

Instanton corrections to the universal hypermultiplet and automorphic forms on $SU(2, 1)$

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The hypermultiplet moduli space in Type IIA string theory compactified on a rigid Calabi–Yau threefold \mathcal{X} , corresponding to the “universal hypermultiplet,” is described at tree level by the symmetric space $SU(2, 1)/(SU(2) \times U(1))$. To determine the quantum corrections to this metric, we posit that a discrete subgroup of the continuous tree level isometry group $SU(2, 1)$, namely the Picard modular group $SU(2, 1; \mathbb{Z}[i])$, must remain unbroken in the exact metric — including all perturbative and non-perturbative quantum corrections. This assumption is expected to be valid when \mathcal{X} admits complex multiplication by $\mathbb{Z}[i]$. Based on this hypothesis, we construct an $SU(2, 1; \mathbb{Z}[i])$ -invariant, non-holomorphic Eisenstein series, and tentatively propose that this Eisenstein series provides the exact contact potential on the twistor space over the universal hypermultiplet moduli space. We analyze its non-Abelian Fourier expansion, and show that the Abelian and non-Abelian Fourier coefficients take the required form for instanton corrections due to Euclidean D2-branes wrapping special Lagrangian submanifolds, and to Euclidean NS5-branes wrapping the entire Calabi–Yau threefold, respectively. While this tentative proposal fails to reproduce the correct one-loop correction, the consistency of the Fourier expansion with physics expectations provides strong support for the usefulness of the Picard modular group in constraining the quantum moduli space.

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1. Introduction and summary

String theory compactified on a manifold \mathcal{X} typically leads to a low-energy effective action with an often large number of massless scalar fields valued in a moduli space \mathcal{M} . In general, the Riemannian metric on \mathcal{M} is deformed by perturbative and non-perturbative quantum corrections, making it very difficult to determine the exact form of the quantum effective action. In this paper we study the particular case of compactifications of type IIA string theory on a rigid Calabi–Yau threefold \mathcal{X} (i.e., with Betti number $h_{2,1}(\mathcal{X}) = 0$). In this case, the hypermultiplet part of the moduli space \mathcal{M} is known to be described, at tree level in the string perturbative expansion, by the symmetric space $\mathcal{M}_{\text{UH}} = SU(2, 1)/(SU(2) \times U(1))$. We analyze the quantum corrections to this classical geometry, and in particular conjecture the form of D2-brane and NS5-brane instanton contributions. Before entering into the details, we begin by discussing some of the ideas leading up to our proposal.

1.1. Rigid moduli spaces for $\mathcal{N} \geq 4$ and Eisenstein series

For compactifications preserving $\mathcal{N} \geq 4$ supersymmetry in $D = 4$, the moduli space is always locally a symmetric space $\mathcal{M} = G/K$, with G being a global symmetry and K , the maximal compact subgroup of G , being a local R-symmetry. In particular, \mathcal{M} has restricted holonomy group K and is rigid (see, e.g., [1] for a nice discussion). Quantum corrections are encoded in the global structure of \mathcal{M} , given by a double coset

$$(1.1) \quad \mathcal{M}_{\text{exact}} = G(\mathbb{Z}) \backslash G/K,$$

where $G(\mathbb{Z})$ is typically an arithmetic subgroup of G , known as the S -, T - or U -duality group, depending on the context [2–4]. For example, M-theory compactified on T^7 (or type IIA/B on T^6) gives rise to $\mathcal{N} = 8$ supergravity in four dimensions, whose exact moduli space is conjectured to be $E_{7(7)}(\mathbb{Z}) \backslash E_{7(7)} / (SU(8)/\mathbb{Z}_2)$ [2]. In such cases, the quantum effective action is expected to be invariant under $G(\mathbb{Z})$, which gives a powerful constraint on possible quantum corrections.

This idea was exploited with great success in the seminal work [5] in the context of type IIB supergravity in 10 dimensions, where the exact coefficient of the higher-derivative \mathcal{R}^4 -type corrections were proposed to be given by a non-holomorphic Eisenstein series $\mathcal{E}_{3/2}^{SL(2, \mathbb{Z})}$ as a function of the “axio-dilaton” $C_{(0)} + ie^{-\phi}$, valued on the fundamental domain $\mathcal{M} = SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R}) / SO(2)$ of the Poincaré upper half plane. This proposal reproduced the known tree-level and one-loop corrections [6, 7], predicted the absence of higher loop corrections, later verified by an explicit two-loop computation [8], and suggested the exact form of D(-1)-instanton contributions, later corroborated by explicit matrix model computations [9, 10]. From the mathematical point of view, perturbative corrections and instanton contributions correspond, respectively, to the constant terms and Fourier coefficients of the automorphic form $\mathcal{E}_{3/2}^{SL(2, \mathbb{Z})}$. This work was extended to toroidal compactifications of M-theory, where the \mathcal{R}^4 -type corrections were argued to be given by Eisenstein series of the respective U-duality group [11–13], predicting the contributions of Euclidean Dp -brane instantons and, when $n \geq 6$, NS5-branes. Unfortunately, extracting the constant terms and Fourier coefficients of Eisenstein series is not an easy task, and it has been difficult to put the conjecture to the test. Part of our motivation is to develop the understanding of Eisenstein series beyond the relatively well understood case of $G(\mathbb{Z}) = SL(n, \mathbb{Z})$.

1.2. The hypermultiplet moduli space of $\mathcal{N} = 2$ supergravity

Compactifications with fewer unbroken supersymmetries ($\mathcal{N} \leq 2$ in $D = 4$) lead to moduli spaces which are generically not symmetric spaces. An interesting example is type IIA string theory compactified on a Calabi–Yau threefold \mathcal{X} , leading to $\mathcal{N} = 2$ supergravity in four dimensions coupled to $h_{1,1}$ vector multiplets and $h_{2,1} + 1$ hypermultiplets. The moduli space locally splits into a direct product $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$, where \mathcal{M}_V is a $2h_{1,1}$ -dimensional special Kähler manifold and \mathcal{M}_H a $4(h_{2,1} + 1)$ -dimensional quaternion-Kähler manifold, respectively. \mathcal{M}_V encodes the (complexified)

Kähler structure of \mathcal{X} , while \mathcal{M}_H encodes deformations of the complex structure. \mathcal{M}_V is exact at tree-level in the perturbative string expansion, and well understood thanks to classical mirror symmetry (see e.g. [14] for an extensive introduction). In this paper we focus on the less understood hypermultiplet moduli space \mathcal{M}_H . Note however that upon further compactification on a circle, \mathcal{M}_V is extended to a $4(h_{1,1} + 1)$ -dimensional quaternion-Kähler manifold by the c -map, and the vector and hypermultiplet moduli spaces become equally complicated, being exchanged under T-duality along the circle [15].

Contrary to \mathcal{M}_V , the hypermultiplet moduli space \mathcal{M}_H receives perturbative and non-perturbative corrections in the string coupling [16–20]. The non-perturbative corrections are due to Euclidean D2-branes wrapping special Lagrangian submanifolds in \mathcal{X} , as well as due to Euclidean NS5-branes wrapping the entire Calabi–Yau threefold [16].¹ It has been an outstanding problem to understand how these effects modify the geometry of the moduli space \mathcal{M}_H , mainly due to the fact that quaternion-Kähler geometry is much more complicated than special Kähler geometry. Recently, however, it has become apparent that twistor techniques can be efficiently applied to quaternion-Kähler geometry. In particular, deformations of the quaternion-Kähler geometry of \mathcal{M}_H are in one-to-one correspondance with deformations of its twistor space $\mathcal{Z}_{\mathcal{M}_H}$, a $\mathbb{C}P^1$ bundle over \mathcal{M}_H [22–24] (see [25–29] for a physics realization of this equivalence). One virtue of this approach is that, contrary to \mathcal{M}_H , the twistor space $\mathcal{Z}_{\mathcal{M}_H}$ is Kähler, and therefore quantum corrections to \mathcal{M}_H can in principle be described in terms of the Kähler potential on its twistor space $\mathcal{Z}_{\mathcal{M}_H}$. Furthermore, $\mathcal{Z}_{\mathcal{M}_H}$ being a complex contact manifold, it can be described by holomorphic data, namely a set of complex symplectomorphisms between complex Darboux coordinate patches.

Using these techniques, much headway has been made in summing up part of the instanton corrections to hypermultiplet moduli spaces in both type IIA and IIB string theory [30–33]. These techniques were combined with the $SL(2, \mathbb{Z})$ -invariance of the four-dimensional effective action in [30], to obtain the quantum corrections to $\mathcal{M}_H^{\text{IIB}}$ due to D(–1), F1 and D1 instantons.² In this context, the Eisenstein series $\mathcal{E}_{3/2}^{SL(2, \mathbb{Z})}$, discussed above, reappears as the D(–1) instanton contribution to the Kähler potential on the twistor space of $\mathcal{M}_H^{\text{IIB}}$. This result was then mapped over to the IIA side using mirror symmetry [31], providing the quantum corrections to $\mathcal{M}_H^{\text{IIA}}$

¹See [21] for a recent analysis of these effects in heterotic compactifications.

²The holomorphic action of $SL(2, \mathbb{Z})$ on the twistor space $\mathcal{Z}_{\mathcal{M}_H}$ has been recently clarified in [33].

from Euclidean D2-branes wrapping A -cycles in \mathcal{X} . Subsequently, the contribution from D2-branes wrapping arbitrary combinations of A and B -cycles was obtained to linear order by covariantizing the result of [31] under “electric-magnetic duality” between A - and B -cycles in the Calabi–Yau [28, 32]. By the T-duality argument mentioned above, this also provides the contributions of 4D BPS black holes to the vector multiplet moduli space in type IIA or IIB string theory compactified on $\mathcal{X} \times S_1$. However, the NS5-brane contributions (or, on the vector multiplet side, the Kaluza–Klein monopole contributions) have so far proven to be considerably more elusive, although they can be in principle reached following the “roadmap” proposed in [31]. By postulating invariance under a larger discrete group $SL(3, \mathbb{Z})$, a subset of the NS5-brane contributions corresponding to the “extended universal hypermultiplet” was conjectured in [34]. This analysis (and presumably also the analysis in [30]) breaks down for rigid Calabi–Yau threefolds, the sector which we address in this work.

1.3. Rigid Calabi–Yau threefolds and the Picard modular group

In the present paper, we study the hypermultiplet moduli space \mathcal{M}_H in a restricted setting, namely for type IIA string theory compactified on a rigid Calabi–Yau threefold \mathcal{X} (i.e., with $h_{2,1}(\mathcal{X}) = 0$). By the T-duality argument indicated above, our analysis applies equally well to the vector multiplet moduli space in type IIB string theory compactified on $\mathcal{X} \times S_1$. Rigid Calabi–Yau threefolds are rare, but examples can be found in the mathematics (see, e.g., [35]) and the physics literature (see, e.g., [36–39]). One of their peculiarities is that they do not admit a mirror in the usual sense, since $h_{1,1} \geq 1$ for any Calabi–Yau threefold.³ Thus, it is no longer clear that \mathcal{M}_H should admit an isometric action of $SL(2, \mathbb{Z})$. Moreover, rigid Calabi–Yau threefolds do not admit a K3 fibration, so are not amenable to heterotic/type II duality [40].

For such rigid Calabi–Yau threefolds then, the hypermultiplet sector consists solely of the “universal hypermultiplet,” given at tree-level by the quaternion–Kähler symmetric space $\mathcal{M}_{UH}(\mathcal{X}) = SU(2, 1)/(SU(2) \times U(1))$. The metric on $\mathcal{M}_{UH}(\mathcal{X})$ is obtained via the c -map procedure from the complex structure moduli space of \mathcal{X} [15, 41]. Since \mathcal{X} has no complex structure

³It is possible that the superconformal field theory on \mathcal{X} admits a mirror description as a Landau–Ginzburg model LG, but it is not obvious that this equivalence should extend at the non-perturbative level. Put differently, it is unclear whether type IIA on LG can still be lifted to M-theory, and whether type IIB on LG still exhibits $SL(2, \mathbb{Z})$ symmetry.

deformations, its prepotential $F(X)$ is determined from the special geometry relations

$$(1.2) \quad X = \int_{\mathcal{A}} \Omega, \quad \frac{\partial F}{\partial X} = \int_{\mathcal{B}} \Omega$$

to be quadratic, namely

$$(1.3) \quad F(X) = \tau X^2/2, \quad \tau := \frac{\int_{\mathcal{B}} \Omega}{\int_{\mathcal{A}} \Omega},$$

where $\Omega \in H^{3,0}(\mathcal{X})$ is the holomorphic 3-form, $(\mathcal{A}, \mathcal{B})$ is an integral symplectic basis of $H_3(\mathcal{X}, \mathbb{Z})$ and τ is a fixed complex number, the period matrix. The c -map then leads to the metric⁴

$$(1.4) \quad ds_{\mathcal{M}_{\text{UH}}}^2(\tau) = d\phi^2 + e^{2\phi} \frac{|d\tilde{\chi} + \tau d\chi|^2}{\Im \tau} + e^{4\phi} (d\psi + \chi d\tilde{\chi} - \tilde{\chi} d\chi)^2.$$

In type IIA string theory compactified on \mathcal{X} , e^ϕ is the four-dimensional string coupling, ψ is the NS–NS axion, dual to the 2-form $B_{(2)}$ in $D = 4$, and $(\chi, \tilde{\chi})$ are the periods of the Ramond–Ramond 3-form $C_{(3)}$:

$$(1.5) \quad \chi = \int_{\mathcal{A}} C_{(3)}, \quad \tilde{\chi} = \int_{\mathcal{B}} C_{(3)}.$$

In the dual-type IIB string theory on $\mathcal{X} \times S_1$, e^ϕ is instead the inverse radius of the circle in 4D Planck units, while $\chi, \tilde{\chi}$ are the components of the 10-dimensional Ramond–Ramond 4-form $C_{(4)}$ on $H^3(\mathcal{X}, \mathbb{R}) \times S^1$ and ψ is the NUT potential (the dual of the Kaluza–Klein gauge field).

Classically, the family of metrics (1.4), parametrized by τ , are all locally isometric to the symmetric space $SU(2, 1)/(SU(2) \times U(1)) = \mathbb{C}\mathbb{H}^2$ (see Section 2 for details). A standard choice is to take $\tau = i$, leading to the familiar form of the left-invariant metric on $\mathbb{C}\mathbb{H}^2$. However, at the quantum level the choice of τ is not innocuous. Indeed, the Ramond–Ramond scalars $(\chi, \tilde{\chi})$ are

⁴Concretely, this metric may be obtained from the general c -map metric given in Equation (4.31) of [28] by setting $h_{2,1} = 0$, $F(X) = \tau X^2/2$ and implementing the change of variables: $r = e^{-2\phi}$, $\zeta^0 = -2\sqrt{2}\chi$, $\tilde{\zeta}_0 = 2\sqrt{2}\tilde{\chi}$, $\sigma = 8\psi$.

known [42] to parametrize the intermediate Jacobian

$$(1.6) \quad J(\mathcal{X}) = \frac{H^3(\mathcal{X}, \mathbb{C})}{H^3(\mathcal{X}, \mathbb{Z})}.$$

Equivalently, they are subject to discrete identifications

$$(1.7) \quad (\chi, \tilde{\chi}) \rightarrow (\chi + a, \tilde{\chi} + b), \quad a, b \in \mathbb{Z}.$$

If \mathcal{X} is a rigid Calabi–Yau threefold, its intermediate Jacobian is an elliptic curve

$$(1.8) \quad J(\mathcal{X}) = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),$$

where τ is the period matrix defined in (1.3). Thus, different choices of τ lead to locally isometric but globally inequivalent metrics. In the present work we shall restrict to the particular choice⁵ $\tau = i$, corresponding to rigid Calabi–Yau threefolds for which the intermediate Jacobian is a square torus $\mathbb{C}/\mathbb{Z}[i]$, where $\mathbb{Z}[i]$ denotes the Gaussian integers $\{z \in \mathbb{C} \mid \Re(z), \Im(z) \in \mathbb{Z}\}$, corresponding to the ring of integers in the imaginary quadratic number field $\mathbb{Q}(i)$. Mathematically, this choice implies in particular that $J(\mathcal{X})$ admits “complex multiplication,” a notion that originates from the study of elliptic curves $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, which are said to admit complex multiplication (or are of “CM-type”) if and only if the modular parameter τ takes values in an imaginary quadratic extension of \mathbb{Q} . Many but not all rigid Calabi–Yau threefolds admit complex multiplication; a necessary and sufficient criterion is that the intermediate Jacobian of the Calabi–Yau threefold is of CM-type (see, e.g., [35] for a review). An example of a rigid Calabi–Yau threefold that does not admit complex multiplication is provided by the hypersurface constructed in [43].⁶ For examples of rigid Calabi–Yau threefolds that admit complex multiplication by $\mathbb{Z}[i]$, as is relevant in the present work, see [35].

While quantum corrections are bound to break the continuous isometric action of $SU(2, 1)$ on the hypermultiplet moduli space \mathcal{M}_{UH} , we posit that they preserve a discrete arithmetic subgroup $G(\mathbb{Z}) \subset SU(2, 1)$ (note however

⁵We are grateful to Jan Stienstra for pointing out this assumption, which was implicit in the first version of this work. We comment on the possible duality symmetries for rigid Calabi–Yau compactifications with $\tau \neq i$ in Section 6.

⁶We also note that complex multiplication has appeared previously in the physics literature in [44, 45].

that we do not assume that $\mathcal{M}_{\text{UH}}^{\text{exact}}$ is a double coset $G(\mathbb{Z})\backslash G/K$.⁷ On physical grounds, the subgroup $G(\mathbb{Z})$ should contain the following action on the moduli:

1. A discrete Heisenberg group $N(\mathbb{Z})$, acting by discrete (Peccei–Quinn) shift symmetries on the axions $\chi, \tilde{\chi}$ and ψ [47]:

$$(1.9) \quad \begin{aligned} \chi &\longmapsto \chi + a, \\ \tilde{\chi} &\longmapsto \tilde{\chi} + b, \\ \psi &\longmapsto \psi + \frac{1}{2}c - a\tilde{\chi} + b\chi, \end{aligned}$$

where $a, b, c \in \mathbb{Z}$, while leaving the dilaton ϕ invariant. In the type IIA setting, the breaking of the continuous shifts of χ and $\tilde{\chi}$ are due to D2-brane instantons, while the breaking of the shift of ψ is due to the combined effects of D2- and NS5-brane instantons. The factor $1/2$ appearing in front of c is in agreement with the quantization condition on the NS5-brane instantons derived in [47]. The Heisenberg symmetry (1.9) should hold more generally for any value of the period matrix τ .

2. The “electric–magnetic duality” R which interchanges the R–R scalars χ and $\tilde{\chi}$ [47]:

$$(1.10) \quad R : (\chi, \tilde{\chi}) \longmapsto (-\tilde{\chi}, \chi).$$

This symmetry is only expected to hold for rigid Calabi–Yau compactifications with $\tau = i$. In that case it amounts to a change of symplectic basis for $H_3(\mathcal{X}, \mathbb{Z})$.

3. We further assume that a discrete subgroup $SL(2, \mathbb{Z})$ of the four-dimensional S -duality (or, on the type IIB side, Ehlers symmetry), acting in the standard non-linear way on the complex parameter $\chi + ie^{-\phi}$ on the slice $\tilde{\chi} = \psi = 0$, is left unbroken by quantum corrections. As in earlier endeavors [48–52], it is difficult to justify this assumption rigorously, but the fact, demonstrated herein, that it leads to physically sensible results can be taken as support for this assumption.⁸

⁷The fate of the continuous symmetry $SU(2, 1)$ at the level of higher derivative corrections to the low energy effective action will be analyzed in a follow-up paper [46].

⁸On non-rigid Calabi–Yau manifolds one can use mirror symmetry to obtain S -duality transformations in IIA from the $SL(2, \mathbb{Z})$ symmetry of type IIB, as has been advocated in literature for example in [19, 53–55].

Based on these assumptions, it follows that, when $\tau = i$, $G(\mathbb{Z})$ must be the *Picard modular group*⁹ $SU(2, 1; \mathbb{Z}[i])$, defined as the intersection (see, e.g., [56, 57]):

$$(1.11) \quad SU(2, 1; \mathbb{Z}[i]) := SU(2, 1) \cap SL(3, \mathbb{Z}[i]).$$

Indeed, the symmetries 1, 2, 3 above reproduce the list of generators of the Picard modular group obtained in [56]. We find it remarkable that adjoining electric–magnetic duality and S-duality to the physically well-established Peccei–Quinn shift symmetries generates an interesting discrete subgroup of $SU(2, 1)$.

We conclude this section by emphasizing that although the moduli $\{\phi, \chi, \tilde{\chi}, \psi\} \in \mathcal{M}_{\text{UH}}$ occur universally for any Calabi–Yau threefold, the “universal hypermultiplet moduli space” (1.4) is not a universal subsector of the hypermultiplet moduli space $\mathcal{M}_{\text{H}}(\mathcal{X})$ for non-rigid \mathcal{X} [1]. (This is in contrast to the “extended universal hypermultiplet sector” introduced in [34].) However, in cases where \mathcal{M} is a symmetric space, it can often be written as a fiber bundle over \mathcal{M}_{UH} . One example is type II string theory compactified on T^7 , where the moduli space can be written as the fiber bundle [1, 15]

$$(1.12) \quad \frac{E_{7(\tau)}}{SU(8)/\mathbb{Z}_2} \rightarrow \frac{SU(2, 1)}{SU(2) \times U(1)} \times \frac{SL(2, \mathbb{R})}{SO(2)}.$$

In this context, it would be interesting to investigate which discrete subgroup of $SU(2, 1)$ is singled out by the intersection with the U-duality group $E_{7(\tau)}(\mathbb{Z})$ [2]. A similar decomposition as (1.12) also occurs for very special $\mathcal{N} = 2$ supergravity theories, where the second factor on the r.h.s. is replaced by a non-compact version of the five-dimensional U-duality group [58, 59]. Thus, it is plausible that the considerations in this paper may have bearing on the more general case of non-rigid Calabi–Yau threefolds.

1.4. Automorphic forms for the Picard modular group

Having identified (1.11) as a candidate symmetry group, we apply standard machinery to construct a simple automorphic form $\mathcal{E}_s(\mathcal{K})$ of $SU(2, 1; \mathbb{Z}[i])$,

⁹The nomenclature “Picard group” is not unique, in fact our Picard group is a member of a family of similar groups $PSU(1, n + 1; \mathbb{Z}[i])$ of which the case $n = 0$, corresponding to $PSL(2, \mathbb{Z}[i])$ is also often called the Picard group. In this paper we will always mean $SU(2, 1; \mathbb{Z}[i])$ when speaking of the Picard group.

which satisfies

$$(1.13) \quad \Delta_{\mathbb{C}\mathbb{H}^2} \mathcal{E}_s(\phi, \lambda, \gamma) = s(s - 2) \mathcal{E}_s(\phi, \lambda, \gamma),$$

where $\Delta_{\mathbb{C}\mathbb{H}^2}$ is the Laplace–Beltrami operator on $\mathbb{C}\mathbb{H}^2$, given in (3.9) below. For the sake of completeness, and because they mutually enlighten each other, we shall present three equivalent constructions of the Eisenstein series for the Picard modular group:

- First, we generalize the construction of non-holomorphic Eisenstein series for real classical groups over the integers in [13] to unitary groups over the Gaussian integers, and consider the constrained lattice sum

$$(1.14) \quad \mathcal{E}_s(\mathcal{K}) := \sum'_{\substack{\vec{\omega} \in \mathbb{Z}[i]^3 \\ \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = 0}} \left[\vec{\omega}^\dagger \cdot \mathcal{K} \cdot \vec{\omega} \right]^{-s},$$

where $\mathcal{K} = \mathcal{V}\mathcal{V}^\dagger$ is a Hermitian matrix parametrizing the coset space $SU(2, 1)/(SU(2) \times U(1))$, and the sum runs over non-zero triplets of Gaussian integers $\vec{\omega}$ subject to a certain quadratic constraint (3.3). The prime indicates that $\vec{\omega} = (0, 0, 0)$ is excluded from the sum. More generally, throughout this paper a prime on a summation symbol will indicate that the zero value is excluded from the sum.

- Secondly, we consider the $SU(2, 1; \mathbb{Z}[i])$ -invariant Poincaré series

$$(1.15) \quad \mathcal{P}_s(\mathcal{Z}) := \sum_{\gamma \in N(\mathbb{Z}) \backslash SU(2, 1; \mathbb{Z}[i])} \mathcal{F}(\gamma \cdot \mathcal{Z})^s,$$

where $\mathcal{F}(\mathcal{Z})$ is a function on $SU(2, 1)/(SU(2) \times U(1))$, which is manifestly invariant under the Heisenberg subgroup $N(\mathbb{Z}) \subset SU(2, 1; \mathbb{Z}[i])$. As shown in Section 3, this construction yields the same automorphic form as before up to an s -dependent factor,

$$(1.16) \quad \mathcal{E}_s(\mathcal{K}) = 4\zeta_{\mathbb{Q}(i)}(s)\mathcal{P}_s(\mathcal{Z}),$$

where $\zeta_{\mathbb{Q}(i)}(s)$ is the Dedekind zeta function for the Gauss field $\mathbb{Q}(i)$, defined in (4.15).

- Thirdly, in Appendix B we show that the same automorphic form can be obtained by the general adelic method explained, e.g., in [60–62],

$$(1.17) \quad \mathcal{E}_s(\mathcal{V}) = \sum'_{\substack{(C_1, C_2) \in P\mathbb{Q}(i)^2 \\ |C_1|^2 - 2\Im(C_2) = 0}} \rho(\mathcal{V}) \cdot \prod_{g \text{ prime}} f_g(C_1, C_2),$$

where the infinite product runs over all Gaussian prime numbers including $g = \infty$. Here, \mathcal{V} is a coset representative of $SU(2, 1)/(SU(2) \times U(1))$, $P\mathbb{Q}(i)^2$ is the projective space $\mathbb{Q}(i)^3/\mathbb{Q}(i)^*$, ρ is the principal continuous series representation of $SU(2, 1)$ on functions on $P\mathbb{Q}(i)^2$, $f_\infty := f_K$ is the $SU(2) \times U(1)$ -invariant spherical vector from [63], and f_g ($g < \infty$) its p -adic counterpart.

Generally, any function f on $\mathbb{C}\mathbb{H}^2$ which obeys the following three conditions:

1. f satisfies the Laplace equation $\Delta_{\mathbb{C}\mathbb{H}^2} f = s(s - 2) f$,
2. f is invariant under the Heisenberg group $N(\mathbb{Z})$ (see (1.9)),
3. f has at most polynomial growth in e^ϕ as $e^\phi \rightarrow 0$,

admits a non-Abelian Fourier expansion of the form¹⁰

$$\begin{aligned}
 f(\phi, \chi, \tilde{\chi}, \psi) &= A(s) e^{-2s\phi} + B(s) e^{-2(2-s)\phi} \\
 &+ e^{-2\phi} \sum'_{(\ell_1, \ell_2) \in \mathbb{Z}^2} C_{\ell_1, \ell_2}^{(A)}(s) K_{2s-2} \left(2\pi e^{-\phi} \sqrt{\ell_1^2 + \ell_2^2} \right) e^{-2\pi i(\ell_1 \chi + \ell_2 \tilde{\chi})} \\
 &+ e^{-\phi} \sum'_{k \in \mathbb{Z}} \sum_{\ell=0}^{4|k|-1} \sum_{n \in \mathbb{Z} + \frac{\ell}{4|k|}} \sum_{r=0}^{\infty} C_{r, k, \ell}^{(\text{NA})}(s) |k|^{1/2-s} e^{-4\pi|k|(\tilde{\chi}-n)^2} \\
 &\times H_r \left(\sqrt{8\pi|k|}(\tilde{\chi} - n) \right) \\
 (1.18) \quad &\times W_{-r-\frac{1}{2}, s-1} \left(4\pi|k|e^{-2\phi} \right) e^{8\pi i k n \chi - 4\pi i k(\psi + \chi \tilde{\chi})},
 \end{aligned}$$

where K_s , H_r and $W_{-r-\frac{1}{2}, s-1}$ denote the modified Bessel function, Hermite polynomial and Whittaker function, respectively. The first line in (1.18) corresponds to the “constant terms”, i.e., the leading terms in an expansion at the cusp $e^\phi \rightarrow 0$. The second line is the Abelian contribution, corresponding to an ordinary Fourier expansion with respect to the Abelianized Heisenberg group N/Z , where N is the Heisenberg subgroup of upper triangular matrices in $SU(2, 1)$ and Z is the center of N (in this particular case it coincides with the commutator subgroup $[N, N]$). Finally, the last two lines represent the non-Abelian contribution, i.e., the part of f which transforms non-trivially under the action of the center Z of the Heisenberg group. We have chosen to write them in the “ $\tilde{\chi}$ -polarization”, where the a and c shifts

¹⁰A similar expansion was given in [64] but we disagree with the details in this reference. For instance, the second constant term appears to be missing in Theorem 5.3.1ii) of [64]. Such a term is required on general grounds [65].

in (1.9) are diagonalized, but it is also possible to use the “ χ -polarization,” where b and c shifts are diagonalized (see Section 4.1 for a detailed discussion of the choice of polarization). The constant terms $A(s), B(s)$ and the numerical Fourier coefficients $C_{\ell_1, \ell_2}^{(A)}(s)$ will be derived in Section 4.

Our main mathematical results may now be summarized in the following propositions and conjecture:

Proposition 1.1 (Constant terms). *The constant terms in the Fourier expansion of the Eisenstein series $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$ in Equation (1.18) are given by*

$$(1.19) \quad \mathcal{E}_s^{(\text{const})} = 4\zeta_{\mathbb{Q}(i)}(s) \left\{ e^{-2s\phi} + \frac{\mathfrak{Z}(2-s)}{\mathfrak{Z}(s)} e^{-2(2-s)\phi} \right\}.$$

Here we have defined the “Picard zeta function” $\mathfrak{Z}(s)$ through

$$(1.20) \quad \mathfrak{Z}(s) := \zeta_{\mathbb{Q}(i)*}(s) \beta_*(2s-1),$$

where $\zeta_{\mathbb{Q}(i)*}(s)$ and $\beta_*(2s-1)$ are the completed Dedekind zeta function (4.16) and Dirichlet beta function (4.22), respectively.

This proposition is proven in Sections 4.2 and 4.4.

Proposition 1.2 (Abelian Fourier coefficients). *The numerical Fourier coefficients $C_{\ell_1, \ell_2}^{(A)}(s)$ in the expansion of $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$ with respect to the abelianized Heisenberg group $N/[N, N]$ are given by*

$$(1.21) \quad C_{\ell_1, \ell_2}^{(A)}(s) = \frac{2\zeta_{\mathbb{Q}(i)}(s)}{\mathfrak{Z}(s)} [\ell_1^2 + \ell_2^2]^{s-1} \sum_{\omega'|\Lambda} |\omega'|^{2-2s} \sum_{z|\frac{\Lambda}{\omega'}} |z|^{4-4s},$$

where $\sum_{\omega'|\Lambda}$ denotes a sum over Gaussian divisors of $\Lambda = \ell_2 - i\ell_1 \in \mathbb{Z}[i]$, and ω' is a Gaussian integer such that $\Re(\omega'), \Im(\omega') = 1$.

This proposition is proven in Section 4.5 and Appendix C.

To have a complete understanding of the Fourier expansion one must also extract the non-Abelian numerical coefficients $C_{r, k, \ell}^{(\text{NA})}(s)$, corresponding to the Fourier expansion of \mathcal{E}_s with respect to the center $Z = [N, N]$ of the Heisenberg group N . Some preliminary results in this direction are presented in Section 4.6.

Conjecture (Functional relation). *The Eisenstein series $\mathcal{P}_s = (4\zeta_{\mathbb{Q}(i)}(s))^{-1} \mathcal{E}_s$ satisfies the following functional relation:*

$$(1.22) \quad \mathfrak{Z}(s) \mathcal{P}_s = \mathfrak{Z}(2-s) \mathcal{P}_{2-s}.$$

The map $s \mapsto 2-s$ corresponds to the action of the restricted Weyl group of $\mathfrak{su}(2, 1)$. The validity of (1.22) at the level of the constant and Abelian terms holds true by the above propositions, but our incomplete understanding of the non-Abelian terms prevents us from stating it as a theorem. We note that a different functional relation for \mathcal{P}_s was stated in [66] but it appears to be inconsistent with the constant terms (1.19).

1.5. Eisenstein series and the exact universal hypermultiplet geometry

As a test of the idea that the Picard modular group should control the quantum corrections to the universal hypermultiplet moduli space, we tentatively propose that the $SU(2, 1; \mathbb{Z}[i])$ -invariant Eisenstein series $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$, for the specific value $s = 3/2$, gives a non-perturbative completion of the contact potential $e^{\Phi(x^\mu, z)}$ restricted to a certain holomorphic section $z = z(x^\mu)$ of the twistor space $\mathcal{Z}_{\mathcal{M}_{\text{UH}}}$ of the universal hypermultiplet moduli space. In this context, the constant terms of the Fourier expansion (1.18) represent the classical and one-loop contributions to the moduli space metric, respectively. Matching the power of the dilaton e^ϕ is what fixes the value of s above. This proposal is tentative however, as it turns out that the numerical coefficient of the one-loop term predicted by (1.18) is inconsistent with its known value in string theory [17] (in particular, it has the wrong sign). We proceed nevertheless, since the structure of the non-Abelian Fourier expansion (1.18) is largely independent of the automorphic form under consideration.

In this scenario, the Abelian terms in (1.18) have the suitable form to describe the effects of D2-brane instantons wrapping supersymmetric 3-cycles in the homology class $\ell_1 \mathcal{A} + \ell_2 \mathcal{B} \in H_3(\mathcal{X}, \mathbb{Z})$, where \mathcal{A} and \mathcal{B} provide a symplectic basis for $H_3(\mathcal{X}, \mathbb{Z})$. The Abelian Fourier coefficients $C_{\ell_1, \ell_2}^{(A)}(3/2)$, related to the instanton measure $\mu_{3/2}(\ell_1, \ell_2)$ via (4.69), should count supersymmetric cycles in the above homology class. The instanton measure $\mu_{3/2}(\ell_1, \ell_2)$ generalizes the familiar D(-1) instanton measure $\mu_{3/2}(N)$ of [5], which is also known to capture the effects of pure charge N A-type D2-brane instantons [31] (corresponding to $\ell_2 = 0$). On the other hand, non-Abelian terms have the suitable form to represent the effects of charge k NS5-brane instantons, possibly bound to D2-branes. We take these encouraging facts

as evidence that the Picard modular group should be a powerful principle in constraining the exact metric on the hypermultiplet moduli space in rigid Calabi–Yau compactifications, and in particular in determining the NS5-brane instantons which have remained elusive for generic Calabi–Yau compactifications. We comment on alternative choices of automorphic form for $SU(2, 1; \mathbb{Z}[i])$, which may alleviate the shortcomings of the Eisenstein series $\mathcal{E}_{3/2}$ in Section 6.

1.6. Outline

In Section 2 we give a detailed description of the group $SU(2, 1)$, the symmetric space $SU(2, 1)/(SU(2) \times U(1))$ and the Picard modular group $SU(2, 1; \mathbb{Z}[i])$. In Section 3 we construct an $SU(2, 1; \mathbb{Z}[i])$ -invariant Eisenstein series $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$ in the principal continuous series of $SU(2, 1)$ by two different methods. We proceed in Section 4 to compute the Fourier expansion of \mathcal{E}_s , extracting explicit forms for the constant terms as well as the Abelian and non-Abelian Fourier coefficients. Finally, in Section 5, we use the automorphic form $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$ at order $s = 3/2$ to conjecture the exact form of the D2-brane and NS5-brane instanton corrections to the universal hypermultiplet moduli space \mathcal{M}_{UH} . In Appendix A, we review some basic facts about Gaussian integers and Gaussian primes as well as a reminder of Dirichlet series. The third construction of the Eisenstein series $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$ using an adelic construction is given for completeness in Appendix B. This construction can be viewed as an extension of the analysis in Section 2 of [63] to the automorphic setting. In Appendix C we provide details on the derivation of the Abelian Fourier coefficients $C_{\ell_1, \ell_2}^{(A)}(s)$.

2. On the Picard modular group $SU(2, 1; \mathbb{Z}[i])$

As indicated in the Introduction, a key character in this paper is the symmetric space $SU(2, 1)/(SU(2) \times U(1))$. This space describes the tree-level moduli space of the universal hypermultiplet in type IIA string theory compactified on a rigid Calabi–Yau threefold.¹¹ In this section, we set up notations for the group $SU(2, 1)$, give two equivalent descriptions of the symmetric space $SU(2, 1)/(SU(2) \times U(1))$ and introduce the Picard modular group $SU(2, 1; \mathbb{Z}[i])$.

¹¹We note that the coset space $SU(2, 1)/(SU(2) \times U(1))$ also appears as the moduli space of the Einstein–Maxwell system when dimensionally reduced from $D = 4$ to $D = 3$ on a spacelike circle.

2.1. The group $SU(2, 1)$ and its Lie algebra $\mathfrak{su}(2, 1)$

The Lie group $SU(2, 1)$ is defined as a subgroup of the group $GL(3, \mathbb{C})$ of invertible (3×3) complex matrices via

$$(2.1) \quad SU(2, 1) = \left\{ g \in GL(3, \mathbb{C}) : g^\dagger \eta g = \eta \text{ and } \det(g) = 1 \right\}.$$

Here, the defining metric η is given by

$$(2.2) \quad \eta = \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix}$$

and has signature $(+ + -)$. The condition $g^\dagger \eta g = \eta$ already implies $|\det(g)| = 1$ and so we can also think of $SU(2, 1)$ as the set of unitary matrices $U(2, 1)$ modulo a pure phase, $SU(2, 1) \cong PU(2, 1)$, with the projectivization P referring to the equivalence relation $g \sim g e^{i\alpha}$ for $\alpha \in [0, 2\pi)$. The diagonal matrices $e^{i\alpha} \text{diag}(1, 1, 1)$ form the center of the group $U(2, 1)$.

The Lie group $SU(2, 1)$ as defined in (2.1) has as Lie algebra of real dimension 8

$$(2.3) \quad \mathfrak{su}(2, 1) = \left\{ X \in \mathfrak{gl}(3, \mathbb{C}) : X^\dagger \eta + \eta X = 0 \text{ and } \text{tr}(X) = 0 \right\}.$$

It consists of four compact and four non-compact generators, and the maximal real torus is one dimensional. We define the non-compact and compact Cartan generators

$$(2.4) \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix},$$

the positive step operators

$$(2.5) \quad \begin{aligned} X_1 &= \begin{pmatrix} 0 & -1+i & 0 \\ 0 & 0 & 1-i \\ 0 & 0 & 0 \end{pmatrix}, & \tilde{X}_1 &= \begin{pmatrix} 0 & 1+i & 0 \\ 0 & 0 & 1+i \\ 0 & 0 & 0 \end{pmatrix}, \\ X_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and the negative step operators

$$(2.6) \quad \begin{aligned} Y_{-1} &= \begin{pmatrix} 0 & 0 & 0 \\ 1+i & 0 & 0 \\ 0 & -1-i & 0 \end{pmatrix}, & \tilde{Y}_{-1} &= \begin{pmatrix} 0 & 0 & 0 \\ -1+i & 0 & 0 \\ 0 & -1+i & 0 \end{pmatrix}, \\ Y_{-2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The subscript refers to the eigenvalue under the adjoint action of the non-compact Cartan generator H , e.g., $[H, X_1] = X_1$; the adjoint action of the compact Cartan generator J is not diagonalizable over the real numbers. Furthermore, the generators satisfy

$$(2.7) \quad [X_1, \tilde{X}_1] = -4X_2,$$

such that the positive step operators form a Heisenberg algebra. Furthermore, the negative step operators Y are minus the Hermitian conjugate of the positive step operators X .

The Lie algebra $\mathfrak{su}(2, 1)$ has a natural five grading by the generator H as a direct sum of vector spaces

$$(2.8) \quad \mathfrak{su}(2, 1) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

with

$$(2.9) \quad \begin{aligned} \mathfrak{g}_{-2} &= \mathbb{R}Y_{-2}, & \mathfrak{g}_{-1} &= \mathbb{R}Y_{-1} \oplus \mathbb{R}\tilde{Y}_{-1}, & \mathfrak{g}_0 &= \mathbb{R}H \oplus \mathbb{R}J, \\ \mathfrak{g}_1 &= \mathbb{R}X_1 \oplus \mathbb{R}\tilde{X}_1, & \mathfrak{g}_2 &= \mathbb{R}X_2. \end{aligned}$$

One sees that the H -eigenspaces with eigenvalue ± 1 are degenerate. This is a characteristic feature of the reduced root system BC_1 underlying the real form $\mathfrak{su}(2, 1)$ of $\mathfrak{sl}(3, \mathbb{C})$. There is a single simple root α since the real rank of $\mathfrak{su}(2, 1)$ is one, and there are non-trivial root spaces \mathfrak{g}_1 and \mathfrak{g}_2 corresponding to α and 2α , respectively.¹² The $\mathfrak{sl}(2, \mathbb{R})$ subalgebra associated with the 2α root space is canonically normalized and can be given a standard basis for example with H , $E = X_2$ and $F = -Y_{-2}$, so that $[E, F] = H$. The corresponding $SL(2, \mathbb{R})$ subgroup of $SU(2, 1)$ is given by matrices of

¹²A discussion of the restricted root system can, for example, be found in [67].

the form

$$(2.10) \quad \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\} \subset SU(2, 1).$$

Under this embedding, the fundamental representation of $SU(2, 1)$ decomposes as $3 = 2 \oplus 1$. There exists a second, non-regular embedding of $SL(2, \mathbb{R})$ inside $SU(2, 1)$, consisting of matrices of the form

$$(2.11) \quad \begin{aligned} & SL(2, \mathbb{R}) \\ & = \left\{ \begin{pmatrix} a^2 & (-1+i)ab & ib^2 \\ (-1-i)ac & ad+bc & (1-i)bd \\ -ic^2 & (1+i)cd & d^2 \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}. \end{aligned}$$

Under this embedding, the fundamental representation of $SU(2, 1)$ remains irreducible. The two subgroups (2.10) and (2.11) together generate the whole of $SU(2, 1)$.

The Iwasawa decomposition of the Lie algebra $\mathfrak{su}(2, 1)$ reads

$$(2.12) \quad \mathfrak{su}(2, 1) = \mathfrak{n}_+ \oplus \mathfrak{a} \oplus \mathfrak{k},$$

where the non-compact (Abelian) Cartan subalgebra is defined as $\mathfrak{a} = \mathbb{R}H$, while the nilpotent subspace $\mathfrak{n}_+ = \mathbb{R}X_1 \oplus \mathbb{R}\tilde{X}_1 \oplus \mathbb{R}X_2$ is spanned by the positive step operators. The compact subalgebra of $\mathfrak{su}(2, 1)$ is $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ as a direct sum of Lie algebras.¹³ The generators of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ are given explicitly by the anti-Hermitian matrices

$$(2.13) \quad \begin{aligned} \hat{K}_1 &= \frac{1}{4}(X_1 + Y_{-1}), & \hat{K}_2 &= \frac{1}{4}(\tilde{X}_1 + \tilde{Y}_{-1}), & \hat{K}_3 &= \frac{1}{4}(X_2 + Y_{-2} + J), \\ \hat{J} &= \frac{3}{4}(X_2 + Y_{-2}) - \frac{1}{4}J. \end{aligned}$$

¹³By contrast, the Iwasawa decomposition (2.12) is only a direct sum of vector spaces and not of Lie algebras.

These satisfy $[\hat{J}, \hat{K}_i] = 0$ and $[\hat{K}_i, \hat{K}_j] = -\epsilon_{ijk} \hat{K}_k$. The Weyl group of the reduced root system BC_1 is

$$(2.14) \quad \mathcal{W}(\mathfrak{su}(2, 1)) = \mathcal{W}(BC_1) \cong \mathbb{Z}_2,$$

corresponding to the Weyl reflection with respect to α .

2.2. Complex hyperbolic space

The group $SU(2, 1)$ acts transitively and isometrically on the complex two-dimensional space¹⁴

$$(2.15) \quad \mathbb{C}\mathbb{H}^2 = \{ \mathcal{Z} = (z_1, z_2) \in \mathbb{C}^2 : \mathcal{F}(\mathcal{Z}) > 0 \},$$

equipped with the Kähler metric

$$(2.16) \quad ds^2 = \frac{1}{4} \mathcal{F}^{-2} [dz_1 d\bar{z}_1 + iz_2 dz_1 d\bar{z}_2 - i\bar{z}_2 dz_2 d\bar{z}_1 + 2\Im(z_1) dz_2 d\bar{z}_2].$$

The “height function” $\mathcal{F} : \mathbb{C}^2 \rightarrow \mathbb{R}$ is defined by

$$(2.17) \quad \mathcal{F}(\mathcal{Z}) := \Im(z_1) - \frac{1}{2}|z_2|^2,$$

and provides a Kähler potential for the metric (2.16),

$$(2.18) \quad K_{\mathbb{C}\mathbb{H}^2}(\mathcal{Z}) = -\log \mathcal{F}(\mathcal{Z}).$$

The action of $SU(2, 1)$ on $\mathcal{Z} \in \mathbb{C}\mathbb{H}^2$ is via fractional linear transformations

$$(2.19) \quad g \cdot \mathcal{Z} = \frac{A\mathcal{Z} + B}{C\mathcal{Z} + D} \quad \text{for} \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the blocks A , B , C and D have the sizes (2×2) , (2×1) , (1×2) and (1×1) , respectively, so that the denominator is a complex number. Since the height function transforms as

$$(2.20) \quad \mathcal{F}(g \cdot \mathcal{Z}) = \frac{\mathcal{F}(\mathcal{Z})}{|C\mathcal{Z} + D|^2},$$

the condition $\mathcal{F}(\mathcal{Z}) > 0$ is preserved and the action is isometric. In fact, when verifying (2.20) one only requires the condition $g^\dagger \eta g = \eta$ so that (2.19)

¹⁴This is referred to as the “unbounded hyperquadric model” in [57].

defines an action of all of $U(2, 1)$ on complex hyperbolic two-space. Since elements from the center act trivially, one can restrict to $PU(2, 1) \cong SU(2, 1)$. We will refer to the space $\mathbb{C}\mathbb{H}^2$ defined in (2.15) as the *complex hyperbolic space*, or the *complex upper half plane*. The slice $z_2 = 0, \Im(z_1) > 0$ inside $\mathbb{C}\mathbb{H}^2$ is preserved by the action of the $SL(2, \mathbb{R})$ subgroup in (2.10), and gives an embedding of the standard Poincaré upper half plane inside $\mathbb{C}\mathbb{H}^2$.

2.3. Relation to the scalar coset manifold $SU(2, 1)/(SU(2) \times U(1))$

The complex hyperbolic upper half plane is isomorphic to the Hermitian symmetric space

$$(2.21) \quad \mathbb{C}\mathbb{H}^2 \cong SU(2, 1)/(SU(2) \times U(1)),$$

where the right-hand side should properly be restricted to the connected component of the identity. This space can be parametrized by four real variables $\{\phi, \chi, \tilde{\chi}, \psi\}$, using the coset representative in the Iwasawa gauge (2.12),

$$(2.22) \quad \mathcal{V} = e^{\chi X_1 + \tilde{\chi} \tilde{X}_1 + 2\psi X_2} e^{-\phi H} = \begin{pmatrix} e^{-\phi} & \tilde{\chi} - \chi + i(\chi + \tilde{\chi}) & e^{\phi} (2\psi + i(\chi^2 + \tilde{\chi}^2)) \\ 0 & 1 & e^{\phi} (\chi + \tilde{\chi} + i(\tilde{\chi} - \chi)) \\ 0 & 0 & e^{\phi} \end{pmatrix}.$$

The symmetric space is a right coset in our conventions, the coset element \mathcal{V} transforming as $\mathcal{V} \rightarrow g\mathcal{V}k^{-1}$ with $g \in SU(2, 1)$ and $k \in SU(2) \times U(1)$. The four scalar fields can take arbitrary real values.

It is convenient to define the Hermitian matrix

$$(2.23) \quad \mathcal{K} = \mathcal{V}\mathcal{V}^\dagger$$

that transforms as $\mathcal{K} \rightarrow g\mathcal{K}g^\dagger$ under the action of $g \in SU(2, 1)$. Explicitly, this matrix reads

$$(2.24) \quad \mathcal{K} = \begin{pmatrix} e^{-2\phi} + |\lambda|^2 + e^{2\phi}|\gamma|^2 & i\bar{\lambda} + e^{2\phi}\bar{\lambda}\gamma & e^{2\phi}\gamma \\ -i\lambda + e^{2\phi}\lambda\bar{\gamma} & 1 + e^{2\phi}|\lambda|^2 & e^{2\phi}\lambda \\ e^{2\phi}\bar{\gamma} & e^{2\phi}\bar{\lambda} & e^{2\phi} \end{pmatrix},$$

where, for later convenience, we have defined the complex variables

$$(2.25) \quad \lambda := \chi + \tilde{\chi} + i(\tilde{\chi} - \chi), \quad \gamma := 2\psi + \frac{i}{2}|\lambda|^2.$$

From \mathcal{K} one can obtain the metric on the symmetric space via

$$(2.26) \quad ds^2 = -\frac{1}{8} \text{tr} (d\mathcal{K} d(\mathcal{K}^{-1})) = \frac{1}{8} \text{tr} \left(\mathcal{V}^{-1} d\mathcal{V} + (\mathcal{V}^{-1} d\mathcal{V})^\dagger \right)^2.$$

Working this out for the coset element (2.22) one finds the following $SU(2, 1)$ invariant metric

$$(2.27) \quad ds^2 = d\phi^2 + e^{2\phi} (d\chi^2 + d\tilde{\chi}^2) + e^{4\phi} (d\psi + \chi d\tilde{\chi} - \tilde{\chi} d\chi)^2.$$

Comparing (2.27) with (2.16) leads to the identification

$$(2.28) \quad \begin{aligned} z_1 &= 2\psi + i \left(e^{-2\phi} + \frac{1}{2} |z_2|^2 \right) = 2\psi + i \left(e^{-2\phi} + \chi^2 + \tilde{\chi}^2 \right), \\ z_2 &= \chi + \tilde{\chi} + i(\tilde{\chi} - \chi). \end{aligned}$$

Note that $z_1 = \gamma + ie^{-2\phi}$, $z_2 = \lambda$, and the condition $0 < \mathcal{F}(\mathcal{Z}) = e^{-2\phi}$ is automatically satisfied. Note that in terms of the real variables $\{\phi, \chi, \tilde{\chi}, \psi\}$ the Kähler potential (2.18) simply becomes $K_{\mathbb{C}\mathbb{H}^2} = 2\phi$.

In the variables $\mathcal{Z} = (z_1, z_2)$ given by (2.28), the matrix \mathcal{K} of (2.23) takes the simple form

$$(2.29) \quad \mathcal{K} = \tilde{\mathcal{K}} + \eta,$$

where η is the defining matrix of $SU(2, 1)$ given in (2.2) and

$$(2.30) \quad \tilde{\mathcal{K}} = e^{2\phi} \begin{pmatrix} |z_1|^2 & z_1 \bar{z}_2 & z_1 \\ \bar{z}_1 z_2 & |z_2|^2 & z_2 \\ \bar{z}_1 & \bar{z}_2 & 1 \end{pmatrix},$$

where one should bear in mind that $e^{2\phi} = 1/\mathcal{F}(\mathcal{Z})$. The relations (2.28) together with (2.19) allow one to determine the action of an element of $SU(2, 1)$ in the real coordinates $\phi, \chi, \tilde{\chi}, \psi$. In particular, one may check that on the slice $\tilde{\chi} = \psi = 0$, the $SL(2, \mathbb{R})$ subgroup (2.11) acts by fractional linear transformations on the complex modulus $\chi + ie^{-\phi}$. This action may be considered as a remnant of the $SL(2, \mathbb{R})$ S-duality in 10-dimensional type IIB string theory. Similarly, on the slice $z_2 = 0$ the $SL(2, \mathbb{R})$ subgroup (2.10) acts by fractional linear transformations on the complex modulus $2\psi + ie^{-2\phi}$, which realizes four-dimensional S-duality.

2.4. Coset transformations and subgroups of $SU(2, 1)$

We now study the effect of some particular elements of $SU(2, 1)$ on complex hyperbolic two-space, which have an immediate physical interpretation.

2.4.1. Heisenberg translations. Let N denote the exponential of the nilpotent algebra of positive step operators \mathfrak{n}_+ . We define the following elements of N :

$$(2.31) \quad T_1 = \begin{pmatrix} 1 & -1+i & i \\ 0 & 1 & 1-i \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{T}_1 = \begin{pmatrix} 1 & 1+i & i \\ 0 & 1 & 1+i \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These are defined such that $T_1 = \exp(X_1)$, etc. Any element $n \in N$ can be written as

$$(2.32) \quad n = (T_1)^a (\tilde{T}_1)^b (T_2)^{c+2ab} = e^{aX_1 + b\tilde{X}_1 + cX_2} \\ = \begin{pmatrix} 1 & a(-1+i) + b(1+i) & c + i(a^2 + b^2) \\ 0 & 1 & a(1-i) + b(1+i) \\ 0 & 0 & 1 \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$. The effect of this transformation on $\mathcal{Z} = (z_1, z_2)$ is

$$(2.33) \quad \begin{aligned} z_1 &\longmapsto z_1 + [a(-1+i) + b(1+i)]z_2 + c + i(a^2 + b^2), \\ z_2 &\longmapsto z_2 + a(1-i) + b(1+i), \end{aligned}$$

or in terms of the four scalars fields of (2.22)

$$(2.34) \quad \begin{aligned} \phi &\longmapsto \phi, \\ \chi &\longmapsto \chi + a, \\ \tilde{\chi} &\longmapsto \tilde{\chi} + b, \\ \psi &\longmapsto \psi + \frac{1}{2}c - a\tilde{\chi} + b\chi. \end{aligned}$$

The appearance of the shift parameters a and b in the transformation of ψ is due to the non-Abelian structure of \mathfrak{n}_+ given by the Heisenberg algebra (2.7). This effect is also evident in the first line of expression (2.32) for the general element of N . From the point of view of the coset, the Heisenberg

translations do not require any compensating transformation as they preserve the Iwasawa gauge.

2.4.2. Rotations. Rotations are generated by the compact Cartan element J of $\mathfrak{su}(2, 1)$ given in (2.4). Let

$$(2.35) \quad R = \exp(\pi J/2) = \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix},$$

then the most general transformation of this type is given by $R^\sigma := e^{\sigma\pi J/2}$, for $\sigma = 0, 1, 2, 3$, and acts on $\mathcal{Z} = (z_1, z_2)$ via

$$(2.36) \quad z_1 \rightarrow z_1, \quad z_2 \rightarrow e^{i\pi\sigma/2} z_2.$$

In terms of the four scalar fields this transformation reads

$$(2.37) \quad \begin{aligned} \phi &\mapsto \phi, \\ \chi &\mapsto \cos(\pi\sigma/2)\chi - \sin(\pi\sigma/2)\tilde{\chi}, \\ \tilde{\chi} &\mapsto \sin(\pi\sigma/2)\chi + \cos(\pi\sigma/2)\tilde{\chi}, \\ \psi &\mapsto \psi \end{aligned}$$

and so rotates the two scalars χ and $\tilde{\chi}$ among each other while leaving the other two invariant. The compensating transformation to restore the Iwasawa gauge for the coset element (2.22) is $k = R^\sigma$.

2.4.3. Involution. The last transformation of interest is the involution

$$(2.38) \quad S = \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

which acts on $\mathcal{Z} = (z_1, z_2)$ according to

$$(2.39) \quad z_1 \mapsto -\frac{1}{z_1}, \quad z_2 \mapsto -i\frac{z_2}{z_1},$$

corresponding to the non-trivial generator in the Weyl group (2.14). For the real scalars themselves we find the following transformation:

$$\begin{aligned}
 (2.40) \quad \phi &\mapsto -\frac{1}{2} \ln \left[\frac{e^{-2\phi}}{4\psi^2 + [e^{-2\phi} + \chi^2 + \tilde{\chi}^2]^2} \right], \\
 \chi &\mapsto \frac{2\psi\tilde{\chi} - (e^{-2\phi} + \chi^2 + \tilde{\chi}^2)\chi}{4\psi^2 + [e^{-2\phi} + \chi^2 + \tilde{\chi}^2]^2}, \\
 \tilde{\chi} &\mapsto \frac{2\psi\chi + (e^{-2\phi} + \chi^2 + \tilde{\chi}^2)\tilde{\chi}}{4\psi^2 + [e^{-2\phi} + \chi^2 + \tilde{\chi}^2]^2}, \\
 \psi &\mapsto -\frac{\psi}{4\psi^2 + [e^{-2\phi} + \chi^2 + \tilde{\chi}^2]^2}.
 \end{aligned}$$

It is straightforward to check that the required compensating transformation in this case indeed belongs to the maximal compact subgroup $SU(2) \times U(1)$.

2.5. The Picard modular group

We finally discuss the Picard modular group $SU(2, 1; \mathbb{Z}[i])$. This group can be defined as the intersection [57]

$$(2.41) \quad SU(2, 1; \mathbb{Z}[i]) := SU(2, 1) \cap SL(3, \mathbb{Z}[i]),$$

where $\mathbb{Z}[i]$ denotes the Gaussian integers

$$(2.42) \quad \mathbb{Z}[i] = \{z \in \mathbb{C} : \Re(z), \Im(z) \in \mathbb{Z}\} = \{m_1 + im_2 : m_1, m_2 \in \mathbb{Z}\}.$$

This definition implies that any element $g \in SU(2, 1)$ which has only Gaussian integer matrix entries belongs to $SU(2, 1; \mathbb{Z}[i])$. In view of the discussion of $PU(2, 1) \cong SU(2, 1)$ the Picard modular group can also be called $PU(2, 1; \mathbb{Z}[i])$.

Let us now examine the particular $SU(2, 1)$ -transformations of the previous subsection to check whether they belong to the Picard group. The Heisenberg group $N \subset SU(2, 1)$ contains a subgroup $N(\mathbb{Z}) := N \cap SU(2, 1; \mathbb{Z}[i])$. By inspection of Equation (2.32) we see that $N(\mathbb{Z})$ must be of the form

$$(2.43) \quad N(\mathbb{Z}) = \{e^{aX_1 + b\tilde{X}_1 + cX_2} : a, b, c \in \mathbb{Z}\}.$$

In view of (2.32), a natural set of generators for $N(\mathbb{Z})$ is given by the three matrices in (2.31) T_1 , \tilde{T}_1 and T_2 . The action of these discrete shifts are then

as given in (2.34) with parameters $a, b, c \in \mathbb{Z}$. The translations (2.31) are of infinite order in the Picard modular group.

The rotation R defined in (2.35) is an element of order 4 in the Picard modular group, and R^σ belongs to $SU(2, 1; \mathbb{Z}[i])$ for the discrete values of the exponent $\sigma = 0, 1, 2, 3$. The action of R on the scalar fields is

$$(2.44) \quad R : (\chi, \tilde{\chi}) \mapsto (-\tilde{\chi}, \chi),$$

while leaving ϕ and ψ invariant. Physically speaking, this corresponds to electric–magnetic duality, which is expected to be preserved in the quantum theory [47].

Finally, we will examine the involution S in Equation (2.38). Clearly, this involution is an element (of order 2) in the Picard modular group. As already noted above, the involution (2.38) corresponds to the Weyl reflection of the restricted root system BC_1 of the non-split real form $\mathfrak{su}(2, 1)$. This reflection is associated with the long root 2α . In this context, we can also give an interpretation of the rotation R . This is a transformation that rotates within the degenerate, two-dimensional α root space, spanned by the generators X_1 and \tilde{X}_1 .

The Picard modular group acts discontinuously on the complex hyperbolic space $\mathbb{C}\mathbb{H}^2$. A fundamental domain for its action has been given by Francsics and Lax in [57]. Recently, together with Falbel and Parker, they have also proven that the Picard modular group $SU(2, 1; \mathbb{Z}[i])$ is generated by the translations T_1 and T_2 , together with the rotation R and the involution S [56].¹⁵

Since the two translations T_1 and \tilde{T}_1 are related through “electric–magnetic duality” by $\tilde{T}_1 = RT_1R^{-1}$, one may equivalently choose either of the translations T_1 or \tilde{T}_1 associated with the α root space in the theorem. Since all three translations T_1 , \tilde{T}_1 and T_2 will turn out to have a clear physical interpretation, we present the Picard modular group as generated (non-minimally) by the following five elements:

$$(2.45) \quad T_1 = \begin{pmatrix} 1 & -1+i & i \\ 0 & 1 & 1-i \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{T}_1 = \begin{pmatrix} 1 & 1+i & i \\ 0 & 1 & 1+i \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R = \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix}.$$

¹⁵We are very grateful to G. Francsics and P. Lax for communicating this result to us prior to publication.

3. Eisenstein series for the Picard modular group

In this section we shall construct Eisenstein series for the Picard modular group in the principal continuous series representation of $SU(2, 1)$. We shall give three different constructions, which, despite being equivalent, mutually enlighten each other. In Section 3.1 we construct a manifestly $SU(2, 1; \mathbb{Z}[i])$ -invariant function on $SU(2, 1)/(SU(2) \times U(1))$ by summing over points in the three-dimensional Gaussian lattice $\mathbb{Z}[i]^3$. This produces a non-holomorphic Eisenstein series \mathcal{E}_s , parametrized by s , which will be the central object of study in the remainder of this paper. In Section 3.2, we use the isomorphism between the coset space $SU(2, 1)/(SU(2) \times U(1))$ and the complex upper half plane $\mathbb{C}\mathbb{H}^2$ to construct a Poincaré series \mathcal{P}_s on $\mathbb{C}\mathbb{H}^2$. This turns out to be identical to \mathcal{E}_s up to an s -dependent Dedekind zeta function factor. For completeness, in Appendix B we give a third construction using standard adelic techniques, which illuminates the representation-theoretic nature of \mathcal{E}_s .

3.1. Lattice construction and quadratic constraint

Following [13], a non-holomorphic function on the double quotient

$$(3.1) \quad SU(2, 1; \mathbb{Z}[i]) \backslash SU(2, 1) / (SU(2) \times U(1))$$

can be constructed as the Eisenstein series¹⁶

$$\begin{aligned} \mathcal{E}_s(\mathcal{K}) &:= \sum'_{\substack{\vec{\omega} \in \mathbb{Z}[i]^3 \\ \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = 0}} \left[\vec{\omega}^\dagger \cdot \mathcal{K} \cdot \vec{\omega} \right]^{-s} \\ (3.2) \quad &= \sum'_{\substack{\vec{\omega} \in \mathbb{Z}[i]^3 \\ \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = 0}} e^{-2s\phi} \left[|\omega_1 + \omega_2\lambda + \omega_3\gamma|^2 + e^{-2\phi} |\omega_2 + i\omega_3\bar{\lambda}|^2 + e^{-4\phi} |\omega_3|^2 \right]^{-s}, \end{aligned}$$

where $\mathcal{K} = \mathcal{V}\mathcal{V}^\dagger$ is the “generalized metric” (2.24), and the variables λ and γ were defined as functions of $\mathcal{Z} = (z_1, z_2)$ in (2.25). In (3.2) the sum runs over 3-vectors of Gaussian integers $\vec{\omega} \neq (0, 0, 0)$ subject to the quadratic constraint

$$(3.3) \quad \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = |\omega_2|^2 - 2\Im(\omega_1\bar{\omega}_3) = 0, \quad \vec{\omega} := \begin{pmatrix} \bar{\omega}_3 \\ \bar{\omega}_2 \\ \bar{\omega}_1 \end{pmatrix}.$$

¹⁶We note that the same summand and constraint appear in the analysis of [68].

Setting

$$(3.4) \quad \omega_1 = m_1 + im_2, \quad \omega_2 = n_1 + in_2, \quad \omega_3 = p_1 + ip_2,$$

this may be rewritten as a sum over six integers m_i, n_i, p_i , not all vanishing, subject to the constraint

$$(3.5) \quad \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = n_1^2 + n_2^2 + 2m_1p_2 - 2m_2p_1 = 0.$$

The Eisenstein series defined in (3.2) converges absolutely for $\Re(s) > 2$.

To explain the role of the quadratic constraint (3.3), it is convenient to utilize the isomorphism between the coset space $SU(2, 1)/(SU(2) \times U(1))$ and the complex hyperbolic space $\mathbb{C}\mathbb{H}^2$, as discussed in Section 2.3. We recall from (2.30) that in terms of the variable $\mathcal{Z} = (z_1, z_2) \in \mathbb{C}\mathbb{H}^2$, the matrix \mathcal{K} reads

$$(3.6) \quad \mathcal{K} = \tilde{\mathcal{K}} + \eta,$$

where η is the $SU(2, 1)$ -invariant metric, Equation (2.2), and the matrix $\tilde{\mathcal{K}}$ is given by

$$(3.7) \quad \tilde{\mathcal{K}} = e^{2\phi} \begin{pmatrix} |z_1|^2 & z_1\bar{z}_2 & z_1 \\ \bar{z}_1z_2 & |z_2|^2 & z_2 \\ \bar{z}_1 & \bar{z}_2 & 1 \end{pmatrix} = \tilde{\mathcal{V}}\tilde{\mathcal{V}}^\dagger \quad \text{for} \quad \tilde{\mathcal{V}} = e^\phi \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this new parametrization, the Eisenstein series becomes

$$(3.8) \quad \begin{aligned} \mathcal{E}_s(\mathcal{Z}) &= \sum'_{\substack{\vec{\omega} \in \mathbb{Z}[i]^3 \\ \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = 0}} \left[\vec{\omega}^\dagger \cdot \tilde{\mathcal{K}} \cdot \vec{\omega} + \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} \right]^{-s} \\ &= \sum'_{\substack{\vec{\omega} \in \mathbb{Z}[i]^3 \\ \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = 0}} e^{-2s\phi} |\omega_1 + \omega_2 z_2 + \omega_3 z_1|^{-2s}. \end{aligned}$$

The constraint (3.3) can now be motivated as follows [13]. Since the coset representative $\mathcal{V} \in SU(2, 1)/(SU(2) \times U(1))$ transforms in the fundamental representation \mathcal{R} of $SU(2, 1)$, the generalized metric $\mathcal{K} = \mathcal{V}\mathcal{V}^\dagger$ transforms in the symmetric tensor product $\mathcal{R} \otimes_s \mathcal{R}$. As reflected in (3.6), this tensor product is not irreducible. In order for \mathcal{E}_s to be an eigenfunction of the Laplacian on $\mathbb{C}\mathbb{H}^2$, it is necessary to project out the singlet component in (3.6), hence to enforce the constraint (3.3) in the sum. To be specific, the

Laplacian on the coset space $\mathbb{C}\mathbb{H}^2$, written in terms of the real variables $\{y = e^{-2\phi}, \chi, \tilde{\chi}, \psi\}$, is given by

$$(3.9) \quad \begin{aligned} \Delta_{\mathbb{C}\mathbb{H}^2} = & \frac{1}{4}y(\partial_\chi^2 + \partial_{\tilde{\chi}}^2) + \frac{1}{4}(y^2 + y(\chi^2 + \tilde{\chi}^2))\partial_\psi^2 \\ & + \frac{1}{2}y(\tilde{\chi}\partial_\chi - \chi\partial_{\tilde{\chi}})\partial_\psi + y^2\partial_y^2 - y\partial_y. \end{aligned}$$

Taking into account the quadratic constraint (3.3), it is straightforward to check that \mathcal{E}_s is an eigenvector of the Laplacian with eigenvalue $s(s - 2)$, as stated in (1.13) above.

Since $SU(2, 1)$ admits two Casimir operators of degree 2 and 3, and since $\Delta_{\mathbb{C}\mathbb{H}^2}$ represents the action of the quadratic Casimir on the space of (square-integrable) functions on $SU(2, 1)/(SU(2) \times U(1))$, one may ask whether \mathcal{E}_s is also an eigenvector of an invariant differential operator of degree 3. It turns out however, as already noticed in [63], that the principal representation of the cubic Casimir in the space of functions on $SU(2, 1)/(SU(2) \times U(1))$ vanishes identically. In terms of the parametrization of the Casimir eigenvalues by the complex variables (p, q) used in [63, 69], the Eisenstein series \mathcal{E}_s is attached to the principal spherical representation with $p = q = s - 2$ (see Appendix B for some details on the principal series of $SU(2, 1)$).

Let us also comment on the functional dimension of the representation associated to \mathcal{E}_s . The summation ranges over six (real) integers coordinating the lattice $\mathbb{Z}[i]^3 \sim \mathbb{Z}^6$. Since both the summand and the constraint are homogeneous in $\vec{\omega}$ one can factor out an overall common Gaussian integer. Among the remaining four real integers the (real) quadratic constraint $|\omega_2|^2 - 2\Im(\omega_1\bar{\omega}_3) = 0$ eliminates one of the summation variables, leaving effectively a sum over three integers only. This is consistent with the functional dimension 3 of the principal continuous series representation of $SU(2, 1)$ and the number of expected different instanton contributions.

3.2. Poincaré series on the complex upper half plane

In the mathematical literature, a standard way of constructing non-holomorphic Eisenstein series on a symmetric space G/K is in terms of Poincaré series. For the case of the coset space $SL(2, \mathbb{R})/SO(2)$, parametrized by a complex coordinate τ , such a Poincaré series is obtained by summing the function $\Im(\gamma \cdot \tau)^s$ over the orbit $\gamma \in \Gamma_\infty \backslash SL(2, \mathbb{Z})$, where Γ_∞ is generated by $T : \tau \mapsto \tau + 1$. This indeed produces a non-holomorphic Eisenstein series on the double quotient $SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R})/SO(2)$ with eigenvalue $s(s - 1)$ under the Laplacian on $SL(2, \mathbb{R})/SO(2)$ (for a very nice treatment, see [70]).

Here we generalize this construction to the case of the complex upper half plane $\mathbb{C}\mathbb{H}^2$, parametrized by the variable $\mathcal{Z} = (z_1, z_2)$. The generalization of $\mathfrak{S}(\tau)$ is then given by the $N(\mathbb{Z})$ -invariant function $\mathcal{F}(\mathcal{Z})$, constructed in (2.17) [71].¹⁷ The invariance of $\mathcal{F}(\mathcal{Z})$ under $N(\mathbb{Z})$ can be checked by direct substitution of the Heisenberg translations in Equation (2.33). As we have seen in Section 2, the Picard modular group $SU(2, 1; \mathbb{Z}[i])$ acts by fractional transformations on $\mathcal{Z} \in \mathbb{C}\mathbb{H}^2$ such that the function $\mathcal{F}(\mathcal{Z})$ transforms as

$$(3.10) \quad \mathcal{F}(\gamma \cdot \mathcal{Z}) = \frac{\mathcal{F}(\mathcal{Z})}{|C\mathcal{Z} + D|^2}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2, 1; \mathbb{Z}[i]).$$

A Poincaré series for the Picard group may now be constructed as follows:

$$(3.11) \quad \mathcal{P}_s(\mathcal{Z}) := \sum_{\gamma \in N(\mathbb{Z}) \backslash SU(2, 1; \mathbb{Z}[i])} \mathcal{F}(\gamma \cdot \mathcal{Z})^s = \sum_{\gamma \in N(\mathbb{Z}) \backslash SU(2, 1; \mathbb{Z}[i])} \left(\frac{\mathcal{F}(\mathcal{Z})}{|C\mathcal{Z} + D|^2} \right)^s.$$

Taking $C := (\omega_3, \omega_2) \in \mathbb{Z}[i]^2$ and $D := \omega_1 \in \mathbb{Z}[i]$, and recalling that $\mathcal{F}(\mathcal{Z}) = e^{-2\phi}$, then reproduces the same form of the Eisenstein series as in Equation (3.8), i.e.,

$$(3.12) \quad \mathcal{P}_s(\mathcal{Z}) = \sum_{\gamma \in N(\mathbb{Z}) \backslash SU(2, 1; \mathbb{Z}[i])} e^{-2s\phi} |\omega_1 + \omega_2 z_2 + \omega_3 z_1|^{-2s}.$$

The sum over orbits in $N(\mathbb{Z}) \backslash SU(2, 1; \mathbb{Z}[i])$ is equivalent to the sum over the Gaussian lattice $\mathbb{Z}[i]^3$ modulo the constraint $\vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = 0$, together with a coprime condition on the summation variables $\vec{\omega}$ [66]:

$$(3.13) \quad \mathcal{P}_s(\mathcal{Z}) = \sum'_{\substack{\vec{\omega} \in \mathbb{Z}[i]^3, \text{gcd}(\omega'_1, \omega'_2, \omega'_3) = 1 \\ \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega}' = 0}} e^{-2s\phi} |\omega'_1 + \omega'_2 z_2 + \omega'_3 z_1|^{-2s}.$$

Defining $\vec{\omega} = \vec{\omega}'\beta$ with $\beta = \text{gcd}(\omega_1, \omega_2, \omega_3) \in \mathbb{Z}[i]$ and inserting this into (3.2) we then have the relation

$$(3.14) \quad \mathcal{E}_s(\phi, \lambda, \gamma) = 4\zeta_{\mathbb{Q}(i)}(s)\mathcal{P}_s(\mathcal{Z}),$$

where $\zeta_{\mathbb{Q}(i)}(s)$ is the Dedekind zeta function for the quadratic extension $\mathbb{Q}(i)$ of the rational numbers, and the overall factor of 4 originates from the four units in $\mathbb{Z}[i]$. This will be discussed in more detail in Section 4.2 (see Equation (4.15)).

¹⁷We are grateful to Genkai Zhang for helpful discussions on this construction.

4. Fourier expansion of $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$

In this section we compute the Fourier expansion of the Eisenstein series (3.2). We begin by recalling the general decomposition with respect to the action of the Heisenberg subgroup $N \subset SU(2, 1)$.

4.1. General structure of the non-Abelian Fourier expansion

The main complication of the Fourier expansion stems from the non-Abelian nature of the nilpotent group $N \subset SU(2, 1)$. N is isomorphic to a three-dimensional Heisenberg group, where the center $Z = [N, N]$ is parametrized by ψ . The Fourier expansion therefore splits into an Abelian part and a non-Abelian part. The Abelian term corresponds to an expansion with respect to the abelianized group N/Z , while the non-Abelian terms represent the expansion with respect to the center Z . This general structure of the Fourier expansion of automorphic forms for the Picard modular group is discussed in detail by Ishikawa [64], to which we refer the interested reader. A similar discussion may also be found in the mathematics [72, 73] and physics [34] literature for the case of automorphic forms on $SL(3, \mathbb{R})/SO(3)$.

We have seen in Section 2 that the action of an arbitrary Heisenberg shift $\mathcal{U}_{a,b;c} \in N(\mathbb{Z}) = N \cap SU(2, 1; \mathbb{Z}[i])$ on $\chi, \tilde{\chi}$ and ψ is given by

$$\begin{aligned}
 \mathcal{U}_{a,b;c} : \chi &\longmapsto \chi + a, \\
 &\tilde{\chi} \longmapsto \tilde{\chi} + b, \\
 &\psi \longmapsto \psi + \frac{1}{2}c - a\tilde{\chi} + b\chi
 \end{aligned}
 \tag{4.1}$$

for $a, b, c \in \mathbb{Z}$. Since the Eisenstein series (3.2) is in particular invariant under $N(\mathbb{Z})$ we can organize the Fourier expansion by diagonalizing different subgroups of the non-Abelian Heisenberg group $N(\mathbb{Z})$.

Explicitly, we write the general form of the Fourier expansion as

$$\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi) = \mathcal{E}_s^{(\text{const})}(\phi) + \mathcal{E}_s^{(\text{A})}(\phi, \chi, \tilde{\chi}) + \mathcal{E}_s^{(\text{NA})}(\phi, \chi, \tilde{\chi}, \psi),
 \tag{4.2}$$

where $\mathcal{E}_s^{(\text{const})}(\phi)$ is the constant term and

$$\begin{aligned}
 \mathcal{E}_s^{(\text{A})}(\phi, \chi, \tilde{\chi}) &= \sum'_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \mathfrak{e}_{\ell_1, \ell_2}^{(\text{A})}(\phi; s) e^{-2\pi i(\ell_1 \chi + \ell_2 \tilde{\chi})}, \\
 \mathcal{E}_s^{(\text{NA})}(\phi, \chi, \tilde{\chi}, \psi) &= \sum'_{k \in \mathbb{Z}} \mathfrak{e}_k^{(\text{NA})}(\phi, \chi, \tilde{\chi}; s) e^{-4\pi i k \psi}
 \end{aligned}
 \tag{4.3}$$

are called the Abelian and non-Abelian terms, respectively. Following [34, 64], we proceed to extract an additional phase factor in the non-Abelian term, which accounts for the shifts of ψ along the non-central directions. This yields the following structure of the non-Abelian term:

$$(4.4) \quad \mathcal{E}_s^{(\text{NA})}(\phi, \chi, \tilde{\chi}, \psi) = \sum'_{k \in \mathbb{Z}} \sum_{\ell=0}^{4|k|-1} \sum_{n \in \mathbb{Z} + \frac{\ell}{4|k|}} \mathfrak{C}_{k,\ell}^{(\text{NA})}(\phi, \tilde{\chi} - n; s) e^{8\pi i k n \chi - 4\pi i k(\psi + \chi \tilde{\chi})}.$$

The Abelian term is manifestly invariant under shifts of the form $\mathcal{U}_{a,b;0} \in N(\mathbb{Z})/Z$. For the non-Abelian term, invariance under

$$(4.5) \quad \begin{aligned} \mathcal{U}_{1,0;0} : \chi &\longmapsto \chi + 1 \\ &: \psi \longmapsto \psi - \tilde{\chi} \end{aligned}$$

is manifest since $4kn \in \mathbb{Z}$. On the other hand, the transformation

$$(4.6) \quad \begin{aligned} \mathcal{U}_{0,1;0} : \tilde{\chi} &\longmapsto \tilde{\chi} + 1 \\ &: \psi \longmapsto \psi + \chi \end{aligned}$$

requires a compensating shift $n \mapsto n + 1$ on the summation, under which the variation of the total phase cancels. Note also the restricted dependence on $\tilde{\chi}$ in the Fourier coefficient; upon shifting $\tilde{\chi} \mapsto \tilde{\chi} + 1$ and compensating $n \mapsto n + 1$ the coefficient is indeed invariant. Finally, invariance under $\mathcal{U}_{0,0;1}$ is manifest since this gives an overall phase $e^{-4\pi i k/2} = 1$.

Note that in writing the non-Abelian term (4.4) we have made an explicit choice of polarization, in the sense that we have manifestly diagonalized the action of Heisenberg shifts of the restricted form $\mathcal{U}_{a,0;c}$. We could have chosen the opposite polarization in which we instead diagonalize the action of $\mathcal{U}_{0,b;c}$. In this case, the non-Abelian term reads

$$(4.7) \quad \mathcal{E}_s^{(\text{NA})}(\phi, \chi, \tilde{\chi}, \psi) = \sum'_{k \in \mathbb{Z}} \sum_{\ell'=0}^{4|k|-1} \sum_{\tilde{n} \in \mathbb{Z} + \frac{\ell'}{4|k|}} \tilde{\mathfrak{C}}_{k,\ell'}^{(\text{NA})}(\phi, \chi - \tilde{n}; s) e^{-8\pi i k \tilde{n} \tilde{\chi} - 4\pi i k(\psi - \chi \tilde{\chi})}.$$

The Fourier coefficients $\mathfrak{C}_{k,\ell}^{(\text{NA})}$ and $\tilde{\mathfrak{C}}_{k,\ell'}^{(\text{NA})}$ in the two different polarizations are related via a Fourier transform (see [34]). In the sequel we work for definiteness with the first polarization defined by (4.4).

Besides invariance under the Heisenberg group we can also use invariance under the electric-magnetic duality transformation $R : (\chi, \tilde{\chi}) \mapsto (-\tilde{\chi}, \chi)$ of (2.44). On the Abelian term this implies that the coefficient $\mathfrak{C}_{\ell_1, \ell_2}^{(A)}$ is invariant under $\pi/2$ rotations of (ℓ_1, ℓ_2) . On the non-Abelian term (4.4) application of R leads to

$$(4.8) \quad \mathcal{E}_s^{(\text{NA})}(\phi, \chi, \tilde{\chi}, \psi) = \sum'_{k \in \mathbb{Z}} \sum_{\ell=0}^{4|k|-1} \sum_{n \in \mathbb{Z} + \frac{\ell}{4|k|}} \mathfrak{C}_{k, \ell}^{(\text{NA})}(\phi, \chi - n; s) e^{-8\pi i k n \tilde{\chi} - 4\pi i k (\psi - \chi \tilde{\chi})}$$

and hence we have $\mathfrak{C}_{k, \ell}^{(\text{NA})} = \tilde{\mathfrak{C}}_{k, \ell}^{(\text{NA})}$, relating the two choices of polarization as to be expected from electric-magnetic duality. Applying R again leads to relations among the coefficients $\mathfrak{C}_{k, \ell}^{(\text{NA})}$ and $\mathfrak{C}_{k, \ell'}^{(\text{NA})}$ for different ℓ and ℓ' .

Finally, we can use the Laplacian condition on the Eisenstein series \mathcal{E}_s (see Equation (1.13)) to further constrain the Fourier coefficients $\mathfrak{C}_{\ell_1, \ell_2}^{(A)}$ and $\mathfrak{C}_{k, \ell}^{(\text{NA})}$ and determine their functional dependence on the moduli. In all cases, we require normalizability of the solution, which physically means a well-behaved “weak-coupling” limit $e^\phi \rightarrow 0$. Plugging in the Abelian term $\mathcal{E}_s^{(A)}$ into the eigenvalue equation (1.13) yields an equation for the ϕ -dependence of the coefficients, which is solved by a modified Bessel function. More precisely, we find that the Abelian term in the expansion takes the form

$$(4.9) \quad \begin{aligned} \mathcal{E}_s^{(A)}(\phi, \chi, \tilde{\chi}, \psi) &= e^{-2\phi} \sum'_{(\ell_1, \ell_2) \in \mathbb{Z}^2} C_{\ell_1, \ell_2}^{(A)}(s) \\ &\times K_{2s-2} \left(2\pi e^{-\phi} \sqrt{\ell_1^2 + \ell_2^2} \right) e^{-2\pi i (\ell_1 \chi + \ell_2 \tilde{\chi})}, \end{aligned}$$

where the remaining coefficients $C_{\ell_1, \ell_2}^{(A)}(s)$ are now independent of ϕ and encode the arithmetic information of the group $SU(2, 1; \mathbb{Z}[i])$. The precise form of these numerical coefficients will be computed in Section 4.5 below.

Turning to the non-Abelian term (4.4), the Laplacian condition on the coefficient separates into a harmonic oscillator equation in the variable $x = \tilde{\chi} - n$, with solution given by a Hermite polynomial H , as well as a hypergeometric equation in the variable $y = e^{-2\phi}$ whose solution can be written in terms of a Whittaker function W . The separation of variables induces a sum over the eigenvalues of the harmonic oscillator, leading to the following

structure for the non-Abelian term:

$$\begin{aligned}
 \mathcal{E}_s^{(\text{NA})}(\phi, \chi, \tilde{\chi}, \psi) &= e^{-\phi} \sum'_{k \in \mathbb{Z}} \sum_{\ell=0}^{4|k|-1} \sum_{n \in \mathbb{Z} + \frac{\ell}{4|k|}} \sum_{r=0}^{\infty} C_{r,k,\ell}^{(\text{NA})}(s) |k|^{1/2-s} e^{-4\pi|k|(\tilde{\chi}-n)^2} \\
 &\times H_r\left(\sqrt{8\pi|k|}(\tilde{\chi}-n)\right) W_{-r-\frac{1}{2}, s-1}\left(4\pi|k|e^{-2\phi}\right) \\
 (4.10) \quad &\times e^{8\pi i k n \chi - 4\pi i k(\psi + \chi \tilde{\chi})},
 \end{aligned}$$

where the numerical coefficients $C_{r,k,\ell}^{(\text{NA})}(s)$ will be further discussed in Section 4.6.

We shall now proceed to compute the explicit form of the Fourier expansion; that is, determine the constant term $\mathcal{E}_s^{(\text{const})}$ as well as the Abelian and non-Abelian numerical Fourier coefficients $C_{\ell_1, \ell_2}^{(\text{A})}(s)$ and $C_{r,k,\ell}^{(\text{NA})}(s)$.

4.2. First constant term

The constant term¹⁸ is defined generally as

$$(4.11) \quad \mathcal{E}_s^{(\text{const})}(\phi) = \int_0^1 d\chi \int_0^1 d\tilde{\chi} \int_0^{1/2} d\psi \mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi),$$

where the integral over the coordinate ψ (physically, the NS-axion modulus) runs from 0 to 1/2 because of the extra factor of 2 in front of ψ in our parametrization of N in Equation (2.22). Since the Cartan subgroup A appearing in the Iwasawa decomposition of $SU(2, 1)$ is one-dimensional, the constant term only depends on the dilatonic scalar ϕ . Moreover, recall from the discussion in Section 2.4 that the Weyl group of $\mathfrak{su}(2, 1)$ is the Weyl group of the restricted root system BC_1 , which is isomorphic with \mathbb{Z}_2 . Hence, the constant term $\mathcal{E}_s^{(\text{const})}(\phi)$ consists of two contributions, $\mathcal{E}_s^{(0)}$ and $\mathcal{E}_s^{(1)}$, which are permuted by \mathbb{Z}_2 [65].¹⁹

The powers of e^ϕ in $\mathcal{E}_s^{(\text{const})}(\phi)$ may be determined by the Laplacian condition on \mathcal{E}_s . In Section 3.2 we have seen that the Eisenstein series is an eigenfunction of the Laplacian $\Delta_{\mathbb{C}\mathbb{H}^2}$ with eigenvalue $s(s-2)$. This implies that all the constant terms must individually be eigenfunctions of $\Delta_{\mathbb{C}\mathbb{H}^2}$ with

¹⁸The terminology constant term is derived from holomorphic Eisenstein series where these terms are truly constant and independent of the scalar fields. For non-holomorphic Eisenstein series, as the one studied here, the constant terms retain a dependence on the fields corresponding to Cartan generators.

¹⁹We are grateful to Pierre Vanhove for helpful discussions on the constant terms.

the same eigenvalue. It turns out that there is a unique solution to this, and we find that $\mathcal{E}_s^{(0)}$ must be of the form

$$(4.12) \quad \mathcal{E}_s^{(\text{const})}(\phi) = \mathcal{E}_s^{(0)} + \mathcal{E}_s^{(1)} = A(s)e^{-2s\phi} + B(s)e^{-2(2-s)\phi}.$$

Below we will compute the coefficients $A(s)$ and $B(s)$. The first constant term $\mathcal{E}_s^{(0)}$ corresponds to the leading order term in an expansion at the cusp $e^\phi \rightarrow 0$, which physically corresponds to the regime of weak coupling.

Our strategy for performing the Fourier expansion is as follows: we first consider the term $\omega_3 = 0$, which by virtue of the constraint (3.5) also requires $\omega_2 = 0$. The remaining sum over $\omega_1 \neq 0$ yields the first constant term $\mathcal{E}_s^{(0)}$. We then consider the case $\omega_3 \neq 0$ and solve the constraint (3.5) explicitly using the Euclidean algorithm, which reduces the remaining sum to one over three integers. On these integers we will perform Poisson resummations to uncover the second constant term, as well the Abelian and non-Abelian Fourier coefficients.

Accordingly, we start by extracting the $\omega_3 = 0$ (implying $\omega_2 = 0$) part of the sum in the Eisenstein series, $\mathcal{E}_s^{(0)}$, leaving a remainder $\mathcal{A}^{(s)}$

$$(4.13) \quad \mathcal{E}_s(\phi, \lambda, \gamma) = \mathcal{E}_s^{(0)} + \mathcal{A}^{(s)}.$$

The first term $\mathcal{E}_s^{(0)}$ is the leading order contribution in the limit $e^\phi \rightarrow 0$ and corresponds to a sum over $\omega_1 = m_1 + im_2$

$$(4.14) \quad \mathcal{E}_s^{(0)} = e^{-2s\phi} \sum'_{(m_1, m_2) \in \mathbb{Z}^2} \frac{1}{(m_1^2 + m_2^2)^s} = 4\zeta_{\mathbb{Q}(i)}(s)e^{-2s\phi},$$

where $\zeta_{\mathbb{Q}(i)}(s)$ is the Dedekind zeta function over the Gaussian integers

$$(4.15) \quad \zeta_{\mathbb{Q}(i)}(s) = \frac{1}{4} \sum'_{\omega \in \mathbb{Z}[i]} |\omega|^{-2s} = \frac{1}{4} \sum'_{(m, n) \in \mathbb{Z}^2} \frac{1}{(m^2 + n^2)^s}.$$

The factor of 4 is related to the units of the Gaussian integers (see Appendix A).

The Dedekind zeta function $\zeta_{\mathbb{Q}(i)}(s)$ satisfies a functional equation, which is most conveniently written in terms of the “completed Dedekind zeta function”

$$(4.16) \quad \zeta_{\mathbb{Q}(i)*}(s) := \pi^{-s} \Gamma(s) \zeta_{\mathbb{Q}(i)}(s),$$

in terms of which one has

$$(4.17) \quad \zeta_{\mathbb{Q}(i)*}(1-s) = \zeta_{\mathbb{Q}(i)*}(s).$$

It is known that the Dedekind function over a quadratic number field can be written as a Dirichlet L-function times the standard Riemann zeta function. In our case this reads (see, e.g., [74] for a proof)

$$(4.18) \quad \zeta_{\mathbb{Q}(i)}(s) = \beta(s)\zeta(s),$$

where the standard Riemann zeta function is defined as

$$(4.19) \quad \zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \quad \text{for } \Re(s) > 1$$

and $\beta(s)$ is the Dirichlet beta function,²⁰

$$(4.20) \quad \beta(s) := \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s} \quad \text{for } \Re(s) > 0.$$

We also note that $\beta(s)$ has an Euler product representation of the form

$$(4.21) \quad \beta(s) = \prod_{p: p \equiv 1 \pmod{4}} \frac{1}{1-p^{-s}} \prod_{p: p \equiv 3 \pmod{4}} \frac{1}{1+p^{-s}},$$

which together with the Euler product form of the Riemann zeta function $\zeta(s)$ above will be useful later. The functional relation for $\beta(s)$ is again best stated using its completion

$$(4.22) \quad \beta_*(s) := \left(\frac{\pi}{4}\right)^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) \beta(s),$$

for which the functional relation takes the simple form

$$(4.23) \quad \beta_*(s) = \beta_*(1-s).$$

In conclusion, we have found that the first coefficient $A(s)$ in (4.12) is given by the Dedekind function $\zeta_{\mathbb{Q}(i)}(s)$ and that it is related to the term $\omega_3 = 0$ in the sum over the Gaussian integers. We will now proceed to evaluate the terms with $\omega_3 \neq 0$, contained in $\mathcal{A}^{(s)}$ of (4.13). We emphasize that the term with $\omega_3 = 0$ and $\omega_2 \neq 0$ vanishes identically because of the quadratic constraint (3.5). Thus, $\mathcal{A}^{(s)}$ only contains terms for which $\omega_3 \neq 0$.

²⁰The Dirichlet beta function is also known as $L(\chi_{-4}, s)$, i.e., it is the L-function associated with the alternating character modulo 4.

4.3. Solution of constraint and Poisson resummation

To solve the constraint (3.5) we shall make use of the Euclidean algorithm, which implies that for integers p_1 and p_2 the equation

$$(4.24) \quad q_1 p_2 - q_2 p_1 = d$$

has integer solutions for q_1 and q_2 if and only if d divides $\gcd(p_1, p_2)$. The most general solution is the sum of a particular solution (q_1, q_2) plus an integer times (p_1, p_2) . More precisely, in the case of our constraint (3.5) we find that for $\omega_3 = p_1 + ip_2 \neq 0$ there are solutions in $\mathbb{Z}[i]^3$ if and only if

$$(4.25) \quad \frac{|\omega_2|^2}{2d} \in \mathbb{Z}, \quad \text{where } d = \gcd(p_1, p_2)$$

and the most general solution for $\omega_1 = m_1 + im_2$ is then

$$(4.26) \quad \begin{aligned} m_1 &= -\frac{|\omega_2|^2}{2d} q_1 + m \frac{p_1}{d}, \\ m_2 &= -\frac{|\omega_2|^2}{2d} q_2 + m \frac{p_2}{d}. \end{aligned}$$

Here, q_1 and q_2 is any particular solution of $q_1 p_2 - q_2 p_1 = d$ and $m \in \mathbb{Z}$ is an *unconstrained* integer. Therefore, we can rewrite the constrained sum as

$$(4.27) \quad \sum_{\omega_3 \neq 0} \sum_{\substack{\omega_2 \in \mathbb{Z}[i] \\ 2d \mid |\omega_2|^2}} \sum_{m \in \mathbb{Z}} (\dots)$$

where in the summand, $\omega_1 = m_1 + im_2$ has to be replaced by the expression from (4.26).

Let us implement this procedure on our Eisenstein series. After solving the constraint, the first term in the bracket of (3.2) becomes

$$(4.28) \quad \begin{aligned} |\omega_1 + \omega_2 \lambda + \omega_3 \gamma|^2 &= \frac{|\omega_3|^2}{d^2} \left[\left(m - \frac{|\omega_2|^2}{2|\omega_3|^2} (q_1 p_1 + q_2 p_2) + \tilde{\ell}_1 \chi + \tilde{\ell}_2 \tilde{\chi} + 2d\psi \right)^2 \right. \\ &\quad \left. + \frac{1}{16d^2} \left((\tilde{\ell}_1 + 2d\tilde{\chi})^2 + (\tilde{\ell}_2 - 2d\chi)^2 \right)^2 \right], \end{aligned}$$

where we defined

$$(4.29) \quad \begin{aligned} \tilde{\ell}_1 &:= \frac{d}{|\omega_3|^2} \left[(p_1 - p_2)n_1 + (p_1 + p_2)n_2 \right], \\ \tilde{\ell}_2 &:= \frac{d}{|\omega_3|^2} \left[(p_1 + p_2)n_1 - (p_1 - p_2)n_2 \right]. \end{aligned}$$

Extracting an overall factor of $|\omega_3|^2/d^2$, the total summand may be written as

$$(4.30) \quad \begin{aligned} \frac{d^2}{|\omega_3|^2} \omega^\dagger \cdot \mathcal{K} \cdot \omega &= \left[m - \frac{|\omega_2|^2}{2|\omega_3|^2} (q_1 p_1 + q_2 p_2) + \tilde{\ell}_1 \chi + \tilde{\ell}_2 \tilde{\chi} + 2d\psi \right]^2 \\ &+ \frac{e^{-4\phi}}{d^2} \left[d^2 + \frac{e^{2\phi}}{4} \left((\tilde{\ell}_1 + 2d\tilde{\chi})^2 + (\tilde{\ell}_2 - 2d\chi)^2 \right) \right]^2. \end{aligned}$$

Using an integral representation for the summand in the remainder $\mathcal{A}^{(s)}$ defined in (4.13),

$$(4.31) \quad \left[\vec{\omega}^\dagger \cdot \mathcal{K} \cdot \vec{\omega} \right]^{-s} = \frac{\pi^s}{\Gamma(s)} \int \frac{dt}{t^{s+1}} e^{-\frac{\pi}{t} \vec{\omega}^\dagger \cdot \mathcal{K} \cdot \vec{\omega}},$$

and performing a Poisson resummation on m using the standard formula

$$(4.32) \quad \sum_{m \in \mathbb{Z}} e^{-\pi x(m+a)^2 + 2\pi i m b} = \frac{1}{\sqrt{x}} \sum_{\tilde{m} \in \mathbb{Z}} e^{-\frac{\pi}{x}(\tilde{m}+b)^2 - 2\pi i(\tilde{m}+b)a},$$

we obtain

$$(4.33) \quad \begin{aligned} \mathcal{A}^{(s)} &= \frac{\pi^s}{\Gamma(s)} e^{-2s\phi} \sum_{\tilde{m} \in \mathbb{Z}} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{\substack{(n_1, n_2) \in \mathbb{Z}^2 \\ 2d|n_1^2 + n_2^2}} \\ &\times \frac{d}{|\omega_3|} e^{-2\pi i \tilde{m} \left(-\frac{|\omega_2|^2}{2|\omega_3|^2} (q_1 p_1 + q_2 p_2) + \tilde{\ell}_1 \chi + \tilde{\ell}_2 \tilde{\chi} + 2d\psi \right)} \\ &\times \int_0^\infty \frac{dt}{t^{s+1/2}} e^{-\pi t \frac{d^2}{|\omega_3|^2} \tilde{m}^2 - \frac{\pi}{t} \frac{|\omega_3|^2}{d^4} e^{-4\phi} \left[d^2 + \frac{e^{2\phi}}{4} \left((\tilde{\ell}_1 + 2d\tilde{\chi})^2 + (\tilde{\ell}_2 - 2d\chi)^2 \right) \right]^2}, \end{aligned}$$

where we have indicated explicitly the constraint from (4.25) that $2d$ must divide $|\omega_2|^2$.

As we will see in Section 5, the Abelian terms in the Fourier expansion correspond physically to instantons with zero NS5-brane charge, independent of the NS–NS scalar ψ . We therefore split off the Abelian contribution

with $\tilde{m} = 0$:

$$(4.34) \quad \mathcal{A}^{(s)} = \mathcal{D}^{(s)} + \mathcal{E}_s^{(\text{NA})},$$

where $\mathcal{E}_s^{(\text{NA})}$ denotes the non-Abelian term with $\tilde{m} \neq 0$, to be considered later. From $\mathcal{D}^{(s)}$ we will be able to extract the second constant term $\mathcal{E}_s^{(1)}$ as well as the Abelian Fourier coefficients $\mathfrak{E}_{\tilde{\ell}_1, \tilde{\ell}_2}^{(A)}(s)$. Explicitly, we have

$$(4.35) \quad \begin{aligned} \mathcal{D}^{(s)} &= \frac{\pi^s}{\Gamma(s)} e^{-2s\phi} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{\substack{(n_1, n_2) \in \mathbb{Z}^2 \\ 2d|n_1^2 + n_2^2}} \frac{d}{|\omega_3|} \\ &\times \int_0^\infty \frac{dt}{t^{s+1/2}} e^{-\frac{\pi}{t} \frac{e^{-4\phi} |\omega_3|^2}{d^4} \left[d^2 + \frac{e^{2\phi}}{4} ((\tilde{\ell}_1 + 2d\tilde{\chi})^2 + (\tilde{\ell}_2 - 2d\chi)^2) \right]^2}. \end{aligned}$$

To get rid of the square in the exponent, we shall perform the integration over t and then choose a new integral representation of the summand. The current form of the exponent will be convenient for the evaluation of the non-Abelian terms in Section 4.6, but for our present purposes we shall rewrite it in the following way:

$$(4.36) \quad \begin{aligned} &\frac{e^{-4\phi} |\omega_3|^2}{d^4} \left[d^2 + \frac{e^{2\phi}}{4} ((\tilde{\ell}_1 + 2d\tilde{\chi})^2 + (\tilde{\ell}_2 - 2d\chi)^2) \right]^2 \\ &= \frac{1}{4|\omega_3|^2} \left[|\mathcal{Y}|^2 + 2e^{-2\phi} |\omega_3|^2 \right]^2, \end{aligned}$$

where we defined the new variable $\mathcal{Y} = \mathcal{Y}_1 + i\mathcal{Y}_2$, with

$$(4.37) \quad \begin{aligned} \mathcal{Y}_1 &:= n_1 + (p_1 - p_2)\tilde{\chi} - (p_1 + p_2)\chi, \\ \mathcal{Y}_2 &:= n_2 + (p_1 + p_2)\tilde{\chi} + (p_1 - p_2)\chi. \end{aligned}$$

Evaluating the integral over t then yields

$$(4.38) \quad \begin{aligned} \mathcal{D}^{(s)} &= \frac{2^{2s-1} \sqrt{\pi} \Gamma(s - 1/2)}{\Gamma(s)} e^{-2s\phi} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{\substack{(n_1, n_2) \in \mathbb{Z}^2 \\ 2d|n_1^2 + n_2^2}} \frac{d}{|\omega_3|^{2-2s}} \left\{ \left[|\mathcal{Y}|^2 + 2e^{-2\phi} |\omega_3|^2 \right] \right\}^{1-2s}. \end{aligned}$$

After replacing the term within brackets by its integral representation, we obtain

$$\begin{aligned}
 \mathcal{D}^{(s)} &= \frac{(2\pi)^{2s-1}\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)\Gamma(2s-1)} e^{-2s\phi} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{\substack{(n_1, n_2) \in \mathbb{Z}^2 \\ 2d | n_1^2 + n_2^2}} \frac{d}{|\omega_3|^{2-2s}} \\
 (4.39) \quad &\times \int_0^\infty \frac{dt}{t^{2s}} e^{-\frac{\pi}{t} [|\mathcal{Y}|^2 + 2e^{-2\phi} |\omega_3|^2]}.
 \end{aligned}$$

Since all values of n_1 and n_2 are almost degenerate we shall perform a further Poisson resummation on these variables. Here we must take into account the remaining constraint that $2d$ divides $n_1^2 + n_2^2$. The set of solutions to this constraint can be written as

$$(4.40) \quad n_1 = n_1^0 + \delta n_1, \quad n_2 = n_2^0 + \delta n_2,$$

where $\delta n_1 + i\delta n_2$ runs over the lattice L

$$(4.41) \quad L = \{d[(k_1 + k_2) + i(k_1 - k_2)] : (k_1, k_2) \in \mathbb{Z}^2\}$$

and (n_1^0, n_2^0) runs over all solutions of the quadratic equation $n_1^2 + n_2^2 = 0 \pmod{2d}$ in a fundamental domain of $\mathbb{Z}[i]/L$, which we take to be $0 \leq n_1^0 < d$ and $0 \leq n_2^0 < 2d$, with area $2d^2$. We denote the set of such solutions as

$$\begin{aligned}
 (4.42) \quad \mathcal{F}(d) &:= \{n_1^0 + in_2^0 : n_1^2 + n_2^2 = 0 \pmod{2d}, 0 \leq n_1^0 < d, 0 \leq n_2^0 < 2d\},
 \end{aligned}$$

and its cardinality by

$$(4.43) \quad N(d) := \#\mathcal{F}(d).$$

As we shall discuss below and in Appendix C, the series $N(d)$ is multiplicative though not completely multiplicative (see [75] for the first few values).

After inserting (4.40) into (4.39) and performing a Poisson resummation on δn_1 and δn_2 , we obtain

$$\begin{aligned}
 \mathcal{D}^{(s)} &= \frac{(2\pi)^{2s-1}\sqrt{\pi}\Gamma(s-1/2)}{2\Gamma(s)\Gamma(2s-1)} e^{-2s\phi} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{\omega_2 \in L^*} \sum_{f \in \mathcal{F}(d)} \frac{1}{d|\omega_3|^{2-2s}} \\
 (4.44) \quad &\times e^{2\pi i\Re(\tilde{\omega}_2 f)} \int_0^\infty \frac{dt}{t^{2s-1}} e^{-\pi t(\tilde{n}_1^2 + \tilde{n}_2^2) - \frac{2\pi}{t} e^{-2\phi} |\omega_3|^2 + 2\pi i(\ell_1 \chi + \ell_2 \tilde{\chi})},
 \end{aligned}$$

where L^* is the lattice dual to L ,

$$(4.45) \quad L^* = \left\{ \tilde{\omega}_2 = \tilde{n}_1 + i\tilde{n}_2 = \frac{1}{2d}[(\tilde{k}_1 + \tilde{k}_2) + i(\tilde{k}_1 - \tilde{k}_2)] : (\tilde{k}_1, \tilde{k}_2) \in \mathbb{Z}^2 \right\},$$

and we defined the new charges

$$(4.46) \quad \begin{aligned} \ell_1 &:= \tilde{n}_1(p_1 + p_2) - \tilde{n}_2(p_1 - p_2), \\ \ell_2 &:= \tilde{n}_1(p_2 - p_1) - \tilde{n}_2(p_1 + p_2). \end{aligned}$$

4.4. Second constant term

We may now extract the second constant term from the $\ell_1 = \ell_2 = 0$ part of the sum, and accordingly we split $\mathcal{D}^{(s)}$ as

$$(4.47) \quad \mathcal{D}^{(s)} = \mathcal{E}_s^{(1)} + \mathcal{E}_s^{(A)},$$

where $\mathcal{E}_s^{(A)}$ is the Abelian term in the Fourier expansion to be considered in the next subsection. The $\ell_1 = \ell_2 = 0$ part arises from the $\tilde{\omega}_2 = 0$ term, which reads

$$(4.48) \quad \begin{aligned} \mathcal{E}_s^{(1)} &= \frac{(2\pi)^{2s-1}\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)\Gamma(2s-1)} e^{-2s\phi} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{f \in \mathcal{F}(d)} \frac{1}{d|\omega_3|^{2-2s}} \\ &\times \int_0^\infty \frac{dt}{t^{2s-1}} e^{-\frac{2\pi}{t} e^{-2\phi} |\omega_3|^2}. \end{aligned}$$

The sum over $f \in \mathcal{F}(d)$ produces the multiplicative function $N(d)$ in (4.43). The integral can be explicitly evaluated with the result

$$(4.49) \quad \mathcal{E}_s^{(1)} = \frac{\pi^{3/2}\Gamma(s-1/2)\Gamma(2s-2)}{\Gamma(s)\Gamma(2s-1)} e^{-2(2-s)\phi} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} N(d) \frac{1}{d|\omega_3|^{2s-2}}.$$

The sum can now be expressed in terms of the Riemann zeta function and Dedekind zeta function (4.15) as follows. Extract the greatest common divisor of p_1 and p_2 , defining $p_1 = dp'_1$ and $p_2 = dp'_2$, with $d = \gcd(p_1, p_2)$ and $\gcd(p'_1, p'_2) = 1$. This yields a sum over d and coprime (p'_1, p'_2)

$$(4.50) \quad \sum'_{(p_1, p_2) \in \mathbb{Z}^2} N(d) d^{-1} |p|^{2-2s} = \left(\sum_{d>0} N(d) d^{1-2s} \right) \left(\sum_{(p'_1, p'_2)=1} \frac{1}{(p_1'^2 + p_2'^2)^{s-1}} \right).$$

The second sum may be rewritten as a ratio of Riemann and Dedekind zeta functions as follows (see Section 4.2):

$$(4.51) \quad \sum_{(p'_1, p'_2)=1} \frac{1}{(p_1'^2 + p_2'^2)^{s-1}} = \frac{4\zeta_{\mathbb{Q}(i)}(s-1)}{\zeta(2s-2)}.$$

Let us now consider the first sum on the right-hand side of (4.50), which involves the combinatorial function $N(d)$ defined in (4.43) (see also [75]). Given $N(d)$ we may construct the Dirichlet series

$$(4.52) \quad L(N, s) := \sum_{d=1}^{\infty} N(d)d^{-s},$$

that converges for $\Re(s) > 2$. Since $N(d)$ is multiplicative, we may evaluate $L(N, s)$ using Euler products. To this end we note that the multiplicative series exhibits the following properties (mentioned in [75], and derived in greater generality in Appendix C):

$$(4.53) \quad N(2^m) = 2^m, \quad N(p^m) = \begin{cases} (m(p-1) + p)p^{m-1}, & p \equiv 1 \pmod{4}, \\ p^{2\lfloor m/2 \rfloor}, & p \equiv 3 \pmod{4}. \end{cases}$$

Therefore, the Dirichlet series (4.52) has an Euler product representation given by (see Appendix A for the derivation)

$$(4.54) \quad L(N, s) = \frac{1}{1-2^{1-s}} \prod_{p: p \equiv 1 \pmod{4}} \frac{1-p^{-s}}{(1-p^{1-s})^2} \prod_{p: p \equiv 3 \pmod{4}} \frac{1+p^{-s}}{(1-p^{1-s})(1+p^{1-s})},$$

where the product runs over all primes $p > 2$. Comparing with (4.19) and (4.21), we deduce that

$$(4.55) \quad L(N, s) = \frac{\beta(s-1)\zeta(s-1)}{\beta(s)}.$$

Putting everything together we then find the following expression for the constant term:

$$(4.56) \quad \mathcal{E}_s^{(1)} = 4 \frac{\pi^{3/2}\Gamma(s-1/2)\Gamma(2s-2)}{\Gamma(s)\Gamma(2s-1)} \frac{L(N, 2s-1)}{\zeta(2s-2)} \zeta_{\mathbb{Q}(i)}(s-1) e^{-2(2-s)\phi}.$$

Referring back to the completed Dedekind zeta function (4.16) and Dirichlet beta function (4.22), we define a completed ‘‘Picard Zeta function’’ by

$$(4.57) \quad \mathfrak{Z}(s) := \zeta_{\mathbb{Q}(i)*}(s) \beta_*(2s-1),$$

in terms of which the two constant terms can be neatly summarized by

$$(4.58) \quad \mathcal{E}_s^{(\text{const})} = \mathcal{E}_s^{(0)} + \mathcal{E}_s^{(1)} = 4\zeta_{\mathbb{Q}(i)}(s) \left\{ e^{-2s\phi} + \frac{\mathfrak{Z}(2-s)}{\mathfrak{Z}(s)} e^{-2(2-s)\phi} \right\}.$$

Equation (4.58) can be viewed as an extension of Langlands’s constant term formula [65] for Eisenstein series associated to special linear groups to the case of the unitary group $SU(2, 1)$. The completed Picard zeta function $\mathfrak{Z}(s)$ plays the same role as the completed Riemann zeta function $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ in Langlands’ formula.

4.5. Abelian Fourier coefficients

We now turn to the Abelian Fourier coefficients, corresponding to the terms $(\tilde{n}_1, \tilde{n}_2) \neq 0$ in (4.44). The integral over t leads to a modified Bessel function,

$$(4.59) \quad \begin{aligned} \mathcal{E}_s^{(A)} &= \frac{2\pi^{2s-1/2}\Gamma(s-1/2)}{\Gamma(s)\Gamma(2s-1)} e^{-2\phi} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum'_{(\tilde{k}_1, \tilde{k}_2) \in \mathbb{Z}^2} \sum_{f \in \mathcal{F}(d)} \frac{1}{d^{2s-1}} |u|^{2s-2} \\ &\times e^{\frac{\pi i}{d} \Re[uf(1-i)]} K_{2s-2} \left(2\pi e^{-\phi} |\Lambda| \right) e^{2\pi i(\ell_1 \chi + \ell_2 \bar{\chi})}, \end{aligned}$$

where we have introduced the following additional notation:

$$(4.60) \quad u = \tilde{k}_1 + i\tilde{k}_2, \quad \Lambda = \ell_2 - i\ell_1$$

for

$$(4.61) \quad \ell_1 = \frac{1}{d}(\tilde{k}_1 p_2 + \tilde{k}_2 p_1), \quad \ell_2 = \frac{1}{d}(\tilde{k}_2 p_2 - \tilde{k}_1 p_1).$$

These charges are manifestly integral since d divides p_1 and p_2 . This last relation can also be written as

$$(4.62) \quad \Lambda = \frac{u \omega_3}{d} = u \omega'_3,$$

where $\omega'_3 = \omega_3/d$ is a primitive Gaussian number (i.e., a Gaussian number whose real and imaginary parts are coprime). To extract the Abelian Fourier coefficients $\mathfrak{C}_{\ell_1, \ell_2}^{(A)}(\phi)$ we therefore replace the sum over ω_3 and u by a sum over

d , Λ and ω'_3 where the primitive Gaussian integer ω'_3 has to be a Gaussian divisor of Λ , to wit

$$\begin{aligned}
 \mathcal{E}_s^{(A)} &= C_s^{(A)} e^{-2\phi} \sum'_{\Lambda \in \mathbb{Z}[i]} \left\{ \sum_{\omega'_3 | \Lambda} \left| \frac{\Lambda}{\omega'_3} \right|^{2s-2} \left(\sum_{d>0} \frac{1}{d^{2s-1}} \sum_{f \in \mathcal{F}(d)} e^{\frac{\pi i}{d} \Re \left[\frac{\Lambda}{\omega'_3} f(1-i) \right]} \right) \right\} \\
 (4.63) \quad &\times K_{2s-2} \left(2\pi e^{-\phi} |\Lambda| \right) e^{2\pi i(\ell_1 \chi + \ell_2 \bar{\chi})},
 \end{aligned}$$

where the coefficient is given by

$$(4.64) \quad C_s^{(A)} = \frac{2\pi^{2s-1/2} \Gamma(s-1/2)}{\Gamma(s) \Gamma(2s-1)} = \frac{8\zeta_{\mathbb{Q}(i)}(s) \beta(2s-1)}{\mathfrak{Z}(s)}.$$

To make contact with the general discussion of Section 4.1, we rewrite this result as a sum over the real variables ℓ_1 and ℓ_2 :

$$\begin{aligned}
 \mathcal{E}_s^{(A)} &= 2\zeta_{\mathbb{Q}(i)}(s) \frac{e^{-2\phi}}{\mathfrak{Z}(s)} \sum'_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \mu_s(\ell_1, \ell_2) [\ell_1^2 + \ell_2^2]^{s-1} \\
 (4.65) \quad &\times K_{2s-2} \left(2\pi e^{-\phi} \sqrt{\ell_1^2 + \ell_2^2} \right) e^{2\pi i(\ell_1 \chi + \ell_2 \bar{\chi})},
 \end{aligned}$$

where we defined the summation measure

$$(4.66) \quad \mu_s(\ell_1, \ell_2) := 4\beta(2s-1) \sum_{\omega'_3 | \Lambda} |\omega'_3|^{2-2s} \left(\sum_{d>0} d^{1-2s} \sum_{f \in \mathcal{F}(d)} e^{\frac{\pi i}{d} \Re \left[\frac{\Lambda}{\omega'_3} f(1-i) \right]} \right),$$

containing the sum over primitive Gaussian divisors of $\Lambda = \ell_2 - i\ell_1$. The sum over d in the parenthesis may be carried out for fixed Λ and ω'_3 to give the Gaussian divisor function (see Appendix C for the derivation)

$$(4.67) \quad \sum_{d>0} d^{1-2s} \sum_{f \in \mathcal{F}(d)} e^{\frac{\pi i}{d} \Re \left[\frac{\Lambda}{\omega'_3} f(1-i) \right]} = \frac{1}{4\beta(2s-1)} \sum_{z | \frac{\Lambda}{\omega'_3}} |z|^{4-4s},$$

whence the instanton measure (4.66) simplifies to

$$(4.68) \quad \mu_s(\ell_1, \ell_2) = \sum_{\omega'_3 | \Lambda} |\omega'_3|^{2-2s} \sum_{z | \frac{\Lambda}{\omega'_3}} |z|^{4-4s}.$$

Thus, the Abelian summation measure (4.68) involves both a sum over primitive divisors of Λ and a sum over all divisors of Λ/ω'_3 . By comparing (4.65)

with (4.9) we may now extract the numerical Abelian Fourier coefficients:

$$(4.69) \quad C_{\ell_1, \ell_2}^{(A)}(s) = \frac{2\zeta_{\mathbb{Q}(i)}(s)}{\mathfrak{Z}(s)} \mu_s(\ell_1, \ell_2) [\ell_1^2 + \ell_2^2]^{s-1}.$$

We note that the Abelian instanton measure (4.68) is multiplicative in a restricted sense: The relation

$$(4.70) \quad \mu_s(\Lambda_1) \mu_s(\Lambda_2) = \mu_s(\Lambda_1 \Lambda_2)$$

holds if (and only if) the Gaussian integers Λ_1 and Λ_2 admit no common prime factor *up to complex conjugation* (this caveat is relevant for split primes, see Appendix A).

4.6. Non-Abelian Fourier coefficients

Finally, we consider the non-Abelian term $\mathcal{E}_s^{(\text{NA})}$ in (4.34). This term reads

$$(4.71) \quad \begin{aligned} \mathcal{E}_s^{(\text{NA})} &= \frac{\pi^s}{\Gamma(s)} e^{-2s\phi} \sum'_{\tilde{m} \in \mathbb{Z}} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{\substack{(n_1, n_2) \in \mathbb{Z}^2 \\ 2d|n_1^2 + n_2^2}} \\ &\times \frac{d}{|\omega_3|} e^{-2\pi i \tilde{m} \left(-\frac{|\omega_2|^2}{2|\omega_3|^2} (q_1 p_1 + q_2 p_2) + \tilde{\ell}_1 \chi + \tilde{\ell}_2 \tilde{\chi} + 2d\psi \right)} \\ &\times \int_0^\infty \frac{dt}{t^{s+1/2}} e^{-\pi t \frac{d^2}{|\omega_3|^2} \tilde{m}^2 - \frac{\pi}{t} \frac{|\omega_3|^2}{d^4} e^{-4\phi} \left[d^2 + \frac{e^{2\phi}}{4} \left((\tilde{\ell}_1 + 2d\tilde{\chi})^2 + (\tilde{\ell}_2 - 2d\chi)^2 \right) \right]^2}. \end{aligned}$$

The integral is of Bessel type and yields

$$(4.72) \quad \begin{aligned} \mathcal{E}_s^{(\text{NA})} &= \frac{2\pi^s}{\Gamma(s)} e^{-2s\phi} \sum'_{\tilde{m} \in \mathbb{Z}} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{\substack{(n_1, n_2) \in \mathbb{Z}^2 \\ 2d|n_1^2 + n_2^2}} \left[\frac{d}{|\omega_3|} \right]^{s+1/2} \left[\frac{|\tilde{m}|^2}{\Re(S_{\ell_1, \ell_2, k})} \right]^{s-1/2} \\ &\times K_{s-1/2} \left(2\pi \Re(S_{\ell_1, \ell_2, k}) \right) e^{-2\pi i \Im(S_{\ell_1, \ell_2, k})} e^{-\frac{\pi i}{2kd} (\ell_1^2 + \ell_2^2) (q_1 p_1 + q_2 p_2)}, \end{aligned}$$

where the real and imaginary parts of $S_{\ell_1, \ell_2, k}$ are given by

$$(4.73) \quad \begin{aligned} \Re(S_{\ell_1, \ell_2, k}) &= |k| e^{-2\phi} + \frac{1}{4|k|} \left[(\ell_1 + 2k\tilde{\chi})^2 + (\ell_2 - 2k\chi)^2 \right], \\ \Im(S_{\ell_1, \ell_2, k}) &= \ell_1 \chi + \ell_2 \tilde{\chi} + 2k\psi, \end{aligned}$$

and we also defined²¹

$$(4.74) \quad \begin{aligned} k &:= \tilde{m}d, \\ \ell_1 &:= \frac{k}{|\omega_3|^2} \left[(p_1 - p_2)n_1 + (p_1 + p_2)n_2 \right], \\ \ell_2 &:= \frac{k}{|\omega_3|^2} \left[(p_1 + p_2)n_1 - (p_1 - p_2)n_2 \right]. \end{aligned}$$

In Gaussian notation, $\Lambda = \ell_2 - i\ell_1$, the last two relations amount to

$$(4.75) \quad \bar{\Lambda} = \frac{(1+i)k\omega_2}{\omega_3}.$$

Comparing the expression (4.72) with the general form of the non-Abelian term (4.10), we see that the former involves a sum of nearly Gaussian wavefunctions peaked around $(\ell_2, -\ell_1)/(2k)$ in the $(\chi, \tilde{\chi})$ plane, while the latter is written in terms of a basis of Landau-type wave functions which are eigenmodes of ∂_χ and $\partial_{\psi+\chi\tilde{\chi}}$, with quantized charges $4kn$ and k . To extract the non-Abelian Fourier coefficients $C_{r,k,\ell}^{(\text{NA})}(s)$ we must therefore transform (4.72) into the correct basis. This can be achieved via Fourier transform along the variable χ (or $\tilde{\chi}$ in the other polarization).

To perform the Fourier transform we go back to the integral representation in (4.71). The integrand is quartic in χ and therefore inconvenient for Fourier transform. To remedy this we make the following change of integration variables:

$$(4.76) \quad t = \frac{t'|\omega_3|^2 A}{k^2},$$

where

$$(4.77) \quad A(y, \chi, \tilde{\chi}) = k \left[y + \left(\tilde{\chi} + \frac{\ell_1}{2k} \right)^2 + \left(\chi - \frac{\ell_2}{2k} \right)^2 \right],$$

and we recall that $y = e^{-2\phi}$. Implementing this in (4.71), and denoting $\tilde{\ell}_i =$

²¹The non-Abelian charges ℓ_i defined in (??) should not be confused with the Abelian charges ℓ_i in (4.46).

l_i/\tilde{m} , we obtain

$$\begin{aligned}
 \mathcal{E}_s^{(\text{NA})} &= \frac{\pi^s}{\Gamma(s)} y^s \sum'_{\tilde{m} \in \mathbb{Z}} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{\substack{(n_1, n_2) \in \mathbb{Z}^2 \\ 2d|n_1^2+n_2^2}} \frac{d|k|^{2s-1}}{|\omega_3|^{2s}} \\
 &\times e^{-2\pi i \tilde{m} \left(-\frac{|\omega_2|^2}{2|\omega_3|^2} (q_1 p_1 + q_2 p_2) + \tilde{l}_1 \chi + \tilde{l}_2 \tilde{\chi} + 2d\psi \right)} \\
 (4.78) \quad &\times \int_0^\infty \frac{dt'}{t'^{s+1/2}} A^{1/2-s} e^{-\pi \left(t' + \frac{1}{t'} \right) A},
 \end{aligned}$$

where the exponent is now quadratic in both χ and $\tilde{\chi}$. Using an integral representation for the factor $A^{1/2-s}$, and dropping the prime on t' , we may rewrite this expression as follows:

$$\begin{aligned}
 \mathcal{E}_s^{(\text{NA})} &= \frac{\pi^{2s-1/2}}{\Gamma(s)\Gamma(s-1/2)} y^s \sum'_{\tilde{m} \in \mathbb{Z}} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{\substack{(n_1, n_2) \in \mathbb{Z}^2 \\ 2d|n_1^2+n_2^2}} \frac{d|k|^{2s-1}}{|\omega_3|^{2s}} e^{\frac{\pi i \tilde{m} |\omega_2|^2}{2|\omega_3|^2} (q_1 p_1 + q_2 p_2)} \\
 &\times e^{-4\pi i k \psi - 2\pi i l_2 \tilde{\chi}} \int_0^\infty \frac{dt du}{t^{s+1/2} u^{3/2-s}} \\
 (4.79) \quad &\times e^{-\pi k \left(u + t + \frac{1}{t} \right) \left[y + \left(\tilde{\chi} + \frac{\ell_1}{2k} \right)^2 \right]} f(y, \chi, \tilde{\chi}; t, u),
 \end{aligned}$$

where all the χ -dependence is contained in the function

$$(4.80) \quad f(y, \chi; t, u) = e^{-\pi k \left(u + t + \frac{1}{t} \right) \left(\chi - \frac{\ell_2}{2k} \right)^2 - 2\pi i l_1 \chi}.$$

The Fourier transform over χ is now implemented by substituting

$$(4.81) \quad f(y, \chi; t, u) = e^{-4\pi i k \chi \tilde{\chi}} \int dn \hat{f}(y, n, \tilde{\chi}; t, u) e^{8\pi i k n \chi},$$

with

$$\begin{aligned}
 \hat{f}(y, n, \tilde{\chi}; t, u) &= 4|k| \int d\xi e^{-8\pi i k n \xi + 4\pi i k \xi \tilde{\chi}} f(y, \xi, \tilde{\chi}; t, u) \\
 (4.82) \quad &= 4|k| e^{-\frac{\pi}{k} \frac{(\ell_1 - 2k \tilde{\chi} + 8\pi k n)^2}{u + t + 1/t}}.
 \end{aligned}$$

After Fourier transform, the non-Abelian term thus becomes

$$\begin{aligned}
 \mathcal{E}_s^{(\text{NA})} &= \frac{4\pi^{2s-1/2}}{\Gamma(s)\Gamma(s-1/2)} y^s \sum'_{\tilde{m} \in \mathbb{Z}} \sum'_{(p_1, p_2) \in \mathbb{Z}^2} \sum_{\substack{(n_1, n_2) \in \mathbb{Z}^2 \\ 2d|n_1^2 + n_2^2}} \frac{d|k|^{2s}}{|\omega_3|^{2s}} e^{\frac{\pi i \tilde{m} |\omega_2|^2}{2|\omega_3|^2} (q_1 p_1 + q_2 p_2)} \\
 &\times \int \frac{dt du}{t^{s+1/2} u^{3/2-s}} e^{-\pi k(u+t+\frac{1}{t})} \left[y + \left(\tilde{\chi} + \frac{\ell_1}{2k} \right)^2 \right] \\
 (4.83) \quad &\times \int dn e^{-\frac{\pi i}{k} \ell_2 (\ell_1 - 4kn)} e^{-\frac{\pi}{k} \frac{(\ell_1 - 2k\tilde{\chi} + 8\pi kn)^2}{u+t+1/t}} e^{8\pi i kn \chi - 4\pi i k(\psi + \chi \tilde{\chi})}.
 \end{aligned}$$

Let us now comment on the structure of Equation (4.83). After Fourier transforming we see that the non-Abelian term indeed corresponds to an expansion in terms of the invariant wavefunctions on the twisted torus as in Equation (4.10). However, we have not been able to further manipulate Equation (4.83) into the form displayed in (4.10) and therefore we cannot extract the numerical Fourier coefficients $C_{k,\ell}^{(\text{NA})}(s)$ in as compact a form as the Abelian coefficients (4.69). Nevertheless, as a consistency check we shall show that the leading order exponential behavior of (4.83) near the cusp $y \rightarrow \infty$ coincides with that of Equation (4.10). To this end we may take the saddle point approximation for the integrals over t and u in (4.83) for which the saddle points are located at $t = 1$ and $u = 0$. We thus find that the leading exponential dependence of (4.83) at the saddle point is given by e^{-S} with

$$(4.84) \quad \Re(S) = 2\pi|k| \left[y + \left(\tilde{\chi} + \frac{\ell_1}{2|k|} \right)^2 \right] + \frac{\pi}{2|k|} (\ell_1 - 2|k|\tilde{\chi} + 4|k|n)^2.$$

Rearranging terms, this can be written as

$$(4.85) \quad \Re(S) = 2\pi|k|y + 4\pi|k|(\tilde{\chi} - n)^2 + 4\pi|k| \left(n + \frac{\ell_1}{2|k|} \right)^2.$$

Using the asymptotic behavior of the Whittaker function $W_{k,m}(x) \sim e^{-x/2}$ one may indeed verify that the first two terms in (4.85) exactly coincide with the leading behavior of the general expression (4.10) in the limit $y \rightarrow \infty$. We further expect that the summation over ℓ_1 and ℓ_2 (or, more precisely, over ω_2 and ω_3) will restrict the integral over n such that it localizes on the points in $\mathbb{Z} + \ell/(4|k|)$ as is expected from the general expression (4.10). We stress that the result (4.85) is valid in the polarization (4.4) we have chosen. There is an analogous result for the other polarization.

Besides the representation of the non-Abelian coefficients in the form (4.83) one could also try to extract the coefficients by other means. One

possibility would be to manipulate the expression (4.72) by expanding out the Bessel function and binomially expanding the resulting power series in $A(y, \chi, \tilde{\chi})$ to make contact with the power series expansions of the Hermite polynomials and Whittaker functions of (4.10). Alternatively, one could try to compute the coefficients by going to a suitably chosen point in moduli space (e.g., the cusp $y = \infty$) or by using $SU(2, 1; \mathbb{Z}[i])$ symmetry or Hecke operators to relate the non-Abelian coefficients to the Abelian ones. We hope to present a complete investigation of the non-Abelian coefficients in a future publication.

4.7. Functional relation

Expression (4.58) for the constant terms of the Eisenstein series \mathcal{E}_s is suggestive of a functional relation most conveniently written in terms of the Poincaré series (3.11) and the Picard Zeta function (4.57),

$$(4.86) \quad \mathfrak{Z}(s)\mathcal{P}_s = \mathfrak{Z}(2-s)\mathcal{P}_{2-s}.$$

Indeed, it is easily checked that both the constant terms (4.58) and the Abelian Fourier coefficients (4.65), (4.68) satisfy this relation, taking into account the symmetry of the modified Bessel function $K_{2s-2}(x) = K_{2-2s}(x)$. Unfortunately, due to the unwieldy form of the non-Abelian terms we are unable to present a full proof of (4.86), which would constitute an analog of the familiar functional relation for Eisenstein series associated to special linear groups [65]. We note that a different functional relation for \mathcal{P}_s has been proposed in [66] but this appears to contradict the constant term formula (4.58).

5. Instanton corrections to the universal hypermultiplet

In this section we propose that the Eisenstein series for the Picard modular group constructed in Section 3, and further analyzed in Section 4, controls the exact metric on the universal hypermultiplet moduli space \mathcal{M}_{UH} , including the D2- and NS5-brane instanton corrections. We start by recalling some aspects of quantum corrections to hypermultiplet moduli spaces in type II Calabi–Yau compactifications, with particular emphasis on recent developments involving twistor techniques.

5.1. Twistor techniques for quaternionic-Kähler spaces

Quantum corrections to the hypermultiplet moduli space are most conveniently described using twistor techniques [22,24,28,29]. Given a quaternion-Kähler space \mathcal{M} , one may construct its twistor space $\mathcal{Z}_{\mathcal{M}}$, a $\mathbb{C}P^1$ bundle over \mathcal{M} which admits a canonical complex structure \mathcal{J} , a complex contact structure \mathcal{C} , a compatible real structure τ and a Kähler-Einstein metric $ds^2_{\mathcal{Z}_{\mathcal{M}}}$ with Kähler potential $K_{\mathcal{Z}_{\mathcal{M}}}$. The contact one-form $\mathcal{C}^{[i]}$ is proportional to the $(1,0)$ -form $Dz := dz + p_+ - ip_3z + p_-z^2$, where z is a complex coordinate on the $\mathbb{C}P^1$ fiber and (p_3, p_+, p_-) is the $SU(2)$ part of the Levi-Civita connection on \mathcal{M} . The metric on \mathcal{M} can be recovered from the Kähler-Einstein metric on $\mathcal{Z}_{\mathcal{M}}$ using

$$(5.1) \quad ds^2_{\mathcal{Z}_{\mathcal{M}}} = \frac{1}{4} (e^{-2K_{\mathcal{Z}_{\mathcal{M}}}} |\mathcal{C}|^2 + \nu ds^2_{\mathcal{M}}),$$

where ν is a numerical constant related to the curvature of the base manifold \mathcal{M} . Locally, any contact structure is trivial, so there exists an open covering U_i of $\mathcal{Z}_{\mathcal{M}}$ and a local complex coordinate system $(\xi_{[i]}^{\Lambda}, \tilde{\xi}_{\Lambda}^{[i]}, \alpha_{[i]})$ on U_i such that the complex contact structure takes the Darboux form

$$(5.2) \quad \mathcal{C}^{[i]} = d\alpha_{[i]} + \xi_{[i]}^{\Lambda} d\tilde{\xi}_{\Lambda}^{[i]} := 2 e^{\Phi_{[i]}} \frac{Dz}{z}.$$

The second equality defines the ‘‘contact potential’’ $\Phi_{[i]}$ in the patch U_i , a complex function on $\mathcal{Z}_{\mathcal{M}}$ holomorphic along the fiber. The contact potential in the patch U_i is related to the Kähler potential $K_{\mathcal{Z}_{\mathcal{M}}}$ in the same patch via

$$(5.3) \quad \mathcal{K}_{\mathcal{Z}_{\mathcal{M}}}^{[i]} = \log \frac{1 + z\bar{z}}{|z|} + \Re[\Phi_{[i]}(x^{\mu}, z)].$$

Globally, the complex contact structure on $\mathcal{Z}_{\mathcal{M}}$ is determined by the complex contact transformations $S^{[ij]}$ between the Darboux coordinate system on the overlap $U_i \cap U_j$. These can be described, e.g., by providing holomorphic generating functions $S_{ij}(\xi_{[i]}^{\Lambda}, \tilde{\xi}_{\Lambda}^{[j]}, \alpha_{[j]})$, subject to compatibility conditions on triple overlaps $U_i \cap U_j \cap U_k$, equivalence under local contact transformations on U_i and U_j , and reality constraints. The quaternion-Kähler metric on \mathcal{M} can then be extracted from these holomorphic data, by determining the contact twistor lines, i.e., expressing $(\xi_{\Lambda}, \tilde{\xi}^{\Lambda}, \alpha, \Phi)$ in some patch U in terms of the coordinates $x^{\mu} \in \mathcal{M}$ on the base manifold and the complex coordinate $z \in \mathbb{C}P^1$ on the fiber. Plugging the solution into (5.2) allows one to extract

the $SU(2)$ connection p_{\pm}, p_3 , the quaternionic 2-forms and finally the metric on \mathcal{M} . More details on this construction can be found in [28, 29]. It should be noted that these twistor techniques for quaternion-Kähler manifolds are related to the more standard twistor techniques for hyperkähler manifolds by the superconformal quotient construction [23, 26, 27].

5.1.1. On the twistor space of the tree-level universal hypermultiplet. We now illustrate this construction in the case of the tree-level universal hypermultiplet moduli space. The twistor space $\mathcal{Z}_{\mathcal{M}_{\text{UH}}}$ of the classical moduli space \mathcal{M}_{UH} can be nicely described group-theoretically as follows. Viewing the $\mathbb{C}P^1$ twistor fiber as $S^2 = SU(2)/U(1)$, the fibration of $SU(2)/U(1)$ over \mathcal{M}_{UH} is such that the $SU(2)$ cancels [63, 76]:

$$(5.4) \quad \mathcal{Z}_{\mathcal{M}_{\text{UH}}} = \frac{SU(2)}{U(1)} \times \frac{SU(2, 1)}{SU(2) \times U(1)} = \frac{SU(2, 1)}{U(1) \times U(1)}.$$

The twistor space $\mathcal{Z}_{\mathcal{M}_{\text{UH}}}$ is a complex three-dimensional contact manifold, with local coordinates $(\xi, \tilde{\xi}, \alpha)$. These coordinates parametrize the complexified Heisenberg group $N_{\mathbb{C}}$, or, equivalently, coordinates on the complex coset space $P_{\mathbb{C}} \backslash SL(3, \mathbb{C})$, where $P_{\mathbb{C}}$ is the complexification of the parabolic subgroup $P \subset SU(2, 1)$ discussed in Appendix B and $SL(3, \mathbb{C})$ is the complexification of $SU(2, 1)$. In terms of the coordinates $(\xi, \tilde{\xi}, \alpha)$ on $P_{\mathbb{C}} \backslash SL(3, \mathbb{C})$ the Kähler potential of $\mathcal{Z}_{\mathcal{M}_{\text{UH}}}$ takes the following form [63]:

$$(5.5) \quad K_{\mathcal{Z}_{\mathcal{M}_{\text{UH}}}} = \frac{1}{2} \log \left[\left((\xi - \bar{\xi})^2 + (\tilde{\xi} - \bar{\tilde{\xi}})^2 \right)^2 + 4 \left(\alpha - \bar{\alpha} + \bar{\xi}\tilde{\xi} - \xi\bar{\tilde{\xi}} \right)^2 \right].$$

As mentioned above, the contact twistor lines for the unperturbed twistor space correspond to the change of variables that relate the coordinates $(\xi, \tilde{\xi}, \alpha)$ on $\mathcal{Z}_{\mathcal{M}_{\text{UH}}}$ to the coordinates $x^{\mu} = \{e^{\phi}, \chi, \tilde{\chi}, \psi\}$ on the base \mathcal{M}_{UH} and the coordinate z on the fiber $\mathbb{C}P^1 = SU(2)/U(1)$. These twistor lines were obtained in [63]. In our notations they read (away from the north pole $z = 0$ and south pole $z = \infty$) as

$$(5.6) \quad \begin{aligned} \xi &= -\sqrt{2}\chi + \frac{1}{\sqrt{2}}e^{-\phi}(z - z^{-1}), \\ \tilde{\xi} &= -\sqrt{2}\tilde{\chi} - \frac{i}{\sqrt{2}}e^{-\phi}(z + z^{-1}), \\ \alpha &= 2\psi - e^{-\phi} \left[z(\tilde{\chi} + i\chi) - z^{-1}(\tilde{\chi} - i\chi) \right]. \end{aligned}$$

Plugging these into (5.2) and (5.5), we find that the contact potential in this patch is simply

$$(5.7) \quad e^{\Phi(x^\mu, z)} = e^{-2\phi},$$

in particular independent of z , and verify that (5.3) is satisfied. We further note that under an action of $SU(2, 1)$, the contact potential and contact one-form transform as

$$(5.8) \quad e^\Phi \mapsto |C + DZ|^{-2} e^\Phi, \quad \mathcal{C} \mapsto (C + DZ)^2 \mathcal{C},$$

which ensure that the Kähler potential $K_{\mathcal{Z}_{\mathcal{M}_{\text{UH}}}}$ transforms by a Kähler transformation, and that $SU(2, 1)$ acts isometrically on both $\mathcal{Z}_{\mathcal{M}_{\text{UH}}}$ and \mathcal{M}_{UH} itself.

5.2. Quantum corrected hypermultiplet moduli spaces in type IIA

Using these and related techniques, much progress has been achieved recently in understanding the hypermultiplet moduli space in type IIA string compactifications on a Calabi–Yau manifold \mathcal{X} . At the perturbative level, the metric on \mathcal{M}_{H} is believed to receive a one-loop correction, but no higher loop corrections [17–20, 77, 78]. For the universal hypermultiplet this was rigorously proven in [77]. The general form of the perturbative corrections can be inferred from compactifications of higher derivative couplings in 10 dimensions [17], or via an explicit string theory calculation in $D = 4$ [20]. As a consequence, the contact potential on the twistor space must reduce at large volume, small coupling to

$$(5.9) \quad e^\Phi = \frac{\tau_2^2 V_{\mathcal{X}}}{2} + \frac{\chi_E}{192\pi} + \dots,$$

where $\tau_2 = 1/g_s$ is the 10-dimensional string coupling, χ_E is the Euler number of \mathcal{X} and $V_{\mathcal{X}}$ is the volume of \mathcal{X} in string units. The complete perturbatively corrected metric corresponding to the contact potential (5.9) can be found in [28, 78]. We note that in the corresponding expression in the type IIB hypermultiplet sector, there are additional contributions arising from α' -corrections and worldsheet instantons. However, due to the fact that the metric on the complex structure moduli space of \mathcal{X} is insensitive to α' -effects, these corrections are absent in the type IIA expression (5.9). The corrections to the contact potential which are non-perturbative in g_s are

due to D2-brane and NS5-brane instantons [16]. Using S-duality and mirror symmetry, the form of the D2-brane instanton corrections was obtained in a series of works [27–31, 79]. To first order away from the one-loop corrected metric, their contribution to the contact potential reads

$$(5.10) \quad e^{\Phi_{(\text{D2})}} = \frac{1}{4\pi^2} \sum_{\gamma} n_{\gamma} \sum_{m>0} \frac{e^{-\phi|Z_{\gamma}|}}{m} K_1 \left(8\pi m e^{-\phi|Z_{\gamma}|} \right) e^{2\pi i m \int_{\gamma} C_{(3)}},$$

where γ runs over the homology classes in $H^3(\mathcal{X}, \mathbb{Z})$, Z_{γ} is the central charge associated to the cycle γ , e^{ϕ} is the 4D string coupling such that $e^{-2\phi} = \tau_2^2 V_{\mathcal{X}}/2$, $C_{(3)}$ is the Ramond–Ramond 3-form and n_{γ} is a numerical factor, which counts the number of BPS states in the homology class γ . NS5-brane contributions have been discussed in [34, 79], but remain largely mysterious in general.

5.3. On the contact potential and the Picard–Eisenstein series

We now restrict to the case of type IIA string theory compactified on a rigid Calabi–Yau threefold, and propose that the Picard–Eisenstein series $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$, for a suitable value of the parameter s , controls the exact, quantum corrected metric on the universal hypermultiplet moduli space. As in [34] (see in particular Section 3.1), we shall restrict our attention to the contact potential $\Phi(x^{\mu}, z)$ on a certain holomorphic section²² $z(x^{\mu})$ of the twistor space $\mathcal{Z}_{\mathcal{M}_{\text{UH}}}$. We also choose variables such that the action of $SU(2, 1)$ on \mathcal{M}_{UH} is the tree-level action (though it is no longer isometric in general), and look for a completion of $\Phi(x^{\mu}, z(x^{\mu}))$, which reproduces the expected perturbative contributions. Determining the specific holomorphic section $z(x^{\mu})$ and the exact twistor lines and hypermultiplet metric are important open problems which lie outside the scope of this work.

²²In the presence of NS5-brane corrections, the contact potential is no longer constant along the fiber. The quaternion-Kähler metric on $\mathcal{M} = \mathcal{M}_{\text{UH}}$ can nevertheless be described, in many different ways, in terms of a single real function $h(x^{\mu})$ on \mathcal{M} subject to a non-linear partial differential equation [79, 80]. The latter can be identified with the Kähler potential $K_{\mathcal{Z}_{\mathcal{M}}}$ on any holomorphic section $z(x^{\mu})$ of $\mathcal{Z}_{\mathcal{M}}$ [81]. Our proposal refers to a specific choice of holomorphic section, which we are not able to specify at this stage. We stress that this technical point plays no role at the level of our present analysis.

Matching the powers of the dilaton, we then propose that, on the holomorphic section $z(x^\mu)$ introduced above, the contact potential for the quantum corrected metric on \mathcal{M}_{UH} is given by²³

$$(5.11) \quad e^{\Phi_{\text{exact}}(x^\mu, z(x^\mu))} = \kappa e^\phi \mathcal{E}_{3/2}(\phi, \chi, \tilde{\chi}, \psi),$$

where \mathcal{E}_s is the Picard–Eisenstein series (3.2), and κ is an adjustable numerical constant. Using the Fourier expansion (1.18), we see that (5.11) predicts

$$(5.12) \quad e^{\Phi_{\text{exact}}} = 4\zeta_{\mathbb{Q}(i)}(3/2)\kappa \left(e^{-2\phi} + \frac{\mathfrak{3}(1/2)}{\mathfrak{3}(3/2)} \right) + e^{\Phi_{(\text{A})}} + e^{\Phi_{(\text{NA})}},$$

where the last two terms correspond to the Abelian and non-Abelian parts of the Fourier expansion, respectively. The two constant terms have the same dependence on the dilaton e^ϕ as the two perturbative contributions in (5.9). We thus want to identify the second constant term at $s = 3/2$ with the one-loop coefficient $\chi_{\text{E}}/192\pi$ in (5.9). Here we run into a problem since for $s = 3/2$ we find

$$(5.13) \quad \frac{\mathfrak{3}(1/2)}{\mathfrak{3}(3/2)} \approx -2.32607,$$

implying that matching with the physical one-loop term requires $\chi_{\text{E}} \sim -1403.05$, a negative, non-integer number. This contradicts the fact that $\chi_{\text{E}} = 2h_{1,1} \in 2\mathbb{N}$ for a rigid Calabi–Yau threefold. Hence, the value of the one-loop coefficient predicted by the second constant term in the Eisenstein series is not physically viable. While this invalidates the proposal that the principal Eisenstein series \mathcal{E}_s describes the exact universal hypermultiplet metric, it does not necessarily ruin the idea that the Picard modular group controls that metric. In the concluding Section 6, we speculate that automorphic forms attached to the quaternionic discrete series of $SU(2, 1, \mathbb{Z}[i])$ may be relevant. We proceed with our current proposal however, as the form of the non-Abelian Fourier expansion is largely independent of the details of the automorphic form under consideration. In particular, we show next that the form of the Abelian and non-Abelian contributions to the Fourier expansion of $\mathcal{E}_{3/2}(\phi, \chi, \tilde{\chi}, \psi)$ agrees with the expected form of D2-brane and NS5-brane instanton contributions, respectively.

²³Due to the different power of e^ϕ , the contact potential Φ_{exact} appears to transform differently from its tree-level counterpart (5.8); this is not a fatal flaw however, since the locus $z(x^\mu)$ is in general not fixed by the action of the Picard modular group.

5.3.1. D2-brane instantons. The Abelian contribution (4.65) at $s = 3/2$ becomes

$$(5.14) \quad e^{\Phi_{(A)}} = \frac{2\kappa \zeta_{\mathbb{Q}(i)}(3/2) e^{-\phi}}{\mathfrak{Z}(3/2)} \sum'_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \mu_{3/2}(\ell_1, \ell_2) [\ell_1^2 + \ell_2^2]^{1/2} \times K_1 \left(2\pi e^{-\phi} \sqrt{\ell_1^2 + \ell_2^2} \right) e^{-2\pi i(\ell_1 \chi + \ell_2 \tilde{\chi})},$$

where the summation measure $\mu_{3/2}(\ell_1, \ell_2)$ is given in (4.68). In the weak-coupling limit $e^\phi \rightarrow 0$ we may use the asymptotic expansion of the modified Bessel function at large x ,

$$(5.15) \quad K_t(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{n \geq 0} \frac{\Gamma(t + n + \frac{1}{2})}{\Gamma(n + 1) \Gamma(t - n + \frac{1}{2})} (2x)^{-n},$$

to approximate

$$(5.16) \quad e^{\Phi_{(A)}} \sim \frac{\kappa \zeta_{\mathbb{Q}(i)}(3/2)}{\mathfrak{Z}(3/2)} e^{-\phi/2} \sum'_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \mu_{3/2}(\ell_1, \ell_2) (\ell_1^2 + \ell_2^2)^{1/4} e^{-2\pi S_{\ell_1, \ell_2}} [1 + \mathcal{O}(e^\phi)].$$

We thus find that $e^{\Phi_{(A)}}$ exhibits exponentially suppressed corrections in the limit $e^\phi \rightarrow 0$, weighted by the instanton action

$$(5.17) \quad S_{\ell_1, \ell_2} = e^{-\phi} \sqrt{\ell_1^2 + \ell_2^2} + i(\ell_1 \chi + \ell_2 \tilde{\chi}).$$

This is recognized as the action for Euclidean D2-branes wrapping special Lagrangian 3-cycles in the homology class $\ell_1 \mathcal{A} + \ell_2 \mathcal{B} \in H_3(\mathcal{X}, \mathbb{Z})$, where $(\mathcal{A}, \mathcal{B})$ provides an integral symplectic basis of $H_3(\mathcal{X}, \mathbb{Z})$. To see this, we note that generally the instanton action for D2-branes wrapping a special Lagrangian submanifold in the homology class $\gamma \in H_3(X, \mathbb{Z})$ inside a Calabi–Yau threefold \mathcal{X} is given by

$$(5.18) \quad S_\gamma = \frac{1}{g_s} \left| \int_\gamma \Omega \right| + i \int_\gamma C_{(3)},$$

where g_s is the 10-dimensional string coupling, $\Omega \in H_{3,0}(X)$ is the holomorphic three-form and $C_{(3)} \in H^3(X, \mathbb{R})/H^3(X, \mathbb{Z})$ is the RR three-form. The real part of the action can further be written in terms of the central charge $Z_\gamma = e^{K/2} \int_\gamma \Omega$ as $\Re(S_\gamma) = e^{-K/2} |Z_\gamma|/g_s$, where $K = -\log \int_X \Omega \wedge \bar{\Omega}$

is the Kähler potential of the complex structure moduli space. Noting that $K = -\log V_{\mathcal{X}}$ we then find

$$(5.19) \quad S_{\gamma} = e^{-\phi} |Z_{\gamma}| + i \int_{\gamma} C_{(3)},$$

where we defined the four-dimensional dilaton by $e^{\phi} := V_{\mathcal{X}}^{-1/2} g_s$. Restricting to a rigid Calabi–Yau threefold \mathcal{X} , we recall from Section 1 that the prepotential is $F = \tau X/2$ with τ being the period “matrix” $\int_{\mathcal{B}} \Omega / \int_{\mathcal{A}} \Omega$. In this case the D2-brane wraps a three-cycle in the homology class $\gamma = \ell_1 \mathcal{A} + \ell_2 \mathcal{B} \in H_3(\mathcal{X}, \mathbb{Z})$, which gives $Z_{\gamma} = (\ell_1 + \tau \ell_2) / \sqrt{\Im \tau}$, so the instanton action reduces to

$$(5.20) \quad S_{\ell_1, \ell_2}(\tau) = e^{-\phi} \frac{|\ell_1 + \tau \ell_2|}{\sqrt{\Im \tau}} + i \int_{\ell_1 \mathcal{A} + \ell_2 \mathcal{B}} C_{(3)}$$

Further setting $\tau = i$, which is the relevant value for our analysis, and using Equation (1.5) for the periods of the Ramond–Ramond three-form $C_{(3)}$, this action indeed coincides with the instanton action (5.17) predicted from $SU(2, 1; \mathbb{Z}[i])$ -invariance. Thus, we may conclude that the Abelian term (5.14) in the Fourier expansion agrees with the general form of D-instanton corrections in (5.10) upon restricting to a rigid Calabi–Yau threefold, which admits complex multiplication by $\mathbb{Z}[i]$.

The infinite series within the brackets in (5.16) should, in the spirit of [5], arise from perturbative contributions around the instanton background. The summation measure is given by specifying (4.68) to $s = 3/2$,

$$(5.21) \quad \mu_{3/2}(\ell_1, \ell_2) = \sum_{\omega'_3 | \Lambda} |\omega'_3|^{-1} \sum_{z | \frac{\Lambda}{\omega'_3}} |z|^{-2},$$

where we recall that $\Lambda = \ell_2 - i\ell_1$ is a complex combination of the electric and magnetic charges (ℓ_1, ℓ_2) . The instanton measure $\mu_{3/2}(\ell_1, \ell_2)$ should count the degeneracy of Euclidean D2-branes in the homology class $\ell_1 \mathcal{A} + \ell_2 \mathcal{B} \in H_3(\mathcal{X}, \mathbb{Z})$. For D2-instantons with A -type charge only, i.e., $\ell_2 = 0$, and such that $\ell := \ell_1$ is a product of inert primes (those of the form $p = 4n + 3$, see Appendix A), the first sum collapses to $\omega'_3 = 1$ and the instanton measure reduces to

$$(5.22) \quad \mu_{3/2}(\ell, 0) = \sum_{z | \ell} |z|^{-2}.$$

This reproduces the instanton measure found on the basis of $SL(2, \mathbb{Z})$ invariance in [29–31, 82], which by analogy with [5, 9, 10] should count ways of splitting a marginal bound state into smaller constituents. However, it is possible that ℓ be prime over the integers but factorizable over the Gaussian integers, e.g., $2 = -i(1 + i)^2$ or $5 = (2 + i)(2 - i)$, in which case the measure (5.21) involves additional contributions compared to [29–31, 82]. This novel feature of compactifications on rigid Calabi–Yau manifolds is an interesting prediction of $SU(2, 1; \mathbb{Z}[i])$ -invariance which deserves further investigation.

5.3.2. NS5-brane instantons. As mentioned above, the non-Abelian term $e^{\Phi_{(\text{NA})}}$ may be interpreted as NS5-brane instanton contributions. Although we have not been able to extract the coefficients $C_{r,k,\ell}^{(\text{NA})}(s)$ in the non-Abelian Fourier expansion (4.10), we can still extract the instanton action by taking the semiclassical limit. This corresponds to the asymptotic behavior of (4.10) in the limit $y \rightarrow \infty$, or, equivalently, to the saddle point approximation of the t -integral in (4.83) as analyzed in Section 4.6. Expanding the Whittaker function around $x = \infty$ yields

$$\begin{aligned} W_{k,m}(x) &\sim e^{-x/2} x^k \sum_{n \geq 0} \frac{\Gamma(m - k + n + \frac{1}{2}) \Gamma(m + k + \frac{1}{2})}{\Gamma(n + 1) \Gamma(m - k + \frac{1}{2}) \Gamma(m + k - n + \frac{1}{2})} x^{-n} \\ (5.23) \quad &\sim e^{-x/2} x^k \left[1 + \mathcal{O}(1/x) \right]. \end{aligned}$$

Implementing this in (4.10) and extracting the leading $r = 0$ term, we deduce that the leading order contribution to $e^{\Phi_{(\text{NA})}}$ is given by

$$(5.24) \quad e^{\Phi_{(\text{NA})}} \sim e^{\phi} \sum'_{k \in \mathbb{Z}} \sum_{\ell=0}^{4|k|-1} \sum_{n \in \mathbb{Z} + \frac{\ell}{4|k|}} C_{r,k,\ell} |k|^{-s} e^{-2\pi S_{k,q}} \left[1 + \mathcal{O}(e^{2\phi}) \right],$$

where we have defined

$$(5.25) \quad S_{k,q} = |k| e^{-2\phi} + 2|k| (\tilde{\chi} - n)^2 - iq\chi + 2ik(\psi + \chi\tilde{\chi}).$$

This reproduces the Euclidean action of k NS5-branes bound to $q := 4nk$ D2-branes. Note that even in the absence of D2-brane instanton contributions, $q = 0$, the real part of the action receives a contribution from the background Ramond–Ramond flux $\tilde{\chi}$, as found previously in [79]. For vanishing $\tilde{\chi}$, this

reduces to the pure NS5-brane instanton action of [16]:

$$(5.26) \quad S_k = |k|e^{-2\phi} + 2ikk\psi.$$

It should be emphasized that the result (5.24) displays the contribution from *A*-type D2-brane instantons only. The *B*-type D2-branes could be exposed by choosing the alternative polarization displayed in (4.7), but then the *A*-type D2-brane effects are not visible. This is in contrast to the situation in [34], where the appearance of an extra summation in the non-Abelian term made it possible to expose the D(-1), D5 and NS5-brane effects simultaneously.²⁴

Finally, we observe that the asymptotic expansion of the Whittaker function predicts an infinite series of perturbative corrections around the NS5-brane instanton background. This is in marked contrast to the case of type IIA Euclidean NS5-branes wrapping $K3 \times T^2$, where the perturbative corrections around the instanton background truncate at one loop [84].

6. Conclusions

In this work we postulated that quantum corrections to the hypermultiplet moduli space in type IIA string theory compactified on a rigid Calabi–Yau threefold with complex multiplication by $\mathbb{Z}[i]$ are controlled by the Picard modular group $SU(2, 1; \mathbb{Z}[i])$. We investigated the consequences of this assumption for the simplest automorphic form, the Eisenstein series (1.14). Despite a serious discrepancy with the sign of the one-loop term, the fact that we were led to D2- and NS5-brane instanton corrections with the correct classical action provides support for our postulate. In the case of D2-brane instantons, the prediction of the Eisenstein series (5.14) is in fact in full agreement with the general form predicted in [29, 31, 82], though one could argue that it is largely a consequence of the Laplace equation (1.13). The instanton measure (5.22) is also similar to the dilogarithm sum found in [29], with additional refinements when the charges include non-inert prime factors. It would be interesting to compare the instanton summation measure with the generalized Donaldson–Thomas invariants of rigid Calabi–Yau manifolds.

²⁴We note that the presence of an extra “theta-angle” in the NS5-brane instanton action of [34], compared to our result (5.25), is related to the fact that the spherical vector f_K in the principal series of $SL(3, \mathbb{R})$ displays a cubic phase factor [83] which is absent in the corresponding spherical vector for $SU(2, 1)$ [63].

While the sign of the one-loop term invalidates our proposal that the Eisenstein series (1.14) governs the exact metric on the hypermultiplet moduli space, and so forbids us to expect a detailed agreement between our summation measure and the generalized Donaldson–Thomas invariants, we do not think that it ruins the basic postulate that the Picard modular group $SU(2, 1; \mathbb{Z}[i])$ should act isometrically on the exact universal hypermultiplet moduli space. Rather, we take it as an incentive to construct a more sophisticated automorphic form, which would produce the correct one-loop term, as well as produce a non-trivial dependence on the coordinate z on the twistor fiber $\mathbb{C}P^1$, which is generally expected when all isometries are broken. In fact, since the twistor space is known to be described by holomorphic contact transformations, it is natural to expect that automorphic forms attached to the quaternionic discrete series of $SU(2, 1)$ should be relevant. Indeed, these forms can be lifted to sections of a certain complex line bundle on the twistor space $\mathcal{Z}_{\mathcal{M}_{\text{UH}}} = SU(2, 1)/(U(1) \times U(1))$ [63, 85]. It is challenging to construct such automorphic forms explicitly, and adapt the analysis in [33] to produce a manifestly $SU(2, 1, \mathbb{Z}[i])$ -invariant description of the twistor space. We anticipate, however, that the resulting instanton corrections will be qualitatively similar to the ones considered here, although the summation measure will certainly be quite different.

In this work we have concentrated exclusively on rigid Calabi–Yau threefolds whose intermediate Jacobian $J(\mathcal{X})$ admits complex multiplication by $\mathbb{Z}[i]$. It is interesting to ask how our construction may generalize to other values of the period matrix τ . When $J(\mathcal{X})$ admits complex multiplication by the ring of integers \mathcal{O}_d in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, $d > 0$, it is natural to conjecture that the relevant arithmetic subgroup of $SU(2, 1)$ would be $SU(2, 1; \mathcal{O}_d)$. For example, choosing $\tau = (1 + i\sqrt{3})/2 := \omega$ should correspond to the “Picard–Eisenstein” modular group $SU(2, 1; \mathbb{Z}[\omega])$, where $\mathbb{Z}[\omega]$ are the Eisenstein integers, corresponding to the ring of integers $\mathcal{O}_3 = \mathbb{Z}[\omega]$ in $\mathbb{Q}(\sqrt{-3})$ [86]. In contrast to the $\tau = i$ case, it is interesting to note that $SU(2, 1; \mathbb{Z}[\omega])$ does not contain the “rotation” generator R in (1.10) [87]. Indeed, one does not generally expect the full electric–magnetic duality group to be a quantum symmetry, but rather its subgroup generated by monodromies in the moduli space of complex structures (which is non-existent in the case of rigid Calabi–Yau threefolds).

From a purely mathematical point of view, we have provided several explicit constructions of an automorphic form for the Picard modular group attached to the principal continuous series of $SU(2, 1)$. In addition, we analyzed its Abelian and non-Abelian Fourier expansion in detail and found

evidence for its functional equation. Sums over Gaussian divisors and Dirichlet L -series for the Gauss field play central roles in the analysis. We expect that our results will be useful in subsequent investigations of automorphic forms for various types of Picard groups.

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Appendix A. Dirichlet series and Gaussian integers

In this appendix, we collect for the reader's convenience some standard facts about Dirichlet series and Gaussian integers.

A.1. Euler products and Dirichlet series

A series $a(n)$ for $n \in \mathbb{N}$ is called multiplicative if and only if $a(n_1 n_2) = a(n_1) a(n_2)$ whenever n_1 and n_2 are coprime [88]. The associated Dirichlet series

$$(A.1) \quad L(a, s) = \sum_{n>0} a(n) n^{-s}$$

constructed from a multiplicative $a(n)$ can be recast as an Euler product over the primes ($p > 1$)

$$(A.2) \quad L(a, s) = \prod_{p \text{ prime}} P(p, s),$$

where

$$(A.3) \quad P(p, s) = \sum_{k \geq 0} a(p^k) p^{-ks}.$$

As an example consider the multiplicative series (4.43). One finds

$$(A.4) \quad P(2, s) = \sum_{k \geq 0} (2^{1-s})^k = \frac{1}{1 - 2^{1-s}}$$

and for Pythagorean primes $p = 1 \pmod{4}$

$$(A.5) \quad P(p, s) = \sum_{k \geq 0} (p^{1-s})^k + \frac{p-1}{1-s} \partial_p \sum_{k \geq 0} (p^{1-s})^k = \frac{1 - p^{-s}}{(1 - p^{1-s})^2}.$$

For primes of the form $p = 3 \pmod{4}$ one has

$$(A.6) \quad P(p, s) = \sum_{k \geq 0} (p^{2-2s})^k + p^{-s} \sum_{k \geq 0} (p^{2-2s})^k = \frac{1 + p^{-s}}{(1 - p^{1-s})(1 + p^{1-s})},$$

whence one recovers (4.54).

A.2. Structure of Gaussian primes

The ring of Gaussian integers $\mathbb{Z}[i]$ forms a principal ideal domain [74, 89], i.e., every element admits a unique prime factorization up to migration of the four units $\pm 1, \pm i$. We will use the notation g for Gaussian primes and p for standard (rational) primes. Gaussian prime numbers $g = a + ib \in \mathbb{Z}[i]$ fall under three different cases, called *ramified*, *inert* and *split*.

- (i) The first case consists solely of $g = 1 + i$. Since $2 = -i(1 + i)^2$, this implies that $p = 2$, despite being prime in \mathbb{Z} , is no longer prime in $\mathbb{Z}[i]$. The rational prime $p = 2$ is said to be ramified over the Gaussian integers.
- (ii) Inert primes are of the form $g = 4n + 3$ for some $n \in \mathbb{N}$ such that $4n + 3$ is prime in \mathbb{Z} . The name inert indicates that such integers are prime both over \mathbb{Z} and $\mathbb{Z}[i]$.
- (iii) Split primes come in complex conjugate pairs g and \bar{g} . Such $g = a + ib, b \neq 0$ are prime if and only if $p := g\bar{g} = a^2 + b^2$ is a standard prime in \mathbb{Z} and $p > 2$. By Fermat's theorem on the sums of squares p must be of the form $p = 4n + 1$ for some $n \in \mathbb{N}$, i.e., a Pythagorean prime.

Appendix B. Spherical vector and p -adic Eisenstein series

Automorphic forms can be constructed quite generally using adelic methods, as explained for the layman, e.g., in [60–62]. In this appendix, we apply this method to recover the Eisenstein series $\mathcal{E}_s(\phi, \lambda, \gamma)$ for the Picard modular group. This alternative approach also sheds light on the relation between the quadratic constraint (3.3) and the representation-theoretic structure underlying the Eisenstein series. This appendix may be viewed as an automorphic extension of the results in Section 2 of [63].

B.1. Formal construction

In general, to construct an automorphic form Ψ on G/K , invariant under a discrete subgroup $G(\mathbb{Z}) \subset G$, we require three ingredients: (1) a K -invariant spherical vector $f_K \in \mathcal{H}$ (\mathcal{H} being a Hilbert space of square integrable functions), (2) a linear representation ρ of G acting on \mathcal{H} , and (3) a $G(\mathbb{Z})$ -invariant distribution $f_{\mathbb{Z}} \in \mathcal{H}^*$ in the dual space of \mathcal{H} . Using the natural pairing $\langle \cdot, \cdot \rangle$ between \mathcal{H} and \mathcal{H}^* , the automorphic form Ψ can then be defined formally as

$$(B.1) \quad \Psi(g) := \langle f_{\mathbb{Z}}, \rho(g) \cdot f_K \rangle,$$

with $g \in G$. By virtue of the Iwasawa decomposition,

$$(B.2) \quad G = NAK,$$

an arbitrary group element $g \in G$ splits as $g = nak := \mathcal{V}k$, and, since f_K is K -invariant, Ψ simplifies to

$$(B.3) \quad \Psi(\mathcal{V}) = \langle f_{\mathbb{Z}}, \rho(\mathcal{V}) \cdot f_K \rangle.$$

The coset representative $\mathcal{V} \in G/K$ transforms by $k^{-1} \in K$ from the right and $\gamma \in G(\mathbb{Z})$ from the left,

$$(B.4) \quad \mathcal{V} \mapsto \gamma \mathcal{V} k^{-1}.$$

From the point of view of $\Psi(\mathcal{V})$ the right action by k^{-1} on $\rho(\mathcal{V})$ becomes a left action on f_K , which is invariant by definition, and the left action of γ becomes a right action on $f_{\mathbb{Z}}$, which is also invariant. Hence, $\Psi(\mathcal{V})$ is by construction a function on the double quotient $G(\mathbb{Z}) \backslash G/K$ as desired.

Although very appealing, this method is often unpractical due to the difficulty of obtaining the invariant distribution. Adelic methods offer a powerful way to obtain $f_{\mathbb{Z}}$, by reducing this problem to that of finding the p -adic spherical vector f_p for all primes p (see [90, 91] for an introduction to p -adic numbers, and [60–62] for illustrations of the adelic method). The distribution $f_{\mathbb{Z}}(x)$ is then obtained as the product of $f_p(x)$ over all prime numbers of the given number field. For our purposes, p runs over the Gaussian primes and we will denote it by g in accordance with Appendix A. Nevertheless, we will refer to the approach as the p -adic approach. The function Ψ then can be rewritten formally as

$$(B.5) \quad \Psi(\mathcal{V}) = \sum'_{\vec{x} \in \mathbb{Q}(i)^n} \rho(\mathcal{V}) \cdot \left[\prod_{g \text{ prime}} f_g(\vec{x}) \right],$$

where \vec{x} is a vector of rational Gaussian numbers in $\mathbb{Q}(i)^n$, the product runs over all Gaussian prime numbers g including the “place at infinity” $g = \infty$, and we defined $f_{\infty} = f_K$. Note that we restrict to the case where $\mathcal{V} \in G(\mathbb{R})$, such that $\rho(\mathcal{V})$ acts only on f_K , but it is natural to extend (B.5) to the case where \mathcal{V} is an element of the adèle group $G(\mathbb{A})$, in which case $\rho(\mathcal{V})$ acts on all the f_g ’s as well. We shall now see that the Eisenstein series $\mathcal{E}_s(\phi, \lambda, \gamma)$, constructed in Section 3.1, can indeed be obtained from this adelic point of view.

B.2. Real and p -Adic spherical vector

To reproduce the Picard–Eisenstein series (3.2) by this method, we consider the principal continuous series representation of $SU(2, 1)$, induced from the Heisenberg parabolic P whose Lie algebra consists of the non-positive grade part of the 5-grading (2.8):

$$(B.6) \quad \mathfrak{p} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \subset \mathfrak{su}(2, 1).$$

The parabolic group P thus corresponds to the subgroup of lower-triangular matrices,

$$(B.7) \quad P = \left\{ \begin{pmatrix} t_1 & & \\ * & t_2 & \\ * & * & t_3 \end{pmatrix} \in SU(2, 1) : t_1 t_2 t_3 = 1 \right\}.$$

The coset space $P \backslash SU(2, 1)$ is isomorphic to the Heisenberg group N , and can be parameterized as follows:

$$(B.8) \quad n = e^{xX_1 + \tilde{x}\tilde{X}_1 + 2yX_2} = \begin{pmatrix} 1 & i\bar{C}_2 & C_1 \\ & 1 & C_2 \\ & & 1 \end{pmatrix} := \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{pmatrix} \in N,$$

where

$$(B.9) \quad C_1 := 2y + \frac{i}{2}|C_2|^2, \quad C_2 := x + \tilde{x} + i(\tilde{x} - x)$$

satisfy the quadratic relation

$$(B.10) \quad |C_2|^2 - 2\Im(C_1) = 0,$$

and the last equality in (B.8) defines the row vectors \vec{r}_i of the Heisenberg group element.

The coset space $N = P \backslash SU(2, 1)$ admits an action of $g \in SU(2, 1)$ by multiplication from the right, followed by a compensating action by $p(g) \in P$ from the left so as to restore the upper triangular gauge (B.8). The principal continuous series representation consists of functions $f(x, \tilde{x}, y)$ on N transforming by the character $\chi_s(p(g))$ under the action of g , where

$$(B.11) \quad \chi_s(p) := t_1^{-2s}, \quad p = \begin{pmatrix} t_1 & & \\ * & t_2 & \\ * & * & t_3 \end{pmatrix} \in P.$$

The spherical vector f_K can be obtained straightforwardly as follows [63], while the compensating left-action of P on the second and third rows, \vec{r}_2 and \vec{r}_3 , of n is quite complicated, the action on the first row \vec{r}_1 is very simple: $p \in P$ simply modifies \vec{r}_1 by an overall factor of t_1 . Moreover, the action of $k \in SU(2) \times U(1)$ leaves invariant the (complex) norms of the rows \vec{r}_i . The spherical vector f_K can therefore be obtained by raising the norm of the first row \vec{r}_1 of n to the appropriate power of s [63]:²⁵

$$\begin{aligned} f_K(x, \tilde{x}, y) &:= |\vec{r}_1|^{-2s} = (1 + |C_1|^2 + |C_2|^2)^{-s} \\ &= \left(1 + 2(x^2 + \tilde{x}^2) + 4y^2 + (x^2 + \tilde{x}^2)^2\right)^{-s}. \end{aligned}$$

This object is indeed invariant under $SU(2) \times U(1)$, since the right action of k on n is a “rotation” that preserves the norm, while the compensating

²⁵See also [83] for a similar construction in the context of $SL(3, \mathbb{R})$.

left action of p merely modifies f_K by an overall factor t_1^{2s} , which in turn is canceled against the character $\chi_s(p) = t_1^{-2s}$ which is present since f_K is in the principal series.

The next step is to compute the action of $\rho(\mathcal{V})$ on f_K . Following the prescription above, this can be done by first computing $n \cdot \mathcal{V} = p_0 \cdot n'$, with

$$(B.12) \quad \begin{aligned} p_0 &= \begin{pmatrix} e^{-\phi} & & & \\ & 1 & & \\ & & & e^{\phi} \\ & & & & \end{pmatrix} \in P, \\ n' &= \begin{pmatrix} 1 & ie^{\phi}(\bar{\lambda} + \bar{C}_2) & e^{2\phi}(\gamma + i\bar{C}_2\lambda + C_1) & & \\ & 1 & e^{\phi}(\lambda + C_2) & & \\ & & & 1 & \\ & & & & \end{pmatrix} \in P \backslash SU(2, 1). \end{aligned}$$

Applying this to the spherical vector $f_K(x, \tilde{x}, y) = f_K(n)$ yields

$$(B.13) \quad \rho(\mathcal{V}) \cdot f_K(n) = f_K(n\mathcal{V}) = f_K(p_0 n') = \chi_s(p_0) f_K(n') = e^{2s\phi} |\vec{r}'_1|^{-2s},$$

which may be written explicitly in the form

$$(B.14) \quad \rho(\mathcal{V}) \cdot f_K(C_1, C_2) = e^{-2s\phi} \left(|\bar{C}_1 - iC_2\bar{\lambda} + \bar{\gamma}|^2 + e^{-2\phi} |C_2 + \lambda|^2 + e^{-4\phi} \right)^{-s}.$$

The p -adic spherical vector $f_p(C_1, C_2)$ can now be found by replacing the Euclidean norm $|\cdot|$ appearing in the real spherical vector f_K by its counterpart over the p -adic Gaussian numbers²⁶ (see, e.g., [92])

$$(B.15) \quad |z|_g^{\mathbb{Q}(i)} := |g|^{-k}, \quad z \in \mathbb{Q}(i),$$

for any Gaussian prime g , with $k \in \mathbb{Z}$ being the maximum power of g appearing in the prime factorization of z in Gaussian primes. The p -adic spherical vector for a Gaussian prime g is then given by

$$(B.16) \quad f_g(C_1, C_2) := \left[|\vec{r}'_1|_g^{\mathbb{Q}(i)} \right]^{-2s} = \max \left(1, |\bar{C}_1|_g^{\mathbb{Q}(i)}, |C_2|_g^{\mathbb{Q}(i)} \right)^{-2s}.$$

²⁶Note that with this definition, $|z|_g^{\mathbb{Q}(i)}$ is not invariant under complex conjugation, as can be seen easily by taking z to be a split prime. The definition of the p -adic norm in [91] differs from the one we use. It is invariant under complex conjugation but misses other desirable properties; in particular it does not reproduce Equation (B.19) correctly.

B.3. Product over primes

The automorphic form $\Psi(\mathcal{V})$ in this representation now reads

$$(B.17) \quad \Psi(\mathcal{V}) = \sum'_{\substack{(C_1, C_2) \in \mathbb{Q}(i)^2 \\ |C_2|^2 - 2\Im(C_1) = 0}} \rho(\mathcal{V}) \cdot \left[\prod_{g < \infty} f_g(C_1, C_2) \right] f_K(C_1, C_2).$$

Next we must evaluate the infinite product over Gaussian prime numbers g . To this end we split the rational variables C_1 and C_2 in the following way:

$$(B.18) \quad C_1 = \frac{\omega_1}{\omega_3}, \quad C_2 = \frac{i\bar{\omega}_2}{\bar{\omega}_3},$$

with $\omega_j \in \mathbb{Z}[i]$, for $j = 1, 2, 3$, $\gcd(\omega_1, \omega_2, \omega_3) = 1$.²⁷ Using the definition (B.15) we can explicitly evaluate the infinite product over primes in (B.17) as

$$(B.19) \quad \prod_{g < \infty} \max \left(1, \left| \frac{\bar{\omega}_1}{\bar{\omega}_3} \right|_g^{\mathbb{Q}(i)}, \left| \frac{\bar{\omega}_2}{\bar{\omega}_3} \right|_g^{\mathbb{Q}(i)} \right)^{-2s} = |\omega_3|^{-2s}.$$

Multiplying the constraint (B.10) further by a factor of $|\omega_3|^2$ one obtains

$$(B.20) \quad |\omega_3|^2 \left(|C_2|^2 - 2\Im(C_1) \right) = |\omega_2|^2 - 2\Im(\omega_1\bar{\omega}_3) = \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = 0.$$

Combining Equations (B.17), (B.19) and (B.20) and adding the contribution at $C_1 = C_2 = \infty$ (i.e., $\omega_3 = 0$) then yields the final form of $\Psi(\mathcal{V})$:

$$(B.21) \quad \Psi(\mathcal{V}) = \sum'_{\substack{\vec{\omega} \in \mathbb{Z}[i]^3, \gcd(\omega_1, \omega_2, \omega_3) = 1 \\ |\omega_2|^2 - 2\Im(\omega_1\bar{\omega}_3) = 0}} \times e^{-2s\phi} \left[|\bar{\omega}_1 + \bar{\omega}_2\bar{\lambda} + \bar{\omega}_3\bar{\gamma}|^2 + e^{-2\phi} |\bar{\omega}_2 - i\bar{\omega}_3\lambda|^2 + e^{-4\phi} |\omega_3|^2 \right]^{-s},$$

which we recognize as the Eisenstein series $\mathcal{P}_s(\phi, \lambda, \gamma)$ constructed in Section 3.2.

²⁷We note that the greatest common divisor in $\mathbb{Z}[i]$ is defined up to Gaussian units which are a subgroup of order 4 in the Gaussian integers $\mathbb{Z}[i]$. See Appendix A for more details on the Gaussian integers.

Appendix C. More on the Abelian measure

This appendix contains a detailed analysis of the norm constraint (4.25) entering the Fourier expansion at various places and the derivation of the Abelian measure (4.68) as a sum over Gaussian divisors.

C.1. Analysis of the norm constraint

The norm constraint (4.25) requires to find, for a fixed integer d , all Gaussian integers with norm squared divisible by $2d$. In this appendix we write this constraint as

$$(C.1) \quad |\alpha|^2 := 0 \pmod{2d}.$$

C.1.1. Multiplicative structure. The solutions to the norm constraint possess a multiplicative structure. Let d_1 and d_2 be coprime integers and let α_1 and α_2 be Gaussian integers such that $2d_i$ divides $|\alpha_i|^2$. Then clearly $\alpha_1\alpha_2$ satisfies the norm constraint for d_1d_2 . Due to prime factorization of Gaussian integers we know that this describes all solutions and it is therefore sufficient to study the solutions to the norm constraint for powers of primes $d = p^k$. There are three qualitatively different cases.

- (i) $p = 2$, whence $d = 2^k$. The structure of the set of solutions looks different for k even and odd. For k even one has that

$$(C.2) \quad \alpha = 2^{k/2}(n_1 + in_2) \quad \text{for } n_1 + n_2 \in 2\mathbb{Z}$$

solves the constraint, whereas for k odd

$$(C.3) \quad \alpha = 2^{(k+1)/2}(n_1 + in_2)$$

solves the constraint without restriction on the integers n_1 and n_2 .

- (ii) $p = 4n + 3$. Again one has to distinguish k even and k odd in solving (C.1) for $d = p^k$. For k even one has

$$(C.4) \quad \alpha = p^{k/2}(n_1 + in_2) \quad \text{for } n_1 + n_2 \in 2\mathbb{Z}$$

solves the constraint, whereas for k odd

$$(C.5) \quad \alpha = p^{(k+1)/2}(n_1 + in_2) \quad \text{for } n_1 + n_2 \in 2\mathbb{Z}$$

solves the constraint. Note that there are restrictions on the integers n_1 and n_2 in both cases.

(iii) $p = 4n + 1$. This case is the most complicated one. Any such prime can be written as $p = a^2 + b^2$ for some integers a and b and we assume $a > b$ without loss of generality. To describe the set of solutions to (C.1) for $d = p^k$ we again distinguish even and odd k . An important auxiliary definition is furnished by

$$(C.6) \quad e_k = (a - ib)^k(1 + i) \quad \Rightarrow \quad |e_k|^2 = 2p^k,$$

providing an elementary solution of the constraint. With the help of the Gaussian integer e_k one can define the following pairs of lattices for k odd and $j = 0, \dots, \frac{k-1}{2}$:

$$(C.7) \quad \begin{aligned} \Lambda_{j+1} &= \{p^j(k_1 e_{k-2j} + k_2 i e_{k-2j}) : k_1, k_2 \in \mathbb{Z}\}, \\ \bar{\Lambda}_{j+1} &= \{p^j(k_1 \bar{e}_{k-2j} + k_2 i \bar{e}_{k-2j}) : k_1, k_2 \in \mathbb{Z}\}. \end{aligned}$$

The set of all solutions for k odd is then given by

$$(C.8) \quad \bigcup_{j=0}^{(k-1)/2} (\Lambda_{j+1} \cup \bar{\Lambda}_{j+1}).$$

For k even one also requires the lattice

$$(C.9) \quad \Lambda_{k+1} = \left\{ p^{k/2}(k_1 + ik_2) : k_1, k_2 \in \mathbb{Z} \text{ and } k_1 + k_2 \in 2\mathbb{Z} \right\}$$

and then all solutions are given by

$$(C.10) \quad \bigcup_{j=0}^{k/2-1} (\Lambda_{j+1} \cup \bar{\Lambda}_{j+1}) \cup \Lambda_{k+1}.$$

Pictures of the three kinds of solution sets will be given momentarily when discussing the restriction to a fundamental domain under the action of a translation group.

C.1.2. Restriction to a fundamental domain. In the Abelian measure we made use of writing the solution to the constraint in terms of solutions in a fundamental domain in (4.40). We denote by

$$(C.11) \quad \mathcal{F}(d) = \{ \alpha \in \mathbb{Z}[i] : |\alpha|^2 := 0 \pmod{2d} \text{ and } 0 \leq \Re(\alpha) < d, 0 \leq \Im(\alpha) < 2d \}$$

the set of solutions to (C.1) in the fundamental domain. From the analysis above we know that for d_1 and d_2 coprime, the following holds:

$$(C.12) \quad \mathcal{F}(d_1 d_2) \cong \mathcal{F}(d_1) \times \mathcal{F}(d_2),$$

where the solutions are of the form $d_2 f_1 + d_1 f_2$ for $f_i \in \mathcal{F}(d_i)$ up to translation by the lattice L of (4.41) defining the fundamental domain. Therefore it is sufficient to restrict to $d = p^k$ being a power of a prime. For describing (C.11) more explicitly we have to make recourse to the results of the preceding section and distinguish three cases.

- (i) $d = 2^k$. Here one simply restricts the integers n_1 and n_2 in (C.2) and (C.3). The number of points in the fundamental domain is

$$(C.13) \quad \sharp \mathcal{F}(2^k) = 2^k = N(2^k)$$

in agreement with (4.53). The easiest way of doing the counting is by computing the sizes of the fundamental cell of the lattices and comparing to the total area of the fundamental domain $2d^2$. A typical lattice is depicted on the left of figure 1. For $k = 1$ there are two points in the fundamental domain.

- (ii) $d = p^k$ for $p = 4n + 3$. The counting works similar, but one has to take into account the additional constraint in (C.5) leading to

$$(C.14) \quad \sharp \mathcal{F}(p^k) = p^{2\lfloor k/2 \rfloor} = N(p^k).$$

The form of the lattice is identical to that of the left part of figure 1. For $k = 1$ there is only a single point $\alpha = 0$ in the fundamental domain.

- (iii) $d = p^k$ for $p = 4n + 1$. The counting of points in the fundamental domain is now more involved since the individual lattices in (C.8), or (C.10), have common points that should not be overcounted. Each lattice has p^k points but, for example, there are p^{2j} common points for the lattices Λ_{j+1} and $\bar{\Lambda}_{j+1}$ forming the square lattice

$$(C.15) \quad \Lambda_{j+1} \cap \bar{\Lambda}_{j+1} = \left\{ p^{k-j}(k_1 + ik_2) : k_1 + k_2 \in 2\mathbb{Z} \right\},$$

implying

$$(C.16) \quad \sharp ((\Lambda_{j+1} \cup \bar{\Lambda}_{j+1}) \cap \mathcal{F}(d)) = 2p^k - p^{2j}.$$

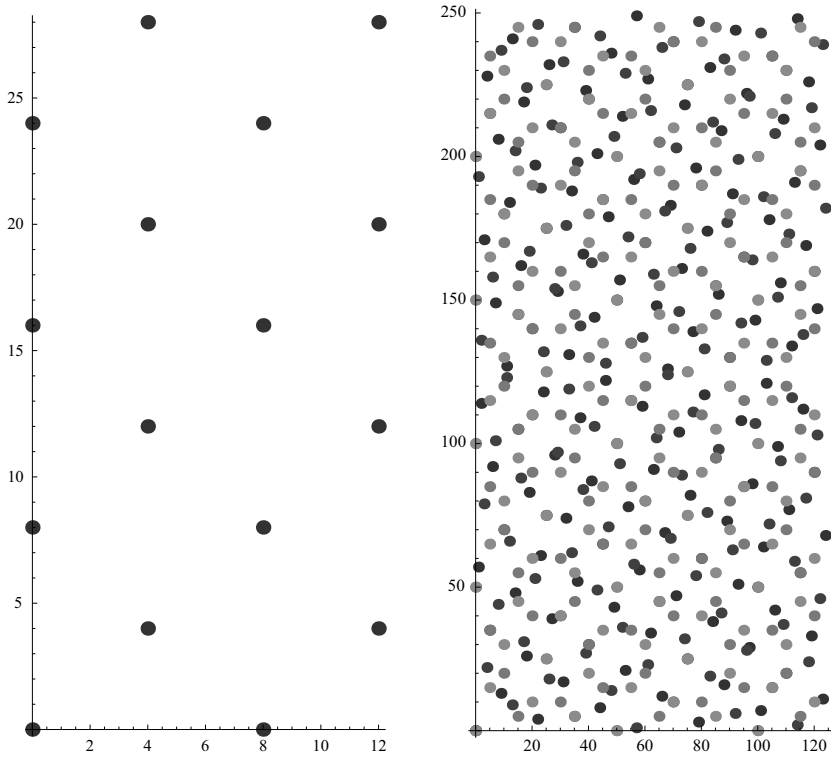


Figure 1: Left: The set $\mathcal{F}(2^4)$ is a $\pi/4$ rotated and rescaled square lattice. Right: The set $\mathcal{F}(5^3)$ as the union of four lattices with common points.

One can also show that

$$(C.17) \quad \# \left((\Lambda_{j+1} \cup \bar{\Lambda}_{j+1}) \cap (\Lambda_{j+2} \cup \bar{\Lambda}_{j+2}) \cap \mathcal{F}(d) \right) = 2p^{k-1} - p^{2j}$$

and that all common points between pairs of lattices whose indices are farther apart are already contained in the intersection above. Putting everything together one arrives at the following count of points in the fundamental domain for k odd:

$$(C.18) \quad \begin{aligned} \# \mathcal{F}(p^k) &= 2p^k - p^0 + \sum_{j=1}^{(k-1)/2} (2p^k - p^{2j} - (2p^{k-1} - p^{2j-2})) \\ &= (k+1)p^k - kp^{k-1} = N(p^k). \end{aligned}$$

For k even the analysis is similar. An example of lattices with intersection points can be found on the right of figure 1.

C.2. Rewriting the Abelian measure

We now turn to deriving (4.68) from the Abelian measure (4.66). This involves mainly demonstrating the equality (4.67). To this end we introduce an additional function on the Gaussian integers

$$\begin{aligned}
 \nu_s(q) &= |q|^{2s-2} \beta(2s-1) \left(\sum_{d>0} d^{1-2s} \sum_{f \in \mathcal{F}(d)} e^{\frac{\pi i}{d} \Re[qf(1-i)]} \right) \\
 \text{(C.19)} \quad &= |q|^{2s-2} \beta(2s-1) \sum_{d>0} d^{1-2s} a_q(d).
 \end{aligned}$$

The function $\nu_s(q)$ is related to the l.h.s. of (4.67) in an obvious way.

C.2.1. Evaluation of $a_q(d)$ and $\nu_s(q)$. The first observation is that the series $a_q(d)$ is multiplicative in d for fixed q (but not in q). This follows from (C.12) and a simple rewriting of the exponent. Therefore it is sufficient to determine $a_q(d)$ for $d = p^k$. This is where the description of the sets $\mathcal{F}(p^k)$ enters. The series $\nu_s(q)$ can be shown to be multiplicative so we only require $a_q(d)$ for $q = g^m$ as a power of a Gaussian prime and $d = p^k$ the power of standard prime.

The next observation is that

$$\text{(C.20)} \quad a_{g^m}(p^k) = \sum_{f \in \mathcal{F}(p^k)} e^{\frac{\pi i}{p^k} \Re[g^m f(1-i)]} = \sum_{f \in g^m \mathcal{F}(p^k)} e^{\frac{\pi i}{p^k} \Re[f(1-i)]}$$

by rotating (and rescaling) the set of fundamental solutions. If g does not divide p then the rotated set is an equivalently good fundamental set of solutions. Hence

$$\text{(C.21)} \quad a_{g^m}(p^k) = a_1(p^k) \quad \text{if } g \text{ does not divide } p.$$

For this reason we will first evaluate $a_1(p^k)$ and treat the case when g divides p afterwards.

It turns out that it suffices to count the number of times the lattice containing only the point $\alpha = 0$ in $\mathcal{F}(p^k)$ appears in the sum over $\mathcal{F}(p^k)$. For

all other lattices the sum over phases is zero. Hence one finds immediately

$$(C.22) \quad a_1(2^k) = 0 \quad \text{for } k > 0$$

and for $p = 4n + 3$ that

$$(C.23) \quad a_1(p^k) = \begin{cases} 1, & k = 1, \\ 0, & k > 1. \end{cases}$$

For $p = 4n + 1$ one has to count more carefully due to the intersection points. For $k = 1$ the origin is the only common point in Λ_1 and $\bar{\Lambda}_1$ and hence is overcounted once leading to $a_1(p) = -1$. For $k > 1$ this is offset by the intersection with $\Lambda_2 \cup \bar{\Lambda}_2$, making $a_1(p^k)$ vanish. In total one has therefore for the Pythagorean primes

$$(C.24) \quad a_1(p^k) = \begin{cases} -1, & k = 1, \\ 0, & k > 1. \end{cases}$$

Constructing the Dirichlet series in (C.19) via its Euler product therefore leads to, after referring back to (4.21),

$$(C.25) \quad \sum_{d>0} d^{1-2s} a_1(d) = \frac{1}{\beta(2s-1)} \quad \Rightarrow \quad \nu_s(1) = 1.$$

For $q = g^m$ one can perform a scaling of the lattices involved. Starting with the case of $g = 1 + i$ the value $p = 2$ is important and one has

$$(C.26) \quad (1 + i)^m \mathcal{F}(2^k) \cong [\mathcal{F}(2^{k-m})]^{2^m}$$

defining the right-hand side to consist only of the origin for $k \leq m$. Counting now the number of times the origin appears leads to

$$(C.27) \quad a_{(1+i)^m}(2^k) = \begin{cases} 0, & \text{for } k > m, \\ 2^k = N(2^k), & \text{for } k \leq m. \end{cases}$$

The corresponding auxiliary series (C.19) is then

$$(C.28) \quad \begin{aligned} \nu_s((1 + i)^m) &= 2^{m(s-1)} \beta(2s-1) \left(1 + \sum_{k=1}^m 2^{k(2-2s)} \right) \sum_{d>0} d^{1-2s} a_1(d), \\ &= \frac{1}{4} 2^{m(1-s)} \sum_{z|(1+i)^m} |z|^{4s-4}, \end{aligned}$$

and so is the usual divisor function multiplied by the proper overall factor to make it symmetric under $s \leftrightarrow 2 - s$.

A similar analysis can be carried out for inert primes leading to

$$(C.29) \quad a_{p^m}(p^k) = \begin{cases} 0 & \text{for } k \geq 2(m+1), \\ p^{2\lfloor k/2 \rfloor} = N(p^k) & \text{for } k < 2(m+1). \end{cases}$$

Therefore the full result for (C.19) for inert primes is

$$(C.30) \quad \begin{aligned} \nu_s(p^m) &= p^{2m(s-1)} \beta(2s-1) \left(1 + \sum_{k=1}^{2m+1} p^{2\lfloor k/2 \rfloor + k(1-2s)} \right) \frac{\sum_{d>0} d^{1-2s} a_1(d)}{1 + p^{1-2s}} \\ &= \frac{1}{4} p^{2m(1-s)} \sum_{z|p^m} |z|^{4s-4}. \end{aligned}$$

For split primes one finds

$$(C.31) \quad a_{g^m}(p^k) = \begin{cases} 0 & \text{for } k > m+1, \\ -p^m & \text{for } k = m+1, \\ p^{k-1}(p-1) & \text{for } k < m+1. \end{cases}$$

Therefore the full result for split primes is

$$(C.32) \quad \begin{aligned} \nu_s(g^m) &= \frac{p^{m(s-1)}}{1 - p^{1-2s}} \left(1 + \sum_{k=1}^m (p-1) p^{k-1+k(1-2s)} - p^{m+(m+1)(1-2s)} \right) \\ &= \frac{1}{4} |g|^{2m(1-s)} \sum_{z|g^m} |z|^{4s-4}. \end{aligned}$$

In summary the function $\nu_s(q)$ defined in (C.19) takes the value

$$(C.33) \quad \nu_s(q) = \frac{1}{4} |q|^{2-2s} \sum_{z|q} |z|^{4s-4}$$

for any Gaussian integer $q \neq 0$ and therefore is a Gaussian divisor function in disguise.

C.2.2. The Abelian instanton measure. The Abelian instanton measure of (4.66) is thus given by a sum over primitive divisors of $\Lambda = \ell_2 - i\ell_1$

$$(C.34) \quad \mu_s(\ell_1, \ell_2) = 4 \sum_{\omega'_3|\Lambda} |\Lambda|^{2-2s} \nu_s \left(\frac{\Lambda}{\omega'_3} \right),$$

which, together with (C.33), leads to (4.68).

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