

RECOVERY OF A DISTRIBUTED ORDER FRACTIONAL DERIVATIVE IN AN UNKNOWN MEDIUM*

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Abstract. In this work, we study an inverse problem of recovering information about the weight in distributed-order time-fractional diffusion from the observation at one single point on the domain boundary. In the absence of an explicit knowledge of the medium, we prove that the one-point observation can uniquely determine the support bound of the weight. The proof is based on asymptotics of the data, analytic continuation and Titchmarsh convolution theorem. When the medium is known, we give an alternative proof of an existing result, i.e., the one-point boundary observation uniquely determines the weight. Several numerical experiments are also presented to complement the analysis.

Keywords. Distributed order; time-fractional diffusion; weight recovery; ultra-slow diffusion; reconstruction.

AMS subject classifications. 35R30; 35R11; 65M32.

1. Introduction

This work is concerned with an inverse problem for distributed-order time-fractional diffusion. Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be an open bounded and connected subset with a $C^{2\lceil \frac{d}{4} \rceil + 2}$ boundary $\partial\Omega$ (the notation $\lceil r \rceil$ denotes the smallest integer $\geq r \in \mathbb{R}$). Let $a := (a_{i,j})_{i,j=1}^d \in C^{1+2\lceil \frac{d}{4} \rceil}(\bar{\Omega}; \mathbb{R}^{d \times d})$ be symmetric and fulfill the standard ellipticity condition, i.e., there exists a $c > 0$ such that

$$\sum_{i,j=1}^d a_{i,j}(x) \xi_i \xi_j \geq c |\xi|^2, \quad \forall x \in \bar{\Omega}, \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, \quad (1.1)$$

and $q \in C^{2\lceil \frac{d}{4} \rceil}(\bar{\Omega})$ is strictly positive in $\bar{\Omega}$. Then we define the operator \mathcal{A} by

$$\mathcal{A}u(x) := - \sum_{i,j=1}^d \partial_{x_i} (a_{i,j}(x) \partial_{x_j} u(x)) + q(x)u(x), \quad x \in \Omega.$$

Finally, $\mu \in C[0, 1]$ is a non-negative function satisfying $\text{supp}(\mu) = [b_1, b_2]$ with $0 \leq b_1 < b_2 < 1$ and b_1 and b_2 being lower and upper bounds of the support of the weight μ . We define the function

$$K_\mu(t) := \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \mu(\alpha) d\alpha,$$

where $\Gamma(z) = \int_0^\infty s^{z-1} e^{-s} ds$, for $\Re(z) > 0$, denotes Euler's Gamma function. Then we

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define an integral operator I_{K_μ} by

$$I_{K_\mu}g(x,t) = \int_0^t K_\mu(t-s)g(x,s)ds, \quad g \in L^1_{loc}(\mathbb{R}_+; L^2(\Omega)), \quad x \in \Omega, \quad t \in \mathbb{R}_+ := (0, +\infty).$$

We define Riemann-Liouville and Djrbashian-Caputo fractional derivatives with the kernel K_μ for $g \in C([0, +\infty); L^2(\Omega))$ by

$$D_t^{[\mu]}g(x,t) = \partial_t I_{K_\mu}g(x,t), \quad \partial_t^{[\mu]}g(x,t) = D_t^{[\mu]}[g - g(\cdot, 0)](x,t), \quad x \in \Omega, \quad t \in \mathbb{R}_+.$$

For $g \in W^{1,1}_{loc}([0, +\infty); L^2(\Omega))$, $\partial_t^{[\mu]}g = I_{K_\mu}\partial_t g$, which coincides with the classical Djrbashian-Caputo fractional derivative for the kernel K_μ . Now we consider the following initial boundary value problem:

$$\begin{cases} (\partial_t^{[\mu]} + \mathcal{A})u(x,t) = F(x,t), & \text{in } \Omega \times \mathbb{R}_+, \\ \mathcal{R}u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \quad (1.2)$$

Here \mathcal{R} denotes either Dirichlet trace $\mathcal{R}u = u|_{\partial\Omega \times \mathbb{R}_+}$ or conormal derivative ∂_{ν_a} associated with a , i.e., $\mathcal{R}u = \partial_{\nu_a}u|_{\partial\Omega \times \mathbb{R}_+} = \sum_{i,j=1}^d a_{ij}\partial_{x_j}u\nu_i|_{\partial\Omega \times \mathbb{R}_+}$, where $\nu \in \mathbb{R}^d$ denotes the unit outward normal to $\partial\Omega$. Throughout, the adjoint trace \mathcal{R}^* is the Dirichlet trace if \mathcal{R} is the Neumann one and vice versa.

Physically, problem (1.2) describes ultra-slow anomalous diffusion processes, in which the mean square displacement admits a logarithmic growth [30, 31]. The model is also called distributed-order fractional diffusion, generalizing time-fractional diffusion [11, 21, 34]. The distributed-order fractional derivative with a constant weight was used by Caputo [2] in 1969 to describe the stress-strain relation of anelastic media. Later it was used to formally determine eigenfunctions of torsional modes of anelastic or dielectric spherical shells and infinite plates and to estimate split periods of their free modes [3]. The related differential equation was first solved by Caputo [4] using Laplace transform. Since then, the model has received much attention in applied disciplines, e.g., polymer physics and kinetics of particles moving in quenched random force fields [6, 33, 36]. There is also a growing body of literature [17–20, 23, 25, 26, 29] on the analysis of the direct problem. Kochubei [17, 18] investigated fundamental solutions to the Cauchy problems for both ODEs and PDEs with $\mu \in C^1[0, 1]$. The long- and short-time asymptotic behaviors were discussed in [25] using an argument similar to the Watson lemma; see also [19] for a more general case using energy estimates.

The weight μ characterizes ultra-slow diffusion processes. In practice, it is determined by the inhomogeneity of the medium, but the relevant physical law is still unknown at present, and moreover, it also cannot be measured directly. This naturally motivates the inverse problem of identifying the weight μ from, e.g., the measurement of the solution u at one point. When the medium is known and $F \equiv 0$, Li et al. [26] proved that the measurement at one interior point can determine the weight μ . The proof is based on analytic properties of solutions, especially time analyticity, and properties of Laplace transform of solutions of (1.2). In the 1D case, Rundell and Zhang [35] studied a similar inverse problem using the value of the solution u in the time interval $(0, \infty)$, where the main tool in the analysis is a fractional version of the classical θ function. More recently, Li et al. [22] studied the inverse problem with nonzero boundary conditions, and generalized the result in [35] using a Harnack-type inequality and strong maximum principle. Also [28] is the only work on numerical recovery, based on

Tikhonov regularization and conjugate gradient method. See [24] for an overview of results on determining the orders / weights in time-fractional models.

In this work, we investigate the inverse problem of recovering the support info of the weight μ , i.e., b_1 and b_2 , from the observation at one point on the boundary $\partial\Omega$, where the medium is unknown. While the approach in [26] can be successfully applied to the case of a known medium, it is unclear if similar results still hold when the medium is unknown. The main goal of this work is to determine the support bounds b_1 and b_2 of the weight μ from the knowledge $\mathcal{R}^*u(x_0, t)$ for $t \in (0, T)$, $x_0 \in \partial\Omega$, when the medium is unknown. We answer the question in the affirmative. See Section 2 for the precise statements of the main results. Further, we provide numerical experiments to complement the analysis for both known and unknown media. The numerical recovery is still under-explored, due to the complexity associated with approximating $\partial_t^{[\mu]}u$. In the appendix, we describe an easy-to-implement algorithm to perform numerical simulation.

The rest of the paper is organized as follows. In Section 2, we present and discuss the main results. Then in Section 3, we recall preliminary results. In Section 4, we present several asymptotic results for $t \rightarrow 0^+$ and $t \rightarrow +\infty$. Then in Section 5 we give the proofs of the main results. We provide several numerical experiments in Section 6. Finally we give concluding remarks in Section 7.

2. Main results and discussions

In this section, we describe the main theoretical results. We define an operator $A = \mathcal{A}$ acting on $L^2(\Omega)$ with its domain $D(A)$ given by $D(A) = \{v \in H^2(\Omega) : \mathcal{A}v \in L^2(\Omega), \mathcal{R}v = 0 \text{ on } \partial\Omega\}$. By [9, Theorem 2.5.1.1], for all $\ell = 1, \dots, \lceil \frac{d}{4} \rceil + 1$,

$$D(A^\ell) = \{v \in H^{2\ell}(\Omega) : \mathcal{R}v = \mathcal{R}(\mathcal{A}v) = \dots = \mathcal{R}(\mathcal{A}^{\ell-1}v) = 0\}. \tag{2.1}$$

The spectrum of A consists of a non-decreasing sequence of strictly positive eigenvalues $(\lambda_n)_{n \geq 1}$ (repeated with multiplicity). We introduce an $L^2(\Omega)$ orthonormal basis of eigenfunctions $(\varphi_n)_{n \geq 1}$ of A associated with the eigenvalues $(\lambda_n)_{n \geq 1}$. For all $s \geq 0$, we define the operator A^s by

$$A^s h = \sum_{n=1}^{+\infty} \langle h, \varphi_n \rangle \lambda_n^s \varphi_n, \quad h \in D(A^s) = \left\{ h \in L^2(\Omega) : \sum_{n=1}^{+\infty} |\langle h, \varphi_n \rangle|^2 \lambda_n^{2s} < \infty \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\Omega)$ inner product, and the graph norm by $\|h\|_{D(A^s)} = (\sum_{n=1}^{+\infty} |\langle h, \varphi_n \rangle|^2 \lambda_n^{2s})^{\frac{1}{2}}$, for $h \in D(A^s)$. It is convenient to introduce the concept of an admissible tuple, to describe regularity conditions on the problem data so that the direct problem has a weak solution u in the sense of Definition 2.2 below.

DEFINITION 2.1. *A tuple (Ω, a, q, f, u_0) is said to be admissible if the following conditions are fulfilled.*

- (i) $\Omega \subset \mathbb{R}^d$ is a $C^{2\lceil \frac{d}{4} \rceil + 2}$ bounded open set, $a := (a_{i,j})_{i,j=1}^d \in C^{1+2\lceil \frac{d}{4} \rceil}(\overline{\Omega}; \mathbb{R}^{d \times d})$ satisfies the ellipticity condition (1.1), and $q \in C^{2\lceil \frac{d}{4} \rceil}(\overline{\Omega})$ is strictly positive on $\overline{\Omega}$.
- (ii) $u_0 \in D(A^{r+1})$, with $r > \frac{d+3}{2}$, $F \in L^1(\mathbb{R}_+; L^2(\Omega))$ and there exists $T \in \mathbb{R}_+$ such that

$$F(x, t) := \sigma(t)f(x), \quad (x, t) \in \Omega \times (0, T), \tag{2.2}$$

with $f \in D(A^r)$ and $\sigma \in L^1(0, T)$ and $\sigma \not\equiv 0$.

Following [15], we define weak solutions of problem (1.2) as follows.

DEFINITION 2.2. *Let the coefficients in (1.2) satisfy (1.1). A function $u \in L^1_{loc}(\mathbb{R}_+; L^2(\Omega))$ is said to be a weak solution to problem (1.2) if it satisfies the following conditions.*

- (i) *The identity $D_t^\mu[u - u_0](x, t) + \mathcal{A}u(x, t) = F(x, t)$, $t \in \mathbb{R}_+$, $x \in \Omega$ holds in the sense of distribution in $\Omega \times \mathbb{R}_+$.*
- (ii) *There hold $I_{K_\mu}[u - u_0] \in W^{1,1}_{loc}(\mathbb{R}_+; H^{-2}(\Omega))$ and $I_{K_\mu}[u - u_0](x, 0) = 0$, $x \in \Omega$.*
- (iii) *There holds $\tau_0 = \inf\{\tau > 0 : e^{-\tau t}u \in L^1(\mathbb{R}_+; L^2(\Omega))\} < \infty$ and there exists $\tau_1 \geq \tau_0$ such that for all $p \in \mathbb{C}$ with $\Re(p) > \tau_1$, $\hat{u}(\cdot, p) := \int_0^\infty e^{-pt}u(\cdot, t) dt \in H^2(\Omega)$, and $\mathcal{R}\hat{u}(\cdot, p) = 0$ on $\partial\Omega$.*

The conditions in Definition 2.2 describe different aspects of (1.2): (i) is associated with the governing equation, (ii) describes the link with the initial condition and (iii) gives the boundary condition.

The main results can be stated as follows. In formula (2.3), δ_{ij} denotes the Kronecker delta symbol.

ASSUMPTION 2.1. *\mathcal{R} is the Dirichlet trace or \mathcal{R} is the Neumann trace and the following condition holds*

$$a_{ij}(x) = \delta_{ij}, \quad x \in \partial\Omega. \tag{2.3}$$

THEOREM 2.1. *Let $(\Omega_k, a_k, q_k, f_k, 0)$, $k = 1, 2$, be two admissible tuples with $f_k \neq 0$, $\mu_k \in C[0, 1]$ non-negative such that $\text{supp}(\mu_k) = [b_1^k, b_2^k]$ with $0 \leq b_1^k < b_2^k < 1$, and u^k the corresponding weak solution of problem (1.2) with $\mu = \mu_k$. If there exist $x_k \in \partial\Omega_k$, $k = 1, 2$, such that the condition*

$$\min(|\mathcal{R}_1^* f_1(x_1)|, |\mathcal{R}_2^* f_2(x_2)|) > 0, \tag{2.4}$$

holds. Then, the condition

$$\mathcal{R}_1^* u^1(x_1, t) = \mathcal{R}_2^* u^2(x_2, t), \quad t \in (0, T) \tag{2.5}$$

implies $b_2^1 = b_2^2$. In addition, if Assumption 2.1 holds for \mathcal{R}_k and the following sign condition

$$f_1 \geq 0 \text{ and } f_2 \geq 0 \quad \text{or} \quad f_1 \leq 0 \text{ and } f_2 \leq 0 \tag{2.6}$$

holds, then condition (2.5) implies also $b_1^1 = b_1^2$.

THEOREM 2.2. *Let $(\Omega_k, a_k, q_k, 0, u_0^k)$, $k = 1, 2$, be two admissible tuples with $u_0^k \neq 0$, $\mu_k \in C[0, 1]$ non-negative such that $\text{supp}(\mu_k) = [b_1^k, b_2^k]$ with $0 \leq b_1^k < b_2^k < 1$, and u^k the corresponding weak solution of problem (1.2) with $\mu = \mu_k$. If there exist $x_k \in \partial\Omega_k$, $k = 1, 2$, such that the condition*

$$\min(|\mathcal{R}_1^* \mathcal{A}_1 u_0^1(x_1)|, |\mathcal{R}_2^* \mathcal{A}_2 u_0^2(x_2)|) > 0 \tag{2.7}$$

holds, then for any $T_1 \in (0, T)$ and $T_2 \in (T_1, T)$, with T given by (2.2), the condition

$$\mathcal{R}_1^* u^1(x_1, t) = \mathcal{R}_2^* u^2(x_2, t), \quad t \in (T_1, T_2) \tag{2.8}$$

implies $b_2^1 = b_2^2$. In addition, if Assumption 2.1 holds for \mathcal{R}_k , and the following sign condition

$$u_0^1 \geq 0 \text{ and } u_0^2 \geq 0 \quad \text{or} \quad u_0^1 \leq 0 \text{ and } u_0^2 \leq 0 \tag{2.9}$$

holds, then condition (2.8) implies also $b_1^1 = b_1^2$.

REMARK 2.1. The proof in Section 5 indicates that the regularity condition $\mu \in C[0, 1]$ can be relaxed to μ being piecewise smooth. Notably Theorems 2.1 and 2.2 do not require a full knowledge of the problem data (Ω, a, q, f, u_0) . Uniqueness in the cases of single-term and multi-term models was established in [13] and [12], respectively; see also [38] for a uniqueness result from interior data.

In the spirit of [26, Theorem 2.2], we can strengthen the results when the medium is known. Then we can show the unique recovery of the weight $\mu \in L^1(0, 1)$.

THEOREM 2.3. Let (Ω, a, q, f, u_0) be an admissible tuple, with T given by condition (2.2), $\mu \in L^1(0, 1)$ a non-negative function satisfying

$$\exists \alpha_0 \in (0, 1), \exists \varepsilon \in (0, \alpha_0), \forall \alpha \in (\alpha_0 - \varepsilon, \alpha_0), \quad \mu(\alpha) \geq \frac{\mu(\alpha_0)}{2} > 0 \tag{2.10}$$

with $\mu = \mu_k$, and u^k the corresponding weak solution of problem (1.2). Let Assumption 2.1 hold for \mathcal{R} . Fix any point $x_0 \in \partial\Omega$. If $u_0 \equiv 0$ and $f \neq 0$ is of constant sign, then the condition

$$\mathcal{R}^* u^1(x_0, t) = \mathcal{R}^* u^2(x_0, t), \quad t \in (0, T) \tag{2.11}$$

implies $\mu_1 = \mu_2$. Moreover, if $f \equiv 0$ and $u_0 \neq 0$ is of constant sign, for any $T_1, T_2 \in [0, T]$ with $T_1 < T_2$, the condition

$$\mathcal{R}^* u^1(x_0, t) = \mathcal{R}^* u^2(x_0, t), \quad t \in (T_1, T_2) \tag{2.12}$$

implies $\mu_1 = \mu_2$.

REMARK 2.2. Note that in all the results, the data actually does not access the source F for $\Omega \times (T, +\infty)$. Thus, the statement of the inverse problem is on the set $\Omega \times (0, T)$ instead of $\Omega \times \mathbb{R}_+$.

3. Preliminaries

In this section we give preliminary results of problem (1.2), e.g., existence of a weak solution and time analyticity. These results will play a crucial role in the uniqueness proof in Section 5. Since $\mu \in C[0, 1]$ is non-negative and $\text{supp}(\mu) = [b_1, b_2]$, for all $\alpha_0 \in [b_1, b_2)$ and all $\varepsilon \in (0, b_2 - \alpha_0]$, there exist $\delta_0 \in (0, \varepsilon)$ and $\delta_1 \in (0, \delta_0)$ such that

$$\mu(\alpha_1) > \frac{\mu(\alpha_0 + \delta_0)}{2} > 0, \quad \forall \alpha_1 \in [\alpha_0 + \delta_1, \alpha_0 + \delta_0]. \tag{3.1}$$

Throughout, we fix $\theta \in (\frac{\pi}{2}, \min(\frac{\pi}{2b_2}, \pi))$, $\delta \in \mathbb{R}_+$ and the contour $\gamma(\delta, \theta) \subset \mathbb{C}$ defined by $\gamma(\delta, \theta) := \{\delta e^{i\beta} : \beta \in [-\theta, \theta]\} \cup \{r e^{\pm i\theta} : r \geq \delta\}$, oriented in the counterclockwise direction. Let $\theta_1 \in (0, \frac{2\theta - \pi}{8})$. For any $p \in \mathbb{C} \setminus (-\infty, 0]$ we set $p^\alpha := e^{\alpha \log(p)}$ with \log being complex logarithm defined on $\mathbb{C} \setminus (-\infty, 0]$. For all $z \in D_{\theta_1} := \{r e^{i\beta} : r > 0, \beta \in (-\theta_1, \theta_1)\}$, we can define two solution operators $S_1(z), S_2(z) \in \mathcal{B}(L^2(\Omega))$ by

$$S_1(z)v = \frac{1}{2i\pi} \int_{\gamma(\delta, \theta_1)} e^{zp} (A + \mathcal{V}(p))^{-1} p^{-1} \mathcal{V}(p)v dp, \tag{3.2}$$

$$S_2(z)v = \frac{1}{2i\pi} \int_{\gamma(\delta, \theta_1)} e^{zp} (A + \mathcal{V}(p))^{-1} v dp, \tag{3.3}$$

with the weight function $\mathcal{V}(p)$ given by $\mathcal{V}(p) = \int_0^1 p^\alpha \mu(\alpha) d\alpha$, for $p \in \mathbb{C} \setminus (-\infty, 0]$. The operators $S_1(z)$ and $S_2(z)$ correspond to the initial data and right-hand side, respectively.

Under condition (3.1), the argument of [23, Theorem 1.4] and [15, Lemma 3.1] gives the following result.

LEMMA 3.1. *For all $\tau \in [0, 1]$, the map $z \mapsto S_j(z)$, $j = 1, 2$, is holomorphic in the set D_{θ_1} as a map taking values in $\mathcal{B}(L^2(\Omega); D(A^\tau))$ and there exist $C > 0$ and $\delta' \in (0, b_2 - b_1)$ such that*

$$\begin{aligned} \|S_1(z)\|_{\mathcal{B}(L^2(\Omega); D(A^\tau))} &\leq C \max(|z|^{(b_1+\delta')(1-\tau)-1}, |z|^{b_2(1-\tau)-1}), \quad z \in D_{\theta_1}, \\ \|S_2(z)\|_{\mathcal{B}(L^2(\Omega); D(A^\tau))} &\leq C \max(|z|^{(b_1+\delta')(1-\tau)-b_2}, |z|^{b_2(1-\tau)-b_1}), \quad z \in D_{\theta_1}. \end{aligned}$$

Lemma 3.1 and [15, Theorem 1.4] imply that for $F \in L^1(\mathbb{R}_+; L^2(\Omega))$ and $u_0 \in L^2(\Omega)$, problem (1.2) admits a unique weak solution $u \in L^1_{loc}(\mathbb{R}_+; D(A^\tau))$, $\tau \in [0, 1]$, given by

$$u(\cdot, t) = S_1(t)u_0 + \int_0^t S_2(t-s)F(\cdot, s) ds, \quad t \in \mathbb{R}_+. \tag{3.4}$$

Similar to [23], for $z \in D_{\theta_1}$ and $j = 1, 2$, we have

$$S_j(z)h = \sum_{n=1}^{+\infty} S_{j,n}(z) \langle h, \varphi_n \rangle \varphi_n,$$

with $S_{1,n}(z) = \frac{1}{2i\pi} \int_{\gamma(\delta, \theta)} e^{z p} (\lambda_n + \mathcal{V}(p))^{-1} p^{-1} \mathcal{V}(p) dp$ and $S_{2,n}(z) = \frac{1}{2i\pi} \int_{\gamma(\delta, \theta)} e^{z p} (\lambda_n + \mathcal{V}(p))^{-1} dp$. The argument for Lemma 3.1 implies also that for all $n \in \mathbb{N}$ and $j = 1, 2$, $S_{j,n}$ is holomorphic on D_{θ_1} . Moreover, we have for all $\tau \in [0, 1]$ [23, Lemmas 2.1 and 2.2]

$$|S_{1,n}(z)| \leq C \lambda_n^{-\tau} \max(|z|^{(b_1+\delta')(1-\tau)-1}, |z|^{b_2(1-\tau)-1}), \quad z \in D_{\theta_1}, \quad n \in \mathbb{N}, \tag{3.5}$$

$$|S_{2,n}(z)| \leq C \lambda_n^{-\tau} \max(|z|^{(b_1+\delta')(1-\tau)-b_2}, |z|^{b_2(1-\tau)-b_1}), \quad z \in D_{\theta_1}, \quad n \in \mathbb{N}, \tag{3.6}$$

with the constant $C > 0$ depending only on θ, \mathcal{A}, μ and Ω . Then we can prove the following representation of the measurement $\mathcal{R}^*u(x, t)$.

LEMMA 3.2. *Let $f, u_0 \in D(A^{\frac{d}{4}})$, $\sigma \in L^1(0, T)$ and $F \in L^1(\mathbb{R}_+; L^2(\Omega))$ satisfy condition (2.2). Then the maps $t \mapsto S_1(t)u_0$ and $t \mapsto S_2(t)f$ are analytic with respect to $t \in \mathbb{R}_+$ as a function taking values in $C^1(\bar{\Omega})$. Moreover, problem (1.2) admits a unique weak solution $u \in L^1_{loc}(\mathbb{R}_+; L^2(\Omega))$ such that the restriction of u to $\Omega \times (0, T)$ belongs to $L^1(0, T; C^1(\bar{\Omega}))$ and*

$$\mathcal{R}^*u(x, t) = \mathcal{R}^*[S_1(t)u_0](x) + \int_0^t \sigma(s) \mathcal{R}^*[S_2(t-s)f](x) ds, \quad t \in (0, T), \quad x \in \partial\Omega. \tag{3.7}$$

Proof. We only prove (3.7) for $u_0 \equiv 0$. By the identity (2.1) and interpolation, the space $D(A^{\frac{d}{4} + \frac{3}{4}})$ embeds continuously into $H^{\frac{d}{2} + \frac{3}{2}}(\Omega)$ and Sobolev embedding theorem implies that $D(A^{\frac{d}{4} + \frac{3}{4}})$ embeds continuously into $C^1(\bar{\Omega})$. In addition, by applying (3.6), for all $z \in D_{\theta_1}$ and all $m_1, m_2 \in \mathbb{N}$, $m_1 < m_2$, we have

$$\begin{aligned} \left\| \sum_{n=m_1}^{m_2} S_{2,n}(z) \langle f, \varphi_n \rangle \varphi_n \right\|_{C^1(\bar{\Omega})} &\leq C \left\| \sum_{n=m_1}^{m_2} S_{2,n}(z) \langle f, \varphi_n \rangle \varphi_n \right\|_{D(A^{\frac{d}{4} + \frac{3}{4}})} \\ &\leq C \max(|z|^{\frac{b_1+\delta'}{4}-1}, |z|^{\frac{b_2}{4}-1}) \left(\sum_{n=m_1}^{m_2} \lambda_n^{\frac{d}{2}} |\langle f, \varphi_n \rangle|^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{3.8}$$

with $C > 0$ independent of z , m_1 and m_2 . This and the condition $f \in D(A^{\frac{d}{4}})$ yield that the sequence $\sum_{n=1}^N S_{2,n}(z) \langle f, \varphi_n \rangle \varphi_n$, $N \in \mathbb{N}$, converges uniformly with respect to z on any compact subset of D_{θ_1} to $S_2(z)f$ as a function taking values in $C^1(\overline{\Omega})$. Thus, the map $D_{\theta_1} \ni z \mapsto S_2(z)f$ is holomorphic as a function taking values in $C^1(\overline{\Omega})$. This shows the first assertion. Next, in view of (2.2) and (3.4), we have

$$u(\cdot, t) = \int_0^t S_2(t-s)F(\cdot, s) ds = \int_0^t S_2(t-s)\sigma(s)f ds = \int_0^t \sigma(s)S_2(t-s)f ds, \quad t \in (0, T).$$

Thus, the estimate (3.8) implies

$$\|u(\cdot, t)\|_{D(A^{\frac{d}{4}+\frac{3}{4}})} \leq C \|f\|_{D(A^{\frac{d}{4}})} \left(\max(t^{\frac{b_1+\delta}{4}-1}, t^{\frac{b_2}{4}-1}) \mathbb{1}_{(0,T)} \right) * (|\sigma| \mathbb{1}_{(0,T)})(t), \quad t \in (0, T),$$

where $\mathbb{1}_{(0,T)}$ denotes the characteristic function of $(0, T)$ and $*$ denotes the convolution product. Hence, Young’s inequality gives $u \in L^1(0, T; D(A^{\frac{d}{4}+\frac{3}{4}})) \subset L^1(0, T; C^1(\overline{\Omega}))$, showing the second assertion. In the same way, by applying (3.8), we deduce the representation (3.7). \square

Below we also use the following result from [12, Lemma 2.3].

LEMMA 3.3. *The following estimate and identity hold*

$$\sum_{n=1}^{+\infty} \lambda_n |\langle v, \varphi_n \rangle| \|\mathcal{R}^* \varphi_n\|_{L^\infty(\partial\Omega)} \leq C \|v\|_{D(A^{\frac{d}{2}+\frac{3}{2}})}, \quad v \in D(A^{\frac{d}{2}+\frac{3}{2}}), \quad (3.9)$$

$$\mathcal{R}^* v(x) = \sum_{n=1}^{+\infty} \langle v, \varphi_n \rangle \mathcal{R}^* \varphi_n(x), \quad x \in \partial\Omega, \quad v \in D(A^{\frac{d}{2}+\frac{1}{2}}). \quad (3.10)$$

4. Asymptotic properties of solutions as $t \rightarrow 0$ and $t \rightarrow +\infty$

In this section, we fix an admissible tuple (Ω, a, q, h, u_0) with $\mu \in C[0, 1]$ a non-negative function satisfying $\text{supp}(\mu) = [b_1, b_2]$, $0 \leq b_1 < b_2 < 1$. Let $w(\cdot, t) = S_2(t)h \in L^1_{loc}(\mathbb{R}_+; C^1(\overline{\Omega})) \cap C(\mathbb{R}_+; C^1(\overline{\Omega}))$, cf. Lemma 3.2. Below we fix $x_0 \in \partial\Omega$ and study the asymptotics of $\mathcal{R}^* w(x_0, t)$ as $t \rightarrow 0^+$ and $t \rightarrow +\infty$. The next result gives the asymptotic behavior of $\mathcal{R}^* w(x_0, t)$ as $t \rightarrow 0^+$.

LEMMA 4.1. *Let $Q(t, p)$ and $P(t)$ be defined, for $t \in \mathbb{R}_+$, $p \in \mathbb{C} \setminus (-\infty, 0]$, respectively by*

$$Q(t, p) := \int_0^1 t^{-\alpha} p^\alpha \mu(\alpha) d\alpha \quad \text{and} \quad P(t) = \int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha. \quad (4.1)$$

Define also

$$\mathfrak{J}(t) := \frac{1}{2i\pi} \int_{\gamma(1, \theta)} e^p Q(t, p)^{-1} dp. \quad (4.2)$$

Then for $h \in D(A^{\frac{d}{2}+\frac{3}{2}})$, the following asymptotic expansion holds

$$\mathcal{R}^* w(x_0, t) = t^{-1} \mathfrak{J}(t) \mathcal{R}^* h(x_0) + O_{t \rightarrow 0^+}(t^{-1} P(t)^{-2}). \quad (4.3)$$

Proof. For any $h \in D(A^{\frac{d}{2}+\frac{1}{2}})$, we have $(A + \mathcal{V}(p))^{-1} h \in D(A^{\frac{d}{2}+\frac{1}{2}})$ and Lemma 3.3 implies

$$\mathcal{R}^* (A + \mathcal{V}(p))^{-1} h(x_0) = \sum_{n=1}^{+\infty} \frac{\langle h, \varphi_n \rangle}{\lambda_n + \mathcal{V}(p)} \mathcal{R}^* \varphi_n(x_0).$$

Then [23, Lemmas 2.1 and 2.2] and Lebesgue dominated convergence theorem [7, Theorem 1.19, p. 28] imply

$$\mathcal{R}^*w(x_0,t) = \mathcal{R}^*S_2(t)h(x_0) = \frac{1}{2i\pi} \int_{\gamma(\delta,\theta)} e^{tp} \left(\sum_{n=1}^{+\infty} \frac{\langle h, \varphi_n \rangle}{\lambda_n + \mathcal{V}(p)} \mathcal{R}^* \varphi_n(x_0) \right) dp,$$

where $\delta > 0$ is arbitrary. By fixing $\delta = t^{-1}$ and changing variables, we deduce

$$\begin{aligned} \mathcal{R}^*w(x_0,t) &= \frac{1}{2i\pi} \int_{\gamma(t^{-1},\theta)} e^{tp} \left(\sum_{n=1}^{+\infty} \frac{\langle h, \varphi_n \rangle}{\lambda_n + \mathcal{V}(p)} \mathcal{R}^* \varphi_n(x_0) \right) dp \\ &= \frac{1}{2i\pi} \int_{\gamma(1,\theta)} e^p \left(\sum_{n=1}^{+\infty} \frac{t^{-1} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{\lambda_n + Q(t,p)} \right) dp. \end{aligned}$$

Next we fix $n \in \mathbb{N}$, $t \in (0, 1)$, $p \in \gamma(1, \theta)$. Direct computation gives

$$\begin{aligned} \frac{t^{-1} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{\lambda_n + Q(t,p)} &= \frac{t^{-1} Q(t,p)^{-1} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{1 + \lambda_n Q(t,p)^{-1}} \\ &= t^{-1} Q(t,p)^{-1} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0) - \int_0^1 \frac{t^{-1} Q(t,p)^{-2} \lambda_n \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{(1 + s \lambda_n Q(t,p)^{-1})^2} ds. \end{aligned}$$

Now summing this expression over $n \in \mathbb{N}$ and applying Lemma 3.3 lead to

$$\begin{aligned} &\sum_{n=1}^{+\infty} \frac{t^{-1} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{\lambda_n + Q(t,p)} \\ &= t^{-1} Q(t,p)^{-1} \sum_{n=1}^{+\infty} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0) - \sum_{n=1}^{+\infty} \int_0^1 \frac{t^{-1} Q(t,p)^{-2} \lambda_n \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{(1 + s \lambda_n Q(t,p)^{-1})^2} ds \\ &= t^{-1} Q(t,p)^{-1} \mathcal{R}^* h(x_0) - \sum_{n=1}^{+\infty} \int_0^1 \frac{t^{-1} Q(t,p)^{-2} \lambda_n \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{(1 + s \lambda_n Q(t,p)^{-1})^2} ds. \end{aligned}$$

Since $\theta \in (\frac{\pi}{2}, \min(\frac{\pi}{2b_2}, \pi))$ and $\text{supp}(\mu) \subset [b_1, b_2]$, for all $s \in [0, 1]$, $r \in [1, +\infty)$ and all $\beta \in (-\theta, \theta)$, we have

$$\begin{aligned} |1 + s \lambda_n Q(t, re^{i\beta})^{-1}| &= |Q(t, re^{i\beta})|^{-1} |Q(t, re^{i\beta}) + s \lambda_n| \\ &\geq |Q(t, re^{i\beta})|^{-1} |s \lambda_n + \Re(Q(t, re^{i\beta}))| \\ &\geq |Q(t, re^{i\beta})|^{-1} \left(s \lambda_n + \int_0^1 t^{-\alpha} r^\alpha \cos(\alpha\beta) \mu(\alpha) d\alpha \right) \\ &\geq |Q(t, re^{i\beta})|^{-1} \cos(b_2\theta) \int_{b_1}^{b_2} t^{-\alpha} r^\alpha \mu(\alpha) d\alpha \\ &\geq \cos(b_2\theta) |Q(t, re^{i\beta})|^{-1} |Q(t, re^{i\beta})| = \cos(b_2\theta). \end{aligned}$$

Therefore, applying again Lemma 3.3 and the above estimate lead to

$$\begin{aligned} &\left| \sum_{n=1}^{+\infty} \frac{t^{-1} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{\lambda_n + Q(t,p)} - t^{-1} Q(t,p)^{-1} \mathcal{R}^* h(x_0) \right| \\ &= \left| \sum_{n=1}^{+\infty} \int_0^1 \frac{t^{-1} Q(t,p)^{-2} \lambda_n \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{(1 + s \lambda_n Q(t,p)^{-1})^2} ds \right| \end{aligned}$$

$$\begin{aligned} &\leq t^{-1} \cos(b_2\theta)^{-2} |Q(t,p)|^{-2} \sum_{n=1}^{+\infty} \lambda_n |\langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)| \\ &\leq Ct^{-1} |Q(t,p)|^{-2} \|h\|_{D(A^{\frac{d+3}{2}})}, \end{aligned}$$

with the constant $C > 0$ depending only on A, θ and μ . This and the definition of $\gamma(\delta, \theta)$ imply

$$\begin{aligned} &\left| \mathcal{R}^* w(x_0, t) - \frac{\mathcal{R}^* h(x_0)}{2i\pi} \int_{\gamma(1, \theta)} e^p t^{-1} Q(t, p)^{-1} dp \right| \\ &\leq Ct^{-1} \|h\|_{D(A^{\frac{d+3}{2}})} \left(\int_{-\theta}^{\theta} e^{\cos(\beta)} |Q(t, e^{i\beta})|^{-2} d\beta + \int_1^{+\infty} e^{r \cos(\theta)} |Q(t, r e^{\pm i\theta})|^{-2} dr \right) \\ &:= Ct^{-1} \|h\|_{D(A^{\frac{d+3}{2}})} (I_1(t) + I_2(t)). \end{aligned}$$

Next we bound the two terms $I_1(t)$ and $I_2(t)$. Since $\theta \in (\frac{\pi}{2}, \min(\frac{\pi}{2b_2}, \pi))$ and $\text{supp}(\mu) \subset [b_1, b_2]$, we deduce that, for all $r \in [1, +\infty)$ and all $\beta \in (-\theta, \theta)$,

$$\begin{aligned} |Q(t, r e^{i\beta})| &\geq |\Re(Q(t, r e^{i\beta}))| = \int_{b_1}^{b_2} t^{-\alpha} r^\alpha \cos(\alpha\beta) \mu(\alpha) d\alpha \\ &\geq \cos(b_2\theta) r^{b_2} \int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha. \end{aligned} \tag{4.4}$$

Then it follows that $I_1(t) \leq C(\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha)^{-2}$ and $I_2(t) \leq C(\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha)^{-2}$. Hence

$$\begin{aligned} &\left| \mathcal{R}^* w(x_0, t) - \frac{\mathcal{R}^* h(x_0)}{2i\pi} \int_{\gamma(1, \theta)} e^p t^{-1} Q(t, p)^{-1} dp \right| \\ &\leq Ct^{-1} \|h\|_{D(A^{\frac{d+3}{2}})} \left(\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha \right)^{-2}. \end{aligned} \tag{4.5}$$

This proves the estimate (4.3), and completes the proof of the lemma. □

Next we consider the asymptotic behavior of $\mathcal{R}^* w(x_0, t)$ as $t \rightarrow +\infty$.

LEMMA 4.2. *Let $Q(t, p)$ and $P(t)$ be defined by (4.1), and let*

$$\mathfrak{K}(t) = \frac{1}{2i\pi} \int_{\gamma(1, \theta)} e^p Q(t, p) dp. \tag{4.6}$$

Then for $h \in D(A^{\frac{d}{2} + \frac{1}{2}})$, the following asymptotic expansion holds

$$\mathcal{R}^* w(x_0, t) = -t^{-1} \mathcal{R}^* A^{-2} h(x_0) \mathfrak{K}(t) + O_{t \rightarrow +\infty}(t^{-1} P(t)^2). \tag{4.7}$$

Proof. Fix $n \in \mathbb{N}, t \in (1, +\infty), p \in \gamma(1, \theta)$. Using Taylor formula, we find

$$\begin{aligned} \frac{t^{-1} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{\lambda_n + Q(t, p)} &= t^{-1} \lambda_n^{-1} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0) - t^{-1} \lambda_n^{-2} Q(t, p) \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0) \\ &\quad + 2 \int_0^1 \frac{t^{-1} Q(t, p)^2 \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0) (1-s)}{(\lambda_n + sQ(t, p))^3} ds. \end{aligned} \tag{4.8}$$

Moreover, for all $s \in [0, 1]$, we have

$$|\lambda_n + sQ(t, re^{i\beta})| \geq \lambda_n + s \int_0^1 \cos(\alpha\beta) r^\alpha t^{-\alpha} \mu(\alpha) d\alpha, \quad r \in [1, +\infty), \beta \in (-\theta, \theta),$$

and since $\text{supp}(\mu) \subset [b_1, b_2]$ and $\theta < \frac{\pi}{2b_2}$, we obtain

$$\begin{aligned} |\lambda_n + sQ(t, re^{i\beta})| &\geq \lambda_n + s \int_{b_1}^{b_2} \cos(\alpha\beta) r^\alpha t^{-\alpha} \mu(\alpha) d\alpha \\ &\geq \lambda_n + s \cos(b_2\theta) \int_{b_1}^{b_2} r^\alpha t^{-\alpha} \mu(\alpha) d\alpha \geq \lambda_n, \quad r \in [1, +\infty), \beta \in (-\theta, \theta). \end{aligned}$$

Now summing (4.8) over $n \in \mathbb{N}$ and applying Lemma 3.3 yield

$$\begin{aligned} &\left| \sum_{n=1}^{+\infty} \frac{t^{-1} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{\lambda_n + Q(t, p)} - t^{-1} \mathcal{R}^* A^{-1} h(x_0) + Q(t, p) t^{-1} \mathcal{R}^* A^{-2} h(x_0) \right| \\ &\leq \sum_{n=1}^{+\infty} \left| \frac{t^{-1} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0)}{\lambda_n + Q(t, p)} - \frac{\langle h, \varphi_n \rangle}{t\lambda_n} \mathcal{R}^* \varphi_n(x_0) + \frac{Q(t, p)}{t\lambda_n^2} \langle h, \varphi_n \rangle \mathcal{R}^* \varphi_n(x_0) \right| \\ &\leq Ct^{-1} |Q(t, p)|^2 \|h\|_{D(A^{\frac{d+1}{2}})} \leq C|p|^{2b_2} t^{-1} \left(\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha \right)^2 \|h\|_{D(A^{\frac{d+1}{2}})}. \end{aligned}$$

Integrating the expression in $p \in \gamma(1, \theta)$ gives

$$\begin{aligned} &\left| \mathcal{R}^* w(x_0, t) - \frac{\mathcal{R}^* A^{-1} h(x_0)}{t} \frac{1}{2i\pi} \int_{\gamma(1, \theta)} e^p dp + \frac{\mathcal{R}^* A^{-2} h(x_0)}{t} \frac{1}{2i\pi} \int_{\gamma(1, \theta)} e^p Q(t, p) dp \right| \\ &\leq Ct^{-1} \|h\|_{D(A^{\frac{d+1}{2}})} \left(\int_{-\theta}^{\theta} e^{\cos(\beta)} |Q(t, e^{i\beta})|^2 d\beta + \int_1^{+\infty} e^{r\cos(\theta)} |Q(t, re^{\pm i\theta})|^2 dr \right) \\ &\leq Ct^{-1} \|h\|_{D(A^{\frac{d+1}{2}})} \left(\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha \right)^2 \left(1 + \int_1^{+\infty} e^{r\cos(\theta)} r^{2b_2} dr \right) \\ &\leq Ct^{-1} \|h\|_{D(A^{\frac{d+1}{2}})} \left(\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha \right)^2. \end{aligned}$$

Meanwhile, the Cauchy formula gives $\frac{1}{2i\pi} \int_{\gamma(1, \theta)} e^p dp = 0$, and upon fixing $\mathfrak{K}(t)$ as (4.6), we get

$$|\mathcal{R}^* w(x_0, t) + t^{-1} \mathcal{R}^* A^{-2} h(x_0) \mathfrak{K}(t)| \leq Ct^{-1} \|h\|_{D(A^{\frac{d+1}{2}})} \left(\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha \right)^2,$$

and thus we obtain the estimate (4.7). □

We will also need the following two intermediate results.

LEMMA 4.3. *Let $\mathfrak{J}(t)$ and $\mathfrak{K}(t)$ be given by (4.2) and (4.6), respectively. Then, there exists a constant $C > 0$ depending only on μ and θ such that*

$$C^{-1} P(t)^{-1} \leq |\mathfrak{J}(t)| \leq C P(t)^{-1}, \quad t \in (0, 1), \tag{4.9}$$

$$C^{-1}P(t) \leq |\mathfrak{K}(t)| \leq CP(t), \quad t \in (1, +\infty). \tag{4.10}$$

Proof. It follows from the proof of Lemmas 4.1 and 4.2 that the following estimate holds $|\mathfrak{J}(t)| \leq CP(t)^{-1}$, for $t \in (0, 1)$. Moreover, we have

$$\begin{aligned} |\mathfrak{K}(t)| &\leq C \left(\int_{-\theta}^{\theta} e^{\cos(\beta)} |Q(t, e^{i\beta})| d\beta + \int_1^{+\infty} e^{r \cos(\theta)} |Q(t, r e^{\pm i\theta})| dr \right) \\ &\leq C \left(\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha \right) \left(1 + \int_1^{+\infty} e^{r \cos(\theta)} r^{b_2} dr \right) \leq C \int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha, \end{aligned}$$

which shows the following estimate

$$|\mathfrak{K}(t)| \leq CP(t), \quad t \in (1, +\infty).$$

So we only need to prove the following two estimates

$$C^{-1}P(t)^{-1} \leq |\mathfrak{J}(t)|, \quad t \in (0, 1) \quad \text{and} \quad C^{-1}P(t) \leq |\mathfrak{K}(t)|, \quad t \in (1, +\infty).$$

Note that

$$|Q(t, p)| \leq |p|^{b_2} \int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha, \quad p \in \gamma(1, \theta),$$

and, by repeating the arguments for Lemmas 4.1 and 4.2, we can prove the first estimate. In the same way, we obtain the second estimate from (4.4). \square

REMARK 4.1. Note that the argument in this section allows deriving asymptotics of the solutions to the distributed order model. Indeed, applying Lemmas 4.2 and 4.3, we have

$$|\mathcal{R}^* w(x_0, t)| \leq Ct^{-1}P(t) \|h\|_{D(A^r)}, \quad t > 1,$$

with $C > 0$ independent of t and h . In addition, we find

$$P(t) \leq \|\mu\|_{L^\infty(0,1)} \int_{b_1}^{b_2} t^{-\alpha} d\alpha \leq \|\mu\|_{L^\infty(0,1)} \left(\frac{t^{-b_1}}{\ln(t)} - \frac{t^{-b_2}}{\ln(t)} \right), \quad t > 1$$

and consequently

$$|\mathcal{R}^* w(x_0, t)| \leq C \frac{t^{-1-b_1}}{\ln(t)} \|h\|_{D(A^r)}, \quad t > 1.$$

Similarly, by fixing $\epsilon \in (0, (b_2 - b_1)/2)$, we obtain

$$P(t) \geq \left(\inf_{[b_1+\epsilon, b_2-\epsilon]} \mu \right) \int_{b_1+\epsilon}^{b_2-\epsilon} t^{-\alpha} d\alpha = \left(\inf_{[b_1+\epsilon, b_2-\epsilon]} \mu \right) \left(\frac{t^{-\epsilon-b_1}}{\ln(t)} - \frac{t^{-b_2+\epsilon}}{\ln(t)} \right), \quad t \in (0, 1)$$

and Lemmas 4.1 and 4.3 imply

$$|\mathcal{R}^* w(x_0, t)| \leq Ct^{-1}P(t)^{-1} \|h\|_{D(A^r)} \leq Ct^{b_2-1+\epsilon} |\ln(t)| \|h\|_{D(A^r)}, \quad t \in (0, 1).$$

Moreover, assuming that $b_2 = 1$ and $\mu(1) > 0$, we obtain

$$|\mathcal{R}^* w(x_0, t)| \leq C |\ln(t)| \|h\|_{D(A^r)}, \quad t \in (0, 1).$$

These estimates can be compared with that in [25] for the solution of problem (1.2) with $F \equiv 0$.

The next result gives the asymptotics of the integral $\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha$ as $t \rightarrow 0^+$ and $t \rightarrow +\infty$, which will play a role in determining the support of the weight μ .

LEMMA 4.4. *The following identities hold*

$$\lim_{t \rightarrow +\infty} t^b \int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha = 0, \quad b \in (-\infty, b_1), \tag{4.11}$$

$$\lim_{t \rightarrow +\infty} t^b \int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha = +\infty, \quad b \in (b_1, +\infty), \tag{4.12}$$

$$\lim_{t \rightarrow 0^+} \frac{t^{-b}}{\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha} = 0, \quad b \in (-\infty, b_2), \tag{4.13}$$

$$\lim_{t \rightarrow 0^+} \frac{t^{-b}}{\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha} = +\infty, \quad b \in (b_2, +\infty). \tag{4.14}$$

Proof. To show the identity (4.11), we fix $b \in (-\infty, b_1)$. Note that

$$\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha \leq t^{-b_1} \int_{b_1}^{b_2} \mu(\alpha) d\alpha, \quad t \in (1, +\infty). \tag{4.15}$$

Moreover, since $\mu \in C[0, 1]$ is non-negative with $\text{supp}(\mu) = [b_1, b_2]$, μ cannot be uniformly vanishing and $\int_{b_1}^{b_2} \mu(\alpha) d\alpha > 0$. Combining this with (4.15) gives (4.11). To show (4.12), we fix $b \in (b_1, +\infty)$. By (3.1) with $\alpha_0 = b_1$ and $\varepsilon = b - b_1$, there exist $\delta_0 \in (0, \varepsilon)$ and $\delta_1 \in (0, \delta_0)$ such that (3.1) is fulfilled. Then, it follows

$$\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha \geq \int_{b_1 + \delta_1}^{b_1 + \delta_0} t^{-\alpha} \mu(\alpha) d\alpha \geq (\delta_0 - \delta_1) \frac{\mu(b_1 + \delta_0)}{2} t^{-b_1 - \delta_1}, \quad t \in (1, +\infty).$$

Since $(\delta_0 - \delta_1) \frac{\mu(b_1 + \delta_0)}{2} > 0$ and $b_1 + \delta_1 < b_1 + \varepsilon = b$, we deduce easily (4.12) from this last estimate. For (4.13), we fix $b \in (b_1, b_2)$ and using (3.1) with $\alpha_0 = b$ and $\varepsilon = b_2 - b$, we obtain

$$\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha \geq \int_{b + \delta_1}^{b + \delta_0} t^{-\alpha} \mu(\alpha) d\alpha \geq (\delta_0 - \delta_1) \frac{\mu(b + \delta_0)}{2} t^{-b - \delta_0}, \quad t \in (0, 1),$$

with $\delta_0 \in (0, b_2 - b)$ and $\delta_1 \in (0, \delta_0)$. This and the inequality $(\delta_0 - \delta_1) \frac{\mu(b + \delta_0)}{2} > 0$ and the condition $\delta_0 > 0$ imply (4.13). Similarly, upon noting $\int_{b_1}^{b_2} t^{-\alpha} \mu(\alpha) d\alpha \leq t^{-b_2} \int_{b_1}^{b_2} \mu(\alpha) d\alpha$ for $t \in (0, 1)$, we deduce (4.14). \square

5. Proof of Theorems 2.1, 2.2 and 2.3

To prove Theorems 2.1 and 2.2, we need an intermediate result.

LEMMA 5.1. *Let (Ω, a, q, f, u_0) be an admissible tuple and let the condition*

$$a_{ij}(x) = \delta_{ij}, \quad x \in \partial\Omega, \tag{5.1}$$

be fulfilled. If $f \not\equiv 0$ and $u_0 \not\equiv 0$ are of constant sign and \mathcal{R} is the Neumann trace. Then, for any $x_0 \in \partial\Omega$ we have $\mathcal{R}^ A^{-2} f(x_0) \neq 0$ and $\mathcal{R}^* A^{-1} u_0(x_0) \neq 0$.*

Proof. We only show the result for $A^{-2}f$ with $f \leq 0$ since that for $A^{-1}u_0$ follows similarly. Let $v = A^{-2}f$. Assume the contradictory, i.e., there exists $x' \in \partial\Omega$ such that $v(x') = \mathcal{R}^*v(x') = 0$. Since $f \in D(A^r)$, $v \in D(A^{r+2})$ and, Sobolev embedding theorem and elliptic regularity theory [9, Theorem 2.5.1.1] imply $v \in C^1(\bar{\Omega})$. The maximum principle [8, Theorem 8.1] implies $A^{-1}f \leq 0$ and since $f \neq 0$, we have $A^{-1}f \neq 0$. If v is constant, then $qv = Av = A^{-1}f$ and since $A^{-1}f \neq 0$, $A^{-1}f \leq 0$ and q is strictly positive, the constant v is negative, contradicting $v(x') = 0$. Hence v is not a constant function. Now we consider the following two cases.

Case 1. $\max_{x \in \partial\Omega} v(x) = 0$. Since v is not constant, the strong maximum principle [8, Theorem 3.5] implies that for all $x \in \Omega$, $v(x) < 0 = v(x')$ and Hopf's lemma [8, Lemma 3.4] yields $\partial_\nu v(x') > 0$. By (5.1) and the condition $v \in D(A)$, this contradicts the condition $\partial_\nu v = \mathcal{R}v = 0$ on $\partial\Omega$.

Case 2. $\max_{x \in \partial\Omega} v(x) \neq 0$. Since $v(x') = 0$, we have $\max_{x \in \partial\Omega} v(x) > 0$. Since $v \in C(\partial\Omega)$, we can find $x_2 \in \partial\Omega$ such that $v(x_2) = \max_{x \in \partial\Omega} v(x) > 0$ and, since v is not constant, the strong maximum principle implies that for all $x \in \Omega$ we have $v(x) < v(x_2)$. Then, Hopf's lemma yields $\partial_\nu v(x_2) > 0$. This also contradicts the condition $\partial_\nu v = \mathcal{R}v = 0$ on $\partial\Omega$.

In both cases we have a contradiction and it follows that for any $x_0 \in \partial\Omega$, we have $\mathcal{R}^*A^{-2}f(x_0) \neq 0$. □

5.1. Proof of Theorem 2.1. We divide the proof into two steps.

Step 1. We prove that (2.5) implies $b_2^1 = b_2^2$. By Lemma 3.2, we have

$$\mathcal{R}_j^*u_j(x_j, t) = \int_0^t \sigma(s)\mathcal{R}_j^*[S_2^j(t-s)f_j](x_j)ds, \quad t \in (0, T), \quad j = 1, 2,$$

with S_2^j corresponding to (3.3) with $\mu = \mu_j$ and with $A = A_j$ (acting in $L^2(\Omega_j)$) and $\mathcal{R} = \mathcal{R}_j$. Let $v_j(t) = \mathcal{R}_j^*[S_2^j(t)f_j](x_j)$ as a function in $L^1(0, T)$, cf. Lemma 3.2. Therefore, condition (2.5) implies

$$\int_0^t \sigma(s)[v_1(t-s) - v_2(t-s)]ds = 0, \quad t \in (0, T).$$

By Titchmarsh convolution theorem [37, Theorem VII], there exist $t_1, t_2 \in [0, T]$ such that $t_1 + t_2 \geq T$, $\sigma|_{(0, t_1)} \equiv 0$ and $(v_1 - v_2)|_{(0, t_2)} \equiv 0$. Since $\sigma|_{(0, T)} \not\equiv 0$, we have $t_1 < T$. This proves $t_2 = t_1 + t_2 - t_1 \geq T - t_1 > 0$ and $\mathcal{R}_1^*[S_2^1(t)f_1](x_1) = \mathcal{R}_2^*[S_2^2(t)f_2](x_2)$ for $t \in (0, t_2)$. The analyticity of the map $\mathbb{R}_+ \ni t \mapsto \mathcal{R}_j^*[S_2^j(t)f_j](x_j)$, cf. Lemma 3.2, implies $\mathcal{R}_1^*[S_2^1(t)f_1](x_1) = \mathcal{R}_2^*[S_2^2(t)f_2](x_2)$ for $t \in \mathbb{R}_+$. Let $w_j(\cdot, t) = S_2^j(t)f_j$. Then

$$\mathcal{R}_1^*w_1(x_1, t) = \mathcal{R}_2^*w_2(x_2, t), \quad t \in \mathbb{R}_+. \tag{5.2}$$

Now we deduce from Lemma 4.1 that

$$\mathcal{R}_j^*w_j(x_j, t) = t^{-1}\mathfrak{J}_j(t)\mathcal{R}_j^*f_j(x_j) + \underset{t \rightarrow 0^+}{O}(t^{-1}P_j(t)^{-2}),$$

with $Q_j(t, p) := \int_0^1 t^{-\alpha} p^\alpha \mu_j(\alpha) d\alpha$, $P_j(t) = \int_{b_2^j}^{b_1^j} t^{-\alpha} \mu_j(\alpha) d\alpha$ and $\mathfrak{J}_j(t) := \frac{1}{2i\pi} \int_{\gamma(1, \theta)} e^p Q_j(t, p)^{-1} dp$. Therefore, (5.2) implies

$$t^{-1}\mathfrak{J}_1(t)\mathcal{R}_1^*f_1(x_1) + \underset{t \rightarrow 0^+}{O}(t^{-1}P_1(t)^{-2}) = t^{-1}\mathfrak{J}_2(t)\mathcal{R}_2^*f_2(x_2) + \underset{t \rightarrow 0^+}{O}(t^{-1}P_2(t)^{-2}). \tag{5.3}$$

Combining this with (2.4), (4.9) and (4.13)-(4.14) yields $b_2^1 = b_2^2$.

Step 2. Now, we fix $b_2 = b_2^1 = b_2^2$, and assume that (2.6) holds. Then we prove that condition (2.5) implies $b_1^1 = b_1^2$. The estimate (5.2) and Lemma 4.2 imply

$$\begin{aligned} & -t^{-1}\mathcal{R}_1^*A_1^{-2}f_1(x_1)\mathfrak{K}_1(t) + \underset{t \rightarrow +\infty}{O} (t^{-1}P_1(t)^2) \\ & = -t^{-1}\mathcal{R}_1^*A_2^{-2}f_2(x_2)\mathfrak{K}_2(t) + \underset{t \rightarrow +\infty}{O} (t^{-1}P_2(t)^2), \end{aligned} \tag{5.4}$$

with $\mathfrak{K}_j(t) = \frac{1}{2i\pi} \int_{\gamma(1,\theta)} e^{pt} Q_j(t,p) dp$. By conditions (4.9) and (4.11)-(4.12), this implies $b_1^1 = b_1^2$ if

$$\mathcal{R}_j^*A_j^{-2}f_j(x_j) \neq 0, \quad j = 1, 2. \tag{5.5}$$

So it remains to prove (5.5). If \mathcal{R}_j is the Neumann trace and condition (2.3) holds, Lemma 5.1 implies (5.5). Similarly, by [13, Lemma 3.1], if \mathcal{R}_j is the Dirichlet trace, condition (2.6) implies

$$\partial_{\nu^j} A_j^{-2} f_j(x_j) \neq 0, \quad j = 1, 2, \tag{5.6}$$

with ν^j the unit outward normal to $\partial\Omega_j$. Meanwhile, since \mathcal{R}_j is the Dirichlet trace, letting the matrix $B_j = (a_{k\ell}^j(x_j))_{k,\ell=1}^d$ and since $A_j^{-2}f_j = 0$ on $\partial\Omega_j$, we get

$$\begin{aligned} \mathcal{R}_j^*A_j^{-2}f_j(x_j) &= \partial_{\nu_{\alpha_j}} A_j^{-2} f_j(x_j) = [\nabla A_j^{-2} f_j(x_j)] \cdot [B_j \nu^j(x_j)] \\ &= [(\partial_{\nu^j} A_j^{-2} f_j(x_j)) \nu^j(x_j)] \cdot [B_j \nu^j(x_j)] = (\partial_{\nu^j} A_j^{-2} f_j(x_j)) \nu^j(x_j)^T B_j \nu^j(x_j). \end{aligned}$$

Moreover, it follows from the ellipticity condition (1.1) that

$$\nu^j(x_j)^T B_j \nu^j(x_j) = \sum_{k,\ell=1}^d a_{k,\ell}^j(x_j) \nu_k^j(x_j) \nu_\ell^j(x_j) \geq c |\nu^j(x_j)|^2 = c > 0.$$

This and (5.6) imply (5.5). Thus, in both cases, (5.5) holds and we deduce $b_1^1 = b_1^2$.

5.2. Proof of Theorem 2.2. Note that condition (2.2) implies $u^j(t, \cdot) = S_1^j(t)u_0^j$ for $t \in (0, T)$ with S_1^j corresponding to (3.2) with $\mu = \mu_j$ and with $A = A_j$ (acting in $L^2(\Omega_j)$) and with $\mathcal{R} = \mathcal{R}_j$. Since $u_0^j \in D(A_j)$, the argument of [15, pp. 21-22] implies $u^j \in W^{1,1}(0, T; L^2(\Omega_j))$ with $\partial_t u^j(t, \cdot) = -S_2^j(t)A_j u_0^j$. Since $A_j u_0^j \in D(A_j^r)$, $j = 1, 2$, repeating the argument of Lemma 3.2 yields $u^j \in W^{1,1}(0, T; D(A_j^r))$. Then Lemma 3.3 implies

$$\partial_t \mathcal{R}_j^* u^j(t, x_j) = \mathcal{R}_j^* \partial_t u^j(t, x_j) = -\mathcal{R}_j^* S_2^j(t) A_j u_0^j(x_j), \quad t \in (0, T).$$

Therefore, differentiating condition (2.5) in t gives

$$\mathcal{R}_1^* S_2^1(t) A_1 u_0^1(x_1) = \mathcal{R}_2^* S_2^2(t) A_2 u_0^2(x_2), \quad t \in (T_1, T_2).$$

This and the time analyticity of the map $t \mapsto \mathcal{R}_j^* S_2^j(t) A_j u_0^j(x_j)$, cf. Lemma 3.2, imply (5.2) with $w_j(\cdot, t) = S_2^j(t) A_j u_0^j$. Then, repeating the argument for Step 1 of the proof of Theorem 2.1 with $f_j = A_j u_0^j$ yields $b_2^1 = b_2^2$. In addition, if condition (2.9) holds and \mathcal{R}_k is the Dirichlet trace, [13, Lemma 3.1] implies that for $f_j = A_j u_0^j$, condition (5.6) holds. Then, similar to Step 2 of the proof of Theorem 2.1, we deduce (5.5). Similarly, if \mathcal{R}_j is the Neumann trace and condition (2.3) holds, Lemma 5.1 implies (5.5). Hence, in both cases we deduce from (5.4) that $b_1^1 = b_1^2$.

5.3. Proof of Theorem 2.3. For $F \in L^1(\mathbb{R}_+; L^2(\Omega))$, problem (1.2) admits a unique weak solution $u \in L^1_{loc}(\mathbb{R}_+; L^2(\Omega))$ given by (3.4) [15, Theorem 1.4], and the first statement of Lemma 3.1 and the results of Lemma 3.2 are still valid ([23, Theorem 1.4] and [15, Lemma 3.1]). Now we can prove Theorem 2.3.

Proof. We prove the result for $u_0 \equiv 0$ and $f \neq 0$ of constant sign. The result for $f \equiv 0$ and $u_0 \neq 0$ of constant sign follows similarly as Theorem 2.2. The argument of Theorem 2.1 yields (5.2) with $x_1 = x_2 = x_0$ and $w_j(\cdot, t) = S_2^j(t)f$, $j = 1, 2$, with S_2^j defined in (3.3) with $\mu = \mu_j$. For all $p \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Re(z) > 0\}$, the Laplace transform $\widehat{w}_j(\cdot, p) = \int_0^{+\infty} e^{-pt} w_j(\cdot, t) dt$ of w_j is well defined as an element of $H^2(\Omega)$ ([15, Proposition 3.2] and [23]), and we have

$$\widehat{w}_j(\cdot, p) = (A + \mathcal{V}_j(p))^{-1} f = \sum_{n=1}^{+\infty} \frac{\langle f, \varphi_n \rangle}{\lambda_n + \mathcal{V}_j(p)} \varphi_n, \quad \text{with } \mathcal{V}_j(p) = \int_0^1 p^\alpha \mu_j(\alpha) d\alpha. \quad (5.7)$$

Moreover, the argument for Lemma 3.2 and [15, Lemma 3.1] give

$$\|w_j(\cdot, t)\|_{C^1(\overline{\Omega})} \leq C \|w_j(\cdot, t)\|_{D(A^r)} \leq C \max(t^{\alpha_0 - \varepsilon - 1}, t^{\alpha_0 - 1}) \|f\|_{D(A^r)}, \quad t > 0,$$

and hence $\mathcal{R}^* \widehat{w}_j(x_0, \cdot)(p)$ is well defined for $p \in \mathbb{C}_+$, and fixing $v_j(t) = \mathcal{R}^* w_j(x_0, t)$ leads to $\widehat{v}_j(p) = \mathcal{R}^* \widehat{w}_j(x_0, p)$ for $p \in \mathbb{C}_+$. This, (5.7) and Lemma 3.3 imply

$$\widehat{v}_j(p) = \sum_{n=1}^{+\infty} \frac{\langle f, \varphi_n \rangle}{\lambda_n + \mathcal{V}_j(p)} \mathcal{R}^* \varphi_n(x_0), \quad p \in \mathbb{C}_+, \quad j = 1, 2.$$

Therefore, applying the Laplace transform in $t \in \mathbb{R}_+$ to the identity (5.2) gives

$$\sum_{n=1}^{+\infty} \frac{\langle f, \varphi_n \rangle}{\lambda_n + \mathcal{V}_1(p)} \mathcal{R}^* \varphi_n(x_0) = \sum_{n=1}^{+\infty} \frac{\langle f, \varphi_n \rangle}{\lambda_n + \mathcal{V}_2(p)} \mathcal{R}^* \varphi_n(x_0), \quad p \in \mathbb{C}_+,$$

which implies

$$(\mathcal{V}_1(p) - \mathcal{V}_2(p)) \sum_{n=1}^{+\infty} \frac{\langle f, \varphi_n \rangle}{(\lambda_n + \mathcal{V}_1(p))(\lambda_n + \mathcal{V}_2(p))} \mathcal{R}^* \varphi_n(x_0) = 0, \quad p \in \mathbb{R}_+. \quad (5.8)$$

Let

$$\mathcal{G}(p) := \sum_{n=1}^{+\infty} \frac{\langle f, \varphi_n \rangle}{(\lambda_n + \mathcal{V}_1(p))(\lambda_n + \mathcal{V}_2(p))} \mathcal{R}^* \varphi_n(x_0), \quad p \in \mathbb{R}_+.$$

By Lemma 3.3, $\mathcal{G} \in C([0 + \infty))$ with

$$\mathcal{G}(0) = \sum_{n=1}^{+\infty} \frac{\langle f, \varphi_n \rangle}{(\lambda_n + \mathcal{V}_1(0))(\lambda_n + \mathcal{V}_2(0))} \mathcal{R}^* \varphi_n(x_0) = \sum_{n=1}^{+\infty} \frac{\langle f, \varphi_n \rangle}{(\lambda_n)^2} \mathcal{R}^* \varphi_n(x_0) = \mathcal{R}^* A^{-2} f(x_0).$$

Then, Lemma 5.1, [13, Lemma 3.1] and the argument at Step 2 of Theorem 2.1 imply $|\mathcal{G}(0)| > 0$ for \mathcal{R} being either Dirichlet or Neumann trace. Hence, there exists $\varepsilon \in \mathbb{R}_+$ such that $|\mathcal{G}(p)| > 0$, $p \in (0, \varepsilon)$. This, (5.8) and fixing $\mu = \mu_1 - \mu_2$ yield

$$\int_0^1 p^\alpha \mu(\alpha) d\alpha = 0, \quad p \in (0, \varepsilon). \quad (5.9)$$

Let μ_* be the extension of μ by zero to \mathbb{R}_+ . Then it follows from (5.9) that

$$\widehat{\mu}_*(-\ln(\tau)) = \int_0^{+\infty} e^{\alpha \ln(\tau)} \mu_*(\alpha) d\alpha = \int_0^1 \tau^\alpha \mu(\alpha) d\alpha = 0, \quad \tau \in (0, \varepsilon),$$

which implies

$$\widehat{\mu}_*(p) = 0, \quad p \in (-\ln(\varepsilon), +\infty). \tag{5.10}$$

Since $\mu_* \in L^1(\mathbb{R}_+)$ is compactly supported, $\mathbb{C} \ni p \mapsto \widehat{\mu}_*(p)$ is holomorphic and (5.10) implies $\widehat{\mu}_*(p) = 0$ for all $p \in \mathbb{C}$. Then, the injectivity of the Laplace transform implies $\mu_* \equiv 0$, i.e., $\mu_1 = \mu_2$. \square

6. Numerical results and discussions

In this section, we numerically illustrate the analysis, including both support recovery in Theorems 2.1–2.2 and weight recovery in Theorem 2.3. The exact data $g^\dagger = \mathcal{R}^* u(x_0, t)$, $x_0 \in \partial\Omega$, is generated by a fully discrete scheme described in the appendix. The noisy data g^δ is generated by $g^\delta(t) = g^\dagger(t) + \epsilon \|g^\dagger\|_{L^\infty(0, T)} \xi(t)$, where $\xi(t)$ follows the standard Gaussian distribution, and $\epsilon \geq 0$ denotes the noise level.

6.1. Bound recovery. First we illustrate Theorems 2.1 and 2.2 with the following example, where $\mathbb{1}_S$ denotes the characteristic function of the set S .

EXAMPLE 6.1. In this example, we take the domain $\Omega = (0, 1)$, with

- (i) $u_0(x) = x(1-x)e^x$, $f \equiv 0$, and $a(x) = 1 + x^2$.
- (ii) $u_0 \equiv 0$, $f(x, t) = x(1-x)e^x \mathbb{1}_{[0, 1]}(t)$, and $a(x) = 1 + \sin(\pi x)$.

The direct problem (1.2) is equipped with a zero Dirichlet boundary condition. The following five weights are tested: $\mu_1 = \mathbb{1}_{[0.2, 0.8]}$, $\mu_2 = \mathbb{1}_{[0.2, 0.6]}$, $\mu_3 = \mathbb{1}_{[0.2, 0.4]}$, $\mu_4 = \mathbb{1}_{[0.4, 0.8]}$, and $\mu_5 = \mathbb{1}_{[0.4, 0.8]}$. The weights share either upper or lower bounds to facilitate the comparison of the asymptotics.

The measured data $g(t) = \partial_{\nu_a} u(x_0, t)$, at $x_0 = 0$, are shown in Figure 6.1. For small time, we plot $|g(t) - g(0)|$ in order to show the asymptotics more clearly. In case (i), the asymptotics of $|g(t) - g(0)|$ (noting $g(0) = 1$) are identical for the weights μ_1 , μ_4 and μ_5 but they differ markedly from that for μ_2 and μ_3 . This agrees well with the analysis: the upper bound b_2 of the weight support determines the small-time asymptotics. Indeed, the upper bounds of μ_1 , μ_4 and μ_5 are identical (all equal to 0.8), whereas that of μ_2 and μ_3 are different. A similar behavior is observed for case (ii). In contrast, for large time, the asymptotic of $|g(t)|$ is nearly identical for the weights μ_1 , μ_2 and μ_3 , which differs from that for μ_4 and μ_5 . This again agrees with the analysis: the lower bound b_1 determines the large-time asymptotics. Indeed, the lower bounds of μ_1 , μ_2 and μ_3 are identical (all equal to 0.2), whereas that of μ_4 and μ_5 are different. This observation is the same for case (ii), except the small transition near $t = 0$. In sum, the results confirm the asymptotic behavior.

Next we recover the support bound. This is more challenging, similar to that for multi-term case [12]. By the asymptotics in Lemma 4.4, we may recover the bound b by minimizing

$$J(b, c_0, c_1) = \|c_0 + c_1 t^{-b} - h(x_0, t)\|_{L^2(t_1, t_2)}^2,$$

with small (t_1, t_2) for the upper bound b_2 , and large t_1, t_2 for the lower bound b_1 . The numerical results for Example 6.1(ii) are given in Table 6.1, where the lower and upper

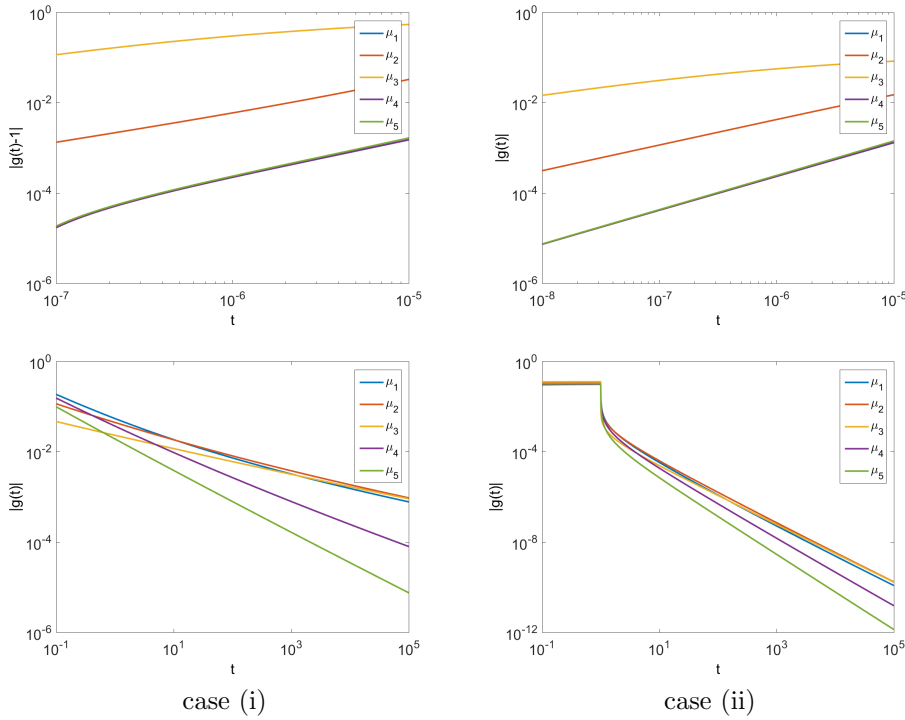


FIG. 6.1. The small (top) and large (bottom) time asymptotics of the data $g(t) = \partial_\nu u(x_0, t)$ for Example 6.1, at $x_0 = 0$.

bounds are estimated from the observation over the intervals $(1e-6, 1e-5)$ and $(1e4, 1e5)$, respectively. Note that the estimated lower bound is slightly more accurate than the estimated upper bound. The results for case (i) are not shown, since the upper bound estimates are very inaccurate, similar to what was observed in the multi-term case [12]. This failure might be attributed to the fact that the interval $(1e-6, 1e-5)$ is still too large. This can also be seen from Figure 6.1, where the curves are nearly parallel to the x -axis, indicating a quasi-steady state. Nonetheless, the lower bound estimates are largely the same with that for case (ii).

	μ_1	μ_2	μ_3	μ_4	μ_5
b_1	0.30 (0.20)	0.30 (0.20)	0.27 (0.20)	0.48 (0.40)	0.66 (0.60)
b_2	0.75 (0.80)	0.55 (0.60)	0.16 (0.40)	0.75 (0.80)	0.76 (0.80)

TABLE 6.1. The upper and lower bounds estimated from $g(t)$ for Example 6.1(ii).

6.2. Weight recovery. Now we recover the weight μ from the observation $\mathcal{R}^*u(\mu)(x_0, t)$, $0 < t < T$, where $u(\mu)$ denotes the solution to problem (1.2), when the medium is known. We employ an iterative method to approximately minimize

$$J(\mu) := \frac{1}{2} \|\mathcal{R}^*u(\mu)(x_0, \cdot) - g^\delta\|_{L^2(0, T)}^2. \tag{6.1}$$

Here one may add a penalty in order to promote distinct features, e.g., total variation for discontinuous weight, which however is not investigated below. To minimize $J(\mu)$, we employ the conjugate gradient method [1], which is a regularization method when equipped with early stopping. To apply the method, one needs the gradient $J'(\mu)$, which can be evaluated efficiently using the adjoint technique [5]. Assuming \mathcal{R} being the Neumann trace, let v be the solution to the following adjoint problem

$$\begin{cases} {}^R\partial_T^{[\mu]}v + \mathcal{A}v = 0, & \text{in } \Omega \times (0, T), \\ \partial_{\nu_a}v(x, t) = (u(\mu) - g^\delta)\delta_{x_0}, & \text{on } \partial\Omega \times (0, T), \\ {}_tI_T^{[\mu]}v(x, T) = 0, & \text{in } \Omega, \end{cases} \tag{6.2}$$

where the notation ${}_tI_T^{[\mu]}v$ and ${}^R\partial_T^{[\mu]}v$ denote the right-sided Riemann-Liouville fractional integral and derivative of v (with respect to μ), defined by ${}_tI_T^{[\mu]}v(t) = \int_t^T K_\mu(s - t)v(s) ds$ and ${}^R\partial_T^{[\mu]}v(t) = -\frac{d}{dt}{}_tI_T^{[\mu]}u(s)$. Note that the Neumann data in (6.2) involves a Dirac delta function $\delta_{x_0}(x)$ for some $x_0 \in \partial\Omega$ and it should be interpreted in the sense of distribution. When implementing the Galerkin FEM, one should evaluate the integral on the boundary carefully, which results in a term supported at x_0 only, cf. (6.6) below. By changing variables and using the zero terminal data, problem (6.2) can be transformed into an initial value problem with a Djrbashian-Caputo type derivative and zero initial data, which can then be solved using the standard scheme in the appendix.

We have the following representation of the gradient $J'(\mu)$.

PROPOSITION 6.1. *The gradient $J'(\mu)$ of the functional $J(\mu)$ is given by*

$$J'(\mu) : (0, 1) \ni \alpha \mapsto - \int_0^T \int_\Omega v \partial_t^\alpha u dx dt, \tag{6.3}$$

where v is the solution to the adjoint problem (6.2) and, for all $\alpha \in (0, 1)$, $\partial_t^\alpha u$ is defined by $\partial_t^\alpha u(x, t) := \partial_t \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} (u(x, s) - u_0(x)) ds$, for $(x, t) \in \Omega \times (0, T)$.

Proof. The derivation follows by a standard procedure. Let $u \equiv u(\mu)$. By definition, the directional derivative $J'(\mu)[h]$ of the functional J with respect to the weight μ in the direction $h \in L^2(0, 1)$ is given by

$$J'(\mu)[h] = (u_h(x_0, t), u(x_0, t) - g^\delta(t))_{L^2(0, T)},$$

where u_h is the derivative of u with respect to the weight μ in the direction h . Clearly u_h satisfies

$$\begin{cases} \partial_t^{[\mu]}u_h + \mathcal{A}u_h = - \int_0^1 h(\alpha) \partial_t^\alpha u d\alpha, & \text{in } \Omega \times (0, T), \\ \partial_{\nu_a}u_h(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \\ u_h(x, 0) = 0, & \text{in } \Omega. \end{cases} \tag{6.4}$$

Multiplying (6.4) with a test function $\phi(x, t)$, integrating over $\Omega \times (0, T)$ and then applying integration by parts yield

$$\int_0^T \int_\Omega \left(\int_0^1 \partial_t^\alpha u_h \mu(\alpha) d\alpha \right) \phi + a \nabla u_h \cdot \nabla \phi dx dt = \int_0^T \int_\Omega \left(- \int_0^1 \partial_t^\alpha u h(\alpha) d\alpha \right) \phi dx dt. \tag{6.5}$$

Meanwhile, the weak formulation for the adjoint solution v is given by

$$\int_0^T \int_{\Omega} \left(\int_0^1 {}^R_t \partial_T^\alpha v \mu(\alpha) d\alpha \right) \phi + a \nabla v \cdot \nabla \phi dx dt = \int_0^T (u(\mu)(x_0, t) - g^\delta) \phi(x_0, t) dt, \quad (6.6)$$

where we recall that for all $\alpha \in (0, 1)$, ${}^R_t \partial_T^\alpha v$ is defined by

$${}^R_t \partial_T^\alpha v(x, t) := -\partial_t \left(\int_t^T \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} v(x, s) ds \right), \quad (x, t) \in \Omega \times (0, T).$$

Since $u_h(0) = 0$ and $v(T) = 0$, the following identity holds ([16, p. 76, Lemma 2.7] or [11, Lemma 2.6])

$$\int_0^T \int_{\Omega} v \partial_t^\alpha u_h dx dt = \int_0^T \int_{\Omega} u_h {}^R_t \partial_T^\alpha v dx dt, \quad \alpha \in (0, 1).$$

Using this identity, taking $\phi = v$ in (6.5) and $\phi = u_h$ in (6.6) and subtracting the resulting identities give

$$\int_0^1 \left(-\int_0^T \int_{\Omega} \partial_t^\alpha uv dx dt \right) h(\alpha) d\alpha = \int_0^T u_h(x_0, t) (u(\mu)(x_0, t) - g^\delta) dt.$$

This and the definition of the derivative $J'(\mu)$ show the desired assertion. □

Now we can describe the conjugate gradient method [1] for minimizing the functional J ; see Algorithm 1 for the complete procedure. In the algorithm, Steps 6-7 compute the conjugate descent direction, and Step 9 computes the optimal step size using the sensitivity problem. The operator P_+ at Step 10 gives the projection into the positive quadrant. For each update, the algorithm involves three forward solves (direct, adjoint and sensitivity problems), which represent the main computational effort. For the stopping criterion at Step 11, one may employ the discrepancy principle [10, 32]. Note that this algorithm has been adopted in [28] recently, where the gradient is computed using a finite difference approximation, which is generally less efficient than the adjoint technique. In the experiments, we smooth the gradient $J'(\mu)$ by the negative Dirichlet Laplacian, which is analogous to the $H^1(0, 1)$ penalty in [28]. Accordingly, at Step 6 of the algorithm, we compute the conjugate coefficient γ in the $H^1(0, 1)$ -seminorm, where ∂_α denotes taking the first-order derivative in α . This approach is suitable for recovering a smooth weight.

The next example illustrates the weight recovery.

EXAMPLE 6.2. In this example, we take the domain $\Omega = (0, 1)$, terminal time $T = 1$, diffusion coefficient $a(x) = 1 + x(1 - x)$, initial condition $u_0(x) = x(1 - x)e^x$. The direct problem (1.2) is equipped with the Neumann boundary condition, which at $x = 0$ and $x = 1$ are taken to be 0 and 1, respectively. Consider the following two weights: (i) $\mu(\alpha) = \alpha(1 - \alpha)^2 e^{2\alpha}$ and (ii) $\mu(\alpha) = 2\min(\alpha, 1 - \alpha)$. The Dirichlet data at the point $x_0 = 0$ is used for the recovery. Algorithm 1 is initialized with $\mu^0 = \frac{1}{100} \sin(\pi\alpha)$.

The numerical results are shown in Table 6.2 and Figures 6.2 and 6.3, where the stopping index k^* (in the bracket) is taken such that the $L^2(0, 1)$ error $e = \|\hat{\mu} - \mu^\dagger\|_{L^2(0, 1)}$ of the recovered weight $\hat{\mu}$ (with respect to the exact one μ^\dagger) is smallest. The weight μ can be fairly reconstructed for the noise level ϵ up to 5%, agreeing with the results in

Algorithm 1 Conjugate gradient method for minimizing the functional J .

- 1: Initialize μ^0 , and set $k=0$.
- 2: **for** $k=0, \dots, K$ **do**
- 3: Solve for u^k from problem (1.2) with $\mu = \mu^k$.
- 4: Solve for v^k from problem (6.2) with $r^k = u^k(x_0, t) - g^\delta$.
- 5: Compute the gradient $J'(\mu^k)$ by (6.3).
- 6: Compute the conjugate coefficient γ^k by $\gamma^0 = 0$ and

$$\gamma^k = \frac{\|\partial_\alpha J'(\mu^k)\|_{L^2(0,1)}^2}{\|\partial_\alpha J'(\mu^{k-1})\|_{L^2(0,1)}^2}, \quad k \geq 1.$$

- 7: Compute the conjugate direction d^k by $d^k = -J'(\mu^k) + \gamma^k d^{k-1}$.
- 8: Solve for u_{d^k} from problem (6.4) with $h = d^k$.
- 9: Compute the step size s^k by

$$s^k = -\frac{(u_{d^k}(x_0, \cdot), r^k)_{L^2(0,T)}}{\|u_{d^k}(x_0, \cdot)\|_{L^2(0,T)}^2}.$$

- 10: Update the source component $\mu^{k+1} = P_+(\mu^k + s^k d^k)$.
 - 11: Check the stopping criterion.
 - 12: **end for**
-

ϵ	0	1e-3	3e-3	1e-2	3e-2	5e-2
(i)	2.50e-3(22)	4.09e-3(22)	8.82e-3(50)	1.62e-2(20)	4.16e-3(17)	9.33e-3(17)
(ii)	4.49e-2(26)	4.48e-2(26)	4.63e-2(20)	5.58e-2(18)	4.96e-2(17)	5.99e-2(14)

TABLE 6.2. The $L^2(0,1)$ error of the recovered weight $\hat{\mu}$ for Example 6.2 at different noise levels.

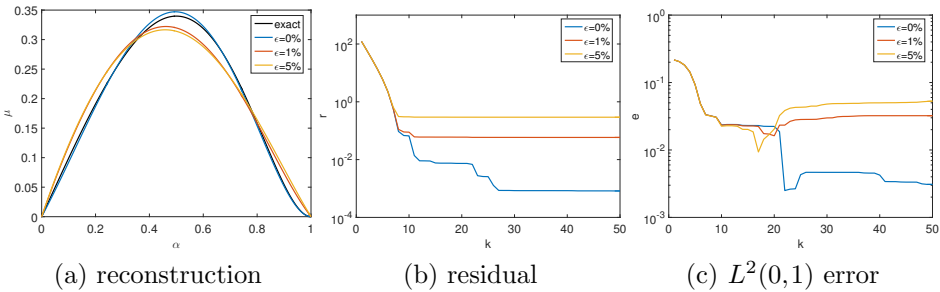


FIG. 6.2. Numerical results for Example 6.2(i) by the conjugate gradient method.

Table 6.2. The conjugate gradient method converges steadily, irrespective of the noise level. However, early stopping is needed in order to obtain high-quality reconstructions, due to the typical semiconvergence behaviour of the method: the error first decreases steadily and then it increases as the iteration further proceeds. This observation holds for both smooth and nonsmooth weights, which shows the feasibility of recovering the weight μ from the observation at one single point, when all other problem data are fully known, confirming the assertion of Theorem 2.3.

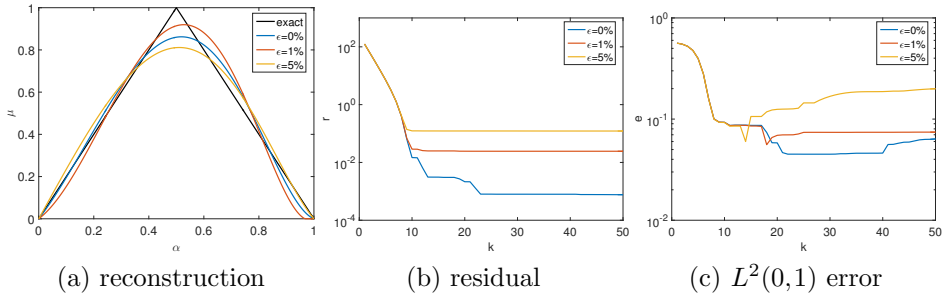


FIG. 6.3. Numerical results for Example 6.2(ii) by the conjugate gradient method.

7. Conclusion

In this work we have investigated the inverse problem of recovering information about the weight in a distributed-order time-fractional diffusion model from one-point observation on the boundary. We prove the support bound recovery when the medium is unknown, and the weight recovery when the medium is known. The latter result complements an existing uniqueness result for a known medium. Further we have also presented numerical experiments to complement the analysis, partly confirming the unique recovery of either support boundary or weight. The experiments show the significant computational challenge with accurately recovering support bounds and weights, especially the upper bound of the support. Theoretically, it is of much interest to study the case of a nonseparable source. This would require alternative analysis techniques than Titchmarsh convolution theorem, which has played a crucial role in the analysis of this work.

Appendix. Numerical scheme for the direct problem (1.2). Now we describe a numerical scheme for approximating the direct problem (1.2) for completeness; see the book [14] for further details on the numerical approximation of the standard time-fractional diffusion model. For the spatial discretization, we employ the standard Galerkin finite element method with the continuous piecewise linear finite element space X_h , subordinated to a shape regular triangulation of the domain Ω . Let M_h and S_h be the corresponding mass and stiffness matrices, respectively. For any $N \in \mathbb{N}$ total number of time steps, let $\tau = N^{-1}T$ be the time step size, and $t_n = n\tau$, $n = 0, \dots, N$, the time grid. The classical L1 scheme (cf. [27] or [14, Section 4.1]) approximates the Djrbashian-Caputo fractional derivative $\partial_t^\alpha u(t_n)$, $n \geq 1$, by

$$\partial_t^\alpha u^n \approx \sum_{j=0}^n b_{j,n}^{(\alpha)} u^{n-j},$$

with the weights $b_{j,n}^{(\alpha)}$ given by

$$b_{j,n}^{(\alpha)} = \frac{1}{\Gamma(2-\alpha)\tau^\alpha} \begin{cases} 1, & j=0, \\ (j+1)^{1-\alpha} + (j-1)^{1-\alpha} - 2j^{1-\alpha}, & j=1, \dots, n-1, \\ n^{1-\alpha} - (n-1)^{1-\alpha}, & j=n. \end{cases}$$

For the distributed-order fractional derivative $\partial_t^{[\mu]} u^n$, one may use quadrature to approximate the integral with respect to the weight $\mu(\alpha)$. For example, the trapezoidal

rule gives

$$\partial_t^{[\mu]} u^n = \int_0^1 \partial_t^\alpha u(t_n) \mu(\alpha) d\alpha \approx \delta_\alpha \sum_{i=0}^{N_\alpha} c_i \mu(\alpha_i) \partial_t^{\alpha_i} u^n,$$

with $c_0 = c_{N_\alpha} = 1/2$, and $c_i = 1$ for $i = 1, \dots, N_\alpha - 1$, and $\delta_\alpha = N_\alpha^{-1}$. These two approximations lead to

$$\partial_t^{[\mu]} u^n \approx \sum_{j=0}^n p_{n-j}^{[\mu]} (u^j - u^0), \quad \text{with } p_j^{[\mu]} = \delta_\alpha \sum_{i=0}^{N_\alpha} c_i \mu(\alpha_i) b_{j,n}^{(\alpha_i)}.$$

Note that the complexity of the scheme is comparable with the standard one, except computing the weights $p_j^{[\mu]}$. The fully discrete scheme reads: with $U_h^0 = P_h u_0$ and $F_h^n = P_h f(t_n)$ (with P_h being the $L^2(\Omega)$ projection on X_h), find $U_h^i \in X_h$ such that for $n = 1, 2, \dots, N$,

$$(p_0^{[\mu]} M_h + S_h) U_h^n = p_0^{[\mu]} U_h^0 - \sum_{j=1}^{n-1} p_j^{[\mu]} (U_h^{n-j} - U_h^0) + F_h^n.$$

Note that the computational complexity per time step of the scheme increases linearly with the time step n , which results in a quadratic complexity of the overall scheme. For large-time simulation, one may use the sum of exponential approximation to reduce the complexity [14, Section 4.2].

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