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$\mathbb{O}P^2$ bundles in M-theory

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Ramond has observed that the massless multiplet of 11-dimensional supergravity can be generated from the decomposition of certain representation of the exceptional Lie group F_4 into those of its maximal compact subgroup Spin(9). The possibility of a topological origin for this observation is investigated by studying Cayley plane, $\mathbb{O}P^2$, bundles over 11-manifolds Y^{11}. The lift of the topological terms gives constraints on the cohomology of Y^{11} which are derived. Topological structures and genera on Y^{11} are related to corresponding ones on the total space M^{27}. The latter, being 27-dimensional, might provide a candidate for "bosonic M-theory." The discussion leads to a connection with an octonionic version of Kreck–Stolz elliptic homology theory.

1. Introduction

The relation between M-theory and type IIA string theory leads to very interesting connections to K-theory [13, 14] and twisted K-theory [8, 9, 34]. Exceptional groups have also long appeared in physics. In particular, the topological piece of the M-theory action is encoded in part by an E_8 gauge theory in eleven dimensions [61]. This captures the cohomology of the C-field. Models for the M-theory C-field were proposed in [13] with and without using E_8. The E_8 bundle leads to a loop bundle on the type IIA base of the circle bundle [2, 34]. The role of E_8 and LE_8 was emphasized in [49, 51]. In particular, in [49] an important role for the *String* orientation was found within the E_8 construction. It is in the case when the base X^{10} is *String*-oriented that the topological action has a WZW-like interpretation and the degree-two component of the eta-form [34] is identified with the Neveu–Schwarz B-field [49].

In this paper we study another side of the problem, by including the whole 11-dimensional supermultiplet (g, C_3, Ψ), i.e., the metric, the C-field and the Rarita–Schwinger field, and not just the C-field. This turns out to be related to another exceptional Lie group, namely F_4, the exceptional Lie group of rank 4. Ramond [39, 41, 42] gave evidence for F_4 coming from the following two related observations:

1. F_4 appears explicitly [42] in the light-cone formulation of supergravity in 11 dimensions [12]. The generators $T^{\mu\nu}$ of the little group $SO(9)$ of the Poincaré group $ISO(1, 10)$ in 11 dimensions and the spinor generators T^a combine to form the 52 operators that generate the exceptional Lie algebra $\mathfrak{f}_4$ such that the constants $f^{\mu\nu ab}$ in the commutation relation

$$[T^{\mu\nu}, T^a] = if^{\mu\nu ab}T^b \tag{1.1}$$

are the structure constants of $\mathfrak{f}_4$. The 36 generators $T^{\mu\nu}$ are in the adjoint of $SO(9)$ and the 16 T^a generate its spinor representation. This can be viewed as the analog of the construction of E_8 out of the generators of $SO(16)$ and of $E_8/SO(16)$ in [19].

2. The identity representation of F_4, i.e., the one corresponding to Dynkin index $[0, 0, 0, 0]$, generates the three representations of Spin(9) [39]

$$\mathrm{Id}(F_4) \longrightarrow (44, 128, 84), \tag{1.2}$$

the numbers on the right-hand side correctly matching the number of degrees of freedoms of the massless bosonic content of 11-dimensional supergravity with the individual summands corresponding, respectively, to the graviton, the gravitino and the C-field.

The purpose of this paper is to expand on Ramond's observations by investigating the possibility of having an actual $\mathbb{O}P^2 = F_4/\mathrm{Spin}(9)$ bundle over Y^{11} through which the above observations can be explained geometrically and topologically. Since F_4 is the isometry group of the Cayley plane, the $\mathbb{O}P^2$ bundle will be the bundle associated to a principal F_4 bundle. We analyze some conditions under which this is possible.

In physics, the lifting of M-theory via the 16-dimensional manifold $\mathbb{O}P^2$ brings us to 27 dimensions. Given a Kaluza–Klein interpretation, this suggests the existence of a theory in 27 dimensions, whose dimensional reduction over $\mathbb{O}P^2$ leads to M-theory. The higher dimensional theory involves spinors, and it is natural to ask whether or not the theory can be supersymmetric. In one form we propose this as a candidate for the "bosonic M-theory" sought after in [25], from gravitational geometric arguments, and in [43], from matrix model arguments.

We consider the point of view of 11-dimensional manifolds in M-theory with extra topological structure, such as a *String* structure. Since any Y^{11} with a *String* structure is zero bordant in the *String* bordism group $\Omega_{11}^{\langle 8\rangle}$ then this raises the question of whether there is an equivalence with a total space of a bundle in which Y^{11} is a base. For the Spin case, Kreck and Stolz [28] constructed an elliptic homology theory in which a spin manifold of dimension $4k$ is *Spin* bordant to the total space of an $\mathbb{H}P^2$ bundle over a zero-bordant base if and only if its elliptic genus $\Phi_{\mathrm{ell}} \in \mathbb{Q}[\delta, \varepsilon]$ vanishes, where the generators δ and ε have degree 4 and 8, respectively. The same authors also expected the existence of a homology theory based on $\mathbb{O}P^2$ bundles for the *String* case, i.e., for manifolds such that $\frac{1}{2}p_1 = 0$, where p_1 is the first Pontrjagin class. So in our case, we ask whether there is a manifold M^{27}, which is an $\mathbb{O}P^2$ bundle over a zero bordant base and what consequence that has on the elliptic and the Witten genus.

Some aspects of the connection to this putative homology theory are

1. The elliptic homology theory requires the fundamental class[1] $[\mathbb{O}P^2]$ of $\mathbb{O}P^2$ to be inverted. This suggests connecting the lower-dimensional theory, in our case 11-dimensional M-theory, to a higher-dimensional one obtained by increasing the dimension by 16.
2. Previous works have used elliptic cohomology. We emphasize that in this paper we make use of a *homology* theory. Thus this not only provides further evidence for the relation between elliptic (co)homology and string/M-theory, but it also provides a new angle on such a relationship.

In previous work [29–31,48,50] evidence from various angles for a connection between string theory and elliptic cohomology was given. These papers relied heavily on analogies with the case in string theory, and were thus not intrinsically M-theoretic. In [45–47] a program was initiated to make the relation directly with M-theory. Thus, from another angle, the general purpose of this paper is two-fold:

- to point out further connections between elliptic cohomology and M-theory;
- to make the connection more M-theoretic, i.e., without reliance on any arguments from string theory.

$\mathbb{O}P^2$ is the Cayley, or octonionic projective, plane. For an extensive description see [7,20,44]. The group F_4 acts transitively on $\mathbb{O}P^2$, from which it follows that $\mathbb{O}P^2 \cong F_4/\mathrm{Spin}(9)$. In fact, F_4 is the isometry group of $\mathbb{O}P^2$. The tangent space to $\mathbb{O}P^2$ at a point is the coset of the corresponding Lie algebras $\mathfrak{f}_4/\mathfrak{so}(9)$, which is $\mathbb{O}^2 \cong \mathbb{R}^{16}$.

2. The fields in M-theory and $\mathbb{O}P^2$ bundles

The low-energy limit of M-theory (cf. [15, 59, 60]) is 11-dimensional supergravity [12], whose field content on an 11-dimensional spin manifold Y^{11} with Spin bundle SY^{11} is

[1] Viewed as a generator.

- *Two bosonic fields*: The metric g and the three-form C_3. It is often convenient to work with Cartan's moving frame formalism so that the metric is replaced by the 11-bein e_M^A such that $e_M^A e_N^B = g_{MN}\eta^{AB}$, where η is the flat metric on the tangent space.
- *One fermionic field*: The Rarita–Schwinger vector-spinor Ψ_1, which is classically a section of $SY^{11} \otimes TY^{11}$, i.e., a spinor coupled to the tangent bundle.

We now give the main theme around which this paper is centered.
Main idea: We interpret Ramond's triplets as arising from $\mathbb{O}P^2$ bundles with structure group F_4 over our 11-dimensional manifold Y^{11}, on which M-theory is 'defined.'

One major advantage of the introduction of an $\mathbb{O}P^2$ bundle is that in this picture the bosonic fields of M-theory, namely the metric and the C-field, can be unified.

Theorem 2.1. *The metric and the C-fields are orthogonal components of the positive spinor bundle of $\mathbb{O}P^2$.*

Proof. The spinor bundle $S^+(\mathbb{O}P^2)$ of the Cayley plane is isomorphic to

$$S^+(\mathbb{O}P^2) = S_0^2(V^9) \oplus \Lambda^3(V^9), \tag{2.1}$$

where V^9 is a nine-dimensional vector space and S_0^2 denotes the space of traceless symmetric 2-tensors. This follows from an application of Proposition 3 in [17], which requires the study the 16-dimensional spin representations $\Delta_{16}^{\pm}$ as Spin(9)-representations. The element $e_1 \cdots e_{16}$ belongs to the subgroup $\widetilde{\mathrm{Spin}}(9) \subset \mathrm{Spin}(16)$ and acts on $\Delta_{16}^{\pm}$ by multiplication by (± 1). This means that Δ_{16}^{+} is an $SO(9)$-representation, but Δ_{16}^{-} is a Spin(9)-representation [1]. Both representations do not contain nontrivial Spin(9)-invariant elements. Such an element would define a parallel spinor on $F_4/$ Spin(9) but, since the Ricci tensor of $\mathbb{O}P^2$ is not zero, the spinor must vanish by the Lichnerowicz formula [32] $D^2 = \nabla^2 + \frac{1}{4}R_{\mathrm{scal}}$. Then Δ_{16}^{+} as a Spin(9) representation is given by Equation (2.1), and Δ_{16}^{-} is the unique irreducible Spin(9)-representation of dimension 128. □

Thus we have

Theorem 2.2. *The massless fields of M-theory are encoded in the spinor bundle of $\mathbb{O}P^2$.*

2.1. $\mathbb{O}P^2$ Bundles

Having motivated $\mathbb{O}P^2$ bundles in M-theory, we now carry on with our proposal and construct such bundles in 11 dimensions. We study the properties of the $\mathbb{O}P^2$ bundle as well as the associated F_4 bundle and give some consistency conditions. As bundles are characterized by characteristic classes and genera, we 'compare' the structure of the base and that of the total space. For that purpose we start with discussing the relevant genera of the fiber.

2.2. Genera of $\mathbb{O}P^2$

A genus is a function on the cobordism ring Ω (see Section 3 for cobordism). More precisely, it is a ring homomorphism $\varphi : \Omega \otimes R \to R$, where R is any integral domain over $\mathbb{Q}$. It could be $\mathbb{Z}$, $\mathbb{Z}_p$ or $\mathbb{Q}$ itself. Genera in general have expressions given in terms of characteristic classes. Two important 'modern' genera are the elliptic genus $\Phi_{\rm ell}$ and the Witten genus $\Phi_{\rm W}$. The first is characterized by two parameters, denoted ε and δ, whose various values give different specializations of $\Phi_{\rm ell}$. Special values of the parameters correspond to more 'classical' genera. The values $\delta = \varepsilon = 1$ leads to the L-genus $L : \Omega \otimes \mathbb{Q} \to \mathbb{Q}$, and the values $\delta = -\frac{1}{8}$, $\varepsilon = 0$ leads to the $\widehat{A}$-genus $\widehat{A} : \Omega \otimes \mathbb{Q} \to \mathbb{Q}$. Depending on the type of cobordism considered, Ω and also R can vary. For instance, when the manifolds are Spin then the $\widehat{A}$-genus is an integer and so $\widehat{A} : \Omega^{\rm Spin} \otimes \mathbb{Z} \to \mathbb{Z}$. The Witten genus is defined for any topological manifold but it becomes a modular form for special manifolds, namely ones with a *String* structure or $BO\langle 8\rangle$-structure, and those are the manifolds that satisfy $\frac{1}{2}p_1 = 0$, where p_1 is the first Pontrjagin class of the tangent bundle. The Witten genus is a map $\Phi_{\rm W} : \Omega^{BO\langle 8\rangle} \otimes R \to {\rm MF} = R[E_4, E_6]$, where MF is the ring of modular forms generated by the Eisenstein series E_4 and E_6, and R is usually $\mathbb{Q}$ or $\mathbb{Z}$. We describe this more precisely below.

It is natural to ask what the values of the elliptic genus and of the Witten genus of $\mathbb{O}P^2$ are. First, however, we consider the classical genera.

2.2.1. The classical genera. We give the following specialization.

Lemma 2.3.

1. *The $\widehat{A}$-genus of $\mathbb{O}P^2$ is zero,* $\widehat{A}[\mathbb{O}P^2] = 0$.
2. *The L-genus of $\mathbb{O}P^2$ is u^2, where u is the generator of $H^8(\mathbb{O}P^2; \mathbb{Z})$.*

2.2.2. The Witten genus Next we consider another genus, the Witten genus, which can be defined in the following way. There is a convenient collection of manifolds $\{M^{4n}\}$ that generate the rational cobordism ring $\Omega \otimes \mathbb{Q}$ [33]. The advantage of this basis is that each M^{4n} has a single non-zero Pontrjagin class, the top one $p_n = d_n(2n-1)!m$ where m generates $H^{4n}(M^{4n})$. On this basis, $\Phi_{\rm W}(M^{4k}) = {\rm num}_k E_{2k}$ for $k > 1$ and $\Phi_{\rm W}(M^4) = 0$, where ${\rm num}_n/d_n = B_{2n}/4n$ is the given numerator, with ${\rm num}_n$ and d_n relatively prime, and B_{2n} the even Bernoulli numbers. The ring of modular forms for the full modular group is (cf. [4]) ${\rm MF} = \mathbb{Z}[E_4, E_6, \Delta]/(E_4^3 - E_6^2 - 1728\Delta)$, where $\Delta = q\prod_n(1-q^n)^{24}$. By inspecting the Bernoulli numbers we can see that the first four terms in d_n are $24, 240, 504, 480$. This is enough for working up until real dimension 16.

Theorem 2.4. *The Witten genus of* $\mathbb{O}P^2$ *is zero,* $\Phi_{\rm W}(\mathbb{O}P^2) = 0$.

Proof. $\mathbb{O}P^2$ has positive scalar curvature, so its $\widehat{A}$-genus is zero $\widehat{A}(\mathbb{O}P^2) = 0$. $\mathbb{O}P^2$ is also a *String* manifold, so its Witten genus $\Phi_{\rm W}(\mathbb{O}P^2) : \Omega_{16}^{BO\langle 8\rangle} = \pi_{16}MO\langle 8\rangle \to \pi_* eo_2 = {\rm MF}_*$ must be a modular form for $SL(2,\mathbb{Z})$ of weight equals to half its dimension [62], i.e., 8. What modular forms do we have? The ring of integral modular forms is (cf. [4])

$$\text{(2.2)} \qquad {\rm MF}_* = \mathbb{Z}[E_4, E_6, \Delta]/(2^6 \cdot 3^3\Delta - E_4^3 + E_6^2),$$

where $E_4 \in {\rm MF}_4$, $E_6 \in {\rm MF}_6$ and $\Delta \in {\rm MF}_{12}$. Thus the only modular form of weight 8 is E_4^2. However, the form of the Eisenstein series is $E_4 = 1+$ higher terms, so that the required modular form does not start with zero. Therefore $\Phi_{\rm W}(\mathbb{O}P^2) = 0$. □

2.2.3. The elliptic genus Next we consider the elliptic genus $\Phi_{\rm ell} : \Omega_*^{BO\langle 8\rangle} \otimes \mathbb{Q} \to \mathbb{Q}[\delta, \varepsilon]$, where the generators δ and ε have degrees 4 and 8, respectively.

Theorem 2.5. *The elliptic genus of the Cayley plane is* $\Phi_{\rm ell}(\mathbb{O}P^2) = \varepsilon^2$.

Proof. There are several ways to prove this. The first one is to use the idea of cobordism as in the proof of the case of the quaternionic projective plane $\mathbb{H}P^2$. However, we can simply apply a result from [22]. Since $\mathbb{O}P^2$ is a connected homogeneous space of a compact connected Lie group F_4, and since $\mathbb{O}P^2$ is oriented and admits a Spin structure, then the normalized elliptic genus $\Phi_{\rm norm} := \Phi_{\rm ell}/\varepsilon^2$ is a constant modular function

$$\text{(2.3)} \qquad \Phi_{\rm norm}(\mathbb{O}P^2) = \sigma(\mathbb{O}P^2).$$

Thus we immediately get the result. □

2.2.4. The Ochanine genus We next consider the Ochanine genus [38], which is a generalization of the elliptic genus in such a way that it involves q-expansions. The Ochanine genus is a ring homomorphism

$$\Phi_{\text{och}} : \Omega_*^{\text{spin}} \longrightarrow KO_*(\text{pt})[[q]], \tag{2.4}$$

from the Spin cobordism ring to the ring of power series with coefficients in

$$KO_*(\text{pt}) = \mathbb{Z}\left[\eta, \omega, \mu, \mu^{-1}\right] / \left(2\eta, \eta^3, \eta\omega, \omega^2 - 2^2\mu\right), \tag{2.5}$$

where $\eta \in KO_1, \omega \in KO_4$ and $\mu \in KO_8$ are generators of degrees 1, 4 and 8, respectively, and are given by the normalized Hopf bundles $\gamma_{_{\mathbb{R}P^1}} - 1$, $\gamma_{_{\mathbb{H}P^1}} - 1$, $\gamma_{_{\mathbb{O}P^1}} - 1$ (viewed as real vector bundles) over the real, quaternion and Cayley projective lines $\mathbb{R}P^1 = S^1$, $\mathbb{H}P^1 = S^4$ and $\mathbb{O}P^1 = S^8$.

For a manifold M^m of dimension m, corresponding to the projection map $\pi^{M^m} : M^m \to \text{pt}$ there is the Gysin map $\pi_!^{M^m} : KO(M^m) \to KO^m(\text{pt}) = KO_m(\text{pt})$. Now consider a real vector bundle E on M^m and form the following combination of exterior powers and symmetric powers of E:

$$\Theta_q(E) = \sum_{i\geq 0} \Theta^i(E) q^i = \bigotimes_{n\geq 1} \left(\Lambda_{-q^{2n-1}}(E) \otimes S_{q^n}(E)\right), \tag{2.6}$$

which, since it is multiplicative under Whitney sum, can be considered as an exponential map $\Theta_q : KO(M^m) \to KO(M^m)[[q]]$. Now specialize E to be the reduced tangent bundle $\widetilde{TM^m}$, which is $TM^m - m$. Then the Ochanine genus is defined to be [28, 38]

$$\begin{aligned}
\Phi_{\text{och}}(M^m) &:= \sum_{i\geq 1} \Phi^i_{\text{och}}(M^m) q^i \\
&= \sum_{i\geq 0} \pi_!^{M^m} \left(\Theta^i(\widetilde{TM^m})\right) q^i \\
&= \theta(q)^{-m} \langle \Theta_q(TM^m), [M^m]_{KO}\rangle \in KO_m(\text{pt})[[q]],
\end{aligned} \tag{2.7}$$

where $[M^m]_{KO} \in KO_m(M^m)$ denotes the Atiyah–Bott–Shapiro orientation [6] of M^m, $\langle , \rangle : KO^i(X) \otimes KO_j(X) \to KO_{j-i}$ is the Kronecker pairing and

$$\theta(q) := \Theta_q(1) = \prod_{n\geq 1} \frac{1 - q^{2n-1}}{1 - q^{2n}} = 1 - q + q^2 - 2q^3 \pm \cdots \quad \in \mathbb{Z}[[q]] \tag{2.8}$$

is the Ochanine genus of the trivial line bundle.

The degree zero part $\Theta^0(E)$ is a trivial real line bundle, and corresponds to the Atiyah invariant $\Phi^0_{\rm och}(M) = \pi_!^{M^m}(1) = \langle 1, [M^m]_{KO}\rangle = \alpha(M^m)$. The cobordism invariant $\alpha \in KO_m$ [5] can be interpreted as the index of a family of operators associated to M^m parametrized by S^m [23]. Thus the α-invariant is the classical value of the Ochanine genus in the same way that the $\widehat{A}$-genus and the L-genus are the classical values of the elliptic genus corresponding, respectively, to

$$\begin{aligned} &\delta = \widehat{A}(\mathbb{C}P^2) = -\frac{1}{8}, \quad \varepsilon = \widehat{A}(\mathbb{H}P^2) = 0, \text{ and} \\ &\delta = L(\mathbb{C}P^2) = 0, \quad \varepsilon = L(\mathbb{H}P^2) = 1. \end{aligned} \tag{2.9}$$

The Ochanine genus is related to the restriction $\Phi_{\rm ell,int}$ to $\Omega_*^{\rm spin}$ of the universal elliptic genus $\Phi_{\rm ell,uni} : \Omega_*^{SO} \to \mathbb{Q}[[q]]$, whose parameters are

$$\begin{aligned} \delta &= -\frac{1}{8} - 3\sum_{n\geq 1}\left(\sum_{d|n,\ d \text{ odd}} d\right) q^n = -\frac{1}{8} + q - \text{expansion}, \\ \varepsilon &= \sum_{n\geq 1}\left(\sum_{d|n,\ \frac{n}{d} \text{ odd}} d^3\right) q^n = 0 + q - \text{expansion}. \end{aligned} \tag{2.10}$$

More precisely, $\Phi_{\rm ell,int} = {\rm Ph} \circ \Phi_{\rm och} : \Omega_*^{\rm spin} \to \mathbb{Z}[[q]]$, where Ph is the Pontrjagin character

$$\text{Ph} : KO^*(X) \xrightarrow{\otimes\mathbb{C}} K^*(X) \xrightarrow{\text{ch}} H^{**}(X;\mathbb{Q}), \tag{2.11}$$

which can be thought of as the analog for real vector bundles of the Chern character for complex vector bundles.

We now check the value of $\Phi_{\rm och}$ for $\mathbb{O}P^2$.

Theorem 2.6. *The Ochanine genus of* $\mathbb{O}P^2$ *is* $\Phi_{\rm och}(\mathbb{O}P^2) = \varepsilon^2\mu^2$.

Proof. The Ochanine genus $\Phi_{\rm och}(\mathbb{O}P^2)$ is the map $\Omega_{16}^{\rm spin} \to KO_{16}[[q]]$. Note that $\Omega_{16}^{\rm spin} = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ and that $KO_{16}({\rm pt}) = \mathbb{Z}$ with generator μ^2. The image $\Phi_{\rm och}(\Omega_{16}^{\rm spin})$ is the set of all modular forms of degree 16 and weight 8 over $KO_{16} = \mathbb{Z}$. Let $M^\Gamma(KO_{16})$ be the graded ring of modular forms over KO_{16} for Γ, a subgroup of finite index in $SL(2;\mathbb{Z})$. For $M_*^\Gamma(\mathbb{Z}) = \mathbb{Z}[\delta_0, \varepsilon]$, where $\delta_0 = -8\delta \in M_2^\Gamma(\mathbb{Z})$ and δ and $\varepsilon \in M_4^\Gamma(\mathbb{Z})$ are the generators in (2.10), we have

$$M^{\Gamma}(KO_{16}) \cong KO_{16} \otimes M_*^{\Gamma}(\mathbb{Z}) \\ = \mathbb{Z} \otimes \mathbb{Z}[\delta_0, \varepsilon]. \tag{2.12}$$

Then a modular form of degree 16 and weight 8 can be written in a unique way as a polynomial $P(\delta_0, \varepsilon)$ of weight 8 with integer coefficients. Still applying the construction in [38], the Ochanine genus in our case is

$$\Phi_{\mathrm{och}}(\mathbb{O}P^2) = \left(a_0(\mathbb{O}P^2)\delta_0^4 + a_1(\mathbb{O}P^2)\delta_0^2\varepsilon + a_2(\mathbb{O}P^2)\varepsilon^2\right)\mu^2, \tag{2.13}$$

with uniquely defined homomorphisms , for $i = 1, 2, 3$,

$$a_i \cdot \mu^2 : \Omega_{16}^{\mathrm{spin}} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \longrightarrow KO_{16} = \mathbb{Z}. \tag{2.14}$$

The integers a_i can be determined as follows. We have already seen that the lowest coefficient is given by the Atiyah invariant. Since $\mathbb{O}P^2$ admits a Riemannian metric of positive scalar curvature then, from [23], $\alpha(\mathbb{O}P^2) = 0$, and hence we have determined that $a_0(\mathbb{O}P^2) = 0$. Another way of seeing this is to notice that for manifolds of dimension $4k$, the Atiyah invariant is essentially the $\widehat{A}$-genus, which, by Lichnerowicz theorem [32], vanishes for a manifold with positive scalar curvature. The highest coefficient, $a_2(\mathbb{O}P^2)$, is given by the Ochanine k-invariant, which in this case is just the signature $a_2(\mathbb{O}P^2) = \sigma(\mathbb{O}P^2) = 1$. It remains to calculate a_1. This is given by the first KO-Pontrjagin class Π_1

$$a_1(\mathbb{O}P^2) = \Pi_1(T\mathbb{O}P^2) = -\Lambda^1(T\mathbb{O}P^2 - 16), \tag{2.15}$$

which is just $-(T\mathbb{O}P^2 - 16)$. The KO-Pontjagin classes are defined as follows [3]. For an n-dimensional vector bundle ξ over a space X, $\Pi_u(\xi) \in KO^0(X)$ are defined by

$$(1+t)^n \sum_{k=0}^{\infty} \frac{t^k}{(1+t)^{2k}} \Pi_k(\xi) = \sum_{k=0}^{\infty} t^k \Lambda^k(\xi). \tag{2.16}$$

For $k = 1$ this gives the first KO-Pontrjagin class used in (2.15). Alternatively, we can look at the q-components of Φ_{och} from the first line of Equation

(2.7) and get

(2.17)
$$\Phi^0_{\mathrm{och}}(\mathbb{O}P^2) = \langle 1, [\mathbb{O}P^2]_{KO}\rangle = \alpha(\mathbb{O}P^2),$$
$$\Phi^1_{\mathrm{och}}(\mathbb{O}P^2) = \langle -\Pi_1(\mathbb{O}P^2), [\mathbb{O}P^2]_{KO}\rangle = \langle -(T\mathbb{O}P^2 - 16), [(\mathbb{O}P^2)]_{KO}\rangle.$$

We still have to calculate a_1. We use the topological Riemann-Roch theorem (see [55]), which states that for M a closed Spin manifold and $x \in \widetilde{KO}^*(M)$, then $Ph\langle x, [M]_{KO}\rangle = \langle \widehat{A}(M)Ph(x), [M]\rangle_H$, where $\langle\ ,\ \rangle_H$ is the Kronecker pairing on cohomology. Taking $M = \mathbb{O}P^2$ and $x = T\mathbb{O}P^2$, we get for a_1

$$\langle \widehat{A}(\mathbb{O}P^2)Ph(T\mathbb{O}P^2), [\mathbb{O}P^2]\rangle_H, \tag{2.18}$$

which is zero because, as we have seen, $\widehat{A}(\mathbb{O}P^2) = 0$. □

2.3. $\mathbb{O}P^2$ bundles over 11-manifolds

Consider the fiber bundle $E \to Y^{11}$ with fiber $\mathbb{O}P^2$ and structure group F_4. There is a universal bundle of this type. $\mathbb{O}P^2$ bundles over Y^{11} are pullbacks of the universal bundle

$$\mathbb{O}P^2 = F_4/\mathrm{Spin}(9) \longrightarrow B\mathrm{Spin}(9) \longrightarrow BF_4 \tag{2.19}$$

by the classifying map $f : Y^{11} \to BF_4$. In this paper we will consider the diagram

$$\begin{array}{ccccc} \mathbb{O}P^2 & \longrightarrow & M^{27} & & \\ & & \downarrow{\scriptstyle \pi} & \searrow & \\ & & Y^{11} & \xrightarrow[f]{} & BF_4. \end{array} \tag{2.20}$$

Note that the map from M^{27} to BF_4 can be $f\pi$ and this will be useful later in Section 3. We first have the following.

Proposition 2.7. *The obstruction to existence of a section of an $\mathbb{O}P^2$ fiber bundle over an 11-dimensional manifold Y^{11} lies in $H^9(Y^{11};\mathbb{Z})$, $H^{10}(Y^{11};\mathbb{Z}_2)$ and $H^{11}(Y^{11};\mathbb{Z}_2)$.*

Proof. For a fiber bundle $F \to E \to B$, the existence to having a section lies in the groups $H^r(B; \pi_{r-1}(F))$ for all nonzero $r \in \mathbb{N}$. In our case, $\mathbb{O}P^2$ has $\pi_i = 0$ for $i \leq 7$, so that the first obstruction is in $H^9\left(Y^{11}; \pi_8(\mathbb{O}P^2)\right)$, which is $H^9(Y^{11}; \mathbb{Z})$. The next two nontrivial homotopy groups of $\mathbb{O}P^2$, both of which are isomorphic to $\mathbb{Z}_2$, occur in dimensions 9 and 10 so that the obstructions are in $H^{10}(Y^{11}; \mathbb{Z}_2)$ and $H^{11}(Y^{11}; \mathbb{Z}_2)$. $\mathbb{O}P^2$ has further nontrivial homotopy groups but that would bring us to $H^{\geq 12}$, which are zero for an 11-manifold. □

Remarks.

1. The first obstruction $H^9(Y^{11}; \mathbb{Z})$ is called the primary obstruction.
2. Note that the primary obstruction is a $\mathbb{Z}$-class whereas the secondary obstructions are $\mathbb{Z}_2$-classes.

In forming bundles with $\mathbb{O}P^2$ as fibers, we are forming bundles of $BO\langle 8\rangle$-manifolds over Y^{11}. We will next investigate the relation between structures on Y^{11}, on the fiber $\mathbb{O}P^2$ and on the total space M^{27}.

2.4. Relating Y^{11} and M^{27}

2.4.1. Topological consequences: the higher structures We ask the question whether topological conditions on Y^{11}, namely having *Spin*, *String* or *Fivebrane* structure [52,53], will lead to (similar) structures on M^{27}. The answer to such a question is possible because we know about the (non-)existence of these structures on $\mathbb{O}P^2$.

The condition $\lambda := \frac{1}{2}p_1 = 0$ for lifting the structure group of the tangent bundle to String(n) is related to the condition $W_7 = 0$ for orientation with respect to either the $p = 2$ integral Morava K-theory $K(2)$ or Landweber's elliptic cohomology theory $E(2)$ [29]. The first condition implies the second, but the converse is not true, a counterexample being $X^{10} = S^2 \times S^2 \times \mathbb{C}P^3$ [29]. Thus if we assume the *String* orientation, then we are already guaranteed the W_7 orientation, and so the discussion and constructions in [29–31, 48] for 10-dimensional string theory apply. The condition $\lambda = 0$ can be extended from 10 to 11 dimensions and vice versa. This is because for $Y^{11} = X^{10} \times S^1$ the first Pontrjagin classes are related as (using bundle notation) $p_1(TX^{10} \oplus TS^1) = p_1(TX^{10}) + p_1(TS^1)$, but for dimensional reasons $p_1(TS^1) = 0$ so that we have $p_1(Y^{11}) = p_1(X^{10})$. Thus the *String* condition can be translated from M-theory to string theory and back as desired.

There is no cohomology in degree four for $\mathbb{O}P^2$, so we immediately have

Proposition 2.8. *$\mathbb{O}P^2$ admits a $BO\langle 8\rangle$-structure.*

Remark. If Y^{11} is a $BO\langle 8\rangle$-manifold, i.e., is $MO\langle 8\rangle$-orientable, then it has an $MO\langle 8\rangle$ homology fundamental class,

$$[Y^{11}]_{MO\langle 8\rangle} \in MO\langle 8\rangle_{11}(Y^{11}). \tag{2.21}$$

Any integral expression will involve this class. This would also enter the construction of the $BO\langle 8\rangle$ partition function.

We would like to check to what extent we can know the cohomology of the total space M^{27} in terms of the cohomology of the base Y^{11}, given that we know the cohomology of the fiber $\mathbb{O}P^2$. One way to detect this is by using the Serre spectral sequence for the bundle

$$E_2^{p,q} = H^p\left(Y^{11}, H^q(\mathbb{O}P^2)\right) \Rightarrow H^{p+q}(M^{27}). \tag{2.22}$$

Consider the case of a product $M^{27} = \mathbb{O}P^2 \times Y^{11}$, i.e., when the bundle is trivial. In this case, using the Künneth theorem and the fact that the cohomology of $\mathbb{O}P^2$ is non-zero only in degrees 8 and 16, we get

Proposition 2.9.

$$H^n(\mathbb{O}P^2 \times Y^{11}; C) \cong H^{n-8}(Y^{11}; C) \oplus H^{n-16}(Y^{11}; C). \tag{2.23}$$

We next consider the case when the bundle is not trivial. A simplification is made if coefficients are taken so that the cohomology of the fiber is trivial in those coefficients. The torsion ('bad') primes for F_4 are 2 and 3, so that one might expect that those are the primes that do not cause such a simplification. It will turn out that this is true only for $p = 3$, as we now show. We first show that $p = 3$ occurs and then that it is the only one.

The cohomology of the classifying spaces of Spin(9) and F_4 with $\mathbb{Z}_p$ coefficients, $p = 2, 3$, are as follows. The cohomology ring of BF_4 with coefficients in $\mathbb{Z}_2$ is given by the polynomial ring [10]

$$H^*(BF_4; \mathbb{Z}_2) = \mathbb{Z}_2\left[x_4, x_6, x_7, x_{16}, x_{24}\right], \tag{2.24}$$

where x_i are polynomial generators of degree i related by the Steenrod square operation $Sq^i : H^n(BF_4; \mathbb{Z}_2) \to H^{n+i}(BF_4; \mathbb{Z}_2)$ as

$$x_6 = Sq^2 x_4, \quad x_7 = Sq^3 x_4, \quad x_{24} = Sq^8 x_{16}. \tag{2.25}$$

$H^*(BF_4;\mathbb{Z}_3)$ is generated by x_i for $i = 4, 8, 9, 20, 21, 25, 26, 36, 48$, with the structure of a polynomial algebra [58]. Considering $p = 3$, this is

$$H^*(BF_4;\mathbb{Z}_3) \cong \mathbb{Z}_3[x_{36}, x_{48}] \otimes \left(\mathbb{Z}_3[x_4, x_8] \otimes \left\{1, x_{20}, x_{20}^2\right\} + \Lambda(x_9) \otimes \mathbb{Z}_3[x_{26}] \otimes \{1, x_{20}, x_{21}, x_{25}\}\right). \tag{2.26}$$

The generators can be chosen to be related by the Steenrod power operations at $p = 3$, $P^i : H^n(BF_4;\mathbb{Z}_3) \to H^{n+4i}(BF_4;\mathbb{Z}_3)$, as

$$\begin{aligned} &x_8 = P^1 x_4, \quad x_9 = \beta x_8 = \beta P^1 x_4, \quad x_{20} = P^3 P^1 x_4, \\ &x_{21} = \beta P^3 P^1 x_4, \quad x_{25} = P^4 \beta P^1 x_4, \quad x_{26} = \beta P^4 \beta P^1 x_4 \end{aligned} \tag{2.27}$$

and $x_{48} = P^3 x_{36}$. If we restrict to degrees ≤ 11 then we have the truncated polynomial

$$H^*(BF_4;\mathbb{Z}_3) \cong \mathbb{Z}_3[x_4, x_8] + \Lambda(x_9). \tag{2.28}$$

The classes coming from $B\mathrm{Spin}(9)$ are just the Stiefel–Whitney classes in the $\mathbb{Z}_2$ case and the Pontrjagin classes (reduced mod 3) in the integral ($\mathbb{Z}_3$ case). These are actually not much different from the classes of $B\mathrm{Spin}(11)$. Explicitly, at $p = 2$ the cohomology ring of $B\mathrm{Spin}(9)$ is given by the polynomial ring [40]

$$H^*(B\mathrm{Spin}(9);\mathbb{Z}_2) = \mathbb{Z}_2\left[w_4, w_6, w_7, w_8, w_{16}'\right], \tag{2.29}$$

where w_i is the restriction of the universal Stiefel–Whitney class, and w_{16}' is the Stiefel–Whitney class $\omega_{16}(\Delta_{\mathrm{Spin}(9)})$ of the spin representation $\Delta_{\mathrm{Spin}(9)}$: $\mathrm{Spin}(9) \to O(16)$. At $p = 3$, $H^*(B\mathrm{Spin}(9);\mathbb{Z}_3)$ is generated by the first four Pontrjagin classes [58]

$$H^*(B\mathrm{Spin}(9);\mathbb{Z}_3) = \mathbb{Z}_3[p_1, p_2, p_3, p_4], \quad \deg(p_i) = 4i. \tag{2.30}$$

Let us look at $\mathbb{Z}_3$ coefficients. From (2.28) and (2.30) we see that $H^9(B\mathrm{Spin}(9);\mathbb{Z}_3) = 0$ while $H^9(BF_4;\mathbb{Z}_3) \neq 0$, which implies that the map $H^9(BF_4;\mathbb{Z}_3) \to H^9(B\mathrm{Spin}(9);\mathbb{Z}_3)$ cannot be injective. Therefore, at $p = 3$ the Serre spectral sequence is not trivial. In the case of $\mathbb{Z}_2$, the situation is reversed, this time in degree eight: $H^8(B\mathrm{Spin}(9);\mathbb{Z}_2) \neq 0$ and $H^8(BF_4;\mathbb{Z}_2) = 0$.

Now we proceed with the uniqueness by applying the results in [27]. The cohomology of $\mathbb{O}P^2$ is $H^*(\mathbb{O}P^2;C) = C[x]/x^3$, $|x| = \deg x = 8$, as an algebra. Then, requiring that the Serre fibering $\mathbb{O}P^2 \to M^{27} \to Y^{11}$ be trivial over C implies for the E_2-term

$$E_2 = H^*(Y^{11}; C) \otimes_C C[x]/x^3. \tag{2.31}$$

Now the E_9 term is $E_{|x|+1} = E_2$ and the fibering is nontrivial if and only if we have a nonzero differential $d_9(1 \otimes x) \neq 0$. If $d_9(1 \otimes x) = a \otimes 1 \neq 0$ then $0 = d_9(1 \otimes x^3) = 3(a \otimes x^2)$. Hence the characteristic of C must not be relatively prime to 3, the degree of the ideal in the cohomology ring of $\mathbb{O}P^2$. Therefore, we have

Proposition 2.10. *The Serre spectral sequence for the fiber bundle $\mathbb{O}P^2 \to M^{27} \to Y^{11}$ is nontrivial only for cohomology with $\mathbb{Z}_3$ coefficients.*

We will make use of this and also say more in Section 2.4.2 – see Theorem 2.15 and the discussion around it.

Proposition 2.11. *If Y^{11} admits a String structure then so does M^{27} provided that there is no contribution from the degree four class from BF_4.*

Proof. We have the $\mathbb{O}P^2$ bundle over Y^{11} with total space M^{27}

$$\begin{array}{ccc} M^{27} & \xrightarrow{\tilde{f}} & B\mathrm{Spin}(9) \\ \pi \downarrow & & \downarrow Bi \\ Y^{11} & \xrightarrow[f]{} & BF_4 \end{array}, \tag{2.32}$$

which gives the decomposition $TM^{27} = \pi^* TY^{11} \oplus \tilde{f}^* T$, and so the tangential Pontrjagin class is

$$p_1(M^{27}) = \pi^* \left(p_1(Y^{11}) + f^* p_1(T) \right). \tag{2.33}$$

In the case Y^{11} is a 3-connected $BO\langle 8\rangle$-manifold, we have that $H^4(Y^{11}; \mathbb{Z})$ is free and $\pi^* : H^4(Y^{11}; \mathbb{Z}) \to H^4(M^{27}; \mathbb{Z})$ is an isomorphism. Thus M^{27} is also a $BO\langle 8\rangle$-manifold if and only if $f^* \overline{x}_4 = 0 \in H^4(Y^{11}; \mathbb{Z})$, where $\overline{x}_4 \in H^4(BF_4; \mathbb{Z})$ is the generator. Therefore we have shown that M^{27} is *String* if and only if G_4 in M-theory gets no contribution from BF_4. □

Remarks.

1. The quantization condition for the field strength G_4 in M-theory is known [61]. Since this field does not seem to get a contribution from

a class in BF_4, the condition in Proposition 2.11 seems reasonable. In some sense we could view the presence of such a degree four class as an anomaly which we have just cured.

2. We connect the above discussion back to cobordism groups. While there is no transfer map from $\Omega_{11}^{\langle 8\rangle}(BF_4)$ to $\Omega_{27}^{\langle 8\rangle}$, there is a transfer map after killing $\overline{x}_4$ [26]. Denoting by[2] $BF_4\langle\overline{x}_4\rangle$ the corresponding classifying space that fibers over BF_4, killing $\overline{x}_4$ is done by pulling back the path fibration $PK(\mathbb{Z},4) \to K(\mathbb{Z},4)$ with a map $\overline{x}_4 : BF_4 \to K(\mathbb{Z},4)$ realizing $\overline{x}_4$. The corresponding transfer map is $\Omega_{11}^{\langle 8\rangle}(BF_4\langle\overline{x}_4\rangle) \to \Omega_{27}^{\langle 8\rangle}$.

Next, for the higher structures we have

Proposition 2.12.

1. *In order for M^{27} to admit a Fivebrane structure, the second Pontrjagin class of Y^{11} should be the negative of that of $\mathbb{O}P^2$, i.e., $p_2(TY^{11}) = -p_2(T\mathbb{O}P^2) = -6u$.*
2. $\widehat{A}(M^{27}) = 0$*, irrespective of whether or not the $\widehat{A}$-genus of Y^{11} is zero.*
3. $\Phi_{\mathrm{W}}(M^{27}) = 0$.
4. $\Phi_{\mathrm{ell}}(M^{27}) = 0$.

Proof. For part (1) note that if Y^{11} admits a Fivebrane structure then M^{27} does not necessarily admit such a structure. This is because the obstruction to having a Fivebrane structure is $\frac{1}{6}p_2$ [53] but we know that $\frac{1}{6}p_2(\mathbb{O}P^2) = u \neq 0$. However, we can choose Y^{11} appropriately so that it conspires with $\mathbb{O}P^2$ to cancel the obstruction and lead to a Fivebrane structure for M^{27}. Noting that the tangent bundles are related as $TM^{27} = TY^{11} \oplus T\mathbb{O}P^2$, then considering the degree eight part of the formula (see [36]) $p(E \oplus F) = \sum p(E)p(F)$ mod 2−torsion, we get for our spaces

$$p_2(TY^{11} \oplus T\mathbb{O}P^2) = p_1(TY^{11})p_1(T\mathbb{O}P^2) + p_2(TY^{11}) + p_2(T\mathbb{O}P^2) \mod 2\text{−torsion}. \tag{2.34}$$

Since we have $p_1(T\mathbb{O}P^2) = 0$, then requiring that $p_2(TM^{27}) = 0$ leads to the constraint that $p_2(TY^{11}) + p_2(T\mathbb{O}P^2) = 0$ modulo 2-torsion.

[2] This is the analog of the *String* group when $G = \mathrm{Spin}$, in the sense that it is the 3-connected cover.

For part (2) we use the multiplicative property of the $\widehat{A}$-genus for Spin fiber bundles to get

$$\widehat{A}(M^{27}) = \widehat{A}(Y^{11})\widehat{A}(\mathbb{O}P^2). \tag{2.35}$$

Since the $\widehat{A}$-genus of $\mathbb{O}P^2$ is zero then the result follows.

For part (3) we use a result of Ochanine [37]. Taking the total space M^{27} and the base Y^{11} to be closed oriented manifolds, and since the fiber $\mathbb{O}P^2$ is a Spin manifold and the structure group F_4 of the bundle is compact, then the multiplicative property of the genus can be applied

$$\Phi_{\mathrm{W}}(M^{27}) = \Phi_{\mathrm{W}}(\mathbb{O}P^2)\Phi_{\mathrm{W}}(Y^{11}). \tag{2.36}$$

We proved in Theorem 2.4 that $\Phi_{\mathrm{W}}(\mathbb{O}P^2) = 0$, so it follows immediately that $\Phi(M^{27})$ is zero regardless of whether or not $\Phi_{\mathrm{W}}(Y^{11})$ vanishes. Even more, $\Phi_{\mathrm{W}}(Y^{11})$ is zero because Y^{11} is odd-dimensional.[3]

For part (4) we use the fact that the fiber is Spin and the structure group F_4 is compact and connected so we can apply the multiplicative property of the elliptic genus [37]

$$\Phi_{\mathrm{ell}}(M^{27}) = \Phi_{\mathrm{ell}}(Y^{11})\Phi_{\mathrm{ell}}(\mathbb{O}P^2). \tag{2.37}$$

In this case the genus for the fiber is not zero (see Proposition 2.5) but the elliptic genus of Y^{11} is zero, again because of dimension. Therefore $\Phi_{\mathrm{ell}}(M^{27}) = 0$. □

We next consider the relation between the Ochanine genera of the base and of the total space.

Having the Ochanine genera for S^1 and X^{10}, we now proceed to determine the corresponding genus for the eleven-dimensional manifold Y^{11}.

Proposition 2.13. *Let Y^{11} be an 11-dimensional Spin manifold, which is the total space of a circle bundle over a 10-dimensional Spin manifold X^{10}. Then the Ochanine genus of Y^{11} is*

$$\Phi_{\mathrm{och}}(Y^{11}) = \Phi_{\mathrm{och}}(X^{10}) \cdot \alpha(S^1). \tag{2.38}$$

[3]However, see the case when Y^{11} is a circle bundle at the end of this section.

Proof. Unlike other genera, the Ochanine genus does not in general enjoy a multiplicative property on fiber bundles. However, in the special case when the fiber is the circle with a $U(1)$ action $\Phi_{\rm och}$ does become multiplicative on the circle bundle [28]. We simply apply the result for $S^1 \to Y^{11} \to X^{10}$ to get

$$\Phi_{\rm och}(Y^{11}) = \Phi_{\rm och}(X^{10}) \cdot \Phi_{\rm och}(S^1). \tag{2.39}$$

With $\Phi_{\rm och}(S^1) = \alpha(S^1)$ the degree one generator in $KO_*(\text{pt})$, the result follows. □

Now that we have the Ochanine genus for Y^{11}, we go back and consider the original questions of finding the Ochanine genus of M^{27}, given that of Y^{11}.

Theorem 2.14. *The Ochanine genus of the total space M^{27} of an $\mathbb{O}P^2$ bundle over an 11-dimensional compact Spin manifold Y^{11}, which is a circle bundle over a 10-dimensional Spin manifold X^{10}, is*

$$\Phi_{\rm och}(M^{27}) = \Phi_{\rm och}(\mathbb{O}P^2) \cdot \Phi_{\rm och}(X^{10}) \cdot \alpha(S^1), \tag{2.40}$$

where $\Phi_{\rm och}(\mathbb{O}P^2)$ is given in Theorem 2.6 and $\Phi_{\rm och}(X^{10})$ is given as follows: If $k(X^{10}) = 0 \in \mathbb{Z}_2$ then $\Phi_{\rm och}(X^{10}) = \alpha(X^{10})$, while if $k(X^{10}) = 1 \in \mathbb{Z}_2$ then in $KO_{10} \otimes \mathbb{Z}_2$ we have

$$\Phi_{\rm och}(X^{10}) = \alpha(X^{10}) + \eta^2\mu(q + q^9 + q^{25} + \cdots). \tag{2.41}$$

Proof. As mentioned in the proof of Proposition 2.13 above, $\Phi_{\rm och}$ is not in general multiplicative for fiber bundles. Again, interestingly, we are in a special case where such a property holds [28]. It is so because the dimension of the fiber $\mathbb{O}P^2$ is a multiple of 4, the structure group F_4 is a compact connected Lie group, and the base Y^{11} is a closed Spin manifold. Applying to the fiber bundle $\mathbb{O}P^2 \to M^{27} \to Y^{11}$, and using Proposition 2.13, then gives the formula in the theorem. □

Remark. The circle in Theorem 2.14 is the one with the nontrivial/non-bounding/supersymmetric/Ramond-Ramond Spin structure.

2.4.2. Topological terms in the lifted action Having motivated and then constructed $\mathbb{O}P^2$ bundles in M-theory, we now turn to the discussion of some of the consequences. The most obvious question from a physics point of view is to characterize the corresponding 'theory' in 27 dimensions. We will not be able to achieve that, but we will be able to characterize some of the terms in the would-be action up in 27 dimensions. In the absence of a clear handle, we take the most economical approach and concentrate on the topological terms, which in any case are the terms we can trust. We also make some remarks on other terms as well.

The simplest topological term coming from $\mathbb{O}P^2$ at the rational level would be some differential form of degree 16. This could also be decomposable, i.e., a wedge product of differential forms of lower degrees such that the total degree is 16. We should seek forms that naturally occur on $\mathbb{O}P^2$. Looking at the question from a 27-dimensional perspective, a Kaluza–Klein mechanism comes to mind. We do not attempt to discuss this problem fully here but merely provide some possibilities that are compatible with the structures that we have. In dimensional reduction from 10 and 11 dimensions to lower dimensions, holonomy plays an important role as it gives some handle on the differential forms involved, as well as on supersymmetry.

From the cohomology of $\mathbb{O}P^2$, the possible topological terms generated from this internal space come from $X_i \in H^i(\mathbb{O}P^2)$ for $i = 8, 16$, so that their linear combination generates a candidate degree 16 term

$$\rho_{16} := aX_{16} + bX_8^2, \tag{2.42}$$

where X_8 and X_{16} are 8- and 16-forms, respectively, and a and b are some parameters.

Remarks.

1. Since the degree 16 generator is built out of the degree 8 generator, namely the first is proportional to u^2 and the second is u, then Equation (2.42) is redundant as X_{16} is really built out of X_8^2. Thus Equation (2.42) should be replaced by $\rho_{16} = bX_8^2$.

2. In terms of the generator u of $H^8(\mathbb{O}P^2;\mathbb{Z})$, the expression at the integral level should be

$$\rho_{16} = \alpha u^2, \tag{2.43}$$

with $\alpha \in \mathbb{Q}$.

3. The term ρ_{16} would be thought of as a degree 16 analog of the one loop term I_8 in M-theory and type IIA string theory from [16]. It would appear as a topological term in the action, rationally as

$$S^{\rm top}_{(27)} = \int_{M^{27}} L^{\rm top}_{(27)} = \int_{M^{27}} \rho_{16} \wedge L_{(11)^{\rm top}}, \tag{2.44}$$

where $L^{\rm top}_{(11)}$ is the topological Lagrangian in 11 dimensions given by

$$L^{\rm top}_{(11)} = \frac{1}{6} G_4 \wedge G_4 \wedge C_3 - I_8 \wedge C_3\,. \tag{2.45}$$

Then we have

$$S^{\rm top}_{(27)} = \int_{Y^{11}} L^{\rm top}_{(11)} \int_{\mathbb{O}P^2} \rho_{16} = \alpha \int_{Y^{11}} L^{\rm top}_{(11)} = \alpha S^{\rm top}_{(11)}.$$

1. At the rational level we can thus use ω_8 to build a Spin(9)-invariant degree 16 expression $\rho_{16}^{\mathbb{R}} = \omega_8 \wedge \omega_8$ that we integrate and insert as part of the action as $\int_{\mathbb{O}P^2} \rho_{16}^{\mathbb{R}}$.

The integration of ρ_{16} over $\mathbb{O}P^2$ in the second step of Equation (2.46) requires the existence of a fundamental class $[\mathbb{O}P^2]$ for the Cayley plane. The Cayley 8-form $\mathcal{J}_8$ allows for such an evaluation at the rational and integral level. The next question is about torsion. The existence of such a fundamental class at that level is neither automatic nor obvious. In order to state the following result we recall some notation. Let $\beta : H^i(Y^{11};\mathbb{Z}_3) \to H^{i+1}(Y^{11};\mathbb{Z})$ be the Bockstein homomorphism corresponding to the reduction modulo 3, $r_3 : \mathbb{Z} \to \mathbb{Z}_3$, i.e., associated to the short exact sequence $0 \to \mathbb{Z}_3 \to \mathbb{Z}_9 \to \mathbb{Z}_3 \to 0$ and $P^1_3 : H^j(Y^{11};\mathbb{Z}_3) \to H^{j+4}(Y^{11};\mathbb{Z}_3)$ be the Steenrod reduced power operation at $p = 3$. Then we have

Theorem 2.15. *A fundamental class exists provided that $\beta P^1_3 x_4 = 0$, where x_4 is the mod 3 class on Y^{11} pulled back from BF_4 via the classifying map.*

Proof. Consider the fiber bundle $E \to Y^{11}$ with fiber $\mathbb{O}P^2$ and structure group F_4. There is a universal bundle of this type. $\mathbb{O}P^2$ bundles over Y^{11} are pullbacks of the universal bundle

$$\mathbb{O}P^2 = F_4/\mathrm{Spin}(9) \longrightarrow B\mathrm{Spin}(9) \longrightarrow BF_4 \tag{2.46}$$

by the classifying map $f : Y^{11} \to BF_4$. Since BF_4 is path-connected and $\mathbb{O}P^2$ is connected then we can apply the Serre spectral sequence to the

fibration (2.46). We consider two cases for the coefficients of the cohomology: $\mathbb{Z}_p$(or any field in general), p a prime and $\mathbb{Z}$ coefficients.
Coefficients in $\mathbb{Z}_p$: The important primes are $p = 2, 3$ as these are the torsion primes of F_4. For $p = 2$ the inclusion map $i : \mathrm{Spin}(9) \hookrightarrow F_4$ induces a map on the classifying spaces so that $H^*(B\mathrm{Spin}(9); \mathbb{Z}_p)$ is a free $H^*(BF_4; \mathbb{Z}_p)$-module on generators $1, x, x^2$ with $x \in H^8(B\mathrm{Spin}(9); \mathbb{Z}_p)$ the universal Leray–Hirsch generator that maps to $x \in H^8(\mathbb{O}P^2; \mathbb{Z}_p)$. Here we use the fact [35] that the Serre spectral sequence for a fiber bundle $F \to E \to B$ collapses if and only if the corresponding Poincaré series $\mathcal{P}(-) := \sum_{n \geq 0} t^n \dim_{\mathbb{Z}_p} H^n(-; \mathbb{Z}_p)$ satisfies $\mathcal{P}(E) = \mathcal{P}(F)\mathcal{P}(B)$. In our case the Serre spectral sequence of (2.46) collapses [26]. This follows from the equality of the corresponding Poincaré polynomials

$$\begin{aligned} \frac{\mathcal{P}(B\mathrm{Spin}(9))}{\mathcal{P}(BF_4)} &= \frac{(1-t^4)^{-1}(1-t^6)^{-1}(1-t^7)^{-1}(1-t^8)^{-1}(1-t^{16})^{-1}}{(1-t^4)^{-1}(1-t^6)^{-1}(1-t^7)^{-1}(1-t^{16})^{-1}(1-t^{24})^{-1}} \\ &= \frac{1-t^{24}}{1-t^8} = 1 + t^8 + t^{16}, \end{aligned} \tag{2.47}$$

which is just the Poincaré polynomial $\mathcal{P}(\mathbb{O}P^2)$ of the Cayley plane. This implies that the Leray–Hirsch theorem holds, i.e., that the map $H^*(\mathbb{O}P^2) \otimes H^*(BF_4) \to H^*(B\mathrm{Spin}(9))$ is an isomorphism of $H^*(BF_4)$-modules. This implies in particular that $H^*(B\mathrm{Spin}(9))$ is a free BF_4-module on $1, x, x^2$, where x is either w_8 or $w_8 + w_4^2$. The Wu formula with $w_1 = w_2 = 0$ for both cases gives that $Sq^1 x = Sq^2 x = Sq^3 x = Sq^5 x = 0$ so that

$$Sqx = x + Sq^4 x + Sq^6 x + Sq^7 x + x^2. \tag{2.48}$$

The elements x_4, $Sq^2 x_4$, $Sq^3 x_4 \in H^* BF_4$ are mapped to the elements w_4, $w_6 = Sq^2 w_4, w_7 = Sq^3 w_4 \in H^* B\mathrm{Spin}(9)$. The Leray–Hirsch theorem holds for the universal bundle, and consequently for all $\mathbb{O}P^2$ bundles [26].

For $p = 3$ the argument is similar except that now the generators in degrees 4 and 8 are related as $p_1 = \overline{p}_1$ and $p_2 = \overline{p}_2 + \overline{p}_1^2$, respectively [58]. Here p_i are the Pontrjagin classes.
Coefficients in $\mathbb{Z}$: We would like to find the differentials for

$$H^*(B\mathrm{Spin}(9); \mathbb{Z}) \Longleftarrow H^*\left(BF_4, H^*(\mathbb{O}P^2; \mathbb{Z})\right). \tag{2.49}$$

The class u maps under the differential to a $\mathbb{Z}_3$ class of degree 9 which we will call α. The lowest degree class on the fiber is x_8, so the differentials begin with d_9. The differential is d_9 on x_8 so that the class is $\beta P_3^1 x_4$, where

x_4 is the mod 3 class on Y^{11} coming from BF_4

$$Y^{11} \longrightarrow BF_4 \longrightarrow K(\mathbb{Z}_3, 9). \tag{2.50}$$

We thus have a 3-torsion class of $\mathbb{O}P^2$ bundles. The obstruction in $H^9(Y^{11};\mathbb{Z})$ coming from $H^9(BF_4;\mathbb{Z})$ is zero if and only if there exists a degree 16 class, say ρ_{16}, that restricts on each fiber to the fundamental class. □

Thus the vanishing of d_9 provides us with a fundamental class which we use to integrate over $\mathbb{O}P^2$.

Remark. The Pontrjagin classes p_2 and p_4 of $\mathbb{O}P^2$ are divisible by three. There is always a class in M^{27} that restricts on the fiber to three times the generator of the cohomology of $\mathbb{O}P^2$.

3. Connection to cobordism and elliptic homology

3.1. Cobordism and boundary theories

In this section we consider the question of extension of the theories in 11 and 27 dimensions to bounding theories in 12 and 28 dimensions, respectively, assuming the spaces to be *String* and taking into account the F_4 bundles. As mentioned in the Introduction, our discussion will make contact with a version of elliptic cohomology constructed by Kreck and Stolz [28]. In that paper the emphasis was on the Spin case corresponding geometrically to quaternionic projective plane $\mathbb{H}P^2$ bundles, but the authors assert the existence of a $BO\langle 8\rangle$ version corresponding to octonionic projective plane $\mathbb{O}P^2$ bundles. Let us denote this theory by $E^{\langle 8\rangle}$ or, equivalently, by $E^{\mathbb{O}}$.

We consider the *String* condition from an 11-dimensional point of view. One point that we utilize is that $\Omega_{11}^{\text{spin}}(\text{pt})$, the Spin cobordism group in 11 dimensions, is zero. This means that any 11-dimensional Spin manifold bounds a 12-dimensional one. It is also the case that the $BO\langle 8\rangle$ cobordism group $\Omega_{11}^{\langle 8\rangle}(\text{pt})$ is zero [18], so that the extension from an 11-dimensional *String* manifold to the corresponding boundary is unobstructed. Thus, if the space Y^{11} in which M-theory is defined admits a *String* structure then this always bounds a 12-dimensional *String* manifold Z^{12}.

Generalized cohomology theories can, in fact, be obtained as quotients of cobordism (see [29] for some exposition on this for physicists) by classic results [11]. For instance, Spin cobordism $\Omega_*^{\text{spin}} = \Omega_*^{\langle 4\rangle}$ is closely related to real K-theory KO, a fact we used in Section 2.1. For a space X, $KO^*(\text{pt})$

can be made into an Ω_*^{spin}-module and there is an isomorphism of $KO^*(X)$ with $\Omega_*^{\mathrm{spin}}(X) \otimes_{\Omega_*^{\mathrm{spin}}} KO^*(\mathrm{pt})$. As we have seen, this is related to the mod 2 index of the Dirac operator with values in real bundles in 10 dimensions which appears in the mod 2 part of the partition function [14]. There is an analogous construction for elliptic cohomology, where the starting point is $\Omega_*^{\langle 8 \rangle}$. This fact is related to the elliptic refinement of the mod 2 index which then has values in a real version of elliptic cohomology [29].

3.2. Cobordism of $BO\langle 8 \rangle$-manifolds with fiber $\mathbb{O}P^2$

Now we go back to our main discussion of relating the cobordisms of the 11- and 27-dimensional theories together with the F_4-$\mathbb{O}P^2$ bundles. Thus we are led to the study of the cobordism groups $\Omega_i^{\langle 8 \rangle}(BF_4)$ for $i = 11$ and 27. We will also be interested in relating these two groups.

We have an 11-dimensional base manifold Y^{11}, assumed to admit a String(11) structure, with an $\mathbb{O}P^2$ bundle such that the total space is M^{27} and the structure group is F_4. Let $\mathcal{I} \in \Omega_{27}^{\langle 8 \rangle}$ be the ideal generated by elements of the form $[M^{27}] - [\mathbb{O}P^2][Y^{11}]$ where, as before, $M^{27} \to Y^{11}$ is a fibration with fiber $\mathbb{O}P^2$ and structure group F_4. We have

Proposition 3.1. *Let Y^{11} be a compact manifold with a String structure on which M-theory is taken, and let M^{27} be the String manifold on which the 27-dimensional theory is taken, realizing the Euler triplets geometrically. Then such 27-manifolds M^{27} are in the ideal $\mathcal{I}$ of $\Omega_{27}^{\langle 8 \rangle}$ generated by $\mathbb{O}P^2$ bundles.*

Our setting is given in the following diagram

$$\begin{array}{ccc} \mathbb{O}P^2 & \longrightarrow & M^{27} & & \\ & & \downarrow \pi & \searrow^{f'} & \\ & & Y^{11} & \xrightarrow{f} & N. \end{array} \tag{3.1}$$

First we ignore the structure group and consider N to be a point. As in Section 3.1, let $\Omega_*^{\langle 8 \rangle}$ be the cobordism ring of manifolds with $w_1 = w_2 = \frac{1}{2}p_1 = 0$. This ring has only 2-torsion and 3-torsion, with the 3-torsion being a $\mathbb{Z}_3$ summand in dimensions 3, 10 and 13 (this is known only up to roughly dimension 16).

Note that cobordim groups $\Omega_*^{\langle n\rangle}$ arise as homotopy groups of the Thom spectra $MO\langle n\rangle$, in the sense that the former groups are the homotopy groups of the spectra (this is general for any type of cobordism). Hence the Thom spectrum for the *String* cobordism ring is $MO\langle 8\rangle$, and $\Omega_*^{\langle 8\rangle} = \pi_*(MO\langle 8\rangle)$. We can actually gain information about $\Omega_*^{\langle 8\rangle}$ by looking at topological modular forms. This is due to the following fact. Let $MO\langle 8\rangle \to tmf$ be any multiplicative map whose underlying genus is the Witten genus. Then the induced map on the homotopy groups $\pi_* MO\langle 8\rangle \to \pi_* tmf$ is surjective [24]. The low-dimensional homotopy groups of tmf are [24][4]

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
π_k tmf	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/6$	0	$\mathbb{Z}$	$\mathbb{Z}/3$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

The 2-primary components ${}_{(2)}\Omega_*^{\langle 8\rangle}$ of $\Omega_*^{\langle 8\rangle}$ are given by [18] (see also [26,57])

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
${}_{(2)}\Omega_k^{\langle 8\rangle}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z})^2$

By comparing the two tables, we can indeed see the 'missing' $\mathbb{Z}/3$ factors.

Note that in dimension 11, the result of [18] implies that $\Omega_{11}^{\langle 8\rangle} = 0$ since the 2-primary part is zero and there is no torsion in that dimension. There does not seem to be a computation for dimensions as high as 27. This implies that the map

$$\varrho : \Omega_{11}^{\langle 8\rangle}(\mathrm{pt}) \longrightarrow \Omega_{27}^{\langle 8\rangle}(\mathrm{pt}) \tag{3.2}$$

is a map whose domain is 0, and is thus not interesting.

We next allow the structure group F_4 so that there is a map from Y^{11} to its classifying space BF_4. Thus we are considering $N = BF_4$ and the classifying map to be f in (3.1). In this case, instead of the map ϱ we will consider the map

$$\varrho' : \Omega_{11}^{\langle 8\rangle}(BF_4) \longrightarrow \Omega_{27}^{\langle 8\rangle}(BF_4), \tag{3.3}$$

$$[Y^{11}, f] \longmapsto [M^{27}, f'], \tag{3.4}$$

which maps bordism classes of 11-manifolds, together with a map f to BF_4, to bordism classes of 27-manifolds together with a map f' to BF_4. Now both

[4]Here we prefer to use the notation for cyclic groups used in homotopy theory, e.g., $\mathbb{Z}/2$ in place of $\mathbb{Z}_2$. We hope this will be clear.

the domain and the range are in general nonempty unless certain conditions are applied.

Remarks.

1. The classifying space BF_4 has at least interesting degree 4 cohomology. However, we have seen that for the *String* condition to be multiplicative on $\mathbb{O}P^2$ bundles then we must kill x_4 coming from BF_4. This would then mean that we should in this case consider $BF\langle x_4\rangle$ instead of BF_4.
2. Killing x_4 as above would lead to the rational homotopy type

$$BF_4\langle x_4\rangle \sim S^{12} \times \text{higher spheres}, \tag{3.5}$$

so that the first homotopy is in dimension 12. This then would mean that should consider $\Omega_{11}^{\langle 8\rangle}(BF_4\langle x_4\rangle)$, which is zero, by dimension.

3. If we use $BF_4\langle x_4\rangle$ instead of BF_4, then this might cause some problems for the description of the fields of M-theory in terms of $\mathbb{O}P^2$ bundles, since there we used the Lie group F_4 on the nose. In other words, unlike the case for compact E_8 in 11 dimensions, F_4 appears not merely topologically, but via representation theory. However, compare the arguments in [49] for the E_8 model of the C-field in M-theory. It should be checked that the representations coming from the Lie 2-group $F_4\langle x_4\rangle$ respect the discussion in Section 2.

We can actually say more about the extensions of the F_4 bundle. We have

Proposition 3.2. *The F_4 bundle on a String manifold Y^{11} can be extended to Z^{12} where $\partial Z^{12} = Y^{11}$.*

Proof. We look for cobordism obstructions. Extending the bundle would be obstructed by $\Omega_{11}^{\langle n\rangle}(BF_4)$. Since the homotopy type of F_4 is $(3, 11, 15, 23)$ then that of BF_4 is $(4, 12, 16, 24)$ so that up to dimension 11 the classifying space BF_4 has the homotopy type of $K(Z, 4)$, much the same as E_8 does (and in fact all exceptional Lie groups except E_6) in that range. Now we reduce the problem to checking whether $\Omega_{11}^{\langle n\rangle}\langle(K(\mathbb{Z}, 4))\rangle$ is zero. This is indeed so by calculations of Stong [56], for $n = 4$, and Hill ([21], motivated by this question), for $n = 8$. ☐

Let $T_{27}^{\langle 8\rangle}(BF_4)$ be the subgroup of $\Omega_{27}^{\langle 8\rangle}(BF_4)$ consisting of bordism classes $[M^{27}, f\circ\pi]$, i.e., the classes that factor through the base Y^{11}. It could happen that some of the classes $[Y^{11}, f]$ of the bordism group of the base are zero. Let $\widetilde{T}_{27}^{\langle 8\rangle}(BF_4)$ be the subgroup whose elements satisfy the additional assumption that $[Y^{11}, f]=0$ in $\Omega_{11}^{\langle 8\rangle}(BF_4)$. Corresponding to the diagram (2.20) there is a classifying map

$$\psi : \Omega_{11}^{\langle 8\rangle}(BF_4) \longrightarrow \Omega_{27}^{\langle 8\rangle}(\mathrm{pt}), \tag{3.6}$$

which takes the class $[Y^{11}, f]$ to the class $[M^{27}=f^*E]$. The image $T_{27}^{\langle 8\rangle} = \mathrm{im}\,\psi$ of this map is the set of total spaces of $\mathbb{O}P^2$ bundles in $\Omega_{27}^{\langle 8\rangle}$. If we forget the classifying map f then instead of (3.6) we can map

$$\lambda : \Omega_{11}^{\langle 8\rangle}(BF_4) \longrightarrow \Omega_{11}^{\langle 8\rangle}(\mathrm{pt}), \tag{3.7}$$

where now the class $[Y^{11}, f]$ lands in the class $[Y^{11}]$ by simply forgetting f. Obviously, the kernel of λ makes up the classes $[Y^{11}, f]$ which map to $[Y^{11}]$ that are zero in $\Omega_{11}^{\langle 8\rangle}$. Such classes $[Y^{11}, f]$ map under ψ to total spaces of $\mathbb{O}P^2$ bundles with zero-bordant bases in $\Omega_{11}^{\langle 8\rangle}$. It is clear that $\psi(\ker\lambda)$ is the subgroup $\widetilde{T}_{27}^{\langle 8\rangle}$. That is, we have

(3.8)
$$T_{27}^{\langle 8\rangle} := \mathrm{im}\,\psi = \left\{\text{total spaces of } \mathbb{O}P^2 \text{ bundles in } \Omega_{27}^{\langle 8\rangle}(\mathrm{pt})\right\},$$
$$\widetilde{T}_{27}^{\langle 8\rangle} := \psi(\ker\lambda)$$
(3.9)
$$= \left\{\text{total spaces of } \mathbb{O}P^2 \text{ bundles with zero bordant base in } \Omega_{27}^{\langle 8\rangle}(\mathrm{pt})\right\}.$$

Note that, as mentioned above, the 2-primary part of $\Omega_n^{\langle 8\rangle}$ for $n\le 16$ is calculated in [18]. For $n=11$ this is zero. This implies that the kernel of λ is all of $\Omega_{11}^{\langle 8\rangle}(BF_4)$, i.e., all cobordism classes of total spaces have zero bordant bases. Then we have

Proposition 3.3. *$T_{27}^{\langle 8\rangle}$ and $\widetilde{T}_{27}^{\langle 8\rangle}$ coincide for base String manifolds of dimension 11.*

There are two cases to consider in order to determine whether or not the above spaces are trivial:

1. If $\Omega_{27}^{\langle 8\rangle}$ turns out to be zero, then the map ψ will be trivial in that degree.
2. If it turns out that $\Omega_{27}^{\langle 8\rangle} \neq 0$, then the map ψ is not trivial. It would then mean that $T_{27}^{\langle 8\rangle} = \widetilde{T}_{27}^{\langle 8\rangle} \neq \emptyset$. However, looking carefully at the map ψ we notice that its domain is zero. This is because the homotopy type of F_4 is $K(\mathbb{Z}, 3)$ up to dimension 10, so that the homotopy type of BF_4 is $K(\mathbb{Z}, 4)$ up to dimension 11. This means that $\Omega_{11}^{\langle 8\rangle}(BF_4) = \Omega_{11}^{\langle 8\rangle}((K\mathbb{Z}, 4)) = 0$. This then implies that the map ψ is trivial. In modding out by the corresponding equivalence to form

$$E_{27}^{\mathbb{O}} = E_{27}^{\langle 8\rangle} = \Omega_{27}^{\langle 8\rangle}/T_{27}^{\langle 8\rangle}, \tag{3.10}$$

we simply get

Proposition 3.4. *The homology theory is just the bordism ring* $E_{27}^{\mathbb{O}} = \Omega_{27}^{\langle 8\rangle}$.

Remarks.

1. Proposition 3.4 implies that in dimension 27 we do not get anything smaller or simpler than bordism.
2. The two spaces (3.8) and (3.9) have been characterized in the quaternionic case, i.e., when the fiber is $\mathbb{H}P^2$ with structure group $PSp(3)$, as

$$T_{27}^{\langle 4\rangle} = \ker(\alpha), \tag{3.11}$$

$$\widetilde{T}_{27}^{\langle 4\rangle} = \ker(\Phi_{\text{och}}), \tag{3.12}$$

   i.e., as the kernels of the Atiyah invariant in [54] and the Ochanine genus in [28], respectively. We see that in our case, $\alpha(M^{27}) = 0$, but $\Phi_{\text{och}}(M^{27})$ is not necessarily zero. This provides another justification for the calculations leading to Theorem 2.14. In fact, we can use the nontriviality of the Ochanine genus to check whether or not the homology theory is empty. Since, using Theorem 2.14, we can find a 27-dimensional manifold M^{27} with $\Phi_{\text{och}}(M^{27}) \neq 0$, the Spin cobordism group is nonzero $\Omega_{27}^{\langle 4\rangle} \neq 0$. Consequently, we have the following result for the corresponding *String* cobordism group.

Theorem 3.5. $\Omega_{27}^{\langle 8\rangle} \neq 0$.

Remark. Alternatively, the theorem can be proved using information about tmf. Since the orientation map from $M\mathrm{String} = MO\langle 8\rangle$ to tmf is surjective [4] then it is enough to know that the homotopy group of tmf in dimension 27 is nonzero. Indeed,[5] at least $\pi_{27}(tmf) \supset \mathbb{Z}/3$, so that $\Omega_{27} = \pi_{27}(M\mathrm{String}) \neq 0$.

In [57], the Witten genus was proposed as a candidate for the replacement of α in the octonionic case, so that

$$T_{27}^{\langle 8\rangle}(\mathrm{pt}) = \ker(\alpha^{\mathbb{O}}) := \ker(\Phi_{\mathrm{W}}). \tag{3.13}$$

Indeed, we have shown in Proposition 2.12 that the Witten genus is zero for our 27-dimensional manifolds, which are $\mathbb{O}P^2$ bundles. The extension of the the 'new Atiyah invariant' $\alpha^{\mathbb{O}}$ would be to a 'new Ochanine genus'

$$\Phi_{\mathrm{och}}^{\mathbb{O}} : \Omega_*^{\langle 8\rangle} \longrightarrow \mathbb{Q}[E_4, E_6][[q]], \tag{3.14}$$

i.e., to the power series ring over rationalized coefficients of level 1 elliptic cohomology, such that the constant term is the Witten genus. We have seen in Theorem 2.4 that the Witten genus of $\mathbb{O}P^2$ is zero, so that in the current context, the constant term is zero. We do not know what the higher terms are, and so they can conceivably be nonzero. The 'new Ochanine genus' is expected to be related to $K3$-cohomology. Such a theory has not yet been explicitly constructed but it should exist.

Define the functor $X \to \Omega_*^{\langle 8\rangle}(X)/\mathcal{I}$, where $\mathcal{I}$ is the ideal introduced in the beginning of this section. The question is whether this is a generalized (co)homology theory. The desired homology theory $E_n^{\mathbb{O}}$ is formed by dividing $\Omega_*^{\langle 8\rangle}$ by $\widetilde{T}_n$ and inverting the primes 2 and 3 [57]. However, there is one extra condition required, which is the invertibility of the element $v = \mathbb{O}P^2$. By taking the limit in

$$E_n^{\mathbb{O}}(X)[\mathbb{O}P^2]^{-1} = \lim_j E_{n+16j}^{\mathbb{O}}(X) \tag{3.15}$$

over the sequence of homomorphisms given by multiplying by $\mathbb{O}P^2$ the resulting theory is

$$e\ell\ell_*^{\mathbb{O}}(X) = E_*^{\mathbb{O}}(X)[\mathbb{O}P^2]^{-1} = \bigoplus_{k\geq 0} \Omega_{*+16k}(X)/\sim, \tag{3.16}$$

[5] I thank Mike Hill for pointing out the $\mathbb{Z}/3$ summand in this homotopy group.

where the equivalence relation $\sim$ is generated by identifying $[Y, f] \in \Omega_*^{\langle 8 \rangle}(X)$ with $[M, f \circ \pi] \in \Omega_{*+16k}^{\langle 8 \rangle}(X)$ for an $\mathbb{O}P^2$ bundle $\pi : M \to Y$, with structure group $\mathrm{Isom}\,\mathbb{O}P^2 = F_4$, i.e., the total space of an $\mathbb{O}P^2$ bundle is identified with its base. A full construction of this theory is not yet achieved by homotopy theorists but it is believed that this should be possible in principle. We mentioned towards the end of Section 3.1 that $KO^*(\mathrm{pt})$ can be made into an Ω_*^{spin}-module and the existence of an isomorphism relating $KO^*(X)$ and $KO^*(\mathrm{pt})$. The octonionic version of Kreck–Stolz theory is arrived at by replacing $KO^*(\mathrm{pt})$ by $e\ell\ell_n^{\mathbb{O}}(\mathrm{pt})$, i.e.,

$$\Omega_*^{\langle 8 \rangle}(X) \otimes_{\Omega_*^{\langle 8 \rangle}} e\ell\ell_*^{\mathbb{O}}(\mathrm{pt}) \longrightarrow \mathrm{e}\ell\ell_*^{\mathbb{O}}(\mathrm{X}) \tag{3.17}$$

is an isomorphism away from the primes 2 and 3 [57].

Remarks.

1. The model for elliptic homology in fact involves indefinitely higher cobordism groups in increments of 16,

$$e\ell\ell_{11}^{\mathbb{O}}(Y^{11}) = \bigoplus_{k \geq 0} \Omega_{11+16k} / \sim, \tag{3.18}$$

   where $\sim$ is an equivalence that provides a correlation between topology in M-theory and topology in dimensions $27, 43, \ldots, 11 + 16k, \ldots, \infty$. We have two points to make:
   - The first bundle with total space an $\mathbb{O}P^2$ bundle over Y^{11} is related to Ramond's Euler multiplet.
   - As the pattern continues in higher and higher dimensions, one is tempted to seek physical interpretations for such theories as well. While this direction is tantalizing, we do not pursue it in this paper.

2. There is another homology theory that one can form, namely by identifying the image of ψ with the trivial bundle as in [28]. The construction is analogous. The advantage here is that we do not kill $\mathbb{O}P^2$, as dividing by T has the effect of killing the fiber.

We have seen connections between 11-dimensional M-theory and the putative theory in 27 dimensions. If the latter theory in 27 dimensions is fundamental, then it should ultimately be studied also without restricting to the relation to M-theory. This is analogous to the case of M-theory itself in relation to 10-dimensional type IIA string theory. Since M-theory is, as far as we know, a fundamental theory, then it should be (and it is being) studied without necessarily assuming a circle bundle for the 11-dimensional manifold. In other words, what about 27-dimensional manifolds that are not the total space of $\mathbb{O}P^2$ bundles over 11-manifolds? Hence

3.2.1. Proposal To study the bosonic theory as a fundamental theory in 27 dimensions we should also consider modding out by the equivalence relation (the ideal).
For example, extension problems can be studied in this way.

3.3. Families

It is desirable to consider the $\mathbb{O}P^2$ bundle as a family problem of objects on the fiber of M^{27} parametrized by points in the base Y^{11}. The family of these 16-dimensional *String* manifolds will define an element of the cobordism group

$$MO\langle 8\rangle^{-16}(Y^{11}). \tag{3.19}$$

Remarks.

1. We have seen in Section 2.4.1 that the total space of an $\mathbb{O}P^2$ bundle is not necessarily *String* even if Y^{11} is *String*. However, we do get a family of *String* manifolds provided we kill the degree four class pulled back from BF_4 (see Proposition 2.11).
2. Unfortunately, genera are multiplicative on fiber bundles so that the vanishing of $\Phi_{\rm W}(\mathbb{O}P^2)$ will force the Witten genus of M^{27} to be zero as well. Also taking higher and higher bundles – so as to get fibers of dimensions higher than 16– as in (3.18) will not help in making the Witten genus nonzero. tmf is the home of the parametrized version of the Witten genus, but we do not see modular forms in this picture. This is to be contrasted with the $\mathbb{H}P^2$ case where the Witten genus is $E_4/288$.
3. Nevertheless, the elliptic genus $\Phi_{\rm ell}$ of $\mathbb{O}P^2$ is not zero, so the total space will not automatically have a zero elliptic genus. However, elliptic genera are defined for Spin manifolds of dimension divisible by 4. Our base space Y^{11} is 11-dimensional and so will automatically have zero elliptic genus. This also applies for the Witten genus. One way out of this is instead to consider the bounding 12-dimensional theory, i.e., the extension of the topological terms from $Y^{11} = \partial Z^{12}$ to Z^{12} as in [61]. If we also take a 28-dimensional coboundary for M^{27}, i.e., $\partial W^{28} = M^{27}$,

we would then have

$$
(3.20) \qquad
\begin{array}{ccccc}
\mathbb{O}P^2 \longrightarrow & M^{27} & \hookrightarrow & W^{28} & \\
& \downarrow \pi & & \downarrow \pi & \searrow \\
& Y^{11} & \hookrightarrow & Z^{12} & \xrightarrow[f]{} BF_4.
\end{array}
$$

Such an extension would involve cobordism obstructions. The manifolds extend nicely, as $\Omega_{11}^{\langle n \rangle} = 0$ for both $n = 4$ (Spin) and $n = 8$ (String). The bundles also extend as shown in Proposition 3.2. It is tempting to propose that the theories should be defined on the $(12 + 16m)$-dimensional spaces, and then restriction to the boundaries would be a special instance.

We have provided evidence for some relations between M-theory and an octonionic version of Kreck–Stolz elliptic homology. Strictly speaking, both theories are conjectural, and we hope that this contribution motivates more active research both on completing the mathematical construction of this elliptic homology theory (part of which is outlined in [57]) as well as making more use of the connection to M-theory. In doing so, we even hope that M-theory itself would in turn give more insights into the homotopy theory.

In closing we hope that further investigation will help shed more light on the mysterious appearance of the exceptional groups E_8 and F_4 and to give a better understanding of their role in M-theory.

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