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UNIFIED ASYMPTOTIC ANALYSIS AND NUMERICAL SIMULATIONS OF SINGULARLY PERTURBED LINEAR DIFFERENTIAL EQUATIONS UNDER VARIOUS NONLOCAL BOUNDARY EFFECTS*

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Abstract. While being concerned with a singularly perturbed linear differential equation subject to integral boundary conditions, the exact solutions, in general, cannot be specified, and the validity of the maximum principle is unassurable. Hence, a problem arises: *how to identify the boundary asymptotics more precisely?* We develop a rigorous asymptotic method involving recovered boundary data to tackle the problem. A key ingredient of the approach is to transform the "nonlocal" boundary conditions into "local" boundary conditions. Then, we perform an "$\varepsilon\log\varepsilon$-estimate" to obtain the refined boundary asymptotics of its solutions with respect to the singular perturbation parameter ε. Furthermore, for the inhomogeneous case, diversified asymptotic behaviors including uniform boundedness and asymptotic blow-up are obtained. Numerical simulations and validations are also presented to further support the corresponding theoretical results.

Keywords. Singular perturbation; Integral boundary condition; Refined asymptotics; Recovered boundary data; Nonlocal boundary effect.

AMS subject classifications. 34B10; 34D15; 34E05; 34K26; 35J25.

1. Introduction

The work treats a class of singularly perturbed linear differential equations with several types of integral boundary conditions involving parameters. To proceed the analysis in an orderly way, we first study a homogeneous equation

$$\varepsilon^2 u''(x)+\varepsilon a(x)u'(x)-b(x)u(x)=0,\ x\in(0,1), \tag{1.1}$$

with the integral boundary condition

$$u(0)=\mu_0+\frac{1}{\varepsilon}\int_{l_0}^{1} g_0(x)u(x)\,\mathrm{d}x, \quad u(1)=\mu_1+\frac{1}{\varepsilon}\int_{0}^{l_1} g_1(x)u(x)\,\mathrm{d}x. \tag{1.2}$$

Here the symbol "$'$" denotes $\frac{\mathrm{d}}{\mathrm{d}x}$ and $0<\varepsilon\ll 1$ is a singular perturbation parameter. It is required that a, b and g_j's defined on $[0,1]$ are as smooth as necessary to carry out the rigorous analysis, where a can change sign and b is always positive. For simplicity we assume that they are independent of ε. Besides, μ_j's and l_j's are constants independent of ε, where $(l_0,l_1)\in[0,1)\times(0,1]$. It should be stressed that the unknown function u depends on ε and should be denoted as u_ε, but throughout the paper we drop its subscript for simplicity.

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Equations (1.1)–(1.2) arise frequently in many practical problems such as linear optimal control theory [9, 12, 23, 24, 32, 42, 44] and the electrical network [35], and more particularly relate to linearized models in population theory [20]; see, e.g., [15, eq. (1.14)]. On the other hand, it is known that under a relation $u(x) = \exp\{\int_0^x \frac{\phi}{\varepsilon}\}$, (1.1) formally becomes a Riccati equation $\varepsilon\phi' = b(x) - a(x)\phi - \phi^2$ which means that (1.1)–(1.2) can be transformed into a singularly perturbed Riccati equation with nonlocal boundary conditions. For more physical background and the corresponding mathematical study, we refer the reader to [6, 7, 16, 19, 27, 30, 41, 43]. We also provide its close relation to Duffing-type equations in Remark 2.4 for the mathematical interest.

The well-known approximate solutions to (1.1)–(1.2) are not rigorous due to the nonlocal dependence of boundary conditions; see, e.g., (1.4). This work shall propose a unified method for asymptotic analysis with respect to this class of integral-type boundary conditions, most notably the nonlocal effects on the asymptotic behavior of solutions; see Section 2.1. Moreover, for the inhomogeneous case $\varepsilon^2 u'' + \varepsilon a(x)u' - b(x)u = f(x)$ with the boundary condition (1.2), we show that various relations among a, b, f and g_j's result in diversified asymptotic behaviors (*uniform boundedness* & *asymptotic blow-up*) of u as $\varepsilon \downarrow 0$, which will be completely classified. For the sake of clarity, after completely introducing the study of (1.1)–(1.2) we will turn the interest on several integral boundary conditions involving various parameters (cf. Section 2.2) and the inhomogeneous case (cf. Section 2.3).

In Section 1.5, we will briefly discuss the difficulty in the asymptotic analysis of the nonlinear equation with the integral boundary condition (1.2). We want to point out that, for the nonlinear case, the refined asymptotics of solutions remains an open problem. This is exactly our ongoing project.

1.1. Background and motivation. Singularly perturbed models with various integral boundary conditions have been investigated numerically (cf. [8, 10, 13, 14, 31, 38–40]). The previous works related to the unique or multiple solutions of linear/nonlinear equations with various integral boundary conditions can be found in [2, 5, 18, 26, 29, 37, 45]. Despite the importance of recent research, when $a(x)$ and $b(x)$ are not constant-valued functions, the *refined asymptotic analysis* of (1.1)–(1.2) remains unclear.

To formulate our study in a more concrete fashion, let us notice that the boundary value of $u(0)$ (resp., $u(1)$) is evaluated by unknown u and a given function g_0 (resp., g_1) in a region $[l_0, 1]$ (resp., $[0, l_1]$) which may be in the vicinity of, or far away from, the boundary point $x = 0$ (resp., $x = 1$). As a consequence, there is a nonlocal interaction between boundary values $u(0)$ and $u(1)$. Although (1.1) is linear, the boundary condition (1.2) makes the asymptotics of u nontrivial. In general, as $\varepsilon > 0$ is sufficiently small, applying the maximum principle to (1.1)–(1.2) is unable to conclude the boundedness of u; see, e.g., [36, Lemma 3.1]. On the other hand, perhaps one would surmise roughly $\frac{1}{\varepsilon}|\int_{l_0}^1 g_0 u| \gg 1$ and $\frac{1}{\varepsilon}|\int_0^{l_1} g_1 u| \gg 1$ and $|u| \to \infty$ near the boundary as $0 < \varepsilon \ll 1$. This is not the case, as will be understood later. As a consequence, the standard singular perturbation analysis may not work on studying the nonlocal model (1.1)–(1.2) and the asymptotic behavior of its solutions seems counterintuitive. In summary, a central question is how to recover more accurately the boundary values of $u(0)$ and $u(1)$ so that we can capture the refined information of u. Accordingly, we shall address the following questions:

- **(Q1)** *How to identify more precisely the asymptotics of $u(0)$ and $u(1)$?*
- **(Q2)** *What roles do the functions a, b and g_j's play in the asymptotic behavior of u?*

The proposed problems above are essentially important for applications since nonlocal boundary conditions "conceal" the accurate boundary information. Developing a method to obtain the more refined boundary asymptotics is meaningful.

Although several useful methods for the singularly perturbed models have been established over the past few decades [3, 4, 11, 17, 21, 22, 25, 28, 33, 34], to the best of our knowledge, these arguments are limited to handle this type of nonlocal equations. A difficulty comes from the fact that the numerical methods for solving such a class of singularly perturbed equations with unknown boundary data might be unreliable as the parameter $\varepsilon>0$ is sufficiently small, and hence fail to provide accurate asymptotic results. Their corresponding "rigorous analysis" for asymptotic solutions usually stay at the preliminary estimation, rather than the refined asymptotics. It should also be stressed that most of the related literature assume $\min\limits_{[0,1]} a>0$ that is exactly a simple situation for investigating the asymptotics of (1.1)–(1.2) with $0<\varepsilon\ll 1$. For the case of $\min\limits_{[0,1]} a<0$, we will point out the difficulty in analysis, and introduce an idea to deal with this case (cf. Section 3.1). The present work shall focus on the more general case that $a(x)$ can be a sign-changing function, and aim to establish the precise leading term (with $0<\varepsilon\ll 1$) of these two nonlocal coefficients so that we can describe the limiting profile of u in the whole domain $[0,1]$. A highlight of this work is to establish the refined asymptotic profile of u on the whole domain, which is uniformly convergent as the perturbation parameter $\varepsilon\downarrow 0$ (see (1.6) for a special case or Theorem 2.1 for the general case). In our opinion, the results are useful for the numerical studies such as stability and convergence analysis; see the numerical results in Section 2.

To be more exact, on one side, our study intends to recover the boundary data $u(0)$ and $u(1)$ which are actually influenced by variable coefficients a, b, g_0 and g_1 as the parameter ε vanishes. To the best of our knowledge, even for the linear Equation (1.1), these questions remain open since $u(0)$ and $u(1)$ relying nonlocally on the behavior of solution u in subdomains of $[0,1]$ are unknown. Note that the asymptotic behavior of solution u and the nonlocal terms in its boundary condition (1.2) are influenced by each other. Thus, "refined apriori estimates" will be established so that we can obtain the precise leading-order terms of $u(0)$ and $u(1)$ with respect to $0<\varepsilon\ll 1$. Our argument is based on the comparison theorem and the so-called **$\boldsymbol{\varepsilon\log\varepsilon}$-estimate** which is useful for the estimate of solutions near the boundary. These estimates will be explained in Section 3.1.

Furthermore, to see the importance and non-triviality of the boundary condition (1.2) with the singularity parameter $\frac{1}{\varepsilon}$, we shall also consider $u=u_\tau$, $\tau\in\mathbb{R}$, satisfying (1.1) with the boundary condition

$$u_\tau(0)=\mu_0+\varepsilon^{-\tau}\int_{l_0}^{1} g_0(x)u_\tau(x)\,\mathsf{d}x, \quad u_\tau(1)=\mu_1+\varepsilon^{-\tau}\int_{0}^{l_1} g_1(x)u_\tau(x)\,\mathsf{d}x. \tag{1.3}$$

For sufficiently understanding the diversification of asymptotic behavior of u_τ, we focus on the nontrivial case $\mu_0\mu_1\neq 0$. We obtain that, as $0<\varepsilon\ll 1$, the boundary values $u_\tau(0)$ and $u_\tau(1)$ have precise leading order $\varepsilon^{\max\{1,\tau\}-1}$. Moreover, when $\tau\neq 1$, the effect of τ on asymptotics of u is summarized as follows (see Theorems 2.1–2.2 in detail):

- **(cf. Theorem 2.2(i)).** When $\tau<1$, the nonlocal effects in (1.3) are quite insignificant since

$$\max\left\{\int_{l_0}^{1} g_0(x)u_\tau(x)\,\mathsf{d}x, \int_{0}^{l_1} g_1(x)u_\tau(x)\,\mathsf{d}x\right\}\ll\varepsilon^{\tau} \quad \text{if } \tau<1.$$

For this case, the boundary condition (1.3) is exactly approximated to the standard Dirichlet type boundary condition since $u_\tau(0) \xrightarrow{\varepsilon\downarrow 0} \mu_0$ and $u_\tau(1) \xrightarrow{\varepsilon\downarrow 0} \mu_1$, and the asymptotic behavior of u_τ is quite trivial compared to the case $\tau=1$; see (1.19)–(1.21) and (2.2).

- **(cf. Theorem 2.2(ii)).** When $\tau>1$, there hold

$$\max_{[0,1]}|u_\tau| \leq \max\{|u_\tau(0)|,|u_\tau(1)|\} \xrightarrow{\varepsilon\downarrow 0} 0,$$

and

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-\tau}\int_{l_0}^{1} g_0(x)u_\tau(x)\,\mathsf{d}x = -\mu_0, \quad \lim_{\varepsilon\downarrow 0}\varepsilon^{-\tau}\int_{0}^{l_1} g_1(x)u_\tau(x)\,\mathsf{d}x = -\mu_1.$$

For this case, although $\varepsilon^{-\tau} \gg \frac{1}{\varepsilon}$ (as $0<\varepsilon\ll 1$) gives a quite strong singular perturbation on boundary data, the asymptotic profile of u_τ becomes flat in the whole domain $(-1,1)$. *This might seem counterintuitive in the standard singular perturbation theory.*

Accordingly, it motivates us to investigate the nontrivial case $\tau=1$.

Finally, we shall emphasize that our analysis can be widely applied to other types of integral boundary conditions. Among the mathematical interest and practical applications for such a class of singular perturbations, Equation (1.1) with various types of integral boundary conditions (cf. (2.10) and (2.11)) will be discussed in more detail. Based upon our arguments dealing with such types of boundary conditions, we further study the asymptotic behavior of u to (1.1) with those boundary conditions; see Section 1.4 and the corresponding results in Section 2.2.

1.2. An overview of (1.1)–(1.2): formal versus rigorous analysis. The parameters ε^2 (in front of u'') and ε (in front of au') are naturally due to the standard length scales. On the other hand, let us consider the equation $\varepsilon^2 u''(x)+\varepsilon^\alpha a(x)u'(x)-b(x)u(x)=0$, i.e., the term $\varepsilon a(x)u'(x)$ in (1.1) is replaced by $\varepsilon^\alpha a(x)u'(x)$. Then, for $\alpha\neq 1$, comparing this new equation with the original Equation (1.1) we have that:

- If $\alpha>1$, then $\varepsilon^{\alpha-1}\max_{[0,1]}|a| \xrightarrow{\varepsilon\downarrow 0} 0$. Employing the method of matched asymptotic expansions to this new equation and taking into account $\alpha>1$, one finds that the effect of a on asymptotics of solutions u is far weaker than the effect of b as $\varepsilon\downarrow 0$.
- If $\alpha<1$, this new equation can further be transformed into

$$\widetilde{\varepsilon}^2 u''(x)+\widetilde{\varepsilon}a(x)u'(x)-\widetilde{\varepsilon}^{\frac{2(1-\alpha)}{2-\alpha}}b(x)u(x)=0 \quad \text{with} \quad \widetilde{\varepsilon}=\varepsilon^{2-\alpha}.$$

The situation about the influences of $a(x)$ and $b(x)$ on asymptotics of solution u is totally turned around since $\widetilde{\varepsilon}^{\frac{2(1-\alpha)}{2-\alpha}}\max_{[0,1]}|b| \xrightarrow{\varepsilon\downarrow 0} 0$.

As a consequence, this motivates us to focus on the case $\alpha=1$, and we are devoted to investigating the effects of both $a(x)$ and $b(x)$ on the asymptotic behavior of u. Despite the linearity of Equation (1.1), the presence of convection term with the order ε plays a crucial role in the asymptotics of u as $\varepsilon\downarrow 0$.

Let us first point out a difficulty in the asymptotics of (1.1)–(1.2). It should be stressed that for (1.1) with the standard Dirichlet boundary condition (i.e, $u(0)$ and

$u(1)$ are given), one obtains a trivial outer solution $\mathbf{u}^{\mathbf{out}}_{\varepsilon}(x)=0$ in $(0,1)$, and a *formal* inner solution

$$\mathbf{u}^{\mathbf{in}}_{\varepsilon}(\mathbf{x}_{\boldsymbol{\varepsilon}})=u(0)\exp\left\{\frac{\Lambda_-(0)}{2\varepsilon}\mathrm{x}_{\varepsilon}\right\}+u(1)\exp\left\{-\frac{\Lambda_+(1)}{2\varepsilon}(1-\mathrm{x}_{\varepsilon})\right\} \tag{1.4}$$

with

$$\Lambda_-(z):=-a(z)-\sqrt{a^2(z)+4b(z)}<0, \qquad \Lambda_+(z):=-a(z)+\sqrt{a^2(z)+4b(z)}>0, \tag{1.5}$$

where $\mathbf{x}_{\boldsymbol{\varepsilon}}$, depending on ε, approaches one of the boundary points $x=0,1$ as $\varepsilon>0$ goes to zero. (Note that $b(z)>0$.) What we want to point out is that the standard method of matched asymptotic expansions may not work on such a linear equation with integral boundary condition (1.2). A reason for the difficulty in studying the asymptotic behavior of u with $\varepsilon\downarrow 0$ comes from a fact that the boundary condition (1.2) gives a functional constraint implicitly between $u(0)$ and $u(1)$, and the outer and inner solutions are formal approximations which are not useful for the rigorous asymptotic analysis of those nonlocal terms with respect to $0<\varepsilon\ll 1$. Furthermore, when taking a formal look at the limiting form of (1.2) with $\varepsilon\downarrow 0$, the terms $\frac{1}{\varepsilon}\int_{l_0}^1 g_0(x)u(x)\,\mathrm{d}x$ and $\frac{1}{\varepsilon}\int_0^{l_1} g_1(x)u(x)\,\mathrm{d}x$ may strongly dominate the boundary asymptotic behavior, and such a nonlocal effect may give a difficulty in analyzing the leading-order asymptotics of u with respect to $0<\varepsilon\ll 1$. In this work we will address this issue; see Proposition 1.1 and Remark 1.1 for the non-existence, uniqueness and multiplicity of (1.1)–(1.2). The main result (cf. Theorem 2.1) focuses on the uniqueness case and establishes rigorously the refined asymptotics of u with respect to $0<\varepsilon\ll 1$. For an introduction of Theorem 2.1, we present an essential case for the reader's understanding. For $\mathrm{C}^2_{\varepsilon}$-norm defined in (2.1), when $l_0\in(0,1]$ and $l_1\in[0,1)$, by (1.20)–(1.21) and (2.3) below, uniform asymptotics of u can be depicted as follows:

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\Bigg\| u(x)-\frac{1}{\mathfrak{D}}\Bigg[&\left(\mu_0+\frac{2g_0(1)}{\Lambda_+(1)}\mu_1\right)\exp\left\{\frac{1}{2\varepsilon}\int_0^x\Lambda_-(z)\,\mathrm{d}z\right\}\\ &+\left(-\frac{2g_1(0)}{\Lambda_-(0)}\mu_0+\mu_1\right)\exp\left\{-\frac{1}{2\varepsilon}\int_x^1\Lambda_+(z)\,\mathrm{d}z\right\}\Bigg]\Bigg\|_{\mathrm{C}^2_{\varepsilon}([0,1])}=0,\end{aligned} \tag{1.6}$$

where $\Lambda_{\pm}(z)$ has been defined by (1.5), and

$$\mathfrak{D}:=1+\frac{4g_0(1)g_1(0)}{\Lambda_-(0)\Lambda_+(1)}\neq 0,$$

see also Remark 1.2 below. It should be emphasized that $\mathfrak{D}\neq 0$ is a sufficient condition for the uniqueness of the Equation (1.1)–(1.2), and the coefficients

$$\frac{1}{\mathfrak{D}}\left(\mu_0+\frac{2g_0(1)}{\Lambda_+(1)}\mu_1\right) \quad \text{and} \quad \frac{1}{\mathfrak{D}}\left(-\frac{2g_1(0)}{\Lambda_-(0)}\mu_0+\mu_1\right)$$

come from the nonlocal effects of (1.2), which cannot be directly observed from the original boundary condition. This indeed shows the role of a, b, g_0 and g_1 in the nontrivial asymptotic behavior of u under (1.2). For the completeness of the study, our result also includes the cases of $l_0=0$ and/or $l_1=1$.

1.3. Methodology and preliminary results. Since (1.1) is linear, for dealing with the nonlocal asymptotics, a key idea is to decompose u into a linear combination of functions v and w with unknown coefficients $u(0)$ and $u(1)$:

$$u(x)=u(0)v(x)+u(1)w(x) \tag{1.7}$$

where v and w satisfy

$$\begin{cases} \varepsilon^2 v''+\varepsilon a(x)v'-b(x)v=0,\ x\in(0,1),\\ v(0)=1,\ v(1)=0, \end{cases} \tag{1.8}$$

and

$$\begin{cases} \varepsilon^2 w''+\varepsilon a(x)w'-b(x)w=0,\ x\in(0,1),\\ w(0)=0,\ w(1)=1. \end{cases} \tag{1.9}$$

(Their subscript ε is omitted for simplicity.) The condition $b(x)\geq\beta>0$ implies the uniqueness of $v,w\in \mathrm{C}^2([0,1])$. We will provide basic asymptotic estimates of v and w with $0<\varepsilon\ll 1$ in Proposition 3.1.

Owing to the fact that (1.1) is linear, the relation (1.7) is useful for the Equation (1.1) with integral boundary conditions (1.3). One can put (1.7) into the boundary condition (1.2) and obtain a linear system

$$\left(\mathcal{I}-\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}\right)\begin{bmatrix} u(0)\\ u(1)\end{bmatrix}=\begin{bmatrix}\mu_0\\ \mu_1\end{bmatrix} \tag{1.10}$$

for boundary values $u(0)$ and $u(1)$, where

$$\mathcal{I}=\begin{bmatrix}1&0\\0&1\end{bmatrix} \quad \text{and} \quad \mathcal{A}_\varepsilon^{(v,w)}=\begin{bmatrix}\int_{l_0}^1 g_0(x)v(x)\,\mathrm{d}x & \int_{l_0}^1 g_0(x)w(x)\,\mathrm{d}x\\ \int_0^{l_1} g_1(x)v(x)\,\mathrm{d}x & \int_0^{l_1} g_1(x)w(x)\,\mathrm{d}x\end{bmatrix}. \tag{1.11}$$

Hence, (1.10)–(1.11) shows that $u(0)$ and $u(1)$ are, in general, influenced by each other. For general a and b, (1.8) and (1.9) do not have explicit forms. Although equations (1.8) and (1.9) are quite simple, the first step in conducting a challenge study is to rigorously obtain the **precise leading order term** of $\varepsilon^{-1}\mathcal{A}_\varepsilon^{(v,w)}$ with respect to $0<\varepsilon\ll 1$. Notice further that v and w are independent of μ_0 and μ_1. Owing to the uniqueness of v and w, the existence, non-existence, uniqueness and multiplicity of solutions to Equation (1.1) with the boundary condition (1.2) follow directly from the analysis of linear system (1.10) of $(u(0),u(1))$. Here we focus mainly on the uniqueness of (1.1)–(1.2).

Proposition 1.1. *Assume that $a(x)$ and $b(x)$ are smooth functions defined in $[0,1]$ with $b(x)\geq\beta$, where β is a positive constant independent of ε. Then, for $\varepsilon>0$, Equation (1.1) with the boundary condition (1.2) has a unique solution $u\in\mathrm{C}^2([0,1])$ if and only if $\mathcal{I}-\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}$ is invertible, i.e.,*

$$\det\left(\mathcal{I}-\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}\right)\neq 0. \tag{1.12}$$

The unique solution is non-trivial if and only if $(\mu_0,\mu_1)\neq(0,0)$.

REMARK 1.1. When $\det\left(\mathcal{I}-\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}\right)=0$, (1.1)–(1.2) has infinitely many solutions (resp., no solution) if μ_0 and μ_1 satisfy

$$\det\begin{bmatrix} \mu_0 & -\dfrac{1}{\varepsilon}\displaystyle\int_{l_0}^1 g_0(x)w(x)\,\mathrm{d}x \\ \mu_1 & 1-\dfrac{1}{\varepsilon}\displaystyle\int_0^{l_1} g_1(x)w(x)\,\mathrm{d}x \end{bmatrix}=0 \quad (\text{resp.}, \neq 0).$$

A more difficult task is to establish the refined asymptotics of $u(0)$ and $u(1)$ with sufficiently small $\varepsilon>0$, since (1.10) only gives formal representations of $u(0)$ and $u(1)$, and so far we do not have any idea on the asymptotics of $\mathcal{A}_\varepsilon^{(v,w)}$ with respect to $0<\varepsilon\ll 1$. Some assumptions of g_0 and g_1 will be made later on, in order to guarantee that (1.12) holds.

Accordingly, we will establish refined asymptotics of (1.8) and (1.9) so that we can obtain the precise leading order terms of those coefficients in (1.10) with respect to $0<\varepsilon\ll 1$; see Proposition 3.1. Based on Proposition 3.1, we introduce the following properties which play a crucial role in dealing with the asymptotics of $u(0)$ and $u(1)$ from (1.10).

PROPOSITION 1.2. *Assume that a and b are smooth functions defined in $[0,1]$ with $b(x)\geq\beta$, where β is a positive constant independent of ε. For $\varepsilon>0$, let v and w be the unique non-negative solutions of (1.8) and (1.9), respectively. Then, the following convergences hold:*

(i) *When $p>1$, we have, for $\phi\in \mathrm{C}([0,1])$, that*

$$\int_0^{\delta_0}\frac{v^p}{\varepsilon}\phi\,\mathrm{d}x \xrightarrow{\varepsilon\downarrow 0} -\frac{2\phi(0)}{p\Lambda_-(0)}, \qquad \int_{\delta_1}^1\frac{w^p}{\varepsilon}\phi\,\mathrm{d}x \xrightarrow{\varepsilon\downarrow 0} \frac{2\phi(1)}{p\Lambda_+(1)}, \tag{1.13}$$

$$\int_0^{\delta_0}\varepsilon|v'|^2\phi\,\mathrm{d}x \xrightarrow{\varepsilon\downarrow 0} -\frac{\Lambda_-(0)\phi(0)}{4}, \qquad \int_{\delta_1}^1\varepsilon|w'|^2\phi\,\mathrm{d}x \xrightarrow{\varepsilon\downarrow 0} \frac{\Lambda_+(1)\phi(1)}{4}, \tag{1.14}$$

and

$$\int_{\delta_0}^1\frac{v^p}{\varepsilon}\phi\,\mathrm{d}x,\ \int_{\delta_0}^1\varepsilon|v'|^2\phi\,\mathrm{d}x,\ \int_0^{\delta_1}\frac{w^p}{\varepsilon}\phi\,\mathrm{d}x,\ \int_0^{\delta_1}\varepsilon|w'|^2\phi\,\mathrm{d}x \xrightarrow{\varepsilon\downarrow 0} 0, \tag{1.15}$$

where Λ has been defined by (1.4), $\delta_0\in(0,1]$ and $\delta_1\in[0,1)$ are independent of ε.

(ii) *For the case $p=1$, the convergences presented in (1.13) and (1.15) also hold for $\phi\in \mathrm{C}^1([0,1])$. If, in addition, we assume $a(0)>0$ and $a(1)>0$, then the convergences presented in (1.13) can hold for $\phi\in\mathrm{C}([0,1])$.*

Proposition 1.2 is based on a series of interior estimates of v and w established in Proposition 3.1. The proof will be stated in Section 3.3.

We want to stress again that even if (1.8) and (1.9) are of linear type, Propositions 1.2 and 3.1 seem to be novel and significant. To the best of our knowledge, such results are useful for dealing with the singularly perturbed Equation (1.1) with the nonlocal boundary conditions, but they do not appear in the previous literature.

By Propositions 1.1 and 1.2, we make a brief discussion about some sufficient conditions for the uniqueness of u with small $\varepsilon>0$ and for the boundary and interior asymptotics of u, respectively.

(I) $\boxed{\textbf{Uniqueness.}}$ Under the same hypotheses as in Proposition 1.2, we assume that g_i's are smooth functions defined in $[0,1]$. Then, by (1.13) and (1.15) with $p=1$, there holds

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}=\mathcal{A}^*:=\begin{bmatrix} -\dfrac{2g_0(0)}{\Lambda_-(0)}\mathbb{1}_{\{0\}}(l_0) & \dfrac{2g_0(1)}{\Lambda_+(1)} \\ -\dfrac{2g_1(0)}{\Lambda_-(0)} & \dfrac{2g_1(1)}{\Lambda_+(1)}\mathbb{1}_{\{1\}}(l_1)\end{bmatrix} \tag{1.16}$$

where $\mathcal{A}_\varepsilon^{(v,w)}$ was defined in (1.11), and $\mathbb{1}_S:[0,1]\to\{0,1\}$ is the standard indicator function defined on a set $S\subset[0,1]$. Particularly,

$$\begin{cases}\mathbb{1}_{\{0\}}(0)=\mathbb{1}_{\{1\}}(1)=1;\\ \mathbb{1}_{\{0\}}(l_0)=\mathbb{1}_{\{1\}}(l_1)=0 \quad \text{for} \quad l_0\in(0,1],\, l_1\in[0,1).\end{cases} \tag{1.17}$$

On the other hand, we obtain from (1.10), Proposition 1.1, Remark 1.1 and (1.16) that if

$$\det(\mathcal{I}-\mathcal{A}^*)\neq 0, \tag{1.18}$$

then as $\varepsilon>0$ is sufficiently small, Equation (1.1) with the boundary condition (1.2) has a unique solution $u\in \mathrm{C}^2([0,1])$.

(II) $\boxed{\textbf{Boundary asymptotics \& a positivity preserving property.}}$ Applying the Cramer's formula to (1.10) and using (1.16), we obtain

$$\lim_{\varepsilon\downarrow 0}u(0)=B_0,\quad \lim_{\varepsilon\downarrow 0}u(1)=B_1, \tag{1.19}$$

where B_0 and B_1 are uniquely determined by $(\mathcal{I}-\mathcal{A}^*)[B_0\,B_1]^{\mathrm{T}}=[\mu_0\,\mu_1]^{\mathrm{T}}$, i.e.,

$$B_0:=\det\left((\mathcal{I}-\mathcal{A}^*)^{-1}\begin{bmatrix}\mu_0 & -\dfrac{2g_0(1)}{\Lambda_+(1)}\\ \mu_1 & 1-\dfrac{2g_1(1)}{\Lambda_+(1)}\mathbb{1}_{\{1\}}(l_1)\end{bmatrix}\right), \tag{1.20}$$

$$B_1:=\det\left((\mathcal{I}-\mathcal{A}^*)^{-1}\begin{bmatrix}1+\dfrac{2g_0(0)}{\Lambda_-(0)}\mathbb{1}_{\{0\}}(l_0) & \mu_0\\ \dfrac{2g_1(0)}{\Lambda_-(0)} & \mu_1\end{bmatrix}\right); \tag{1.21}$$

see (1.16) for $\mathcal{A}^*$. Moreover, when both B_0 and B_1 are positive, one can use the result (1.19) and apply the strong maximum principle to (1.1) and obtain a positivity preserving property of u with sufficiently small ε. This is quite nontrivial since there is no intuitive way to verify conditions of μ_j's and g_j's which guarantees $u>0$ in $[0,1]$, i.e.,

$$u>0 \text{ in } [0,1] \text{ for sufficiently small } \varepsilon>0 \text{ if and only if } B_0 \text{ and } B_1 \text{ are positive.}$$

This points out the importance of precise leading terms of boundary values with respect to $0<\varepsilon\ll 1$.

(III) $\boxed{\textbf{Interior estimates.}}$ There exist positive constants M and $\widetilde{\gamma}$ independent of ε such that (see Theorem 2.1):

$$|u(x)|+\varepsilon|u'(x)|\le M\left(\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}x\right\}+\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}(1-x)\right\}\right). \tag{1.22}$$

As a consequence, the boundary asymptotics (1.19) with (1.20) and (1.21) answer (Q1) in the affirmative, but an answer to question (Q2) is not obvious; see questions (Q1) and (Q2) in Section 1.1. To avoid the trivial case, by Proposition 1.1 and (1.18), it suffices to investigate the asymptotics of u for the case $B_0B_1\neq 0$. Without loss of generality, in what follows we focus on two cases for B_0 and B_1:

$$B_0<0<B_1 \tag{1.23}$$

and

$$B_0>0, B_1>0. \tag{1.24}$$

Later on, we will establish the asymptotic profiles of u under (1.23) and (1.24). The study for other cases is analogous.

Remark 1.2. Although (1.20) and (1.21) show that the boundary asymptotics of u as ε approaches zero is exactly due to the nonlocal effects of boundary conditions (1.2), their formulas are usually a little complicated. Here we give a simple situation when $l_0\in(0,1]$ and $l_1\in[0,1)$. For this case, by (1.16), (1.17) and (1.20)–(1.21) we have

$$B_0=\left(\mu_0+\frac{2g_0(1)}{\Lambda_+(1)}\mu_1\right)\det(\mathcal{I}-\mathcal{A}^*)^{-1} \quad \textit{and} \quad B_1=\left(-\frac{2g_1(0)}{\Lambda_-(0)}\mu_0+\mu_1\right)\det(\mathcal{I}-\mathcal{A}^*)^{-1},$$

where $\det(\mathcal{I}-\mathcal{A}^*)=1+\frac{4g_0(1)g_1(0)}{\Lambda_-(0)\Lambda_+(1)}\neq 0$. This shows how those variable coefficients a, b, g_0 and g_1 affect the boundary asymptotics of u.

1.4. On more general cases. Thanks to Proposition 1.2, we can study the asymptotic behavior of (1.1) with various boundary conditions which are more general than (1.2).

Firstly, to stress the nontriviality of the boundary condition (1.2), we also consider the boundary condition (1.3) with $\tau\neq 1$. Then we have

$$\left(\varepsilon^{\tau-1}\mathcal{I}-\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}\right)\begin{bmatrix}\varepsilon^{1-\tau}u_\tau(0)\\ \varepsilon^{1-\tau}u_\tau(1)\end{bmatrix}=\begin{bmatrix}\mu_0\\ \mu_1\end{bmatrix}. \tag{1.25}$$

According to Proposition 1.1, we shall focus on the situation

$$\det\left(\varepsilon^{\tau-1}\mathcal{I}-\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}\right)\neq 0 \quad \& \quad (\mu_0,\mu_1)\neq(0,0) \tag{1.26}$$

(see Remark 1.3 below). Then, for the case $\tau<1$, we obtain $(u_\tau(0),u_\tau(1))\xrightarrow{\varepsilon\downarrow 0}(\mu_0,\mu_1)$, while for the case $\tau>1$, there holds $\limsup_{\varepsilon\downarrow 0}\varepsilon^{1-\tau}\max_{[0,1]}|u_\tau|<\infty$, and, in particular, $\max_{[0,1]}|u_\tau|\xrightarrow{\varepsilon\downarrow 0}0$. To highlight the difference, we will only discuss the boundary asymptotics of u for simplicity; see Theorem 2.2.

Remark 1.3. By Proposition 1.2, (1.16) and (1.25)–(1.26), we have the following uniqueness result for $\tau\neq 1$ and sufficiently small $\varepsilon>0$.

(i) For the case $\tau<1$, we have $\mathcal{I}-\varepsilon^{-\tau}\mathcal{A}_\varepsilon^{(v,w)}\approx\mathcal{I}$ as $0<\varepsilon\ll 1$. It immediately concludes that Equation (1.1) with the boundary condition (1.3) has a unique solution u as $\varepsilon>0$ is sufficiently small.

(ii) For the case $\tau>1$, we have $\mathcal{I}-\varepsilon^{-\tau}\mathcal{A}_\varepsilon^{(v,w)}\approx-\varepsilon^{-\tau}\mathcal{A}_\varepsilon^{(v,w)}$ as $0<\varepsilon\ll 1$. Hence, it suffices to assume $\det\mathcal{A}^*\neq 0$ (cf. (1.16)) so that Equation (1.1) with the boundary condition (1.3) has a unique solution u as $\varepsilon>0$ is sufficiently small.

For the sake of completeness of this paper, we further consider the Equation (1.1) with several boundary effects that are more general than (1.2). The motivation of this study is quite straightforward since the boundary condition (1.2) is a special case of the form

$$u(i)=\mu_i+\int_{\Omega_i}G_i(x,q(\varepsilon)u,\widetilde{q}(\varepsilon)u')\,\mathsf{d}x,\quad i\in\{0,1\},\tag{1.27}$$

where $q(\varepsilon)$ and $\widetilde{q}(\varepsilon)$ are functions of ε and $\Omega_0=(l_0,1)$, $\Omega_1=(0,l_1)$. To the best of our knowledge, for general $G_i:[0,1]\times\mathbb{R}^2\to\mathbb{R}$, the asymptotic analysis under boundary condition (1.27) remains open and it is expected that nontrivial boundary concentration phenomena may occur; see, e.g., Proposition 1.2 for the case $G_i=\varepsilon^{-1}g_iu$. Based on the investigation of the Equation (1.1) with the boundary condition (1.2), we will focus on some types of boundary conditions with the corresponding $q(\varepsilon)$ and $\widetilde{q}(\varepsilon)$ as the first step in our research in this topic (see (2.10) and (2.11) below). We will state the corresponding preliminary knowledge and the main result in Section 2.2.

1.5. The nonlinear case: an open problem and a future plan. Before proceeding to the asymptotic analysis for the linear Equation (1.1) with integral boundary conditions (1.2), we shall propose its nonlinear analogue

$$\varepsilon^2u''(x)+\varepsilon a(x)u'(x)-b(x)h(u(x))=0,\quad x\in(0,1),\tag{1.28}$$

where the nonlinear source $h:\mathbb{R}\to\mathbb{R}$ is smooth. When $b(x)$ is positive and $h:\mathbb{R}\to\mathbb{R}$ is strictly increasing, the second author in his recent work [29] considers the Equation (1.28) with some more general nonlocal boundary conditions. He applies the fixed point argument and establishes asymptotic estimates of the solution u with respect to sufficiently small $\varepsilon>0$ to show the uniqueness of solutions as $\varepsilon\in(0,\varepsilon_0)$. In particular, when $h(s)=s$ is linear, he also provides an example to show that there exists $\varepsilon_1>\varepsilon_0$ depending on μ_j's and g_j's such that Equation (1.28) with the boundary condition (1.2) has multiple solutions; see [29, Example 1.3]. According to our understanding, for each $\varepsilon>0$ (which may not be small), the existence and uniqueness of the Equation (1.28) with nonlocal type boundary conditions remains an open problem.

As our first step in the investigation of this problem, this work focuses on the linear homogeneous/inhomogeneous case and establishes the sufficient and necessary conditions for the existence and uniqueness; see Proposition 1.1 in Section 1.3. Furthermore, as $\varepsilon>0$ approaches zero, we obtain the refined asymptotic expansions of the uniqueness solution of the Equation (1.1) under various nonlocal boundary effects; see the main results presented in Section 2.

However, to the best of our knowledge, the refined asymptotic expansions (as $\varepsilon\downarrow 0$) of solutions to Equation (1.28) with the boundary condition (1.2) are not yet obtained. The main difficulty lies in a fact that the boundary asymptotics of $u(0)$ and $u(1)$ involves nonlocal effects. On the other hand, when $x\in(0,1)$ is close to the boundary points, the three terms $\varepsilon^2u''(x)$, $\varepsilon a(x)u'(x)$ and $h(u(x))$ in (1.28) enjoy the same order of ε (as $\varepsilon\downarrow 0$), and their refined asymptotics are influenced by each other. In general, we could not establish precise quantities (similar to $\Lambda_-(0)$ and $\Lambda_+(1)$) to obtain their precise asymptotics. This project will be our future research directions.

Outline and notations. The rest of this paper outlines the following structure. In Section 2 we state the main analytical results (see Theorems 2.1–2.4) and their corresponding numerical results (see Figures 2.1–2.5 and Tables 2.1–2.4), where Theorem 2.1 presents the refined asymptotic profile of the unique solution u to the Equation (1.1) with the boundary condition (1.2), and Theorems 2.2–2.3 focus on the asymptotic behavior of solutions u to the Equation (1.1) with other types of boundary conditions (1.3), (2.10) and (2.11), respectively. For the inhomogeneous case, the corresponding asymptotic results are stated in Theorem 2.4; see Section 2.3. To prove the main results, we introduce basic properties of v and w in Section 3 and complete the proof of Proposition 1.2 in Section 3.3. Then we state the proof of Theorems 2.1 and 2.2 in Section 4. In Section 5 we will give the proof of Theorem 2.3. Finally, the proof of Theorem 2.4 will be stated in Section 6. In our proofs, we will frequently abbreviate "$\leq C$" to "$\lesssim$", where $C>0$ is a generic constant independent of the parameter ε.

2. Theoretical results and corresponding numerical examinations

To describe the asymptotic profiles of u with respect to $0<\varepsilon\ll 1$, let us define

$$||F||_{\mathrm{C}^2_\varepsilon([0,1])} := \max_{[0,1]}|F| + \varepsilon\max_{[0,1]}|F'| + \varepsilon^2\max_{[0,1]}|F''|, \quad F\in \mathrm{C}^2([0,1]). \tag{2.1}$$

In this section, we state the main results Theorems 2.1–2.4. Moreover, extensive numerical results are provided to demonstrate our theoretical analysis.

2.1. The main results for the homogeneous case. Assume (1.18) with $\mathcal{A}^*$ defined by (1.16). Let B_0 and B_1 be defined in (1.20) and (1.21), respectively. As mentioned previously, for the Equation (1.1) with the boundary condition (1.2), we shall consider two cases (1.23) and (1.24) for B_0 and B_1. The main result is stated as follows.

Theorem 2.1. *Under the same hypotheses as in Proposition 1.1, we assume (1.18). Then there exists $\eta>0$ such that as $\varepsilon\in(0,\eta)$, Equation (1.1) with the boundary condition (1.2) has a unique solution $u\in \mathrm{C}^2([0,1])$, which satisfies (1.19), (1.22) and*

$$\lim_{\varepsilon\downarrow 0}\varepsilon u'(0) = \frac{B_0}{2}\Lambda_-(0), \quad \lim_{\varepsilon\downarrow 0}\varepsilon u'(1) = \frac{B_1}{2}\Lambda_+(1), \tag{2.2}$$

and fulfills the asymptotics

$$\left\| u(x) - B_0\exp\left\{\int_0^x \frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\} - B_1\exp\left\{-\int_x^1 \frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\right\|_{\mathrm{C}^2_\varepsilon([0,1])} \leq M^*\varepsilon^{\theta_1^*}, \tag{2.3}$$

as $0<\varepsilon\ll 1$, where Λ, B_0 and B_1 were defined in (1.4) and (1.20)–(1.21). Besides, M^* is a positive constant independent of ε and $\theta_1^*\in(0,\frac{1}{2}]$ depending mainly on $a(x)$ satisfies (see the exact value of θ_1^* in Claim 1 and Remark 3.3 of Section 3.3):

$$\theta_1^* = \frac{1}{2} \quad \text{if} \quad \min_{[0,1]} a \geq 0; \qquad \theta_1^* \in (0,\frac{1}{2}) \quad \text{if} \quad \min_{[0,1]} a < 0. \tag{2.4}$$

Moreover,

(i) *if B_0 and B_1 satisfy (1.23), then u is strictly increasing on $(0,1)$;*

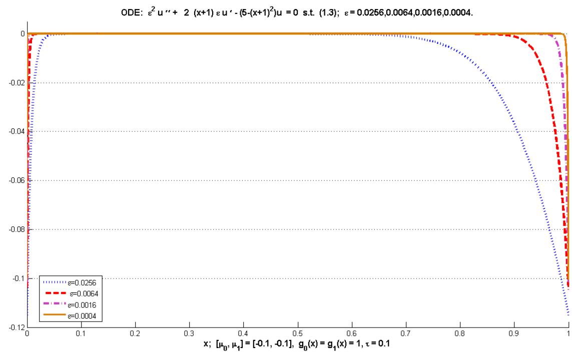

FIG. 2.1. *A numerical examination for Theorem 2.2(i).*

(ii) *if* B_0 *and* B_1 *satisfy* (1.24), *then there uniquely exists* $\mathfrak{p}_\varepsilon \in (0,1)$ *satisfying*

$$\frac{\mathfrak{p}_\varepsilon}{\varepsilon} \xrightarrow{\varepsilon\downarrow 0} \infty \quad \text{and} \quad \frac{1-\mathfrak{p}_\varepsilon}{\varepsilon} \xrightarrow{\varepsilon\downarrow 0} \infty \tag{2.5}$$

such that u *is strictly decreasing on* $(0,\mathfrak{p}_\varepsilon)$ *and strictly increasing on* $(\mathfrak{p}_\varepsilon,1)$.

For the other two cases $B_0>0>B_1$ and $B_0, B_1<0$, we can obtain the similar results but we omit the details here.

(2.3) plays a crucial role in describing the interior asymptotic behavior of $u(x_\varepsilon)$ when $x_\varepsilon \in (0,1)$ depending on ε is close to a boundary point. To be more precise, for $k\geq 0$, we denote by

$$E^-_{j,k} = \exp\left\{\frac{k}{2}\Lambda_-(j)\right\} \quad \text{and} \quad E^+_{j,k} = \exp\left\{-\frac{k}{2}\Lambda_+(j)\right\}, \quad j=0,1.$$

Then for $z_\varepsilon^{0,k}, z_\varepsilon^{1,k} \in (0,1)$ satisfying

$$\frac{z_\varepsilon^{0,k}}{\varepsilon} \xrightarrow{\varepsilon\downarrow 0} k \quad \text{and} \quad \frac{1-z_\varepsilon^{1,k}}{\varepsilon} \xrightarrow{\varepsilon\downarrow 0} k,$$

(2.3) immediately implies the pointwise asymptotics

$$\begin{aligned} u(z_\varepsilon^{0,k}) &\xrightarrow{\varepsilon\downarrow 0} B_0 E^-_{0,k}, \quad \varepsilon u'(z_\varepsilon^{0,k}) \xrightarrow{\varepsilon\downarrow 0} \frac{B_0 E^-_{0,k}}{2}\Lambda_-(0), \\ u(z_\varepsilon^{1,k}) &\xrightarrow{\varepsilon\downarrow 0} B_1 E^+_{1,k}, \quad \varepsilon u'(z_\varepsilon^{1,k}) \xrightarrow{\varepsilon\downarrow 0} \frac{B_1 E^+_{1,k}}{2}\Lambda_+(1). \end{aligned}$$

It should also be stressed that when $x\in(0,1)$ is independent of ε, the estimate (1.22) shows that $u(x)$ and $u'(x)$ exponentially decay to zero, which is better than the estimate (2.3). As a consequence, Theorem 2.1 completely describes the asymptotic profile of u with respect to $0<\varepsilon\ll 1$. The proof of Theorem 2.1 will be stated in Section 4.1.

We next consider the boundary condition (1.3) with $\tau\neq 1$. Comparing with the asymptotics for the case $\tau=1$ presented in Theorem 2.1, the following result focuses on illustrating a significant difference between their boundary asymptotics.

THEOREM 2.2. *Under the same hypotheses as in Proposition 1.1, we assume* (1.26). *For* $\varepsilon>0$ *sufficiently small and* $\tau\neq 1$, *let* $u_\tau \in C^2([0,1])$ *be the unique solution of* (1.1) *with the boundary condition* (1.3) *(cf. Remark 1.3). Then, as* ε *approaches zero, for each interior point* $x\in(0,1)$ *both* $u_\tau(x)$ *and* $u'_\tau(x)$ *exponentially decay to zero, while at the boundary points, we have that:*

(i) *When* $\tau<1$*, there hold* $(u_\tau(0),u_\tau(1))\xrightarrow{\varepsilon\downarrow 0}(\mu_0,\mu_1)$*, and*

$$\varepsilon u'_\tau(0)\xrightarrow{\varepsilon\downarrow 0}\frac{\mu_0}{2}\Lambda_-(0)\quad \textit{and}\quad \varepsilon u'_\tau(1)\xrightarrow{\varepsilon\downarrow 0}\frac{\mu_1}{2}\Lambda_+(1). \tag{2.6}$$

(ii) *When* $\tau>1$ *and* $\det\mathcal{A}^*\neq 0$*, there hold* $\max\limits_{[0,1]}|u|\xrightarrow{\varepsilon\downarrow 0}0$ *with*

$$\varepsilon^{1-\tau}u_\tau(0)\xrightarrow{\varepsilon\downarrow 0}-\det\left((A^*)^{-1}\begin{bmatrix}\mu_0 & \dfrac{2g_0(1)}{\Lambda_+(1)}\\ \mu_1 & \dfrac{2g_1(1)}{\Lambda_+(1)}\mathbb{1}_{\{1\}}(l_1)\end{bmatrix}\right):=\widetilde{B}_0, \tag{2.7}$$

$$\varepsilon^{1-\tau}u_\tau(1)\xrightarrow{\varepsilon\downarrow 0}-\det\left((A^*)^{-1}\begin{bmatrix}-\dfrac{2g_0(0)}{\Lambda_-(0)}\mathbb{1}_{\{0\}}(l_0) & \mu_0\\ -\dfrac{2g_1(0)}{\Lambda_-(0)} & \mu_1\end{bmatrix}\right):=\widetilde{B}_1, \tag{2.8}$$

and

$$\varepsilon^{2-\tau}u'_\tau(0)\xrightarrow{\varepsilon\downarrow 0}\frac{\widetilde{B}_0}{2}\Lambda_-(0),\quad \varepsilon^{2-\tau}u'_\tau(1)\xrightarrow{\varepsilon\downarrow 0}\frac{\widetilde{B}_1}{2}\Lambda_+(1). \tag{2.9}$$

The proof of Theorem 2.2 is stated in Section 4.2 (see Table 2.1 for a numerical result supporting (2.6)–(2.9)).

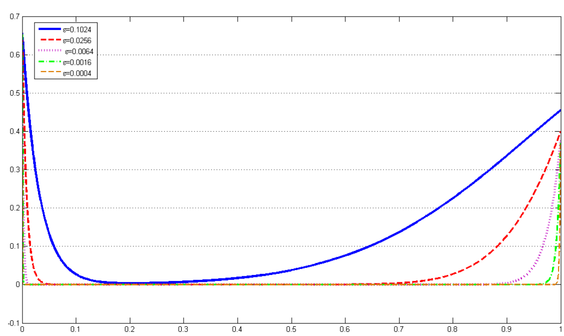

FIG. 2.2. *Numerical solutions of equation* $\varepsilon^2u''(x)+2\varepsilon(x+1)u'(x)-[5-(x+1)^2]u(x)=0$ *with the boundary condition* (2.10)*, where* $(\mu_0,\mu_1)=(1,1.2)$*,* $(\ell_0,\ell_1)=(0,1)$*,* $(g_0(x),g_1(x))=(-1,-e^x)$ *and* $(p_0,p_1)=(2,2)$*.*

2.2. Asymptotic analysis for (1.1) with various boundary effects. In this section, we are devoted to investigating the Equation (1.1) with the following boundary conditions that are more general than (1.2):

$$u(0)=\mu_0+\frac{1}{\varepsilon}\int_{l_0}^{1}g_0(x)|u(x)|^{p_0}\,\mathsf{d}x,\quad u(1)=\mu_1+\frac{1}{\varepsilon}\int_0^{l_1}g_1(x)|u(x)|^{p_1}\,\mathsf{d}x, \tag{2.10}$$

$$u(0)=\mu_0+\varepsilon\int_{l_0}^{1}g_0(x)|u'(x)|^2\,\mathsf{d}x,\quad u(1)=\mu_1+\varepsilon\int_0^{l_1}g_1(x)|u'(x)|^2\,\mathsf{d}x, \tag{2.11}$$

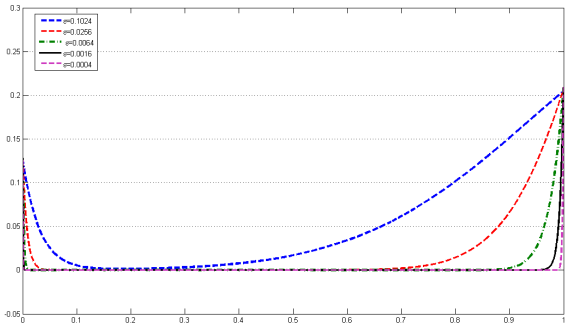

FIG. 2.3. *Numerical solutions of equation* $\varepsilon^2 u''(x)+2\varepsilon(x+1)u'(x)-[5-(x+1)^2]u(x)=0$ *with the boundary condition* (2.11), *where* $(g_0(x),g_1(x))=(-1,-e^x)$, $(\mu_0,\mu_1)=(0.16,0.25)$ *and* $(\ell_0,\ell_1)=(0,1)$.

Case 1: $\tau=0.1$				
ε	$\lvert u_\tau(0)-\mu_0\rvert$	$\lvert u_\tau(1)-\mu_1\rvert$	$\lvert \varepsilon u_\tau'(0)-\frac{\mu_0\Lambda_-(0)}{2}\rvert$	$\lvert \varepsilon u_\tau'(1)-\frac{\mu_1\Lambda_+(1)}{2}\rvert$
0.0256	0.01492	0.01492	0.048949	0.0041799
0.0064	0.0046	0.0046	0.01505	0.0012366
0.0016	0.00137	0.00137	0.0044591	0.0003589
0.0004	0.0004	0.0004	0.0012917	0.00010255
0.0001	0.00011	0.00011	0.0003717	2.9191e-005

Case 2: $\tau=2$					
ε	$\lvert\frac{u_\tau(0)}{\varepsilon}-\widetilde{B}_0\rvert$	$\lvert\frac{u_\tau(1)}{\varepsilon}-\widetilde{B}_1\rvert$	$\max\limits_{[0,1]}\lvert u\rvert$	$\lvert u_\tau'(0)-\frac{\widetilde{B}_0\Lambda_-(0)}{2}\rvert$	$\lvert u_\tau'(1)-\frac{\widetilde{B}_1\Lambda_+(1)}{2}\rvert$
0.1024	0.059783	0.015693	0.0030527	0.19421	0.0042813
0.0256	0.014915	0.0045842	0.00038543	0.048181	0.0011634
0.0064	0.0044883	0.0013726	0.00016309	0.014488	0.00033975
0.0016	0.0012428	0.00037571	4.5965e-005	0.0040116	9.2267e-005
0.0004	0.00033774	9.6848e-005	1.1853e-005	0.0010903	2.3731e-005

TABLE 2.1. *Focusing on the equation* $\varepsilon^2 u_\tau''(x)+2\varepsilon(x+1)u_\tau'(x)-[5-(x+1)^2]u_\tau(x)=0$ *under the boundary condition* (1.3) *with* $(\ell_0,\ell_1)=(0,1)$, *we consider two cases* $\tau=0.1$ *and* $\tau=2$ *to examine the results in Theorem* 2.2 *numerically. For* $\tau=0.1$, *we set* $(\mu_0,\mu_1)=(-0.1,-0.1)$ *and* $(g_0(x),g_1(x))=(1,1)$ *(cf. Figure* 2.1*). For* $\tau=2$, *we set* $(\mu_0,\mu_1)=(-0.1,-0.12)$ *and* $(g_0(x),g_1(x))=(2,e^x)$, *and get* $(\widetilde{B}_0,\widetilde{B}_1)\approx(-0.029971,-0.009617)$ *from* (2.7) *and* (2.8). *Moreover,* $\varepsilon^{1-\tau}u_\tau(0)$ *and* $\varepsilon^{1-\tau}u_\tau(1)$ *can be obtained by* (1.25). *The following numerical results with respect to various* ε *support* (2.6)–(2.9).

ε	$u(0)-B_0^\vee$	$u(1)-B_1^\vee$	$\lvert\varepsilon u'(0)-\frac{B_0^\vee\Lambda_-(0)}{2}\rvert$	$\lvert\varepsilon u'(1)-\frac{B_1^\vee\Lambda_+(1)}{2}\rvert$	$\lVert F_\varepsilon^\vee\rVert_{C^2_\varepsilon([0,1])}$
0.1024	0.00450496	0.0891017	0.028282	0.031389	0.261398
0.0256	0.00128422	0.0332327	0.007888	0.010124	0.0809492
0.0064	0.000342411	0.0102049	0.0020407	0.0029465	0.0227721
0.0016	8.21154e-005	0.00275294	0.00049884	0.00078161	0.00590314
0.0004	-6.65396e-005	0.000678467	0.00015706	0.00019291	0.00119104

TABLE 2.2. *Corresponding to numerical solutions shown in Figure* 2.2, *we have used* (2.13) *to obtain* $(B_0^\vee,B_1^\vee)\approx(0.65153,0.36546)$ *and the following errors with respect to* ε, *where* $F_\varepsilon^\vee(x)=u(x)-\Big(B_0^\vee\exp\Big\{\int_0^x\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\Big\}+B_1^\vee\exp\Big\{-\int_x^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\Big\}\Big)$ *(cf. Theorem* 2.3*(i),* (2.1) *and* (2.3)*).*

ε	$u(0)-B_0^\wedge$	$u(1)-B_1^\wedge$	$\lvert\varepsilon u'(0)-\frac{B_0^\wedge\Lambda_-(0)}{2}\rvert$	$\lvert\epsilon u'(1)-\frac{B_1^\wedge\Lambda_+(1)}{2}\rvert$	$\lvert\lvert F_\varepsilon^\wedge\rvert\rvert_{C_\varepsilon^2([0,1])}$
0.1024	0.00235323	0.00478142	0.0053066	0.0035313	0.0198415
0.0256	0.00077098	0.00179304	0.0017663	0.00076325	0.00756108
0.0064	0.000215058	0.000533187	0.00051281	0.00017285	0.0022747
0.0016	5.82104e-005	0.000143458	0.00014252	4.0972e-005	0.000637809
0.0004	1.44721e-005	3.6228e-005	3.5365e-005	1.0168e-005	0.00015871

TABLE 2.3. *Corresponding to numerical solutions shown in Figure 2.3, we have used (2.14) to obtain* $(B_0^\wedge, B_1^\wedge)\approx(0.12822, 0.20934)$ *and the following errors with respect to* ε*, where* $F_\varepsilon^\wedge(x)=u(x)-\left(B_0^\wedge\exp\left\{\int_0^x\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}+B_1^\wedge\exp\left\{-\int_x^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\right)$ *(cf. Theorem 2.3(ii)).*

where $l_0\in[0,1)$ and $l_1\in(0,1]$, and $p_0,p_1>1$. To study the nontrivial case, for (2.10) and (2.11) we shall assume

$$g_0,g_1\in\mathrm{C}([0,1])\quad\text{with}\quad g_0(1)g_1(0)\neq 0. \tag{2.12}$$

Recall that the existence and uniqueness of (1.1) is determined by the existence and uniqueness of the boundary values $u(0)$ and $u(1)$. However, it should be stressed that under boundary conditions (2.10) or (2.11), various situations (including non-existence, uniqueness and multiplicity) for solutions u of (1.1) will occur. More precisely, by a similar argument as in (1.10) and (1.16), we shall apply (1.7) and Proposition 1.2 to deal with the boundary conditions (2.10) and (2.11), respectively. After making appropriate manipulations, we obtain the corresponding systems (see, also, Theorem 2.3):

$$\begin{bmatrix}\mathcal{X}_0\\ \mathcal{X}_1\end{bmatrix}-\begin{bmatrix}-\dfrac{2g_0(0)}{p_0\Lambda_-(0)}\mathbb{1}_{\{0\}}(l_0) & \dfrac{2g_0(1)}{p_0\Lambda_+(1)}\\ -\dfrac{2g_1(0)}{p_1\Lambda_-(0)} & \dfrac{2g_1(1)}{p_1\Lambda_+(1)}\mathbb{1}_{\{1\}}(l_1)\end{bmatrix}\begin{bmatrix}|\mathcal{X}_0|^{p_0}\\ |\mathcal{X}_1|^{p_1}\end{bmatrix}=\begin{bmatrix}\mu_0\\ \mu_1\end{bmatrix} \tag{2.13}$$

for (2.10), and

$$\begin{bmatrix}\mathcal{X}_0\\ \mathcal{X}_1\end{bmatrix}-\begin{bmatrix}-\dfrac{\Lambda_-(0)g_0(0)}{4}\mathbb{1}_{\{0\}}(l_0) & \dfrac{\Lambda_+(1)g_0(1)}{4}\\ -\dfrac{\Lambda_-(0)g_1(0)}{4} & \dfrac{\Lambda_+(1)g_1(1)}{4}\mathbb{1}_{\{1\}}(l_1)\end{bmatrix}\begin{bmatrix}\mathcal{X}_0^2\\ \mathcal{X}_1^2\end{bmatrix}=\begin{bmatrix}\mu_0\\ \mu_1\end{bmatrix} \tag{2.14}$$

for (2.11). We assume that each of these two systems has at least one solution $(\mathcal{X}_0,\mathcal{X}_1)$ (cf. Remark 2.1) which asserts the existence of u as $\varepsilon>0$ is sufficiently small. We will establish the refined asymptotic behavior of u to the Equation (1.1) with the boundary conditions (2.10) and (2.11) separately.

REMARK 2.1. We provide examples for the existence of systems (2.13) and (2.14), where we consider the case $l_0\in(0,1]$, $l_1\in[0,1)$, $\mu_i>0$, $i=0,1$, and g_j's satisfy $\Lambda_+(1)g_0(1)<0<\Lambda_-(0)g_1(0)$ for simplicity.

(i) When $\left(\mu_0+\frac{2g_0(1)}{p_0\Lambda_+(1)}\mu_1^{p_1}\right)\left(\mu_1-\frac{2g_1(0)}{p_1\Lambda_-(0)}\mu_0^{p_0}\right)>0$, (2.13) has a positive solution $(\mathcal{X}_0,\mathcal{X}_1)$ satisfying $\mathcal{X}_0^{p_1}<-\frac{\mu_0p_0\Lambda_+(1)}{2g_0(1)}$ and $\mathcal{X}_1^{p_0}<\frac{\mu_1p_1\Lambda_-(0)}{2g_1(0)}$.

(ii) When $\left(2\sqrt{\mu_0}-\mu_1\sqrt{-\Lambda_+(1)g_0(1)}\right)\left(2\sqrt{\mu_1}-\mu_0\sqrt{\Lambda_-(0)g_1(0)}\right)>0$, (2.14) has two solutions $(\mathcal{X}_{0,-},\mathcal{X}_{1,-})$ and $(\mathcal{X}_{0,+},\mathcal{X}_{1,+})$ satisfying $\mathcal{X}_{0,-}<0<\mathcal{X}_{0,+}$, $\mathcal{X}_{1,-}<0<\mathcal{X}_{1,+}$ and $\mathcal{X}_{0,\pm}^2<-\frac{4\mu_0}{\Lambda_+(1)g_0(1)}$ and $\mathcal{X}_{1,\pm}^2<\frac{4\mu_1}{\Lambda_-(0)g_1(0)}$.

(i) and (ii) are easy to check via the intermediate value theorem so we omit the proof here.

Note that the solution of (1.1) is uniquely determined by boundary values $u(0)$ and $u(1)$. Under the situation (see, e.g., Remark 2.1) that both systems (2.13) and (2.14) have at least one solution $(\mathcal{X}_0,\mathcal{X}_1)$, our conclusions are stated as follows:

THEOREM 2.3. *Under the same hypotheses as in Proposition 1.1, we assume* (2.12). *Then we have the following results.*

(i) *If* $(\mathcal{X}_0,\mathcal{X}_1)=(B_0^{\vee},B_1^{\vee})$ *is a solution of system* (2.13), *then as* $\varepsilon>0$ *is sufficiently small,* (1.1) *with the boundary condition* (2.10) *has a solution satisfying* $(u(0),u(1))\xrightarrow{\varepsilon\downarrow 0}(B_0^{\vee},B_1^{\vee})$ *and*

$$\varepsilon u'(0)\xrightarrow{\varepsilon\downarrow 0}\frac{B_0^{\vee}}{2}\Lambda_-(0),\quad \varepsilon u'(1)\xrightarrow{\varepsilon\downarrow 0}\frac{B_1^{\vee}}{2}\Lambda_+(1). \tag{2.15}$$

Moreover, the asymptotics of u *in* $\mathrm{C}^2([0,1])$ *satisfies* (2.3) *with* $(B_0,B_1)=(B_0^{\vee},B_1^{\vee})$.

(ii) *If* $(\mathcal{X}_0,\mathcal{X}_1)=(B_0^{\wedge},B_1^{\wedge})$ *is a solution of system* (2.14), *then as* $\varepsilon>0$ *is sufficiently small,* (1.1) *with the boundary condition* (2.11) *has a solution satisfying* $(u(0),u(1))\xrightarrow{\varepsilon\downarrow 0}(B_0^{\wedge},B_1^{\wedge})$ *and*

$$\varepsilon u'(0)\xrightarrow{\varepsilon\downarrow 0}\frac{B_0^{\wedge}}{2}\Lambda_-(0),\quad \varepsilon u'(1)\xrightarrow{\varepsilon\downarrow 0}\frac{B_1^{\wedge}}{2}\Lambda_+(1). \tag{2.16}$$

Moreover, the asymptotics of u *in* $\mathrm{C}^2([0,1])$ *satisfies* (2.3) *with* $(B_0,B_1)=(B_0^{\wedge},B_1^{\wedge})$.

We will state the proof of Theorem 2.3 in Section 5.

Numerically, once the boundary data $(\mathcal{X}_0,\mathcal{X}_1)$ is recovered from the 2-variable algebraic system (2.13) or (2.14), then a numerical solution $u(x)$ of the Equation (1.1) with the nonlinear boundary conditions (2.10) or (2.11) can be easily obtained from (1.7), i.e., $u(x)=\mathcal{X}_0 v(x)+\mathcal{X}_1 w(x)$, where $v(x)$ and $w(x)$ are solutions of (1.8) and (1.9), respectively. It is known that for the standard 2-point boundary value problems, a wide variety of numerical methods such as the Finite Difference/Element Method can be applied. Based on this concept, we use the subroutine BVP4C in Matlab to solve (1.8) and (1.9) and the subroutine FSOLVE in Matlab to solve (2.13) or (2.14), respectively. Before the rigorous asymptotic analysis, we check the results of the estimate (2.3) and Theorem 2.3 numerically in Tables 2.2–2.3, where the values of $u(i)$ and $u'(i)$, $i=0,1$, are obtained via numerical solutions shown in Figures 2.2 and 2.3.

2.3. The main results for the inhomogeneous case. The homogeneous case is completely studied so that we now can apply the results to the asymptotic behavior of an inhomogeneous equation

$$\varepsilon^2 u''(x)+\varepsilon a(x)u'(x)-b(x)u(x)=f(x),\quad x\in(0,1), \tag{2.17}$$

with the boundary condition (1.2), where the source term $f:[0,1]\to\mathbb{R}$ is a smooth function. To depict the asymptotic profile of u as $\varepsilon>0$ approaches zero, we introduce the homogeneous equations

$$\begin{cases}\varepsilon^2 U_0''(x)+\varepsilon a(x)U_0'(x)-b(x)U_0(x)=0,\quad x\in(0,1),\\ U_0(0)=\mathfrak{m}_0(0)+\dfrac{1}{\varepsilon}\displaystyle\int_{l_0}^{1}g_0U_0\,\mathsf{d}x,\quad U_0(1)=\mathfrak{m}_0(1)+\dfrac{1}{\varepsilon}\displaystyle\int_0^{l_1}g_1U_0\,\mathsf{d}x,\end{cases} \tag{2.18}$$

and

$$\begin{cases} \varepsilon^2 U_1''(x)+\varepsilon a(x)U_1'(x)-b(x)U_1(x)=0, \quad x\in(0,1), \\ U_1(0)=\mathfrak{m}_1(0)+\frac{1}{\varepsilon}\int_{l_0}^1 g_0U_1\,\mathsf{d}x, \quad U_1(1)=\mathfrak{m}_1(1)+\frac{1}{\varepsilon}\int_0^{l_1} g_1U_1\,\mathsf{d}x, \end{cases} \tag{2.19}$$

where each $\mathfrak{m}_i(j)$, $i,j\in\{0,1\}$, is given as follows:

$$\begin{bmatrix} \mathfrak{m}_0(0) & \mathfrak{m}_0(1) \\ \mathfrak{m}_1(0) & \mathfrak{m}_1(1) \end{bmatrix} := \begin{bmatrix} -\int_{l_0}^1 \frac{g_0 f}{b}\,\mathsf{d}x & -\int_0^{l_1}\frac{g_1 f}{b}\,\mathsf{d}x \\ \mu_0+\frac{f(0)}{b(0)}-\int_{l_0}^1 \frac{g_0 a}{b}\left(\frac{f}{b}\right)'\mathsf{d}x & \mu_1+\frac{f(1)}{b(1)}-\int_0^{l_1}\frac{g_1 a}{b}\left(\frac{f}{b}\right)'\mathsf{d}x \end{bmatrix}. \tag{2.20}$$

Note that (2.18) *and* (2.19) *share the same form with* (1.1)–(1.2). Similar to (1.20) and (1.21), let us define

$$\mathfrak{B}_0 := \det\left((\mathcal{I}-\mathcal{A}^*)^{-1}\begin{bmatrix} \mathfrak{m}_0(0) & -\frac{2g_0(1)}{\Lambda_+(1)} \\ \mathfrak{m}_0(1) & 1-\frac{2g_1(1)}{\Lambda_+(1)}\mathbb{1}_{\{1\}}(l_1) \end{bmatrix}\right), \tag{2.21}$$

$$\mathfrak{B}_1 := \det\left((\mathcal{I}-\mathcal{A}^*)^{-1}\begin{bmatrix} 1+\frac{2g_0(0)}{\Lambda_-(0)}\mathbb{1}_{\{0\}}(l_0) & \mathfrak{m}_0(0) \\ \frac{2g_1(0)}{\Lambda_-(0)} & \mathfrak{m}_0(1) \end{bmatrix}\right). \tag{2.22}$$

Then the asymptotic behavior of solutions to (2.17) with the boundary condition (1.2) is mainly described by that of U_0 and U_1. Moreover, when U_0 is non-vanishing near boundary points, we use the asymptotic analysis to show that as $0<\varepsilon\ll 1$, u asymptotically blows up near boundary points $x=0$ and $x=1$ (see also Figures 2.4–2.5 and Table 2.4) and we are interested in the refined blow-up rate when the position x is sufficiently close to the boundary points, which is stated as follows.

THEOREM 2.4. *Let $f:[0,1]\to\mathbb{R}$ be a smooth function. Under the same hypotheses as in Proposition 1.1, we assume* (1.18). *Then as $\varepsilon\in(0,\eta)$ (with η defined in Theorem 2.1), Equation* (2.17) *with the boundary condition* (1.2) *has a unique solution $u\in\mathrm{C}^2([0,1])$ and fulfills the asymptotics*

$$\left\| u-\frac{U_0}{\varepsilon}-U_1+\frac{f}{b}\right\|_{\mathrm{C}^2_\varepsilon([0,1])} \lesssim \varepsilon, \quad \textit{as } 0<\varepsilon\ll 1. \tag{2.23}$$

Therefore, the asymptotics of u with $0<\varepsilon\ll 1$ can be obtained via (2.23) *with the direct application of Theorem 2.1 to* (2.18) *and* (2.19). *In particular,*

(i) *if $(\mathfrak{B}_0,\mathfrak{B}_1)=(0,0)$, then u is uniformly bounded in $[0,1]$, and $\left\| u-U_1+\frac{f}{b}\right\|_{\mathrm{C}^2_\varepsilon([0,1])} \xrightarrow{\varepsilon\downarrow 0} 0$;*

(ii) *if $(\mathfrak{B}_0,\mathfrak{B}_1)\neq(0,0)$, then u is uniformly bounded in any compact subset $\mathbf{K}\Subset(0,1)$ as $0<\varepsilon\ll 1$. However, for $\mathfrak{B}_0\neq 0$ (resp., $\mathfrak{B}_1\neq 0$), $|u(z_\varepsilon)|\xrightarrow{\varepsilon\downarrow 0}\infty$ as z_ε is sufficiently close to the boundary point $x=0$ (resp., $x=1$). Moreover, the pointwise blow-up rate of $u(z_\varepsilon)$'s varies with the order $\varepsilon^{-\xi(z_\varepsilon)}$ with $\xi(z_\varepsilon)\in(1-$*

$\theta_1^*,1]$ *depending sensitively on the position* z_ε, *which can be precisely described as follows:*

$$\begin{cases} \varepsilon^{1+\frac{\zeta_0}{2}\Lambda_-(0)}u(\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}) & \xrightarrow{\varepsilon\downarrow 0} \mathfrak{B}_0\mathfrak{q}^{\frac{\zeta_0}{2}\Lambda_-(0)}, \qquad \textit{for } \zeta_0\in(0,\dfrac{2\theta_1^*}{|\Lambda_-(0)|}), \\ \varepsilon u(\zeta_0\varepsilon) & \xrightarrow{\varepsilon\downarrow 0} \mathfrak{B}_0\exp\left\{\frac{\zeta_0}{2}\Lambda_-(0)\right\}, \textit{for } \zeta_0>0, \end{cases} \tag{2.24}$$

and

$$\begin{cases} \varepsilon^{1-\frac{\zeta_1}{2}\Lambda_+(1)}u(1-\zeta_1\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}) & \xrightarrow{\varepsilon\downarrow 0} \mathfrak{B}_1\mathfrak{q}^{-\frac{\zeta_1}{2}\Lambda_+(1)}, \qquad \textit{for } \zeta_1\in(0,\dfrac{2\theta_1^*}{\Lambda_+(1)}), \\ \varepsilon u(1-\zeta_1\varepsilon) & \xrightarrow{\varepsilon\downarrow 0} \mathfrak{B}_1\exp\left\{-\frac{\zeta_1}{2}\Lambda_+(1)\right\}, \textit{for } \zeta_1>0, \end{cases} \tag{2.25}$$

where $\theta_1^*\in(0,\frac{1}{2}]$ *(defined by* (2.4)*),* $\mathfrak{q}>0$ *and* ζ_j*'s are independent of* ε.

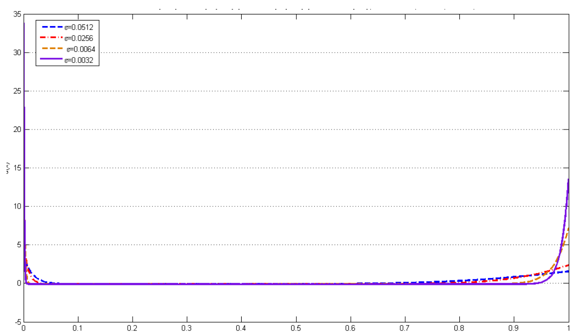

FIG. 2.4. **Asymptotic profile of u.** *Numerical solutions of equation* $\epsilon^2u''(x)+\epsilon 2\epsilon(x+1)u'(x)-[5-(x+1)^2]u(x)=0.5-0.1(x+1)^2$ *with boundary condition (1.2), where* $(\mu_0,\mu_1)=(-0.2,-0.8),(g_0(x),g_1(x))=(1,\cos(x))$ *and* $(\ell_0,\ell_1)=(0,1)$ *(cf. Theorem 2.4(ii)).*

Theorem 2.4 focusing mainly on (2.23) shows that as $\varepsilon\downarrow 0$, u develops diversified asymptotic behaviors which are determined by the relation between f and those variable coefficients a, b and g_j's, which are precisely presented in (2.20)–(2.22). For other types of boundary conditions such as (2.10) and (2.11), the studies are analogous. In this paper, we omit the details for brevity.

The proof of Theorem 2.4 will be stated in Section 6. A crucial step for the proof is to demonstrate the asymptotic expansion $u=\frac{U_0}{\varepsilon}+\left(U_1-\frac{f}{b}\right)+\varepsilon\left[\mathsf{U}_\varepsilon-\frac{a}{b}\left(\frac{f}{b}\right)'\right]+O(\varepsilon^2)$ uniformly in $[0,1]$ as $0<\varepsilon\ll 1$ (cf. (6.6)–(6.9)), where U_ε satisfies $\varepsilon^2\mathsf{U}_\varepsilon''+\varepsilon a\mathsf{U}_\varepsilon'-b\mathsf{U}_\varepsilon=0$ in $(0,1)$ and both $|\mathsf{U}_\varepsilon(0)-\frac{1}{\varepsilon}\int_{l_0}^1 g_0\mathsf{U}_\varepsilon\,\mathrm{d}x|$ and $|\mathsf{U}_\varepsilon(1)-\frac{1}{\varepsilon}\int_0^{l_1} g_1\mathsf{U}_\varepsilon\,\mathrm{d}x|$ are uniformly bounded as $0<\varepsilon\ll 1$. (Hence, by Theorem 2.1, U_ε is uniformly bounded as $\varepsilon\downarrow 0$.) We shall stress that the term $\frac{a}{b}\left(\frac{f}{b}\right)'$ appears in the boundary condition of U_1.

Finally, we state several remarks as follows.

REMARK 2.2 (A challenge in numerical simulation for the boundary blow-up rate). According to Theorem 2.4, the blow-up rate at the boundary points $x=0$ and $x=1$ is of order ε^{-1}. However, due to the limitation of computer's accuracy, it will be challenging to capture the blow-up when $\varepsilon>0$ is extremely small. In Figures 2.4–2.5, we numerically approximate the blow-up scenario for the case $\varepsilon=0.0016$; see also Table 2.4.

ε	$u(0)$	$u(1)$
0.0512	3.5461	1.5596
0.0256	5.5358	2.3649
0.0064	17.651	7.1881
0.0032	33.82	13.547
0.0016	66.155	26.222

TABLE 2.4. *Asymptotic boundary blowing-up of solutions $u(x)$ to equation $\varepsilon^2u''(x)+2\varepsilon(x+1)u'(x)-[5-(x+1)^2]u(x)=0.5-0.1(x+1)^2$ with the boundary condition (1.2) and $\varepsilon=0.0512,0.0256,0.0064,0.0032,0.0016$, where $(\mu_0,\mu_1)=(-0.2,-0.8)$, $(g_0(x),g_1(x))=(1,\cos x)$ and $(\ell_0,\ell_1)=(0,1)$; see also Figures 2.4–2.5 and Theorem 2.4(ii).*

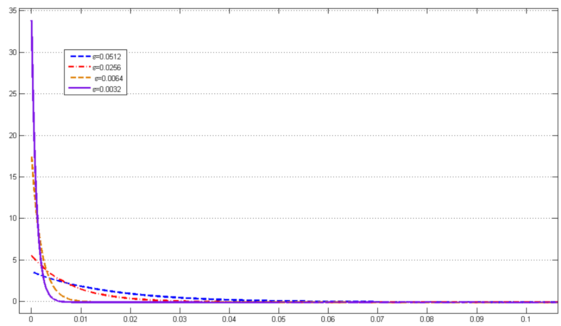

FIG. 2.5. **Asymptotic blowing-up of u near $x=0$ (zoom in)**. *Numerical solutions of equation $\varepsilon^2u''(x)+2\varepsilon(x+1)u'(x)-[5-(x+1)^2]u(x)=0.5-0.1(x+1)^2$ with the boundary condition (1.2), where $(\mu_0,\mu_1)=(-0.2,-0.8)$, $(g_0(x),g_1(x))=(1,\cos x)$ and $(\ell_0,\ell_1)=(0,1)$ (cf. Theorem 2.4(ii)).*

REMARK 2.3. (2.24) and (2.25) also present the precise leading order term of z_ε with respect to $0<\varepsilon\ll 1$ as $u(z_\varepsilon)\sim\varepsilon^{-\xi(z_\varepsilon)}$ with $\xi(z_\varepsilon)\in(1-\theta_1^*,1]$. When $\xi(z_\varepsilon)\in(0,1-\theta_1^*]$, the refined asymptotics of z_ε with respect to $0<\varepsilon\ll 1$ is difficult to verify because it involves the second order terms of $\frac{1}{\varepsilon}\int_{l_0}^1 g_0U_0\,\mathrm{d}x$ and $\frac{1}{\varepsilon}\int_0^{l_1} g_1U_0\,\mathrm{d}x$ (see (2.18) and (2.23)).

REMARK 2.4 (Closely related to Duffing-type equations). Equation (2.17) with the boundary condition (1.2) has a close relation with the following nonlocal equation with standard Dirichlet boundary conditions (e.g., a Duffing-type equation involving an integral forcing term [1]):

$$\begin{cases} \varepsilon^2\widetilde{U}_\varepsilon''(x)+\varepsilon a(x)\widetilde{U}_\varepsilon'(x)-b(x)\left(\widetilde{U}_\varepsilon(x)-\displaystyle\int_0^1\widetilde{g}_\varepsilon(z)\widetilde{U}_\varepsilon(z)\,\mathrm{d}z\right)=f(x)\text{ in }(0,1), \\ \widetilde{U}_\varepsilon(0)=\mu_0,\ \widetilde{U}_\varepsilon(1)=\mu_1, \end{cases} \tag{2.26}$$

where we assume that $\{\widetilde{g}_\varepsilon\}_{\varepsilon>0}$ satisfying $\int_0^1\widetilde{g}_\varepsilon\xrightarrow{\varepsilon\downarrow 0}1$, and $\varepsilon\widetilde{g}_\varepsilon/\int_0^1(\widetilde{g}_\varepsilon-1)\xrightarrow{\varepsilon\downarrow 0}g$ a smooth function defined in $[0,1]$. (For example, $\widetilde{g}_\varepsilon=\frac{(1+\varepsilon)g}{(1+\varepsilon)\int_0^1 g-\varepsilon}$ with $\int_0^1 g\geq 1$ and $\varepsilon>0$.) Indeed, let us set $u_\varepsilon^*=\widetilde{U}_\varepsilon-\int_0^1\widetilde{g}_\varepsilon\widetilde{U}_\varepsilon$. After making simple calculations we obtain that, as $0<\varepsilon\ll 1$, the equation of u_ε^* formally approaches Equation (2.17) with the boundary condition (1.2) of u, where $(l_0,l_1)=(0,1)$ and $g_0=g_1=g$. For such a linear Equation (2.26), the term $\int_0^1\widetilde{g}_\varepsilon\widetilde{U}_\varepsilon$ exactly gives a nonlocal perturbation with respect to ε, and the more refined asymptotic behavior of $\widetilde{U}_\varepsilon$ with $\varepsilon\downarrow 0$, to the best of our knowledge, remains to be unknown. What we want to point out is that Theorem 2.4 can be applied directly to studying this model.

3. Preliminaries: Basic properties of v and w

Firstly, let us define $\gamma > 0$ and $\widetilde{\gamma} > 0$ which satisfy

$$\gamma^2 = \gamma \min_{x\in[0,1]} \{0, a(x)\} + \beta, \tag{3.1}$$

and

$$\widetilde{\gamma}^2 = \widetilde{\gamma}\left(\min_{x\in[0,1]} \{0, a(x)\} - 1\right) + \frac{\beta}{2} \quad \text{with} \quad 0 < \widetilde{\gamma} < \gamma. \tag{3.2}$$

The existence of $\gamma > 0$ and $\widetilde{\gamma} \in (0,\gamma)$ is trivial since $\beta > 0 \geq \min\limits_{[0,1]}\{0,a\}$. Now we are ready to establish interior estimates for v and w provided that $\varepsilon > 0$ is sufficiently small.

Proposition 3.1. *Assume that a and b are smooth functions defined in $[0,1]$ with $b(x) \geq \beta$, where β is a positive constant independent of ε. Then, for $\varepsilon > 0$, both (1.8) and (1.9) have unique solutions. v is strictly decreasing on $[0,1]$, and w is strictly increasing on $[0,1]$. Moreover, for $x \in (0,1)$ there hold the following estimates.*

(i) *As $\varepsilon > 0$ is sufficiently small, v and w satisfy the exponentially decaying estimates*

$$0 \leq v(x) \leq \exp\left\{-\frac{\gamma}{\varepsilon}x\right\}, \quad 0 \leq w(x) \leq \exp\left\{-\frac{\gamma}{\varepsilon}(1-x)\right\}, \tag{3.3}$$

and

$$-\frac{C_{a,b}}{\varepsilon}\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}x\right\} \leq v'(x) < 0 < w'(x) \leq \frac{C_{a,b}}{\varepsilon}\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}(1-x)\right\}, \tag{3.4}$$

where $C_{a,b} = \max\{|\Lambda_-(0)|, \Lambda_+(1)\} > 0$ (cf. (1.4)).

(ii) *There exist $\varepsilon^\star > 0$ and $C^\star > 0$ independent of ε such that as $0 < \varepsilon < \varepsilon^\star$, the following estimates hold:*

$$\left|\sqrt{\varepsilon}v'(x) - \frac{\Lambda_-(x)}{2\sqrt{\varepsilon}}v(x)\right| \leq C^\star \exp\left\{-\frac{1}{2\varepsilon}\int_0^x a(z)\,\mathrm{d}z\right\}, \tag{3.5}$$

and

$$\left|\sqrt{\varepsilon}w'(x) - \frac{\Lambda_+(x)}{2\sqrt{\varepsilon}}w(x)\right| \leq C^\star \exp\left\{\frac{1}{2\varepsilon}\int_x^1 a(z)\,\mathrm{d}z\right\}, \tag{3.6}$$

for $x \in [0,1]$.

The proof of Proposition 3.1 will be stated in Section 3.2.

Remark 3.1. By (3.3) and (3.4), one obtains that, as $\varepsilon > 0$ is sufficiently small,

$$0 \leq v(x)w(x) \leq \exp\left\{-\frac{\gamma}{\varepsilon}\right\}, \quad 0 < |v'(x)w'(x)| \leq \exp\left\{-\frac{\widetilde{\gamma}}{2\varepsilon}\right\}, \forall x \in [0,1]. \tag{3.7}$$

3.1. Uniform asymptotics of v and w using $\varepsilon\log\varepsilon$-estimate. In this section we shall explain how Proposition 3.1 plays a crucial role in the pointwise asymptotics of v and w and Proposition 1.2, and we point out a difficulty in the refined asymptotic analysis of $v(x_1)$ and $w(x_2)$ with x_1 and $1-x_2$ sufficiently near 0.

We first point out that Proposition 1.2 is based on the uniform asymptotics of v and w in the whole domain $[0,1]$. Although (3.3) shows that $v(x_1)$ and $w(x_2)$ exponentially decay to zero as $\lim_{\varepsilon\downarrow 0}\frac{x_1}{\varepsilon}=\infty$ and $\lim_{\varepsilon\downarrow 0}\frac{1-x_2}{\varepsilon}=\infty$, and such exponentially decaying estimates are more refined than the standard outer expansions for solutions v (of Equation (1.8)) and w (of Equation (1.9)), when $\lim_{\varepsilon\downarrow 0}\left(\frac{x_1}{\varepsilon}+\frac{1-x_2}{\varepsilon}\right)<\infty$, (3.3)–(3.4) are not able to imply the refined asymptotic behavior of $v(x_1)$ and $w(x_2)$ with respect to $0<\varepsilon\ll 1$. But, for this case, by (3.5) and (3.6) we have

$$\lim_{\varepsilon\downarrow 0}\left(\left|\varepsilon v'(x_1)-\frac{\Lambda_-(x_1)}{2}v(x_1)\right|+\left|\varepsilon w'(x_2)-\frac{\Lambda_+(x_2)}{2}w(x_2)\right|\right)=0, \tag{3.8}$$

since $\exp\left\{-\frac{1}{2\varepsilon}\int_0^{x_1}a(z)\,\mathsf{d}z\right\}\le\exp\left\{\frac{x_1}{2\varepsilon}\max_{[0,1]}|a|\right\}$ and $\exp\left\{\frac{1}{2\varepsilon}\int_{x_2}^1 a(z)\,\mathsf{d}z\right\}\le \exp\left\{\frac{1-x_2}{2\varepsilon}\max_{[0,1]}|a|\right\}$ are uniformly bounded as $0<\varepsilon\ll 1$. Accordingly, we can apply (3.8) to deal with the boundary asymptotics of v and w as ε goes to zero. Moreover, we shall introduce a so-called "$\varepsilon\log\varepsilon$-estimate" to establish the refined boundary asymptotics of solutions.

Precisely speaking, when $\min_{[0,1]}a\ge 0$, (3.5) and (3.6) are indeed good estimates for asymptotic analysis of v and w in the whole domain $[0,1]$ since their right-hand sides provide uniform upper bound as $0<\varepsilon\ll 1$. On the contrary, when $\min_{[0,1]}a<0$, the situation becomes tricky, and an idea is to consider intervals $I_{\varepsilon,j}$'s with $0\in I_{\varepsilon,1}$ and $1\in I_{\varepsilon,2}$ such that

$$\ell(I_{\varepsilon,j})\xrightarrow{\varepsilon\downarrow 0}0,\qquad \frac{\ell(I_{\varepsilon,j})}{\varepsilon}\xrightarrow{\varepsilon\downarrow 0}\infty,$$

$$\sup_{I_{\varepsilon,1}}\sqrt{\varepsilon}\exp\left\{-\frac{1}{2\varepsilon}\int_0^x a(z)\,\mathsf{d}z\right\}+\sup_{I_{\varepsilon,2}}\sqrt{\varepsilon}\exp\left\{-\frac{1}{2\varepsilon}\int_x^1 a(z)\,\mathsf{d}z\right\}\xrightarrow{\varepsilon\downarrow 0}0,$$

where $\ell(I_{\varepsilon,j})$ means the length of $I_{\varepsilon,j}$. It is feasible when $\ell(I_{\varepsilon,j})$ is the order of $\varepsilon\log\frac{1}{\varepsilon}$. In doing so, by (3.3)–(3.6) we can obtain uniform asymptotics of v and w in the whole domain $[0,1]$. Although such an idea seems intuitive, it is quite inconvenient to handle. Various estimates will be established. The uniform estimate of $\mathrm{C}^2_\varepsilon([0,1])$-norm of v and w with respect to $0<\varepsilon\ll 1$ will be established in Proposition 4.1, which will be used to prove Proposition 1.2.

3.2. Proof of Proposition 3.1. Since $b(x)\ge\beta>0$, the uniqueness of (1.8) and (1.9) is obvious. By applying the maximum principle to (1.8) and (1.9), we obtain $0\le v(x),w(x)\le 1$ for $x\in[0,1]$. In particular, by the fact that v and w are nontrivial, there hold $v'(1)<0<w'(0)$.

Multiplying the equation of v in (1.8) by $\exp\left\{\int_1^x\frac{a(z)}{\varepsilon}\,\mathsf{d}z\right\}$, one finds

$$\left(v'(x)\exp\left\{\int_1^x\frac{a(z)}{\varepsilon}\,\mathsf{d}z\right\}\right)'=\frac{b(x)}{\varepsilon^2}v(x)\exp\left\{\int_1^x\frac{a(z)}{\varepsilon}\,\mathsf{d}z\right\}\ge 0,\ x\in[0,1]. \tag{3.9}$$

Thus, $v'(x)\exp\left\{\int_1^x \frac{a(z)}{\varepsilon}\,\mathrm{d}z\right\}$ is increasing and attains the maximum value at $x=1$. As a consequence, for $x\in[0,1]$, it holds $v'(x)\exp\left\{\int_1^x \frac{a(z)}{\varepsilon}\,\mathrm{d}z\right\}\le v'(1)<0$. This shows that v is strictly decreasing on $[0,1]$. Using the similar argument, we obtain $w'(x)\exp\left\{\int_0^x \frac{a(z)}{\varepsilon}\,\mathrm{d}z\right\}\ge w'(0)>0$, and w is strictly increasing on $[0,1]$. We first prove (3.3) as follows.

Proof. **(Proof of (3.3)).** Define

$$\Gamma(x):=v(x)-\exp\left\{-\frac{\gamma}{\varepsilon}x\right\},$$

where $\gamma>0$ satisfies (3.1) which asserts

$$\gamma^2=\gamma\min_{x\in[0,1]}\{0,a(x)\}+\beta\le\gamma a(x)+b(x).$$

Hence, by (1.8) one obtains, for $x\in(0,1)$, that

$$\varepsilon^2\Gamma''(x)+\varepsilon a(x)\Gamma'(x)-b(x)\Gamma(x)=(-\gamma^2+\gamma a(x)+b(x))\exp\left\{-\frac{\gamma}{\varepsilon}x\right\}\ge 0.$$

This along with $\Gamma(0)=0>\Gamma(1)$ immediately implies $\Gamma(x)\le 0$, i.e., $v(x)\le\exp\left\{-\frac{\gamma}{\varepsilon}x\right\}$. Similarly, we can obtain $w(x)\le\exp\left\{-\frac{\gamma}{\varepsilon}(1-x)\right\}$ and complete the proof of (3.3). ◻

For the sake of convenience, we will prove (3.4) after we complete the proof of (ii). The proof of (3.5) and (3.6) are stated as follows.

Proof. **(Proof of (3.5) and (3.6)).** To prove (3.5), we apply the standard transformation

$$V(x)=v(x)\exp\left\{\frac{1}{2\varepsilon}\int_0^x a(z)\,\mathrm{d}z\right\} \tag{3.10}$$

to (1.8), which transforms Equation (1.8) into an equation of V without a convection term:

$$\varepsilon^2V''(x)=\left(\frac{a^2(x)}{4}+\frac{\varepsilon a'(x)}{2}+b(x)\right)V(x)\text{ in }(0,1);\ V(0)=1,\ V(1)=0. \tag{3.11}$$

Since $a'(x)$ is uniformly bounded in $[0,1]$ and $b(x)\ge\beta>0$, from (3.11) we may set

$$\varepsilon^\star=\min_{[0,1]}\frac{\beta}{1+|a'|} \tag{3.12}$$

such that, as $0<\varepsilon<\varepsilon^\star$, we arrive at

$$\varepsilon^2V''(x)\ge\beta\left(1-\frac{\varepsilon}{\beta}\max_{[0,1]}|a'|\right)^2V(x)\quad\text{in }(0,1). \tag{3.13}$$

Here we have used the property $0\le V(x)\le 1$ and a suitable lower bound for $\frac{a^2(x)}{4}+\frac{\varepsilon a'(x)}{2}+b(x)$ (with respect to $0<\varepsilon<\varepsilon^\star$) as follows:

$$\frac{a^2(x)}{4}+\frac{\varepsilon a'(x)}{2}+b(x)\ge-\frac{\varepsilon}{2}\max_{[0,1]}|a'|+\beta$$

$$\geq \beta\left(1-\frac{\varepsilon}{\beta}\max_{[0,1]}|a'|\right)^2+\varepsilon\max_{[0,1]}|a'|\left(1-\frac{\varepsilon}{\beta}\max_{[0,1]}|a'|\right)$$
$$\geq \beta\left(1-\frac{\varepsilon}{\beta}\max_{[0,1]}|a'|\right)^2.$$

It should be stressed that (3.12) is a sufficient condition for the above estimate, where we consider $\frac{\beta}{1+|a'|}$ instead of $\frac{\beta}{|a'|}$ since $a(x)$ may be a constant-valued function. As a consequence, by (3.13) and the boundary condition of V in (3.11), we obtain a super-solution

$$V_{\text{sup}}(x):=\exp\left\{-\sqrt{\beta}\left(1-\frac{\varepsilon}{\beta}\max_{[0,1]}|a'|\right)\frac{x}{\varepsilon}\right\} \tag{3.14}$$

for Equation (3.11). Hence, $0\leq V(x)\leq V_{\text{sup}}(x)$ for $x\in[0,1]$.

Now we shall prove (3.5). Firstly, we can choose $x_\varepsilon\in(0,1)$ such that $V'(x_\varepsilon)=V(1)-V(0)=-1$. Notice that for $0<\varepsilon<\varepsilon^\star$, by (3.11), we have $V''\geq 0$. In particular, it implies $-1=V'(x_\varepsilon)\leq V'(1)\leq 0$. Furthermore, multiplying (3.11) by $V'(x)$ yields

$$\left\{\varepsilon^2V'^2(x)-\left(\frac{a^2(x)}{4}+b(x)+\varepsilon a'(x)\right)V^2(x)\right\}'=-\left(\frac{a(x)a'(x)}{2}+b'(x)+\varepsilon a''(x)\right)V^2(x).$$

As a consequence, for $x\in[0,1]$, by (3.14) one may obtain the following estimate from the above equation:

$$\begin{aligned}&\left|\varepsilon^2V'^2(x)-\left(\frac{a^2(x)}{4}+b(x)+\varepsilon a'(x)\right)V^2(x)\right|\\ &\leq\varepsilon^2V'^2(1)+\int_x^1\left|\frac{a(z)a'(z)}{2}+b'(z)+\varepsilon a''(z)\right|V_{\text{sup}}^2(z)\,\mathsf{d}z\\ &\leq\varepsilon^2+\frac{\varepsilon}{\sqrt{\beta}}\max_{[0,1]}\left|\frac{aa'}{2}+b'+\varepsilon a''\right|\leq M\varepsilon,\end{aligned} \tag{3.15}$$

where M is a positive constant independent of ε. Since $V'\leq 0\leq V\leq 1$ and $\frac{a^2}{4}+b>0$, we have $\left|\varepsilon V'(x)+\sqrt{\frac{a^2(x)}{4}+b(x)}V(x)\right|\leq\left|\varepsilon V'(x)-\sqrt{\frac{a^2(x)}{4}+b(x)}V(x)\right|$. This along with (3.15) immediately implies

$$\left|\varepsilon V'(x)+\sqrt{\frac{a^2(x)}{4}+b(x)}V(x)\right|\leq\sqrt{(a'(x)V^2(x)+M)\varepsilon}\leq\sqrt{\left(\max_{[0,1]}|a'|+M\right)\varepsilon}. \tag{3.16}$$

Therefore, (3.5) follows from (3.10), (3.16) and a simple calculation $V'(x)=(v'(x)+\frac{a(x)}{2\varepsilon}v(x))\exp\{\frac{1}{2\varepsilon}\int_0^x a(z)\,\mathsf{d}z\}$.

We next consider

$$W(x)=w(x)\exp\left\{-\frac{1}{2\varepsilon}\int_x^1 a(z)\,\mathsf{d}z\right\}.$$

Note that w satisfies (1.9). Thus, we have

$$\varepsilon^2W''(x)=\left(\frac{a^2(x)}{4}+\frac{\varepsilon a'(x)}{2}+b(x)\right)W(x)\text{ in }(0,1);\ W(0)=0,\ W(1)=1. \tag{3.17}$$

In particular, there hold $W(x)\geq 0$ and $W'(x)\geq 0$, $x\in[0,1]$. Following a similar argument of (3.11)–(3.16) on (3.17), we can obtain

$$\varepsilon^2 W''(x)\geq\beta\Big(1-\frac{\varepsilon}{\beta}\max_{[0,1]}|a'|\Big)^2 W(x)$$

and

$$\left|\varepsilon W'(x)-\sqrt{\frac{a^2(x)}{4}+b(x)}W(x)\right|\leq\sqrt{\Big(\max_{[0,1]}|a'|+M\Big)\varepsilon},\qquad x\in[0,1],$$

as $0<\varepsilon<\varepsilon^\star$. Thus (3.6) immediately follows from these estimates. □

Completion of the proof of Proposition 3.1. It suffices to prove (3.4). Recall $v(0)=1$ and $w(1)=1$. By (3.5) and (3.6) we obtain boundary asymptotics

$$\lim_{\varepsilon\downarrow 0}\varepsilon v'(0)=\frac{\Lambda_-(0)}{2},\quad \lim_{\varepsilon\downarrow 0}\varepsilon w'(1)=\frac{\Lambda_+(1)}{2}. \tag{3.18}$$

By (3.3) there exists $x_\varepsilon^*\in(1-\varepsilon,1)$ such that $\varepsilon|v'(x_\varepsilon^*)|=|v(1)-v(1-\varepsilon)|\leq\exp\{-\frac{\gamma}{\varepsilon}+\gamma\}$. Along with (3.9) we know that

$$|v'(1)|\leq|v'(x_\varepsilon^*)|\exp\left\{\int_{x_\varepsilon^*}^1\frac{a(z)}{\varepsilon}\mathrm{d}z\right\}\leq\frac{1}{\varepsilon}\exp\left\{-\frac{\gamma}{\varepsilon}+\gamma+\max_{[0,1]}|a|\right\} \tag{3.19}$$

since $|1-x_\varepsilon^*|<\varepsilon$. Applying the same argument, we can obtain the same upper bound of $w'(0)$ as in (3.19). As a consequence,

$$v'(1)\xrightarrow{\varepsilon\downarrow 0}0\quad\text{and}\quad w'(0)\xrightarrow{\varepsilon\downarrow 0}0\quad\text{exponentially.} \tag{3.20}$$

Differentiating Equation (1.8) with respect to x, we obtain $\varepsilon^2 v'''(x)+\varepsilon a(x)v''(x)-(b(x)-\varepsilon a'(x))v'(x)=b'(x)v(x)$. On the other hand, multiplying Equation (1.8) by $\frac{b'(x)}{b(x)}$ and combining the result with the previous equation, we arrive at

$$\varepsilon^2 v'''(x)+\varepsilon\left(a(x)-\varepsilon\frac{b'(x)}{b(x)}\right)v''(x)-\left(b(x)-\varepsilon a'(x)+\varepsilon a(x)\frac{b'(x)}{b(x)}\right)v'(x)=0. \tag{3.21}$$

Now we define

$$v_\star(x)=v'(x)-v'(0)\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}x\right\}, \tag{3.22}$$

where $\widetilde{\gamma}>0$ was defined in (3.2). Firstly, we claim that there exists $\varepsilon_\star(\gamma,\widetilde{\gamma})>0$ depending on γ and $\widetilde{\gamma}$ such that as $0<\varepsilon<\varepsilon_\star(\gamma,\widetilde{\gamma})$,

$$v_\star(0)=0,\ v_\star(1)>0. \tag{3.23}$$

Proof. **(Proof of (3.23)).** Obviously, $v_\star(0)=0$. Note that $v'(0),v'(1)<0$. Since $\widetilde{\gamma}<\gamma$ and $b(0)\geq\beta>0$, by (3.18), (3.19) and (3.22), one may check that

$$v_\star(1)=v'(1)-v'(0)\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}\right\}$$

$$\geq -\frac{1}{\varepsilon}\exp\left\{-\frac{\gamma}{\varepsilon}+\gamma+\max_{[0,1]}|a|\right\}+\frac{|\Lambda_-(0)|}{4\varepsilon}\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}\right\}$$
$$=\frac{1}{\varepsilon}\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}\right\}\left(-\exp\left\{-\frac{\gamma-\widetilde{\gamma}}{\varepsilon}+\gamma+\max_{[0,1]}|a|\right\}+\frac{|\Lambda_-(0)|}{4}\right)>0,$$

where the term in the last parentheses is positive provided that $0<\varepsilon<\varepsilon_\star(\gamma,\widetilde{\gamma})$ with sufficiently small $\varepsilon_\star(\gamma,\widetilde{\gamma})>0$. This completes the proof. ∎

By (3.21) and (3.22), a direct calculation yields

$$\begin{aligned}
&\varepsilon^2 v_\star''(x)+\varepsilon\left(a(x)-\varepsilon\frac{b'(x)}{b(x)}\right)v_\star'(x)-\left(b(x)-\varepsilon a'(x)+\varepsilon a(x)\frac{b'(x)}{b(x)}\right)v_\star(x)\\
&=-v'(0)\left[\widetilde{\gamma}^2-\left(a(x)-\varepsilon\frac{b'(x)}{b(x)}\right)\widetilde{\gamma}-\left(b(x)-\varepsilon a'(x)+\varepsilon a(x)\frac{b'(x)}{b(x)}\right)\right]\\
&\leq -v'(0)\left[\left(\min_{x\in[0,1]}\{0,a(x)\}-1-a(x)+\varepsilon\frac{b'(x)}{b(x)}\right)\widetilde{\gamma}+\left(\frac{\beta}{2}-b(x)+\varepsilon a'(x)-\varepsilon a(x)\frac{b'(x)}{b(x)}\right)\right].
\end{aligned}$$

Here we have used the fact $-v'(0)>0$. Moreover, since $a'(x)$ and $b'(x)$ are bounded and $b(x)\geq\beta>0$, there exists $\varepsilon^\star(\gamma,\widetilde{\gamma})\in(0,\varepsilon_\star(\gamma,\widetilde{\gamma}))$ such that

$$\varepsilon\max_{[0,1]}\frac{|b'|}{b}<1 \quad\text{and}\quad b(x)-\varepsilon a'(x)+\varepsilon a(x)\frac{b'(x)}{b(x)}>\frac{\beta}{2}$$

as $0<\varepsilon<\varepsilon^\star(\gamma,\widetilde{\gamma})$. As a consequence, for $0<\varepsilon<\varepsilon^\star(\gamma,\widetilde{\gamma})$, we have

$$\varepsilon^2 v_\star''(x)+\varepsilon\left(a(x)-\varepsilon\frac{b'(x)}{b(x)}\right)v_\star'(x)-\left(b(x)-\varepsilon a'(x)+\varepsilon a(x)\frac{b'(x)}{b(x)}\right)v_\star(x)<0. \tag{3.24}$$

Applying the maximum principle to (3.24) and using (3.23), we obtain $v_\star(x)\geq 0$. By (3.18) and (3.22), we thus arrive at $\Lambda_-(0)\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}x\right\}\leq\varepsilon v'(x)<0$ as $0<\varepsilon<\varepsilon^\star(\gamma,\widetilde{\gamma})$. Similarly, we can prove $0<\varepsilon w'(x)\leq\Lambda_+(1)\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}(1-x)\right\}$ as $0<\varepsilon<\varepsilon^\star(\gamma,\widetilde{\gamma})$. Therefore, we obtain (3.4) and the proof of Proposition 3.1 is indeed complete.

3.3. Proof of Proposition 1.2. In what follows we let $\phi\in C([0,1])$ except when it is specifically emphasized otherwise.

By (3.3) and (3.4), we have, for $p\geq 1$, $\frac{v^p}{\varepsilon},\varepsilon|v'|^2\to 0$ exponentially in $(0,1]\supseteq[\delta_0,1]$, and $\frac{w^p}{\varepsilon},\varepsilon|w'|^2\to 0$, exponentially in $[0,1)\supseteq[0,\delta_1]$ as $\varepsilon\downarrow 0$. Hence, (1.15) immediately follows.

We first consider the case $p>1$. Due to (1.15), it suffices to prove (1.13) and (1.14) for the cases $\delta_0=1$ and $\delta_1=0$. We shall give more refined estimates as follows.

Claim 1. For $p>1$, let

$$\theta_p^*=\begin{cases}\dfrac{1}{2}, & \text{if } \min\limits_{[0,1]}a\geq 0,\\[2ex] \min\limits_{[0,1]}\dfrac{p\gamma}{2p\gamma+|a|}\in(0,\dfrac{1}{2}), & \text{if } \min\limits_{[0,1]}a<0.\end{cases}$$

Then for $\phi\in C([0,1])$, there holds

$$\limsup_{\varepsilon\downarrow 0}\varepsilon^{-\theta_p^*}\left(\left|\int_0^1\frac{v^p(x)}{\varepsilon}\phi(x)\,\mathrm{d}x+\frac{2\phi(0)}{p\Lambda_-(0)}\right|+\left|\int_0^1\frac{w^p(x)}{\varepsilon}\phi(x)\,\mathrm{d}x-\frac{2\phi(1)}{p\Lambda_+(1)}\right|\right)<\infty. \tag{3.25}$$

REMARK 3.2. *Although for Proposition 1.2, such a refined estimate is not required, (3.25) will be used to prove (2.3) and Theorem 2.4.*

Proof. **(Proof of Claim 1).** Observe first that for any fixed number $\theta > 0$, by (3.3), we have

$$\left|\int_0^1 \frac{v^p(x)}{\varepsilon}\phi(x)\,\mathrm{d}x - \int_0^{\theta\varepsilon\log\frac{1}{\varepsilon}} \frac{v^p(x)}{\varepsilon}\phi(x)\,\mathrm{d}x\right| \le \int_{\theta\varepsilon\log\frac{1}{\varepsilon}}^1 \frac{v^p(x)}{\varepsilon}|\phi(x)|\,\mathrm{d}x \le \frac{\varepsilon^{p\theta\gamma}}{p\gamma}\max_{[0,1]}|\phi|. \tag{3.26}$$

On the other hand, from (3.3) and (3.5) one may observe that, under a suitable choice of θ, the difference between $\frac{v^p}{\varepsilon}$ and $-\frac{2}{\Lambda_-}v^{p-1}v'$ in $[0,\theta\varepsilon\log\frac{1}{\varepsilon}]$ tends to zero as $\varepsilon\downarrow 0$. Hence, for the purpose of dealing with (3.26) we take an estimate

$$\begin{aligned}
&\left|\int_0^{\theta\varepsilon\log\frac{1}{\varepsilon}} \left(\frac{v^p(x)}{\varepsilon} - \frac{2}{\Lambda_-(x)}v^{p-1}(x)v'(x)\right)\phi(x)\,\mathrm{d}x\right| \\
\le & C^\star \int_0^{\theta\varepsilon\log\frac{1}{\varepsilon}} \frac{2v^{p-1}(x)}{\sqrt{\varepsilon}}\left|\frac{\phi(x)}{\Lambda_-(x)}\right| \exp\left\{-\frac{1}{2\varepsilon}\int_0^x a(z)\,\mathrm{d}z\right\}\mathrm{d}x, \\
\le & \frac{C^\star}{\beta}\left(\max_{[0,1]}|\phi|\left(|a|+\sqrt{\beta}\right)\right)\exp\left\{\frac{\theta}{2}\max_{[0,1]}|a|\log\frac{1}{\varepsilon}\right\}\int_0^{\theta\varepsilon\log\frac{1}{\varepsilon}} \frac{v^{p-1}(x)}{\sqrt{\varepsilon}}\,\mathrm{d}x \\
\le & \frac{C^\star}{\beta\gamma(p-1)}\left(\max_{[0,1]}|\phi|\left(|a|+\sqrt{\beta}\right)\right)\varepsilon^{\frac{1}{2}\left(1-\theta\max_{[0,1]}|a|\right)},
\end{aligned} \tag{3.27}$$

which exactly tends to zero provided that $0<\theta<\frac{1}{\max_{[0,1]}|a|}$. Here we have used an elementary inequality[1]

$$\min_{[0,1]}\frac{1}{2\left(|a|+\sqrt{b}\right)} \le \frac{1}{|\Lambda_-(x)|} \le \max_{[0,1]}\frac{|a|+\sqrt{\beta}}{2\beta}, \quad x\in[0,1], \tag{3.28}$$

to get the third line of (3.27). For the last line of (3.27) we have used (3.3) to obtain $\int_0^{\theta\varepsilon\log\frac{1}{\varepsilon}} v^{p-1}\,\mathrm{d}x \le \frac{\varepsilon}{\gamma(p-1)}$. In particular, setting

$$I_{\varepsilon\log\frac{1}{\varepsilon}} := [0,\theta\varepsilon\log\frac{1}{\varepsilon}] \quad \text{with} \quad \theta = \min_{[0,1]}\frac{1}{2p\gamma+|a|},$$

i.e., $p\theta\gamma = \frac{1}{2}\left(1-\theta\max_{[0,1]}|a|\right) := \theta_p^*$, we obtain, from (3.26) and (3.27), that

$$\left|\int_0^1 \frac{v^p(x)}{\varepsilon}\phi(x)\,\mathrm{d}x - \int_{I_{\varepsilon\log\frac{1}{\varepsilon}}} \frac{2}{\Lambda_-(x)}v^{p-1}(x)v'(x)\phi(x)\,\mathrm{d}x\right| \lesssim \varepsilon^{\theta_p^*}. \tag{3.29}$$

It remains to deal with (3.29). Let $D(x) := -\frac{2\phi(x)}{\Lambda_-(x)} + \frac{2\phi(0)}{\Lambda_-(0)}$. Notice first that the con-

[1]The left-hand estimate of (3.28) is trivial. The right-hand estimate is due to $\frac{1}{|\Lambda_-|} = \frac{1}{a+\sqrt{a^2+4\beta}} \le \frac{|a|+2\sqrt{\beta}-a}{4\beta} \le \frac{|a|+\sqrt{\beta}}{2\beta}$.

tinuous differentiability of $\frac{2\phi(x)}{\Lambda_-(x)}$ indicates $\max\limits_{I_{\varepsilon\log\frac{1}{\varepsilon}}}|D| \lesssim \varepsilon\log\frac{1}{\varepsilon} \xrightarrow{\varepsilon\downarrow 0} 0$. Hence,

$$
\begin{aligned}
&\left|\int\limits_{I_{\varepsilon\log\frac{1}{\varepsilon}}} \frac{2}{\Lambda_-(x)} v^{p-1}(x)v'(x)\phi(x)\,\mathrm{d}x + \frac{2\phi(0)}{p\Lambda_-(0)}\right| \\
\leq & 2\left|\frac{\phi(0)}{\Lambda_-(0)}\left(\int\limits_{I_{\varepsilon\log\frac{1}{\varepsilon}}} v^{p-1}(x)v'(x)\,\mathrm{d}x + \frac{1}{p}\right)\right| + \left|\int\limits_{I_{\varepsilon\log\frac{1}{\varepsilon}}} D(x)v^{p-1}(x)v'(x)\,\mathrm{d}x\right| \\
= & \frac{2|\phi(0)|v^p(\theta\varepsilon\log\frac{1}{\varepsilon})}{p|\Lambda_-(0)|} + \left|\int\limits_{I_{\varepsilon\log\frac{1}{\varepsilon}}} D(x)v^{p-1}(x)v'(x)\,\mathrm{d}x\right| \lesssim \varepsilon^{\theta_p^*}.
\end{aligned} \tag{3.30}
$$

Here we have used the facts that $v(\theta\varepsilon\log\frac{1}{\varepsilon}) = O(\varepsilon^{\theta\gamma})$ (by (3.3)), and

$$
\left|\int\limits_{I_{\varepsilon\log\frac{1}{\varepsilon}}} D(x)v^{p-1}(x)v'(x)\,\mathrm{d}x\right| \leq \frac{1}{p}\max_{I_{\varepsilon\log\frac{1}{\varepsilon}}}|D| \lesssim \varepsilon\log\frac{1}{\varepsilon} \ll \varepsilon^{\theta_p^*},
$$

since $\theta_p^* \leq \frac{1}{2}$. Therefore, by (3.29) and (3.30), we obtain (3.25). Similarly, we have

$$
\limsup_{\varepsilon\downarrow 0} \varepsilon^{-\theta_p^*}\left|\int_0^1 \frac{w^p(x)}{\varepsilon}\phi(x)\,\mathrm{d}x - \frac{2\phi(1)}{p\Lambda_+(1)}\right| < \infty, \qquad p > 1.
$$

Thus, we prove (1.13) and complete the proof of Claim 1. ∎

Next we deal with the estimate of $\int_0^1 \varepsilon v'^2\phi\,\mathrm{d}x$. By (3.4) we have

$$
\begin{aligned}
\int\limits_{[0,1]\setminus I_{\varepsilon\log\frac{1}{\varepsilon}}} \varepsilon v'^2(x)\phi(x)\,\mathrm{d}x &\leq \frac{1}{\varepsilon}\max_{[0,1]}|\phi| \int\limits_{[0,1]\setminus I_{\varepsilon\log\frac{1}{\varepsilon}}} \exp\left\{-\frac{2\widetilde{\gamma}}{\varepsilon}x\right\}\mathrm{d}x \\
&\leq \frac{\varepsilon^{2\theta\widetilde{\gamma}}}{2\widetilde{\gamma}}\max_{[0,1]}|\phi|.
\end{aligned} \tag{3.31}
$$

Note also that $v' \leq 0$. Thus, by (3.5), one has

$$
\begin{aligned}
&\int\limits_{I_{\varepsilon\log\frac{1}{\varepsilon}}} \left|\left(\varepsilon v'^2(x) - \frac{\Lambda_-(x)}{2}v(x)v'(x)\right)\phi(x)\right|\mathrm{d}x \\
\leq & -\sqrt{\varepsilon}C^\star \int\limits_{I_{\varepsilon\log\frac{1}{\varepsilon}}} \exp\left\{\frac{x}{2\varepsilon}\max_{[0,1]}|a|\right\} v'(x)|\phi(x)|\,\mathrm{d}x \\
\leq & -\varepsilon^{\frac{1}{2}(1-\theta\max_{[0,1]}|a|)}C^\star\max_{[0,1]}|\phi| \int\limits_{I_{\varepsilon\log\frac{1}{\varepsilon}}} v'(x)\,\mathrm{d}x \leq \varepsilon^{\theta_p^*}C^\star\max_{[0,1]}|\phi|,
\end{aligned} \tag{3.32}
$$

and

$$\left|\int_{I_{\varepsilon\log\frac{1}{\varepsilon}}}\Bigg(\underbrace{-\frac{\Lambda_-(x)}{2}\phi(x)+\frac{\Lambda_-(0)}{2}\phi(0)}_{:=\widetilde{D}(x)}\Bigg)v(x)v'(x)\,\mathrm{d}x\right|$$

$$\leq -\left(\max_{I_{\varepsilon\log\frac{1}{\varepsilon}}}|\widetilde{D}|\right)\int_{I_{\varepsilon\log\frac{1}{\varepsilon}}} v(x)v'(x)\,\mathrm{d}x \leq \frac{1}{2}\max_{I_{\varepsilon\log\frac{1}{\varepsilon}}}|\widetilde{D}| \xrightarrow{\varepsilon\downarrow 0} 0. \tag{3.33}$$

Here we have used $\max\limits_{I_{\varepsilon\log\frac{1}{\varepsilon}}}\exp\left\{\frac{x}{2\varepsilon}\max\limits_{[0,1]}|a|\right\}\leq\varepsilon^{-\frac{\theta}{2}\max\limits_{[0,1]}|a|}$ to obtain the last inequality of (3.32). Combining (3.31)–(3.33), we arrive at

$$\lim_{\varepsilon\downarrow 0}\int_0^1 \varepsilon v'^2(x)\phi(x)\,\mathrm{d}x=\frac{\Lambda_-(0)\phi(0)}{2}\lim_{\varepsilon\downarrow 0}\int_{I_{\varepsilon\log\frac{1}{\varepsilon}}} v(x)v'(x)\,\mathrm{d}x=-\frac{\Lambda_-(0)\phi(0)}{4}.$$

Following the same argument, we can prove $\lim\limits_{\varepsilon\downarrow 0}\int_0^1\varepsilon w'^2\phi\,\mathrm{d}x=\frac{\Lambda_+(1)\phi(1)}{4}$ and complete the proof of (1.14).

It remains to prove (ii). Let $\psi\in \mathrm{C}^1([0,1])$. Multiplying (1.8) by $\frac{\psi}{b}$ and integrating the expression from 0 to 1, one may check that

$$\begin{aligned}
&\int_0^1 \frac{v(x)}{\varepsilon}\psi(x)\,\mathrm{d}x\\
&=\varepsilon\int_0^1\frac{\psi(x)}{b(x)}v''(x)\,\mathrm{d}x+\int_0^1\frac{a(x)}{b(x)}v'(x)\psi(x)\,\mathrm{d}x\\
&=\varepsilon\left(\frac{\psi(1)}{b(1)}v'(1)-\frac{\psi(0)}{b(0)}v'(0)-\int_0^1\left(\frac{\psi}{b}\right)'v'\,\mathrm{d}x\right)-\frac{a(0)\psi(0)}{b(0)}-\int_0^1\left(\frac{a\psi}{b}\right)'v\,\mathrm{d}x\\
&\xrightarrow{\varepsilon\downarrow 0}-\frac{\psi(0)}{b(0)}\left(-\frac{a(0)+\sqrt{a^2(0)+4b(0)}}{2}+a(0)\right)=-\frac{2\psi(0)}{\Lambda_-(0)}.
\end{aligned}\tag{3.34}$$

Here we have applied (3.3), (3.4), (3.18) and (3.19) to the second line of (3.34). Similarly, we can prove $\int_0^1\frac{w}{\varepsilon}\psi\,\mathrm{d}x\xrightarrow{\varepsilon\downarrow 0}\frac{2\psi(1)}{\Lambda_+(1)}$. Hence, we obtain that the convergences presented in (1.13) hold.

Now we assume $a(0)>0$ and $\phi\in\mathrm{C}([0,1])$. This implies that for fixed $\theta>0$, there holds $a(z)>\frac{a(0)}{2}>0$ for $z\in I_{\varepsilon\log\frac{1}{\varepsilon}}=[0,\theta\varepsilon\log\frac{1}{\varepsilon}]$ as $\varepsilon>0$ is sufficiently small. Then by (3.5), for $\phi\in\mathrm{C}([0,1])$, one may use the estimate $\frac{1}{|\Lambda_-(x)|}\leq\frac{|a(x)|+\sqrt{\beta}}{2\beta}$ to check that

$$\left|\int_{I_{\varepsilon\log\frac{1}{\varepsilon}}}\left(-\frac{2v'(x)}{\Lambda_-(x)}\phi(x)+\frac{v(x)}{\varepsilon}\phi(x)\right)\mathrm{d}x\right|\leq\frac{2C^\star}{\sqrt{\varepsilon}}\int_{I_{\varepsilon\log\frac{1}{\varepsilon}}}\left|\frac{\phi(x)}{\Lambda_-(x)}\right|\exp\left\{-\frac{1}{2\varepsilon}\int_0^x a(z)\,\mathrm{d}z\right\}\mathrm{d}x,$$

$$\leq\frac{C^\star}{\beta}\left(\max_{[0,1]}|\phi|\left(|a|+\sqrt{\beta}\right)\right)\varepsilon^{\frac{\theta}{4}a(0)-\frac{1}{2}},\quad \text{as } 0<\varepsilon\ll 1. \tag{3.35}$$

Note also that, for the case $p=1$, (3.26) and (3.30) still hold. Combining these two estimates with (3.35) and particularly taking $\theta=\frac{4}{a(0)}>0$ and $I_{\varepsilon\log\frac{1}{\varepsilon}}=[0,\frac{4\varepsilon}{a(0)}\log\frac{1}{\varepsilon}]$, we thus arrive at

$$\lim_{\varepsilon\downarrow 0}\int_0^1\frac{v(x)}{\varepsilon}\phi(x)\,\mathrm{d}x=2\lim_{\varepsilon\downarrow 0}\int_{I_{\varepsilon\log\frac{1}{\varepsilon}}}\frac{v'(x)}{\Lambda_-(x)}\phi(x)\,\mathrm{d}x=-\frac{2\phi(0)}{\Lambda_-(0)}.$$

On the other hand, if $a(1)>0$, we can apply the same argument to $\frac{w}{\varepsilon}$ and obtain $\lim_{\varepsilon\downarrow 0}\int_0^1\frac{w}{\varepsilon}\phi\,\mathrm{d}x=\frac{2\phi(1)}{\Lambda_+(1)}$. Therefore, the proof of Proposition 1.2 is complete.

Remark 3.3. (3.26) and (3.30) hold for $p=1$. By (3.26), (3.30) and (3.35) with $\theta=\frac{4}{a(0)}>0$, this implies $\frac{\theta}{4}a(0)-\frac{1}{2}=\frac{1}{2}$ in (3.35). Hence, Claim 1 still holds for $p=1$.

4. Proof of Theorems 2.1 and 2.2

4.1. Proof of Theorem 2.1. By (1.16) and (1.18), we know that there exists $\eta^*>0$ such that (1.12) holds as $\varepsilon\in(0,\eta^*)$. Hence, as $\varepsilon\in(0,\eta^*)$, the uniqueness of u follows immediately from Proposition 1.1. Next, we shall claim (1.19), (1.22) and (2.2) as follows. Since (1.12) implies that $u(0)$ and $u(1)$ are uniquely determined by system (1.10) with (1.11), for smooth functions g_0 and g_1, applying Proposition 1.2(ii) to (1.11) gives (1.19). The interior estimate (1.22) of u is a direct consequence of (1.7), (1.19) and Proposition 3.1(i) with (3.1)–(3.2). Moreover, by (1.7), (1.19) and (3.18), we obtain (2.2). As a consequence, a constant $\eta<\min\{\eta^*,\varepsilon^\star(\gamma,\widetilde{\gamma})\}$ can be determined so that for $\varepsilon\in(0,\eta)$ there hold the following properties:

$$u(0)<0<u(1)\text{ and }u'(0),u'(1)>0\text{ if }B_0\text{ and }B_1\text{ satisfy (1.23)},\tag{4.1}$$
$$u(0),u(1)>0\text{ and }u'(0)<0<u'(1)\text{ if }B_0\text{ and }B_1\text{ satisfy (1.24)},\tag{4.2}$$

where $\varepsilon^\star(\gamma,\widetilde{\gamma})$ was defined in Proposition 3.1(i).

Recall that $u(x)=u(0)v(x)+u(1)w(x)$ (see (1.7)) and, for $x\in(0,1)$, $v(x)$ and $w(x)$ exponentially decay to zero as ε approaches zero. Before proving (i)–(iii), we have to establish pointwise asymptotics of v and w.

Proposition 4.1. *Let $\kappa=\frac{1}{2}$ for the case $a(x)\geq 0$ and $\kappa\in(0,\frac{1}{2})$ be arbitrary for the case $\min_{[0,1]}a<0$. Under the same hypotheses as in Proposition 1.2, there exists a positive constant C_κ depending on κ (independent of ε) such that, for $\varepsilon>0$,*

$$\left\|v(x)-\exp\left\{\int_0^x\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}\right\|_{\mathrm{C}^2_\varepsilon([0,1])}+\left\|w(x)-\exp\left\{-\int_x^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\right\|_{\mathrm{C}^2_\varepsilon([0,1])}\leq C_\kappa\varepsilon^\kappa,\tag{4.3}$$

where $\Lambda_\pm(z)$ was defined in (1.5) and the C^2_ε-norm was defined by (2.1).

Proposition 4.1 seems to be well known, but we could not find a suitable reference for it. We prudently provide a proof here for the reader's satisfaction. The proof will be stated in Section 4.3 for the sake of convenience.

Using this preliminary step, we are in a position to deal with (2.3). Firstly, by (1.7), (1.19) and (2.1) we have

$$\begin{aligned}&u(x)-\left(B_0\exp\left\{\int_0^x\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}+B_1\exp\left\{-\int_x^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\right)\\&=u(0)\left(v(x)-\exp\left\{\int_0^x\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}\right)+(u(0)-B_0)\exp\left\{\int_0^x\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}\end{aligned}$$

$$+u(1)\left(w(x)-\exp\left\{-\int_x^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\right)+(u(1)-B_1)\exp\left\{-\int_x^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}. \quad (4.4)$$

By (4.3) with a fixed κ and (4.4), one may check that

$$\begin{aligned}
&\left\|u(x)-\left(B_0\exp\left\{\int_0^x\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}+B_1\exp\left\{-\int_x^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\right)\right\|_{\mathrm{C}^2_\varepsilon([0,1])}\\
\leq&|u(0)|\left\|v(x)-\exp\left\{\int_0^x\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}\right\|_{\mathrm{C}^2_\varepsilon([0,1])}+|u(0)-B_0|\left\|\exp\left\{\int_0^x\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}\right\|_{\mathrm{C}^2_\varepsilon([0,1])}\\
&+|u(1)|\left\|w(x)-\exp\left\{-\int_x^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\right\|_{\mathrm{C}^2_\varepsilon([0,1])}\\
&+|u(1)-B_1|\left\|\exp\left\{-\int_x^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\right\|_{\mathrm{C}^2_\varepsilon([0,1])}\\
\leq&\max\{|u(0)|,|u(1)|\}C_\kappa\varepsilon^\kappa+\left(\sup_{\varepsilon>0}\mathfrak{M}_\varepsilon(\Lambda)\right)(|u(0)-B_0|+|u(1)-B_1|)\\
\leq&\widetilde{C}_\kappa(\varepsilon^\kappa+|u(0)-B_0|+|u(1)-B_1|).
\end{aligned}\quad (4.5)$$

Here $\mathfrak{M}_\varepsilon(\Lambda)$ is defined as the maximum of $\left\|\exp\left\{\int_0^x\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}\right\|_{\mathrm{C}^2_\varepsilon([0,1])}$ and $\left\|\exp\left\{-\int_x^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\right\|_{\mathrm{C}^2_\varepsilon([0,1])}$, and we have verified via (2.1) that $\mathfrak{M}_\varepsilon(\Lambda)$ is uniformly bounded for $\varepsilon>0$. Since both $u(0)$ and $u(1)$ are uniformly bounded for $\varepsilon>0$, $\widetilde{C}_\kappa>0$ can be chosen so that it is independent of ε.

To complete the proof of (2.3), we need to estimate $|u(0)-B_0|$ and $|u(1)-B_1|$ with respect to ε as $0<\varepsilon\ll1$. By (1.10) and (1.20)–(1.21), we can obtain

$$\begin{bmatrix}u(0)-B_0\\u(1)-B_1\end{bmatrix}=\left(\mathcal{I}-\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}\right)^{-1}\left(\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}-\mathcal{A}\right)\begin{bmatrix}B_0\\B_1\end{bmatrix}. \quad (4.6)$$

Applying the Cauchy–Schwarz inequality to (4.6), one may make appropriate manipulations to arrive at

$$|u(0)-B_0|+|u(1)-B_1|\leq\sqrt{2}(|B_0|+|B_1|)\left\|\left(\mathcal{I}-\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}\right)^{-1}\right\|_{\mathrm{HS}}\left\|\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}-\mathcal{A}\right\|_{\mathrm{HS}}, \quad (4.7)$$

where $||A||_{\mathrm{HS}}:=\sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$ (the standard Hilbert–Schmidt norm). On the other hand, by (1.11), (1.16), (3.25) and Remark 3.3, $\left\|\left(\mathcal{I}-\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}\right)^{-1}\right\|_{\mathrm{HS}}$ is uniformly bounded and

$$\left\|\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}-\mathcal{A}\right\|_{\mathrm{HS}}\lesssim\varepsilon^{\theta_p^*},\quad \text{as}\quad 0<\varepsilon\ll1.$$

As a consequence, by (4.7) we obtain $|u(0)-B_0|+|u(1)-B_1|\lesssim\varepsilon^{\theta_p^*}$. This along with (4.5) yields (2.3), where we choose $\kappa=\theta_p^*=\frac{1}{2}$ if $a\geq0$, and $\kappa=\theta_p^*\in(0,\frac{1}{2})$ if $\min_{[0,1]}a<0$.

It remains to prove (i) and (ii).

Proof. **(Proof of (i) and (ii)).** Following the same argument as (3.9), one obtains

$$\left(u'(x)\exp\left\{\int_x^1\frac{a(z)}{\varepsilon}\mathrm{d}z\right\}\right)'=\frac{b(x)}{\varepsilon^2}u(x)\exp\left\{\int_x^1\frac{a(z)}{\varepsilon}\mathrm{d}z\right\},\ x\in[0,1]. \quad (4.8)$$

We now assume (1.23). By (4.1) and (4.8), we get $u'>0$ on $(0,1)$. This completes the proof of (i). To prove (ii), we assume (1.24). Then, by the strong maximum principle, Equation (1.1)–(1.2) with $b(x)\geq\beta>0$ and (4.2) implies that u attains its minimum value at an interior point $\mathfrak{p}_\varepsilon\in(0,1)$ and $u(\mathfrak{p}_\varepsilon)>0=u'(\mathfrak{p}_\varepsilon)$. Along with (4.8), it yields $u'(x_1)<0<u'(x_2)$ for $x_1\in(0,\mathfrak{p}_\varepsilon)$ and $x_2\in(\mathfrak{p}_\varepsilon,1)$. This also obtains the uniqueness of $\mathfrak{p}_\varepsilon$.

We shall next claim (2.5) by contradiction. Firstly, setting $x=\mathfrak{p}_\varepsilon$ in (1.7) and using (4.2) and $u'(\mathfrak{p}_\varepsilon)=0$, we obtain

$$|v'(\mathfrak{p}_\varepsilon)|=\frac{u(1)}{u(0)}w'(\mathfrak{p}_\varepsilon). \tag{4.9}$$

Note also that $\frac{u(1)}{u(0)}\sim\frac{B_1}{B_0}$ and w' are positive if $\varepsilon>0$ is sufficiently small. Suppose that, on the contrary, $\limsup\limits_{\varepsilon\downarrow 0}\frac{\mathfrak{p}_\varepsilon}{\varepsilon}=\mathfrak{c}<\infty$. Then by (1.19), (3.4) and (4.9), we have $|v'(\mathfrak{p}_\varepsilon)|\lesssim\varepsilon^{-1}\exp\{-\frac{\widetilde{\gamma}}{\varepsilon}\}$ as $0<\varepsilon\ll 1$. Along with (3.5), one may check that

$$\begin{aligned}
0<v(\mathfrak{p}_\varepsilon)&\leq\frac{2}{\Lambda_-(\mathfrak{p}_\varepsilon)}\left(\sqrt{\varepsilon}C^\star\exp\left\{-\frac{1}{2\varepsilon}\int_0^{\mathfrak{p}_\varepsilon}a(z)\,\mathrm{d}z\right\}+\varepsilon|v'(\mathfrak{p}_\varepsilon)|\right)\\
&\leq\frac{\max_{[0,1]}|a|+\sqrt{\beta}}{\beta}\left(\sqrt{\varepsilon}C^\star\exp\left\{\mathfrak{c}\max_{[0,1]}|a|\right\}+O(1)\exp\left\{-\frac{\widetilde{\gamma}}{\varepsilon}\right\}\right)\\
&\lesssim\sqrt{\varepsilon},\qquad \text{as}\,0<\varepsilon\ll 1.
\end{aligned}\tag{4.10}$$

Here we have used the inequality (3.28). Moreover, by (4.3) with a fixed $\kappa\in(0,\frac{1}{2})$ and (4.10), we arrive at

$$\exp\left\{\int_0^{\mathfrak{p}_\varepsilon}\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}\lesssim\varepsilon^\kappa\qquad\text{as}\,0<\varepsilon\ll 1.$$

Particularly, $\int_0^{\mathfrak{p}_\varepsilon}\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\lesssim\kappa\log\varepsilon\xrightarrow{\varepsilon\downarrow 0}-\infty$ which gives a contradiction since $0>\int_0^{\mathfrak{p}_\varepsilon}\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\geq\frac{\mathfrak{p}_\varepsilon}{2\varepsilon}\min\limits_{[0,1]}\Lambda_-$ and $\limsup\limits_{\varepsilon\downarrow 0}\frac{\mathfrak{p}_\varepsilon}{\varepsilon}=\mathfrak{c}<\infty$. Hence, we obtain $\lim\limits_{\varepsilon\downarrow 0}\frac{\mathfrak{p}_\varepsilon}{\varepsilon}=\infty$. Similarly, we can prove $\lim\limits_{\varepsilon\downarrow 0}\frac{1-\mathfrak{p}_\varepsilon}{\varepsilon}=\infty$. Therefore, we rigorously arrive at (2.5) and complete the proof of (ii). □

The proof of Theorem 2.1 is thus complete.

4.2. Proof of Theorem 2.2. In this section we assume (2.12). Note that for the solution u_τ of Equation (1.1) equipped with the boundary condition (1.3), (1.7) and (3.3)–(3.4) still hold. Thus, it suffices to prove (2.6)–(2.9). We first assume $\tau<1$. Then by (1.7), Proposition 1.2(ii) and (3.7), one finds estimates

$$\begin{aligned}
\varepsilon^{-\tau}\left|\int_{l_0}^1 g_0(x)u_\tau(x)\,\mathrm{d}x\right|&\leq\varepsilon^{-\tau}\left(|u_\tau(0)|\left|\int_{l_0}^1 g_0(x)v(x)\,\mathrm{d}x\right|+|u_\tau(1)|\left|\int_{l_0}^1 g_0(x)w(x)\,\mathrm{d}x\right|\right)\\
&\lesssim\varepsilon^{1-\tau}(|u_\tau(0)|+|u_\tau(1)|),
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon^{-\tau}\left|\int_0^{l_1} g_1(x)u_\tau(x)\,\mathrm{d}x\right|&\leq\varepsilon^{-\tau}\left(|u_\tau(0)|\left|\int_0^{l_1} g_1(x)v(x)\,\mathrm{d}x\right|+|u_\tau(1)|\left|\int_0^{l_1} g_1(x)w(x)\,\mathrm{d}x\right|\right)\\
&\lesssim\varepsilon^{1-\tau}(|u_\tau(0)|+|u_\tau(1)|),
\end{aligned}$$

as $0<\varepsilon\ll 1$. Applying these two estimates to (1.3) gives

$$\sum_{i=0,1}|u_\tau(i)-\mu_i|\lesssim \varepsilon^{1-\tau}\sum_{i=0,1}(|u_\tau(i)-\mu_i|+|\mu_i|).$$

This, along with $1-\tau>0$, directly implies $\sum_{i=0,1}|u_\tau(i)-\mu_i|\lesssim \frac{\varepsilon^{1-\tau}}{1-O(1)\varepsilon^{1-\tau}}\sum_{i=0,1}|\mu_i|$, i.e., $(u_\tau(0),u_\tau(1))\xrightarrow{\varepsilon\downarrow 0}(\mu_0,\mu_1)$. Then, using (3.18) and (3.20) and following the similar argument as (2.2), we obtain (2.6) and complete the proof of (i).

To prove (ii), we now assume $\tau>1$. Then for (1.25), we have

$$\det\left(\varepsilon^{\tau-1}\mathcal{I}-\frac{1}{\varepsilon}\mathcal{A}_\varepsilon^{(v,w)}\right)\xrightarrow{\varepsilon\downarrow 0}-\det\mathcal{A}^*,$$

where $\mathcal{A}^*$ was defined in (1.16). As a consequence, solving $u_\tau(0)$ and $u_\tau(1)$ via (1.25) and taking the limit with $\varepsilon\downarrow 0$, we obtain (2.7) and (2.8), which along with the maximum principle yields $\max_{[0,1]}|u_\tau|\leq\max\{|u_\tau(0)|,|u_\tau(1)|\}\xrightarrow{\varepsilon\downarrow 0}0$. Moreover, (2.9) follows from (1.7), (2.7)– (2.8) and (3.18), and we prove (ii). Therefore, the proof of Theorem 2.2 is complete.

4.3. Proof of Proposition 4.1. In this proof, we will not use the definition of (1.4) for the sake of clarity in all estimates.

When $a\geq 0$, the proof is trivial. In what follows we focus on the case that $a(x)$ changes its sign on $[0,1]$. Firstly, we set

$$V_0(x)=\exp\left\{-\int_0^x\frac{a(z)+\sqrt{a^2(z)+4b(z)}}{2\varepsilon}\mathrm{d}z\right\},$$

which satisfies

$$\varepsilon V_0'+\frac{a+\sqrt{a^2+4b}}{2}V_0=0,\ V_0(0)=1\text{ and }V_0(1)\leq\exp\left\{-\frac{\beta}{(\max_{[0,1]}|a|+\sqrt{\beta})\varepsilon}\right\}. \tag{4.11}$$

Here we have again used the inequality (3.28) for estimating $V_0(1)$. Along with (3.5), one can check via simple calculations that

$$\left|\left(\frac{v}{V_0}\right)'(x)\right|\leq\frac{C^\star}{\sqrt{\varepsilon}V_0(x)}\exp\left\{-\int_0^x\frac{a(z)}{2\varepsilon}\mathrm{d}z\right\}. \tag{4.12}$$

By this differential inequality we can infer an estimate

$$\begin{aligned}|v(x)-V_0(x)|\leq&\frac{C^\star}{\sqrt{\varepsilon}}\int_0^x\exp\left\{-\int_0^y\frac{a(z)}{2\varepsilon}\mathrm{d}z\right\}\frac{V_0(x)}{V_0(y)}\mathrm{d}y\\ \leq&\frac{C^\star}{\sqrt{\varepsilon}}\left(\max_{y\in[0,x]}\exp\left\{-\int_0^y\frac{a(z)}{2\varepsilon}\mathrm{d}z\right\}\right)\int_0^x\exp\left\{-\int_y^x\frac{\sqrt{a^2(z)+4b(z)}}{2\varepsilon}\mathrm{d}z\right\}\mathrm{d}y\\ \leq&C^\star\sqrt{\frac{\varepsilon}{\beta}}\max_{y\in[0,x]}\exp\left\{-\int_0^y\frac{a(z)}{2\varepsilon}\mathrm{d}z\right\}\end{aligned} \tag{4.13}$$

since $\sqrt{a^2(x)+4b(x)}\geq 2\sqrt{\beta}$. In particular, for each $\kappa\in(0,\frac{1}{2})$, let us define

$$\widetilde{\theta}(\kappa)=\max\left\{\frac{1-2\kappa}{1+\max\limits_{[0,1]}|a|},\frac{\kappa}{\beta}\left(\max_{[0,1]}|a|+\sqrt{\beta}\right),\frac{\kappa}{\gamma}\right\}. \tag{4.14}$$

Then for $0\leq y\leq x\leq\widetilde{\theta}(\kappa)\varepsilon\log\frac{1}{\varepsilon}$, one obtains

$$\left|\int_0^y \frac{a(z)}{2\varepsilon}\mathrm{d}z\right|\leq\left(\kappa-\frac{1}{2}\right)\log\varepsilon.$$

Along with (4.13), one can use the estimate $\exp\left\{-\frac{\beta}{\left(\max\limits_{[0,1]}|a|+\sqrt{\beta}\right)\varepsilon}\right\}\ll\varepsilon^\kappa$ to obtain

$$\max_{x\in[0,\widetilde{\theta}(\kappa)\varepsilon\log\frac{1}{\varepsilon}]}|v(x)-V_0(x)|\leq\frac{1+C^\star}{\sqrt{\beta}}\varepsilon^\kappa, \tag{4.15}$$

as $\varepsilon>0$ is sufficiently small. On the other hand, we notice that v and V_0 are strictly decreasing on $[0,1]$. Thus, by (3.3) and (4.14) we have

$$\begin{aligned}\max_{x\in[\widetilde{\theta}(\kappa)\varepsilon\log\frac{1}{\varepsilon},1]}|v(x)-V_0(x)|&\leq v\left(\widetilde{\theta}(\kappa)\varepsilon\log\frac{1}{\varepsilon}\right)+V_0\left(\widetilde{\theta}(\kappa)\varepsilon\log\frac{1}{\varepsilon}\right)\\&\leq\varepsilon^{\widetilde{\theta}(\kappa)\gamma}+\varepsilon^{\widetilde{\theta}(\kappa)\beta/\left(\max\limits_{[0,1]}|a|+\sqrt{\beta}\right)}\leq 2\varepsilon^\kappa,\quad\text{as } 0<\varepsilon\ll 1.\end{aligned} \tag{4.16}$$

As a consequence, by (4.15)–(4.16) we obtain $\max\limits_{[0,1]}|v-V_0|\leq\max\{\frac{1+C^\star}{\sqrt{\beta}},2\}\varepsilon^\kappa$ as $0<\varepsilon\ll 1$. Furthermore, since $0\leq v,V_0\leq 1$, there must exist a constant $C_{1,\kappa}\geq\max\{\frac{1+C^\star}{\sqrt{\beta}},2\}$ depending on κ (independent of ε) such that

$$\max_{[0,1]}|v-V_0|\leq C_{1,\kappa}\varepsilon^\kappa,\quad\text{for }\varepsilon>0. \tag{4.17}$$

Moreover, by (4.11) we have the identity

$$\varepsilon(v-V_0)'=\varepsilon V_0\left(\frac{v}{V_0}\right)'-\frac{a+\sqrt{a^2+4b}}{2}(v-V_0),$$

together with (3.4), (4.12) and (4.17), one can follow the similar arguments as in (4.14)–(4.16) to arrive at

$$\begin{aligned}\varepsilon\max_{[0,1]}|(v-V_0)'|\leq C^\star\sqrt{\varepsilon}\exp\left\{\int_0^{\widetilde{\theta}(\kappa)\varepsilon\log\frac{1}{\varepsilon}}\frac{|a(z)|}{2\varepsilon}\mathrm{d}z\right\}+\varepsilon\max_{[\widetilde{\theta}(\kappa)\varepsilon\log\frac{1}{\varepsilon},1]}|(v-V_0)'|\\+\max_{[0,1]}\frac{a+\sqrt{a^2+4b}}{2}|v-V_0|\leq C_{2,\kappa}\varepsilon^\kappa,\quad\text{for }\varepsilon>0,\end{aligned} \tag{4.18}$$

where $C_{2,\kappa}$ relying on κ is a positive constant independent of ε. Combining (4.17) with (4.18) and using the equation of v in (1.8), we can deal with the estimate of $\varepsilon^2(v-V_0)''$ and arrive at

$$||v-V_0||_{\mathrm{C}^2_\varepsilon([0,1])}\leq C_{3,\kappa}\varepsilon^\kappa,\quad\text{for }\varepsilon>0, \tag{4.19}$$

with an ε-independent constant $C_{3,\kappa} > C_{1,\kappa} + C_{2,\kappa}$.

Applying the same argument to $|w - W_0|$ with $W_0 = \exp\left\{ -\int_x^1 \frac{a(z)+\sqrt{a^2(z)+4b(z)}}{2\varepsilon} \mathrm{d}z \right\}$, we can prove

$$||w - W_0||_{C^2_\varepsilon([0,1])} \le C_{3,\kappa}\varepsilon^\kappa, \quad \text{for } \varepsilon > 0. \tag{4.20}$$

(4.3) follows immediately from (4.19)–(4.20) with $C_\kappa = 2C_{3,\kappa}$, and the proof of Proposition 4.1 is therefore complete.

5. Proof of Theorem 2.3

5.1. Proof of Theorem 2.3(i). Recall a property that $\max_{[0,1]} vw$ exponentially decays to zero as $\varepsilon \downarrow 0$ (cf. (3.7)). This along with (1.7) gives a formal intuition that $|u(x)|^p \approx |u(0)|^p v^p(x) + |u(1)|^p w^p(x)$, $x \in [0,1]$, as $0 < \varepsilon \ll 1$. However, a main difficulty is to rigorously investigate the asymptotic behavior of $u(0)$ and $u(1)$. We first establish apriori asymptotic estimates for nonlocal coefficients in (2.10) with $0 < \varepsilon \ll 1$.

Lemma 5.1 (Apriori asymptotic estimate). *Under the same hypotheses as in Theorem 2.3(i), for $\mathcal{G} \in C([0,1])$ with $\mathcal{G}(0)\mathcal{G}(1) \neq 0$ and $p > 1$, we have*

$$\begin{aligned} &\left| \int_0^1 \mathcal{G}(x)\left(|u(x)|^p - |u(0)|^p v^p(x) - |u(1)|^p w^p(x)\right) \mathrm{d}x \right| \\ \le & p2^p \max_{[0,1]} |\mathcal{G}| \max\{|u(0)|^p, |u(1)|^p\} \exp\left\{ -\frac{\gamma}{2\varepsilon} \right\}, \end{aligned} \tag{5.1}$$

as $0 < \varepsilon \ll 1$.

Proof. For the convenience in the argument later, let us set

$$\begin{cases} \mathcal{J}_1(p) = \displaystyle\int_0^{\frac{1}{2}} \mathcal{G}(x)\left(|u(x)|^p - |u(0)|^p v^p(x) - |u(1)|^p w^p(x)\right) \mathrm{d}x, \\ \mathcal{J}_2(p) = \displaystyle\int_{\frac{1}{2}}^1 \mathcal{G}(x)\left(|u(x)|^p - |u(0)|^p v^p(x) - |u(1)|^p w^p(x)\right) \mathrm{d}x. \end{cases} \tag{5.2}$$

Then, for $x \in [0, \frac{1}{2}]$, one obtains

$$\begin{aligned} ||u(x)|^p - |u(0)|^p v^p(x)| &\le p\left(\max\left\{\max_{[0,\frac{1}{2}]}|u|, |u(0)|\max_{[0,\frac{1}{2}]} v\right\}\right)^{p-1} |u(x) - u(0)v(x)| \\ &\le p(|u(0)| + |u(1)|)^{p-1} |u(1)| \exp\left\{-\frac{\gamma}{2\varepsilon}\right\} \\ &\le p2^{p-1} \max\{|u(0)|^p, |u(1)|^p\} \exp\left\{-\frac{\gamma}{2\varepsilon}\right\}. \end{aligned} \tag{5.3}$$

Here we have used an elementary inequality

$$\left||A|^p - |B|^p\right| \le p\max\{|A|, |B|\}^{p-1}|A - B|$$

and (1.7). This along with (3.3) yields the estimate $\max\limits_{[0,\frac{1}{2}]} |u(x)| \le |u(0)| + |u(1)|$ and we thus arrive at the last estimate of (5.3). As a consequence,

$$\begin{aligned} |\mathcal{J}_1(p)| &\le \max_{[0,\frac{1}{2}]} |\mathcal{G}| \left(p2^{p-2} \max\{|u(0)|^p, |u(1)|^p\} \exp\left\{-\frac{\gamma}{2\varepsilon}\right\} + \frac{|u(1)|^p}{2} \exp\left\{-\frac{p\gamma}{2\varepsilon}\right\} \right) \\ &\le p2^{p-1} \max_{[0,\frac{1}{2}]} |\mathcal{G}| \max\{|u(0)|^p, |u(1)|^p\} \exp\left\{-\frac{\gamma}{2\varepsilon}\right\}. \end{aligned} \tag{5.4}$$

Similarly, we have

$$|\mathcal{J}_2(p)| \le p2^{p-1} \max_{[\frac{1}{2},1]} |\mathcal{G}| \max\{|u(0)|^p, |u(1)|^p\} \exp\left\{-\frac{\gamma}{2\varepsilon}\right\}. \tag{5.5}$$

Combining (5.2) with (5.4)–(5.5), we therefore arrive at (5.1) and complete the proof of Lemma 5.1. □

Combining (1.13) with (5.1) gives an estimate as follows

$$\left|\frac{1}{\varepsilon}\int_0^1 \mathcal{G}(x)|u(x)|^p \,\mathrm{d}x - \mathcal{J}_3(p)\right| \le \frac{p2^p}{\varepsilon} \max_{[0,1]} |\mathcal{G}| \max\{|u(0)|^p, |u(1)|^p\} \exp\left\{-\frac{\gamma}{2\varepsilon}\right\} + \mathcal{O}_\varepsilon, \tag{5.6}$$

where

$$\mathcal{J}_3(p) = \frac{2}{p}\left(-\frac{\mathcal{G}(0)}{\Lambda_-(0)}|u(0)|^p + \frac{\mathcal{G}(1)}{\Lambda_+(1)}|u(1)|^p\right)$$

and $\mathcal{O}_\varepsilon$ denotes a quantity approaching to zero as $\varepsilon \downarrow 0$. In particular, by applying (5.6) to the boundary condition (2.10) of $u(0)$ and $u(1)$, one immediately obtains that $u(0)$ and $u(1)$ are uniformly bounded as $0 < \varepsilon \ll 1$, and

$$\lim_{\varepsilon\downarrow 0}\left(u(0) - \frac{2}{p_0}\left(-\frac{g_0(0)}{\Lambda_-(0)}|u(0)|^{p_0}\mathbb{1}_{\{0\}}(l_0) + \frac{g_0(1)}{\Lambda_+(1)}|u(1)|^{p_0}\right)\right) = \mu_0 \tag{5.7}$$

and

$$\lim_{\varepsilon\downarrow 0}\left(u(1) - \frac{2}{p_1}\left(-\frac{g_1(0)}{\Lambda_-(0)}|u(0)|^{p_1} + \frac{g_1(1)}{\Lambda_+(1)}|u(1)|^{p_1}\mathbb{1}_{\{1\}}(l_1)\right)\right) = \mu_1. \tag{5.8}$$

Note that when $u(0)$ and $u(1)$ are determined, Equation (1.1) has a unique solution u. Since $u(0)$ and $u(1)$ satisfying (5.7) and (5.8) are uniformly bounded to ε, and (2.13) has a solution $(\mathcal{X}_0, \mathcal{X}_1) = (B_0^\vee, B_1^\vee)$, we conclude that there exists a solution u to the Equation (1.1) with the boundary condition (2.10), and this u satisfies $(u(0), u(1)) \xrightarrow{\varepsilon\downarrow 0} (B_0^\vee, B_1^\vee)$. This along with (1.7), (3.18) and (3.20), gives (2.15). Moreover, (2.3) with $(B_0, B_1) = (B_0^\vee, B_1^\vee)$ immediately follows, and the proof of Theorem 2.3(i) is therefore complete.

5.2. Proof of Theorem 2.3(ii). Following similar argument of Lemma 5.1, we shall deal with nonlocal terms in the boundary condition (2.11) as follows.

LEMMA 5.2. *Under the same hypotheses as in Theorem 2.3(ii), for $g \in \mathrm{C}([0,1])$ with $g(0)g(1) \neq 0$ there holds*

$$\varepsilon\int_0^1 g(x)\left(u'(x)\right)^2 \mathrm{d}x = \frac{1}{4}\Big(-\Lambda_-(0)g(0)u^2(0) + \Lambda_+(1)g(1)u^2(1)\Big)(1 + \mathcal{O}_\varepsilon) \tag{5.9}$$

as $0 < \varepsilon \ll 1$, where the quantity $\mathcal{O}_\varepsilon \xrightarrow{\varepsilon\downarrow 0} 0$ was defined in (5.6).

Proof. Let us define

$$\mathcal{J}_4 = \frac{1}{4}\left(-\Lambda_-(0)g(0)u^2(0) + \Lambda_+(1)g(1)u^2(1)\right).$$

By (1.7) and (3.7), we have

$$\max_{x\in[0,1]}\left|(u'(x))^2(x)-\left(u^2(0)(v'(x))^2(x)+u^2(1)(w'(x))^2(x)\right)\right|\leq\left(u^2(0)+u^2(1)\right)\exp\left\{-\frac{\widetilde{\gamma}}{2\varepsilon}\right\},$$

together with (1.14) subject to $\phi=g$, it yields

$$\left|\varepsilon\int_0^1 g(x)(u'(x))^2\,\mathrm{d}x-\boldsymbol{\mathcal{J}_4}\right|\leq\varepsilon\max_{[0,1]}|g|\left(u^2(0)+u^2(1)\right)\exp\left\{-\frac{\widetilde{\gamma}}{2\varepsilon}\right\}+\mathcal{O}_\varepsilon\xrightarrow{\varepsilon\downarrow 0}0.$$

Since $g(0)g(1)\neq 0$, we have $\left(u^2(0)+u^2(1)\right)\varepsilon\exp\left\{-\frac{\widetilde{\gamma}}{2\varepsilon}\right\}\ll|\boldsymbol{\mathcal{J}_4}|$ as $0<\varepsilon\ll 1$. Hence, we arrive at (5.9) and complete the proof of Lemma 5.2. ◻

Applying (5.9) to the boundary condition (2.11), one further obtains that $u(0)$ and $u(1)$ are uniformly bounded as $0<\varepsilon\ll 1$, and

$$\lim_{\varepsilon\downarrow 0}\left(u(0)-\frac{1}{4}\left(-\Lambda_-(0)g_0(0)u^2(0)\mathbb{1}_{\{0\}}(l_0)+\Lambda_+(1)g_0(1)u^2(1)\right)\right)=\mu_0$$

and

$$\lim_{\varepsilon\downarrow 0}\left(u(1)-\frac{1}{4}\left(-\Lambda_-(0)g_1(0)u^2(0)+\Lambda_+(1)g_1(1)u^2(1)\mathbb{1}_{\{1\}}(l_1)\right)\right)=\mu_1.$$

Since (2.14) has a solution $(\mathcal{X}_0,\mathcal{X}_1)=(B_0^\wedge,B_1^\wedge)$, by following a similar argument as in the proof of Theorem 2.3, we obtain that there exists a solution u to the Equation (1.1) with the boundary condition (2.11), and this u satisfies $(u(0),u(1))\xrightarrow{\varepsilon\downarrow 0}(B_0^\wedge,B_1^\wedge)$. It follows that u satisfies (2.16) and (2.3) with $(B_0,B_1)=(B_0^\wedge,B_1^\wedge)$. This completes the proof of Theorem 2.3(ii).

6. Asymptotics of the inhomogeneous case — Proof of Theorem 2.4

We shall first prove the existence and uniqueness of (2.17) with the boundary condition (1.2) as $\varepsilon\in(0,\eta)$, where $\eta>0$ has been defined in Theorem 2.1. For this end, let us consider the equation

$$\begin{cases}\varepsilon^2Q_\varepsilon''+\varepsilon aQ_\varepsilon'-bQ_\varepsilon=f \quad \text{in } (0,1),\\ Q_\varepsilon|_{x=0,1}=-\left(\dfrac{f}{b}+\varepsilon\dfrac{a}{b}\left(\dfrac{f}{b}\right)'\right)\Bigg|_{x=0,1}.\end{cases}\tag{6.1}$$

Since $b>0$, for each $\varepsilon>0$ Equation (6.1) has a unique solution. Setting

$$\widetilde{\mathsf{U}}_\varepsilon=u-Q_\varepsilon,\tag{6.2}$$

one may check that $\widetilde{\mathsf{U}}_\varepsilon$ is a solution of (1.1) with the boundary condition

$$\widetilde{\mathsf{U}}_\varepsilon(0)=\widetilde{\mu}_\varepsilon(0)+\frac{1}{\varepsilon}\int_{l_0}^1 g_0\widetilde{\mathsf{U}}_\varepsilon\,\mathrm{d}x,\quad \widetilde{\mathsf{U}}_\varepsilon(1)=\widetilde{\mu}_\varepsilon(1)+\frac{1}{\varepsilon}\int_0^{l_1} g_1\widetilde{\mathsf{U}}_\varepsilon\,\mathrm{d}x,\tag{6.3}$$

where

$$\widetilde{\mu}_\varepsilon(0)=\mu_0+\left(\frac{f}{b}+\varepsilon\frac{a}{b}\left(\frac{f}{b}\right)'\right)(0)+\frac{1}{\varepsilon}\int_{l_0}^1 g_0Q_\varepsilon\,\mathrm{d}x,$$

$$\widetilde{\mu}_\varepsilon(1)=\mu_1+\left(\frac{f}{b}+\varepsilon\frac{a}{b}\left(\frac{f}{b}\right)'\right)(1)+\frac{1}{\varepsilon}\int_0^{l_1}g_1Q_\varepsilon\,\mathsf{d}x.$$

Note that the equation of $\widetilde{\mathsf{U}}_\varepsilon$ with (6.3) has the same form as (1.1)–(1.2), and $\widetilde{\mu}_\varepsilon(j)$'s are independent of $\widetilde{\mathsf{U}}_\varepsilon$. Thus, under the assumption (1.18), by Proposition 1.1 and Theorem 2.1 we obtain the existence and uniqueness of $\widetilde{\mathsf{U}}_\varepsilon$ for $\varepsilon\in(0,\eta)$. This implies the existence and uniqueness of (2.17) with the boundary condition (1.2).

The proof of (2.23) follows several steps. We first set

$$\mathbf{Q}_\varepsilon(x)=Q_\varepsilon(x)+\frac{f(x)}{b(x)}+\varepsilon\frac{a(x)}{b(x)}\left(\frac{f(x)}{b(x)}\right)',\quad x\in[0,1].$$

Along with (6.1), we calculate directly to obtain the equation of $\mathbf{Q}_\varepsilon$ as

$$\varepsilon^2\mathbf{Q}_\varepsilon''+\varepsilon a\mathbf{Q}_\varepsilon'-b\mathbf{Q}_\varepsilon=\varepsilon^2\left[\left(\frac{f}{b}\right)''+a\left(\frac{a}{b}\left(\frac{f}{b}\right)'\right)'\right]+\varepsilon^3\left(\frac{a}{b}\left(\frac{f}{b}\right)'\right)''\quad\text{in }(0,1),\tag{6.4}$$

and $\mathbf{Q}_\varepsilon(0)=\mathbf{Q}_\varepsilon(1)=0$. Since a, b and f are smooth functions defined on $[0,1]$ and they are independent of ε, by (6.4) we have $\sup\limits_{(0,1)}|\varepsilon^2\mathbf{Q}_\varepsilon''+\varepsilon a\mathbf{Q}_\varepsilon'-b\mathbf{Q}_\varepsilon|=\mathcal{O}(1)\varepsilon^2$. Applying the maximum principle to (6.4) and using (6.2) immediately yields

$$\max_{[0,1]}\left|u-\widetilde{\mathsf{U}}_\varepsilon+\frac{f}{b}+\varepsilon\frac{a}{b}\left(\frac{f}{b}\right)'\right|=\max_{[0,1]}|\mathbf{Q}_\varepsilon|\lesssim\varepsilon^2,$$

which implies the precise first two-order terms of $\widetilde{\mu}_\varepsilon(j)$, $j=0,1$, with respect to $0<\varepsilon\ll1$:

$$\left|\widetilde{\mu}_\varepsilon(0)-\frac{\mathfrak{m}_0(0)}{\varepsilon}-\mathfrak{m}_1(0)\right|+\left|\widetilde{\mu}_\varepsilon(1)-\frac{\mathfrak{m}_0(1)}{\varepsilon}-\mathfrak{m}_1(1)\right|\lesssim\varepsilon,\tag{6.5}$$

where $\mathfrak{m}_i(j)$'s have been defined in (2.20). Since $\max\limits_{[0,1]}|\mathbf{Q}_\varepsilon|$ and the right-hand side of (6.4) are uniformly bounded by ε^2, by applying the standard elliptic estimate to the Equation (6.1), we obtain $||\mathbf{Q}_\varepsilon||_{\mathrm{C}^2_\varepsilon([0,1])}\lesssim\varepsilon^2$, i.e.,

$$\left\|u-\widetilde{\mathsf{U}}_\varepsilon+\frac{f}{b}+\varepsilon\frac{a}{b}\left(\frac{f}{b}\right)'\right\|_{\mathrm{C}^2_\varepsilon([0,1])}\lesssim\varepsilon^2,\tag{6.6}$$

where the $\mathrm{C}^2_\varepsilon([0,1])$-norm has been defined by (2.1).

Although the equation of $\widetilde{\mathsf{U}}_\varepsilon$ has the same form as (1.1), $\widetilde{\mu}_\varepsilon(0)$ and $\widetilde{\mu}_\varepsilon(1)$ in (6.3) depend on ε and the asymptotics of $\widetilde{\mathsf{U}}_\varepsilon$ should be dealt with carefully. Let

$$\mathsf{U}_\varepsilon=\frac{1}{\varepsilon}\left(\widetilde{\mathsf{U}}_\varepsilon-\frac{U_0}{\varepsilon}-U_1\right).\tag{6.7}$$

Then, by (2.18), (2.19), (6.3) and (6.5) we have

$$\varepsilon^2\mathsf{U}_\varepsilon''(x)+\varepsilon a(x)\mathsf{U}_\varepsilon'(x)-b(x)\mathsf{U}_\varepsilon(x)=0,\quad x\in(0,1),\tag{6.8}$$

and

$$\limsup_{\varepsilon\downarrow0}\left(\left|\mathsf{U}_\varepsilon(0)-\frac{1}{\varepsilon}\int_{l_0}^1g_0\mathsf{U}_\varepsilon\,\mathsf{d}x\right|+\left|\mathsf{U}_\varepsilon(1)-\frac{1}{\varepsilon}\int_0^{l_1}g_1\mathsf{U}_\varepsilon\,\mathsf{d}x\right|\right)<\infty.\tag{6.9}$$

Comparing (1.1)–(1.2) with (6.8)–(6.9), we can apply Theorem 2.1 directly to (6.8)–(6.9) (see, mainly, (1.19), (1.20), (1.21) and (2.3)), and conclude that $||\mathsf{U}_\varepsilon||_{\mathrm{C}^2_\varepsilon([0,1])}$ is uniformly bounded as $0<\varepsilon\ll 1$. This along with (6.7) arrives at

$$\left\|\widetilde{\mathsf{U}}_\varepsilon - \frac{U_0}{\varepsilon} - U_1\right\|_{\mathrm{C}^2_\varepsilon([0,1])} \lesssim \varepsilon, \tag{6.10}$$

as $0<\varepsilon\ll 1$. Combining (6.6) and (6.10), we therefore arrive at (2.23).

Now we state the proof of (i). Note that $\det(\mathcal{I}-\mathcal{A}^*)\neq 0$. Thus, by (1.16) and (2.21)–(2.22), it is easy to check that $\mathfrak{B}_0=\mathfrak{B}_1=0$ implies $\mathfrak{m}_0(0)=\mathfrak{m}_0(1)=0$. Hence, for $\varepsilon\in(0,\eta)$, by the uniqueness of (cf. Theorem 2.1), $\mathfrak{m}_0(0)=\mathfrak{m}_0(1)=0$ asserts $U_0\equiv 0$ on $[0,1]$. Along with (2.23), we obtain

$$\left\|u-U_1+\frac{f}{b}\right\|_{\mathrm{C}^2_\varepsilon([0,1])} \lesssim \varepsilon \qquad \text{as} \quad 0<\varepsilon\ll 1.$$

The uniform boundedness of u with respect to $0<\varepsilon\ll 1$ follows directly from the uniform boundedness of U_1 since $\mathfrak{m}_1(0)$ and $\mathfrak{m}_1(1)$ are finite. This completes the proof of (i).

It remains to prove (ii). Since equations (2.18) and (2.19) have the same forms as (1.1)–(1.2), by the result of (1.1)–(1.2) we know that U_0 and U_1 are uniformly bounded as $0<\varepsilon\ll 1$. Applying the estimate (2.3) to U_0 and U_1, we obtain $\max\limits_{\mathbf{K}}\left(\frac{|U_0|}{\varepsilon}+|U_1|\right)\xrightarrow{\varepsilon\downarrow 0} 0$ exponentially. Here we have used the properties $|\Lambda_-(x)|\geq 2\beta\left(\max\limits_{[0,1]}|a|+\sqrt{\beta}\right)^{-1}$ (cf. (3.28)) and $\min\limits_{x\in\mathbf{K}}\{x,1-x\}>0$ (independent of ε). Hence, by (2.23) we obtain $\max\limits_{\mathbf{K}}|u+\frac{f}{b}|\xrightarrow{\varepsilon\downarrow 0} 0$ and prove the uniform boundedness of u in $\mathbf{K}$ as $0<\varepsilon\ll 1$.

Now we want to prove (2.24) and (2.25). Assume $\mathfrak{B}_0\neq 0$. Then, obviously $\frac{|U_0(0)|}{\varepsilon}\to\infty$ as $\varepsilon\downarrow 0$, and by (6.6) and (6.10), $u(x)$ and $\frac{U_0(x)}{\varepsilon}$ both share the same leading order term as x is sufficiently close to the left boundary point. Hence, it suffices to deal with the asymptotics of $\frac{U_0}{\varepsilon}$ with respect to $0<\varepsilon\ll 1$, since U_1 is uniformly bounded. Applying (2.3) to (2.18), one arrives at

$$\left|U_0(\zeta_0\varepsilon\log\tfrac{\mathfrak{q}}{\varepsilon})-\left(\mathfrak{B}_0\exp\left\{\int\limits_0^{\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}}\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}+\mathfrak{B}_1\exp\left\{-\int\limits_{\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}}^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\right)\right|\lesssim\varepsilon^{\theta_1^*} \tag{6.11}$$

with $\theta_1^*\in(0,\frac{1}{2}]$ defined by (2.4). Since $\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}\to 0$ as $\varepsilon\downarrow 0$ and Λ_- is continuous at $x=0$, there holds $\frac{1}{\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}}\int_0^{\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}}\Lambda_-(z)\mathrm{d}z\xrightarrow{\varepsilon\downarrow 0}\Lambda_-(0)$. Along with (6.11), we thus obtain

$$\begin{aligned} u(\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}) &\approx \varepsilon^{-1}U_0(\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}) \\ &\approx \mathfrak{B}_0\varepsilon^{-1}\exp\left\{\int_0^{\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}}\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\} \\ &\approx \mathfrak{B}_0\mathfrak{q}^{\frac{\zeta_0}{2}\Lambda_-(0)}\varepsilon^{-\frac{\zeta_0}{2}\Lambda_-(0)-1}, \qquad \text{for} \quad \zeta_0\in(0,\frac{2\theta_1^*}{|\Lambda_-(0)|}). \end{aligned} \tag{6.12}$$

Here, for $\zeta_0\in(0,\frac{2\theta_1^*}{|\Lambda_-(0)|})$, we have used (2.23) and the following results to verify the above asymptotics:

$$\exp\left\{-\int_{\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}}^1\frac{\Lambda_+(z)}{2\varepsilon}\mathrm{d}z\right\}\xrightarrow{\varepsilon\downarrow 0}0 \qquad\qquad \text{exponentially,}$$

$$\exp\left\{\int_0^{\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon}}\frac{\Lambda_-(z)}{2\varepsilon}\mathrm{d}z\right\}\gg\varepsilon^{\theta_1^*},\quad \text{for}\qquad\qquad \zeta_0\in(0,\frac{2\theta_1^*}{|\Lambda_-(0)|}).$$

As a consequence, there holds

$$\varepsilon^{1+\frac{\zeta_0}{2}\Lambda_-(0)}u(\zeta_0\varepsilon\log\frac{\mathfrak{q}}{\varepsilon})\xrightarrow{\varepsilon\downarrow 0}\mathfrak{B}_0\mathfrak{q}^{\frac{\zeta_0}{2}\Lambda_-(0)},\quad \text{for}\quad \zeta_0\in(0,\frac{2\theta_1^*}{|\Lambda_-(0)|}).$$

Using (2.23) and following the same argument utilized in (6.12), for $\zeta_0>0$ we can prove

$$\varepsilon u(\zeta_0\varepsilon)\xrightarrow{\varepsilon\downarrow 0}\mathfrak{B}_0\exp\left\{\frac{\zeta_0}{2}\Lambda_-(0)\right\}$$

and finish the proof of (2.24). When $\mathfrak{B}_1\neq 0$, by the same argument we can obtain (2.25). Therefore, the proof of Theorem 2.4 is complete.

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