

Hecke correspondences and Feynman graphs

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We consider natural representations of the Connes–Kreimer Lie algebras on rooted trees/Feynman graphs arising from Hecke correspondences in the categories $\mathcal{LRF}, \mathcal{C}_{\phi^3}$ constructed by K. Kreimnizer and the author. We thus obtain the insertion/elimination representations constructed by Connes–Kreimer as well as an isomorphic pair that we term top-insertion/top-elimination. We also construct graded finite-dimensional sub/quotient representations of these arising from “truncated” correspondences.

1. Introduction

The Connes–Kreimer Hopf algebras on rooted trees and Feynman graphs $\mathcal{H}_{\mathbb{T}}, \mathcal{H}_{\text{FG}}$, introduced in [2, 5], describe the algebraic structure of the BPHZ algorithm in the renormalization of perturbative quantum field theories. If we let \mathbb{T} denote the set of (non-planar) rooted trees, and $\mathbb{Q}\{\mathbb{T}\}$ the \mathbb{Q} -vector space spanned by these, then as an algebra, $\mathcal{H}_{\mathbb{T}} = \text{Sym}(\mathbb{Q}\{\mathbb{T}\})$, and the coalgebra structure is given by the coproduct

$$\Delta(T) = \sum_{C \text{ admissible cut}} P_C(T) \otimes R_C(T),$$

where $P_C(T)$ is the forest of branches resulting from the cut C , and $R_C(T)$ is the root component remaining “above” the cut (see [2] for a more detailed definition).

\mathcal{H}_{FG} is defined analogously, with Feynman graphs in place of rooted trees. More precisely, given a perturbative QFT, and denoting by $\mathbb{Q}\{\Gamma\}$ the vector space spanned by the one-particle irreducible graphs of the theory (1PI graphs), $\mathcal{H}_{\text{FG}} = \text{Sym}(\mathbb{Q}\{\Gamma\})$ as an algebra. Its coalgebra structure is given by

$$\Delta(\Gamma) = \sum_{\gamma \in \Gamma} \gamma \otimes \Gamma/\gamma,$$

where the sum is over all (not necessarily connected) subgraphs of Γ , and Γ/γ denotes the graph obtained from Γ by shrinking each connected component of γ to a point.

$\mathcal{H}_{\mathbb{T}}$ and \mathcal{H}_{FG} are graded connected commutative Hopf algebras, and so by the Milnor–Moore theorem, their duals $\mathcal{H}_{\mathbb{T}}^*$ and $\mathcal{H}_{\text{FG}}^*$ are isomorphic to the universal enveloping algebras $\mathcal{U}(\mathfrak{n}_{\mathbb{T}})$, $\mathcal{U}(\mathfrak{n}_{\text{FG}})$ of the Lie algebras $\mathfrak{n}_{\mathbb{T}}$, \mathfrak{n}_{FG} of their primitive elements. We refer to $\mathfrak{n}_{\mathbb{T}}$ and \mathfrak{n}_{FG} as the Connes–Kreimer Lie algebras on rooted trees and Feynman graphs, respectively.

In [6], a categorification of the Hopf algebras $\mathcal{U}(\mathfrak{n}_{\mathbb{T}})$, $\mathcal{U}(\mathfrak{n}_{\text{FG}})$ was obtained, by showing that they arise naturally as the Ringel–Hall algebras of certain categories \mathcal{LRF} , \mathcal{C}_{ϕ^3} of labeled rooted forests and Feynman graphs, respectively. We briefly recall this notion. Let \mathcal{C} be one of these categories. Its Ringel–Hall algebra $\mathbb{H}_{\mathcal{C}}$ is the vector space of \mathbb{Q} -valued functions on $\text{Iso}(\mathcal{C})$ (the set of isomorphism classes of \mathcal{C}), with finite support. The algebra structure is given by the associative product

$$f \times g(M) := \sum_{N \subset M} f(N)g(M/N), \quad f, g \in \mathbb{H}_{\mathcal{C}}, \quad M \in \text{Iso}(\mathcal{C}),$$

and the coalgebra structure by the co-commutative coproduct

$$\begin{aligned} \Delta : \mathbb{H}_{\mathcal{C}} &\rightarrow \mathbb{H}_{\mathcal{C}} \otimes \mathbb{H}_{\mathcal{C}} \\ \Delta(f)(M, N) &:= f(M \oplus N), \end{aligned}$$

where \oplus denotes disjoint union of forests/graphs. $\mathbb{H}_{\mathcal{C}}$ is a graded connected co-commutative Hopf algebra, and so the enveloping algebra of the Lie algebra $\mathfrak{n}_{\mathcal{C}}$ of its primitive elements. In this light, the main result of [6] is the construction of categories \mathcal{LRF} , \mathcal{C}_{ϕ^3} such that $\mathfrak{n}_{\mathcal{LRF}} \simeq \mathfrak{n}_{\mathbb{T}}$, and $\mathfrak{n}_{\mathcal{C}_{\phi^3}} \simeq \mathfrak{n}_{\text{FG}}$.

Ringel–Hall algebras provide a very useful perspective in the study of the representations of $\mathfrak{n}_{\mathcal{C}}$. In particular, we may construct representations of $\mathfrak{n}_{\mathcal{C}}$ via *Hecke correspondences*. Let

$$\text{Hecke}_{\mathcal{C}} := \{(A, B) \mid B \in \text{Iso}(\mathcal{C}), A \subset B\}$$

(i.e., the set of pairs consisting of an isomorphism class B in \mathcal{C} , and a sub-object A of B). $\text{Hecke}_{\mathcal{C}}$ comes with three maps to $\text{Iso}(\mathcal{C})$: π_1, π_2 and π_q , where $\pi_1(A, B) = A$, $\pi_2(A, B) = B$ and $\pi_q(A, B) = B/A$. As $\mathbb{H}_{\mathcal{C}}$ is the space of functions on $\text{Iso}(\mathcal{C})$, we obtain the diagram

$$\begin{array}{ccc} & \mathcal{F}(\text{Hecke}_{\mathcal{C}}) & \\ & \nearrow \pi_q^* & \nwarrow \pi_2^* \\ \mathbb{H}_{\mathcal{C}} & \uparrow \pi_1^* & \mathbb{H}_{\mathcal{C}} \end{array}$$

where $\mathcal{F}(\mathcal{H}ecke_{\mathcal{C}})$ denotes the space of functions on $\mathcal{H}ecke_{\mathcal{C}}$. We also have maps $\pi_{1*}, \pi_{2*}, \pi_{q*}$ corresponding to integration along the fiber:

$$\pi_{r*}(f)(y) = \sum_{x_y \in \pi_r^{-1}(y)} f(x_y), \quad r = 1, 2, q,$$

which are well defined on finitely supported functions. By taking r, s, t to be some permutation of $1, 2, q$, we now obtain convolutions:

$$\begin{aligned} \pi_{r,s,t} : \mathbb{H}_{\mathcal{C}} \otimes \mathbb{H}_{\mathcal{C}} &\rightarrow \mathbb{H}_{\mathcal{C}} \\ f \otimes g &\rightarrow \pi_{r*}(\pi_s^*(f)\pi_t^*(g)) \end{aligned}$$

$\pi_{r,s,t}$ can be viewed as a map

$$\mathbb{H}_{\mathcal{C}} \rightarrow \text{Hom}(\mathbb{H}_{\mathcal{C}}, \mathbb{H}_{\mathcal{C}}),$$

and provided that $\pi_{r,s,t}$ is compatible with the algebra structure on $\mathbb{H}_{\mathcal{C}}$, this yields a representation of $\mathfrak{n}_{\mathcal{C}}$ on $\mathbb{H}_{\mathcal{C}}$. Representations constructed in this manner are naturally graded by $K_0(\mathcal{C})$. One checks easily that among them are the left and right actions of $\mathbb{H}_{\mathcal{C}}$ on itself as well as their graded duals.

The utility of this perspective extends far beyond recovering the action of $\mathfrak{n}_{\mathcal{C}}$ on its enveloping algebra. We may consider various subcorrespondences of $\mathcal{H}ecke_{\mathcal{C}}$. For example, we may choose an isomorphism class $M \in \text{Iso}(\mathcal{C})$, and consider

$$\mathcal{H}ecke_{\leq M} := \{(A, B) \mid A \subset B \subset M\}$$

Convolution now yields an action of $\mathbb{H}_{\mathcal{C}}$ on the finite-dimensional space $\mathbb{H}_{\leq M}$ of functions supported on isomorphism classes of sub-objects of M . One may similarly obtain representations in $\mathbb{H}_{\text{quot}(M)}$ -functions supported on isomorphism classes of quotient objects of M .

These techniques are well known in the case when \mathcal{C} is a finitary abelian category, such as the category of modules over a finite-dimensional algebra. In this paper, we apply them in the case when \mathcal{C} is one of the categories $\mathcal{LRF}, \mathcal{C}_{\phi^3}$, to study the representation theory of the Connes–Kreimer Lie algebras $\mathfrak{n}_{\mathbb{T}}, \mathfrak{n}_{\mathbb{FG}}$. These categories are not abelian, but share many properties of finitary abelian categories. We show that the insertion/elimination actions of $\mathfrak{n}_{\mathcal{C}}$ introduced in [3] arise in this way, as do two "new" (but isomorphic to the previous two) representations of $\mathfrak{n}_{\mathcal{C}}$, which we call "top insertion" and "top elimination." While the elimination and top-elimination (resp. insertion and top-insertion) representations are isomorphic, they have non-isomorphic finite-dimensional sub/quotient representations arising from

correspondences of the form $\mathcal{H}ecke_{\leq M}, \mathcal{H}ecke_{\text{quot}(M)}$ as above, and have a combinatorially distinct flavor. One of the advantages of this approach is that it allows us to treat both the cases of rooted trees and Feynman graphs in one language.

This paper is structured as follows. In sections 2, 3 and 4 we recall the construction of categories $\mathcal{LRF}, \mathcal{C}_{\phi^3}$ and their general properties. Section 5 introduces the notion of Ringel–Hall algebra. Section 6 gives explicit combinatorial formulas for the left/right actions of $\mathfrak{n}_{\mathcal{C}}$ on $\mathcal{H}_{\mathcal{C}} \simeq U(\mathfrak{n}_{\mathcal{C}})$ and its graded dual. Sections 7.1 and 7.2 examine finite-dimensional graded representations derived from these. In Section 8 we study Hecke correspondences in $\mathcal{LRF}, \mathcal{C}_{\phi^3}$, and show that we recover the representations from 6. We also show how to use “truncated” correspondences to construct the finite-dimensional representations from 7.1 and 7.2.

2. The category \mathcal{LRF}

In this section, we briefly recall the construction of the category \mathcal{LRF} of labeled rooted forests from [6]. We begin by reviewing some notions related to rooted trees. Let S be an infinite set. For a tree T , denote by $V(T), E(T)$ the vertex and edge sets of T , respectively.

Definition 2.1. (1) A *rooted tree labeled by S* is a tree T , with a distinguished vertex $r(T) \in V(T)$, and an injection $l : V(T) \hookrightarrow S$. Denote by $RT(S)$ the set of all such.

(2) A *rooted forest labeled by S* is a set F of rooted trees labeled by S , whose labels are disjoint, i.e.,

$$F = \{T_1, T_2, \dots, T_k\}, \quad T_i \in RT(S), \text{im}(l_i) \cap \text{im}(l_j) = \emptyset \quad \text{if } i \neq j.$$

(3) An *admissible cut* of a labeled tree T is a subset $C(T) \subset E(T)$ such that at most one member of $C(T)$ is encountered along any path joining a leaf to the root. Removing the edges in an admissible cut divides T into a labeled rooted forest $P_C(T)$ and a labeled rooted tree $R_C(T)$, where the latter is the component containing the root. The *empty* and *full* cuts $C_{\text{null}}, C_{\text{full}}$, where

$$(P_{C_{\text{null}}}(T), R_{C_{\text{null}}}(T)) = (\emptyset, T) \quad \text{and} \quad (P_{C_{\text{full}}}(T), R_{C_{\text{full}}}(T)) = (T, \emptyset)$$

respectively, are considered admissible.

- (4) An *admissible cut* on a labeled forest $F = \{T_1, \dots, T_k\}$ is a collection of cuts $C = \{C_1, \dots, C_k\}$, with C_i an admissible cut on T_i . Let

$$R_C(F) := \{R_{C_1}(T_1), \dots, R_{C_k}(T_k)\},$$

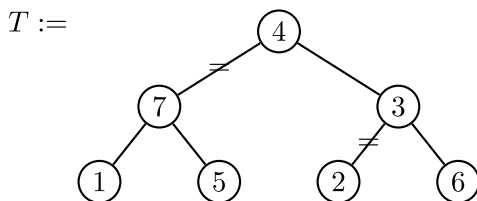
$$P_C(F) := P_{C_1}(T_1) \cup P_{C_2}(T_2) \cup \dots \cup P_{C_k}(T_k).$$

- (5) The *maximum* of two admissible cuts C, D on a rooted tree T , denoted by $\max(C, D)$, is the admissible cut obtained by taking the cut edge closer to the root along any path from leaf to root. Similarly, we define $\min(C, D)$ by taking the cut edge further from the root along any path from leaf to root.
- (6) Two labeled rooted forests $F_1 = \{T_1, \dots, T_k\}$ and $F_2 = \{T'_1, \dots, T'_m\}$ are isomorphic if $k = m$, and there is a permutation $\sigma \in S_k$ and bijections

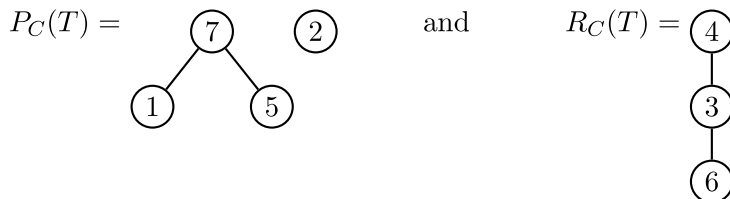
$$f_i : V(T_i) \rightarrow V(T'_{\sigma(i)}), \quad i = 1, \dots, k,$$

which preserve roots and all other incidences.

Example 2.1. Consider the labeled rooted forest consisting of a single tree T , with root drawn at the top.



and the cut edges are indicated with "=", then



We are now ready to define the category $\mathcal{LR}\mathcal{F}$.

Definition 2.2. The category \mathcal{LRF} is defined as follows:

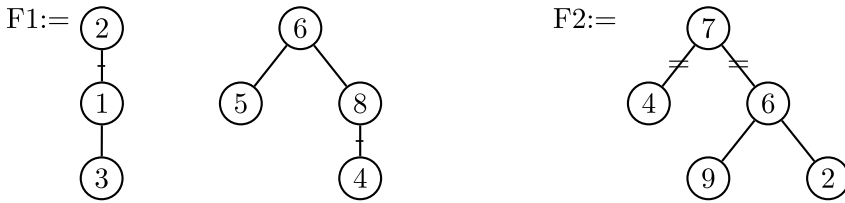
$$\text{Ob}(\mathcal{LRF}) := \{ \text{labeled rooted forests} \} \cup \{\emptyset\}$$

where \emptyset denotes the *empty forest*, which plays the role of zero object.

$$\text{Hom}(F_1, F_2) := \{(C_1, C_2, f) \mid C_i \text{ is an admissible cut of } F_i, \\ f : R_{C_1}(F_1) \cong P_{C_2}(F_2), F_i \in \text{Ob}(\mathcal{LRF})\}.$$

For $F \in \mathcal{C}_{\phi^3}$, $(C_{\text{null}}, C_{\text{full}}, id) : F \rightarrow F$ is the identity morphism in $\text{Hom}(F, F)$.

Example: if



then a morphism is given by the triple (C_1, C_2, f) , where:

- C_1 is indicated by “–”, and C_2 is indicated by “=”.
- $f : R_{C_1}(F_1) \cong P_{C_2}(F_2)$ is defined by $f(2) = 4$, $f(5) = 9$, $f(6) = 6$, $f(8) = 2$.

We recall the definition of the the composition of morphisms

$$\text{Hom}(F_1, F_2) \times \text{Hom}(F_2, F_3) \rightarrow \text{Hom}(F_1, F_3).$$

Suppose that $(C_1, C_2, f) \in \text{Hom}(F_1, F_2)$ and $(D_2, D_3, g) \in \text{Hom}(F_2, F_3)$. The cut $\min(C_2, D_2)$ induces a cut E_1 on F_1 lying above C_1 , and $\max(C_2, D_2)$ a cut E_3 on F_3 lying below D_3 . The restriction of $g \circ f$ gives an isomorphism $R_{E_1}(F_1) \cong P_{E_3}(F_3)$. We define the composition above to be $(E_1, E_3, g \circ f)$. The associativity of composition follows from the associativity of \max and \min .

2.1. The Connes–Kreimer Lie algebra on rooted trees

In this section, we recall the definition of the Connes–Kreimer Lie algebra on rooted trees $\mathfrak{n}_{\mathbb{T}}$ (see [3]). Let \mathbb{T} denote the collection of unlabeled rooted

trees. As a vector space,

$$\mathfrak{n}_{\mathbb{T}} = \mathbb{Q}\{\mathbb{T}\},$$

i.e., the span of unlabeled rooted trees. On $\mathfrak{n}_{\mathbb{T}}$, we have a *pre-Lie* product “*”, given, for $T_1, T_2 \in \mathbb{T}$ by

$$T_1 * T_2 = \sum_{T \in \mathbb{T}} a(T_1, T_2; T)T,$$

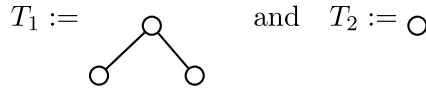
where

$$a(T_1, T_2; T) := |\{e \in E(T) \mid P_{C_e}(T) = T_1, R_{C_e}(T) = T_2\}|$$

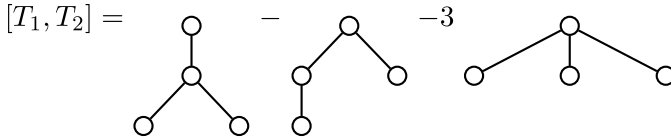
and C_e denotes the cut severing the edge e . The Lie bracket on $\mathfrak{n}_{\mathbb{T}}$ is given by

$$(2.1) \quad [T_1, T_2] := T_1 * T_2 - T_2 * T_1.$$

Thus, for example if



then



3. The category \mathcal{C}_{ϕ^3}

In this section we review the construction of the category of Feynman graphs following [6], with the small modification that here we consider unlabeled graphs. Our treatment of the combinatorics of graphs is taken from [8]. In order to not get bogged down in notation, we focus on the special case of ϕ^3 theory (the case of trivalent graphs with only one edge-type). The results of this section extend to the general case in a completely straightforward manner.

Definition 3.1. A *graph* Γ consists of a set $H = H(\Gamma)$ of half-edges, a set $V = V(\Gamma)$ of vertices, a set of vertex-half edge adjacency relations ($\subset V \times H$), and a set of half edge-half edge adjacency relations ($\subset H \times H$),

with the requirements that each half edge is adjacent to at most one other half edge and to exactly one vertex. Note that graphs may not be connected.

Half edges which are not adjacent to another half edge are called *external edges*, and denoted $\text{Ex}(\Gamma) \subset H = H(\Gamma)$. Pairs of adjacent half edges are called *internal edges*, and denoted $\text{Int}(\Gamma)$.

An *isomorphism* between graphs Γ_1 and Γ_2 is a pair of bijections

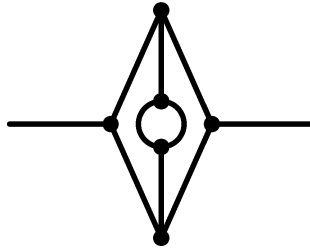
$$\begin{aligned} f_H : H(\Gamma_1) &\rightarrow H(\Gamma_2), \\ f_V : V(\Gamma_1) &\rightarrow V(\Gamma_2), \end{aligned}$$

which preserve all adjacency relations. We write $f : \Gamma_1 \simeq \Gamma_2$ for short. The isomorphisms from a graph Γ to itself form a group, which we denote $\text{Aut}(\Gamma)$.

Definition 3.2. (1) A *Feynman graph* is either the empty graph, or one where each vertex is incident to exactly three half-edges, and each connected component has two or three external edges and at least one loop. We denote the set of Feynman graphs by \mathbb{FG} .

(2) A graph is *1-particle irreducible* (1PI) if it is connected, and remains connected under the removal of an arbitrary internal edge.

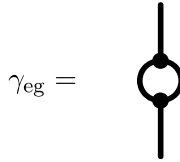
Example: The graph Γ_{eg}



is a three-loop 1PI Feynman graph with two external edges. $V(\Gamma_{\text{eg}})$ are indicated by \bullet .

Definition 3.3. Given a Feynman graph Γ , a *subgraph* γ is a Feynman graph such that $V(\gamma) \subset V(\Gamma)$, $H(\gamma) \subset H(\Gamma)$, and such that if $v \in V(\gamma)$, and $(v, e) \in V(\Gamma) \times H(\Gamma)$, then $e \in H(\gamma)$ (i.e., the subgraph has to contain all half-edges incident to its vertices). This definition implies that $\dim_{\mathbb{Q}}(H_1(\gamma, \mathbb{Q})) > 0$ (i.e., that a subgraph contain at least one loop). We write $\gamma \subset \Gamma$.

Example:



is a subgraph of Γ_{eg} above.

We proceed to define the contraction of subgraphs of Feynman graphs.

Definition 3.4. Let Γ be a Feynman graph, and $\gamma \subset \Gamma$ a connected subgraph. The *quotient graph* Γ/γ is defined as follows. If γ has three external edges, then Γ/γ is the Feynman graph with

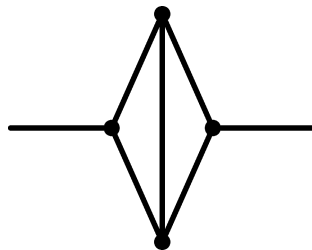
- (1) $V(\Gamma/\gamma)$ set the vertex set of Γ with all vertices of γ removed, and a new trivalent vertex v added.
- (2) $H(\Gamma/\gamma)$ the half edge set of Γ , with all half edges corresponding to internal half edges of γ removed.
- (3) All adjacencies inherited from Γ , and the external half edges of γ joined to v .

If γ has two external edges, then Γ/γ is the Feynman graph with

- (1) $V(\Gamma/\gamma)$ is $V(\Gamma)$ with all the vertices of γ removed.
- (2) $H(\Gamma/\gamma)$ is $H(\Gamma)$ with all half edges of γ removed.
- (3) All adjacencies inherited from Γ , as well as the adjacency of the half-edges incident to the external half-edges of γ .

Finally, if $\gamma \subset \Gamma$ is an arbitrary (not necessarily connected) Feynman subgraph, then Γ/γ is defined to be the Feynman graph obtained by performing successive quotients by each connected component. Note that the order of collapsing does not matter.

Example: With Γ_{eg}, γ_{eg} as above, Γ_{eg}/γ_{eg} is



The set of Feynman subgraphs of a Feynman graph Γ forms a poset under graph inclusion. This poset is a distributive lattice, with $\Gamma_1 \vee \Gamma_2$ the smallest subgraph containing Γ_1 and Γ_2 , and $\Gamma_1 \wedge \Gamma_2$ the largest subgraph contained in $\Gamma_1 \cap \Gamma_2$.

Remark 3.1. If $\gamma \subset \Gamma$ is a subgraph of a Feynman graph, then there is a bijection between subgraphs of Γ/γ , and subgraphs γ' of Γ such that $\gamma \subset \gamma' \subset \Gamma$. This bijection is compatible with shrinking subgraphs, in the sense that if $\gamma' \in \Gamma$ corresponds to $\bar{\gamma}' \in \Gamma/\gamma$, then $\Gamma/\gamma' \simeq (\Gamma/\gamma)/(\bar{\gamma}')$.

3.1. The category \mathcal{C}_{ϕ^3} of Feynman graphs in ϕ^3 -theory

We proceed to define a category \mathcal{C}_{ϕ^3} of Feynman graphs following [6]. Let

$$\begin{aligned} \text{Ob}(\mathcal{C}_{\phi^3}) &:= \{\Gamma \in \text{FG}\} \\ \text{Hom}(\Gamma_1, \Gamma_2) &:= \{(\gamma_1, \gamma_2, f) \mid \gamma_i \text{ is a subgraph of } \Gamma_i, f : \Gamma_1/\gamma_1 \cong \gamma_2\} \end{aligned}$$

For $\Gamma \in \mathcal{C}_{\phi^3}$, (\emptyset, Γ, id) is the identity map in $\text{Hom}(\Gamma, \Gamma)$.

The composition of morphisms in \mathcal{C}_{ϕ^3}

$$\text{Hom}(\Gamma_1, \Gamma_2) \times \text{Hom}(\Gamma_2, \Gamma_3) \rightarrow \text{Hom}(\Gamma_1, \Gamma_3)$$

is defined as follows. Suppose that $(\gamma_1, \gamma_2, f) \in \text{Hom}(\Gamma_1, \Gamma_2)$, and $(\tau_2, \tau_3, g) \in \text{Hom}(\Gamma_2, \Gamma_3)$. By Remark 3.1, the subgraph $\gamma_2 \wedge \tau_2$ corresponds to a subgraph ξ of Γ_1 containing γ_1 , and $\gamma_2 \vee \tau_2$ corresponds to a subgraph $\rho \subset \tau_3 \subset \Gamma_3$. We define the composition $(\tau_2, \tau_3, g) \circ (\gamma_1, \gamma_2, f)$ to be $(\xi, \rho, g \circ f)$. It is straightforward to see that the composition thus defined is associative.

3.2. The Connes–Kreimer Lie algebra on Feynman graphs

In order to define the Connes–Kreimer Lie algebra structure on Feynman graphs, we must first introduce the notion of inserting a graph into another graph. Let

$$\mathbb{Q}\{\text{FG}\}$$

denote the \mathbb{Q} -vector space spanned by Feynman graphs.

Definition 3.5. Let $\Gamma_1, \Gamma_2 \in \mathbb{F}\mathbb{G}$. If Γ_1 has three external edges, $v \in V(\Gamma_2)$, and $f : \text{Ex}(\Gamma_1) \rightarrow H(v)$ is a bijection (where $H(v)$ are the half-edges incident to the vertex v), then let $\Gamma_2 \circ_{v,f} \Gamma_1$ be the Feynman graph such that

- $V(\Gamma_2 \circ_{v,f} \Gamma_1) = V(\Gamma_2) \cup V(\Gamma_1) \setminus v$.
- $H(\Gamma_2 \circ_{v,f} \Gamma_1) = H(\Gamma_1) \cup_f H(\Gamma_2)$ — i.e., the unions of the half-edges of each graph, with the identifications induced by f .
- The adjacencies induced from those of Γ_1 and Γ_2 .

If Γ_1 has two external edges, $\{e_1, e_2\} \in \text{Int}(\Gamma_2) \subset H \times H$, and f is a bijection between $\text{Ex}(\Gamma_1)$ and $\{e_1, e_2\}$ (there are two of these), then $\Gamma_2 \circ_{e,f} \Gamma_1$ is the Feynman graph such that

- $V(\Gamma_2 \circ_{e,f} \Gamma_1) = V(\Gamma_1) \cup V(\Gamma_2)$.
- $H(\Gamma_2 \circ_{e,f} \Gamma_1) = H(\Gamma_1) \cup H(\Gamma_2)$.
- The adjacency induced by f as well as those induced from Γ_1 and Γ_2 .

Let $\mathfrak{n}_{\mathbb{F}\mathbb{G}}$ denote the \mathbb{Q} -vector space spanned by connected Feynman graphs. We equip $\mathfrak{n}_{\mathbb{F}\mathbb{G}}$ with the pre-Lie product “ \star ”, defined by

$$\Gamma_1 \star \Gamma_2 := \sum_{v \in V(\Gamma_2), f: \text{Ex}(\Gamma_1) \rightarrow H(v)} \Gamma_2 \circ_{v,f} \Gamma_1$$

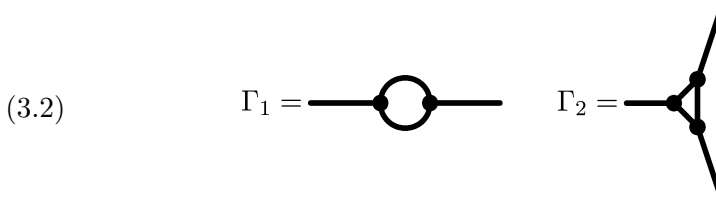
if Γ_1 has three external edges, and

$$\Gamma_1 \star \Gamma_2 := \sum_{e \in \text{Int}(\Gamma_2), f: \text{Ex}(\Gamma_1) \rightarrow \{e_1, e_2\}} \Gamma_2 \circ_{e,f} \Gamma_1$$

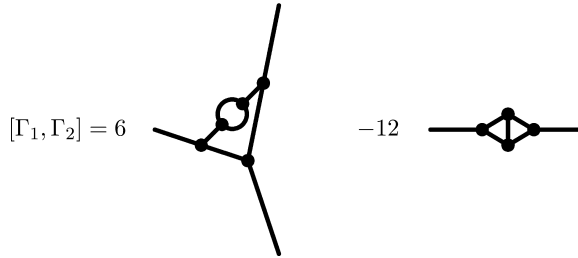
if Γ_1 has two external edges, and extended linearly. Finally, we can define the Lie bracket on $\mathfrak{n}_{\mathbb{F}\mathbb{G}}$ by

$$(3.1) \quad [\Gamma_1, \Gamma_2] := \Gamma_1 \star \Gamma_2 - \Gamma_2 \star \Gamma_1.$$

Example: Suppose



then



4. Properties of the categories $\mathcal{LRF}, \mathcal{C}_{\phi^3}$

In this section, we summarize some of the properties of $\mathcal{LRF}, \mathcal{C}_{\phi^3}$ that will allow us to define the Ringel–Hall algebras of these categories in the next section. For more details, please consult [6].

Let \mathcal{C} denote either of the categories $\mathcal{LRF}, \mathcal{C}_{\phi^3}$. Then \mathcal{C} has the following properties:

- (1) The empty forest/graph $\{\emptyset\}$ is a null object.
- (2) Disjoint union of forests/graphs equips \mathcal{C} with a symmetric monoidal structure, which we denote by \oplus . We will refer to objects of the form $A \oplus B$, with neither A nor B the null object \emptyset as *decomposable*, and those not admitting such a splitting as *indecomposable*.
- (3) Every morphism in \mathcal{C} possesses a kernel. If $\mathcal{C} = \mathcal{LRF}$, and

$$\Psi = (C_1, C_2, f) : F_1 \rightarrow F_2,$$

then

$$\text{Ker}(\Psi) = (C_{\text{null}}, C_1, id) : P_{C_1}(F_1) \rightarrow F_1.$$

If $\mathcal{C} = \mathcal{C}_{\phi^3}$, and

$$\Phi = (\gamma_1, \gamma_2, f) : \Gamma_1 \rightarrow \Gamma_2,$$

then

$$\text{Ker}(\Phi) = (\emptyset, \gamma_1, id) : \gamma_1 \rightarrow \Gamma_1.$$

- (4) Every morphism in \mathcal{C} possesses a cokernel. If $\mathcal{C} = \mathcal{LRF}$, and Ψ is the above morphism, then

$$\text{Coker}(\Psi) = (C_2, C_{\text{full}}, id) : F_2 \rightarrow R_{C_2}(F_2).$$

If $\mathcal{C} = \mathcal{C}_{\phi^3}$, and Φ is as above, then

$$\text{Coker}(\Phi) = (\gamma_2, \Gamma_2/\gamma_2, id) : \Gamma_2 \rightarrow \Gamma_2/\gamma_2.$$

- (5) By the previous two remarks, every morphism in \mathcal{C} has a mono-epi factorization and, moreover, the notion of exactness makes sense.
- (6) For a pair of objects $A, B \in \mathcal{C}$ such that $A \subset B$, there is an inclusion-preserving bijection between subobjects of B containing A and subobjects of B/A such that if $\bar{S} \subset B/A$ corresponds to $S \subset B$, we have $(B/A)/\bar{S} \cong B/S$.
- (7) The Grothendieck group $K_0(\mathcal{C})$ may be defined as usual (as the quotient of the free abelian group of objects by the subgroup of relations coming from short exact sequences). We have $K_0(\mathcal{LRF}) = \mathbb{Z}$, while $K_0(\mathcal{C}_{\phi^3})$ is spanned over \mathbb{Z} by primitive graphs.

Definition 4.1. Let \mathcal{C} denote one of the categories $\mathcal{LRF}, \mathcal{C}_{\phi^3}$ and $M \in \mathcal{C}$. We define the *length* of M to be length of the longest ascending chain of subobjects of M . It is easy to see that for \mathcal{LRF} , this coincides with the number of vertices in the forest.

5. Ringel–Hall algebras

In this section, we define the Ringel–Hall algebra of the categories $\mathcal{LRF}, \mathcal{C}_{\phi^3}$. For an introduction to Ringel–Hall algebras of abelian categories, please see [1, 7]. Let \mathcal{C} denote one of the categories $\mathcal{LRF}, \mathcal{C}_{\phi^3}$ and denote by $\text{Iso}(\mathcal{C})$ the set of isomorphism classes of objects in \mathcal{C} .

Definition 5.1. The Ringel–Hall algebra of \mathcal{C} , denoted $H_{\mathcal{C}}$, is the vector space of finitely supported \mathbb{Q} -valued functions on $\text{Iso}(\mathcal{C})$. I.e.,

$$H_{\mathcal{C}} := \{f : \text{Iso}(\mathcal{C}) \rightarrow \mathbb{Q} \mid |\text{supp}(f)| < \infty\}.$$

$H_{\mathcal{C}}$ is equipped with an associative convolution product given by

$$f \times g(M) := \sum_{N \subset M} f(N)g(M/N) \quad f, g \in H_{\mathcal{C}}, \quad M \in \text{Iso}(\mathcal{C}),$$

and a co-commutative coproduct

$$\begin{aligned} \Delta : H_{\mathcal{C}} &\rightarrow H_{\mathcal{C}} \otimes H_{\mathcal{C}}, \\ \Delta(f)(M, N) &:= f(M \oplus N). \end{aligned}$$

The two structures are easily seen to be compatible, and $H_{\mathcal{C}}$ is a co-commutative Hopf algebra graded by $K_0(\mathcal{C})$ (via the natural map $\text{Iso}(\mathcal{C}) \rightarrow K_0(\mathcal{C})$), and connected. By the Milnor–Moore theorem, it is the enveloping algebra of the Lie algebra $\mathfrak{n}_{\mathcal{C}}$ of its primitive elements, called the *Ringel–Hall Lie algebra* of \mathcal{C} . The following theorem is proved in [6].

Theorem 5.1. *The Ringel–Hall Lie algebras of the categories $\mathcal{LR}\mathcal{F}, \mathcal{C}_{\phi^3}$ are isomorphic to the Connes–Kreimer Lie algebras on rooted trees and Feynman graphs respectively. i.e.,*

$$H_{\mathcal{LR}\mathcal{F}} = U(\mathfrak{n}_{\mathbb{T}}) \quad \text{and} \quad H_{\mathcal{C}_{\phi^3}} = U(\mathfrak{n}_{\mathbb{FG}}).$$

The Ringel–Hall algebra $H_{\mathcal{C}}$ has a natural basis $\{\delta_A | A \in \text{Iso}(\mathcal{C})\}$ consisting of delta functions supported on isomorphism classes of \mathcal{C} . More precisely, for $B \in \text{Iso}(\mathcal{C})$,

$$\delta_A(B) := \begin{cases} 1 & \text{if } B = A, \\ 0 & \text{otherwise.} \end{cases}$$

We may equip $H_{\mathcal{C}}$ with a symmetric bilinear form κ :

$$(5.1) \quad \kappa(f, g) := \sum_{A \in \text{Iso}(\mathcal{C})} f(A)g(A)$$

We will use κ to identify $H_{\mathcal{C}}$ with its graded dual below. The dual of δ_A with respect to κ is denoted ϕ_A .

6. Four natural representations of \mathfrak{n}_C

Let \mathfrak{g} be a complex Lie algebra, and let \mathfrak{g}^{op} denote the opposite Lie algebra, with bracket $[X, Y]_{\text{op}} := [Y, X] = -[X, Y]$. The map

$$\begin{aligned} {}^t : \mathfrak{g} &\rightarrow \mathfrak{g}^{\text{op}} \\ X &\rightarrow X^t := -X \end{aligned}$$

is a Lie algebra isomorphism of \mathfrak{g} with \mathfrak{g}^{op} . The map t extends to an isomorphism

$$(6.1) \quad {}^t : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}^{\text{op}}) \simeq U(\mathfrak{g})^{\text{op}}.$$

Recall that the structure of a \mathfrak{g} -module is equivalent to the structure of a left module over the enveloping algebra $U(\mathfrak{g})$. The isomorphism 6.1 implies that a right $U(\mathfrak{g})$ -module structure on M is equivalent to a left $U(\mathfrak{g})$ (and hence \mathfrak{g})-module structure on M via the twisted action $X \cdot m := m \cdot X^t$. To rephrase, the map ${}^t : v \rightarrow v^t$ yields an isomorphism of \mathfrak{g} -modules between $U(\mathfrak{g})$ and $U(\mathfrak{g})^{\text{op}}$, after identifying \mathfrak{g} with \mathfrak{g}^{op} using t .

For the rest of the paper, \mathcal{C} will denote one of $\mathcal{LRF}, \mathcal{C}_{\phi^3}$. As proven in [6], we have $H_C \simeq U(\mathfrak{n}_C)$. We may now consider the left and right actions of H_C on itself. We will write X_A for the element δ_A viewed as an element of $U(\mathfrak{n}_C)$. We have:

Left action:

$$(6.2) \quad X_A \cdot \delta_B := \delta_A \times \delta_B = \sum_{\substack{C \in \text{Iso}(\mathcal{C}), E \subset C \\ E \simeq A, C/E \simeq B}} \delta_C.$$

This is the action of \mathfrak{n}_C by *insertions* considered in [3]. We denote by H_{ins} the vector space H_C with the \mathfrak{n}_C -module structure given by (6.2).

Remark 6.1. Let $H_{\text{ins}}^{\text{dec}}$ denote the subspace of H_{ins} consisting of functions supported on decomposable objects, i.e.,

$$H_{\text{ins}}^{\text{dec}} = \text{span}\{\delta_A \mid A \in \text{Iso}_C, A = B \oplus C, B \neq \emptyset, C \neq \emptyset\}.$$

This is a left ideal of H_C (i.e., a subrepresentation of H_{ins}), and the quotient $H_{\text{ins}} / H_{\text{ins}}^{\text{dec}}$ is spanned by delta functions supported on indecomposable objects. This is the insertion representation considered in [3].

Right action:

$$(6.3) \quad \delta_B \cdot X_A := \delta_B \times \delta_A = \sum_{\substack{\{C \in \text{Iso}(\mathcal{C}), E \subset C\} \\ E \simeq B, C/E \simeq A}} \delta_C$$

We call this the *top insertion* action. It is closely related to the operator B_+ considered in [3]. We denote by $H_{\text{top-ins}}$ the vector space $H_{\mathcal{C}}$ with the $\mathfrak{n}_{\mathcal{C}}$ -module structure given by (6.3).

Example 6.1. If $\mathcal{C} = \mathcal{LR}\mathcal{F}$, we have:

$$\delta_{\bullet} \cdot X_{\bullet} = \delta_{\bullet} + \delta_{\bullet} + 3\delta_{\bullet}$$

By the observations at the beginning of this section,

$$H_{\text{ins}} \simeq H_{\text{top-ins}}$$

as $\mathfrak{n}_{\mathcal{C}}$ modules. Note however, that at a combinatorial level, the isomorphism t is non-trivial. For instance,

Example 6.2.

$$\delta_{\bullet\bullet}^t = \delta_{\bullet\bullet} + \delta_{\bullet}$$

We may also consider the dual representations of (6.2) and (6.3). Let $H_{\mathcal{C}}^*$ denote the restricted dual of $H_{\mathcal{C}}$ with basis $\{\phi_A | A \in \text{Iso}(\mathcal{C})\}$ dual to $\{\delta_A | A \in \text{Iso}(\mathcal{C})\}$ (equivalently, ϕ_A is the image of δ_A under κ).

- The representation dual to 6.2 is determined by

$$\begin{aligned} X_A \cdot \phi_B(\delta_C) &= -\phi_B(X_A \cdot \delta_C) \\ &= -\phi_B \left(\sum_{\substack{\{D \in \text{Iso}(\mathcal{C}), F \subset D\} \\ F \simeq A, D/F \simeq C}} \delta_D \right) \\ &= -|\{E \subset B | E \simeq A, B/E \simeq C\}|, \end{aligned}$$

which implies that

$$(6.4) \quad X_A \cdot \phi_B = - \sum_{\{E \subset B | E \simeq A\}} \phi_{B/E}.$$

This is the *elimination* action considered in [3] (up to a sign corresponding to an isomorphism between $\mathfrak{n}_{\mathcal{C}}$ and $\mathfrak{n}_{\mathcal{C}}^{\text{op}}$). We denote by $\mathbf{H}_{\text{elim}}^*$ the vector space $\mathbf{H}_{\mathcal{C}}^*$ equipped with the action (6.4).

- The representation dual to (6.3) is determined by

$$\begin{aligned} (\phi_B \cdot X_A)(\delta_C) &= -\phi_B(\delta_C \cdot X_A) \\ &= -\phi_B \left(\sum_{\substack{\{D \in \text{Iso}(\mathcal{C}), F \subset D\} \\ F \simeq C, D/F \simeq A}} \delta_D \right) \\ &= -|\{F \subset B \mid F \simeq C, B/F \simeq A\}|, \end{aligned}$$

which implies that

$$(6.5) \quad \phi_B \cdot X_A = - \sum_{\substack{\{E \subset B\} \\ B/E \simeq A}} \phi_E.$$

We call this the *top elimination* action. We denote by $\mathbf{H}_{\text{top-elim}}^*$ the vector space $\mathbf{H}_{\mathcal{C}}^*$ equipped with the action (6.5).

Example 6.3. If $\mathcal{C} = \mathcal{LR}\mathcal{F}$, we have,

$$\phi \cdot X = 2\phi + \phi$$

Again, we have

$$\mathbf{H}_{\text{elim}}^* \simeq \mathbf{H}_{\text{top-elim}}^* .$$

Remark 6.2. It is a feature of Ringel–Hall algebras that $\mathbf{H}_{\mathcal{C}}$ carries a natural $K_0(\mathcal{C})$ -grading. The representations considered in this section are therefore $K_0(\mathcal{C})$ -graded representations of $\mathfrak{n}_{\mathcal{C}}$. For $B \in \text{Iso}(\mathcal{C})$, the gradation is given by assigning degree $[B] \in K_0(\mathcal{C})$ to δ_B, ϕ_B . Given $A \in \text{Iso}_{\mathcal{C}}$, the corresponding element $X_A \in \mathfrak{n}_{\mathcal{C}}$ has degree $[A]$ in the representations $\mathbf{H}_{\text{ins}}, \mathbf{H}_{\text{top-ins}}$, and degree $-[A]$ in $\mathbf{H}_{\text{elim}}^*, \mathbf{H}_{\text{top-elim}}^*$.

7. Finite-dimensional representations

In this section, we consider finite-dimensional sub/quotients of the four representations $\mathbf{H}_{\text{ins}}, \mathbf{H}_{\text{top-ins}}, \mathbf{H}_{\text{elim}}^*$ and $\mathbf{H}_{\text{top-elim}}^*$ considered in the previous section.

7.1. Finite-dimensional representations from $\mathbf{H}_{\text{elim}}^*, \mathbf{H}_{\text{ins}}$

Let us fix an isomorphism class $M \in \text{Iso}(\mathcal{C})$, and let

$$\text{Quot}(M) := \{E \in \text{Iso}(\mathcal{C}) \mid E \text{ is a quotient of } M\}$$

(i.e., isomorphism classes of quotients of M). Let $\mathbf{H}_{\text{quot}(M)}$ denote the subspace of $\mathbf{H}_{\text{elim}}^*$ spanned by ϕ_A , $A \in \text{Quot}(M)$. This is clearly a finite-dimensional subspace of $\mathbf{H}_{\text{elim}}^*$. We have the following:

Theorem 7.1.

- (a) For every $M \in \text{Iso}(\mathcal{C})$, the subspace $\mathbf{H}_{\text{quot}(M)}$ is a finite-dimensional representation of $\mathfrak{n}_{\mathcal{C}}$ with respect to the action (6.4).
- (b) For every $M \in \text{Iso}(\mathcal{C})$, $\mathbf{H}_{\text{quot}(M)}$ is indecomposable.
- (c) If N is a quotient of M , then $\mathbf{H}_{\text{quot}(N)}$ is an $\mathfrak{n}_{\mathcal{C}}$ submodule of $\mathbf{H}_{\text{quot}(M)}$.

Proof. (a) This follows from the fact that a quotient of a quotient of M is another quotient of M .

- (b) Suppose that $v \in \mathbf{H}_{\text{quot}(M)}$ is a non-zero vector. We claim that there exists an $X \in U(\mathfrak{n}_{\mathcal{C}}) \simeq \mathbf{H}_{\mathcal{C}}$ such that $X \cdot v = \delta_{\emptyset}$. We can write $v = \sum_{A \in \text{Iso}(\mathcal{C})} c_A \phi_A$, $c_A \in \mathbb{Q}$. Let Z be an isomorphism class of maximal length occurring in the sum with $c_Z \neq 0$. We have $X_Z \cdot v = -c_Z \phi_{\emptyset}$. This implies that any subrepresentation of $\mathbf{H}_{\text{quot}(M)}$ contains ϕ_{\emptyset} , and so $\mathbf{H}_{\text{quot}(M)}$ is indecomposable.
- (c) If N is a quotient of M , then any quotient of N is automatically a quotient of M , which implies that $\text{Quot}(N) \subset \text{Quot}(M)$. We thus obtain an inclusion of $\mathfrak{n}_{\mathcal{C}}$ -modules $\mathbf{H}_{\text{quot}(N)} \subset \mathbf{H}_{\text{quot}(M)}$.

□

We can give an explicit description of the dual representation $\mathbf{H}_{\text{quot}(M)}^*$. Since $\mathbf{H}_{\text{quot}(M)} \subset \mathbf{H}_{\text{elim}}^*$, we can identify $\mathbf{H}_{\text{quot}(M)}^*$ with a quotient of $\mathbf{H}_{\mathcal{C}}$ by

the subspace spanned by $\{\delta_E | E \notin \text{Quot}(M)\}$. If $B \in \text{Quot}(M)$, then

$$X_A \cdot \delta_B := \delta_A \times \delta_B = \sum_{\substack{C \in \text{Quot}(M), E \subset C \\ E \simeq A, C/E \simeq B}} \delta_C.$$

i.e., we insert A into B , but sum only over those isomorphism classes which are quotients of M .

7.2. Finite-dimensional representations from $\mathbf{H}_{\text{top-elim}}^*$, $\mathbf{H}_{\text{top-ins}}$

Let us fix an isomorphism class $M \in \text{Iso}(\mathcal{C})$, and let

$$\leq M := \{E \in \text{Iso}\mathcal{C} | E \text{ is a subobject of } M\}$$

(i.e., isomorphism classes of subobjects of M .) Let $\mathbf{H}_{\leq M}$ denote the subspace of $\mathbf{H}_{\text{top-elim}}^*$ spanned by ϕ_A , $A \in \leq M$. This is a finite-dimensional subspace of $\mathbf{H}_{\text{top-elim}}^*$. We have the following analogue of Theorem 7.1

Theorem 7.2.

- (a) For every $M \in \text{Iso}(\mathcal{C})$, the subspace $\mathbf{H}_{\leq M}$ is a finite-dimensional representation of $\mathfrak{n}_{\mathcal{C}}$ with respect to the action (6.5).
- (b) For every $M \in \text{Iso}(\mathcal{C})$, $\mathbf{H}_{\leq M}$ is indecomposable.
- (c) If $N \subset M$, $\mathbf{H}_{\leq N}$ is an $\mathfrak{n}_{\mathcal{C}}$ -submodule of $\mathbf{H}_{\leq M}$.

Proof. (a) This follows from the fact that a subobject of a subobject of M is another subobject of M .

(b) Suppose that $v \in \mathbf{H}_{\leq M}$ is a non-zero vector. We claim that there exists an $X \in U(\mathfrak{n}_{\mathcal{C}}) \simeq \mathbf{H}_{\mathcal{C}}$ such that $v \cdot X = \phi_{\emptyset}$. We can write $v = \sum_{A \in \text{Iso}(\mathcal{C})} c_A \phi_A$. Let Z be an isomorphism class of maximal length occurring in the sum with $c_Z \neq 0$. We have $v \cdot X_Z = -c_Z \phi_{\emptyset}$. This implies that any subrepresentation of $\mathbf{H}_{\leq M}$ contains ϕ_{\emptyset} , and so $\mathbf{H}_{\leq M}$ is indecomposable.

(c) If $N \subset M$, then $\leq N \subset \leq M$, and so we obtain an inclusion of $\mathfrak{n}_{\mathcal{C}}$ -modules $\mathbf{H}_{\leq N} \subset \mathbf{H}_{\leq M}$.

□

Since $\mathbf{H}_{\leq M} \subset \mathbf{H}_{\text{top-elim}}^*$, we can identify $\mathbf{H}_{\leq M}^*$ with a quotient of $\mathbf{H}_{\mathcal{C}}$ by the subspace spanned by $\{\delta_E | E \notin \leq M\}$. The action of $\mathfrak{n}_{\mathcal{C}}$ on $\mathbf{H}_{\leq M}^*$ is explicitly

given by

$$\delta_B \cdot X_A := \delta_B \times \delta_A = \sum_{\substack{\{C \in \leq M, E \subset C\} \\ E \simeq B, C / E \simeq A}} \delta_C.$$

i.e., we top-insert A onto B , but take only those isomorphism classes which are subobjects of M .

Remark 7.1. The finite-dimensional representations of $\mathfrak{n}_{\mathbb{T}}$ were classified in [4], and it would be interesting to determine how the representations above fit into the classification.

Remark 7.2. Although $H_{\text{elim}}^* \simeq H_{\text{top-elim}}^*$, the subrepresentations $H_{\text{quot}(M)}$ and $H_{\leq N}$ do not correspond under this isomorphism. i.e., for a given $M \in \text{Iso}(\mathcal{C})$, there is in general no $N \in \text{Iso}(\mathcal{C})$ such that $H_{\text{quot}(M)} \simeq H_{\leq N}$ as $\mathfrak{n}_{\mathcal{C}}$ -modules.

8. Hecke correspondences and convolutions

In this section, we show that the representations considered above arise through the actions of Hecke correspondences in the categories $\mathcal{LRF}, \mathcal{C}_{\phi^3}$. Let \mathcal{C} denote either of these categories, and let

$$\mathcal{H}ecke_{\mathcal{C}} := \{(A, B) \mid B \in \text{Iso}(\mathcal{C}), A \subset B\}$$

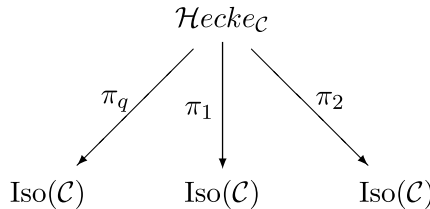
(i.e., the set of pairs consisting of an isomorphism class B in \mathcal{C} , and a subobject A of B). There are natural maps

$$\pi_i : \mathcal{H}ecke_{\mathcal{C}} \rightarrow \text{Iso}(\mathcal{C}), \quad i = 1, 2,$$

where $\pi_1(A, B) = A$ and $\pi_2(A, B) = B$. We also have a natural map

$$\pi_q : \mathcal{H}ecke_{\mathcal{C}} \rightarrow \text{Iso}(\mathcal{C})$$

with $\pi_q(A, B) = B/A$. This can be summarized in the following diagram:



Let $H_{\mathcal{C}}$ denote the Hall algebra of \mathcal{C} , i.e., the set of finitely supported \mathbb{Q} -valued functions on isomorphism classes of \mathcal{C} with its associative convolution product, and $\mathcal{F}(\mathcal{H}ecke_{\mathcal{C}})$ denote the ring of \mathbb{Q} -valued functions on $\mathcal{H}ecke_{\mathcal{C}}$, with the usual commutative multiplication.

Taking the spaces of \mathbb{Q} -valued functions on the sets appearing in the above diagram, we obtain the following dual diagram:

$$\begin{array}{ccc}
 & \mathcal{F}(\mathcal{H}ecke_{\mathcal{C}}) & \\
 \nearrow^{\pi_q^*} & \uparrow^{\pi_1^*} & \nwarrow^{\pi_2^*} \\
 H_{\mathcal{C}} & & H_{\mathcal{C}}
 \end{array}$$

We also have maps $\pi_{q*}, \pi_{1*}, \pi_{2*}$ in the opposite direction, given by integrating along the fiber. i.e.,

$$\pi_{r*}(f)(y) = \sum_{x_y \in \pi_r^{-1}(y)} f(x_y)$$

which are well defined on functions with finite support. We can now form three different convolutions:

$$\begin{aligned}
 H_{\mathcal{C}} \otimes H_{\mathcal{C}} &\rightarrow H_{\mathcal{C}}, \\
 f \otimes g &\rightarrow \pi_{q*}(\pi_1^*(f)\pi_2^*(g)), \\
 H_{\mathcal{C}} \otimes H_{\mathcal{C}} &\rightarrow H_{\mathcal{C}}, \\
 f \otimes g &\rightarrow \pi_{2*}(\pi_q^*(f)\pi_1^*(g)), \\
 H_{\mathcal{C}} \otimes H_{\mathcal{C}} &\rightarrow H_{\mathcal{C}}, \\
 f \otimes g &\rightarrow \pi_{1*}(\pi_q^*(f)\pi_2^*(g)).
 \end{aligned}$$

We proceed to analyze the action of each of them. Since every function on $\text{Iso}(\mathcal{C})$ can be written as a linear combination of delta-functions supported on a single isomorphism class, it suffices to analyze what happens under convolution of two such delta-functions. We denote by δ_{ACB} the delta-function supported on the pair $A \subset B$ in $\mathcal{H}ecke_{\mathcal{C}}$.

(1) *First convolution:* For $A, B \in \text{Iso}(\mathcal{C})$, we have

$$\begin{aligned}
 (8.1) \quad \pi_{q*}(\pi_1^*(\delta_A)\pi_2^*(\delta_B)) &= \pi_{q*} \left(\sum_{\{C|A \subset C\}} \delta_{A \subset C} \times \sum_{\{D|D \subset B\}} \delta_{D \subset B} \right) \\
 &= \pi_{q*} \left(\sum_{A \subset B} \delta_{A \subset B} \right) \\
 &= \sum_{\{E \subset B | E \simeq A\}} \delta_{B/E}.
 \end{aligned}$$

Let us now define a right action (denoted \bullet) of $\mathbb{H}_{\mathcal{C}}$ on itself by

$$(8.2) \quad g \bullet f := \pi_{q*}(\pi_1^*(f)\pi_2^*(g)).$$

It follows from (6.4) that after identifying $\mathbb{H}_{\mathcal{C}}$ with its restricted dual $\mathbb{H}_{\mathcal{C}}^*$ and switching from right to left actions, this coincides with the *elimination* representation (6.4).

(2) *Second convolution:* For $A, B \in \text{Iso}(\mathcal{C})$, we have

$$\begin{aligned}
 (8.3) \quad \pi_{2*}(\pi_q^*(\delta_A)\pi_1^*(\delta_B)) &= \pi_{1*} \left(\sum_{\{C \subset D | D/C \simeq A\}} \delta_{C \subset D} \times \sum_{\{B \subset E\}} \delta_{B \subset E} \right) \\
 &= \pi_{2*} \left(\sum_{\{B \subset E | E/B \simeq A\}} \delta_{B \subset E} \right) \\
 &= \sum_{\{B \subset E | E/B \simeq A\}} \delta_E.
 \end{aligned}$$

We define a right action (denoted \square) of $\mathbb{H}_{\mathcal{C}}$ on itself by

$$(8.4) \quad g \square f := \pi_{2*}(\pi_q^*(f)\pi_1^*(g))$$

It follows from (6.3) that this coincides with the *top insertion* action. If instead we view this as a left action of $\mathbb{H}_{\mathcal{C}}$ on itself (i.e., g acting on f rather than vice versa), we recover the *insertion* action (6.2).

(3) *Third convolution:* For $A, B \in \text{Iso}(\mathcal{C})$, we have

$$\begin{aligned} \pi_{1*}(\pi_q^*(\delta_A)\pi_2^*(\delta_B)) &= \pi_{1*} \left(\sum_{\{C \subset D \mid D/C \simeq A\}} \delta_{C \subset D} \times \sum_{\{E \subset B\}} \delta_{E \subset B} \right) \\ &= \pi_{1*} \left(\sum_{\{E \subset B \mid B/E \simeq A\}} \delta_E \right) \\ &= \sum_{\{E \subset B \mid B/E \simeq A\}} \delta_E. \end{aligned}$$

We define a left action (denoted \star) of $H_{\mathcal{C}}$ on itself by

$$(8.5) \quad f \star g := \pi_{1*}(\pi_q^*(f)\pi_2^*(g)).$$

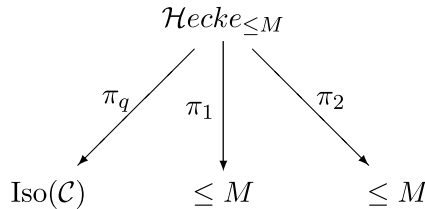
It follows from (6.5) that after identifying $H_{\mathcal{C}}$ with its restricted dual $H_{\mathcal{C}}^*$ and switching from right to left actions, this coincides with the *top-elimination* representation.

8.1. Finite-dimensional representations

The finite-dimensional truncations studied in Sections 7.1, 7.2 can also be described in the setup of Hecke correspondences. We discuss here the construction of $H_{\leq M}$ — that of $H_{\text{quot}(M)}$ is completely analogous. For $M \in \text{Iso}(\mathcal{C})$, let,

$$\mathcal{H}ecke_{\leq M} := \{(A, B) \mid B \in \text{Iso}(\mathcal{C}), A \subset B \subset M\}.$$

As in the previous section, we have a diagram



and the dual diagram

$$\begin{array}{ccccc}
 & & \mathcal{F}(\mathit{Hecke}_{\leq M}) & & \\
 & \nearrow^{\pi_q^*} & \uparrow^{\pi_1^*} & \nwarrow_{\pi_2^*} & \\
 \mathbb{H}_{\mathcal{C}} & & \mathbb{H}_{\leq M} & & \mathbb{H}_{\leq M}
 \end{array}$$

where $\mathcal{F}(\mathit{Hecke}_{\leq M})$ denotes the space of \mathbb{Q} -valued functions on $\mathit{Hecke}_{\leq M}$. The left action of $\mathbb{H}_{\mathcal{C}}$ on $\mathbb{H}_{\leq M}$ given by

$$f \star g := \pi_{1*}(\pi_q^*(f)\pi_2^*(g)), \quad f \in \mathbb{H}_{\mathcal{C}}, g \in \mathbb{H}_{\leq M}$$

is the opposite of the *top-elimination* representation from Section 7.2 (i.e., coincides with it after the isomorphism ${}^t : \mathfrak{n}_{\mathcal{C}} \simeq \mathfrak{n}_{\mathcal{C}}^{\text{op}}$), whereas the right action

$$g \square f := \pi_{2*}(\pi_q^*(f)\pi_1^*(g)), \quad f \in \mathbb{H}_{\mathcal{C}}, g \in \mathbb{H}_{\leq M}$$

is opposite to its dual.

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