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Feynman amplitudes and Landau singularities for one-loop graphs

Spencer Bloch and Dirk Kreimer

We introduce the parametric representation for Feynman integrals via configuration polynomials. We then deduce the well-known fact that massive one-loop Feynman integrals for n-particle scattering evaluate in terms of dilogarithms from the structure of such amplitudes as mixed Hodge structures. Analytic properties of such amplitudes are related to variations of Hodge structures and discussed in examples.

1. Introduction

The subject of Feynman amplitudes with variable momenta and non-zero masses has been studied by physicists since the 1950s. In the interim, new mathematical methods involving Hodge structures and variations of Hodge structures have been developed. The purpose of this paper is to apply these techniques to the study of amplitudes and Landau singularities in kinematic (mass and momentum) space. While the techniques we develop bear on the general case here, we will mainly focus on the one-loop case. In this case, for general values of masses and external momenta, the polar locus of the integrand (written in Feynman coordinates) is a smooth quadric. (Exceptionally, in the "triangle case," the polar locus is a union of a hyperplane and a quadric.) Mathematically, the polar loci form a degenerating family of such objects, which is a familiar and well-studied situation in algebraic geometry. Our objective is firstly to explain motivically the known fact [4] that dilogarithms are ubiquitous in this situation, and secondly to show how the motivic and Hodge-theoretic framework is a powerful way to study thresholds and Landau singularities.

In Section 2, we sketch briefly what we will need from the theory of (mixed) Hodge structures. The Hodge structures which arise in the context of one-loop graphs are quite simple, but it is important to understand how to pass to limits in order to study thresholds in physics. In Section 3, we develop the basic properties of the second Symanzik polynomial that is treated as a quaternionic pfaffian in the sense of E. H. Moore [9]. The motives we need to study are extensions of motives associated to hypersurfaces defined by a linear combination of the first and second Symanzik polynomials.

Section 5 develops the basic calculus of differential forms on projective space that is necessary to calculate the de Rham realizations of our motives. Section 6 is devoted to the essential technical result, Lemma 6.3, which determines the structure of all the one-loop motives. Section 7 defines the relevant motives. We show (formula (7.6)) that the weight graded object is a sum of Tate motives $\mathbb{Q}(i)$ for $i = 0, -1, -2, -3$. In Section 8, we consider the amplitude itself and show it is a period of a sub-Hodge structure (dilogarithm Hodge structure) involving only $\mathbb{Q}(0), \mathbb{Q}(-1), \mathbb{Q}(-2)$. Completing this chain of ideas, we show in Section 9 that the motive of such a dilog Hodge structure is always a sum of dilogs and quadratic expressions in logarithms (cf. (2.5) below). The argument is variational and uses Griffiths transversality. The authors learned it from [2].

Section 10 discusses the motive of the one-loop graph with three edges, the *triangle graph*. It turns out that $gr^W H = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)^{5-\nu} \oplus \mathbb{Q}(-2)$ where ν is the number of masses $m_i = 0$. In an appendix, we discuss a duality theorem which is natural mathematically but does not have any obvious physical interpretation. Section 12 is a general discussion from a mathematical viewpoint of Landau poles and thresholds, and Section 13 discusses the limiting mixed Hodge structures associated to various degenerations. Finally, the last Section 14 offers a physical interpretation of the period matrix in the triangle case via Cutkosky rules.

2. Hodge structures

A pure $\mathbb{Q}$-Hodge structure of weight $n \in \mathbb{Z}$ is a finite-dimensional $\mathbb{Q}$-vector space $H_\mathbb{Q}$ together with a decreasing (Hodge) filtration $F^*H_\mathbb{C}$ defined on $H_\mathbb{C} := H_\mathbb{Q} \otimes \mathbb{C}$. F^* is required to be n-opposite to its complex conjugate in the sense that for any i

$$H_\mathbb{C} \cong F^i H_\mathbb{C} \oplus \overline{F}^{n+1-i} H_\mathbb{C}, \tag{2.1}$$

Here $\overline{F}^j$ is obtained by applying complex conjugation to F^j. It is straightforward to check that if we define $H^{i,j} := F^i \cap \overline{F}^j$, then (2.1) is equivalent to the direct sum decomposition

$$H_\mathbb{C} = \bigoplus_i H_\mathbb{C}^{i,n-i}. \tag{2.2}$$

A $\mathbb{Q}$-mixed Hodge structure is a finite-dimensional $\mathbb{Q}$-vector space with an increasing filtration (weight filtration) W_*H as well as a Hodge filtration $F^*H_\mathbb{C}$. We require that the induced Hodge filtration on $gr_n^W H_\mathbb{C}$ give $gr_n^W H$, the structure of a pure Hodge structure of weight n for each n.

The only pure Hodge structures of dimension 1 are the Tate Hodge structures $\mathbb{Q}(n)$. By definition, $\mathbb{Q}(n)$ has weight $-2n$. We have

$$F^i\mathbb{Q}(n)_\mathbb{C} = \begin{cases} 0, & i > -n, \\ \mathbb{Q}(n)_\mathbb{C}, & i \leq -n. \end{cases} \tag{2.3}$$

In other words, $\mathbb{Q}(n)_\mathbb{C} = (\mathbb{Q}(n)_\mathbb{C})^{-n,-n}$.

A mixed Hodge structure H is called *mixed Tate* if

$$gr_n^W H = \begin{cases} 0, & n = 2m-1, \\ \bigoplus \mathbb{Q}(-m), & n = 2m. \end{cases} \tag{2.4}$$

The central result of this paper is that Feynman amplitudes at one loop involve only mixed Tate Hodge structures. Moreover, the Hodge structures that arise have only three non-trivial weights which we can take to be $0, 2, 4$. We refer to them as *dilogarithm Hodge structures.*

Definition 2.1. A dilogarithm mixed Hodge structure H is a mixed Tate Hodge structure such that for some integer n, we have $gr_{2p}^W H = 0$ for $p \neq n, n+1, n+2$.

We will see in Section 9 that periods of dilogarithm Hodge structures have the form

$$\sum_\mu \int \log f_\mu \frac{dg_\mu}{g_\mu}, \tag{2.5}$$

where the f_μ, g_μ are rational functions.

For a mixed Tate Hodge structure, the weight and Hodge filtrations are opposite in the sense that

$$F^{p+1}H_\mathbb{C} \cap W_{2p}H_\mathbb{C} = (0); \quad H_\mathbb{C} = \bigoplus_p (F^p H_\mathbb{C} \cap W_{2p}H_\mathbb{C}). \tag{2.6}$$

We may choose a basis $\{e_i^{p,p}\}$ of $H_\mathbb{C}$ with $e_i^{p,p} \in F^pH_\mathbb{C} \cap W_{2p}H_\mathbb{C}$.

Example 2.2 (Kummer extensions)**.** To an element $x \in \mathbb{C}^\times$, we can associate a mixed Tate Hodge structure E_x with $gr^W E_x = \mathbb{Q}(1) \oplus \mathbb{Q}(0)$.

Define a free rank 2 $\mathbb{Z}$-module $E_{x,\mathbb{Z}} = \mathbb{Z}\varepsilon_{-1} \oplus \mathbb{Z}\varepsilon_0$ with weight filtration $W_{-2}E_{x,\mathbb{Z}} = \mathbb{Z}\varepsilon_{-1} = W_{-1} \subset W_0 = E_{x,\mathbb{Z}}$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1\mapsto 2\pi \mathrm{i}} & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times & \longrightarrow & 0 \\ & & \| & & \psi\uparrow & & \varepsilon_0\mapsto x\uparrow & & \\ 0 & \longrightarrow & \mathbb{Z}\varepsilon_{-1} & \longrightarrow & E_{x,\mathbb{Z}} & \longrightarrow & \mathbb{Z}\varepsilon_0 & \longrightarrow & 0. \end{array} \tag{2.7}$$

Here $\psi(\varepsilon_{-1}) = 2\pi\mathrm{i}$ and $\psi(\varepsilon_0) = \log(x)$ for some branch of the logarithm.

By linearity, ψ extends to a $\mathbb{C}$-linear map $\psi : E_{x,\mathbb{C}} = E_{x,\mathbb{Z}} \otimes \mathbb{C} \to \mathbb{C}$. Define

$$F^0E_{x,\mathbb{C}} := \ker(\psi_{\mathbb{C}}) = \mathbb{C}\cdot\left(\varepsilon_0 - \frac{\log x}{2\pi\mathrm{i}}\varepsilon_{-1}\right) \subset F^{-1}E_{x,\mathbb{C}} = E_{x,\mathbb{C}}. \tag{2.8}$$

We take $e^{0,0} = \varepsilon_0 - \frac{\log x}{2\pi\mathrm{i}}\varepsilon_{-1}$ and $e^{-1,-1} = \frac{1}{2\pi\mathrm{i}}\varepsilon_{-1}$. It is traditional for mixed Tate Hodge structures to consider the matrix where the columns, interpreted as coefficients of the $e^{i,i}$ form a basis for the $\mathbb{Q}$-structure. In this case $\varepsilon_0 = e^{0,0} + (\log x)e^{-1,-1}, \varepsilon_{-1} = 2\pi\mathrm{i}e^{-1,-1}$ so the matrix is

$$\begin{pmatrix} 1 & 0 \\ \log x & 2\pi\mathrm{i} \end{pmatrix}. \tag{2.9}$$

The category of mixed Hodge structures is Abelian, and

$$\mathrm{Ext}^1(\mathbb{Q}(-1), \mathbb{Q}) \cong \mathbb{C}^\times; \quad E_x \mapsto x. \tag{2.10}$$

We remark that the category of Hodge structures has a tensor product. The definitions follow easily from the definition of a tensor product of filtered vector spaces. One has for example $\mathbb{Q}(m) \otimes \mathbb{Q}(n) = \mathbb{Q}(m+n)$. Tensoring with $\mathbb{Q}(n)$ for a suitable n, we may if we like arrange that any given mixed Tate Hodge structure has weights $0, 2, \ldots, 2r$ for some r.

The central point is that the Betti cohomology of any complex variety (indeed, more generally any diagram of complex algebraic varieties) carries a canonical and functorial mixed Hodge structure. Because Betti groups can be computed using differential forms (de Rham cohomology) our Hodge structures will often have another rational structure coming from algebraic de Rham cohomology. From this viewpoint, the periods, which are the numbers of arithmetic and physical interest, are the entries in a matrix transforming the Betti rational structure to the de Rham rational structure. (As a small example of the power of the method in physics, one can often check the powers of $2\pi\mathrm{i}$ occurring in formulas without doing any computation.)

Example 2.3. Consider the Hodge structure $H := H^1(\mathbb{P}^1 - \{0, \infty\}, \mathbb{Q})$. (One writes abusively $H = H_B = H_{\mathbb{Q}}$ to denote the $\mathbb{Q}$-vector space of $\mathbb{Q}$-Betti cohomology. H also denotes the corresponding object in the category of Hodge structures.) By standard topology this group is one dimensional, dual to the first homology that is spanned by a small circle S around 0 oriented in a counterclockwise direction. Let $z \in H_{\mathbb{Q}}$ be a generator with $\langle z, S\rangle = 1$. As a Hodge structure, $H = \mathbb{Q}(-1)$. On the other hand, the corresponding de Rham cohomology $H^1_{\mathrm{DR}}(\mathbb{P}^1 - \{0, \infty\})$ is the $\mathbb{Q}$-vector space defined by the 1-form dt/t where t is the coordinate on $\mathbb{P}^1$. The pairing with homology is given by integration, and since $\int_S dt/t = 2\pi\mathrm{i}$, it follows that $dt/t = 2\pi\mathrm{i}z \in H$. The $\mathbb{Q}$-vector space H is the Betti cohomology, $H = H^1_B(\mathbb{P}^1 - \{0, \infty\}, \mathbb{Q})$ and the de Rham $\mathbb{Q}$-structure is given in this case by $H_{\mathrm{DR}} = 2\pi\mathrm{i}H_B$.

The referee points out a possible confusion. In Example 2.2, one obtains periods $(1, 2\pi\mathrm{i}, \log x)$ seemingly without fixing a de Rham structure. In fact, however, $\varepsilon_{-1} \in \mathbb{Q}(1)_{\mathbb{C}} \subset E_{x,\mathbb{C}}$ is part of a de Rham basis. Also $\varepsilon_0 \in \mathbb{Q}(0)_{\mathbb{C}}$ is a de Rham basis for $\mathbb{Q}(0)$. The map $F^0E_{x,\mathbb{C}} \xrightarrow{\cong} \mathbb{Q}(0)_{\mathbb{C}}$ is compatible with the de Rham structures, so one has $E_{x,\mathrm{DR}} = \mathbb{Q}\varepsilon_{-1} + \mathbb{Q}(\varepsilon_0 - \frac{\log x}{2\pi\mathrm{i}}\varepsilon_{-1})$. Note that the de Rham rational structure is compatible with the Hodge filtration, but the Betti rational structure is not.

Families of varieties give rise to families of Hodge structures. Of particular interest is the *nilpotent orbit theorem* that is the basic tool in describing degenerations. In the physics surrounding one-loop Feynman graphs, these degenerations (thresholds) are not as well understood as they might be, and we will show how the nilpotent orbit theorem can be applied.

To avoid complications, we focus on a 1-parameter degeneration $\{H_t\}$ parametrized by $t \in D^* = \{t \in \mathbb{C} \mid 0 < |t| < \varepsilon\}$. This means that we are given a local system $\mathcal{H}$ over D^* with fibre H_t. We have a weight filtration that is an increasing filtration $W_*\mathcal{H}$ on the local system, and a Hodge filtration that is a decreasing filtration by coherent subbundles $F^*(\mathcal{H} \otimes_{\mathbb{C}} \mathcal{O}_{D^*})$. The point is that the Hodge filtration is not horizontal for the flat structure determined by the local system $\mathcal{H}$. However, *Griffiths transversality* says

$$\frac{d}{dt}F^i(\mathcal{H} \otimes_{\mathbb{C}} \mathcal{O}_{D^*}) \subset F^{i-1}(\mathcal{H} \otimes_{\mathbb{C}} \mathcal{O}_{D^*}). \tag{2.11}$$

A very general result in algebraic geometry gives that the monodromy on our local system is quasi-unipotent. In other words, replacing t by $u = t^n$ for some n, the action σ of winding around the puncture in D^* on a fibre of $\mathcal{H}$ will be unipotent. This allows us to deal for example with the square roots which one faces in one-loop computations early on. Assuming this has been done, we write

$$N := \log(\sigma), \tag{2.12}$$

so N is a nilpotent endomorphism of a fibre.

If we choose $t_0 \in D^*$, we can identify our variation of Hodge structure as a single $\mathbb{Q}$-vector space $H = \mathcal{H}_{t_0}$ with a weight filtration W_*H, a nilpotent endomorphism $N : H \to H$ stabilizing W_*, and a variable Hodge filtration F_t^*H. In this situation, the nilpotent orbit theorem gives a decreasing filtration $F_{\lim}H_{\mathbb{C}}$ such that the orbit of the one parameter subgroup $\exp(N\frac{\log t}{2\pi \mathrm{i}})$ acting on the filtration $F_{\lim}H_{\mathbb{C}}$ approximates the given F_t^*H:

$$\exp\left(N\frac{\log t}{2\pi \mathrm{i}}\right) F_{\lim} \sim F_t. \tag{2.13}$$

Another way to think about (2.13) is to note (again by a general result in algebraic geometry) that the coherent sheaf $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{O}_{D^*}$ extends to a coherent sheaf $\widetilde{\mathcal{H}}$ on $D = D^* \cup \{0\}$ in such a way that the connection on $\mathcal{H}$ extends to a connection with log poles on $\widetilde{\mathcal{H}}$, i.e., we have

$$\widetilde{\nabla} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}} \cdot \frac{dt}{t}. \tag{2.14}$$

Choose a basis γ_i for $H^\vee$ that we then view as a multi-valued basis of the local system $\mathcal{H}^\vee$. We view the γ_i as homology classes. For ω_t a section of $\widetilde{\mathcal{H}}$, we write $\langle \gamma_i, \omega_t \rangle = \int_{\gamma_i} \omega_t$ as an integral. Then the entries of

$$\exp\left(-N\frac{\log t}{2\pi \mathrm{i}}\right) \begin{pmatrix} \vdots \\ \int_{\gamma_i} \omega_t \\ \vdots \end{pmatrix} \tag{2.15}$$

are single-valued functions on D^*, and the limit $|t| \to 0$ exists.

Furthermore, the filtration F_t is meromorphic with respect to the extension in the sense that we can find a basis of the global sections of $\mathcal{H} \otimes \mathcal{O}_{D^*}$ which is compatible with the filtration $F^*(\mathcal{H} \otimes \mathcal{O}_{D^*})$ and which lies in $t^{-M}\widetilde{\mathcal{H}}$ for $M > 0$. This means there exists a unique saturated filtration $F^*\widetilde{\mathcal{H}}$ inducing F_t on $\mathcal{H} \otimes \mathcal{O}_{D^*}$. If we choose a basis $\omega_{t,j}$ of $\widetilde{\mathcal{H}}$ compatible with the filtration and compute the limits in (2.15), we obtain a concrete matrix representation for $F_{\lim}$. Note that $F_{\lim}$ depends on the choice of a parameter t. For example, if we replace t by ct with $c \in \mathbb{C}^\times$ then the limit in (2.15) is multiplied by $\exp(-N\frac{\log c}{2\pi \mathrm{i}})$.

In the context of the limiting Hodge filtration, Griffiths transversality (2.11) becomes the condition

$$NF^i_{\lim} \subset F^{i-1}_{\lim}. \tag{2.16}$$

The limiting filtration $F_{\lim}$ is itself the Hodge filtration for the *limiting mixed Hodge structure* $H_{\lim}$. There is a weight filtration $W_{\lim}H_{\lim}$ such that the pair $(F_{\lim}, W_{\lim})$ define a mixed Hodge structure as in Section 2. In general, the weight filtration on $H_{\lim}$ is not the limit of the weight filtrations on the H_t. For example, the classical situation is when H_t are *pure*, i.e., have a single weight. In that case, the limit weight filtration is determined in a canonical way by the nilpotent endomorphism N. For applications to one-loop amplitudes, we are interested in limits of dilogarithm mixed Tate Hodge structures. The action of monodromy on $gr^W H_t$ will factor through a finite quotient. (This is true quite generally because monodromy will stabilize both an integral and a unitary structure and hence lie in the intersection of a discrete group and a compact group. Such an intersection is necessarily finite.) Typically, in our examples, the eigenvalues of monodromy on $gr^W H_t$ will be ± 1, so it may be necessary to replace σ by σ^2. This explains the presence of $\sqrt{t}$ in formulas found by physicists. Once σ is unipotent, however, in our examples, the weight filtration on $H_{\lim}$ will be the given weight filtration WH_t. In other words, the weight filtration will not change in the limit, but the Hodge filtration will.

The following example may clarify the picture.

Example 2.4. We consider a family of Kummer Hodge structures as in Example 2.2. In (2.7), take $\varepsilon_0 \mapsto x(t)$ for $x(t)$ a meromorphic function on the disk D, holomorphic away from 0. Write $x(t) = t^M u(t)$ with $u(0) \neq 0, \infty$. With notation as in that example, we take $\gamma_i = \varepsilon_i^\vee, i = 0, -1$ (dual basis) and $\omega_i = \mathrm{e}^{i,i}$. We obtain

$$\begin{pmatrix} \int_{\gamma_0} \omega_0 & \int_{\gamma_0} \omega_{-1} \\ \int_{\gamma_{-1}} \omega_0 & \int_{\gamma_{-1}} \omega_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-\log(t^M u(t))}{2\pi \mathrm{i}} & \frac{1}{2\pi \mathrm{i}} \end{pmatrix}. \tag{2.17}$$

The monodromy is given by

$$N = \begin{pmatrix} 0 & 0 \\ -M & 0 \end{pmatrix}. \tag{2.18}$$

Clearly, we should take

$$F_{\lim} = \lim_{t\to 0} \exp \begin{pmatrix} 0 & 0 \\ +M\log(t)/2\pi\mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-\log(t^M u(t))}{2\pi\mathrm{i}} & \frac{1}{2\pi\mathrm{i}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{-\log(u(0))}{2\pi\mathrm{i}} & \frac{1}{2\pi\mathrm{i}} \end{pmatrix}. \tag{2.19}$$

In this example $F^0 = \mathbb{C}\cdot(\varepsilon_0 - \frac{\log(u(0))}{2\pi\mathrm{i}}\varepsilon_{-1}) = \mathbb{C}e^{0,0}_{\lim}$ (defining $e^{0,0}_{\lim}$). We take $e^{-1,-1}_{\lim} = \frac{\varepsilon_{-1}}{2\pi\mathrm{i}}$.

It is a straightforward exercise to extend this construction to mixed Tate variations H_t with $gr^W H_t = \mathbb{Q}(0)^p \oplus \mathbb{Q}(1)^q$ for arbitrary $p, q \geq 1$.

Example 2.5. Suppose now $gr^W H_t = \mathbb{Q}(0) \oplus \mathbb{Q}(1) \oplus \mathbb{Q}(2)$. We associate to H_t the two Kummer extensions $H'_t = W_{-2}H_t$ and $H''_t = H_t/W_{-4}H_t$. We assume as in Example 2.4 above that we have calculated the logarithms of monodromy N', N''. We can write $H_{\lim,\mathbb{C}} = \mathbb{C}e^{0,0}_{\lim} \oplus \mathbb{C}e^{-1,-1}_{\lim} \oplus \mathbb{C}e^{-2,-2}_{\lim}$ in such a way that $W_{i,\mathbb{C}} = \sum_{j\leq i}\mathbb{C}e^{j,j}_{\lim}$ is stable under N and $NF^i_{\lim} = N(\sum_{j\geq i}\mathbb{C}e^{j,j}_{\lim}) \subset NF^{i-1}_{\lim}$. This implies

$$Ne^{-2,-2}_{\lim} = 0; \quad Ne^{-1,-1}_{\lim} = a'e^{-2,-2}_{\lim}; \quad Ne^{0,0}_{\lim} = a''e^{-1,-1}_{\lim}. \tag{2.20}$$

Griffiths transversality for N (2.16) implies that $Ne^{0,0}_{\lim}$ does not involve $e^{-2,-2}_{\lim}$. As a consequence, N for the dilogarithm motive is determined by N' and N'' for the Kummer sub and quotient motives. These are usually straightforward to calculate. A similar analysis holds for variations of mixed Tate Hodge structures more generally.

3. The second Symanzik polynomial

The second Symanzik polynomial is used in the calculation of the Feynman amplitude associated to a graph G with possibly non-trivial external momenta. In the physics literature, it is usually derived directly from the linear algebra of Feynman coordinates [11, 12]. We will show in this section that it can also be interpreted as a pfaffian in the sense of E. H. Moore [9] associated to a quaternionic Hermitian matrix. We give the pfaffian construction, but our proof that the resulting polynomial coincides with the second Symanzik polynomial is not particularly elegant, so it is relegated to an appendix in the following section. Two consequences of the pfaffian viewpoint that we do not pursue further, are firstly that the polynomial is a *configuration polynomial* for quaternionic subspaces of a based quaternionic vector space, and hence the techniques of [10] should apply to the study of the singularities, and secondly that for each loop number there is a universal family. For example, with one loop and six edges, the hypersurface defined by the second Symanzik is the complement in $\mathbb{P}^5$ of the complex points of a coset space $GL_2(A)/U_2(A)$ where A is the quaternions and $U_2(A)$ is the subgroup of 2×2 quaternionic matrices M satisfying $\overline{M}^t = M^{-1}$. The Betti numbers of these coset spaces are known, and one may hope to better understand the motives of physical interest from this viewpoint. It will be interesting to study the corresponding family at two loops in future work.

Note that in the presence of non-trivial masses m_i, the actual polynomial of physical interest is

$$\Phi(A,q) - \left(\sum m_i^2 A_i\right)\Psi(A), \tag{3.1}$$

where Ψ and Φ are the first and second Symanzik polynomials, respectively.

We write the quaternions $\mathcal{A} = \mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot i \oplus \mathbb{R}\cdot j \oplus \mathbb{R}\cdot k$ as usual, and we embed $\mathcal{A} \hookrightarrow M_2(\mathbb{C})$ by

$$\begin{aligned} 1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\ j &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad k \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \end{aligned} \tag{3.2}$$

Let $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. One checks that the anti-involution $x \mapsto \bar{x}$ on $\mathcal{A}$ given by $\bar{\varepsilon} = -\varepsilon$ for $\varepsilon = i, j, k$ corresponds to $m \mapsto u^{-1}m^t u$ on $M_2(\mathbb{C})$. More generally, we may embed $M_n(\mathcal{A}) \hookrightarrow M_n(M_2(\mathbb{C})) \hookrightarrow M_{2n}(\mathbb{C})$ and the anti-involution $x \mapsto \bar{x}^t$ on $M_n(\mathcal{A})$ corresponds to $M \mapsto U^{-1}M^tU$ where U is the diagonal matrix in $M_n(M_2(\mathbb{C}))$ with u along the diagonal. Note that U is skew-symmetric, $U^t = -U$.

The reduced norm, $Nrd : M_n(\mathcal{A}) \to \mathbb{R}$ is a polynomial of degree $2n$, which corresponds to the determinant on $M_{2n}(\mathbb{C})$.

Let $Herm \subset M_n(\mathcal{A})$ be the $\mathbb{R}$-vector space of *Hermitian* elements, which we can think of as all elements of the form $x + \bar{x}^t$.

Proposition 3.1 (Moore, Tignol). *There exists a unique polynomial map, the pfaffian norm or Moore determinant* $Nrp : Herm \to \mathbb{R}$ *such that* $Nrp(I) = 1$ *and* $Nrp(y)^2 = Nrd(y)$.

Proof. We can compute in $M_{2n}(\mathbb{C})$. *We have*

$$\begin{aligned}\det(M + UM^tU^{-1}) &= \det((MU - (MU)^t)U^{-1})\\ &= \det(MU - (MU)^t)\det(U^{-1})\\ &= (\mathrm{pfaff}(MU - (MU)^t))^2 \cdot \mathrm{pfaff}(U^{-1})^2,\end{aligned} \tag{3.3}$$

using the fact that the determinant of a skew matrix is the square of the pfaffian. ☐

Corollary 3.2. *Suppose M in the above proposition is block diagonal with quaternionic Hermitian matrices $M_1, \dots, M_p$ along the diagonal. Then $Nrp(M) = \prod Nrp(M_j)$.*

Proof. The assertion is true for the usual pfaffians for skew matrices, and all the matrices in the proof of Proposition 3.1 are in block diagonal form. ☐

One way to construct elements in *Herm* is to take $\mathbb{R}$-linear combinations of rank 1 Hermitian elements $x = \bar{x}^t$. The latter are given by

$$x = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \cdots \\ \bar{a}_n \end{pmatrix} \cdot \big(a_1, a_2 \dots, a_n\big) = (\bar{a}_i a_j)_{1\le i,j\le n}, \tag{3.4}$$

where $a_1, \dots, a_n \in \mathcal{A}$. Given a collection $x_1, \dots, x_p$ of such Hermitian elements, we can construct a polynomial of degree n in $A_1, \dots, A_p$ by taking

$$\Phi(A_1, \dots, A_p) := Nrp\left(\sum_{i=1}^{p} A_i x_i\right). \tag{3.5}$$

View $\mathcal{A}^p$ as a right $\mathcal{A}$-vector space of column vectors. Let $\mathcal{H} \subset \mathcal{A}^p$ be a subspace with $\dim_{\mathcal{A}} \mathcal{H} = n$. Choose a basis $\alpha_1, \dots, \alpha_n$ for $\mathcal{H}$ with $\alpha_i = (a_{1i}, \dots, a_{pi})^t$. Define

$$e_j^\vee = (a_{j1}, \dots, a_{jn}), \quad 1 \le j \le p. \tag{3.6}$$

Take $x_j = \bar{e}_j^{\vee,t} \cdot e_j^\vee$ and define

$$\Phi_{\mathcal{H}} := Nrp\left(\sum A_i x_i\right), \tag{3.7}$$

as in (3.5). Writing $\alpha = (a_{ij})$, a $p \times n$ matrix, one sees that a different choice of basis for $\mathcal{H}$ yields a matrix $\beta = (b_{ij}) = \alpha M$, where M is $n \times n$ and invertible. We have $(b_{j1}, \dots, b_{jn}) = e_j^\vee M$ so x_j is replaced by $\overline{M}^t x M$.

Lemma 3.3. *Let M, N be $n \times n$ matrices with entries in $\mathcal{A}$. Assume $N = \overline{N}^t$ and M is invertible. then*

$$Nrp(\overline{M}^t N M) = Nrd(M) Nrp(N). \tag{3.8}$$

Proof. Both sides of (3.8) are polynomial maps in the entries of M and N, and they have the same square. It follows that the ratio is constant. For M the identity matrix, the ratio is 1. (Note $Nrd(M) = \det(\iota(M))$ where $\iota : Mat_n(\mathcal{A}) \hookrightarrow Mat_{2n}(\mathbb{C})$ is defined via the embedding $\mathcal{A} \hookrightarrow Mat_2(\mathbb{C})$. In particular, $Nrd(M) = Nrd(-M)$.) $\square$

As a consequence of the lemma, $\Phi(\sum A_i \overline{M}^t x_i M) = Nrd(M)\Phi(\sum A_i x_i)$ so $\Phi_{\mathcal{H}}$ is well defined upto a non-zero constant factor.

Consider a graph Γ with edge set E and vertex set V. Let $H = H_1(\Gamma, \mathbb{Q})$, and choose a basis $H \cong \mathbb{Q}^r$. We have

$$0 \to \mathbb{Q}^r \to \mathbb{Q}^E \xrightarrow{\partial} \mathbb{Q}^{V,0} \to 0, \tag{3.9}$$

where $\mathbb{Q}^{V,0} \subset \mathbb{Q}^V$ is the image of the boundary map ∂. If we tensor with $\mathcal{A}$ we obtain

$$0 \to \mathcal{A}^r \to \mathcal{A}^E \to \mathcal{A}^{V,0} \to 0 \tag{3.10}$$

Suppose we are given $q := (\dots q_v, \dots) \in \mathcal{A}^{V,0}$. Let $\mathcal{H}_q \subset \mathcal{A}^E$ be the sub right $\mathcal{A}$-module in $\mathcal{A}^E$ spanned by $\mathcal{A}^r$ and a lifting $\tilde{q}$ of q. To each $e \in E$ we define an $r+1$-vector $w_e = (w_{e,1}, \dots, w_{e,r+1})$ by looking at the e-th coordinate of the r basis vectors for $H \otimes \mathcal{A}$ together with $\tilde{q}$. Note $w_{e,1}, \dots, w_{e,r} \in \mathbb{R}$. Define (quaternionic) Hermitian matrices

$$x_e := \overline{w_e}^t \cdot w_e. \tag{3.11}$$

The *second Symanzik polynomial* is the configuration polynomial (3.7) for $\mathcal{H} = \mathcal{H}_q$

$$\Phi(A)_{\Gamma,q} := Nrp(\sum_E A_e x_e). \tag{3.12}$$

Example 3.4. Take $r = 1$ and $H = \mathbb{Q}(e_1 + \cdots + e_n)$. (This is the one-loop case.) Let $\tilde{q} = \sum \mu_e e \in \mathcal{A}^E$. Then

$$N := \sum_E A_e x_e = \begin{pmatrix} \sum_E A_e & \sum_E A_e \mu_e \\ \sum_E A_e \bar{\mu}_e & \sum_E A_e \bar{\mu}_e \mu_e \end{pmatrix}. \tag{3.13}$$

We will see that in this case

$$\begin{aligned} \Phi(A) = Nrp(N) &= -\left(\sum A_e \bar{\mu}_e\right)\left(\sum A_e \mu_e\right) + \left(\sum A_e\right)\left(\sum A_e \bar{\mu}_e \mu_e\right) \\ &= \sum_{i<j} \overline{(\mu_i - \mu_j)}(\mu_i - \mu_j) A_{e_i} A_{e_j}. \end{aligned} \tag{3.14}$$

The physics convention would write $\mu_i = \sum_{j=i}^n q_j$ with $\mu_1 = 0$. The result in (3.14) becomes

$$\Phi(A)_{\Gamma,q} = \sum_{i<j} \overline{(q_i + \cdots + q_{j-1})}(q_i + \cdots + q_{j-1}) A_i A_j. \tag{3.15}$$

Again as in (3.1), the polynomial of physical interest is

$$D(q, A) := \sum_{i<j} \overline{(q_i + \cdots + q_{j-1})}(q_i + \cdots + q_{j-1}) A_i A_j - \left(\sum m_i^2 A_i\right)\left(\sum A_i\right). \tag{3.16}$$

Example 3.5. Let us actually discuss one more example. Consider the three-edge banana. We take as a basis the two independent cycles $\{e_1, e_2\}$ and $\{e_2, e_3\}$. The matrix is then given as

$$\begin{aligned} N &:= A_1(1, 0, \bar{\mu_1})^T \cdot (1, 0, \mu_1) + A_2(1, 1, \bar{\mu_2})^T \cdot (1, 1, \mu_2) \\ &\quad + A_3(0, 1, \bar{\mu_3})^T \cdot (0, 1, \mu_3), \\ N &= \begin{pmatrix} A_1 + A_2 & A_2 & A_1\mu_1 + A_2\mu_2 \\ A_2 & A_2 + A_3 & A_2\mu_2 + A_3\mu_3 \\ A_1\bar{\mu_1} + A_2\bar{\mu_2} & A_2\bar{\mu_2} + A_3\bar{\mu_3} & A_1\bar{\mu_1}\mu_1 + A_2\bar{\mu_2}\mu_2 + A_3\bar{\mu_3}\mu_3 \end{pmatrix}. \end{aligned} \tag{3.17}$$

We have $NRP(N) = A_1 A_2 A_3 \overline{(\mu_1 - \mu_2 + \mu_3)}(\mu_1 - \mu_2 + \mu_3)$.

4. Appendix to Section 3

It remains to show that our definition of the second Symanzik polynomial coincides with the classical physical definition [5], formulas 6–87 and 6–88.

(The argument which follows parallels the argument for scalar momenta given in [10].)

Lemma 4.1. *Let $\mathcal{H} \subset \mathcal{A}^p$ be a subspace as above. Then $\Phi_{\mathcal{H}}$ has degree ≤ 1 in each A_i.*

Proof. First note that if $\alpha_1, \ldots, \alpha_n \in \mathcal{A}^p$ satisfy a linear relation $\sum \alpha_i a_i = 0$, then the x_i in (3.7), viewed as map of row vectors $\mathcal{A}^n \to \mathcal{A}^n$ by multiplication on the right, kills the row vector $((\bar{a})_1, \ldots, (\bar{a})_n)$. It follows that the matrix $\sum A_i x_i$ does not have maximal rank, so $\Phi(\sum A_i x_i) = 0$.

If some A_i appears to degree ≥ 2 in some monomial in $\Phi_{\mathcal{H}}$, then the monomial can contain at most $\dim \mathcal{H} - 1$ distinct A_j. Let $T \subset \{1, \ldots, p\}$ be the indices occurring in this monomial. By assumption, $\#T < \dim \mathcal{H}$. Consider the diagram

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{A}^p \\ \Big\| & & \Big\downarrow \text{proj.} \\ \mathcal{H} & \xrightarrow{\iota} & \mathcal{A}^T. \end{array} \tag{4.1}$$

It is immediate that $\Phi_{\mathcal{H}}|_{A_k=0, k\notin T}$ is the configuration polynomial for the bottom row in (4.1). If this is non-zero, then by the above, the map ι must be injective. In particular, $\#T \geq \dim \mathcal{H}$, a contradiction. $\square$

We now consider $\mathcal{H}_q \subset \mathcal{A}^E$ as above. Let $C \subset E$ with $\#C = r+1$. A necessary condition for the monomial $\prod_{e\in C} A_e$ to appear in $\Phi_{\Gamma,q}$ is that $H \hookrightarrow \mathbb{Q}^E/\mathbb{Q}^{E-C}$. Such a set C of edges is called a *cut set*. For a cut set C, there exists a spanning tree T and an edge $e \in T$ such that $T - e = E - C$. We choose an $\mathcal{A}$-basis $h_1, \ldots, h_{r+1}$ for $\mathcal{H}_q$ such that $h_1, \ldots, h_r \in H_1(\Gamma, \mathbb{Q}) \subset \mathcal{H}_q$. For $c \in C$ let $w_c : \mathcal{H}_q \to \mathcal{A}^C$ be the map $w_c(h) = c^\vee(h)c$. Let $\bar{w}_c^t : \mathcal{A}^C \to \mathcal{H}_q$ be the map $\mathcal{A}^C \twoheadrightarrow c\mathcal{A} \to H \cong \mathcal{A}^{r+1}$ given by $c \mapsto (c^\vee(h_1), \ldots, c^\vee(h_{r+1}))$. Here we identify $\mathcal{H}_q = \mathcal{A}^{r+1}$ using the basis $\{h_i\}$. Note that $\bar{w}_c^t w_d = 0$ for $c \neq d$. It follows that writing $R_C = \sum_{c\in C} w_c$ we have $\bar{R}_C^t R_C = \sum_{c\in C} x_c$ with x_c as in (3.11). Thus

$$\begin{aligned} Nrp(\bar{R}_C^t R_C) &= Nrp(\sum A_e x_e)|_{A_e=0, e\notin C, A_e=1,\ e\in C} \\ &= \text{coefficient of } \prod_{c\in C} A_c \text{ in } \Phi_{\Gamma,q}. \end{aligned} \tag{4.2}$$

It follows from Lemma 3.3 that this coefficient equals $Nrd(R_C)$. Note that the $(r+1)\times(r+1)$-matrix R_C has real entries except for the last column.

Let us define the $\mathcal{A}$-determinant to be the expansion in the last column

$$\det_{\mathcal{A}}(R_C) := (-1)^{r+1}\sum_i(-1)^i\det(R_C^{i,r+1})(R_C)_{i,r+1}, \tag{4.3}$$

where $(R_C)^{i,r+1}$ denotes the minor. Note this matrix has $\mathbb{R}$-coefficients, so the determinant is defined.

Lemma 4.2. *With notation as above, we have* $Nrd(R_C) = (\det_{\mathcal{A}}(R_C))\overline{(\det_{\mathcal{A}}(R_C))}$.

Proof. By definition $Nrd(R_C)$ is calculated using the embedding $\mathcal{A} \hookrightarrow M_2(\mathbb{C})$ to view R_C as a $(2r+2)\times(2r+2)$-complex matrix and then taking the usual determinant. In other words, one views R_C as a map $(\mathbb{C}^2)^{r+1} \to (\mathbb{C}^2)^{r+1}$. The assertion is thus clear if the entries of R_C all lie in $\mathbb{R} \subset \mathcal{A}$. In that case, all the 2×2-matrices are real scalar and we just get the square of the usual determinant. For the general case, it suffices to consider a $2N \times 2N$ complex matrix with entries 2×2 scalar diagonal except for the last two columns. Then one checks that the determinant is computed by interpreting the last two columns as a single column of N 2×2-matrices, expanding as above (4.3) and then taking the determinant. (Note that under the embedding $\mathcal{A} \hookrightarrow M_2(\mathbb{C})$ the determinant corresponds to $x\bar{x}$.) □

Finally, to identify Nrp with the second Symanzik polynomial, we have to show the expansion (4.3) coincides with the usual combinatorial description in terms of cut sets. Fix an orientation and an ordering for the edges of Γ. Let C be a cut set as above. Let F_i, $i = 1, 2$ be disjoint with $\Gamma - C = F_1 \amalg F_2$. Note that one of the F_i may be an isolated vertex. Let $\Gamma//F$ denote the 2-vertex graph obtained by shrinking the two components of $F \subset \Gamma$ to two (separate) vertices v_1, v_2. For $e \in E(\Gamma)$ not an edge of F, the image $\bar{e}$ of e in $\Gamma//F$ is either a loop (tadpole) or has boundary the difference of the two vertices, $\partial e = \pm(v_2 - v_1)$. We have also $H_1(\Gamma) \cong H_1(\Gamma//F)$. As above we enumerate the edges $e_1, \ldots, e_{r+1}$ in $\Gamma - C$. Let $\Gamma_i = (\Gamma//F)/e_i$ be obtained by contracting e_i. Then $\det(R_C^{i,r+1})$ is the determinant of the map from $H_1(\Gamma)$ with basis $h_1, \ldots, h_r$ to $\mathbb{Z}^{E-C-\{e_i\}}$ with basis $e_1, \ldots, \widehat{e_i} \ldots, e_{r+1}$. Define

$$a(i) := \begin{cases} +1 & \partial\bar{e}_i = v_2 - v_1, \\ -1 & \partial\bar{e}_i = v_1 - v_2, \\ 0 & \partial\bar{e}_i = 0. \end{cases} \tag{4.4}$$

The key point then is

$$(4.5) \qquad (-1)^i \det(R_C^{i,r+1}) = a(i)b,$$

where $b = \pm 1$ is independent of i. This can be seen as follows. Let $W = \bigoplus_1^{r+1} \mathbb{Q}e_i,$. The composition $H_1(\Gamma) \subset W \twoheadrightarrow W/\mathbb{Q}e_i$ is an isomorphism. The evident basis $\{e_k, k \neq i\}$ of $W/\mathbb{Q}e_i$ induces a basis of $H_1(\Gamma)$. For two different choices of i, say i_1, i_2, the determinant of the change of basis matrix is $(-1)^{i_1-i_2}$. Indeed, writing $\varepsilon = \frac{1}{r+1}\sum e_i \in W$ and letting $\det_1, \det_2 \in$ det
$H_1(\Gamma)$ be the exterior powers of the basis vectors for the two bases, one has in $\det W$ that $\varepsilon \wedge \det_1 = (-1)^{i_1-i_2}\varepsilon \wedge \det_2$. (Compare both sides with $e_1 \wedge \cdots \wedge e_{r+1}$.)

Finally, we deduce from this and (4.2) the classical combinatorial description of the second Symanzik polynomial, viz. the coefficient of $\prod_{e\in C} A_e$ is given by

$$(4.6) \qquad \overline{\Big(\sum_{\partial\bar{e}_i = v_2 - v_1} e_i^\vee(h_{r+1}) - \sum_{\partial\bar{e}_i = v_1 - v_2} e_i^\vee(h_{r+1})\Big)} \times \Big(\sum_{\partial\bar{e}_i = v_2 - v_1} e_i^\vee(h_{r+1}) - \sum_{\partial\bar{e}_i = v_1 - v_2} e_i^\vee(h_{r+1})\Big).$$

5. Differential forms on projective space

We turn now to the study of motives associated to one-loop graphs. We recall first the structure of differential forms on projective space. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$ be the sheaf of (algebraic) functions on projective n-space, and let $\Omega^i = \bigwedge^i \Omega^1$ denote the sheaf of algebraic differential i-forms. Fix a basis $A_0, \ldots, A_n$ for the linear homogeneous forms on $\mathbb{P}^n$. One has an exact sequence

$$(5.1) \qquad 0 \to \Omega^1 \to \bigoplus_{i=0}^{n} \mathcal{O}(-1)dA_i \xrightarrow{p} \mathcal{O} \to 0.$$

(Here the dA_i are just labels for the various summands of the direct sum.) Twisting by 1, the map $p(1)$ maps dA_i to $A_i \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$. For example, $p(2)(A_j dA_i - A_i dA_j) = A_j A_i - A_i A_j = 0$, so $A_j dA_i - A_i dA_j \in \Gamma(\mathbb{P}^n, \Omega^1(2))$. It follows that $dA_i/A_i - dA_j/A_j$ is a (meromorphic) section of Ω^1.

We are interested in Ω^{n-1}. By standard Koszul algebra we get from (5.1) an exact sequence

$$0 \to \Omega^{n-1}(n-1) \to \bigwedge^{n-1}\Big(\bigoplus_0^n \mathcal{O}\cdot dA_i\Big) \to \bigwedge^{n-2}\Big(\bigoplus_0^n \mathcal{O}\Big)(1), \tag{5.2}$$

the map on the right is given by

$$dA_{i_1}\wedge\cdots\wedge dA_{i_{n-1}} \mapsto \sum_{j=1}^{n-1}(-1)^{j-1}A_{i_j}dA_{i_1}\wedge\cdots\wedge\widehat{dA_{i_j}}\wedge\cdots\wedge dA_{i_{n-1}}. \tag{5.3}$$

Again by standard Koszul stuff we have an exact sequence (we have dropped the labels dA_i)

$$\bigwedge^{n}\Big(\bigoplus_0^n \mathcal{O}\Big) \to \bigwedge^{n-1}\Big(\bigoplus_0^n \mathcal{O}\Big)(1) \to \bigwedge^{n-2}\Big(\bigoplus_0^n \mathcal{O}\Big)(2), \tag{5.4}$$

where the maps are as in (5.3). For $0 \le j \le n$, the section

$$\tau_j := dA_0\wedge\cdots\wedge\widehat{dA_j}\wedge\cdots\wedge dA_n, \tag{5.5}$$

on the left maps to

$$\Theta_j := \sum_{i\neq j} \pm A_i dA_0\wedge\cdots\wedge\widehat{dA_i}\wedge\cdots\wedge\widehat{dA_j}\wedge\cdots\wedge dA_n \in \Gamma(\mathbb{P}^n, \Omega^{n-1}(n)). \tag{5.6}$$

(The sign in the sum is $(-1)^i$ for $i<j$ and $(-1)^{i-1}$ for $i>j$.) Treating these expressions as differential forms in the evident way, we have

$$d\Theta_j = n\tau_j. \tag{5.7}$$

In particular, if $F = G/H$ is a ratio of homogeneous polynomials with $\deg G - \deg H = n$, then we compute

$$\begin{aligned} d(\Theta_j/F) &= n\tau_j/F - dF\wedge\Theta_j/F^2 = \frac{nF\tau_j - (\sum \partial F/\partial A_k dA_k)\Theta_j}{F^2} \\ &= \frac{(nF - \sum_{k\neq j}\partial F/\partial A_k A_k)\tau_j - \partial F/\partial A_j dA_j \wedge\Theta_j}{F^2} \\ &= \frac{\partial F/\partial A_j(A_j\tau_j - dA_j\Theta_j)}{F^2} = \frac{(-1)^j \partial F/\partial A_j \Omega_n}{F^2}. \end{aligned} \tag{5.8}$$

Here $\Omega_n = \sum(-1)^i A_i dA_0 \wedge \cdots \wedge \widehat{dA_i} \wedge \cdots \wedge dA_n$. Note that (5.8) is an identity between meromorphic n-forms on $\mathbb{P}^n$.

Replacing, if necessary, F by a power of F, we have proven

Lemma 5.1. *Let $\omega = \frac{P\Omega_n}{F^p}$ be an n-form on $U := \mathbb{P}^n - \{F = 0\}$. Assume $G = \sum G_i \frac{\partial F}{\partial A_i}$ lies in the ideal generated by the partial derivatives of F. Then we can reduce the order p of pole of $[\omega] \in H^n_{DR}(U)$, i.e., there exists a form $\omega' = \frac{G'\Omega}{F^{p-1}}$ which is cohomologous to ω, $[\omega] = [\omega']$. (Here H_{DR} is algebraic de Rham cohomology calculated using algebraic differential forms. It coincides with Betti cohomology on U because U is affine.)*

6. Complex Poincaré group invariants

Feynman integrals, after integration, are functions of external momenta. If the whole integral transforms as a Lorentz scalar, the integral is a function of Lorentz invariant scalar products of external momenta. The number of and type of these invariants are exhibited here from a mathematical viewpoint, incorporating momentum conservation and the finite dimension of spacetime.

The fact that the amplitudes for one-loop graphs are dilogarithms is a consequence of some basic facts about the invariants of the orthogonal group. Let $O_{\mathbb{C}}(r)$ be the subgroup of $GL(\mathbb{C}^r)$ leaving invariant a non-degenerate inner product $(p, q) \mapsto p \cdot q$. Let $G = \mathbb{C}^r \rtimes O_{\mathbb{C}}(r)$ be the "complex Poincaré group" generated by orthogonal transformations and translations. As an algebraic group over $\mathbb{C}$, G has dimension $r + \frac{r(r-1)}{2}$. Let G act diagonally on $(\mathbb{C}^r)^{r+2}$. The quotient $(\mathbb{C}^r)^{r+2}/G$ has dimension

$$\dim_{\mathbb{C}}(\mathbb{C}^r)^{r+2}/G = r(r+2) - \left(r + \frac{r(r-1)}{2}\right) = \binom{r+2}{2} - 1. \tag{6.1}$$

Let $p_j : (\mathbb{C}^r)^{r+2} \to \mathbb{C}^r, 1 \le j \le r+2$ be the projections. Following physics notation we write

$$(p_j - p_k)^2 := (p_j - p_k) \cdot (p_j - p_k), \tag{6.2}$$

with the inner product as above. We obtain in this way $\binom{r+2}{2}$ G-invariant functions on $(\mathbb{C}^r)^{r+2}$. It follows from (6.1) that there is an algebraic relation between these functions.

To understand this relation, we change bases in $\mathbb{C}^r$ so the inner product is the sum of squares of coordinates (Euclidean inner product). We can view $P := (p_1, \dots, p_{r+2})$ as an $r \times (r+2)$ matrix. The $(r+2) \times (r+2)$-symmetric matrix

$$N := (p_j \cdot p_k) = P^t P \tag{6.3}$$

has rank $\leq r$. It is convenient at this point to introduce masses $m_j, 1 \leq j \leq r+2$. Consider the $(r+2) \times (r+2)$-symmetric matrix

$$\begin{aligned} M(m) &:= (m_i^2 + m_j^2 + (p_i - p_j)^2) = (m_i^2 + p_i^2)_{ij} + (m_j^2 + p_j^2)_{ij} - 2N \\ &= M_1 + M_2 - 2N. \end{aligned} \tag{6.4}$$

View $\mathbb{C}^{r+2}$ as column vectors, and let $H \subset \mathbb{C}^{r+2}$ be the codimension 1 subspace defined by setting the sum of the coordinates to zero. Note that M_1 has all columns the same, so for $h \in H$ we have $M_1 h = 0$. Similarly $h^t M_2 = 0$. It follows that the quadratic form given by the symmetric matrix $M(m)$ is necessarily degenerate when restricted to H, i.e., $\exists\, 0 \neq k \in H$ with $h^t M(m) k = 0$ for all $h \in H$.

Lemma 6.1. *For general values of the p_i we have* $\det(M(0)) \neq 0$.

Proof. Take $p_1, \dots, p_r$ to be the usual orthonormal basis of $\mathbb{C}^r$, and take $p_{r+1} = 0$. One easily checks in this case that the coefficient of $p_{r+2,1}^4$ in $\det(M(0))$ is plus or minus a power of 2. In particular, it is non-zero, and the lemma follows. □

Remark 6.2. *Of course, it follows from the lemma that* $\det(M(m)) \neq 0$ *for general m and p as well.*

Assume now that $M(m)$ is invertible. Write $\vec{1} = (1, \dots, 1) \in \mathbb{C}^{r+2}$. It follows from the above that $M(m)k = \kappa\vec{1}$; $\kappa \neq 0$. Scaling k, we may assume $k = M(m)^{-1}\vec{1}$. Thus

$$(M(m)^{-1}\vec{1}) \cdot \vec{1} = 0. \tag{6.5}$$

When the masses are zero, (6.5) yields the non-trivial algebraic relation between the $(p_i - p_j)^2$. We will interpret (6.5) in the case $r = 4$ as determining where in the weight filtration of a Hodge structure the Feynman integrand lies. In physics terms, it is the statement that for one-loop graphs, the amplitude is expressed in terms of logarithms and dilogarithms of Lorentz-invariant rational functions of momenta, [4], [12].

In physical situations, of course, the p_i are 4-vectors.

Lemma 6.3. *Fix $n \geq 6$. Let $p_i \in \mathbb{C}^4$, $1 \leq i \leq n$, and let $m_i \in \mathbb{C}, 1 \leq i \leq n$. Let $H \subset \mathbb{C}^n$ be the codimension 1 linear subspace defined by setting the sum of the coordinates to 0. The matrix $M(m) = (m_i^2 + m_j^2 + (p_i - p_j)^2)_{ij}$ has rank ≤ 6. For general values of m_i, p_i the rank is exactly 6 and the vector $(1, \ldots, 1)$ lies in the image $M(m)(H) \subset \mathbb{C}^n$.*

Proof. As in (6.4) $M(m)$ is a sum of three matrices. The matrices M_1, M_2 have rank 1. The matrix N has rank 4 (for general p_i) as in (6.3). It follows that $M(m)$ has rank ≤ 6, and it is easy to see the rank is exactly 6 for general values of the parameters. To show the vector $(1, \ldots, 1) \in M(m)(H)$, it suffices to solve the equations

$$\sum_{i=1}^{n} a_i(p_i^2 + m_i^2) = 1; \quad \sum_{i=1}^{n} a_i p_i = 0; \quad \sum_{i=1}^{n} a_i = 0. \tag{6.6}$$

These equations clearly admit a solution in the a_i for general values of the parameters when $n \geq 6$. □

7. The motive

Let $X : Q = 0$ be a rank $\min(6, n+1)$ quadric in $\mathbb{P}^n$. Let $A_0, \ldots, A_n$ be homogeneous coordinates, and write $\Delta : \prod A_i = 0$ for the reference simplex. We will be interested in the "motive" (or more concretely, the Hodge structure)

$$H^n(\mathbb{P}^n - X, \Delta - X \cap \Delta, \mathbb{Q}). \tag{7.1}$$

(In the case $n = 2$, the triangle graph, the motive of physical interest is slightly different. We treat it separately in Section 10.)

We assume that X is in good position with respect to Δ in the sense that for any face $F \cong \mathbb{P}^i \subset \Delta$ the intersection $X \cap F$ has rank $\min(6, i+1)$. In particular, if $\dim F < 6$ then $X \cap F$ is smooth. (The nullspace $L \subset X$ is a linear space of dimension $n - 6$, and our assumption is that L meets all faces of Δ properly.)

Lemma 7.1. (i) *We have*

$$H^n(\mathbb{P}^n - X, \mathbb{Q}) \cong \begin{cases} 0 & n > 5, \\ \mathbb{Q}(-m-1) & n = 2m+1 \leq 5, \\ 0 & n = 2m > 0, \\ \mathbb{Q}(0) & n = 0. \end{cases} \tag{7.2}$$

(ii) $H^k(\mathbb{P}^n - X, \mathbb{Q}) = (0)$ *if* $0 < k \neq n \leq 5$ *or if* $n > 5$ *and* $k \neq 0, 5$.

Proof. Suppose first $n > 5$. Let $p : \mathbb{P}^n - L \to \mathbb{P}^5$ be the projection with center L. We have $X - L = p^{-1}(Y)$, where $Y \subset \mathbb{P}^5$ is a smooth quadric. It follows that $\mathbb{P}^n - X$ is a fibre bundle over $\mathbb{P}^5 - Y$ with fibre $\mathbb{A}^{n-5}$. A standard result for fibrations with contractible fibres yields $H^*(\mathbb{P}^5 - Y, \mathbb{Q}) \cong H^*(\mathbb{P}^n - X, \mathbb{Q})$. Since $\mathbb{P}^5 - Y$ is affine of dimension 5, it has cohomological dimension 5 so $H^n(\mathbb{P}^n - X, \mathbb{Q}) = (0)$.

Quite generally, for X a smooth hypersurface in $\mathbb{P}^n$, the Gysin sequence (given by residues of differential forms) yields an exact sequence which reads in part

$$0 \to H^n(\mathbb{P}^n - X, \mathbb{Q}) \to H^{n-1}(X, \mathbb{Q}(-1)) \to H^{n+1}(\mathbb{P}^n, \mathbb{Q}) \to 0. \tag{7.3}$$

We now assume $n \leq 5$ so X is a smooth quadric. The middle dimensional cohomology of a smooth quadric of dimension d is known to be rank 2 generated by algebraic cycles for d even and zero for d odd. Part (i) of the lemma follows.

The proof of (ii) is similar, but easier. One can identify $H^k(\mathbb{P}^n - X, \mathbb{Q}) \cong H_{2n-k-1}(X, \mathbb{Q}(-n))_{prim}$. When X is a smooth quadric, the primitive homology is zero except when $k = n$ and $n = 2m + 1$ is odd. The singular case is treated as above by writing X as a cone over a smooth quadric. Details are left to the reader. ☐

For an index set $I = \{i_0, \ldots, i_p\} \subset \{0, \ldots, n\}$ write $|I| = p + 1$ and let $\Delta_I \subset \Delta \subset \mathbb{P}^n$ be defined by the vanishing of the homogeneous coordinates A_{i_j}. The motive (7.1) is the hypercohomology of the complex of sheaves

$$\mathbb{Q}_{\mathbb{P}^n - X} \to \bigoplus_{|I|=1} \mathbb{Q}_{\Delta_I - X\cap\Delta_I} \to \cdots \to \bigoplus_{|I|=n} \mathbb{Q}_{\Delta_I}. \tag{7.4}$$

There is a spectral sequence $E_1^{p,q} = H^q(\bigoplus_{|I|=p} \mathbb{Q}_{\Delta_I - X\cap\Delta_I}) \Rightarrow H^{p+q}$ $(\mathbb{P}^n - X, \Delta - X \cap \Delta, \mathbb{Q})$. (For simplicity we write $\mathbb{P}^n - X = \bigoplus_{|I|=0} \Delta_I - X \cap \Delta_I$.) The differentials $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ are zero except when $q = 0$.

Suppose first $n \leq 5$. The weight graded cohomology in these cases is

$$gr^W H^n(\mathbb{P}^n - X, \Delta - X \cap \Delta, \mathbb{Q})$$
$$= \begin{cases} \mathbb{Q}(0) \oplus \bigoplus_{15} \mathbb{Q}(-1) \oplus \bigoplus_{15} \mathbb{Q}(-2) \oplus \mathbb{Q}(-3), & n = 5, \\ \mathbb{Q}(0) \oplus \bigoplus_{10} \mathbb{Q}(-1) \oplus \bigoplus_{5} \mathbb{Q}(-2), & n = 4, \\ \mathbb{Q}(0) \oplus \bigoplus_{6} \mathbb{Q}(-1) \oplus \mathbb{Q}(-2), & n = 3, \\ \mathbb{Q}(0) \oplus \bigoplus_{3} \mathbb{Q}(-1), & n = 2, \\ \mathbb{Q}(0) \oplus \mathbb{Q}(-1), & n = 1. \end{cases} \tag{7.5}$$

For $n \geq 6$ the differential $d_1^{n-6,5} : E_1^{n-6,5} \to E_1^{n-5,5}$ is non-trivial. One finds for the weight graded

$$gr^W H^n(\mathbb{P}^n - X, \Delta - X \cap \Delta, \mathbb{Q})$$
$$= \mathbb{Q}(0) \oplus \bigoplus_{\binom{n+1}{n-1}} \mathbb{Q}(-1) \oplus \bigoplus_{\binom{n+1}{n-3}} \mathbb{Q}(-2) \bigoplus_{c_n} \mathbb{Q}(-3). \tag{7.6}$$

Here c_n is the dimension of $\mathrm{coker}(\bigoplus_{|I|=n-6} \mathbb{Q} \xrightarrow{\partial} \bigoplus_{|I|=n-5} \mathbb{Q})$. In fact, the weight 6 part of these motives will not play a role in our amplitude calculations. This is because (as we will see in Proposition 8.3) the differential form given by the Feynman integrand (8.1) below lies in $W_4 H^n(\mathbb{P}^n - X, \Delta - X \cap \Delta, \mathbb{Q})$.

8. The amplitude

Associated to a one-loop graph with n internal edges and incoming momenta (4-vectors summing to 0) p_i at the vertices we have the second Symanzik polynomial $D(p, A)$ (3.16) which is a homogeneous quadric in the variables $A_1, \ldots, A_n$. The associated amplitude is

$$\int_\sigma \frac{(\sum A_i)^{n-4} \Omega_{n-1}}{D(p, A)^{n-2}}. \tag{8.1}$$

(In the physics literature, this is frequently written as an affine integral over a simplex, cf. [6], formula 1.5.22. The transition to a projective integral more familiar to algebraic geometers is straightforward.) Here the first Symanzik polynomial is just $\sum A_i$, and Ω_{n-1} is as in Section 5. Note if $n \leq 3$ then $\sum A_i$ appears in the denominator. We will focus on the case $n \geq 4$, leaving the *triangle graph* case $n = 3$ (we are now counting edges from $1, \ldots, n$, not from $0, \ldots, n-1$, as in the previous section) to Section 10.

Lemma 8.1. *Assume $n \geq 5$. Let $\mathbb{P}^{n-2} \cong \Delta_i,\ 0 \leq i \leq n-1 \subset \mathbb{P}^{n-1}$, be the maximal faces (facets) of the coordinate simplex $\Delta \subset \mathbb{P}^{n-1}$. Let $X : D(p,A) \subset \mathbb{P}^{n-1}$ be the quadric. Assume momenta and masses are general. Then the form $\eta_{n-1} := \frac{(\sum A_i)^{n-4}\Omega_{n-1}}{D(p,A)^{n-2}}$ on $\mathbb{P}^{n-1} - X$ is exact. We can find an $(n-2)$-form w_{n-1} on $\mathbb{P}^{n-1} - X$ and constants $a_j \in \mathbb{C}$ such that (i) $dw_{n-1} = \eta_{n-1}$; (ii) $w_{n-1}|\Delta_j = \pm a_j \eta_{n-2}$; (iii) $\sum a_j = 0$.*

Proof. Let $M(m) = (m_i^2 + m_j^2 + (p_i - p_j)^2)_{1 \leq i,j \leq n}$ be the symmetric matrix corresponding to $D(p,A)$. From Lemma 6.3 there exists a column vector $\vec{a} = (a_1, \ldots, a_n)$ such that $M(m)\vec{a} = (1, \ldots, 1)$ and $\sum a_i = 0$. Note the first identity yields $\sum a_k \partial D/\partial A_k = \sum A_k$.

Define

$$w_{n-1} := \frac{(\sum A_i)^{n-5} \sum_j (-1)^j a_j \Theta_j}{2(n-3)D(p,A)^{n-3}}, \tag{8.2}$$

where Θ_j is as in (5.6). Using (5.8) with $F = \frac{D(p,A)^{n-3}}{(\sum A_i)^{n-5}}$ we compute

$$\begin{aligned} dw_{n-1} &= \sum_j a_j \left(\frac{(n-3)(\sum A_k)^{n-5} \frac{\partial D}{\partial A_j}}{2(n-3)D^{n-2}} - \frac{(n-5)(\sum A_k)^{n-6}}{2(n-3)D^{n-3}} \right) \Omega_{n-1} \\ &= \frac{(n-3)(\sum A_k)^{n-5} \sum_j a_j \frac{\partial D}{\partial A_j}}{2(n-3)D^{n-2}} \Omega_{n-1} = \frac{(\sum A_k)^{n-4}}{D^{n-2}} \Omega_{n-1} = \eta_{n-1}. \end{aligned} \tag{8.3}$$

Note finally that $\Theta_j|\Delta_k = \pm \delta_{jk}\Omega_{n-2}$, proving (ii). □

Remark 8.2. In response to a question of the referee, we remark that the a_j are not themselves in $\mathbb{Q}$. Rather they are universal rational functions in the momenta. This follows from (6.5).

Proposition 8.3. *With notation as in the lemma, we have $\eta_{n-1} \in W_4 H^{n-1}(\mathbb{P}^{n-1} - X, \Delta - X \cap \Delta)$. The Feynman amplitude for any one-loop graph is a period of a dilogarithm mixed Hodge structure as in Definition 2.1.*

Proof. If $n \geq 6$, the faces $\Delta_j \cong \mathbb{P}^{n-2}$ have dimension ≥ 4 and we can apply the lemma again to the forms $w_{n-1}|\Delta_j = \pm a_j \eta_{n-2}$. In this way, we can build a sort of cascade

(8.4)

$$\begin{array}{ccccccc} & & & & \Omega^{n-2}_{\mathbb{P}^{n-1}-X} & \xrightarrow{d} & \Omega^{n-1}_{\mathbb{P}^{n-1}-X} \\ & & & & \downarrow & & \\ & & \bigoplus_i \Omega^{n-3}_{\Delta_i - X\cap\Delta_i} & \xrightarrow{d} & \bigoplus_i \Omega^{n-2}_{\Delta_i - X\cap\Delta_i} & & \\ & & \downarrow & & & & \\ & & \vdots & & & & \\ \bigoplus_{|I|=n-5} \Omega^3_{\Delta_I - X\cap\Delta_I} & \xrightarrow{d} & \cdots & & & & \\ \downarrow & & & & & & \\ \bigoplus_{|I|=n-4} \Omega^3_{\Delta_I - X\cap\Delta_I}, & & & & & & \end{array}$$

where the vertical maps are restrictions on faces (with appropriate signs). (We simplify notation by writing Ω^i_Z for the sections of the sheaf Ω^i over Z rather than the sheaf itself.) What this means is that the de Rham cohomology of our motive, $H^{n-1}_{DR}(\mathbb{P}^{n-1} - X, \Delta - X \cap \Delta)$ is calculated by a double complex of algebraic differential forms $C^{a,b} = \bigoplus_{|I|=a} \Omega^b_{\Delta_I - X\cap\Delta_I}$. (This double complex is the de Rham complex associated to the complex (7.4) of constant sheaves.) The differential $d' : C^{a,b} \to C^{a+1,b}$ (resp. $d'' : C^{a,b} \to C^{a,b+1}$) is given by restriction to faces of Δ with appropriate signs (resp. exterior differentiation.) The total differential $d = d' + d''$. We have

$$\begin{aligned} H^{n-1}_{\mathrm{DR}}(\mathbb{P}^{n-1} - X, \Delta - X \cap \Delta) &= H^{n-1}(C^{**}, d) \\ &= \left(\bigoplus_{a+b=n-1} C^{a,b} \right) \Big/ d \left(\bigoplus_{a+b=n-2} C^{a,b} \right). \end{aligned} \tag{8.5}$$

(Note that in total degree $n-1$, all cochains are closed.) The cochain $(0, \ldots, 0, \eta_{n-1}) \in C^{0,n-1}$ represents a relative de Rham class whose period integrated against the homology chain given by $\sigma_{n-1} = \{(a_1, \ldots, a_n) \mid a_i \geq 0, \forall i\}$ is the Feynman amplitude. The content of Proposition 8.1 is that we can construct a form $w \in \bigoplus_{a+b=n-2} C^{a,b}$ such that

$$(0, \ldots, 0, \eta_{n-1}) - dw \in C^{n-4,3} \subset \bigoplus_{a+b=n-1} C^{a,b}. \tag{8.6}$$

Note that in evaluating the double complex $C^{*,*}$ in total degree $n-1$, the contributions will come from $C^{n-4,3}, C^{n-2,1}, C^{n-1,0}$. There is no contribution from $C^{n-3,2}$ because $H^{2k}(\mathbb{P}^{2k} - X) = (0)$ for a smooth quadric in even dimensional projective space of any dimension. The argument is the same as in Lemma 8.1.)

Finally, it follows from Lemma 7.1 that the filtration in equation (8.5) coming from the filtration $W_rC^{**} = \bigoplus_{a,b;\ a\geq n-2-r} C^{a,b}$ is the weight filtration $W_rH^{n-1}_{\mathrm{DR}}(\mathbb{P}^{n-1} - X, \Delta - X \cap \Delta)$. □

Note that this is reproves an old result of Nickel, who first studied the dependencies between one-loop graphs in a fixed dimension [8]. The basic content of Proposition 8.3 is that in the one-loop case, Stoke's formula can be used to reduce the Feynman integral to the case $n = 4$.

9. Dilogarithm motives

We have seen (7.6), (8.6) (note also the last comment in Section 7) that motives H arising from one-loop amplitudes satisfy $gr^W H = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)^b \oplus \mathbb{Q}(-2)^c$. (Technically, $H = W_4H^{n-1}(\mathbb{P}^{n-1} - X, \Delta - X \cap \Delta)$ with notation as in the previous section.) H will be a mixed Tate motive with weights $0, 2, 4$. In this section, we show how periods of such motives are related to dilogarithms. A general reference is [3]. We will follow the standard convention and trivialize the one-dimensional vector space $\mathbb{Q}(n) = \mathbb{Q}$ in such a way that the Betti structure is $(2\pi\mathrm{i})^n\mathbb{Q}$ so the DR-structure is $\mathbb{Q}$.

First, let us reduce to the case $c = 1$. We assume for simplicity that our de Rham structure is defined over $\mathbb{Q}$. (If not, one need simply extend the field of coefficients of H.) The quotient pure Hodge structure $H/W_2H \cong \bigoplus \mathbb{Q}(-2)$ satisfies

$$(H/W_2H)_{\mathrm{DR}} = (2\pi\mathrm{i})^2(H/W_2H)_B \subset (H/W_2H)_{\mathbb{C}}. \tag{9.1}$$

Further, the Hodge filtration has a single non-trivial piece in degree -2 and hence is defined already over $\mathbb{Q}$. What this means is that we can take our Feynman integrand η, which we view as lying in H_{DR} and project it to $(H/W_2H)_{\mathrm{DR}}$. The $\mathbb{C}$-line spanned by this image is canonically identified with $\mathbb{Q}(-2)_{\mathbb{C}}$, where $\mathbb{Q}(-2)_{DR} = \mathbb{Q} \cdot \eta$ and $\mathbb{Q}(-2)_B = \mathbb{Q} \cdot (\eta/(2\pi\mathrm{i})^2)$. The preimage $H' \subset H$ of this copy of $\mathbb{Q}(-2)$ has weight graded $gr^W H' = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)^b \oplus Q(-2)$, and it suffices to compute the periods for this Hodge structure.

Because H is a mixed Tate Hodge structure, there will exist a basis $e_{-2}, e_{-1,\mu}, e_0$ of $H_{\mathbb{C}}$ $(1 \leq \mu \leq b)$ such that the weight (resp. Hodge) filtration on $H_{\mathbb{C}} = \mathbb{C}^{[-2,0]}$ is given by $\mathbb{C}^{[-i,0]} = W_{2i}H_{\mathbb{C}}$ and $F^jH_{\mathbb{C}} = \mathbb{C}^{[-2,-j]}$, and such that the trivialization given by the e's identifies $gr^W H = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)^b \oplus \mathbb{Q}(-2)$. Here $\mathbb{C}^{[r,s]}$ is the span of the $e_{p,q}$ with $r \leq p \leq s$.

We consider first the sub-Hodge structure W_2H and the quotient H/W_0H, which are mixed Tate with weights $0, 2$ (resp. $2, 4$). There will exist b-tuples $(f_1,\dots,f_b)$ and $(g_1,\dots,g_b)$ in $(\mathbb{C}^\times)^b$ such that the Betti structures on $W_2H_\mathbb{C}=\mathbb{C}e_0\oplus\bigoplus_{\mu=1}^b\mathbb{C}e_{-1,\mu}$ (resp. $(H/W_0H)_\mathbb{C}=\bigoplus_{\mu=1}^b\mathbb{C}e_{-1,\mu}\oplus\mathbb{C}e_{-2}$) are given by the $\mathbb{Q}$-spans of the elements $\frac{1}{2\pi\mathrm{i}}(e_{-1,\mu}+\log g_\mu e_0)$ (resp. $\frac{1}{(2\pi\mathrm{i})^2}(e_{-2}+\log f_\mu e_{-1,\mu})$. The Betti structure on H will then be the $\mathbb{Q}$-span of the columns of a matrix

$$\frac{1}{(2\pi\mathrm{i})^2}\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0\\ \log f_1 & 2\pi\mathrm{i} & 0 & \dots & 0 & 0\\ \log f_2 & 0 & 2\pi\mathrm{i} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \dots & 0 & 0\\ \log f_b & 0 & 0 & \dots & 2\pi\mathrm{i} & 0\\ h & 2\pi\mathrm{i}\log g_b & 2\pi\mathrm{i}\log g_{b-1} & \dots & 2\pi\mathrm{i}\log g_1 & (2\pi\mathrm{i})^2\end{pmatrix}, \tag{9.2}$$

and the challenge is to compute h.

We can compute h upto a constant by using *Griffiths transversality.* We treat the f_μ, g_μ, h as functions and consider the variation of Hodge structure given by (9.2). It carries a connection ∇ for which the columns are horizontal. The transversality condition says $\nabla(F^i)\subset F^{i-1}$ where F^* is the Hodge filtration.

Write C_i for the columns in (9.2). We have

$$e_{-2}=C_1-\sum_\mu\frac{\log f_\mu}{2\pi\mathrm{i}}C_{\mu+1}-\frac{1}{(2\pi\mathrm{i})^2}\Big(h-\sum\log f_\mu\log g_\mu\Big)C_{b+2}. \tag{9.3}$$

Transversality says that $\nabla e_{-2}=Ae_{-2}+Be_{-1}$ for suitable A,B. On the other hand,

$$\begin{aligned}\nabla e_{-2}&=-\frac{1}{2\pi\mathrm{i}}\sum\frac{df_\mu}{f_\mu}C_{\mu+1}\\ &\quad-\frac{1}{(2\pi\mathrm{i})^2}\Big(dh-\sum(\log f_\mu dg_\mu/g_\mu+\log g_\mu df_\mu/f_\mu)\Big)C_{b+2}\\ &=-\frac{1}{2\pi\mathrm{i}}\sum\frac{df_\mu}{f_\mu}e_{-1,\mu}-\Big(dh-\sum\Big(\log f_\mu\frac{dg_\mu}{g_\mu}\Big)\Big)e_0.\end{aligned} \tag{9.4}$$

We conclude that the Betti structure on H is given upto a constant of integration by setting

$$h=\sum_\mu\int\log f_\mu\frac{dg_\mu}{g_\mu}, \tag{9.5}$$

in (9.2).

Remark 9.1. Note that the actual entries in (9.2) depend on the scaling of the $e_{i,\mu}$, which are given by algebraic de Rham classes. The actual values determined by the Feynman integrand will differ by an algebraic function of masses and momenta (eventually involving square roots) from the logs and dilogs in (9.2). For an example of how this works, see Section 14.

10. The triangle graph

The amplitude associated to the *triangle graph* with one loop, three vertices and three internal edges, is of interest both physically and mathematically. Let $C, D \subset \mathbb{P}^2$ be rational curves. We assume they are reduced but not necessarily irreducible. *Rational* in this context simply means that the normalization of each irreducible component is $\mathbb{P}^1$. Assume further that the intersection $C \cap D$ is transverse. In particular, $C \cap D$ is a finite set of smooth points in C and in D. Let $C^0 = C - (C \cap D)$ (resp. $D^0 = D - (C \cap D)$). The triangle graph yields a motive (10.1), which has the form $H^2(\mathbb{P}^2 - D, C^0)$ for suitable C, D.

We are particularly interested in the case when $C = L_0 \cup L_1 \cup L_2$ is the coordinate simplex (with homogeneous coordinates A_i and $L_i : A_i = 0$) and $D = L \cup X$ with $L : A_0 + A_1 + A_2 = 0$ and $X \subset \mathbb{P}^2$ a conic. We write for simplicity (see Fig. (1))

(10.1)
$$H := H^2\Big(\mathbb{P}^2 - (L \cup X), (L_0 \cup L_1 \cup L_2) - \big((L \cup X) \cap (L_0 \cup L_1 \cup L_2)\big), \mathbb{Q}\Big).$$

For the moment we assume that X is a smooth conic in general position with respect to the other lines.

Remark 10.1. In the proof of the following proposition, we use a generalized Poincaré duality. For P smooth and proper of complex dimension n, and $Z \subset P$ a subvariety which is not necessarily smooth, this identifies cohomology with supports in Z with the homology of Z. More precisely

$$H^k_Z(X, \mathbb{Q}) \cong H_{2n-k}(Z, \mathbb{Q}(-n)). \tag{10.2}$$

This is a powerful tool, but one pays a price in that the familiar residue map becomes

$$H^{k-1}(P - Z, \mathbb{Q}) \to H^k_Z(P, \mathbb{Q}) \cong H_{2n-k}(Z, \mathbb{Q}(-n)). \tag{10.3}$$

When Z has singularities, it can be difficult to understand this map in terms of the familiar residue. For us, P is a surface and Z a curve with normal

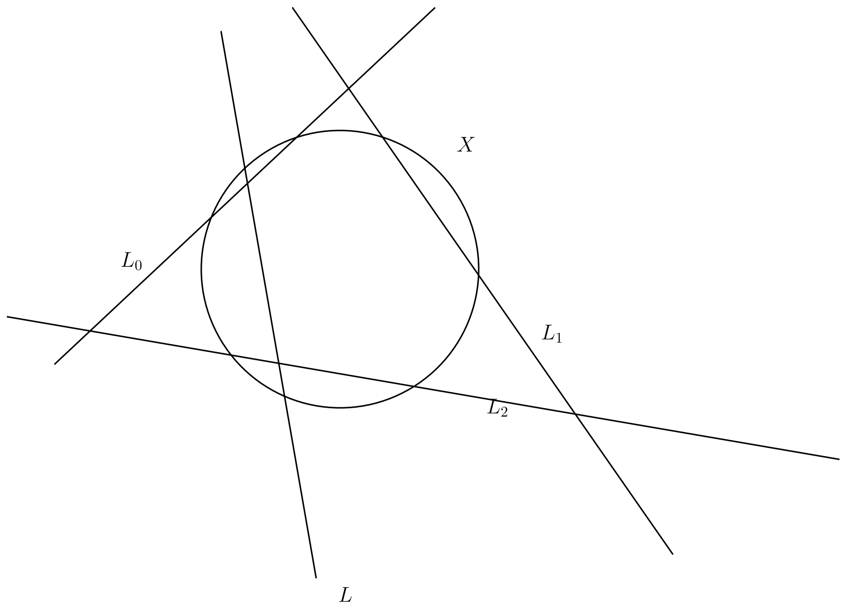


Figure 1: The geometry of the triangle graph. We indicate the three lines L_0, L_1, L_2, the line L and the conic X. The latter is given by $X : q_0^2 A_1 A_2 + q_1^2 A_2 A_0 + q_2^2 A_0 A_1 - (m_0^2 A_0 + m_1^2 A_1 + m_2^2 A_2)(A_0 + A_1 + A_2) = 0$. They are all in general position. There are many degeneracies possible: for example, the conic would go through the three corners $L_i = L_j$ in the massless case, or the conic can become tangential to one of those lines.

crossings. In this basic case, the reader will have no difficulty giving a more concrete construction of the relevant maps.

For example, in (10.6) below, D has normal crossings. The identification $H^2(\mathbb{P}^2 - D, \mathbb{Q}) \cong H_1(D, \mathbb{Q}(-2))$ can be understood as associating to a 2-form on $\mathbb{P}^2 - D$ its residues on the components of D. These 1-forms on the components will themselves have poles with canceling residues at the singular points of D, and it is a straightforward exercise to associate to such a form an element in $H_1(D)$. To check the twists in this case one can show directly that $H^2(\mathbb{P}^2 - D)$ has weight 4.

Proposition 10.2. *The mixed Hodge structure on H is mixed Tate, given by $W_0 H \subset W_2 H \subset W_4 H$ with*

$$gr_0^W H = \mathbb{Q}(0); \quad gr_2^W H = \mathbb{Q}(-1)^5; \quad gr_4^W H = \mathbb{Q}(-2). \tag{10.4}$$

Proof. Write $C = L_0 \cup L_1 \cup L_2$ and $D = L \cup X$. We have

$$H^1(\mathbb{P}^2 - D) \to H^1(C - C \cap D) \to H \to H^2(\mathbb{P}^2 - D) \to 0. \tag{10.5}$$

We have by Poincaré duality (formulated algebro-geometrically using cohomology with support)

$$H^2(\mathbb{P}^2 - D, \mathbb{Q}) \cong H^3_D(\mathbb{P}^2, \mathbb{Q}) \cong H_1(D, \mathbb{Q}(-2)) \cong \mathbb{Q}(-2). \tag{10.6}$$

Note that topologically, D is a union of two Riemann spheres S^2 meeting at two distinct points p_1, p_2: $D = S^2 \amalg S^2 / \sim$. The resulting long exact sequence of homology yields

$$\begin{aligned} H_1(S^2, \mathbb{Q}(-2))^{\oplus 2} \to H_1(D, \mathbb{Q}(-2)) &\to H_0(\{p_1, p_2\}, \mathbb{Q}(-2)) \\ &\to H_0(S^2, \mathbb{Q}(-2))^{\oplus 2} \end{aligned} \tag{10.7}$$

from which one deduces $H_1(D, \mathbb{Q}(-2)) \cong \mathbb{Q}(-2)$. We have again by duality a diagram

$$\begin{array}{ccc} H^1(\mathbb{P}^2 - D) & \longrightarrow & H^1(C - C \cap D) \\ \Big\downarrow \partial_D & & \Big\downarrow \partial_C \\ H_2(D, \mathbb{Q}(-2)) & \longrightarrow & H_0(C \cap D, \mathbb{Q}(-1)). \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ \mathbb{Q}(-1)^{\oplus 2} & \stackrel{\hookrightarrow}{\longrightarrow} & \mathbb{Q}(-1)^{\oplus 9} \end{array} \tag{10.8}$$

The map ∂_D is injective and has image the kernel of $H_2(D, \mathbb{Q}(-2)) \to H_2(\mathbb{P}^2, \mathbb{Q}(-2))$ which is one-dimensional. The image of ∂_C consists of all elements in $H_0(C \cap D, \mathbb{Q}(-1))$ which have degree 0 on each irreducible component of C. The kernel of ∂_C is $H^1(C, \mathbb{Q}) \cong \mathbb{Q}(0)$. The proposition follows. (A detailed proof that $\dim gr_2^W H = 5$ is given in Remark 10.3 below.) □

Remark 10.3. We continue to assume that X, L, L_i are in general position. Let $\ell_i = L_i \cap L$ and $\{m_i, n_i\} = X \cap L_i,\ i = 0, 1, 2$. We can identify $gr_2^W H$ with a subquotient of the Hodge structure $\mathbb{Q}(-1)^9$ with basis indexed by the ℓ_i, m_i, n_i as follows:

$$
\begin{aligned}
gr_2^W H &\cong \mathbb{Q}(-1)^5 \\
&\cong \left\{ \sum_{i=0}^{2} a_i\ell_i + b_i m_i + c_i n_i \mid a_i, b_i, c_i \in \mathbb{Q}(-1), a_i + b_i + c_i = 0 \right\} \Big/ \\
&\quad \times \left\{ \mathbb{Q}(-1) \cdot \sum_{i=0}^{2} (2\ell_i - m_i - n_i) \right\}.
\end{aligned}
\tag{10.9}
$$

Alternatively, consider zero cycles $z = \sum_0^2 a_i\ell_i + \sum_0^2 b_i m_i + \sum_0^2 c_i n_i$ with a_i, $b_i, c_i \in \mathbb{Q}$. We impose the condition that for any one of the irreducible components L, L_0, L_1, L_2, X, the "piece" of z supported on that component has degree 0. This amounts to the linear conditions

$$
0 = \sum a_i = \sum (b_i + c_i) = a_0 + b_0 + c_0 = a_1 + b_1 + c_1 = a_2 + b_2 + c_2. \tag{10.10}
$$

The vector space A of such cycles has dimension 5 and is identified with $\mathbb{Q}(1) \otimes gr_2^W H$.

Remark 10.4. Let $M_i, i = 1, 2, 3$ be the masses associated to the edges of the triangle graph. There is physical interest in the situation when one or more of the $M_i = 0$. With reference to (3.16), we see that setting $M_i = 0$ amounts to having the conic pass through the ith vertex of the triangle. The curves C, D in the above discussion no longer meet transversally, so we must blow up some of the vertices. Let $\pi : P \to \mathbb{P}^2$ be the blowup of $\nu = 1, 2, 3$ of the three points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Assume the other parameters are generic and let E_i be the exceptional divisors. In our motive H (10.1) we must replace $\mathbb{P}^2$ with P. The curve $L \cup X$ is replaced by the strict transform in P of $L \cup X$, and the other rational curve becomes the total inverse image in P of the triangle, a $(3 + \nu)$-gon comprising the strict transforms in P of the three lines $A_j = 0$ and the exceptional divisors E_i. One checks that each blowup drops the rank of $gr_2^W H$ by one. Thus $gr_2^W H = \mathbb{Q}(-1)^b$ with $b = 5 - \nu$, ν being the number of zero masses.

We will not compute the amplitude, as this has been done very nicely in [4]. Instead we will look more closely at qualitative results we can deduce about the motive. In particular, these will help to frame a (future) study of Landau poles for one-loop graphs.

11. Appendix on duality for the triangle graph motive

The following duality result, Proposition 11.2, will not be used in the sequel. The result holds in a general normal crossings situation, but we just state it for the case at hand.

Lemma 11.1. *Consider the diagram of varieties over $\mathbb{C}$ endowed with the complex topology.*

$$\begin{array}{ccc} \mathbb{P}^2 - D & \xrightarrow{j_D} & \mathbb{P}^2 \\ {\scriptstyle k_C}\uparrow & & \uparrow{\scriptstyle j_C} \\ \mathbb{P}^2 - (D \cup C) & \xrightarrow{k_D} & \mathbb{P}^2 - C \end{array}, \tag{11.1}$$

where the maps are the evident inclusions. We continue to assume $C \cap D$ is transverse. Let A be a constant sheaf on $\mathbb{P}^2$. Then

$$j_{D*}k_{C!}A_{\mathbb{P}^2-(D\cup C)} = j_{C!}k_{D*}A_{\mathbb{P}^2-(D\cup C)}. \tag{11.2}$$

Here the lower ! and the lower $*$ are extension by zero and direct image extension viewed as acting on the derived category. (That is, we write, e.g., j_{D*} in place of Rj_{D*}.)

Proof. There is a natural morphism of functors $j_{C!}k_{D*} \to j_{D*}k_{C!}$. Indeed, $j_{C!}$ is left adjoint to $j_C^! = j_C^*$, so it suffices to define a map $k_{D*} \to j_C^* j_{D*}k_{C!} = k_{C*}$, and we can take the identity. To show the map is an isomorphism, it suffices to look at the stalks at points of $C \cap D$. We can coordinatize a small complex neighbourhood of such a point so locally $\mathbb{P}^2 - (C \cap D)$ looks like $(U - \{0\}) \times (U - \{0\})$ where $U \subset \mathbb{C}$ is the open unit disk about 0. Locally, $D = \{0\} \times U$ and $C = U \times \{0\}$. The stalk at $(0, 0)$ on both sides of (11.2) is $j_{C!}A \boxtimes j_{D*}A$ by Kunneth. $\square$

Recall we have a Verdier duality functor on the derived category of constructible sheaves on a reasonable topological space. For $\mathbb{P}^2_{\mathbb{C}}$ it takes the form (to simplify we work with sheaves of $\mathbb{Q}$-vector spaces) $\mathbb{D}F = \mathit{Hom}(F, \mathbb{Q}_{\mathbb{P}^2}(2)[4])$ where the Hom is in the derived category. Verdier duality yields an isomorphism

$$R\Gamma(\mathbb{P}^2, \mathbb{D}F) \cong \mathrm{Hom}_{\mathbb{Q}}(R\Gamma(\mathbb{P}^2, F), \mathbb{Q}). \tag{11.3}$$

We have $\mathbb{D}j_{C!}A = j_{C*}\mathbb{D}A$ and $\mathbb{D}j_{C*}A = j_{C!}\mathbb{D}A$. Using the lemma we find for $A = \mathbb{Q}$

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Q}}(R\Gamma(\mathbb{P}^2, j_{D*}k_{C!}\mathbb{Q}_{\mathbb{P}^2-(D\cup C)}), \mathbb{Q}) &\cong \mathrm{Hom}_{\mathbb{Q}}(R\Gamma(\mathbb{P}^2, j_{C!}k_{D*}\mathbb{Q}_{\mathbb{P}^2-(D\cup C)}), \mathbb{Q}) \\ &\cong R\Gamma(\mathbb{P}^2, j_{C*}k_{D!}\mathbb{Q}_{\mathbb{P}^2-(D\cup C)}(2)[4]).\end{aligned} \tag{11.4}$$

Taking H^{-2} on both sides yields an isomorphism (duality)

Proposition 11.2. *With notation as above, we have*

$$Hom_{\mathbb{Q}}(H^2(\mathbb{P}^2 - D, C - (C\cap D), \mathbb{Q}), \mathbb{Q}) \cong H^2(\mathbb{P}^2 - C, D - (C\cap D), \mathbb{Q}(2)). \tag{11.5}$$

12. Landau poles and thresholds

In this section, we examine the phenomenon of Landau poles and normal and anomalous thresholds.

Example 12.1. Consider the case of the triangle graph. Changing notation slightly, we can rewrite (3.16) in this case

$$D(q,A) = |q_0|^2 A_1 A_2 + |q_1|^2 A_0 A_2 + |q_2|^2 A_0 A_1 - \left(\sum_0^2 m_i^2 A_i\right)\left(\sum_0^2 A_i\right). \tag{12.1}$$

Fix an i and suppose $|q_i|^2 = (m_j \pm m_k)^2$. (Here $i, j, k = 0, 1, 2$.) Then

$$\begin{aligned}D(q,A)|_{A_i=0} &= -(A_j + A_k)(m_j^2 A_j + m_k^2 A_k) + (m_j \pm m_k)^2 A_j A_k \\ &= -(m_j A_j \mp m_k A_k)^2.\end{aligned} \tag{12.2}$$

Geometrically, our conic becomes tangent to the line $L_i : A_i = 0$. If we look at the motive (10.1), we see that this corresponds to a degenerate configuration, and we might reasonably expect the amplitude to become singular. In fact, a moment's reflection reveals a vast number of possible degenerations including situations where the conic itself degenerates to a union of two lines through a point p, which may lie on one of the L_i, and situations where the conic passes through the point $L_i \cap L$ where $L : A_0 + A_1 + A_2 = 0$. (The referee points out that these configurations admit an interpretation in terms of the moduli of pointed genus 0 curves. Such moduli spaces are well understood, and perhaps a classification of the various limiting amplitudes is more accessible than the word "vast" might suggest.) One would like to better understand the behaviour of the amplitude near these singularities.

Let us consider a generalization due to Cutkosky [5] of the above example to more general graphs. If we write the amplitude in its usual (non-parameterized) form, we find an integral over $\mathbb{R}^q$ where q is the loop number of the graph. The integrand has in the denominator a product of rank 4 affine quadrics of the form (roughly) $(\vec{x}-\vec{q})^2+m^2$. These quadrics determine the polar locus of the integrand, and hence the motive whose realization will contain the amplitude as a period. In fact, the motive can be taken to be the union of the projective closures of the quadrics. If we ignore what is happening at infinity and just consider the affine quadrics, we might expect degeneracies to occur for values of the parameter $\vec{q}$ where some subset of the affine quadrics do not meet transversally. In general, the locus of such $\vec{q}$ will form a divisor in the space of momenta, and our first job is to use elimination theory to find this divisor.

To formulate things precisely, we fix a graph Γ. We write $H=H_1(\Gamma,\mathbb{Q})$ and $E=\mathrm{Edge}(\Gamma)$. As in (3.9), (3.10) we have

$$0\to H\otimes_{\mathbb{Q}}\mathcal{A}\to\mathcal{A}^E\xrightarrow{\partial}\mathcal{A}^{V,0}\to 0. \tag{12.3}$$

For $e\in E$ write $e^\vee:\mathcal{A}^E\to\mathcal{A}$ for the evident functional. Let me write (abusively) $e^{\vee,2}:\mathcal{A}^E\to\mathbb{R}$, $\sum_\varepsilon a_\varepsilon\varepsilon\mapsto a_e\bar{a}_e=a_{e,0}^2+a_{e,1}^2+a_{e,2}^2+a_{e,3}^2$. Given $q=\sum q_v v\in\mathcal{A}^{V,0}$ (so $\sum_v q_v=0$) consider the set

$$H(q):=\partial^{-1}(q)\subset\mathcal{A}^E. \tag{12.4}$$

Note that if $q\neq 0$ then technically $H(q)$ is not a vector space but a torsor under the vector space $H\otimes_{\mathbb{Q}}\mathcal{A}$. In fact, $H(q)$ embodies the Feynman rule imposing relations for each vertex. In other words, if vertex v lies on edges $e_1,\dots,e_p$ oriented to point toward v, and if $h\in H(q)$, then $\sum_i e_i^\vee(h)=q_v$.

Let $S\subset E$ be a subset of edges, and let $T\subset S$ be such that the $\{e^\vee|H(q)\}_{e\in T}$ form a basis for the vector space spanned by $\{e^\vee|H(q)\}_{e\in S}$. To make life interesting, assume $T\neq S$. Let $m_e\in\mathbb{R}$ be a collection of masses, and consider the quadrics

$$e^{\vee,2}-m_e^2:H(q)\to\mathbb{R};\quad e\in S. \tag{12.5}$$

Let $X(S,q)\subset H(q)$ be defined by the vanishing of the quadrics (12.5). We want to identify the set of $q\in\mathcal{A}^{V,0}$ such that $X(S,q)$ is not a smooth subvariety of $H(q)$ of codimension equal to $\#S$.

For each $e \in T$ we get 4 $\mathbb{R}$-linear functionals $e^{\vee,(0)}, \ldots, e^{\vee,(3)} : \mathcal{A}^E \to \mathbb{R}$. If we consider the Jacobian matrix for the map

$$(\ldots, e^{\vee,2} - m_e^2, \ldots)_{e \in S} : H(q) \to \mathbb{R}^{\#S}. \tag{12.6}$$

It will have $\#S$ rows and $4 \cdot \#T$ columns. For the row given by $e \in S$ we write $e^\vee | H(q) = \sum_{\varepsilon \in T} c_{e,\varepsilon} \varepsilon^\vee + a_e(q)$ with $a_e(q) \in \mathcal{A}$ and $c_{e,\varepsilon} \in \mathbb{R}$. We have

$$e^{\vee,2} | H(q) = \sum_{i=0}^{3} \left(\sum_{\varepsilon \in T} c_{e,\varepsilon} \varepsilon^{\vee,(i)} + a_e(q)^{(i)} \right)^2. \tag{12.7}$$

For $\tau \in T$ it follows that the entry of the jacobian matrix corresponding to $\partial/\partial\tau^{(i)}$ is

$$2c_{e,\tau} \left(\sum_{\varepsilon \in T} c_{e,\varepsilon} \varepsilon^{\vee,(i)} + a_e(q)^{(i)} \right). \tag{12.8}$$

It follows that the 4 entries of the e-row corresponding to $\partial/\partial\tau$ yield exactly

$$2c_{e,\tau} e^\vee | H(q). \tag{12.9}$$

If we assume S ordered so the elements of $T = \{\tau_1, \ldots, \tau_p\}$ come first, the matrix (12.9) evaluated at $h \in H(q)$ will look like

$$2 \cdot \begin{pmatrix} \tau_1^\vee(h) & 0 & 0 & \ldots \\ 0 & \tau_2^\vee(h) & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & \tau_p^\vee(h) \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{12.10}$$

The point h will be singular in $X(S,q)$ if first of all the quadrics (12.5) vanish at h (i.e., $h \in X(S,q)$) and secondly there exists a real non-zero row vector $\vec{b} = (b_1, \ldots, b_{\#S})$ of length $\#S$ which dies under right multiplication by (12.10). Since the $e^\vee | H(q)$ are affine linear combinations of the $\tau_i^\vee | H(q)$ we can use such a vector, which we treat as a vector with unknown entries b_i, to write p affine linear equations

$$\sum_{i=1}^{p} \alpha_{ij}(\vec{b}) \tau_i^\vee(h) = \beta_j(\vec{b}, q); \quad j = 1, \ldots, p. \tag{12.11}$$

We then solve these equations:

$$\tau_i^\vee(h) = \gamma_i(\vec{b}, q) \tag{12.12}$$

and substitute into the quadrics (12.5) (again using that $e^\vee|H(q)$ are affine linear combinations of the $\tau_i^\vee|H(q)$). Note that the γ_i are homogeneous of degree 0 in the b_j. The quadrics yield $\#S$ equations $F_i(\vec{b}, q) = 0$ which are homogeneous of degree 0 in the b_i. Write $\mathbb{A}^{V,0}$ for the affine space associated to $\mathcal{A}^{V,0} \cong (\mathbb{R}^{V,0})^4$. We can view the b_i as homogeneous coordinates on $\mathbb{P}^{\#S-1}$. In this way we get $\#S$ equations in $\mathbb{P}^{\#S-1} \times \mathbb{A}^{V,0}$. Projecting down to $\mathbb{A}^{V,0}$ amounts to eliminating the variables b_i. The image is a closed subvariety $Z \subset \mathbb{A}^{V,0}$ with the property that $q \in Z \Leftrightarrow$ the intersection of the quadrics in (12.5) is not transverse. Note that in general we expect Z is a hypersurface in $\mathbb{A}^{V,0}$ though of course degeneracies can occur. This Z is our divisor.

Example 12.2. Suppose elements of S form a cut, i.e., that $\Gamma - \bigcup_{e\in S} e$ is disconnected but that S is minimal in the sense that no proper subset of S disconnects Γ. (In removing edges, we do not remove vertices, so one of the connected components may be an isolated vertex.) It is easy to see in this case that $S - T = \{e\}$ is a single edge, so $S = \{\tau_1, \ldots, \tau_p, e\}$ and $\#S = p + 1$. For a suitable edge orientation we obtain

$$e^\vee = \sum_{i=1}^{p} \tau_i^\vee + a(q), \tag{12.13}$$

where $a(q)$ is some fixed linear combination (depending on Γ, S, T) of the q_v. (Recall $q = \sum q_v v$.) The matrix (12.10) in this case is

$$2 \cdot \begin{pmatrix} \tau_1^\vee & 0 & 0 & \ldots \\ 0 & \tau_2^\vee & 0 & \ldots \\ \vdots & \vdots & \vdots & \ldots \\ 0 & \ldots & 0 & \tau_p^\vee \\ \sum_{i=1}^p \tau_i^\vee + a(q) & \sum_{i=1}^p \tau_i^\vee + a(q) & \ldots & \sum_{i=1}^p \tau_i^\vee + a(q) \end{pmatrix}. \tag{12.14}$$

The linear equations and their solutions become

$$(b_i + b_{p+1})\tau_i^\vee + b_{p+1}\left(\sum_{j\neq i} \tau_j^\vee + a(q)\right) = 0; \;\; i = 1, \ldots, p, \tag{12.15}$$

$$\tau_i^\vee = a(q) D_i(b)/D(b). \tag{12.16}$$

It is easy to see the determinant $D(b)$ in the denominator does not identically vanish because the term $b_1 b_2 \cdots b_p$ cannot cancel.

The quadrics in this case become after substitution ($|a|^2 = a\bar{a}, a \in \mathcal{A}$.)

$$|a(q)|^2 D_i(b)^2/D(b)^2 = m_i^2; \quad 1 \leq i \leq p, \tag{12.17}$$

$$|a(q)|^2 \left(1 + \sum_{i=1}^{p} D_i(b)/D(b)\right)^2 = m_{p+1}^2. \tag{12.18}$$

Combining these, we deduce finally

$$|a(q)| = \sum_{i=1}^{p+1} \mu(i) m_i; \ \ \mu(i) = \pm 1. \tag{12.19}$$

Note (12.19) is necessary and sufficient for the intersection of the Feynman quadrics on $H(q)$ indexed by S to be non-transverse. Indeed, if (12.19) holds, we can solve for the $\tau_i^\vee$ as multiples of $a(q)$ using (12.16). The resulting point will lie on $X(S,q)$. The matrix (12.14) can then be treated as a matrix of scalars (more precisely, all entries lie on the same line). It has $p+1$ rows and p columns, so there is necessarily a non-trivial solution $\vec{b}$ and the point on $X(S,q)$ is not a point of transverse intersection.

The values of q where (12.19) hold are called *normal thresholds.* We have seen (12.2) that in the case of the triangle graph, normal thresholds correspond to values of external momenta where the polar conic (12.1) becomes tangent to one of the $L_i : A_i = 0$.

13. Thresholds for the triangle graph

In this section, we outline the use of limiting mixed Hodge structures (cf. Section 2) to study thresholds. We take the very basic case of the triangle graph with zero masses. The second Symanzik polynomial has the form

$$Q = q_0^2 A_1 A_2 + q_1^2 A_0 A_2 + q_2^2 A_0 A_1. \tag{13.1}$$

We will eventually want to assume $q_i = q_i(t)$ for t in a small disk about 0 and that two of the $q_i(t)^2$ tend to 0 as $t \to 0$. (As usual, q_i is a complex 4-vector and q_i^2 is the sum of the squares of the coordinates.) Our objective will be to show in this case that (subject to a certain condition on the degeneration $\rho \neq 0$ where ρ is given in (13.21)) the logarithm of monodromy N (2.12) satisfies $N^2 \neq 0$ and that as a consequence the leading term for the expansion of the amplitude as $t \to 0$ is a non-zero multiple of $(\log t)^2$.

The differential form we need to integrate is

$$\eta(q) := \frac{\Omega_2}{(A_0 + A_1 + A_2)(q_0^2 A_1 A_2 + q_1^2 A_0 A_2 + q_2^2 A_0 A_1)}. \tag{13.2}$$

Let $\pi : P \to \mathbb{P}^2$ be the blowup of the three vertices $A_i = A_j = 0$. We take q general so the singularities of the polar locus of $\eta(q)$ do not fall at the vertices. Let $Z(q) \subset P$ be the strict transform of this polar locus. Let $E_0, E_1, E_2 \subset P$ be the exceptional divisors, so E_i lies over $A_j = A_k = 0$, and let $F_i \subset P$ be the strict transform of the locus $\{A_i = 0\}$. The union

$$\Sigma := \pi^* \Delta = E_0 \cup E_1 \cup E_2 \cup F_0 \cup F_1 \cup F_2 \subset P \tag{13.3}$$

forms a hexagon. Note that $Z(q) = L' \cup Y(q)$ where L' is the strict transform of the line $L : A_0 + A_1 + A_2 = 0$ in $\mathbb{P}^2$ and $Y(q)$ is the strict transform of the conic. L' meets each F_i in a single point, and $Y(q)$ meets each of the E_i in a single point. Let $\Sigma^0 := \Sigma - \Sigma \cap Z(q)$. Then Σ^0 is a hexagon of affine lines E_i^0, F_j^0, so $H^1(\Sigma^0) = \mathbb{Q}(0)$. The motive we need to study is

$$H := H^2(P - Z(q), \Sigma^0). \tag{13.4}$$

We have seen in Remark 10.4 that $gr^W H = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)^2 \oplus Q(-2)$. The next step is to construct the Kummer motives $W_2 H$ and $H/W_0 H$ (see Example 2.2). Let $S = \sum n_i s_i$ be a 0-cycle (formal linear combination of smooth points) on Σ^0. We define a Kummer extension K_S by pullback as follows:

(13.5)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\Sigma^0) & \longrightarrow & H^1(\Sigma^0 - \{s_i\}) & \longrightarrow & \bigoplus_i \mathbb{Q}(-1) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle s} & & \\ 0 & \longrightarrow & \mathbb{Q}(0) & \longrightarrow & K_S & \longrightarrow & \mathbb{Q}(-1) & \longrightarrow & 0. \end{array}$$

Recall (2.10) that Kummer extensions $\leftrightarrow \mathbb{C}^\times$. Let $[S] \in \mathbb{C}^\times$ correspond to K_S as above. It is an easy exercise to check that the mapping

$$\{\text{0-cycles on } \Sigma^0\} \to \mathbb{C}^\times; \quad S \mapsto [S] \tag{13.6}$$

is a homomorphism of groups. To compute this map, we note that the E_i and F_i are projective lines with natural projective coordinates a_j, a_k. We have

$$F_i^0 = F_i - \{-1\}; \quad E_i^0 = E_i - \{|q_j|^2 a_k + |q_k|^2 a_j = 0\}. \tag{13.7}$$

Suppose $S = s$ is a single point. If $s \in E_i^0$ (resp. $s \in F_i^0$) we choose f a regular function on E_i^0 (resp. F_i^0) with a simple zero at s and no other zeroes. We then orient our hexagon by ordering the edges $E_0, F_1, E_2, F_0, E_1, F_2$. Let j, k be such that F_j, E_i, F_k (resp. E_j, F_i, E_k) is part of the ordered string of edges. Define $[s] = f(E_i \cap F_k)/f(E_i \cap F_j)$ (resp. $[s] = f(F_i \cap E_k)/f(F_i \cap E_j)$.)

We will be interested in the case $S = H_i' \cdot \Sigma^0$ where H_i' is the strict transform of the line $H_i : A_j - A_k = 0$ in $\mathbb{P}^2$. Thus $S = \{1 \in E_i^0\} + \{1 \in F_i^0\}$. On F_i we take $f = a_j - a_k/a_j + a_k$, so $[1 \in F_i^0] = -1$. On E_i, let $f = a_j - a_k/q_j^2 a_k + q_k^2 a_j$, so $[1 \in E_i^0] = -q_j^2/q_k^2$. Taken together, we see

$$[H_i' \cdot \Sigma^0] = \frac{q_j^2}{q_k^2}. \tag{13.8}$$

Lemma 13.1. *$W_2 H$ is an extension of $\mathbb{Q}(-1)^2$ by $\mathbb{Q}(0)$ corresponding to extension classes*

$$\frac{q_j^2}{q_k^2}, \frac{q_i^2}{q_k^2} \in \mathbb{C}^\times. \tag{13.9}$$

Here i, j, k are all distinct.

Proof. We have a diagram

(13.10)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\Sigma^0) & \longrightarrow & H^2(P - Z(q), \Sigma^0) & \longrightarrow & H^2(P - Z(q)) & \longrightarrow & 0 \\
 & & \| & & a\uparrow & & d\uparrow & & \\
0 & \longrightarrow & H^1(\Sigma^0) & \longrightarrow & H^2(P - Z(q) \cap \Sigma^0, \Sigma^0) & \longrightarrow & H^2(P - Z(q) \cap \Sigma^0) & \longrightarrow & 0 \\
 & & & & c\uparrow & & b\uparrow & & \\
 & & & & \mathbb{Q}(-1)^2 & = & \mathbb{Q}(-1)^2 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

$W_2 H$ is the image of the map a. Also

$$H^2(P - Z(q) \cap \Sigma^0) \cong H^2(P) \cong \mathbb{Q}(-1)^4 \tag{13.11}$$

generated by the 4 divisor classes $[L'], [E_0], [E_1], [E_2]$. The map b has image generated by the two divisor classes $[L'], [Y(q)] = 2[L'] - [E_0] - [E_1] - [E_2]$. Note that b lifts to a map c as indicated because the divisors $L', Y(q)$ do not meet Σ^0. It follows that the image of the map d is generated by the divisor classes H'_i where the H'_i are as above. The lemma follows from (13.9). □

It is convenient to work with $H'_i - H'_j$. If, e.g., we restrict the extension given by the top line of (13.10) to $\mathbb{Q}(-1)(H'_0 - H'_2) \subset H^2(P - Z(q))$, the resulting Kummer extension by the lemma is $q_2^2 q_0^2/q_1^4$. Similarly, the extension class after restriction to $\mathbb{Q}(-1)(H'_0 - H'_1)$ is $q_2^4/q_0^2 q_1^2$. (We are using here the orientation of the hexagon as fixed above.)

Lemma 13.2. *H/W_0 is an extension of $\mathbb{Q}(-2)$ by $\mathbb{Q}(-1)^2$ corresponding to extension classes given by formulas* (13.18) *and* (13.19) *below.*

Proof. We can identify

$$H/W_0 \cong H^2(P - Z(q)). \tag{13.12}$$

Here $Z(q)$ is isomorphic to the union of the conic $X(q) : q_0^2 A_1 A_2 + q_1^2 A_0 A_2 + q_2^2 A_0 A_1 = 0$ and the projective line $A_0 + A_1 + A_2 = 0$. We take the coefficients q_i^2 to be general, so these two plane curves meet in two distinct points:

$$p_\pm : A_0 = -A_2 - A_1; \tag{13.13}$$

$$A_1 = \frac{q_0^2 - q_1^2 - q_2^2 \pm \sqrt{q_0^4 + q_1^4 + q_2^4 - 2q_0^2 q_1^2 - 2q_0^2 q_2^2 - 2q_1^2 q_2^2}}{2q_2^2} A_2.$$

We will also need (straightforward check) that the function $f_{01} := 1 + q_2^2 A_1/q_1^2 A_2$ on $X(q)$ has divisor $(f_{01}) = (1,0,0) - (0,1,0)$. Similarly, $f_{02} = A_2 f_{01}/A_1$ has divisor $(f_{02}) = (1,0,0) - (0,0,1)$.

Using the techniques of Section 10 and (13.12), we can identify H/W_0 with the extension

$$\begin{aligned} &0 \to \Big(\mathbb{Q}(-1)v_0 + \mathbb{Q}(-1)v_1 + \mathbb{Q}(-1)v_2\Big)\Big/\mathbb{Q}(-1)\Big(v_0 + v_1 + v_2\Big) \\ &\to H_1\Big(Z(q) - \{v_0, v_1, v_2\}, \mathbb{Q}(-2)\Big) \to H_1(Z(q), \mathbb{Q}(-2)) \to 0. \end{aligned} \tag{13.14}$$

More directly, if we identify

(13.15)
$$\mathrm{Image}(H^2(P) \to H^2(P - Z(q))) \cong \mathbb{Q}(-1)(H'_0 - H'_2) \oplus \mathbb{Q}(-1)(H'_1 - H'_2)$$

we can deduce from (13.12) an exact sequence

(13.16)
$$0 \to \mathbb{Q}(-1)(H_0' - H_2') \oplus \mathbb{Q}(-1)(H_1' - H_2') \to H/W_0 \to H_1(Z(q), \mathbb{Q}(-2)) \to 0$$

We have $[H_i' - H_j'] = [E_j - E_i]$, and the identification of (13.16) with (13.14) sends $[H_i' - H_j'] \mapsto v_j - v_i = [E_j - E_i] \cdot Z(q)$.

Twisting and dualizing (13.14), we obtain the extension

$$\begin{aligned} 0 &\to H^1(Z(q), \mathbb{Q}) \to H^1(Z(q) - \{v_0, v_1, v_2\}, \mathbb{Q}) \\ &\to \Big(\mathbb{Q}(-1)v_0 + \mathbb{Q}(-1)v_1 + \mathbb{Q}(-1)v_2\Big)^{\deg 0} \to 0. \end{aligned} \tag{13.17}$$

The class of the extension obtained by restricting on the right to $\mathbb{Q}(-1)$ $(v_i - v_j)$ is calculated by the ratio $f_{ij}(p_+)/f_{ij}(p_-) \in \mathbb{C}^\times$. We have, e.g.,

$$\begin{aligned} & f_{01}(p_+)/f_{01}(p_-) \\ &\quad = \frac{\left(q_0^2 + q_1^2 - q_2^2 + \sqrt{q_0^4 + q_1^4 + q_2^4 - 2q_0^2q_1^2 - 2q_0^2q_2^2 - 2q_1^2q_2^2}\right)^2}{4q_0^2q_1^2}. \end{aligned} \tag{13.18}$$

$$\begin{aligned} & f_{02}(p_+)/f_{02}(p_-) \\ &\quad = \frac{q_0^2 + q_2^2 - q_1^2 - \sqrt{q_0^4 + q_2^4 + q_1^4 - 2q_0^2q_2^2 - 2q_0^2q_1^2 - 2q_2^2q_1^2}}{q_0^2 + q_2^2 - q_1^2 + \sqrt{q_0^4 + q_2^4 + q_1^4 - 2q_0^2q_2^2 - 2q_0^2q_1^2 - 2q_2^2q_1^2}} \\ &\quad\quad \times (f_{01}(p_+)/f_{01}(p_-)). \end{aligned} \tag{13.19}$$

□

Suppose now that the $q_i^2 = q_i(t)^2$ are analytic functions of a parameter t with $|t| < \varepsilon$. After scaling we may suppose $\lim_{t\to 0} q_0(t)^2 = 1$. We will suppose further that $\mathrm{ord}_0(q_i^2) > 0$ for $i = 1, 2$. Replacing t by a power if necessary, we can arrange that the monodromy σ as t winds around 0 acts trivially on $gr^W H$. We want to compute $N^2 = (\log \sigma)^2$. For the family of Kummer extensions $E_{x(t)}$ as in Example 2.2 with $gr^W E = \mathbb{Q}(1) \oplus \mathbb{Q}(0)$, one sees easily from (2.8) that N, viewed as a map $\mathbb{Q}(0) \to \mathbb{Q}(1)(-1) = \mathbb{Q}(0)$ is multiplication by $\mathrm{ord}_0(x(t))$. Similarly, $N^2 : H \to H(-2)$ factors as $N^2 : \mathbb{Q}(-2) = H/W_2H \to (W_0H)(-2) = \mathbb{Q}(-2)$. This in turn can be factored

$$\mathbb{Q}(-2) \xrightarrow{N_{H/W_0}} \mathbb{Q}(-2)(H_0' - H_2') \oplus \mathbb{Q}(-2)(H_0' - H_1') \xrightarrow{N_{W_2}} \mathbb{Q}(-2). \tag{13.20}$$

Set

$$b = \mathrm{ord}_0\Big(q_0^2 + q_2^2 - q_1^2 - \sqrt{q_0^4 + q_2^4 + q_1^4 - 2q_0^2q_2^2 - 2q_0^2q_1^2 - 2q_2^2q_1^2}\Big) > 0.$$
$$c = \mathrm{ord}_0 f_{01}(p_+)/f_{01}(p_-).$$

Formula (13.19) and Lemma 13.1 (see also the discussion below that lemma) one sees that $N^2 : \mathbb{Q}(-2) \to \mathbb{Q}(-2)$ is multiplication by

(13.21)
$$\rho := (4 \cdot \mathrm{ord}_0(q_2^2) - 2 \cdot \mathrm{ord}_0(q_1^2))c + (-4 \cdot \mathrm{ord}_0(q_1^2) + 2 \cdot \mathrm{ord}_0(q_2^2))(b + c).$$

If for example we take $\mathrm{ord}_0(q_1^2) = \mathrm{ord}_0(q_2^2) > 0$, we obtain

$$N^2 = \text{multiplication by} - 2b \cdot \mathrm{ord}_0(q_2^2) \neq 0. \tag{13.22}$$

In order to explicit the limiting behaviour of the amplitude, we consider (2.15), which in the current setup ($\eta(q)$ as in (13.2)) looks like

$$\lim_{t\to 0} \exp\left(-N\frac{\log t}{2\pi \mathrm{i}}\right)\begin{pmatrix} \int_{\gamma_0} \eta(q) \\ \int_{\gamma_{-2,1}} \eta(q) \\ \int_{\gamma_{-2,2}} \eta(q) \\ \int_{\gamma_{-4}} \eta(q) \end{pmatrix} = \begin{pmatrix} a_0 \\ a_{1,1} \\ a_{1,2} \\ a_2 \end{pmatrix}. \tag{13.23}$$

Here, the γ_j for $j \leq i$ form a basis for the homology $H^\vee_{\mathbb{Q}}$. Our limiting approximation for $\int_{\gamma_0} \eta(q)$ is therefore the top entry in the column vector

$$\exp\left(+N\frac{\log t}{2\pi \mathrm{i}}\right)\begin{pmatrix} a_0 \\ a_{1,1} \\ a_{1,2} \\ a_2 \end{pmatrix}. \tag{13.24}$$

Since N^2 has all entries 0 except for ρ (13.21) in the upper right corner, we conclude

$$\int_{\gamma_0} \eta(q) \sim \frac{\rho}{2}\Big(\lim_{t\to 0}\int_{\gamma_{-4}} \eta(q(t))\Big)(\log t/2\pi \mathrm{i})^2 + B\log t/2\pi \mathrm{i} + C, \tag{13.25}$$

for suitable constants B, C.

It remains, finally to compute the limit in (13.25). The cycle γ_{-4} is a generator of the image $(W_4H)^\vee \hookrightarrow H^\vee$. By adjunction, we can compute the integral by a suitable residue computation on $\eta(q)$. In affine coordinates $a_i = A_i/A_0$ we find

$$\pm\eta(q) = \frac{da_1 \wedge da_2}{(1 + a_1 + a_2)(q_0^2 a_1 a_2 + q_1^2 a_2 + q_2^2 a_1)}. \tag{13.26}$$

Let b_2 be the coordinate a_2 restricted to the line $a_1 + a_2 + 1 = 0$. The residue yields

$$\frac{db_2}{-q_0^2 b_2^2 + (q_1^2 - q_0^2 - q_2^2) b_2 - q_2^2}. \tag{13.27}$$

For a suitable choice of γ_{-4}, $\int_{\gamma_{-4}} \eta(q(t))$ will be the difference of the two residues of (13.27). Since the sum of the two residues is zero, it will suffice to show that an individual residue does not tend to 0. With our assumptions that $q_0^2 \to 1, q_i^2 \to 0, i = 1, 2$, this is straightforward. We have proved

Theorem 13.3. *Consider the triangle graph with zero masses and momenta $q_i(t), i = 0, 1, 2$. We treat the momenta as complex 4-vectors, so $q_i(t)^2 = \sum_{j=1}^4 q_i^{(j)}(t)^2$ is analytic in a complex parameter t for $|t| \to 0$. Assume $q_0(0)^2 = 1$ and $q_i(0)^2 = 0,\ \ i = 1, 2$. Assume further that the $\mathrm{ord}_0(q_i^2)$ are such that ρ in (13.21) is non-zero. (E.g., q_1^2 and q_2^2 vanish to equal order at $t = 0$.) Then if we take γ_0 to be the chain $\{(x, y, z) \in \mathbb{P}^2(\mathbb{R}) \mid x, y, z \geq 0\}$ in $\mathbb{P}^2$ then*

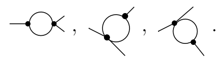
$$\int_{\gamma_0} \eta(q) \sim A(\log t/2\pi\mathrm{i})^2 + B \log t/2\pi\mathrm{i} + C \tag{13.28}$$

for suitable constants A, B, C with $A \neq 0$.

Remark 13.4. The referee inquires about the case when $N^2 = 0$. We have not worked this out. Note in general that the datum ρ is discrete, depending only of the order of vanishing at $t = 0$ of various meromorphic functions. When $\rho = 0$ the limiting mixed Hodge structure collapses and becomes simpler. We would be interested to know if there was any physical significance to this phenomenon.

14. Physics

Let us now try to understand the above considerations from a physicists viewpoint. Setting an edge variable to zero turns the triangle graphs into three reduced diagrams



Each of them is a function of a single invariant q_1^2, q_2^2 or q_3^2. The computation of these reduced diagrams is straightforward and delivers (in the equal mass case, otherwise the Kallen function replaces the square root) a result of the form

$$\sqrt{1-\frac{4m^2}{q^2}}\ln\frac{\sqrt{1-\frac{4m^2}{q^2}}-1}{\sqrt{1-\frac{4m^2}{q^2}}+1}, \tag{14.1}$$

which has, as a function of q^2, a branchcut from $[4m^2, +\infty[$ and a variation there $\sim 2\pi\mathrm{i}\sqrt{1-\frac{4m^2}{q^2}}$.

This gives us the equivalent of the functions f_i above. Note, however, that the expected log is multiplied by an algebraic function. The problem comes in the normalization of e_{-1}, or in other words the choice of differential form to represent a class in de Rham cohomology. Essentially, the Feynman integrand in this case has the form [1] (in the equal mass case, and for the example that edge 2 shrinks)

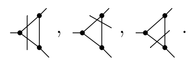
$$\omega = \frac{\ln\frac{m^2(A_0^2+A_1^2)+(2m^2-q_2^2)A_0A_1}{m^2(A_0^2+A_1^2)+(2m^2-\mu^2)A_0A_1}\Omega_1}{(A_0+A_1)^2}, \tag{14.2}$$

where we renormalize at a renormalization point $q_2^2=\mu^2$ by a simple subtraction in this one-loop example.

An easy partial integration (the boundary terms do not contribute as we have a renormalized integrand) determines the result Equation (14.1), with the square root determined from the two solutions of the quadric.

The issue for the functions g_j in (9.2) is more subtle. While the above functions were obtained as restrictions to a face of the simplex, equivalent to contracting an edge, the quotient motive is now obtained by studying the equivalent of the functions g_j before, which leads us to a physicists' notion of Cutkosky cuts of the triangle function.

The idea then is that the span of columns in (9.2) starting from the right hand column is supposed to be invariant under monodromy. In particular, the monodromy of the h is supposed to be a linear combination of the $2\pi\mathrm{i}\log g_j$. To mimic this in physics we may use Cutkosky cuts. Concretely, we look at



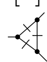
They correspond to integrals

$$\int d^4k \Theta(k_0 + q_{i,0})\Theta(k_0 + k_{j,0})\delta((k + q_i)^2 - m^2)\delta((k + q_j)^2 - m^2)\frac{1}{k^2 - m^2},$$

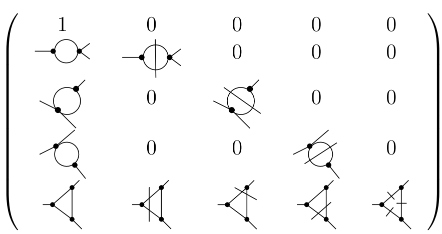
which readily integrate to

$$\int_a^b \frac{du}{cu + d},$$

for suitable a, b, c, d depending on masses and external momenta (these a, b, c, d are in the literature, in [7] for example).

Finally, the completely cut  leaves no integral to be done, but gives a known rational function of the q_i^2, m_j^2.

Hence, from a physicists viewpoint, the above structure looks like

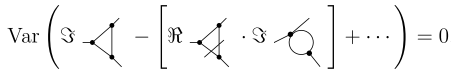


$$= (C_1, C_2, C_3, C_4, C_5),$$

nicely expressed in terms of reduced diagrams, Cutkosky cuts, and a traingle with all edges cut, which delivers a momentum and mass-dependent constant as the right lowermost entry, corresponding to the entry $(2\pi \mathrm{i})^2$ in the classical dilog case. This justifies recent practice in physics to put more internal edges on the mass-shell than prescribed by Cutkosky.

Note that we present here the middle part of the matrix as having three entries. This is a pedagogical simplification. The data of [7] provide six entries, two for each of the indicated cuts. That makes six entries, with momentum conservation we then get indeed five independent entries, in accordance with our previous considerations (see Remark 10.4).

A challenge for the future is to identify the correct differential equations and the connection with Griffith's transversality, so that it makes sense to discuss constructs like



as functions of complex external momenta.

We believe it is basically the presence of such invariant functions in the complex domain, which allows one to analytically continue Feynman diagrams in a way which will make the analytic requirements on Green functions more transparent once the Hodge structures of terms in the perturbative expansion are under control.

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Department of Mathematics,
University of Chicago,
Chicago IL 60637, USA
E-mail address: bloch@math.uchicago.edu

CNRS-IHES, 91440 Bures sur
Yvette, France and Center for Math. Phys,
Boston University,
Boston MA 02215,
USA
E-mail address: kreimer@ihes.fr

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