_1-\rho_2)\|_{L^2}\le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{k+1}{2}},$$

for $1\le p<\frac{3}{2}$ and $0\le k\le l-1$, which is faster than those of themselves.

Now, let us sketch the strategy of proving Theorem 1.1 and explain some of the main difficulties and techniques involved in the process. As mentioned before, the main ideas of [21, 32, 38] are based on reformulating the system (1.1) by taking linear combination of the system (1.1) and employing similar arguments as in [3, 22, 23] for the Navier–Stokes system and in [20, 29] for the UNSP system. Therefore, the methods in [21, 32, 38], depending essentially on this reformulation, do not work here. The main idea here is that instead of using the reformulation, we will work on the system (1.1) directly. So, compared to [21, 32, 38], we need to develop new ingredients to overcome the difficulties arising from unequal viscosities, the non–conservative structure of the system (1.1) and the interaction of two fluids through the electric field. More precisely, we will employ “div–curl” decomposition, the low–frequency and high–frequency decomposition, delicate spectral analysis and energy estimates. Roughly speaking, our proof mainly involves the following five steps.

First, we rewrite the Cauchy problem (1.1)–(1.2) into the perturbation form (2.1) and analyze the spectrum of the solution semigroup to the corresponding linear system. To derive time–decay estimates of the linear system of (2.1), it requires us to make a detailed analysis on the properties of the semigroup. To avoid some complicated analysis of the Green's function, which is a $8\times 8$ system, we will employ the "div–curl" decomposition technique developed in [6–8, 34] to split the linear system into three systems. One has four distinct eigenvalues F and the other two are classic heat equations. Then, by making careful analysis on the Fourier transform of Green's function to the linear equations, we can obtain the desired linear decay rates. More exactly, from the elaborate expression (2.9) of the solution semigroup, we find that the low frequency of the electrostatic potential plays crucial role in the decay rate estimate. Although the solution semigroup of the BNSP system is much more complicated than that of the UNSP system, explicit spectral analysis on the semigroup means that the influence of the electrostatic potential on these two systems is almost the same. Meanwhile, it is also the most important difference between the NS system and the NSP system (see Propositions 2.1 and 2.2 for details).

Second, we deduce low frequency decay estimates on the solution and its highest–order derivatives. In the process of deducing the low frequency decay estimates, the main difficulty lies in deriving the decay rates of the densities, which are faster than those of the momentums. Indeed, noting the expression of the solution (2.14), (2.10) and the estimate of the nonlinear term (3.4), one can get

$$\begin{aligned}\|(\varrho_1^L,\varrho_2^L)(t)\|_{L^2} &\lesssim (1+t)^{-\frac{3}{4}}\|U_0\|_{L^1} + \underbrace{\int_0^t \|((S_1^{\varrho})^L,(S_2^{\varrho})^L)\|_{L^2}d\tau}_{\mathcal{I}(t)} \\ &\lesssim (1+t)^{-\frac{3}{4}}\|U_0\|_{L^1} + \int_0^t (1+t-\tau)^{-\frac{3}{4}}\|\mathcal{N}^L(\tau)\|_{L^1}d\tau \\ &\lesssim (1+t)^{-\frac{3}{4}}\|U_0\|_{L^1} + \mathcal{M}^2(t)\int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{3}{4}}d\tau,\end{aligned} \tag{1.13}$$

where $\mathcal{M}(t)$ is defined in (3.1). However, by taking $p=\frac{3}{2}$, it is clear that

$$\int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{3}{4}}d\tau = O((1+t)^{-\frac{3}{4}}\ln(1+t)). \tag{1.14}$$

In view of (1.13) and (1.14), it seems impossible to obtain the decay estimate of $(\varrho_1^L,\varrho_2^L)$ as in (3.2), which is crucial for the proof of Theorem 1.1. Our key idea here is to split $\mathcal{I}(t)$ into two parts:

$$\begin{aligned}\mathcal{I}(t) &= \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)\|((S_1^{\varrho})^L,(S_2^{\varrho})^L)\|_{L^2}d\tau \\ &:= \mathcal{I}_1(t)+\mathcal{I}_2(t),\end{aligned}$$

and then estimate the terms $\mathcal{I}_1(t)$ and $\mathcal{I}_2(t)$ respectively. For the term $\mathcal{I}_2(t)$, one can easily get

$$\mathcal{I}_2(t) \lesssim \mathcal{M}^2(t)\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{3}{4}}d\tau \lesssim \mathcal{M}^2(t)(1+t)^{-\frac{3}{4}},$$

for any $p\in[1,\frac{3}{2}]$. However, we need to develop new thoughts to deal with the term $\mathcal{I}_1(t)$. More precisely, we have to figure out how to improve the estimates on the terms $\mathrm{div}\left(\frac{m_i\otimes m_i}{\rho_i}\right)$ in (2.2), which cause the slowest decay rate in $\mathcal{I}_1(t)$. To this end, we try our best to explore the nonlinear part in the expression of the solution (2.14). Fortunately, by delicate calculations, we surprisingly find that the nonlinear terms in the expressions of the Fourier transformations of the densities $(\varrho_1^L,\varrho_2^L)$ can be rewritten in divergent forms (see the proof of Proposition 2.2 for details). Note that this is one of the differences between the "densities–momentums" system (2.1) and the "densities–velocities" system (2.3). As a result, one can shift the derivative onto the solution semigroup to obtain the desired decay estimates. With the help of this key observation, one can obtain the same decay rate of the term $\mathcal{I}_1(t)$ as that of $\mathcal{I}_2(t)$. Consequently, we can get the decay estimate of $(\varrho_1^L,\varrho_2^L)$ in (3.2). To obtain the highest–order low frequency decay estimates, we encounter a new difficulty that it requires us to control the terms involving $(l+1)$-th–order or $(l+2)$-th–order spatial derivatives of the solutions which do not belong to the solution space. To get around this difficulty, we separate the time interval into two parts and make full use of the benefit of the low–frequency and high–frequency decomposition to get our desired convergence rates (see the proof of Lemma 3.3 for details).

Third, we deduce the highest–order high frequency decay estimates. In this step, we cannot work directly on the system of the variables $(\varrho_1,m_1,\varrho_2,m_2)$ as in (2.1). Indeed, by noting the nonlinear terms in (2.2), we fail to deal with the trouble terms involving $\nabla^{l+1}(\varrho_i m_i)$ with $i=1,2$. The main observation here is that instead of the variables $(\varrho_1,m_1,\varrho_2,m_2)$, we study the system of the variables $(\varrho_1,u_1,\varrho_2,u_2)$ as in (2.3). Then, the corresponding trouble terms in (2.4) become ones involving $\nabla^{l-1}(\varrho_i\Delta u_i)$ with $i=1,2$, which can be tackled (see the proof of Lemma 3.4 for details).

Fourth, we prove the upper optimal decay rates of the solutions. Combining the low frequency decay estimates on the solution and its highest–order derivatives and the high frequency decay estimates on the highest–order derivatives of the solution obtained in the previous steps, we can get the decay rates of the solution and its spatial derivatives of all orders by taking full advantage of the good properties of the low–frequency and high–frequency decomposition. Then, by Sobolev interpolation and the definition of $\mathcal{M}(t)$ in (3.1), we can get the key time–independent bound on $\mathcal{M}(t)$, and this implies the upper optimal decay rates of the solutions in Theorem 1.1 immediately.

In the last step, we show the lower optimal decay rates of the solutions. To do this, we first employ Duhamel's principle, the lower decay rates of the linear system in (2.20) and (2.21), and Proposition 2.3 to get the lower optimal decay rates of the solution as well as its first–order spatial derivative. Then, for $2\le k\le l$, we can prove the lower optimal decay rates on the $k$th–order spatial derivative by an interpolation trick, and thus this completes the proof of the lower optimal decay rates in Theorem 1.1.

## 2. Reformulation of original problem

**2.1. Linearized system.** Let $\varrho_1=\rho_1-\frac{1}{Z}$ and $\varrho_2=\rho_2-1$. Then by using the fact that $\phi=\Delta^{-1}(Z\varrho_1-\varrho_2)$, the Cauchy problem (1.1)–(1.2) can be rewritten as

$$\begin{cases}\partial_t\varrho_1+\mathrm{div}m_1=0,\\ \partial_t m_1+P_1'\left(\frac{1}{Z}\right)\nabla\varrho_1-\nabla\Delta^{-1}(Z\varrho_1-\varrho_2)-\mu_1 Z\Delta m_1-\nu_1 Z\nabla\mathrm{div}m_1=N_1^m,\\ \partial_t\varrho_2+\mathrm{div}m_2=0,\\ \partial_t m_2+P_2'(1)\nabla\varrho_2+\nabla\Delta^{-1}(Z\varrho_1-\varrho_2)-\mu_2\Delta m_2-\nu_2\nabla\mathrm{div}m_2=N_2^m,\\ (\varrho_1,m_1,\varrho_2,m_2)(x,0)=(\rho_{10}-\frac{1}{Z},m_{10},\rho_{20}-1,m_{20})(x):=(\varrho_{10},m_{10},\varrho_{20},m_{20})(x)\end{cases}\tag{2.1}$$

with

$$\begin{cases} N_1^m = & Z\varrho_1\nabla\phi-\mathrm{div}\mathbb{F}_1 \\ \quad := & Z\varrho_1\nabla\phi-\mathrm{div}\Big(\frac{m_1\otimes m_1}{\rho_1}+\big(P_1(\rho_1)-P_1\left(\frac{1}{Z}\right)-P_1'\left(\frac{1}{Z}\right)\rho_1\big)\mathbb{I}_3 \\ & \qquad +\mu_1 Z\nabla\left(\frac{\varrho_1 m_1}{\rho_1}\right)+\nu_1 Z\mathrm{div}\left(\frac{\varrho_1 m_1}{\rho_1}\right)\mathbb{I}_3\Big), \\ N_2^m = & -\varrho_2\nabla\phi-\mathrm{div}\mathbb{F}_2 \\ \quad := & -\varrho_2\nabla\phi-\mathrm{div}\Big(\frac{m_2\otimes m_2}{\rho_2}+(P_2(\rho_2)-P_2(1)-P_2'(1)\rho_2)\mathbb{I}_3+\mu_2\nabla\left(\frac{\varrho_2 m_2}{\rho_2}\right) \\ & \qquad +\nu_2\mathrm{div}\left(\frac{\varrho_2 m_2}{\rho_2}\right)\mathbb{I}_3\Big). \end{cases} \tag{2.2}$$

We will do the decay rate on the lower–frequency part of the solution to the system (2.1). Exactly speaking, thanks to the divergent form of the nonlinear terms in (2.2), we can estimate the convergence rate of the solution by shifting differential operator. On the contrary, we would fail to estimate the decay rate on the higher–frequency part of the highest–order derivatives of the solution due to the divergent form. To this end, we should introduce the following system of the densities $\varrho_i$ and velocities $u_i=\frac{m_i}{\rho_i}$ with $i=1,2$

$$\begin{cases} \partial_t\varrho_1+\frac{1}{Z}\mathrm{div}u_1=N_1^\varrho, \\ \partial_t u_1+\frac{1}{Z}P_1'\left(\frac{1}{Z}\right)\nabla\varrho_1-Z\nabla\Delta^{-1}(Z\varrho_1-\varrho_2)-\mu_1 Z\Delta u_1-\nu_1 Z\nabla\mathrm{div}u_1=N_1^u, \\ \partial_t\varrho_2+\mathrm{div}u_2=N_2^\varrho, \\ \partial_t u_2+P_2'(1)\nabla\varrho_2+\nabla\Delta^{-1}(Z\varrho_1-\varrho_2)-\mu_2\Delta u_2-\nu_2\nabla\mathrm{div}u_2=N_2^u, \\ (\varrho_1,u_1,\varrho_2,u_2)(x,0)=(\varrho_{10},\frac{m_{10}}{\rho_{10}},\varrho_{20},\frac{m_{20}}{\rho_{20}})(x):=(\varrho_{10},u_{10},\varrho_{20},u_{20})(x) \end{cases} \tag{2.3}$$

with

$$\begin{cases} N_1^\varrho=-\mathrm{div}(\varrho_1 u_1), \\ N_1^u=-u_1\cdot\nabla u_1-\left(\frac{P_1'(\rho_1)}{\rho_1}-ZP_1'\left(\frac{1}{Z}\right)\right)\nabla\varrho_1-\frac{\mu_1 Z\varrho_1}{\rho_1}\Delta u_1-\frac{\nu_1 Z\varrho_1}{\rho_1}\nabla\mathrm{div}u_1, \\ N_2^\varrho=-\mathrm{div}(\varrho_2 u_2), \\ N_2^u=-u_2\cdot\nabla u_2-\left(\frac{P_2'(\rho_2)}{\rho_2}-P_2'(1)\right)\nabla\varrho_2-\frac{\mu_2\varrho_2}{\rho_2}\Delta u_2-\frac{\nu_2\varrho_2}{\rho_2}\nabla\mathrm{div}u_2. \end{cases} \tag{2.4}$$

**2.2. “div–curl” Decomposition.** For $i=1,2$, let $n_i=\Lambda^{-1}\mathrm{div}m_i$ be the “compressible part” of $m_i$ and $M_i=\Lambda^{-1}\mathrm{curl}m_i$ (with $\mathrm{curl}z=(\partial_{x_2}z^3-\partial_{x_3}z^2,\partial_{x_3}z^1-\partial_{x_1}z^3,\partial_{x_1}z^2-\partial_{x_2}z^1)^T$) be the “incompressible part” of $m_i$ respectively. Then the system (2.1) can be rewritten as two parts in the following

$$\begin{cases} \partial_t\varrho_1+\Lambda n_1=0, \\ \partial_t n_1-P_1'\left(\frac{1}{Z}\right)\Lambda\varrho_1-\Lambda^{-1}(Z\varrho_1-\varrho_2)-(\mu_1+\nu_1)Z\Delta n_1=\Lambda^{-1}\mathrm{div}N_1^m, \\ \partial_t\varrho_2+\Lambda n_2=0, \\ \partial_t n_2-P_2'(1)\Lambda\varrho_2+\Lambda^{-1}(Z\varrho_1-\varrho_2)-(\mu_2+\nu_2)\Delta n_2=\Lambda^{-1}\mathrm{div}N_2^m, \\ (\varrho_1,n_1,\varrho_2,n_2)(x,0)=(\varrho_{10},\Lambda^{-1}\mathrm{div}m_{10},\varrho_{20},\Lambda^{-1}\mathrm{div}m_{20})(x):=(\varrho_{10},n_{10},\varrho_{20},n_{20})(x) \end{cases} \tag{2.5}$$

and

$$\begin{cases} \partial_t M_1-\mu_1 Z\Delta M_1=\Lambda^{-1}\mathrm{curl}N_1^m, \\ \partial_t M_2-\mu_2\Delta M_2=\Lambda^{-1}\mathrm{curl}N_2^m, \\ (M_1,M_2)(x,0)=(\Lambda^{-1}\mathrm{curl}m_{10},\Lambda^{-1}\mathrm{curl}m_{20})(x):=(M_{10},M_{20})(x). \end{cases} \tag{2.6}$$

Note that (2.5) is a hyperbolic–parabolic system in that the structure of the solution semigroup is simpler than that of (2.1), and (2.6) are mere heat equations on the $M_i$. Moreover, by the relationship

$$m_i = -\Lambda^{-1}\nabla n_i - \Lambda^{-1}\mathrm{div} M_i, \quad i=1,2$$

involving pseudo–differential operators of degree zero, the estimates in the space $H^l$ for the original function $m_i$ can be derived from $n_i$ and $M_i$. Hence we will focus on the spectral analysis on the solution semigroups of (2.5)–(2.6).

**2.3. Spectral analysis.** Let $U=(\varrho_1,n_1,\varrho_2,n_2)^T$. Due to the semigroup theory for evolutionary equations, we will study the following initial value problem for the linear system

$$\begin{cases} U_t = \mathbb{B}U, \\ U|_{t=0} = U_0 = (\varrho_{10},n_{10},\varrho_{20},n_{20})^T, \end{cases} \tag{2.7}$$

where the operator $\mathbb{B}$ is given by

$$\mathbb{B} = \begin{pmatrix} 0 & -\Lambda & 0 & 0 \\ Z\Lambda^{-1} + P_1'\left(\frac{1}{Z}\right)\Lambda & (\mu_1+\nu_1)Z\Delta & -\Lambda^{-1} & 0 \\ 0 & 0 & 0 & -\Lambda \\ -Z\Lambda^{-1} & 0 & \Lambda^{-1}+P_2'(1)\Lambda & (\mu_2+\nu_2)\Delta \end{pmatrix}.$$

Taking the Fourier transform to the system, we have

$$\begin{cases} \widehat{U}_t = \mathbb{A}(\xi)\widehat{U}, \\ \widehat{U}|_{t=0} = \widehat{U}_0, \end{cases}$$

where $\widehat{U}(\xi,t) = \mathfrak{F}(U(x,t))$ and $\mathbb{A}(\xi)$ is given by

$$\mathbb{A}(\xi) = \begin{pmatrix} 0 & -|\xi| & 0 & 0 \\ Z|\xi|^{-1} + P_1'\left(\frac{1}{Z}\right)|\xi| & -(\mu_1+\nu_1)Z|\xi|^2 & -|\xi|^{-1} & 0 \\ 0 & 0 & 0 & -|\xi| \\ -Z|\xi|^{-1} & 0 & |\xi|^{-1}+P_2'(1)|\xi| & -(\mu_2+\nu_2)|\xi|^2 \end{pmatrix}.$$

The eigenvalues of the matrix $\mathbb{A}(\xi)$ can be solved from the determinant

$$\begin{aligned} &\det\{\mathbb{A}(\xi)-\lambda\mathbb{I}\} \\ &= \lambda^4 + (Z(\mu_1+\nu_1)+\mu_2+\nu_2)|\xi|^2\lambda^3 \\ &\quad + \left(Z(\mu_1+\nu_1)(\mu_2+\nu_2)|\xi|^4 + \left(P_1'\left(\frac{1}{Z}\right)+P_2'(1)\right)|\xi|^2 + 1 + Z\right)\lambda^2 \\ &\quad + \left(\left(Z(\mu_1+\nu_1)P_2'(1) + (\mu_2+\nu_2)P_1'\left(\frac{1}{Z}\right)\right)|\xi|^4 + Z(\mu_1+\nu_1+\mu_2+\nu_2)|\xi|^2\right)\lambda \\ &\quad + P_1'\left(\frac{1}{Z}\right)P_2'(1)|\xi|^4 + \left(P_1'\left(\frac{1}{Z}\right)+ZP_2'(1)\right)|\xi|^2 \\ &= 0. \end{aligned}$$

By direct calculation and delicate analysis on the roots of the above quartic equation, we can deduce that the matrix $\mathbb{A}(\xi)$ has four different eigenvalues $\lambda_i = \lambda_i(\xi)$ with

$i=1,2,3,4$ while $|\xi|\ll 1$; the details can also be seen in [5]. Hence we can decompose the semigroup $e^{t\mathbb{A}(\xi)}$ in the following:

$$e^{t\mathbb{A}(\xi)}=\sum_{i=1}^{4}e^{\lambda_i t}\mathbb{P}_i(\xi)$$

with the projector $\mathbb{P}_i(\xi)$ given by

$$\mathbb{P}_i(\xi)=\prod_{j\neq i}\frac{\mathbb{A}(\xi)-\lambda_j I}{\lambda_i-\lambda_j},\quad i,j=1,2,3,4.$$

Then we can represent the solution of the problem as

$$\widehat{U}(\xi,t)=e^{t\mathbb{A}(\xi)}\widehat{U}_0(\xi)=\left(\sum_{i=1}^{4}e^{\lambda_i t}\mathbb{P}_i(\xi)\right)\widehat{U}_0(\xi). \tag{2.8}$$

LEMMA 2.1. *There exists a positive constant $\eta\ll 1$ such that, for $|\xi|\le\eta$, the eigenvalues have the following Taylor series expansions*

$$\begin{cases}\lambda_{1,2}=-\kappa_1|\xi|^2+O(|\xi|^4)\pm i\left(\sqrt{1+Z}+\dfrac{\sigma_1^2}{2\sqrt{1+Z}}|\xi|^2+O(|\xi|^4)\right)\\ \lambda_{3,4}=-\kappa_2|\xi|^2+O(|\xi|^4)\pm i\left(\sigma_2|\xi|+O(|\xi|^3)\right)\end{cases}$$

*with* $\kappa_1=\frac{Z^2(\mu_1+\nu_1)+\mu_2+\nu_2}{2(1+Z)}$, $\kappa_2=\frac{Z(\mu_1+\nu_1+\mu_2+\nu_2)}{2(1+Z)}$, $\sigma_1=\sqrt{\frac{ZP_1'(\frac{1}{Z})+P_2'(1)}{1+Z}}$, *and* $\sigma_2=\sqrt{\frac{P_1'(\frac{1}{Z})+ZP_2'(1)}{1+Z}}$.

We establish the following estimates for the low–frequency part of the solutions $\widehat{U}(\xi,t)$ to the problem (2.5) and (2.6) while $N_i=0$ with $i=1,2,3,4$:

LEMMA 2.2.
*(i) For $|\xi|\le\eta$, we have*

$$|\widehat{\varrho}_1(\xi,t)|,\quad |\widehat{\varrho}_2(\xi,t)|\lesssim\left(e^{-\frac{\kappa_1}{2}|\xi|^2t}+e^{-\frac{\kappa_2}{2}|\xi|^2t}\right)|\widehat{U}_0(\xi)|,$$

$$|\widehat{n}_1(\xi,t)|\lesssim e^{-\frac{\kappa_1}{2}|\xi|^2t}\frac{|\sin(t\mathrm{Im}\lambda_1)|}{\sqrt{1+Z}}|\xi|^{-1}|Z\widehat{\varrho_{10}}(\xi)-\widehat{\varrho_{20}}(\xi)|+\left(e^{-\frac{\kappa_1}{2}|\xi|^2t}+e^{-\frac{\kappa_2}{2}|\xi|^2t}\right)|\widehat{U}_0(\xi)|,$$

$$|\widehat{n}_2(\xi,t)|\lesssim e^{-\frac{\kappa_1}{2}|\xi|^2t}\frac{|\sin(t\mathrm{Im}\lambda_1)|}{\sqrt{1+Z}}|\xi|^{-1}||Z\widehat{\varrho_{10}}(\xi)-\widehat{\varrho_{20}}(\xi)|+\left(e^{-\frac{\kappa_1}{2}|\xi|^2t}+e^{-\frac{\kappa_2}{2}|\xi|^2t}\right)|\widehat{U}_0(\xi)|.$$

*and*

$$|\widehat{\nabla\phi}(\xi,t)|\lesssim e^{-\frac{\kappa_1}{2}|\xi|^2t}|\xi|^{-1}|Z\widehat{\varrho_{10}}(\xi)-\widehat{\varrho_{20}}(\xi)|+\left(e^{-\frac{\kappa_1}{2}|\xi|^2t}+e^{-\frac{\kappa_2}{2}|\xi|^2t}\right)|\widehat{U}_0(\xi)|.$$

*(ii) For any $\xi$, we have*

$$|\widehat{M}_1(\xi,t)|\sim e^{-\mu_1Z|\xi|^2t}|\widehat{M_{10}}(\xi)|$$

*and*

$$|\widehat{M}_2(\xi,t)|\sim e^{-\mu_2|\xi|^2t}|\widehat{M_{20}}(\xi)|.$$

*Proof.* Part $(ii)$ can be easily derived from the standard heat equation for $M_1$ and $M_2$. In order to prove part $(i)$, we express $\mathbb{P}_i(i=1,2,3,4)$ as

$$\mathbb{P}_1(\xi)=\begin{pmatrix} \frac{Z}{2(1+Z)} & 0 & -\frac{1}{2(1+Z)} & 0 \\ -\frac{iZ|\xi|^{-1}}{2\sqrt{1+Z}} & \frac{Z}{2(1+Z)} & \frac{i|\xi|^{-1}}{2\sqrt{1+Z}} & -\frac{1}{2(1+Z)} \\ -\frac{Z}{2(1+Z)} & 0 & \frac{1}{2(1+Z)} & 0 \\ \frac{iZ|\xi|^{-1}}{2\sqrt{1+Z}} & -\frac{Z}{2(1+Z)} & -\frac{i|\xi|^{-1}}{2\sqrt{1+Z}} & \frac{1}{2(1+Z)} \end{pmatrix}+(O(|\xi|)+iO(|\xi|))\mathbb{J},$$

$$\mathbb{P}_2(\xi)=\begin{pmatrix} \frac{Z}{2(1+Z)} & 0 & -\frac{1}{2(1+Z)} & 0 \\ \frac{iZ|\xi|^{-1}}{2\sqrt{1+Z}} & \frac{Z}{2(1+Z)} & -\frac{i|\xi|^{-1}}{2\sqrt{1+Z}} & -\frac{1}{2(1+Z)} \\ -\frac{Z}{2(1+Z)} & 0 & \frac{1}{2(1+Z)} & 0 \\ -\frac{iZ|\xi|^{-1}}{2\sqrt{1+Z}} & -\frac{Z}{2(1+Z)} & \frac{i|\xi|^{-1}}{2\sqrt{1+Z}} & \frac{1}{2(1+Z)} \end{pmatrix}+(O(|\xi|)+iO(|\xi|))\mathbb{J},$$

$$\mathbb{P}_3(\xi)=\begin{pmatrix} \frac{1}{2(1+Z)} & \frac{i}{2(1+Z)\sigma_2} & \frac{1}{2(1+Z)} & \frac{i}{2(1+Z)\sigma_2} \\ -\frac{i\sigma_2}{2(1+Z)} & \frac{1}{2(1+Z)} & -\frac{i\sigma_2}{2(1+Z)} & \frac{1}{2(1+Z)} \\ \frac{Z}{2(1+Z)} & \frac{iZ}{2(1+Z)\sigma_2} & \frac{Z}{2(1+Z)} & \frac{iZ}{2(1+Z)\sigma_2} \\ -\frac{iZ\sigma_2}{2(1+Z)} & \frac{Z}{2(1+Z)} & -\frac{iZ\sigma_2}{2(1+Z)} & \frac{Z}{2(1+Z)} \end{pmatrix}+(O(|\xi|)+iO(|\xi|))\mathbb{J},$$

and

$$\mathbb{P}_4(\xi)=\begin{pmatrix} \frac{1}{2(1+Z)} & -\frac{i}{2(1+Z)\sigma_2} & \frac{1}{2(1+Z)} & -\frac{i}{2(1+Z)\sigma_2} \\ \frac{i\sigma_2}{2(1+Z)} & \frac{1}{2(1+Z)} & \frac{i\sigma_2}{2(1+Z)} & \frac{1}{2(1+Z)} \\ \frac{Z}{2(1+Z)} & -\frac{iZ}{2(1+Z)\sigma_2} & \frac{Z}{2(1+Z)} & -\frac{iZ}{2(1+Z)\sigma_2} \\ \frac{iZ\sigma_2}{2(1+Z)} & \frac{Z}{2(1+Z)} & \frac{iZ\sigma_2}{2(1+Z)} & \frac{Z}{2(1+Z)} \end{pmatrix}+(O(|\xi|)+iO(|\xi|))\mathbb{J},$$

where $\mathbb{J}$ is a 4–order matrix with all elements equal to 1.

Then we can conclude that

$$\begin{aligned}\sum_{i=1}^{4}e^{\lambda_i t}\mathbb{P}_i(\xi)=&\begin{pmatrix} \frac{Zg_+^{1,2}+g_+^{3,4}}{2(1+Z)} & \frac{ig_-^{3,4}}{2(1+Z)\sigma_2} & \frac{g_+^{3,4}-g_+^{1,2}}{2(1+Z)} & \frac{ig_-^{3,4}}{2(1+Z)\sigma_2} \\ -\frac{iZ|\xi|^{-1}g_-^{1,2}}{2\sqrt{1+Z}}-\frac{i\sigma_2 g_-^{3,4}}{2(1+Z)} & \frac{Zg_+^{1,2}+g_+^{3,4}}{2(1+Z)} & \frac{i|\xi|^{-1}g_-^{1,2}}{2\sqrt{1+Z}}-\frac{i\sigma_2 g_-^{3,4}}{2(1+Z)} & \frac{g_+^{3,4}-g_+^{1,2}}{2(1+Z)} \\ \frac{Z(g_+^{3,4}-g_+^{1,2})}{2(1+Z)} & \frac{iZg_-^{3,4}}{2(1+Z)\sigma_2} & \frac{g_+^{1,2}+Zg_+^{3,4}}{2(1+Z)} & \frac{iZg_-^{3,4}}{2(1+Z)\sigma_2} \\ \frac{iZ|\xi|^{-1}g_-^{1,2}}{2\sqrt{1+Z}}-\frac{iZ\sigma_2 g_-^{3,4}}{2(1+Z)} & \frac{Z(g_+^{3,4}-g_+^{1,2})}{2(1+Z)} & -\frac{i|\xi|^{-1}g_-^{1,2}}{2\sqrt{1+Z}}-\frac{iZ\sigma_2 g_-^{3,4}}{2(1+Z)} & \frac{g_+^{1,2}+Zg_+^{3,4}}{2(1+Z)} \end{pmatrix}\\ &+(O(|\xi|)+iO(|\xi|))\left(e^{-\kappa_1|\xi|^2t+O(|\xi|^4)t}+e^{-\kappa_2|\xi|^2t+O(|\xi|^4)t}\right)\mathbb{J}\end{aligned} \tag{2.9}$$

with

$$g_{\pm}^{i,j}=e^{t\lambda_i}\pm e^{t\lambda_j},\quad \mathit{for}\quad i,j=1,2,3,4.$$

By plugging (2.9) to (2.8), we can complete the proof of part $(i)$. ◻

**2.4. Upper decay rate for the linear system.** Thanks to Lemma 2.2, we can estimate the decay rates on the lower–frequency part of the solutions to the linear systems (2.5) and (2.6) while $N_1^m = N_2^m = 0$ as follows:

PROPOSITION 2.1. *For any $1 \le p \le 2$ and $k = 0,1,\cdots,l$, there exists a positive constant $C$ which is independent of $t$ such that*

$$\|\nabla^k(\varrho_1^L,\varrho_2^L)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U_0^L\|_{L^1}, \tag{2.10}$$

$$\|\nabla^k(n_1^L,n_2^L,(\nabla\phi)^L)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U_0^L\|_{L^1} + C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|(\nabla\phi_0)^L\|_{L^p}, \tag{2.11}$$

*and*

$$\|\nabla^k(M_1,M_2)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U_0\|_{L^1}. \tag{2.12}$$

*Proof.* Here we only prove the decay on $\nabla^k(\nabla\phi)^L$ for $1<p<2$. The other cases can be proved in a similar way or the proof can be seen in our previous work [3, 34]. Indeed, by Lemma 2.2, Plancherel theorem and the Hausdorff–Young inequality, we have that for each $0 \le k \le l$ and $|\xi| \le \eta$,

$$\begin{aligned}
&\|\nabla^k(\nabla\phi)^L\|_{L^2}^2 = \||\xi|^k\widehat{(\nabla\phi)^L}(\xi)\|_{L^2}^2 \\
&\lesssim \int_{|\xi|\le\eta} |\xi|^{2k}\left(e^{-\frac{\kappa_1}{2}|\xi|^2 t}|\xi|^{-1}|Z\widehat{\varrho_{10}}(\xi)-\widehat{\varrho_{20}}(\xi)| + \left(e^{-\frac{\kappa_1}{2}|\xi|^2 t}+e^{-\frac{\kappa_2}{2}|\xi|^2 t}\right)|\widehat{U_0}(\xi)|\right)^2 d\xi \\
&\lesssim \left\||\xi|^{-1}(Z\widehat{\varrho_{10}}(\xi)-\widehat{\varrho_{20}}(\xi))\right\|_{L^q}^2 \left(\int_{|\xi|\le\eta}|\xi|^{\frac{2qk}{q-2}}e^{-\frac{q\kappa_1}{q-2}|\xi|^2 t}d\xi\right)^{\frac{q-2}{q}} \\
&\quad + \|\widehat{U_0}\|_{L^\infty(|\xi|\le\eta)}^2 \int_{|\xi|\le\eta}|\xi|^{2k}\left(e^{-\kappa_1|\xi|^2 t}+e^{-\kappa_2|\xi|^2 t}\right)d\xi \\
&\lesssim (1+t)^{-3\left(\frac{1}{2}-\frac{1}{q}\right)-k}\|\widehat{\nabla\phi_0}\|_{L^q(|\xi|\le\eta)}^2 + (1+t)^{-\frac{3}{2}-k}\|\widehat{U_0}\|_{L^\infty(|\xi|\le\eta)}^2 \\
&\lesssim (1+t)^{-3\left(\frac{1}{p}-\frac{1}{2}\right)-k}\|(\nabla\phi_0)^L\|_{L^p}^2 + (1+t)^{-\frac{3}{2}-k}\|U_0^L\|_{L^1}^2,
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. ∎

In order to estimate the convergence rate for the system (2.1), we also need to analyze the decay of the nonlinear terms. To this end, we rewrite the system (2.5) as

$$\begin{cases} U_t = BU + \mathcal{N}, \\ U|_{t=0} = U_0, \end{cases} \tag{2.13}$$

with

$$\mathcal{N} = (0, \Lambda^{-1}\mathrm{div}N_1^m, 0, \Lambda^{-1}\mathrm{div}N_2^m)^T.$$

Then the solution of (2.13) can be expressed as

$$U = e^{tB} * U_0 + \int_0^t e^{(t-\tau)B} * \mathcal{N}(\tau)d\tau. \tag{2.14}$$

Define

$$\mathcal{S}(x,\tau) := (S_1^\varrho(x,\tau), S_1^n(x,\tau), S_2^\varrho(x,\tau), S_2^n(x,\tau))^T = e^{(t-\tau)B} * \mathcal{N}(\tau)$$

and its Fourier transform

$$\mathfrak{F}[\mathcal{S}(\tau)]:=(\widehat{S_1^\varrho}(\xi,\tau),\widehat{S_1^n}(\xi,\tau),\widehat{S_2^\varrho}(\xi,\tau),\widehat{S_2^n}(\xi,\tau))^T.$$

Now we complement the decay estimates on the nonlinear term of the expression (2.14) of the solution $U(x,t)$ in the following:

PROPOSITION 2.2. *It holds for $k=0,1,\cdots,l$ that*

$$\begin{aligned}&\|\nabla^k\left((S_1^\varrho)^L,(S_2^\varrho)^L\right)(\tau)\|_{L^2}\\ \lesssim&(1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}}(\|(\nabla\phi)^L(\tau)\|_{L^2}^2+\|(\mathcal{N}^L,\mathbb{F}_1^L,\mathbb{F}_2^L)(\tau)\|_{L^1}),\end{aligned}\tag{2.15}$$

$$\|\nabla^k\left((S_1^\varrho)^L,(S_2^\varrho)^L\right)(\tau)\|_{L^2}\lesssim(1+t-\tau)^{-\frac{1}{2}}\|\nabla^k\mathcal{N}^L(\tau)\|_{L^2},\tag{2.16}$$

$$\begin{aligned}&\|\nabla^k\left((S_1^\varrho)^L,(S_2^\varrho)^L,(S_1^n)^L,(S_2^n)^L,\nabla\Delta^{-1}\left(Z(S_1^\varrho)^L-(S_2^\varrho)^L\right)\right)(\tau)\|_{L^2}\\ \lesssim&(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|\mathcal{N}^L(\tau)\|_{L^1},\end{aligned}\tag{2.17}$$

*and*

$$\begin{aligned}&\|\nabla^k\left((S_1^\varrho)^L,(S_2^\varrho)^L,(S_1^n)^L,(S_2^n)^L,\nabla\Delta^{-1}\left(Z(S_1^\varrho)^L-(S_2^\varrho)^L\right)\right)(\tau)\|_{L^2}\\ \lesssim&(1+t-\tau)^{-\frac{3}{4}}\|\nabla^k\mathcal{N}^L(\tau)\|_{L^1}.\end{aligned}\tag{2.18}$$

*Proof.* We can derive from the expression (2.9) of $e^{t\mathbb{A}(\xi)}$ that for $|\xi|\le\eta$,

$$\begin{aligned}\mathfrak{F}[\mathcal{S}(\tau)]=&\begin{pmatrix}\frac{ig_-^{3,4}}{2(1+Z)\sigma_2}\mathfrak{F}\left[\Lambda^{-1}\mathrm{div}(Z\varrho_1\nabla\phi-\mathrm{div}\mathbb{F}_1-\varrho_2\nabla\phi-\mathrm{div}\mathbb{F}_2)\right]\\ \frac{Zg_+^{1,2}+g_+^{3,4}}{2(1+Z)}\mathfrak{F}[\Lambda^{-1}\mathrm{div}N_1^m]+\frac{g_+^{3,4}-g_+^{1,2}}{2(1+Z)}\mathfrak{F}[\Lambda^{-1}\mathrm{div}N_2^m]\\ \frac{iZg_-^{3,4}}{2(1+Z)\sigma_2}\mathfrak{F}\left[\Lambda^{-1}\mathrm{div}(Z\varrho_1\nabla\phi-\mathrm{div}\mathbb{F}_1-\varrho_2\nabla\phi-\mathrm{div}\mathbb{F}_2)\right]\\ \frac{Z(g_+^{3,4}-g_+^{1,2})}{2(1+Z)}\mathfrak{F}[\Lambda^{-1}\mathrm{div}N_1^m]+\frac{g_+^{1,2}+Zg_+^{3,4}}{2(1+Z)}\mathfrak{F}[\Lambda^{-1}\mathrm{div}N_2^m]\end{pmatrix}\\ &+(O(|\xi|)+iO(|\xi|))\left(e^{-\kappa_1|\xi|^2(t-\tau)+O(|\xi|^4)(t-\tau)}+e^{-\kappa_2|\xi|^2(t-\tau)+O(|\xi|^4)(t-\tau)}\right)\mathbb{J}\widehat{\mathcal{N}}\\ =&\begin{pmatrix}\frac{ig_-^{3,4}}{2(1+Z)\sigma_2}\mathfrak{F}\left[\Lambda^{-1}\mathrm{divdiv}\left(\nabla\phi\otimes\nabla\phi-\frac{|\nabla\phi|^2}{2}\mathbb{I}-\mathbb{F}_1-\mathbb{F}_2\right)\right]\\ \frac{Zg_+^{1,2}+g_+^{3,4}}{2(1+Z)}\mathfrak{F}[\Lambda^{-1}\mathrm{div}N_1^m]+\frac{g_+^{3,4}-g_+^{1,2}}{2(1+Z)}\mathfrak{F}[\Lambda^{-1}\mathrm{div}N_2^m]\\ \frac{iZg_-^{3,4}}{2(1+Z)\sigma_2}\mathfrak{F}\left[\Lambda^{-1}\mathrm{divdiv}\left(\nabla\phi\otimes\nabla\phi-\frac{|\nabla\phi|^2}{2}\mathbb{I}-\mathbb{F}_1-\mathbb{F}_2\right)\right]\\ \frac{Z(g_+^{3,4}-g_+^{1,2})}{2(1+Z)}\mathfrak{F}[\Lambda^{-1}\mathrm{div}N_1^m]+\frac{g_+^{1,2}+Zg_+^{3,4}}{2(1+Z)}\mathfrak{F}[\Lambda^{-1}\mathrm{div}N_2^m]\end{pmatrix},\\ &+(O(|\xi|)+iO(|\xi|))\left(e^{-\kappa_1|\xi|^2(t-\tau)+O(|\xi|^4)(t-\tau)}+e^{-\kappa_2|\xi|^2(t-\tau)+O(|\xi|^4)(t-\tau)}\right)\mathbb{J}\widehat{\mathcal{N}}\end{aligned}\tag{2.19}$$

where $\mathbb{F}_1$ and $\mathbb{F}_2$ are defined in (2.2). Hence by (2.19), Plancherel theorem and the Hausdorff–Young inequality, we have that for each $0\leq k\leq l$ and $|\xi|\leq\eta$,

$$
\begin{aligned}
&\|\nabla^k\left((S_1^\varrho)^L,(S_2^\varrho)^L\right)\|_{L^2}^2\\
=&\||\xi|^k\left(\mathfrak{F}[(S_1^\varrho)^L],\mathfrak{F}[(S_2^\varrho)^L]\right)\|_{L^2}^2\\
\lesssim&\int_{|\xi|\leq\eta}|\xi|^{2k}e^{-2\kappa_2|\xi|^2(t-\tau)+O(|\xi|^4)(t-\tau)}|\xi|^2\mathfrak{F}\left[\left(\nabla\phi\otimes\nabla\phi-\frac{|\nabla\phi|^2}{2}\mathbb{I}-\mathbb{F}_1-\mathbb{F}_2\right)\right]^2(\tau)\\
&\quad+|\xi|^{2k}(O(|\xi|)+iO(|\xi|))^2\left(e^{(-\kappa_1|\xi|^2+O(|\xi|^4))(t-\tau)}+e^{(-\kappa_2|\xi|^2+O(|\xi|^4))(t-\tau)}\right)^2|\widehat{\mathcal{N}}(\tau)|^2d\xi\\
\lesssim&(1+t-\tau)^{-\frac{5}{2}-k}\left(\left\|\left(\nabla\phi\otimes\nabla\phi-\frac{|\nabla\phi|^2}{2}\mathbb{I}-\mathbb{F}_1-\mathbb{F}_2\right)^L(\tau)\right\|_{L^1}^2+\|\mathcal{N}^L(\tau)\|_{L^1}^2\right),
\end{aligned}
$$

which gives rise to (2.15). Obviously, (2.16), (2.17) and (2.18) can be obtained in the same way. □

**2.5. Lower decay rate for the linear system.** The lower–bounds on the decay rates for the above linear system are given in the following proposition:

PROPOSITION 2.3. *Assume that* $(\varrho_{10},n_{10},\varrho_{20},n_{20},\nabla\phi_0)\in L^1$ *satisfies*

$$\widehat{\varrho_{10}}(\xi)=\widehat{n_{10}}(\xi)=\widehat{M_{10}}(\xi)=\widehat{\varrho_{20}}(\xi)=\widehat{M_{10}}(\xi)=0,\quad \textit{and}\quad |\widehat{n_{20}}|\geq C\delta_0^{\frac{3}{2}}$$

*for any* $|\xi|\leq\eta$. *Then there exists a positive constant* $C_2$, *which is independent of* $t$, *such that the global solution* $(\varrho_1,n_1,\varrho_2,n_2)$ *of the IVP* (2.7) *satisfies*

$$\min\left\{\|\varrho_1^L\|_{L^2},\|n_1^L\|_{L^2},\|\varrho_2^L\|_{L^2},\|n_2^L\|_{L^2}\right\}\geq C_2\delta_0^{\frac{3}{2}}(1+t)^{-\frac{3}{4}},\tag{2.20}$$

*and*

$$\min\left\{\|\nabla\varrho_1^L\|_{L^2},\|\nabla n_1^L\|_{L^2},\|\nabla\varrho_2^L\|_{L^2},\|\nabla n_2^L\|_{L^2}\right\}\geq C_2\delta_0^{\frac{3}{2}}(1+t)^{-\frac{5}{4}}\tag{2.21}$$

*for any large enough* $t$.

*Proof.* Here we only prove the lower decay rate on $n_2$. From (2.8) and (2.9), we have that for $|\xi|\leq\eta$,

$$
\begin{aligned}
&\widehat{n_2}(t)\\
=&\frac{g_+^{1,2}+Zg_+^{3,4}}{2(1+Z)}\widehat{n_{20}}+(O(|\xi|)+iO(|\xi|))\left(e^{-\kappa_1|\xi|^2t+O(|\xi|^4t)}+e^{-\kappa_2|\xi|^2t+O(|\xi|^4t)}\right)\widehat{n_{20}}\\
=&\left(\frac{1}{1+Z}e^{-\kappa_1|\xi|^2t}\cos\left(\sqrt{1+Z}t+\frac{\sigma_1^2}{2\sqrt{1+Z}}|\xi|^2t\right)+\frac{Z}{1+Z}e^{-\kappa_2|\xi|^2t}\cos(\sigma_2|\xi|t)\right)\widehat{n_{20}}\\
&\quad+\left(O(|\xi|^4t)+O(|\xi|^3t)+O(|\xi|)+i\left(O(|\xi|^4t)+O(|\xi|^3t)+O(|\xi|)\right)\right)\\
&\quad\times\left(e^{-\kappa_1|\xi|^2t}+e^{-\kappa_2|\xi|^2t}\right)\widehat{n_{20}}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\|n_2^L\|_{L^2}^2=\|\widehat{n_2^L}\|_{L^2}^2\\
\geq&\int_{|\xi|\leq\eta}\left(\frac{e^{-\kappa_1|\xi|^2t}}{1+Z}\cos\left(\sqrt{1+Z}t+\frac{\sigma_1^2}{2\sqrt{1+Z}}|\xi|^2t\right)+\frac{Ze^{-\kappa_2|\xi|^2t}}{1+Z}\cos(\sigma_2|\xi|t)\right)^2|\widehat{n_{20}}|^2d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{|\xi|\le\eta}\left(O(|\xi|^4t)+O(|\xi|^3t)+O(|\xi|)\right)^2\left(e^{-\kappa_1|\xi|^2t}+e^{-\kappa_2|\xi|^2t}\right)^2|\widehat{n_{20}}|^2d\xi\\
\ge&\int_{|\xi|\le\eta}\frac{e^{-2\kappa_1|\xi|^2t}}{(1+Z)^2}\cos^2\left(\sqrt{1+Z}t+\frac{\sigma_1^2}{2\sqrt{1+Z}}|\xi|^2t\right)|\widehat{n_{20}}|^2d\xi\\
&+\int_{|\xi|\le\eta}\frac{Z^2e^{-2\kappa_2|\xi|^2t}}{(1+Z)^2}\cos^2(\sigma_2|\xi|t)|\widehat{n_{20}}|^2d\xi\\
&+\int_{|\xi|\le\eta}\frac{2Ze^{-(\kappa_1+\kappa_2)|\xi|^2t}}{(1+Z)^2}\cos\left(\sqrt{1+Z}t+\frac{\sigma_1^2}{2\sqrt{1+Z}}|\xi|^2t\right)\cos(\sigma_2|\xi|t)|\widehat{n_{20}}|^2d\xi\\
&-C(1+t)^{-\frac{5}{2}}\|\widehat{n_{20}}\|_{L^\infty}^2\\
:=&I_1+I_2+I_3-C(1+t)^{-\frac{5}{2}}\|n_{20}\|_{L^1}^2. \qquad (2.22)
\end{aligned}
$$

In spirit of [20] and [34], we can estimate the first two terms in the right-hand side of (2.22) as

$$
\begin{aligned}
I_1=&\int_{|\xi|\le\eta}\frac{e^{-2\kappa_1|\xi|^2t}}{(1+Z)^2}\cos^2\left(\sqrt{1+Z}t+\frac{\sigma_1^2}{2\sqrt{1+Z}}|\xi|^2t\right)|\widehat{n_{20}}|^2d\xi\\
\ge&C\delta_0^3t^{-\frac{3}{2}}\int_0^{\eta\sqrt{t_0}}e^{-2\kappa_1r^2}\cos^2\left(\sqrt{1+Z}t+\frac{\sigma_1^2}{2\sqrt{1+Z}}r^2\right)dr\\
\ge&C\delta_0^3t^{-\frac{3}{2}} \qquad (2.23)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2=&\int_{|\xi|\le\eta}\frac{Z^2e^{-2\kappa_2|\xi|^2t}}{(1+Z)^2}\cos^2(\sigma_2|\xi|t)^2|\widehat{n_{20}}|^2d\xi\\
=&O(1)t^{-\frac{3}{2}}\int_0^{\eta\sqrt{t}}e^{-2\kappa_2r^2}(\cos(2\sigma_2|\xi|t)+1)|\widehat{n_{20}}|^2dr\\
\ge&C\delta_0^3t^{-\frac{3}{2}}\int_0^{\eta\sqrt{t}}e^{-2\kappa_2r^2}dr-C\|\widehat{n_{20}}\|_{L^\infty}^2t^{-\frac{3}{2}}\int_0^{\eta\sqrt{t}}e^{-2\kappa_2r^2}\cos(2\sigma_2|\xi|t)dr\\
\ge&C\delta_0^3t^{-\frac{3}{2}}-C\|n_{20}\|_{L^1}^2t^{-2} \qquad (2.24)
\end{aligned}
$$

for any $t\ge t_0$ with some sufficiently large time $t_0$ dependent of $\|v_{20}\|_{L^1}$.

Moreover, we have that for $t\ge t_0$,

$$
\begin{aligned}
I_3=&\int_{|\xi|\le\eta}\frac{2Ze^{-(\kappa_1+\kappa_2)|\xi|^2t}}{(1+Z)^2}\cos\left(\sqrt{1+Z}t+\frac{\sigma_1^2}{2\sqrt{1+Z}}|\xi|^2t\right)\cos(\sigma_2|\xi|t)|\widehat{n_{20}}|^2d\xi\\
=&O(1)t^{-\frac{3}{2}}\int_0^{\eta\sqrt{t}}e^{-(\kappa_1+\kappa_2)r^2}\cos\left(\sqrt{1+Z}t+\frac{\sigma_1^2}{2\sqrt{1+Z}}r^2\right)\cos\left(\sigma_2r\sqrt{t}\right)|\widehat{n_{20}}|^2dr\\
=&O(1)t^{-\frac{3}{2}}\int_0^{\eta\sqrt{t}}e^{-(\kappa_1+\kappa_2)r^2}\cos\left(\sqrt{1+Z}t+\frac{\sigma_1^2}{2\sqrt{1+Z}}r^2\right)|\widehat{n_{20}}|^2\frac{1}{\sigma_2\sqrt{t}}d\sin\left(\sigma_2r\sqrt{t}\right)\\
\ge&-Ct^{-2}\|n_{20}\|_{L^1}^2, \qquad (2.25)
\end{aligned}
$$

thus plugging (2.23)–(2.25) into (2.22), we can obtain that

$$
\|n_2^L\|_{L^2}\ge C\delta_0^{\frac{3}{2}}(1+t)^{-\frac{3}{4}}-C\|n_{20}\|_{L^1}(1+t)^{-1}-C\|n_{20}\|_{L^1}(1+t)^{-\frac{5}{4}},
$$

which implies that (2.20) holds for some appropriately positive constant $C_2$ and any large enough time $t$. □

## 3. Proof of upper decay estimates

This section is devoted to prove the optimal decay rates of the solution stated in Theorem 1.1. First by the general energy estimate method, we can derive the following energy inequality, whose proof can also be seen in [33]:

PROPOSITION 3.1. *Under the assumption* (1.8) *of Theorem 1.1, the Cauchy problem* (2.3) *admits a unique globally classical solution* $(\varrho_1,u_1,\varrho_2,u_2)$ *such that for any* $t\in[0,\infty)$,

$$\|(\varrho_1,u_1,\varrho_2,u_2,\nabla\phi)(t)\|_{H^l}^2+\int_0^t\left(\|\nabla(\varrho_1,\varrho_2,\nabla\phi)(\tau)\|_{H^{l-1}}^2+\|\nabla(u_1,u_2)(\tau)\|_{H^l}^2\right)d\tau$$
$$\le C\|(\varrho_{10},u_{10},\varrho_{20},u_{20})\|_{H^l}^2\le CC_0.$$

Define the time–weighted energy functional

$$\mathcal{M}(t)=\sup_{0\le\tau\le t}\sum_{k=0}^{l}\Big((1+\tau)^{-\frac{3}{4}-\frac{k}{2}}\|\nabla^k(\varrho_1,\varrho_2)\|_{L^2}+(1+\tau)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{2})+\frac{k}{2}}\|\nabla^k(m_1,m_2,\nabla\phi)\|_{L^2}\Big). \tag{3.1}$$

We shall prove the following proposition to achieve the second part of Theorem 1.1.

PROPOSITION 3.2. *Under the assumptions* (1.8) *and* (1.9) *of Theorem 1.1, it holds*

$$\mathcal{M}(t)\le C(C_0+K_0).$$

Next we divide the proof of Proposition 3.2 into the following steps: deriving the optimal decay rates on the low–frequency parts and the high–frequency parts of solution and its highest–order derivatives separately. To this end, we first need some tools to deal with the integration on the time [11].

LEMMA 3.1. *Assume* $s_1>1$, $s_2\in[0,s_1]$, *then we have*

$$\int_0^t(1+t-\tau)^{-s_1}(1+\tau)^{-s_2}d\tau\le C(s_1,s_2)(1+t)^{-s_2}.$$

Next we estimate the decay rate on the lower–frequency part of the solution as:

LEMMA 3.2. *Assume that the assumptions of Proposition 3.2 are in force. Then it holds*

$$\|(\varrho_1^L,\varrho_2^L)(t)\|_{L^2}\le\left(CK_0+C\mathcal{M}^2(t)\right)(1+t)^{-\frac{3}{4}}, \tag{3.2}$$

*and*

$$\|(m_1^L,m_2^L,(\nabla\phi)^L)(t)\|_{L^2}\le C\left(K_0+\mathcal{M}^2(t)\right)(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}. \tag{3.3}$$

*Proof.* To derive the decay on $\varrho_i^L$ and $m_i^L$ with $i=1,2$, we need to estimate the nonlinear terms in (2.2) as follows. By the Definition (3.1) of $\mathcal{M}(t)$, the Hölder inequality and the fact that $1\le p\le\frac{3}{2}$, we have from (2.2) that

$$\begin{aligned}\|\mathcal{N}^L\|_{L^1}&\le C\|(\varrho_1,m_1,\varrho_2,m_2,\nabla\phi)\|_{L^2}\|(\varrho_1,\nabla m_1,\varrho_2,\nabla m_2)\|_{H^1}\\&\le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\mathcal{M}(t)(1+t)^{-\frac{3}{4}}\mathcal{M}(t)\\&\le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{4}}\mathcal{M}^2(t),\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\|(\mathbb{F}_1^L,\mathbb{F}_2^L)\|_{L^1}\leq& C\|(\varrho_1,m_1,\varrho_2,m_2)\|_{L^2}\|(\varrho_1,m_1,\varrho_2,m_2)\|_{H^1}\\ \leq& C(1+t)^{-3\left(\frac{1}{p}-\frac{1}{2}\right)}\mathcal{M}^2(t),\end{aligned}\tag{3.5}$$

By (2.10), (2.15), (2.17) with $k=0$ and (3.4)–(3.5), we have from (2.14) that

$$\begin{aligned}&\|(\varrho_1^L,\varrho_2^L)(t)\|_{L^2}=\|(\widehat{\varrho_1^L},\widehat{\varrho_2^L})(t)\|_{L^2}\\ \leq& C(1+t)^{-\frac{3}{4}}\|U_0\|_{L^1}+\int_0^{\frac{t}{2}}\|((S_1^\varrho)^L,(S_2^\varrho)^L)\|_{L^2}d\tau+\int_{\frac{t}{2}}^t\|((S_1^\varrho)^L,(S_2^\varrho)^L)\|_{L^2}d\tau\\ \leq& C(1+t)^{-\frac{3}{4}}\|U_0\|_{L^1}+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{4}}\left(\|(\nabla\phi)^L(\tau)\|_{L^2}^2+\|(\mathcal{N}^L,\mathbb{F}_1^L,\mathbb{F}_2^L)(\tau)\|_{L^1}\right)d\tau\\ &+C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{3}{4}}\|\mathcal{N}^L(\tau)\|_{L^1}d\tau\\ \leq&(1+t)^{-\frac{3}{4}}\|U_0\|_{L^1}+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{4}}\left((1+\tau)^{-3\left(\frac{1}{p}-\frac{1}{2}\right)}+(1+\tau)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{4}}\right)\mathcal{M}^2(\tau)d\tau\\ &+C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{4}}\mathcal{M}^2(\tau)d\tau\\ \leq& C(1+t)^{-\frac{3}{4}}\|U_0\|_{L^1}+C\mathcal{M}^2(t)(1+t)^{-\frac{5}{4}}\int_0^{\frac{t}{2}}(1+\tau)^{-3\left(\frac{1}{p}-\frac{1}{2}\right)}d\tau\\ &+C\mathcal{M}^2(t)(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{4}}\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{3}{4}}d\tau\\ \leq& C(K_0+\mathcal{M}^2(t))(1+t)^{-\frac{3}{4}},\end{aligned}\tag{3.6}$$

where the monotonicity of $\mathcal{M}(t)$ is used. And (3.6) yields (3.2).

Next by (2.6), (2.11), (2.12), (2.17) with $k=0$ and (3.4), we have from (2.14) that

$$\begin{aligned}&\|(m_1^L,m_2^L,(\nabla\phi)^L)(t)\|_{L^2}=\|(n_1^L,n_2^L,M_1^L,M_2^L,(\nabla\phi)^L)(t)\|_{L^2}\\ \leq& C(1+t)^{-\frac{3}{4}}\|U_0\|_{L^1}+C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\|\nabla\phi_0\|_{L^p}+C\int_0^t(1+t-\tau)^{-\frac{3}{4}}\|\mathcal{N}^L(\tau)\|_{L^1}d\tau\\ \leq& C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}K_0+C\mathcal{M}^2(t)(1+t)^{-\frac{3}{4}}\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{4}}d\tau\\ &+C\mathcal{M}^2(t)(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{4}}\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{3}{4}}d\tau\\ \leq& C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}(K_0+\mathcal{M}^2(t)),\end{aligned}$$

which gives rise to (3.3). Hence we finish the proof of Lemma 3.2. □

LEMMA 3.3. *Assume that the assumptions of Proposition 3.2 are in force. Then it holds*

$$\|\nabla^l(\varrho_1^L,\varrho_2^L)(t)\|_{L^2}\leq C\left(K_0+\mathcal{M}^2(t)\right)(1+t)^{-\frac{3}{4}-\frac{l}{2}}$$

*and*

$$\left\|\nabla^l(m_1^L,m_2^L,(\nabla\phi)^L)(t)\right\|_{L^2}\leq C(K_0+\mathcal{M}^2(t))(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{l}{2}}.$$

*Proof.* Since by Lemma 5.1, Lemma 5.2 and Lemma 5.4, we can estimate the derivatives of the nonlinear term (2.2) as

$$
\begin{aligned}
&\|\nabla^l\mathcal{N}^L\|_{L^2}\\
&\lesssim \|\nabla^l(\varrho_1\nabla\phi,\varrho_2\nabla\phi)\|_{L^2}+\left\|\nabla^l\left(\frac{m_1\otimes m_1}{\varrho_1},\left(P_1(\rho_1)-P_1\left(\frac{1}{Z}\right)-P_1'\left(\frac{1}{Z}\right)\varrho_1\right)\mathbb{I}_3,\right.\right.\\
&\qquad\qquad\left.\left.\frac{m_2\otimes m_2}{\varrho_2},(P_2(\rho_2)-P_2(1)-P_2'(1)\varrho_2)\mathbb{I}_3\right)\right\|_{L^2}\\
&\quad+\left\|\nabla^{l-1}\left(\nabla\left(\frac{\varrho_1 m_1}{\rho_1}\right),\mathrm{div}\left(\frac{\varrho_1 m_1}{\rho_1}\right),\nabla\left(\frac{\varrho_2 m_2}{\rho_2}\right),\mathrm{div}\left(\frac{\varrho_2 m_2}{\rho_2}\right)\right)\right\|_{L^2}\\
&\lesssim \|(\varrho_1,m_1,\varrho_2,m_2,\nabla\phi)\|_{L^\infty}\|\nabla^l(\varrho_1,m_1,\varrho_2,m_2,\nabla\phi)\|_{L^2}\\
&\lesssim \|(\varrho_1,m_1,\varrho_2,m_2,\nabla\phi)\|_{L^2}^{\frac14}\|\nabla^2(\varrho_1,m_1,\varrho_2,m_2,\nabla\phi)\|_{L^2}^{\frac34}\|\nabla^l(\varrho_1,m_1,\varrho_2,m_2,\nabla\phi)\|_{L^2}\\
&\lesssim (1+t)^{-3\left(\frac1p-\frac12\right)-\frac34-\frac l2}\mathcal{M}^2(t).
\end{aligned}\tag{3.7}
$$

Here the fact that $\|\nabla f^L\|_{L^2}\lesssim\|f\|_{L^2}$ is used to deal with the $(l+1)$-th–order derivatives. And similarly we have

$$
\begin{aligned}
\|\nabla^l\mathcal{N}^L\|_{L^1}\lesssim &\|\nabla^l(\varrho_1\nabla\phi,\varrho_2\nabla\phi)\|_{L^1}\\
&+\left\|\nabla^{l-1}\mathrm{div}\left(\frac{m_1\otimes m_1}{\varrho_1},\left(P_1(\rho_1)-P_1\left(\frac{1}{Z}\right)-P_1'\left(\frac{1}{Z}\right)\varrho_1\right)\mathbb{I}_3,\right.\right.\\
&\qquad\qquad\left.\left.\frac{m_2\otimes m_2}{\varrho_2},(P_2(\rho_2)-P_2(1)-P_2'(1)\varrho_2)\mathbb{I}_3\right)\right\|_{L^1}\\
&+\left\|\nabla^{l-2}\mathrm{div}\left(\nabla\left(\frac{\varrho_1 m_1}{\rho_1}\right),\mathrm{div}\left(\frac{\varrho_1 m_1}{\rho_1}\right),\nabla\left(\frac{\varrho_2 m_2}{\rho_2}\right),\mathrm{div}\left(\frac{\varrho_2 m_2}{\rho_2}\right)\right)\right\|_{L^1}\\
\lesssim &\|(\varrho_1,m_1,\varrho_2,m_2,\nabla\phi)\|_{L^2}\|\nabla^l(\varrho_1,m_1,\varrho_2,m_2,\nabla\phi)\|_{L^2}\\
\lesssim &(1+t)^{-3\left(\frac1p-\frac12\right)-\frac l2}\mathcal{M}^2(t).
\end{aligned}\tag{3.8}
$$

Thus by using (2.10), (2.15), (2.16) with $k=l$, (3.4), (3.5), and (3.7), we have from (2.14) that

$$
\begin{aligned}
&\|\nabla^l(\varrho_1^L,\varrho_2^L)(t)\|_{L^2}\\
&\le C(1+t)^{-\frac34-\frac l2}\|U_0\|_{L^1}+\int_0^{\frac t2}\|\nabla^l((S_1^\varrho)^L,(S_2^\varrho)^L)\|_{L^2}d\tau+\int_{\frac t2}^t\|\nabla^l((S_1^\varrho)^L,(S_2^\varrho)^L)\|_{L^2}d\tau\\
&\le C(1+t)^{-\frac34-\frac l2}\|U_0\|_{L^1}+C\int_0^{\frac t2}(1+t-\tau)^{-\frac54-\frac l2}\left(\|(\nabla\phi)^L(\tau)\|_{L^2}^2+\|(\mathcal{N},\mathbb{F}_1,\mathbb{F}_2)(\tau)\|_{L^1}\right)d\tau\\
&\quad+C\int_{\frac t2}^t(1+t-\tau)^{-\frac12}\|\nabla^l\mathcal{N}^L(\tau)\|_{L^2}d\tau\\
&\le (1+t)^{-\frac34-\frac l2}\|U_0\|_{L^1}+C\int_0^{\frac t2}(1+t-\tau)^{-\frac54-\frac l2}(1+\tau)^{-3\left(\frac1p-\frac12\right)}\mathcal{M}^2(\tau)d\tau\\
&\quad+C\int_{\frac t2}^t(1+t-\tau)^{-\frac12}(1+\tau)^{-3\left(\frac1p-\frac12\right)-\frac34-\frac l2}\mathcal{M}^2(\tau)d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le C(1+t)^{-\frac{3}{4}-\frac{l}{2}}\|U_0\|_{L^1}+C\mathcal{M}^2(t)(1+t)^{-\frac{5}{4}-\frac{l}{2}}\int_0^{\frac{t}{2}}(1+\tau)^{-3(\frac{1}{p}-\frac{1}{2})}d\tau\\
&\quad +C\mathcal{M}^2(t)(1+t)^{-3(\frac{1}{p}-\frac{1}{2})-\frac{3}{4}-\frac{l}{2}}\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}d\tau\\
&\le C\left(K_0+\mathcal{M}^2(t)\right)(1+t)^{-\frac{3}{4}-\frac{l}{2}}.
\end{aligned}
$$

Moreover, by using (2.6), (2.11), (2.12), (2.17), (2.18) with $k=l$, and (3.4)–(3.8), we have from (2.14) that

$$
\begin{aligned}
&\|\nabla^l(m_1^L,m_2^L,(\nabla\phi)^L)(t)\|_{L^2}=\|\nabla^l(n_1^L,n_2^L,M_1^L,M_2^L,(\nabla\phi)^L)(t)\|_{L^2}\\
&\le C(1+t)^{-\frac{3}{4}-\frac{l}{2}}\|U_0\|_{L^1}+C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{l}{2}}\|\nabla\phi_0\|_{L^p}\\
&\quad +\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{l}{2}}\|\mathcal{N}^L(\tau)\|_{L^1}d\tau+\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{4}}\|\nabla^l\mathcal{N}^L(\tau)\|_{L^1}d\tau\\
&\le CK_0(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{l}{2}}+C\mathcal{M}^2(t)(1+t)^{-\frac{3}{4}-\frac{l}{2}}\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{3}{4}}d\tau\\
&\quad +C\mathcal{M}^2(t)(1+t)^{-3(\frac{1}{p}-\frac{1}{2})-\frac{l}{2}}\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{4}}d\tau\\
&\le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{l}{2}}(K_0+\mathcal{M}^2(t)).
\end{aligned}
$$

Thus we finish the proof of Lemma 3.3. ◻

Now we turn to estimate the decay rate on the higher–frequency of the highest–order derivative of the solution, and we state the result in the following:

**Lemma 3.4.** *Assume that the assumptions of Proposition 3.2 are in force. Then it holds*

$$
\|\nabla^l(\varrho_1^H,m_1^H,\varrho_2^H,m_2^H,(\nabla\phi)^H)(t)\|_{L^2}\le C\left(C_0+\mathcal{M}^2(t)\right)(1+t)^{-\frac{3}{4}-\frac{l}{2}}. \tag{3.9}
$$

*Proof.* Applying the operator $\mathfrak{F}^{-1}[(1-\varphi(\xi))\mathfrak{F}(\cdot)]$ to the system (2.3) gives rise to

$$
\begin{cases}
\partial_t\varrho_1^H+\frac{1}{Z}\mathrm{div}u_1^H=(N_1^\varrho)^H,\\
\partial_t u_1^H+P_1'\left(\frac{1}{Z}\right)\nabla\varrho_1^H-Z\nabla\Delta^{-1}(Z\varrho_1^H-\varrho_2^H)-\mu_1Z\Delta u_1^H-\nu_1Z\nabla\mathrm{div}u_1^H=(N_1^u)^H,\\
\partial_t\varrho_2^H+\mathrm{div}u_2^H=(N_2^\varrho)^H,\\
\partial_t u_2^H+P_2'(1)\nabla\varrho_2^H+\nabla\Delta^{-1}(Z\varrho_1^H-\varrho_2^H)-\mu_2\Delta u_2^H-\nu_2\nabla\mathrm{div}u_2^H=(N_2^u)^H,\\
(\varrho_1^H,u_1^H,\varrho_2^H,u_2^H)(x,0)=(\varrho_{10}^H,u_{10}^H,\varrho_{20}^H,u_{20}^H)(x).
\end{cases} \tag{3.10}
$$

Multiplying $\nabla^l(3.10)_1$, $\nabla^{l-1}(3.10)_2$, $\nabla^l(3.10)_3$ and $\nabla^{l-1}(3.10)_4$ by $\nabla^{l-1}u_1^H$, $\nabla^l\varrho_1^H$, $\nabla^{l-1}u_2^H$ and $\nabla^l\varrho_2^H$ respectively, and then summing up, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\langle\nabla^l\varrho_1^H,\nabla^{l-1}u_1^H\rangle+\langle\nabla^l\varrho_2^H,\nabla^{l-1}u_2^H\rangle\right)\\
&\quad +P_1'\left(\frac{1}{Z}\right)\|\nabla^l\varrho_1^H\|_{L^2}^2+P_2'(1)\|\nabla^l\varrho_2^H\|_{L^2}^2+\|\nabla^l(\nabla\phi)^H\|_{L^2}^2\\
&=\left\langle\nabla^l\left(-\frac{1}{Z}\mathrm{div}u_1^H+(N_1^\varrho)^H\right),\nabla^{l-1}u_1^H\right\rangle+\langle\nabla^l\left(-\mathrm{div}u_2^H+(N_2^\varrho)^H\right),\nabla^{l-1}u_2^H\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\langle\nabla^l\varrho_1^H,\nabla^{l-1}\left(\mu_1 Z\Delta u_1^H+\nu_1 Z\nabla\mathrm{div}u_1^H+(N_1^u)^H\right)\rangle\\
&+\langle\nabla^l\varrho_2^H,\nabla^{l-1}\left(\mu_2\Delta u_2^H+\nu_2\nabla\mathrm{div}u_2^H+(N_2^u)^H\right)\rangle\\
\le&\left\langle\nabla^l\left(\frac{1}{Z}u_1^H+(\varrho_1u_1)^H\right),\nabla^l u_1^H\right\rangle+\langle\nabla^l\left(u_2^H+(\varrho_2u_2)^H\right),\nabla^l u_2^H\rangle\\
&+C\left\|\nabla^l\left(\varrho_1^H,\varrho_2^H\right)\right\|_{L^2}\left(\left\|\nabla^{l+1}\left(u_1^H,u_2^H\right)\right\|_{L^2}+\left\|\nabla^{l-1}\left((N_1^u)^H,(N_2^u)^H\right)\right\|_{L^2}\right)\\
\le&\frac{P_1'\left(\frac{1}{Z}\right)}{2}\|\nabla^l\varrho_1^H\|_{L^2}^2+\frac{P_2'(1)}{2}\|\nabla^l\varrho_2^H\|_{L^2}^2+C\|\nabla^l\left(u_1^H,u_2^H,\nabla u_1^H,\nabla u_2^H\right)\|_{L^2}^2\\
&+C\left(\left\|\nabla^l\left((\varrho_1u_1)^H,(\varrho_2u_2)^H\right)\right\|_{L^2}^2+\left\|\nabla^{l-1}\left((N_1^u)^H,(N_2^u)^H\right)\right\|_{L^2}\right)\\
\le&\frac{P_1'\left(\frac{1}{Z}\right)}{2}\|\nabla^l\varrho_1^H\|_{L^2}^2+\frac{P_2'(1)}{2}\|\nabla^l\varrho_2^H\|_{L^2}^2+C\|\nabla^{l+1}\left(u_1^H,u_2^H\right)\|_{L^2}^2\\
&+C\left(\left\|\nabla^l(\varrho_1u_1,\varrho_2u_2)\right\|_{L^2}^2+\left\|\nabla^{l-1}(N_1^u,N_2^u)\right\|_{L^2}^2\right),
\end{aligned}\tag{3.11}
$$

where the inequality $\|f^H\|_{L^2}\le\|f\|_{L^2}$ is used. Since by Lemma 5.2 and Lemma 5.4, we have

$$
\left\|\nabla^l(\varrho_1u_1,\varrho_2u_2)\right\|_{L^2}\le C\|(\varrho_1,u_1,\varrho_2,u_2)\|_{L^\infty}\|\nabla^l(\varrho_1,u_1,\varrho_2,u_2)\|_{L^2},\tag{3.12}
$$

and

$$
\begin{aligned}
&\left\|\nabla^{l-1}(N_1^u,N_2^u)\right\|_{L^2}\\
\lesssim&\left\|\nabla^{l-1}\left(u_1\cdot\nabla u_1,\left(\frac{P_1'(\rho_1)}{\rho_1}-ZP_1'\left(\frac{1}{Z}\right)\right)\nabla\varrho_1,u_2\cdot\nabla u_2,\left(\frac{P_2'(\rho_2)}{\rho_2}-P_2'(1)\right)\nabla\varrho_2\right)\right\|_{L^2}\\
&+\left\|\nabla^{l-1}\left(\frac{\varrho_1}{\rho_1}\Delta u_1,\frac{\varrho_1}{\rho_1}\nabla\mathrm{div}u_1,\frac{\varrho_2}{\rho_2}\Delta u_2,\frac{\varrho_2}{\rho_2}\nabla\mathrm{div}u_2\right)\right\|_{L^2}\\
\lesssim&\|(\varrho_1,u_1,\varrho_2,u_2)\|_{L^\infty}\|\nabla^l(\varrho_1,u_1,\varrho_2,u_2)\|_{L^2}+\|\nabla(\varrho_1,u_1,\varrho_2,u_2)\|_{L^3}\|\nabla^{l-1}(\varrho_1,u_1,\varrho_2,u_2)\|_{L^6}\\
&+\|(\varrho_1,\varrho_2)\|_{L^\infty}\|\nabla^{l+1}(u_1,u_2)\|_{L^2}+\|\nabla^2(u_1,u_2)\|_{L^3}\|\nabla^{l-1}(\varrho_1,\varrho_2)\|_{L^6}\\
\lesssim&\left(\|(\varrho_1,u_1,\varrho_2,u_2)\|_{L^\infty}+\|\nabla(\varrho_1,u_1,\varrho_2,u_2)\|_{1,3}\right)\|\nabla^l(\varrho_1,u_1,\varrho_2,u_2)\|_{L^2}\\
&+\|(\varrho_1,\varrho_2)\|_{L^\infty}\left(\|\nabla^{l+1}(u_1^H,u_2^H)\|_{L^2}+\|\nabla^l(u_1,u_2)\|_{L^2}\right)\\
\lesssim&\left(\|(\varrho_1,u_1,\varrho_2,u_2)\|_{L^\infty}+\|\nabla(\varrho_1,u_1,\varrho_2,u_2)\|_{1,3}\right)\|\nabla^l(\varrho_1,u_1,\varrho_2,u_2)\|_{L^2}\\
&+\|(\varrho_1,\varrho_2)\|_{L^\infty}\|\nabla^{l+1}(u_1^H,u_2^H)\|_{L^2},
\end{aligned}\tag{3.13}
$$

where $f=f^L+f^H$ and $\|\nabla f^L\|_{L^2}\lesssim\|f^L\|_{L^2}\lesssim\|f\|_{L^2}$ are used again. Thus plugging (3.12) and (3.13) into (3.11), we can arrive at

$$
\begin{aligned}
&\frac{d}{dt}\left(\langle\nabla^l\varrho_1^H,\nabla^{l-1}u_1^H\rangle+\langle\nabla^l\varrho_2^H,\nabla^{l-1}u_2^H\rangle\right)\\
&\quad+\frac{P_1'\left(\frac{1}{Z}\right)}{2}\|\nabla^l\varrho_1^H\|_{L^2}^2+\frac{P_2'(1)}{2}\|\nabla^l\varrho_2^H\|_{L^2}^2+\|\nabla^l(\nabla\phi)^H\|_{L^2}^2\\
\le&C\left(\|(\varrho_1,u_1,\varrho_2,u_2)\|_{L^\infty}+\|\nabla(\varrho_1,u_1,\varrho_2,u_2)\|_{1,3}\right)^2\|\nabla^l(\varrho_1,u_1,\varrho_2,u_2)\|_{L^2}^2\\
&\quad+C_3\|\nabla^{l+1}\left(u_1^H,u_2^H\right)\|_{L^2}^2
\end{aligned}\tag{3.14}
$$

with some positive constant $C_3$.

Multiplying $ZP_1'\left(\frac{1}{Z}\right)\nabla^l(3.10)_1^H$, $\nabla^l(3.10)_2^H$, $P_2'(1)\nabla^l(3.10)_3^H$ and $\nabla^l(3.10)_4^H$ by $\nabla^l\varrho_1^H$, $\nabla^l u_1^H$, $\nabla^l\varrho_1^H$ and $\nabla^l u_2^H$ respectively, and then summing up, we have

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{ZP_1'\left(\frac{1}{Z}\right)}{2}\|\nabla^l\varrho_1^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l u_1^H\|_{L^2}^2+\frac{P_2'(1)}{2}\|\nabla^l\varrho_2^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l u_2^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l\nabla\phi\|_{L^2}^2\right\}\\
&\quad+\mu_1\|\nabla^{l+1}u_1^H\|_{L^2}^2+\nu_1\|\nabla^l\mathrm{div}u_1^H\|_{L^2}^2+\mu_2\|\nabla^{l+1}u_2^H\|_{L^2}^2+\nu_2\|\nabla^l\mathrm{div}u_2^H\|_{L^2}^2\\
&=ZP_1'\left(\frac{1}{Z}\right)\langle\nabla^l(N_1^\varrho)^H,\nabla^l\varrho_1^H\rangle+\langle\nabla^l(N_1^u)^H,\nabla^l u_1^H\rangle\\
&\quad+P_2'(1)\langle\nabla^l(N_2^\varrho)^H,\nabla^l\varrho_2^H\rangle+\langle\nabla^l(N_2^u)^H,\nabla^l u_2^H\rangle.
\end{aligned}\tag{3.15}$$

Here we only estimate the first two terms in the right-hand side of (3.15) as follows. By taking full use of the properties of the low–frequency and high–frequency decomposition, Proposition 3.1, the smallness of $C_0$, the Cauchy inequality and Lemma 5.3, we can arrive at

$$\begin{aligned}
&\langle\nabla^l(N_1^\varrho)^H,\nabla^l\varrho_1^H\rangle\\
&=-\left\langle\nabla^l\mathrm{div}\left(\left(\varrho_1^H+\varrho_1^L\right)u_1-(\varrho_1u_1)^L\right),\nabla^l\varrho_1^H\right\rangle\\
&=-\left\langle\nabla^l\left(\nabla\varrho_1^H\cdot u_1+\varrho_1^H\mathrm{div}u_1\right)+\nabla^l\mathrm{div}\left(\varrho_1^L u_1-(\varrho_1u_1)^L\right),\nabla^l\varrho_1^H\right\rangle\\
&=-\left\langle[\nabla^l,u_1]\cdot\nabla\varrho_1^H+\nabla^l\left(\varrho_1^H\mathrm{div}u_1\right)+\nabla^l\mathrm{div}\left(\varrho_1^L u_1-(\varrho_1u_1)^L\right),\nabla^l\varrho_1^H\right\rangle\\
&\qquad+\int_{\mathbb{R}^3}(\mathrm{div}u_1)|\nabla^l\varrho_1^H|^2dx\\
&\le\|\nabla u_1\|_2\|\nabla^l\varrho_1^H\|_{L^2}^2+C\left(\|(\varrho_1,u_1)\|_{1,\infty}\|\nabla^l(\varrho_1,u_1)\|_{L^2}+\|\varrho_1\|_{L^\infty}\|\nabla^{l+1}u_1^H\|_{L^2}\right)\|\nabla^l\varrho_1^H\|_{L^2}\\
&\le\|(\varrho_1,u_1)\|_3\|\nabla^l\varrho_1^H\|_{L^2}^2+C\|\varrho_1\|_{L^\infty}\|\nabla^{l+1}u_1^H\|_{L^2}^2+C\|(\varrho_1,u_1)\|_{1,\infty}\|\nabla^l(\varrho_1,u_1)\|_{L^2}^2.
\end{aligned}$$

And by (3.13), we can get

$$\begin{aligned}
&\langle\nabla^l(N_1^u)^H,\nabla^l u_1^H\rangle\\
&=-\langle\nabla^{l-1}(N_1^u)^H,\nabla^{l-1}\Delta u_1^H\rangle\\
&\le\|\nabla^{l-1}(N_1^u)^H\|_{L^2}\|\nabla^{l+1}u_1^H\|_{L^2}\\
&\le\frac{\mu_1}{4}\|\nabla^{l+1}u_1^H\|_{L^2}+C\|\nabla^{l-1}(N_1^u)^H\|_{L^2}^2\\
&\le\left(\frac{\mu_1}{4}+\|\varrho_1\|_{L^\infty}^2\right)\|\nabla^{l+1}u_1^H\|_{L^2}^2+C\left(\|(\varrho_1,u_1)\|_{L^\infty}+\|\nabla(\varrho_1,u_1)\|_{1,3}\right)\|\nabla^l(\varrho_1,u_1)\|_{L^2}^2.
\end{aligned}$$

Similarly we can get the estimates for $\langle\nabla^l(N_2^\varrho)^H,\nabla^l\varrho_2^H\rangle$ and $\langle\nabla^l(N_2^u)^H,\nabla^l u_2^H\rangle$. Thus combining these estimates with (3.15) gives rise to

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{ZP_1'\left(\frac{1}{Z}\right)}{2}\|\nabla^l\varrho_1^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l u_1^H\|_{L^2}^2+\frac{P_2'(1)}{2}\|\nabla^l\varrho_2^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l u_2^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l\nabla\phi\|_{L^2}^2\right\}\\
&\qquad+\frac{\mu_1}{2}\|\nabla^{l+1}u_1^H\|_{L^2}^2+\frac{\mu_2}{2}\|\nabla^{l+1}u_2^H\|_{L^2}^2\\
&\le C\|(\varrho_1,u_1,\varrho_2,u_2)\|_3\|\nabla^l(\varrho_1^H,\varrho_2^H)\|_{L^2}^2\\
&\qquad+C\left(\|(\varrho_1,u_1,\varrho_2,u_2)\|_{1,\infty}+\|\nabla(\varrho_1,u_1,\varrho_2,u_2)\|_{1,3}\right)\|\nabla^l(\varrho_1,u_1,\varrho_2,u_2)\|_{L^2}^2.
\end{aligned}\tag{3.16}$$

Now multiplying (3.14) with some positive number $C_4=\min\left\{\frac{\mu_1}{4C_3},\frac{\mu_2}{4C_3}\right\}$, and plus (3.16) leads to

$$\begin{aligned}
&\frac{d}{dt}\left\{C_4\langle\nabla^l\varrho_1^H,\nabla^{l-1}u_1^H\rangle+C_4\langle\nabla^l\varrho_2^H,\nabla^{l-1}u_2^H\rangle+\frac{ZP_1'\left(\frac{1}{Z}\right)}{2}\|\nabla^l\varrho_1^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l u_1^H\|_{L^2}^2\right.\\
&\quad\left.+\frac{P_2'(1)}{2}\|\nabla^l\varrho_2^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l u_2^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l(\nabla\phi)^H\|_{L^2}^2\right\}+\frac{C_4P_1'\left(\frac{1}{Z}\right)}{4}\|\nabla^l\varrho_1^H\|_{L^2}^2\\
&\quad+\frac{C_4P_2'(1)}{4}\|\nabla^l\varrho_2^H\|_{L^2}^2+C_4\|\nabla^l(\nabla\phi)^H\|_{L^2}^2+\frac{\mu_1}{4}\|\nabla^{l+1}u_1^H\|_{L^2}^2+\frac{\mu_2}{4}\|\nabla^{l+1}u_2^H\|_{L^2}^2\\
&\le C\left(\|(\varrho_1,u_1,\varrho_2,u_2)\|_{1,\infty}+\|\nabla(\varrho_1,u_1,\varrho_2,u_2)\|_{1,3}\right)\|\nabla^l(\varrho_1,u_1,\varrho_2,u_2)\|_{L^2}^2. \qquad (3.17)
\end{aligned}$$

Define

$$\begin{aligned}
\mathcal{L}(t)=&C_4\langle\nabla^l\varrho_1^H,\nabla^{l-1}u_1^H\rangle+C_4\langle\nabla^l\varrho_2^H,\nabla^{l-1}u_2^H\rangle+\frac{ZP_1'\left(\frac{1}{Z}\right)}{2}\|\nabla^l\varrho_1^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l u_1^H\|_{L^2}^2\\
&+\frac{P_2'(1)}{2}\|\nabla^l\varrho_2^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l u_2^H\|_{L^2}^2+\frac{1}{2}\|\nabla^l(\nabla\phi)^H\|_{L^2}^2.
\end{aligned}$$

By the Cauchy inequality and the fact that $\|f^H\|_{L^2}\le\|\nabla f^H\|_{L^2}$, we can get the following equivalent relationship

$$\mathcal{L}(t)\approx\|\nabla^l(\varrho_1^H,u_1^H,\varrho_2^H,u_2^H,(\nabla\phi)^H)\|_{L^2}^2.$$

Hence by using $\|f^H\|_{L^2}\le\|\nabla f^H\|_{L^2}$ again we have from (3.17) that for some positive constant $C_5$,

$$\frac{d}{dt}\mathcal{L}(t)+C_5\mathcal{L}(t)\le C\left(\|(\varrho_1,u_1,\varrho_2,u_2)\|_{1,\infty}+\|\nabla(\varrho_1,u_1,\varrho_2,u_2)\|_{1,3}\right)\|\nabla^l(\varrho_1,u_1,\varrho_2,u_2)\|_{L^2}^2.$$

Then by the Gronwall inequality and Lemma 3.1, we can arrive at

$$\begin{aligned}
\mathcal{L}(t)\le&e^{-C_5t}\mathcal{L}(0)+\int_0^t e^{-C_5(t-\tau)}C\left(\|(\varrho_1,u_1,\varrho_2,u_2)(\tau)\|_{1,\infty}+\|\nabla(\varrho_1,u_1,\varrho_2,u_2)(\tau)\|_{1,3}\right)\\
&\qquad\times\|\nabla^l(\varrho_1,u_1,\varrho_2,u_2)(\tau)\|_{L^2}^2d\tau\\
\le&e^{-C_5t}\mathcal{L}(0)+C\int_0^t e^{-C_5(t-\tau)}\|(\varrho_1,u_1,\varrho_2,u_2)(\tau)\|_1^{\frac{1}{4}}\|\nabla^2(\varrho_1,u_1,\varrho_2,u_2)(\tau)\|_1^{\frac{3}{4}}\\
&\qquad\times\|\nabla^l(\varrho_1,u_1,\varrho_2,u_2)(\tau)\|_{L^2}^2d\tau\\
\le&e^{-C_5t}C_0^2+C\int_0^t e^{-C_5(t-\tau)}(1+\tau)^{-\frac{9}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{4}-l}\mathcal{M}^4(\tau)d\tau\\
\le&e^{-C_5t}C_0^2+C\int_0^t(1+t-\tau)^{-\frac{9}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{4}-l}(1+\tau)^{-\frac{9}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{4}-l}\mathcal{M}^4(\tau)d\tau\\
\le&(1+t)^{-\frac{9}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{4}-l}(C_0^2+C\mathcal{M}^4(t))\\
\le&C(C_0^2+\mathcal{M}^4(t))(1+t)^{-\frac{3}{2}-l}.
\end{aligned}$$

This together with the fact that

$$\|\nabla^l(m_1^H,m_2^H)\|_{L^2}\lesssim\|(\varrho_1^H,u_1^H,\varrho_2^H,u_2^H)\|_{L^\infty}\|\nabla^l(\varrho_1^H,u_1^H,\varrho_2^H,u_2^H)\|_{L^2}$$

yields (3.9). □

*Proof.* **(Proof of Proposition 3.2.)** Combining Lemma 3.2, Lemma 3.3 and Lemma 3.4, the following estimates can be obtained:

$$\begin{aligned}\|(\varrho_1,\varrho_2)\|_{L^2} &\le \|(\varrho_1^H,\varrho_2^H)\|_{L^2}+\|(\varrho_1^L,\varrho_2^L)\|_{L^2}\\ &\le \|\nabla^l(\varrho_1^H,\varrho_2^H)\|_{L^2}+\|(\varrho_1^L,\varrho_2^L)\|_{L^2}\\ &\le C(C_0+K_0+\mathcal{M}^2(t))(1+t)^{-\frac{3}{4}},\end{aligned} \tag{3.18}$$

$$\begin{aligned}\|(m_1,m_2,\nabla\phi)\|_{L^2} &\le \|(m_1^H,m_2^H,(\nabla\phi)^H)\|_{L^2}+\|(m_1^L,m_2^L,(\nabla\phi)^L)\|_{L^2}\\ &\le \|\nabla^l(m_1^H,m_2^H,(\nabla\phi)^H)\|_{L^2}+\|(m_1^L,m_2^L,(\nabla\phi)^L)\|_{L^2}\\ &\le C(C_0+K_0+\mathcal{M}^2(t))(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})},\end{aligned} \tag{3.19}$$

$$\begin{aligned}\|\nabla^l(\varrho_1,\varrho_2)\|_{L^2} &\le \|\nabla^l(\varrho_1^H,\varrho_2^H)\|_{L^2}+\|\nabla^l(\varrho_1^L,\varrho_2^L)\|_{L^2}\\ &\le C(C_0+K_0+\mathcal{M}^2(t))(1+t)^{-\frac{3}{4}-\frac{l}{2}},\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\|\nabla^l(m_1,m_2,\nabla\phi)\|_{L^2} &\le \|\nabla^l(m_1^H,m_2^H,(\nabla\phi)^H)\|_{L^2}+\|\nabla^l(m_1^L,m_2^L,(\nabla\phi)^L)\|_{L^2}\\ &\le C(C_0+K_0+\mathcal{M}^2(t))(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{l}{2}}.\end{aligned} \tag{3.21}$$

Finally, by the Definition (3.1) of $\mathcal{M}(t)$, and using (3.18)–(3.21) and the Sobolev interpolation inequality, we can get

$$\mathcal{M}(t)\le C(C_0+K_0)+C\mathcal{M}^2(t),$$

which together with the smallness of $C_0$ and $K_0$ implies that $\mathcal{M}(t)\le C(C_0+K_0)$. This completes the proof of Proposition 3.2. □

## 4. Proof of lower decay estimates

In this section, we prove the lower decay estimates of the solution and its derivatives to the system (1.1).

PROPOSITION 4.1. *Under the Assumptions (1.8), (1.9) for $p=1$ and (1.12) of Theorem 3.1, it holds that for $0\le k\le l$ and some positive constant $C_6$,*

$$\min\{\|\nabla^k\varrho_1\|_{L^2},\|\nabla^k m_1\|_{L^2},\|\nabla^k\varrho_2\|_{L^2},\|\nabla^k m_2\|_{L^2},\|\nabla^k\nabla\phi\|_{L^2}\}\ge C_6\delta_0^{\frac{3}{2}}(1+t)^{-\frac{3}{4}-\frac{k}{2}}.$$

*Proof.* First by using (2.6), the lower decay estimate (2.20) for the linear system and (3.4), the decay estimate (2.18) with $k=0$ on the nonlinear term, Proposition 3.2 and Lemma 3.1, we have from (2.14) that

$$\begin{aligned}&\min\{\|\varrho_1\|_{L^2},\|m_1\|_{L^2},\|\varrho_2\|_{L^2},\|m_2\|_{L^2},\|\nabla\phi\|_{L^2}\}\\ &\ge \min\{\|\varrho_1^L\|_{L^2},\|m_1^L\|_{L^2},\|\varrho_2^L\|_{L^2},\|m_2^L\|_{L^2},\|(\nabla\phi)^L\|_{L^2}\}\\ &\ge C\delta_0^{\frac{3}{2}}(1+t)^{-\frac{3}{4}}-C\left|\int_0^t(1+t-\tau)^{-\frac{3}{4}}\|\mathcal{N}^L(\tau)\|_{L^1}d\tau\right|\\ &\ge C\delta_0^{\frac{3}{2}}(1+t)^{-\frac{3}{4}}-C\int_0^t(1+t-\tau)^{-\frac{3}{4}}\|(\varrho_1,m_1,\varrho_2,m_2,\nabla\phi)\|_{L^2}\|(\varrho_1,\nabla m_1,\varrho_2,\nabla m_2)\|_{H^1}d\tau\\ &\ge C\delta_0^{\frac{3}{2}}(1+t)^{-\frac{3}{4}}-CC_0^{\frac{1}{3}}\int_0^t(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{5}{4}}\mathcal{M}^{\frac{5}{3}}(\tau)d\tau\end{aligned}$$

$$\geq (C\delta_0^{\frac{3}{2}} - CC_0^{\frac{1}{3}}(C_0+K_0)^{\frac{5}{3}})(1+t)^{-\frac{3}{4}},$$

this, combined with the fact that $C_0, K_0 \leq \delta_0$ is small, implies that

$$\min\{\|\varrho_1\|_{L^2}, \|m_1\|_{L^2}, \|\varrho_2\|_{L^2}, \|m_2\|_{L^2}, \|\nabla\phi\|_{L^2}\} \geq C_7\delta_0^{\frac{3}{2}}(1+t)^{-\frac{3}{4}} \tag{4.1}$$

for some positive constant $C_7$.

As in the proof of (4.1), by using (2.17) with $k=1$ and (2.21), we have from (2.14) that

$$\begin{aligned}
&\min\{\|\nabla\varrho_1\|_{L^2}, \|\nabla m_1\|_{L^2}, \|\nabla\varrho_2\|_{L^2}, \|\nabla m_2\|_{L^2}, \|\nabla(\nabla\phi)\|_{L^2}\} \\
&\geq \min\{\|\nabla\varrho_1^L\|_{L^2}, \|\nabla m_1^L\|_{L^2}, \|\nabla\varrho_2^L\|_{L^2}, \|\nabla m_2^L\|_{L^2}, \|\nabla(\nabla\phi)^L\|_{L^2}\} \\
&\geq C\delta_0^{\frac{3}{2}}(1+t)^{-\frac{5}{4}} - C\left|\int_0^t (1+t-\tau)^{-\frac{5}{4}}\|\mathcal{N}(\tau)\|_{L^1}d\tau\right| \\
&\geq C\delta_0^{\frac{3}{2}}(1+t)^{-\frac{5}{4}} - CC_0^{\frac{1}{3}}\mathcal{M}^{\frac{5}{3}}\int_0^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{5}{4}}d\tau \\
&\geq (C\delta_0^{\frac{3}{2}} - CC_0^{\frac{1}{3}}(C_0+K_0)^{\frac{5}{3}})(1+t)^{-\frac{5}{4}},
\end{aligned}$$

which also yields that

$$\|\nabla(\varrho_1, u_1, \varrho_2, u_2)\|_{L^2} \geq C_8\delta_0^{\frac{3}{2}}(1+t)^{-\frac{5}{4}} \tag{4.2}$$

for some positive constant $C_8$.

Finally, by using (4.1), (4.2) and the Sobolev interpolation inequality, we can deduce that for $2 \leq k \leq l$,

$$\|\nabla^k(\varrho_1, u_1, \varrho_2, u_2)\|_{L^2} \geq C\|\nabla(\varrho_1, u_1, \varrho_2, u_2)\|_{L^2}^k \|(\varrho_1, u_1, \varrho_2, u_2)\|_{L^2}^{-(k-1)} \geq C_9\delta_0^{\frac{3}{2}}(1+t)^{-\frac{3}{4}-\frac{k}{2}}$$

for some positive constant $C_9$. This finishes the proof of Proposition 4.1. □

## 5. Analytic tools

We will extensively use the Sobolev interpolation of the Gagliardo–Nirenberg inequality; the proof can be seen in [24].

LEMMA 5.1. *Let $0 \leq i, j \leq k$, then we have*

$$\|\nabla^i f\|_{L^p} \lesssim \|\nabla^j f\|_{L^q}^{1-a}\|\nabla^k f\|_{L^r}^a$$

*where $a$ belongs to $\left[\frac{i}{k}, 1\right]$ and satisfies*

$$\frac{i}{3} - \frac{1}{p} = \left(\frac{j}{3} - \frac{1}{q}\right)(1-a) + \left(\frac{k}{3} - \frac{1}{r}\right)a.$$

*Especially, while $p=q=r=2$, we have*

$$\|\nabla^i f\|_{L^2} \lesssim \|\nabla^j f\|_{L^2}^{\frac{k-i}{k-j}}\|\nabla^k f\|_{L^2}^{\frac{i-j}{k-j}}.$$

To estimate the product of two functions, we shall record the following estimate, cf. [18]:

LEMMA 5.2. *It holds that for $k \geq 0$,*

$$\|\nabla^k(gh)\|_{L^{p_0}} \lesssim \|g\|_{L^{p_1}}\|\nabla^k h\|_{L^{p_2}} + \|\nabla^k g\|_{L^{p_3}}\|h\|_{L^{p_4}}.$$

*Here* $p_0, p_2, p_3 \in (1,\infty)$ *and*

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Thus we can easily deduce from Lemma 5.2 the following commutator estimate:

LEMMA 5.3. *Let $f$ and $g$ be smooth functions belonging to $H^k \cap L^\infty$ for any integer $k \geq 1$ and define the commutator*

$$[\nabla^k, f]g = \nabla^k(fg) - f\nabla^k g.$$

*Then we have*

$$\|[\nabla^k, f]g\|_{L^{p_0}} \lesssim \|\nabla f\|_{L^{p_1}} \|\nabla^{k-1} g\|_{L^{p_2}} + \|\nabla^k f\|_{L^{p_3}} \|g\|_{L^{p_4}}.$$

*Here $p_i (i = 0,1,2,3,4)$ are defined in Lemma 5.2.*

Next, to estimate the $L^2$–norm of the spatial derivatives of some smooth function $F(f)$, we shall introduce some estimates which follow from Lemma 5.1 and Lemma 5.2:

LEMMA 5.4. *Let $F(f)$ be a smooth function of $f$ with bounded derivatives of any order and $f$ belong to $H^k$ for any integer $k \geq 3$ , then we have*

$$\|\nabla^k(F(f))\|_{L^2} \lesssim \sup_{0\leq i\leq k} \|F^{(i)}(f)\|_{L^\infty} \left( \sum_{j=2}^{k} \|f\|_{L^2}^{j-1-\frac{3(j-1)}{2k}} \|\nabla^k f\|_{L^2}^{1+\frac{3(j-1)}{2k}} + \|\nabla^k f\|_{L^2} \right).$$

*Moreover, if $f$ has the lower and upper bounds, and $\|f\|_k \leq 1$, we have*

$$\|\nabla^k(F(f))\|_{L^2} \lesssim \|\nabla^k f\|_{L^2}.$$

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