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A new look at one-loop integrals in string theory

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We revisit the evaluation of one-loop modular integrals in string theory, employing new methods that, unlike the traditional 'orbit method', keep T-duality manifest throughout. In particular, we apply the Rankin–Selberg–Zagier approach to cases where the integrand function grows at most polynomially in the infrared. Furthermore, we introduce new techniques in the case where "unphysical tachyons" contribute to the one-loop couplings. These methods can be viewed as a modular invariant version of dimensional regularization. As an example, we treat one-loop BPS-saturated couplings involving the d-dimensional Narain lattice and the invariant Klein j-function, and relate them to (shifted) constrained Epstein Zeta series of $\mathrm{O}(d,d;\mathbb{Z})$. In particular, we recover the well-known results for $d=2$ in a few easy steps.

1. Introduction

In essence, closed string theory is a quantum field theory of infinitely many fields, obtained by tensoring two infinite towers of left and right-moving excitations subject to a level-matching constraint [1]. As a result, any closed scattering amplitude at one-loop is an integral over two parameters, the Lagrange multiplier $\tau_1 \in [-\frac{1}{2}, \frac{1}{2}]$ for the level-matching constraint and the Schwinger time $\tau_2 > 0$ parameterizing the loop. Due to diffeomorphism invariance on the string world-sheet, the identification of the proper time is not unique and the integrand $F(\tau)$ is a modular function of the complex parameter $\tau \equiv \tau_1 + \mathrm{i}\tau_2$ (in general not holomorphic). To avoid an infinite over-counting, the domain of integration is restricted to a fundamental domain $\mathcal{F}$ for the modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, thereby removing the ultraviolet divergences from the region $\tau_2 \to 0$ which usually arise in quantum field theory. The usual field theoretical infrared (IR) divergences from the region $\tau_2 \to \infty$ are in general still present, due the existence of massless particles in the spectrum[1].

In general, computing such "one-loop modular integrals" is a daunting task, (in part) due to the unwieldy shape of the domain $\mathcal{F}$. For specific amplitudes describing BPS-saturated couplings in the low-energy effective action; however, the integrand simplifies and the modular integral can be computed

[1] In the following, we restrict our attention to closed string theories without physical tachyons, but allow for "unphysical tachyons", i.e., relevant operators which do not satisfy the level-matching condition, such as those present in heterotic models.

explicitly. One of the simplest instances occurs for certain anomaly related couplings in the 10-dimensional heterotic string theory [2]. In this case the integrand is the elliptic genus $F = \Phi(\tau)$, a weak holomorphic modular form with a singularity at the boundary of $\mathcal{F}$, and the modular integral can be evaluated by applying Stokes' theorem [2, 3].

In lower dimensions, however, the low-energy couplings depend non-trivially on the geometric moduli of an internal d-dimensional torus T^d. Indeed, the integrand function is typically of the form $\Phi(\tau)\Gamma_{(d+k,d)}$, where $\Phi(\tau)$ is again a weak holomorphic modular form and $\Gamma_{(d+k,d)}$ is the partition function of the Narain lattice associated to the torus compactification. The integrand is not holomorphic, so the integral cannot be reduced to a line integral on the boundary of $\mathcal{F}$ via Stokes' theorem. Rather, the main technique for dealing with such modular integrals in the physics literature has been the 'unfolding trick' or 'orbit method', pioneered in [4, 5] and generalized in many subsequent works [3, 6–14]. In a nutshell, this method consists in extending the integration domain $\mathcal{F}$ to a simpler region (the strip $\tau_1 \in [-1/2, 1/2], \tau_2 > 0$, or the full upper half plane $\tau_1 \in \mathbb{R}, \tau_2 > 0$) at the cost of restricting the sum over momenta and windings to suitable orbits of lattice vectors. While leading to a very useful expansion at large volume, this method has the drawback of obscuring the invariance of the resulting low-energy coupling under the automorphism group $\mathrm{O}(d+k, d; \mathbb{Z})$ of the Narain lattice. Although in some simple cases it is still possible to rewrite the result in terms of known automorphic forms of $\mathrm{O}(d+k, d; \mathbb{Z})$, this is in general not easy to achieve. For $k = 0$, $\varphi = 1$, it was conjectured in [15] that the result of the one-loop modular integral could in fact be expressed (in several different ways) as a constrained Epstein zeta series, manifestly invariant under $\mathrm{O}(d, d; \mathbb{Z})$.

The purpose of this note is to introduce a procedure for evaluating modular integrals that keeps manifest any (additional) symmetry of the integrand function. To this end we apply the Rankin–Selberg–Zagier (RSZ) method, a close relative of the 'unfolding method' which has been a standard tool in the mathematics literature (see e.g., [16] for a survey). In short, the main idea is to insert a non-holomorphic Eisenstein series in the modular integral of interest[2],

$$\int_{\mathcal{F}} d\mu\, F(\tau) \quad \longrightarrow \quad \int_{\mathcal{F}} d\mu \sum_{\substack{(m,n)\in\mathbb{Z}^2,\\ \gcd(m,n)=1}} \left(\frac{\tau_2}{|m-n\tau|^2}\right)^s F(\tau), \tag{1.1}$$

[2]Throughout the paper, $d\mu = \tau_2^{-2}\, d\tau_1\, d\tau_2$ denotes the $\mathrm{SL}(2;\mathbb{Z})$ invariant measure on the hyperbolic plane, normalized so that $\int_{\mathcal{F}} d\mu = \pi/3$.

compute the latter 'deformed' integral by applying the 'unfolding trick' to the sum over m, n for large $\Re(s)$, and finally obtain the desired integral by analytically continuing the result to $s = 0$ (where the Eisenstein series is known to reduce to a constant). Due to the growth of $F(\tau)$ as $\tau \to \mathrm{i}\infty$ in cases of physical interest, the integral in (1.1) is infrared divergent, while the original Rankin–Selberg method was restricted to functions $F(\tau)$ of rapid decay at the cusp. Fortunately, the Rankin–Selberg method was extended by Zagier [17] to allow for functions of moderate growth at the cusp, by introducing a hard infrared cut-off $\tau_2 < \mathcal{T}$ on the Schwinger time, unfolding the sum over m, n at finite $\mathcal{T}$, and giving a prescription for renormalizing the integral as the infrared cut-off is removed. With this renormalization prescription, one can view the replacement (1.1) as a stringy analogue of dimensional regularization, which preserves modular invariance.[3] Moreover, the integral (1.1) for $s \neq 0$ literally arises for certain BPS couplings, e.g., the $D^4 R^4$ couplings studied in [20]. Using the RSZ method, we shall evaluate the renormalized modular integral $\int_{\mathcal{F}} d\mu \varGamma_{(d,d)}$ exactly for any value of d, in terms of a constrained Epstein zeta series of $\mathrm{O}(d, d, \mathbb{Z})$, thereby proving the conjecture in [15]. We shall also recover the celebrated result of [6] for $d = 2$ in just in a few easy steps, illustrating the power of this approach.

While the RSZ method outlined above is very efficient for functions $F(\tau)$ of polynomial growth, which is typically the case for BPS couplings in type II string theory, it is unfortunately inadequate for functions $F(\tau)$ of exponential growth, which typically arise in heterotic amplitudes, due to the ubiquitous unphysical tachyon. More specifically, we are interested in modular integrals of the form

$$\int_{\mathcal{F}} d\mu \, \varPhi(\tau) \, \varGamma_{(d+k,d)}, \tag{1.2}$$

where $\varPhi(\tau)$ is a weak holomorphic modular form of weight $w = -k/2$ with an essential singularity at the cusp, $\varPhi(\tau) \sim \mathrm{e}^{-2\pi \mathrm{i}\kappa\tau} + \mathcal{O}(1)$ with $\kappa > 0$ (for heterotic strings, $\kappa = 1$ but our method works equally well for any $\kappa > 0$). Since the τ_1-average of the polar part of $F(\tau) = \varPhi(\tau) \, \varGamma_{(d+k,d)}$ vanishes, the divergence of the integral (1.2) is not worse for $\kappa > 0$ than for $\kappa = 0$, but the RSZ method nevertheless fails and one must resort to different techniques.

[3] Other modular invariant infrared regulators have been proposed, (e.g., [18]). It is also possible to regulate the integral by subtracting the non-decaying part of $F(\tau)$, as in [6]. As we shall see, it is straightforward to relate these different regularization schemes. For an early use of hard infrared cut-off regularization methods in string theory (see e.g., [19]).

In the second part of this work we shall develop a new procedure for dealing with the above class of modular integrals, which relies on representing the holomorphic part $\Phi(\tau)$ of the integrand in terms of a Poincaré series[4]

$$\Phi(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi(\gamma \cdot \tau), \tag{1.3}$$

for a suitable function $\psi(\tau)$, and then applying the unfolding trick to the sum in (1.3). Our procedure is close in spirit to the Rankin–Selberg method, in particular it keeps manifest the $\mathrm{O}(d+k, d; \mathbb{Z})$ symmetry of the Narain lattice; however, note that it does not involve any auxiliary modular function for unfolding the fundamental domain as in (1.1), rather it uses the function F itself or part of it.

A naive implementation of this idea, however, is hampered by the fact that the Poincaré series of a modular form of non-positive weight is not absolutely convergent and thus its unfolding is not justified. We circumvent this problem by deforming $\Phi(\tau)$ to a non-holomorphic Poincaré series $E(\tau_1, \tau_2; s)$ such that

$$\Phi(\tau) = \lim_{s \to 0} E(\tau_1, \tau_2; s), \tag{1.4}$$

applying the unfolding trick for large $\Re(s)$ and recovering the desired integral in the limit $s \to 0$. To illustrate our method, we shall compute the modular integral $\int d\mu \Gamma_{(d,d)}\, j(\tau)$ and represent it in terms of a 'shifted constrained Epstein zeta series', which (unlike the treatment in [3,8]) makes its invariance property under $\mathrm{O}(d, d, \mathbb{Z})$ manifest.

Before closing this introduction, we note that the RSZ method has already been useful in string theory for studying the distribution of the graded degrees of freedom in tachyon-free oriented closed string vacua, and their connection to the one-loop free energy [21]. Using this method it was shown that non-tachyonic string configurations are characterized by a spectrum of physical excitations that not only must enjoy asymptotic supersymmetry but actually, at very large mass, bosonic and fermionic states are bound to follow a universal oscillating pattern, whose frequencies are related to the non-trivial zeroes of the Riemann ζ-function. Similar studies have then been generalized to higher genus [22–24] where similar constraints on the interactions of physical states are expected to emerge.

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[4]Here, $\Gamma_\infty \subset \Gamma$ is the stabilizer of the cusp $\mathrm{i}\infty$, generated by the triangular matrices $\gamma = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, $n \in \mathbb{Z}$.

Furthermore, we anticipate that further development of the techniques presented in this work will allow to compute a variety of other modular integrals of interest in string theory. In particular, in vacua with broken supersymmetry, it would be interesting to study the behaviour of the one-loop effective potential near points of symmetry enhancement, where extra massless states appear. These typically lead to singularities in the vacuum energy that are difficult to analyse using the standard 'unfolding method' [25, 26], whereas they should be fully captured by our new approach. This would allow one to probe the stringy behaviour around the points of enhanced gauge symmetry. It would also be interesting to generalize our methods to higher genus.

The outline of the paper is as follows. In Section 2, we recall the original Rankin–Selberg method for functions of rapid decay, and its generalization to functions of moderate growth by Zagier. Our paraphrasing of [17] is mainly due to the 'basic identity' which appears in the body of the proof of [17] which we later require. In Section 3, we apply the RSZ method to one-loop modular integrals of symmetric lattice partition functions and recover the celebrated result of [6] in very few steps. We further give a proof of a conjecture in [15], clarifying the relation between the constrained Epstein zeta series of [15] and the Langlands–Eisenstein series studied in [20]. In Section 4, we develop a variation on the method of Rankin–Selberg and Zagier for functions of rapid growth at the cusp but with finite, or at most power-like divergent, modular integrals. We apply our procedure to the modular integral of symmetric lattice partition functions times the modular j-invariant (and its images under the action of the Hecke operators), and we express it in terms of a novel shifted, constrained Epstein Zeta series of $\mathrm{O}(d,d;\mathbb{Z})$. The appendix collects some properties of Kloosterman sums used in the text.

2. A brief review of the Rankin–Selberg–Zagier method

2.1. Rankin–Selberg method for functions of rapid decay

Assume that $F(\tau)$ is an automorphic function of the complex variable $\tau = \tau_1 + \mathrm{i}\tau_2$ of rapid decay at the cusp, i.e., such that $F(\tau)$ vanishes faster that any power of τ_2 at $\tau_2 \to \infty$. The Rankin–Selberg transform $\mathcal{R}^\star(F,s)$ of F is defined as the Petersson product:

$$\mathcal{R}^\star(F,s) \equiv \tfrac{1}{2}\zeta^\star(2s) \sum_{\substack{(c,d)\in\mathbb{Z},\\ (c,d)=1}} \int_{\mathcal{F}} d\mu\, \frac{\tau_2^s}{|c\tau+d|^{2s}}\, F(\tau_1,\tau_2), \tag{2.1}$$

between F and the non-holomorphic Eisenstein series

$$E^\star(\tau;s) \equiv \zeta^\star(2s) E(\tau;s), \qquad (2.2)$$
$$E(\tau;s) \equiv \tfrac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2,\\ (c,d)=1}} \frac{\tau_2^s}{|c\,\tau+d|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left[\Im\left(\gamma\cdot\tau\right)\right]^s .$$

Here $\zeta^\star(s) \equiv \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$ is the completed zeta function with simple poles at $s=1,0$ and zeroes in the strip $0<\Re(s)<1$. The integral over the fundamental domain $\mathcal{F}$ can be unfolded on the strip $\mathcal{S}=\{\tau_2>0, -\frac{1}{2}<\tau_1<\frac{1}{2}\}$, so that

$$\mathcal{R}^\star(F;s) = \zeta^\star(2s) \int_{\mathcal{S}} \frac{d\tau_1\, d\tau_2}{\tau_2^{2-s}}\, F(\tau) = \zeta^\star(2s) \int_0^\infty d\tau_2\, \tau_2^{s-2}\, F_0(\tau_2), \qquad (2.3)$$

where $F_0(\tau_2)$ is the constant term of F,

$$F_0(\tau_2) = \int_{-1/2}^{1/2} d\tau_1\, F(\tau). \qquad (2.4)$$

Thus, $\mathcal{R}^\star(F,s)$ is proportional to the Mellin transform of F_0. Now, $E^\star(\tau;s)$ is well-known to be a meromorphic function in s, with simple poles at $s=0,1$, satisfying the first Kronecker limit formula

$$E^\star(\tau;s) = \frac{1}{2(s-1)} + \tfrac{1}{2}\left(\gamma - \log(4\pi\,\tau_2\,|\eta(\tau)|^4)\right) + \mathcal{O}(s-1), \qquad (2.5)$$

where γ is the Euler–Mascheroni constant and $\eta(\tau)=q^{1/24}\prod_{n=1}^\infty(1-q^n)$ is the Dedekind function (as usual, $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$). These statements follow from the Chowla–Selberg formula

$$E^\star(\tau;s) = \zeta^\star(2s)\,\tau_2^s + \zeta^\star(2s-1)\,\tau_2^{1-s} \qquad (2.6)$$
$$+ 2\sum_{N\neq 0} |N|^{s-\frac{1}{2}} \sigma_{1-2s}(N)\,\tau_2^{1/2} K_{s-\frac{1}{2}}(2\pi|N|\tau_2)\,\mathrm{e}^{2\pi\mathrm{i}N\tau_1},$$

where $\sigma_t(N)$ is the divisor function

$$\sigma_t(N) \equiv \sum_{0<d|N} d^t, \qquad (2.7)$$

and $K_t(z)$ is the modified Bessel function of the second kind. The properties

$$\zeta^\star(s) = \zeta^\star(1-s), \quad K_t(x) = K_{-t}(x), \quad \sigma_t(n) = n^t\,\sigma_{-t}(n), \qquad (2.8)$$

ensure that $E^\star(\tau;s)$ satisfies the functional equation

$$E^\star(\tau;s) = E^\star(\tau;1-s). \tag{2.9}$$

It then follows that $\mathcal{R}^\star(F;s)$ inherits the same analytic and functional properties of $E^\star(\tau;s)$, i.e., it is a meromorphic function of s with simple poles at $s=0,1$ and symmetric with respect to the critical axis $\Re(s)=\frac{1}{2}$,

$$\mathcal{R}^\star(F;s) = \mathcal{R}^\star(F;1-s). \tag{2.10}$$

This method was used in the mathematics literature [27,28] to establish analytic properties of certain Dirichlet L-series (see e.g., [16] for a survey). More importantly for our purposes, the residue at $s=1$ of the Rankin–Selberg transform is equal to (half) the average value of F on the fundamental domain $\mathcal{F}$,

$$\text{Res}\ \mathcal{R}^\star(F;s)\big|_{s=1} = \tfrac{1}{2}\int_{\mathcal{F}} d\mu\, F = -\text{Res}\ \mathcal{R}^\star(F;s)\big|_{s=0}\,. \tag{2.11}$$

This in principle provides a way to evaluate the integral of the automorphic function F on the fundamental domain, from the Mellin transform of the constant term F_0 [21–23]. In particular, the residue at $s=1$ depends only on the behaviour of $F_0(\tau_2)$ near $\tau_2=0$.

2.2. Rankin–Selberg method for functions of moderate growth

In physics applications, F is rarely of rapid decay. Fortunately, the Rankin–Selberg method has been adapted to the case of automorphic functions of moderate growth by Zagier [17], which we paraphrase below.

Let $F(\tau)$ be an automorphic function whose behaviour at the cusp $\tau = \mathrm{i}\infty$ is of the form

$$F(\tau) \sim \varphi(\tau_2) + O(\tau_2^{-N}) \quad (\forall N>0), \tag{2.12}$$

where

$$\varphi(\tau_2) = \sum_{i=1}^{\ell} \frac{c_i}{n_i!}\,\tau_2^{\alpha_i}\,\log^{n_i}\tau_2 \tag{2.13}$$

for suitable $c_i \in \mathbb{C}, \alpha_i \in \mathbb{C}, n_i \in \mathbb{N}$. For this class of functions, following Zagier [17], we *define* the Rankin–Selberg transform as

$$\mathcal{R}^\star(F;s) = \zeta^\star(2s) \int_0^\infty d\tau_2\, \tau_2^{s-2}\, (F_0 - \varphi)\,, \tag{2.14}$$

where $F_0(\tau_2)$ is the τ_1-constant term (2.4) of F. The integral (2.14) converges absolutely when $\Re(s)$ is large enough (namely, $\Re(s) > |\Re(\alpha_i)|$ for all i). As we shall see,[5] $\mathcal{R}^\star(F;s)$ can be meromorphically continued to all s, with possible poles at $s = 0, 1, \alpha_i$ and $1-\alpha_i$, and is invariant under $s \mapsto 1-s$. Moreover, (half) the residue of $\mathcal{R}^\star(F;s)$ at $s=1$ gives a prescription of the (otherwise divergent) renormalized integral of F on the fundamental domain. To establish this, we shall use a combination of 'hard infrared cut-off' and 'zeta function regularization', i.e., consider the (manifestly finite) integral

$$\mathcal{R}^\star_{\mathcal{T}}(F;s) \equiv \int_{\mathcal{F}_\mathcal{T}} d\mu\, F(\tau)\, E^\star(\tau;s) \tag{2.15}$$

on the "cut-off fundamental domain" $\mathcal{F}_\mathcal{T} = \mathcal{F} \cap \{\tau_2 \le \mathcal{T}\}$. It is a fundamental domain for the "cut-off Poincaré upper half plane"

$$\mathcal{H}_\mathcal{T} = \mathcal{H} \cap \{\tau_2 \le \mathcal{T}\} \; - \bigcup_{\substack{(a,c)\in\mathbb{Z}^2,\\ c\ge 1,(a,c)=1}} S_{a/c}, \tag{2.16}$$

where $S_{a/c}$ is the disk of radius $1/(2c^2\mathcal{T}))$ tangent to the real axis at a/c. Defining $\chi_\mathcal{T}$ to be the characteristic function of $\mathcal{H}_\mathcal{T}$ and performing the same unfolding trick as in (2.3) with $F \cdot \chi_\mathcal{T}$ in place of F, we obtain

$$\mathcal{R}^\star_{\mathcal{T}}(F;s) = \zeta^\star(2s) \int_{\Gamma_\infty \backslash \mathcal{H}_\mathcal{T}} d\tau_1 d\tau_2\, F(\tau)\, \tau_2^{s-2}, \tag{2.17}$$

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[5] This statement can be seen right away by noticing that $\varphi(\tau_2)$ is annihilated by the differential operator $\Box \equiv \prod_{i=1}^{l} [\Delta - \alpha_i(\alpha_i - 1)]^{n_i+1}$, where Δ is the Laplacian on $\mathcal{H}$, and applying the standard Rankin–Selberg method to the rapidly decaying function $\Box F$. We are grateful to D. Zagier for pointing this out.

Using (2.16), this may be rewritten as

(2.18)
$$\mathcal{R}^{\star}_{\mathcal{T}}(F;s) = \zeta^{\star}(2s) \times \left(\int_0^{\mathcal{T}} d\tau_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 F(\tau)\tau_2^{s-2} - \sum_{\substack{c\geq 1, (a,c)=1, \\ a \mod c}} \int_{S_{a/c}} d\tau_1 d\tau_2 F(\tau)\tau_2^{s-2} \right).$$

Now, the disc $S_{a/c}$ is mapped to $\mathcal{H} \cap \{\tau_2 > \mathcal{T}\}$ by any element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. For fixed a/c, all such elements are related by right multiplication by Γ_∞. Thus, the last term in the bracket in (2.18) can be rewritten as

$$\int_{\mathcal{F}-\mathcal{F}_{\mathcal{T}}} d\mu\, F(\tau) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma, c\geq 1} [\Im(\gamma\cdot\tau)]^s \, d\mu. \tag{2.19}$$

The sum over γ reproduces the Eisenstein series $E(\tau;s)$, modulo the term τ_2^s due to the restriction $c \geq 1$. Putting everything together, we find

$$\begin{aligned} \mathcal{R}^{\star}_{\mathcal{T}}(F;s) = \zeta^{\star}(2s) & \int_0^{\mathcal{T}} d\tau_2\, F_0(\tau_2)\, \tau_2^{s-2} \\ & - \int_{\mathcal{F}-\mathcal{F}_{\mathcal{T}}} d\mu\, F\, \left(E^{\star}(\tau;s) - \zeta^{\star}(2s)\, \tau_2^s\right). \end{aligned} \tag{2.20}$$

The symmetry $s \mapsto 1-s$ may be restored by further subtracting the second constant term from $E^{\star}(\tau;s)$, so that, after multiplying by $\pi^{-s}\Gamma(s)$ and rearranging terms,

(2.21)
$$\begin{aligned} & \int_{\mathcal{F}_{\mathcal{T}}} d\mu\, F(\tau_1,\tau_2)\, E^{\star}(\tau;s)\, d\mu + \int_{\mathcal{F}-\mathcal{F}_{\mathcal{T}}} d\mu\, F(\tau_1,\tau_2) \left(E^{\star}(\tau;s) - E^{\star(0)}(\tau;s)\right) \\ & \quad = \zeta^{\star}(2s) \int_0^{\mathcal{T}} d\tau_2\, F_0(\tau_2)\, \tau_2^{s-2} - \zeta^{\star}(2s-1) \int_{\mathcal{T}}^{\infty} d\tau_2\, F_0(\tau_2)\, \tau_2^{-1-s}, \end{aligned}$$

where $E^{\star(0)}(\tau;s) = \zeta^{\star}(2s)\, \tau_2^s + \zeta^{\star}(2s-1)\, \tau_2^{1-s}$ is the constant term in the Fourier expansion of $E^{\star}(\tau;s)$. Since this is τ_1-independent, the product $F\, E^{\star(0)}(\tau;s)$ appearing in the first line can be replaced by $F_0\, E^{\star(0)}(\tau;s)$ without changing the result of the integral over the strip $\mathcal{F}-\mathcal{F}_{\mathcal{T}}$. Now, the terms in the second line evaluate to

$$\begin{aligned}(2.22)\qquad \int_0^{\mathcal{T}} d\tau_2\, F_0(\tau_2)\, \tau_2^{s-2} &= \mathcal{R}^\star(F;s)/\zeta^\star(2s) \\ &\quad - \int_{\mathcal{T}}^\infty d\tau_2\, (F_0-\varphi)\, \tau_2^{s-2} + h_{\mathcal{T}}(s), \\ \int_{\mathcal{T}}^\infty d\tau_2\, F_0(\tau_2)\, \tau_2^{-1-s} &= \int_{\mathcal{T}}^\infty d\tau_2\, (F_0-\varphi)\, \tau_2^{-1-s} + h'_{\mathcal{T}}(s),\end{aligned}$$

where $h_{\mathcal{T}}$ and $h'_{\mathcal{T}}$ are incomplete Mellin transforms of φ,

$$(2.23)\qquad h_{\mathcal{T}}(s) = \int_0^{\mathcal{T}} d\tau_2\, \varphi(\tau_2)\, \tau_2^{s-2}, \quad h'_{\mathcal{T}}(s) = \int_{\mathcal{T}}^\infty d\tau_2\, \varphi(\tau_2)\, \tau_2^{-1-s}.$$

A key fact about the class of functions φ in (2.13) is that their (complete) Mellin transform vanishes, therefore $h'_{\mathcal{T}}(s) = -h_{\mathcal{T}}(1-s)$. Moreover, integrating $\varphi\, \tau_2^{s-2}$ once,

$$(2.24)\qquad h_{\mathcal{T}}(s) = \sum_{i=1}^{\ell} \frac{c_i}{n_i!} \sum_{m=0}^{n_i} \frac{(-1)^{n_i-m}}{m!}\, \frac{\mathcal{T}^{s+\alpha_i-1} \log^m \mathcal{T}}{(s+\alpha_i-1)^{n_i-m+1}}.$$

Using this and rearranging terms, we arrive at Zagier's basic identity, Equation (27) in [17],

$$\begin{aligned}(2.25)\qquad \mathcal{R}^\star(F;s) &= \int_{\mathcal{F}_{\mathcal{T}}} d\mu\, F\, E^\star(\tau;s) \\ &\quad + \int_{\mathcal{F}-\mathcal{F}_{\mathcal{T}}} d\mu\, \Big(F\, E^\star(\tau;s) - \varphi\, E^{\star(0)}(\tau;s) \Big) \\ &\quad - \zeta^\star(2s)\, h_{\mathcal{T}}(s) - \zeta^\star(2s-1)\, h_{\mathcal{T}}(1-s).\end{aligned}$$

Evidently, the r.h.s. of (2.25) is independent of $\mathcal{T}$, meromorphic in s, invariant under $s \mapsto 1-s$, and analytic away from $s = 0, 1$ (where $E^\star(\tau;s)$ has a simple pole) and from $s = \alpha_i, 1-\alpha_i$ (where $h_{\mathcal{T}}(1-s)$, respectively $h_{\mathcal{T}}(s)$, has a pole of degree $n_i + 1$)[6]. Thus, when no α_i coincides with $0, 1$,

$$(2.26)\quad \mathcal{R}^\star(F;s) = \sum_{i=1}^{l} c_i \left(\frac{\zeta^\star(2s)}{(1-\alpha_i-s)^{n_i+1}} + \frac{\zeta^\star(2s-1)}{(s-\alpha_i)^{n_i+1}} \right) + \frac{\Phi(s)}{s(s-1)},$$

[6] The apparent pole at $s = 1/2$ cancels between the last two terms in (2.25).

where $\Phi(s)$ is an entire function of s. Moreover, the residue at $s=1$ is

$$\begin{aligned}\text{Res }\mathcal{R}^\star(F;s)\big|_{s=1} &= -\text{Res }\left[\zeta^\star(2s)\,h_{\mathcal{T}}(s)\right]_{s=1} - \text{Res }\left[\zeta^\star(2s-1)\,h_{\mathcal{T}}(1-s)\right]_{s=1} \\ &\quad + \tfrac{1}{2}\left[\int_{\mathcal{F}_{\mathcal{T}}} d\mu\, F(\tau_1,\tau_2) + \int_{\mathcal{F}-\mathcal{F}_{\mathcal{T}}} d\mu\, \left(F(\tau_1,\tau_2)-\varphi(\tau_2)\right)\right].\end{aligned} \tag{2.27}$$

Letting $\hat\varphi(\tau_2)$ be an anti-derivative of $\varphi(\tau_2)$,

$$\hat\varphi(\tau_2) = \sum_{\substack{1\le i\le \ell\\ \alpha_i\neq 1}} c_i \sum_{m=0}^{n_i} \frac{(-1)^{n_i-m}}{m!}\,\frac{\tau_2^{\alpha_i-1}\,\log^m\tau_2}{(\alpha_i-1)^{n_i-m+1}} + \sum_{\substack{1\le i\le \ell\\ \alpha_i=1}} c_i\,\frac{\log^{n_i+1}\tau_2}{(n_i+1)!}, \tag{2.28}$$

and defining the *renormalized integral* as

$$\begin{aligned}\text{R.N.}\int_{\mathcal{F}} d\mu\, F(\tau_1,\tau_2) &= \int_{\mathcal{F}_{\mathcal{T}}} d\mu\, F(\tau_1,\tau_2) \\ &\quad + \int_{\mathcal{F}-\mathcal{F}_{\mathcal{T}}} d\mu\, \left(F(\tau_1,\tau_2)-\varphi(\tau_2)\right) - \hat\varphi(\mathcal{T}) \\ &= \lim_{\mathcal{T}\to\infty}\left[\int_{\mathcal{F}_{\mathcal{T}}} d\mu\, F(\tau_1,\tau_2) - \hat\varphi(\mathcal{T})\right],\end{aligned} \tag{2.29}$$

which is by construction $\mathcal{T}$-independent, Equation (2.27) may be rewritten as

$$\begin{aligned}\text{R.N.}\int_{\mathcal{F}} d\mu\, F(\tau_1,\tau_2) &= 2\,\text{Res}[\mathcal{R}^\star(F;s) + \zeta^\star(2s)\,h_{\mathcal{T}}(s) \\ &\quad + \zeta^\star(2s-1)\,h_{\mathcal{T}}(1-s)]_{s=1} - \hat\varphi(\mathcal{T}).\end{aligned} \tag{2.30}$$

In fact, the r.h.s. of (2.25) is itself the renormalized integral

$$\mathcal{R}^\star(F;s) = \text{R.N.}\int_{\mathcal{F}} d\mu\, F(\tau_1,\tau_2)\,E^\star(\tau;s), \tag{2.31}$$

with $F(\tau_1,\tau_2)\,E^\star(s,\tau)$, $\varphi(\tau_2)\,E^{\star(0)}(\tau;s)$ and $\zeta^\star(2s)\,h_{\mathcal{T}}(s)+\zeta^\star(2s-1)$ $h_{\mathcal{T}}(1-s)$ playing the rôle of $F(\tau_1,\tau_2)$, $\varphi(\tau_2)$ and $\hat\varphi(\mathcal{T})$ in (2.29), respectively.

For functions of rapid decay, the renormalized integral reduces to the usual integral and $h_{\mathcal{T}}, \hat\varphi(\mathcal{T})$ vanish, hence (2.30) reduces to (2.11). More generally, if $\Re(\alpha_i)<1$ for all i, the integral $\int_{\mathcal{F}} d\mu\, F$ still converges, and one can take the limit $\mathcal{T} \to \infty$ in (2.30) and recover (2.11). If however one of the $\Re(\alpha_i) \geq 1$, the integral $\int_{\mathcal{F}_\mathcal{T}} F \, d\mu$ diverges like $\hat{\varphi}(\mathcal{T})$ as a function of the infrared cut-off, and (2.30) provides a renormalization prescription which depends only on the divergent terms with $\Re(\alpha_i) \geq 1$ in (2.13).

Of course, the renormalization prescription (2.29) is by no means the only possible one. For example, one may decide to subtract the non-decaying part from F and integrate the remainder on the fundamental domain, defining

$$\text{R.N.}' \int_{\mathcal{F}} d\mu \, F(\tau_1, \tau_2) = \int_{\mathcal{F}} d\mu \, \left(F(\tau_1, \tau_2) - \varphi([\tau_2]) \right) . \tag{2.32}$$

Here we denoted by $[\tau_2]$ the imaginary part of $\gamma \cdot \tau$, where γ is an element of Γ which maps τ into the standard fundamental domain $\mathcal{F}$ (so $[\tau_2] = \tau_2$ if $\tau \in \mathcal{F}$). The renormalized integrals (2.32) and (2.29) differ by a finite quantity

$$\begin{aligned} \Delta \equiv \text{R.N.} \int_{\mathcal{F}} d\mu \, F(\tau_1, \tau_2) - \text{R.N.}' \int_{\mathcal{F}} d\mu \, F(\tau_1, \tau_2) \\ = \lim_{\mathcal{T} \to \infty} \left[\int_{\mathcal{F}_\mathcal{T}} d\mu \, \varphi([\tau_2]) - \hat{\varphi}(\mathcal{T}) \right] , \end{aligned} \tag{2.33}$$

which can be computed explicitly. For example, if all the n_i's are zero,

$$\Delta = \sum_{i=1}^{\ell} c_i \, \frac{\delta_{\alpha_i, 1} - {}_2F_1(\frac{1}{2}, \frac{1-\alpha_i}{2}; \frac{3}{2}; \frac{1}{4})}{\alpha_i - 1} , \tag{2.34}$$

with $\delta_{a,b}$ being the Kronecker symbol, and ${}_2F_1(a, b; c; z)$ the standard hypergeometric function. Note that the case $\alpha_i = 1$ is well encoded in (2.34) since, in the limit $\alpha_i \to 1$,

$${}_2F_1 \left(\frac{1}{2}, \frac{1-\alpha_i}{2}; \frac{3}{2}; \frac{1}{4} \right) = 1 - \left(1 - \log \frac{3\sqrt{3}}{2} \right) (\alpha_i - 1) + \mathcal{O}((\alpha_i - 1)^2), \tag{2.35}$$

and thus $\Delta = 1 - \log 3\sqrt{3}/2$.

3. Lattice modular integrals and constrained Epstein zeta series

In this section, we apply the Rankin–Selberg method to the evaluation of the integral of the lattice partition function[7]

$$\begin{aligned} \varGamma_{(d,d)}(g,B) &= \tau_2^{d/2} \sum q^{\frac{1}{2} p_{L\,i} g^{ij} p_{L\,j}}\, \bar{q}^{\frac{1}{2} p_{R\,i} g^{ij} p_{R\,j}} \\ &= \tau_2^{d/2} \sum_{(m_i,n^i)\in\mathbb{Z}^{2d}} \mathrm{e}^{-\pi\tau_2\,\mathcal{M}^2}\, \mathrm{e}^{2\pi\mathrm{i}\tau_1\, m_i\, n^i} \end{aligned} \tag{3.1}$$

on the fundamental domain. The left-handed and right-handed momenta

$$p_{L(R)\,i} = \frac{1}{\sqrt{2}} \left(m_j \pm (g_{ij} \mp B_{ij})\, n^j \right), \tag{3.2}$$

depend on the geometric data of the compactification torus, i.e., the metric g_{ij} of the T^d and the NS-NS two-form B_{ij}, that together parameterize the symmetric space $\mathrm{O}(d,d)/\mathrm{O}(d)\times\mathrm{O}(d)$, also known as the Narain moduli space. In the last equality of (3.1),

$$\mathcal{M}^2 = (m_i + B_{ik}n^k) g^{ij} (m_j + B_{jl}n^l) + n^i g_{ij} n^j = p_L^2 + p_R^2 \tag{3.3}$$

is the mass-squared of a string ground state with Kaluza–Klein momentum m_i and winding number n^i, with g^{ij} being the inverse metric. The lattice partition function is manifestly invariant under $\mathrm{O}(d,d;\mathbb{Z})$, but also under $\mathrm{SL}(2;\mathbb{Z})_\tau$. This invariance is exposed after Poisson resummation with respect to m_i,

$$\begin{aligned} \varGamma_{(d,d)}(g,B) = \sqrt{\det g} &\sum_{(m^i,n^i)\in\mathbb{Z}^{2d}} \\ &\times \exp\left[-\pi\, \frac{(m^i+n^i\tau)\, g_{ij} (m^j + n^j\bar\tau)}{\tau_2} + 2\pi\mathrm{i}\, B_{ij}\, m^i n^j \right]. \end{aligned} \tag{3.4}$$

The partition function for the Narain lattice clearly belongs to the class of functions considered in Section 2.2 since, in the limit $\tau_2 \to \infty$,

$$\varGamma_{(d,d)}(g,B) \sim \tau_2^{d/2} = \varphi(\tau_2), \tag{3.5}$$

that matches Equation (2.13) with with $\ell = 1$, $\alpha_i = d/2$ and $n_i = 0$. Using Zagier's extension of the Rankin–Selberg method, as summarized in the

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[7]Here and in the following we set $\alpha' = 1$, and we suppress the explicit dependence of the lattice on τ_1 and τ_2.

previous section, we can then compute the renormalized integral

$$\mathcal{R}^\star(\Gamma_{(d,d)};s) = \text{R.N.} \int_{\mathcal{F}} d\mu\, \Gamma_{(d,d)}(g,B)\, E^\star(s,\tau), \tag{3.6}$$

and especially its residue at $s=0$, which is proportional to the IR finite one-loop integral of the lattice partition function,

$$I_d(g,B) = \text{R.N.} \int_{\mathcal{F}} d\mu\, \Gamma_{(d,d)}(g,B). \tag{3.7}$$

Before proceeding with the explicit calculation of the integral (3.6), we notice that the Rankin–Selberg transform $\mathcal{R}^\star(\Gamma_{(d,d)},s)$ is an eigenfunction of the Laplace operator acting on the moduli space $\mathrm{O}(d,d)/\mathrm{O}(d)\times\mathrm{O}(d)$

$$\Delta_{\mathrm{SO}(d,d)}\, \mathcal{R}^\star(\Gamma_{(d,d)},s) = \tfrac14\,(2s-d)(2s+d-2)\, \mathcal{R}^\star(\Gamma_{(d,d)},s), \tag{3.8}$$

with

$$\Delta_{\mathrm{SO}(d,d)} = \tfrac14\, g^{ik}\, g^{jl} \left(\frac{\partial}{\partial g_{ij}}\, \frac{\partial}{\partial g_{kl}} + \frac{\partial}{\partial B_{ij}}\, \frac{\partial}{\partial B_{kl}} \right) + \tfrac12\, g^{ij}\, \frac{\partial}{\partial g_{ij}}. \tag{3.9}$$

This follows straightforwardly from the following differential equations satisfied by the partition function of the Narain lattice [15] and by the Eisenstein series

$$0 = \left[\Delta_{\mathrm{SO}(d,d)} - 2\,\Delta_{\mathrm{SL}(2)} + \tfrac14\, d(d-2)\right]\, \Gamma_{(d,d)}(g,B), \tag{3.10a}$$

$$0 = \left[\Delta_{\mathrm{SL}(2)} - \tfrac12\, s(s-1)\right]\, E^\star(\tau;s), \tag{3.10b}$$

where

$$\Delta_{\mathrm{SL}(2)} = \tfrac12\, \tau_2^2 \left(\frac{\partial^2}{\partial\tau_1^2} + \frac{\partial^2}{\partial\tau_2^2} \right) \tag{3.11}$$

is the Laplace operator on the hyperbolic plane.

3.1. Constrained Epstein zeta series in dimension $d > 2$

For a generic d-dimensional lattice (3.1),

$$h_{\mathcal{T}}(s) = \frac{\mathcal{T}^{s+d/2-1}}{s+\frac12\,d-1}, \quad \hat\varphi(\tau_2) = \begin{cases} \tau_2^{d/2-1}/(\frac12\,d-1) & \text{if } d\neq 2, \\ \log\,\tau_2 & \text{if } d=2, \end{cases} \tag{3.12}$$

and as expected, the integral

$$\int_{\mathcal{F}} d\mu\, \Gamma_{(d,d)}(g,B)\, E^\star(s,\tau) \tag{3.13}$$

is power-divergent, logarithmically divergent and absolutely convergent for $s+\frac{1}{2}\,d-1>0$, $s+\frac{1}{2}\,d-1=0$ and $s+\frac{1}{2}\,d-1<0$, respectively. In all cases, however, the renormalized integral (3.6) is finite and is given by the Mellin transform

$$\begin{aligned} \mathcal{R}^\star(\Gamma_{(d,d)};s) &= \zeta^\star(2s)\int_0^\infty d\tau_2\, \tau_2^{s+d/2-2} \sum_{m_i,n^i}{}' \,\mathrm{e}^{-\pi\tau_2\,\mathcal{M}^2}\,\mathrm{e}^{2\pi\mathrm{i}\tau_1\, m\cdot n} \\ &= \frac{\zeta^\star(2s)\,\Gamma(s+\frac{1}{2}\,d-1)}{\pi^{s+d/2-1}}\,\mathcal{E}_V^d(g,B;s+\tfrac{1}{2}\,d-1). \end{aligned} \tag{3.14}$$

Here, $m\cdot n = m_i\, n^i$, and we have denoted by $\mathcal{E}_V^d(g,B;s)$ the constrained Epstein zeta series in the vectorial representation of $\mathrm{O}(d,d;\mathbb{Z})$, introduced in [15]

$$\mathcal{E}_V^d(g,B;s) \equiv \sum_{m,n}{}' \frac{\delta(m\cdot n)}{\mathcal{M}^{2s}}, \tag{3.15}$$

which converges absolutely for $s>d$ (as usual, a primed sum does not involve the contribution from $m_i=n^i=0$). It is useful to define the *completed constrained Epstein zeta series*

$$\mathcal{E}_V^{d\,\star}(g,B;s) \equiv \pi^{-s}\,\Gamma(s)\,\zeta^\star(2s-d+2)\,\mathcal{E}_V^d(g,B;s), \tag{3.16}$$

so that

$$\mathcal{R}^\star(\Gamma_{(d,d)};s) = \mathcal{E}_V^{d\,\star}(g,B;s+\tfrac{1}{2}\,d-1). \tag{3.17}$$

From Equation (2.10) it follows that $\mathcal{E}_V^{d\,\star}(g,B;s)$ satisfies the functional equation

$$\mathcal{E}_V^{d\star}(g,B;s) = \mathcal{E}_V^{d\,\star}(g,B;d-1-s), \tag{3.18}$$

in agreement with Equation (3.8).

These properties, together with the invariance of $\mathcal{E}_V^{d\star}(g,\,B;\,s)$ under the ring of $\mathrm{O}(d,d)$-invariant differential operators [15], implies that the constrained Epstein zeta series coincides with the degenerate Langlands–Eisenstein series for $\mathrm{O}(d,d)$ based on the parabolic subgroup P with Levi subgroup $\mathbb{R}^+ \times \mathrm{SO}(d-1,d-1)$ [20, 29, 30]. Moreover, from the general statement below (2.25), it follows that, for $d>2$, $\mathcal{E}_V^{d\star}(g,B;s)$ has simple poles at $s=0, d/2-1, d/2$ and 1, as indicated in [29].

Using the functional equation (3.18), we see that (3.14) is equivalent to

$$\text{R.N.} \int_{\mathcal{F}} d\mu\, \Gamma_{(d,d)}(g,B)\, E^\star(s,\tau) = \mathcal{E}_V^{d\star}\left(g,B;\tfrac{1}{2}\,d-s\right). \tag{3.19}$$

For $d>2$ one can easily extract the residue at $s=1$ to get

$$\begin{aligned} I_d &= \frac{\Gamma(d/2-1)}{\pi^{d/2-1}}\; \mathcal{E}_V^{d}\left(g,B;\tfrac{1}{2}\,d-1\right) \\ &= \frac{\pi}{3}\,\frac{\Gamma(d/2)}{\pi^{d/2}}\,\text{Res}\, \mathcal{E}_V^{d}\left(g,B;s+\tfrac{1}{2}\,d-1\right)\Big|_{s=1}, \end{aligned} \tag{3.20}$$

where in writing the last expression we have made use of the functional Equation (3.18).

Note that the first equality in (3.20) establishes Theorem 4 in [15] rigorously. Moreover, comparing with Equation (C.2) in [20] (and dropping the superfluous volume subtraction), we recognize that the constrained Epstein zeta series is in fact equal to

$$\mathcal{E}_V^{d}(g,B;s) = E^{\mathrm{SO}(d,d)}_{[1,0^{d-1}];s}, \tag{3.21}$$

where $E^{\mathrm{SO}(d,d)}_{[1,0^{d-1}];s}$ is the Langlands–Eisenstein series, introduced in [20]. This relation can also be checked by comparing the large volume expansions given in Equation (C.7) in [20] and in Appendix C.1 of [15].

3.2. Low dimension

The cases $d \le 2$ are special, since the integrals are at most logarithmically divergent, and the delta-function constraints in the definitions of the constrained Epstein zeta functions can be explicitly solved.

For $d=1$, the lattice partition function reduces to

$$\Gamma_{(1,1)}(R) = \sqrt{\tau_2}\sum_{m,n} \mathrm{e}^{-\pi\tau_2[(m/R)^2+(nR)^2]}\,\mathrm{e}^{2i\pi\tau_1\, mn}, \tag{3.22}$$

with $\varphi(\tau_2) = \sqrt{\tau_2}$. The modular integral I_1 is finite and coincides with the renormalized one. The constrained Epstein zeta series (3.15) evaluates to

$$\mathcal{E}_V^1(g, B; s) = 2\,\zeta(2s)\,(R^{2s} + R^{-2s}), \tag{3.23}$$

and thus

$$\mathcal{R}^\star(\Gamma_{(1,1)}; s) = \mathcal{E}_V^{1,\star}(g, B; s - \tfrac{1}{2}) = 2\,\zeta^\star(2s)\,\zeta^\star(2s-1)\,\left(R^{1-2s} + R^{2s-1}\right). \tag{3.24}$$

The r.h.s. has simple poles at $s = 0, 1$ and a double pole at $s = 1/2$, in agreement with the general statement below (2.25). The residue at $s = 1$ produces then the standard result for the modular integral

$$I_1 = \int_{\mathcal{F}} d\mu\, \Gamma_{(1,1)}(R) = \frac{\pi}{3}\,\left(R + R^{-1}\right), \tag{3.25}$$

that is invariant under $R \mapsto 1/R$ as required by the $\mathrm{O}(1,1;\mathbb{Z})$ symmetry of the lattice. Clearly, the same result arises by unfolding the sum over m, n in (3.22).

For $d = 2$, the modular integral $\int_{\mathcal{F}_\mathcal{T}} d\mu\, \Gamma_{(2,2)}$ is logarithmically divergent as $\mathcal{T} \to \infty$. The Rankin–Selberg transform is however finite for large $\Re(s)$ and still given by the constrained Epstein zeta series (3.15). Once more, the constraint can be explicitly solved to get the standard expression for the integral. To this end, it is convenient to parameterize the two-torus in terms of the complex structure modulus $U = U_1 + iU_2$ and Kähler modulus $T = T_1 + iT_2$, so that

$$p_L = \frac{m_1 + Um_2 + \bar{T}(n^2 - Un^1)}{\sqrt{2T_2U_2}}, \quad p_R = \frac{m_1 + Um_2 + T(n^2 - Un^1)}{\sqrt{2T_2U_2}}. \tag{3.26}$$

Now, we note that the most general solution of the constraint $m_1 n^1 + m_2 n^2 = 0$ is given by the elements in the disjoint union

$$(m_1, m_2, n^1, n^2) \in \mathcal{S}_1 \cup \mathcal{S}_2, \tag{3.27}$$

with $\mathcal{S}_1, \mathcal{S}_2$ being the sets:

$$\begin{aligned} \mathcal{S}_1 &= \{(m_1, m_2, 0, 0), (m_1, m_2) \in \mathbb{Z}^2\}, \\ \mathcal{S}_2 &= \{(c\tilde{m}_1, c\tilde{m}_2, -d\tilde{m}_2, d\tilde{m}_1), (\tilde{m}_1, \tilde{m}_2) \in \mathbb{Z}^2, \gcd(\tilde{m}_1, \tilde{m}_2) = 1, \\ &\qquad (c, d) \in \mathbb{Z}, d \geq 1\}. \end{aligned} \tag{3.28}$$

The contribution of the first set to $\mathcal{E}_V^{2\star}(T,U;s)$ easily gives

$$\zeta^\star(2s) \sum_{\substack{(m_1,m_2)\in\mathbb{Z}^2\\(m_1,n_1)\neq(0,0)}} \left[\frac{T_2U_2}{|m_1+Um_2|^2}\right]^s = 2T_2^s\,\zeta(2s)\,E^\star(U;s), \tag{3.29}$$

where $E^\star(\tau;s)$ is the $\mathrm{SL}(2,\mathbb{Z})$ invariant Eisenstein series (2.2). For solutions belonging to the second set, the mass-squared factorizes as

$$\mathcal{M}^2 = |p_L|^2 + |p_R|^2 = \frac{|\tilde m_1 + U\tilde m_2|^2}{U_2} \times \frac{|c+Td|^2}{T_2}, \tag{3.30}$$

so that the contribution of $\mathcal{S}_2$ gives

$$\begin{aligned}\zeta^\star(2s) &\sum_{\substack{(c,d)\in\mathbb{Z}^2\\ d\geq 1}} \left[\frac{T_2}{|c+Td|^2}\right]^s \sum_{\substack{(\tilde m_1,\tilde m_2)\in\mathbb{Z}^2\\(\tilde m_1,\tilde m_2)=1}} \left[\frac{U_2}{|\tilde m_1+U\tilde m_2|^2}\right]^s\\ &= 2E^\star(U;s) \sum_{\substack{(c,d)\in\mathbb{Z}^2\\ d\geq 1}} \left[\frac{T_2}{|c+Td|^2}\right]^s.\end{aligned} \tag{3.31}$$

These two results combine into a simple expression for the constrained Epstein zeta series,

$$\mathcal{R}^\star(\Gamma_{(2,2)};s) = \mathcal{E}_V^{2\star}(T,U;s) = 2\,E^\star(T;s)\,E^\star(U;s). \tag{3.32}$$

This relation is actually a consequence of the group isomorphism

$$\mathrm{O}(2,2;\mathbb{Z}) \sim \mathrm{SL}(2;\mathbb{Z})_T \times \mathrm{SL}(2;\mathbb{Z})_U \ltimes \mathbb{Z}_2, \tag{3.33}$$

and of the decomposition of the vectorial representation of $\mathrm{O}(2,2)$ in terms of the bi-fundamental $(2,2)$ of $\mathrm{SL}(2)\times\mathrm{SL}(2)$.

It is clear from (3.32) that $\mathcal{E}_V^{2\star}(T,U,s)$ has a double pole at $s=0$ and $s=1$. Once more, this is in agreement with (2.25), since, upon using $\zeta^\star(s) = 1/(s-1) + \frac{1}{2}(\gamma - \log 4\pi) + \mathcal{O}(s-1)$ and $h(\mathcal{T}) = \mathcal{T}^s/s$, the second line in this equation has a double pole at $s=0,1$,

$$\begin{aligned}&\zeta^\star(2s)\,h_{\mathcal{T}}(s) + \zeta^\star(2s-1)\,h_{\mathcal{T}}(1-s)\\ &\quad= -\frac{1}{2(s-1)^2} + \frac{1}{2(s-1)}\,[\log(4\pi\,\mathcal{T}) - \gamma] + \dots.\end{aligned} \tag{3.34}$$

To compute the renormalized one-loop integral I_2, one then needs to extract the residue at $s = 1$. Using the first Kronecker limit formula (2.5), one finds

$$\mathcal{R}^\star(\mathit{\Gamma}_{(2,2)}; s) = \frac{1}{2(s-1)^2} + \frac{1}{s-1} \times \left[\gamma - \tfrac{1}{2}\log\left(16\pi^2\, T_2\, U_2\, |\eta(T)\,\eta(U)|^4\right)\right] + \dots . \tag{3.35}$$

Combining (3.34), (3.35) with (2.30) and using $\hat{\varphi}(\mathcal{T}) = \log(\mathcal{T})$, one arrives at the following expression for the *renormalized integral* of the two-dimensional Narain lattice partition function

$$I_2 = \text{R.N.} \int_{\mathcal{F}} \mathit{\Gamma}_{(2,2)}(T,U)\, d\mu = -\log\left(4\pi\, \mathrm{e}^{-\gamma}\, T_2\, U_2\, |\eta(T)\,\eta(U)|^4\right). \tag{3.36}$$

We stress that Equation (3.36) is the result of the renormalization prescription (2.29). It differs by a finite constant from the renormalized integral computed in [6] using the renormalization prescription (2.32). Indeed, using (2.34) one arrives at

$$\int_{\mathcal{F}} \left(\mathit{\Gamma}_{(2,2)}(T,U) - \tau_2\right)\, d\mu = -\log\left(\frac{8\pi \mathrm{e}^{1-\gamma}}{3\sqrt{3}}\, T_2\, U_2\, |\eta(T)\,\eta(U)|^4\right), \tag{3.37}$$

in agreement with [6]. In our opinion, the derivation of (3.37) via the RSZ method is considerably simpler than the original derivation in [6].

Finally, we comment on the properties of the Rankin–Selberg transform (3.32) and the renormalized integral (3.36) of the lattice partition function $\mathit{\Gamma}_{(2,2)}$ under the action of T-duality invariant differential operators. For $d = 2$, Equation (3.10a) becomes

$$\left[-2\Delta^{(\tau)}_{\mathrm{SL}(2)} + \Delta^{(T)}_{\mathrm{SL}(2)} + \Delta^{(U)}_{\mathrm{SL}(2)}\right] \mathit{\Gamma}_{(2,2)} = 0, \tag{3.38}$$

where $\Delta^{(T,U)}_{\mathrm{SL}(2)}$ are the analogue of the Laplacian (3.11) acting on the hyperbolic T, U-plane. However, one may check that $\mathit{\Gamma}_{(2,2)}$ satisfies the stronger equations

$$\Delta^{(\tau)}_{\mathrm{SL}(2)}\, \mathit{\Gamma}_{(2,2)} = \Delta^{(T)}_{\mathrm{SL}(2)}\, \mathit{\Gamma}_{(2,2)} = \Delta^{(U)}_{\mathrm{SL}(2)}\, \mathit{\Gamma}_{(2,2)}. \tag{3.39}$$

Combining these equations with (3.10b), we find that $\mathcal{R}^\star(\mathit{\Gamma}_{(2,2)}; s)$ must be an eigenmode of $\Delta^{(T)}_{\mathrm{SL}(2)}$ and $\Delta^{(U)}_{\mathrm{SL}(2)}$ separately, with the same eigenvalue

$\frac{1}{2}s(s-1)$. This is of course manifest from the explicit result (3.32). Moreover, due to the subtraction of the second order pole in (3.35), the renormalized integral is a quasi-harmonic function of the moduli, namely it satisfies

$$\Delta^{(T)}_{\mathrm{SL}(2)}\, I_2 = \Delta^{(U)}_{\mathrm{SL}(2)}\, I_2 = \tfrac{1}{2}. \tag{3.40}$$

This is again manifest from the explicit result (3.36). These considerations will become fruitful when computing integrals with unphysical tachyons in Section 4.4.

3.3. Decompactification

It is useful to ask about the behaviour of the modular integral (3.6) when the radius R of one circle in the d-dimensional torus is sent to infinity. For simplicity, we restrict to the subspace of the Narain moduli space where the lattice partition function factorizes into

$$\Gamma_{(d,d)}(g,B) = \Gamma_{(1,1)}(R)\times \Gamma_{(d-1,d-1)}(\tilde g,\tilde B), \tag{3.41}$$

where $\tilde g$, $\tilde B$ are the metric and NS-NS two-form on the remaining $(d-1)$-dimensional torus. To investigate the limit $R\to\infty$, we consider the modular integral with a hard cut-off, and unfold the lattice sum $\Gamma_{(1,1)}$. Following the same reasoning as in Equations (2.18) to (2.20) one finds, for large enough $\Re(s)$,

$$\begin{aligned}\mathcal{R}^\star_{\mathcal{T}}(\Gamma_{(d,d)};s) = {}& R\int_{\mathcal{F}_{\mathcal{T}}} d\mu\, \Gamma_{(d-1,d-1)}(\tilde g,\tilde B;\tau)\, E^\star(\tau;s)\\ &+\int_0^{\mathcal{T}}\int_{-1/2}^{1/2}\frac{d\tau_1 d\tau_2}{\tau_2^2}\left(\sum_{m\neq 0} R\,\mathrm{e}^{-\frac{\pi R^2 m^2}{\tau_2}}\right)\\ &\times \Gamma_{(d-1,d-1)}(\tilde g,\tilde B;\tau)\, E^\star(\tau;s)\\ &-\int_{\mathcal{T}}^{\infty}\int_{-1/2}^{1/2}\frac{d\tau_1 d\tau_2}{\tau_2^2}\left(\sum_{m\in\mathbb{Z},n\neq 0} R\,\mathrm{e}^{-\frac{\pi R^2|m-n\tau|^2}{\tau_2}}\right)\\ &\times \Gamma_{(d-1,d-1)}(\tilde g,\tilde B;\tau)\, E^\star(\tau;s).\end{aligned} \tag{3.42}$$

In the limit $R\to\infty$, the second and third lines are exponentially suppressed, except for the contribution of the massless sector of $\Gamma_{(d-1,d-1)}$ and the zero-mode part of $E^\star(\tau;s)$ to the second line, which may be replaced by

(3.43)
$$\int_0^{\mathcal{T}}\int_{-1/2}^{1/2} d\mu \left(\sum_{m\neq 0} R\,\mathrm{e}^{-\frac{\pi R^2 m^2}{\tau_2}}\right)\left[\zeta^\star(2s)\,\tau_2^{s+\frac{d-1}{2}}+\zeta^\star(2s-1)\,\tau_2^{1-s+\frac{d-1}{2}}\right].$$

The renormalized integral $\mathcal{R}^\star(\Gamma_{(d,d)};s)$ may therefore be written as

(3.44)
$$\begin{aligned}
&\mathcal{R}^\star(\Gamma_{(d,d)};s)\\
&\quad= R\,\mathcal{R}^\star(\Gamma_{(d-1,d-1)};s)\\
&\qquad+2\,\zeta^\star(2s)\lim_{\mathcal{T}\to\infty}\left[\int_0^{\mathcal{T}}\int_{-1/2}^{1/2} d\mu\,\tau_2^{s+\frac{d-1}{2}}\left(\sum_{m\in\mathbb{Z}} R\,\mathrm{e}^{-\frac{\pi R^2 m^2}{\tau_2}}-\tau_2^{1/2}\right)\right]\\
&\qquad+2\,\zeta^\star(2s-1)\lim_{\mathcal{T}\to\infty}\left[\int_0^{\mathcal{T}}\int_{-1/2}^{1/2} d\mu\,\tau_2^{\frac{d+1}{2}-s}\left(\sum_{m\in\mathbb{Z}} R\,\mathrm{e}^{-\frac{\pi R^2 m^2}{\tau_2}}-\tau_2^{1/2}\right)\right]\\
&\qquad+\dots,
\end{aligned}$$

where the ellipses denote exponentially suppressed corrections in the limit $R\to\infty$. A Poisson resummation over m shows that the term in round brackets is exponentially suppressed as $\tau_2\to\infty$, and the limit $\mathcal{T}\to\infty$ is therefore finite, and given by

(3.45)
$$\begin{aligned}
\mathcal{R}^\star(\Gamma_{(d,d)};s) &= R\,\mathcal{R}^\star(\Gamma_{(d-1,d-1)};s)\\
&\quad+2\,\zeta^\star(2s)\,\zeta^\star(2s+d-2)\,R^{2s+d-2}\\
&\quad+2\,\zeta^\star(2s-1)\,\zeta^\star(2s-d+1)\,R^{d-2s}+\dots.
\end{aligned}$$

up to exponentially suppressed corrections as $R\to\infty$. This formula is manifestly invariant under $s\mapsto 1-s$, and provides the constant term for the constrained Epstein zeta series (3.16) with respect to the parabolic subgroup with Levi component $\mathrm{SO}(d-1,d-1)\times\mathbb{R}^+$ inside $\mathrm{SO}(d,d)$. It is easy to check that it is satisfied for $d=1,2$ using the explicit results in the previous subsection, and that it agrees with Equation (D.16) in [20] for $d=3$. One may also check that (3.44) is consistent with the Laplace Equation (3.8) using Equation (A.30) in [15].

3.4. Another modular invariant regulator

In the context of threshold corrections in four-dimensional heterotic vacua, Kiritsis and Kounnas [18] have proposed a different modular invariant regularization for on shell infrared divergences, based on replacing Minkowski space by a superconformal field theory with the same central charge, but depending on an infrared cut-off Λ. For non-zero value of Λ, the threshold correction is given by a modular integral

$$I_d^{KK}(g,B) = \int_{\mathcal{F}} d\mu\, \Gamma_{(d,d)}(g,B;\tau)\, Z(\Lambda,\tau), \tag{3.46}$$

where $Z(\Lambda,\tau)$ is related to the partition function $\Gamma_{(1,1)}(R,\tau)$ of a compact boson of radius $R = 1/\Lambda$ by

$$Z(\Lambda,\tau) = D \cdot \Gamma_{(1,1)}(1/\Lambda,\tau), \quad [D\cdot f](\Lambda) \equiv 2\,\Lambda^2 \partial_\Lambda\, [f(2\Lambda) - f(\Lambda)]\,. \tag{3.47}$$

The integral is manifestly finite, since the cut-off function $Z(\Lambda,\tau)$ decays exponentially at $\tau_2 \to \infty$. For fixed τ, the integrand in (3.46) agrees with the usual one in the limit $\Lambda \to 0$, since $Z(\Lambda \to 0, \tau) = 1$ up to exponential corrections.

To relate this prescription to ours, we apply the Ranking–Selberg method for functions of rapid decay, and write

$$I_d^{KK}(g,B) = 2\,\mathrm{Res}_{s=1}\, \mathcal{R}^\star(\Gamma_{(d,d)} Z(\Lambda,\tau), s), \tag{3.48}$$

where

$$\mathcal{R}^\star(\Gamma_{(d,d)} Z(\Lambda,\tau), s) = \text{R.N.} \int_{\mathcal{F}} d\mu\, \Gamma_{(d,d)}(g,B;\tau)\, Z(\Lambda,\tau)\, E^\star(s,\tau). \tag{3.49}$$

Since $Z(\Lambda,\tau)$ is of rapid decay, the sign R.N. is superfluous, however introducing it allows us to take the operator D out of the integral and obtain

(3.50)
$$\mathcal{R}^\star(\Gamma_{(d,d)} Z(\Lambda,\tau), s) = D \cdot \left[\text{R.N.} \int_{\mathcal{F}} d\mu\, \Gamma_{(d,d)}(g,B;\tau)\, \Gamma_{(1,1)}(1/\Lambda)\, E^\star(s,\tau)\right].$$

Using the results of the previous subsection (with d, R replaced by $d+1, 1/\Lambda$), we find that in the limit where the infrared cut-off Λ is removed,

(3.51)
$$\mathcal{R}^\star(\Gamma_{(d,d)} Z(\Lambda,\tau), s) \stackrel{\Lambda\to 0}{\longrightarrow} \mathcal{R}^\star(\Gamma_{(d,d)}; s) + 2\,\zeta^\star(2s)\,\zeta^\star(2s+d-1)\,D\cdot\Lambda^{1-d-2s} + 2\,\zeta^\star(2s-1)\,\zeta^\star(2s-d)\,D\cdot\Lambda^{2s-d-1}.$$

Extracting the residue at $s=1$, we find that for $d=0,1$, (3.46) is finite in the limit $\Lambda\to 0$ and reduces to the standard results. For $d>2$, (3.46) agrees with (3.20) after subtracting the order $\mathcal{O}(\Lambda^{2-d})$ divergent term. Finally, for $d=2$ (3.46) diverges logarithmically as $\Lambda\to 0$, and agrees with (3.36) after subtracting $-2\log(2\mathrm{e}\Lambda)$.

4. Modular integrals with unphysical tachyons

In heterotic string vacua with a 'spectator' d-dimensional torus T^d, an interesting class of couplings in the low energy effective action is determined by the one-loop integral of the elliptic genus [2],

$$\int_{\mathcal{F}} \Gamma_{(d+k,d)}(g,B,y)\,\Phi(\tau)\,d\mu, \tag{4.1}$$

where

$$\Gamma_{(d+k,d)}(g,B,y) \;= \tau_2^{d/2} \sum_{p_L,p_R} q^{\frac{1}{2}p_L^2}\,\bar{q}^{\frac{1}{2}p_R^2} \tag{4.2}$$

is the partition function of the Narain lattice, parameterized by the metric g_{ij}, two-form B_{ij} and by Wilson lines y_i^a $(a=1\ldots k)$, and $\Phi(\tau)$ is a weak holomorphic modular form of negative weight $w=-k/2$ with a simple pole at the cusp $q\equiv \mathrm{e}^{2\mathrm{i}\pi\tau}=0$. This singular behaviour is associated with the 'unphysical tachyons' which are ubiquitous in heterotic vacua, i.e., relevant operators in the $(0,2)$-superconformal world-sheet theory which do not respect the level-matching condition. Although $\Phi(\tau)$ grows exponentially as $\tau\to \mathrm{i}\infty$, the integral (4.1) is at most power-like divergent, under the condition that one integrates first on τ_1 and then on τ_2. We refer to integrals of the type (4.1) as 'modular integrals with unphysical tachyons'.

For simplicity, we shall restrict ourselves to points in the Narain moduli space where the Wilson lines y_i^a vanish, so that the lattice partition function factorizes as $\Gamma_{(d+k,d)}(g,B,y) = \Gamma_{(d,d)}(g,B)\,\Gamma_{(k,0)}$ where $\Gamma_{(k,0)}$ is a holomorphic modular form of weight $k/2$. At the cost of absorbing[8] $\Gamma_{(k,0)}$ into Φ, we can therefore assume that $k=0$ and Φ is a weak holomorphic modular function (i.e., a weak holomorphic modular form of weight zero with trivial multiplier system)

$$\Phi(\tau) = \sum_{n\geq -\kappa} a_n\, q^n. \tag{4.3}$$

For physics applications, we are only interested in having a simple pole at $q=0$ (i.e., $\kappa=1$); however, our mathematical construction works equally well for any non-negative integer κ.

For $d=0$, the integrand function is holomorphic and the integral (4.1) is easily computed by representing the integrand as a total derivative and using Stokes' theorem [2, 3]:

$$\int_{\mathcal{F}} d\mu\, \Phi(\tau) = \frac{\pi}{3}\int_{-1/2}^{1/2} d\tau_1\, G_2(\tau)\, \Phi(\tau) = \frac{\pi}{3}\left(a_0 - 24\sum_{-\kappa\leq n<0} a_n \sigma(-n)\right), \tag{4.4}$$

where G_2 is the holomorphic quasi-modular Eisenstein series of weight two, and $\sigma(n)=\sigma_1(n)$ is the sum of divisors of n:

$$G_2(\tau) = 1 - 24\sum_{m=1}^{\infty}\frac{m\, q^m}{1-q^m} = 1 - 24\sum_{n=1}^{\infty}\sigma(n)\, q^n. \tag{4.5}$$

For $d>0$, however, the integrand function is no longer holomorphic, and cannot be written as a total derivative. The standard approach consists instead in unfolding the lattice sum [3, 8], at the cost of obscuring the T-duality symmetry $\mathrm{O}(d,d,\mathbb{Z})$ of the Narain partition function. In this section, we shall develop techniques for computing integrals of the type (4.1) while keeping this symmetry manifest.

Our basic strategy will be to represent the weak holomorphic modular function Φ in (4.1) as a Poincaré series, and to unfold *it* rather than the

[8] Note that, in general, the limit $y_i^a \to 0$ is singular due to the appearance of extra massless gauge bosons. However, in the case at hand these massless states do not contribute to the IR behaviour of the the modular integral, since we subtract their infrared divergences. Singularities do however appear at points of symmetry enhancement consistent with the factorization of the Narain lattice.

lattice sum.[9] While straightforward in principle, the main difficulty in implementing this idea is the fact that the standard Poincaré series representation of weak holomorphic modular functions (see e.g. [31, 32]):

$$\Phi(\tau) = \tfrac{1}{2}\, a_0 + \tfrac{1}{2} \sum_{-\kappa \le n < 0} a_n \lim_{K\to\infty} \times \sum_{|c|\le K} \sum_{|d|<K;(c,d)=1} \mathrm{e}^{2\pi \mathrm{i} n\, \frac{a\tau+b}{c\tau+d}} \left(1 - \mathrm{e}^{\frac{2\pi\mathrm{i}|n|}{c(c\tau+d)}}\right), \tag{4.6}$$

is not absolutely convergent, so the unfolding cannot be justified.

This problem can be circumvented if the weak holomorphic modular form $\Phi(\tau)$ can be obtained as a suitable limit of an absolutely convergent non-holomorphic Poincaré series (or a linear combination thereof),

$$E(s,\kappa) \equiv E(\tau_1,\tau_2;s,\kappa) = \tfrac{1}{2} \sum_{(c,d)=1} \frac{\tau_2^s}{|c\tau+d|^{2s}}\, \mathrm{e}^{-2\pi\mathrm{i}\kappa\, \frac{a\tau+b}{c\tau+d}}. \tag{4.7}$$

Here the integers a and b are a solution of $ad - bc = 1$. The sum over c, d is absolutely convergent for $\Re(s) > 1$, and defines an automorphic form of weight 0. It may be analytically continued to $s = 0$, where the summand of (4.7) becomes holomorphic, and the Poincaré series (4.7) formally defines a weak holomorphic modular function which behaves as $1/q^\kappa + \mathcal{O}(1)$ at $q = 0$. On the other hand, for any modular function $F(\tau_1, \tau_2)$, the integral over the cut-off fundamental domain

$$R_{\mathcal{T}}(F,s,\kappa) \equiv \int_{\mathcal{F}_{\mathcal{T}}} d\mu\, F(\tau_1,\tau_2)\, E(s,\kappa) \tag{4.8}$$

can be computed for $\Re(s) > 1$ by unfolding the sum over c, d, and analytically continuing to $s = 0$ as well. Choosing $F(\tau_1, \tau_2) = \Gamma_{(d,d)}(g, B; \tau)$, and assuming that $\Phi(\tau)$ can be represented as a linear combination

$$\Phi(\tau) = \lim_{s\to 0} \sum_{-\kappa<n<0} a_n\, E(s,-n) \tag{4.9}$$

---

[9] This is similar in spirit to the Rankin–Selberg method, however note that the Poincaré series is no longer an auxiliary function which reduces to a constant (or a pole with constant residue) at a special value of the parameter, but rather represents part of the modular function to be integrated.

of the Poincaré series (4.12) in the limit $s \to 0$, in principle gives a way to compute the modular integral (4.1) of interest, while keeping T-duality manifest. To carry out this programme, we shall compute the Fourier expansion of (4.7), and study its limit as $s \to 0$.

Before doing so however, let us first note that the non-holomorphic Poincaré series (4.7) is not an eigenmode of the hyperbolic Laplacian

$$\Delta = 2\,\tau_2^2\,\partial_{\bar\tau}\,\partial_\tau, \tag{4.10}$$

but, rather, satisfies [33]

$$\left[\Delta + \tfrac{1}{2}\,s(1-s)\right]\,E(s,\kappa) = 2\pi\,\kappa\,s\,E(s+1,\kappa). \tag{4.11}$$

This equation in principle allows one to determine the analytic properties of $E(s,\kappa)$ for $\Re(s) \le 1$ from the knowledge of the spectrum of the Laplacian Δ.

4.1. Fourier expansion of the non-holomorphic Eisenstein series

The Fourier expansion of $E(s,\kappa)$ can be computed by standard methods [33–37]. After extracting the term with $c = 0, d = 1$, and setting $d = d' + nc$ in the remaining sum, a Poisson re-summation over n yields

$$E(s,\kappa) = \tau_2^s\,\mathrm{e}^{-2\pi\mathrm{i}\kappa\tau} + \sum_{n\in\mathbb{Z}} \tilde{E}_n(s,\kappa)\,\mathrm{e}^{2\pi\mathrm{i}n\tau}, \tag{4.12}$$

where the nth Fourier coefficient

$$\tilde{E}_n(s,\kappa) = \tau_2^{1-s}\sum_{c=1}^{\infty}\frac{S(n,-\kappa;c)}{c^{2s}}\,A_n(\tau_2,c;s,\kappa) \tag{4.13}$$

is expressed through the integral

(4.14)
$$A_n(\tau_2,c;s,\kappa) = \int_{-\infty}^{+\infty} dt\,(t^2+1)^{-s}\ \exp\left[2\pi\mathrm{i}\left(\frac{\kappa}{c^2\tau_2(t+\mathrm{i})} - n\,\tau_2\,(t+\mathrm{i})\right)\right]$$

and the Kloosterman sum

$$S(a,b;c) = \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*}\exp\left(\frac{2\pi\mathrm{i}}{c}(a\,d + b\,d^{-1})\right), \tag{4.15}$$

where d^{-1} stands for the inverse of d modulo c (see Appendix A for some relevant properties of the Kloosterman sums). The Kloosterman–Selberg zeta function

$$Z(a,b;s) = \sum_{c>0} \frac{S(a,b;c)}{c^{2s}}, \tag{4.16}$$

which is holomorphic for $\Re(s) > 1$ and admits a meromorphic continuation to all complex values of s, will play a central rôle in what follows.

The integral (4.14) can be computed by Taylor expanding $\exp\left(\frac{2\pi \mathrm{i}\kappa}{c^2\tau_2(t+\mathrm{i})}\right)$, and using the standard formula for the integral

$$\begin{aligned} &\int_{-\infty}^{\infty} (t+\mathrm{i})^{-\alpha}\,(t-\mathrm{i})^{-\beta}\,\mathrm{e}^{-2\pi\mathrm{i}ut}\,dt \\ &= \begin{cases} \dfrac{2^{2-\alpha-\beta}\pi\,(-\mathrm{i})^{\alpha-\beta}\,\Gamma(\alpha+\beta-1)}{\Gamma y(\alpha)\,\Gamma(\beta)} & \text{if } u=0, \\ \dfrac{(2\pi)^{\alpha+\beta}\,(-\mathrm{i})^{\alpha-\beta}\,u^{\alpha+\beta-1}}{\Gamma(\alpha)\,\Gamma(\beta)}\,\mathrm{e}^{-2\pi u}\,\sigma(4\pi u,\alpha,\beta) & \text{if } u>0, \\ \dfrac{(2\pi)^{\alpha+\beta}\,(-\mathrm{i})^{\alpha-\beta}\,(-u)^{\alpha+\beta-1}}{\Gamma(\alpha)\,\Gamma(\beta)}\,\mathrm{e}^{2\pi u}\,\sigma(-4\pi u,\beta,\alpha) & \text{if } u<0, \end{cases} \end{aligned} \tag{4.17}$$

valid for $u \in \mathbb{R}$, $\Re(\alpha+\beta) > 1$. Here, the function $\sigma(\eta,\alpha,\beta)$ is related to the Whittaker function $W_{\mu,\nu}(x)$ by

$$\sigma(\eta,\alpha,\beta) = \Gamma(\beta)\,\eta^{-(\alpha+\beta)/2}\,\mathrm{e}^{\eta/2}\,W_{\frac{\alpha-\beta}{2},\frac{\alpha+\beta-1}{2}}(\eta). \tag{4.18}$$

In this fashion, we find that the constant term is given by

$$\tilde{E}_0(s,\kappa) = \sum_{m=0}^{\infty} \frac{2^{2(1-s)}\,\pi\,(\pi\kappa)^m\,\Gamma(2s+m-1)}{m!\,\Gamma(s+m)\,\Gamma(s)}\,Z(0,-\kappa;s+m)\,\tau_2^{1-s-m}, \tag{4.19}$$

while the positive-frequency Fourier coefficients are given by

$$\tilde{E}_{n>0}(s,\kappa) = \sum_{m=0}^{\infty} \frac{(2\pi)^{2(s+m)}\;n^{2s+m-1}\;\kappa^m}{m!\,\Gamma(s)\,\Gamma(s+m)}\,\sigma\left(4\pi n\tau_2, s+m, s\right)\;Z(n,-\kappa;s)\,\tau_2^{s} \tag{4.20}$$

and the negative-frequency Fourier coefficients (besides the first term in (4.12)) are given by

(4.21)
$$\tilde{E}_{n<0}(s,\kappa) = \sum_{m=0}^{\infty} \frac{(2\pi)^{2(s+m)}\,(-n)^{2s+m-1}\,\kappa^m}{m!\,\Gamma(s)\,\Gamma(s+m)}\;\mathrm{e}^{4\pi n\tau_2}\,\sigma\left(-4\pi n\tau_2, s, s+m\right) \times Z(n,-\kappa;s+m)\,\tau_2^s.$$

Using standard estimates on the Whittaker function and on the Kloosterman–Selberg zeta function, one may check that all these series are absolutely convergent for $\Re(s) > 1$, and therefore $E(s,\kappa)$ is a meromorphic function of s, analytic in the half-plane $\Re(s) > 1$.

It will be useful to rewrite these expressions in a different form. First, using the fact that the Kloosterman–Selberg zeta function $Z(a,b;s)$ at $a = 0$ can be expressed in terms of the Riemann Zeta function via (A.4), the constant term (4.19) may be written as

$$\tilde{E}_0(s,\kappa) = \sum_{m=0}^{\infty} \frac{2^{2(1-s)}\,\pi\,(\pi\kappa)^m\,\Gamma(2s+m-1)\,\sigma_{1-2s-2m}(\kappa)}{m!\,\Gamma(s+m)\,\Gamma(s)\,\zeta(2s+2m)}\,\tau_2^{1-s-m}. \tag{4.22}$$

It may be checked that (4.22) satisfies the differential Equation (4.11) with $E(s,\kappa)$ replaced by $\tilde{E}_0(s,\kappa)$. Second, using the standard series representation of the Whittaker and Bessel functions

(4.23)
$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)},\quad \sigma(\eta,\alpha,\beta) = \eta^{-\beta} \sum_{p=0}^{\infty} \frac{\Gamma(\beta+p)\,\Gamma(1-\alpha+p)}{p!\,\Gamma(1-\alpha)\,(-\eta)^p},$$

one may carry out the sum over m and obtain, for $n > 0$,

$$\tilde{E}_n(s,\kappa) = 2\pi \sum_{p=0}^{\infty} \frac{\Gamma(s+p)}{\Gamma(s)\,p!}\,\frac{2^{-s-p}}{\tau_2^p}\,\left(\frac{n}{\kappa}\right)^{\frac{s-p-1}{2}} \times \sum_{c>0} \frac{S(n,-\kappa;c)}{c^{s+1+p}}\,I_{s-p-1}\left(\frac{4\pi}{c}\sqrt{\kappa n}\right). \tag{4.24}$$

As an example, we see that the Poincaré series $E(s,\kappa)$ can be analytically continued to $s=1/2$, where its Fourier expansion becomes

(4.25)
$$E(\tfrac{1}{2},\kappa) = \sqrt{\tau_2}\,\mathrm{e}^{-2\pi\mathrm{i}\kappa\tau} + \left(2\sqrt{\tau_2}\,\sigma_0(\kappa) + \frac{4\pi\kappa}{\zeta(3)}\frac{\sigma_{-2}(\kappa)}{\sqrt{\tau_2}} + \frac{4\pi^2\kappa^2}{3\zeta(5)}\frac{\sigma_{-4}(\kappa)}{\tau_2^{3/2}} + \cdots\right) + \cdots .$$

For $\kappa=1$, this reproduces Equations (3.45) in [38].

4.2. Holomorphic limit

Let us now study the limit $s\to 0$ of the non-holomorphic Poincaré series $E(s,\kappa)$. At this value, the summand in (4.7) becomes a holomorphic function of τ. Thus, one expects that the analytic continuation $E(0,\kappa)$ will be a holomorphic modular function[10]. In this limit, the constant Fourier coefficient (4.22) reduces to

$$\tilde{E}_0(0,\kappa) = 12\,\sigma(\kappa). \tag{4.26}$$

Moreover, the negative frequency Fourier coefficients (4.21) (besides the first term in (4.12)) vanish in this limit, while the positive frequency Fourier coefficients become

$$\tilde{E}_n(\kappa) = 2\pi\,\sqrt{\frac{\kappa}{n}}\,\sum_{c>0}\frac{S(n,-\kappa;c)}{c}\,I_1\left(\frac{4\pi}{c}\sqrt{\kappa n}\right). \tag{4.27}$$

Thus, the analytic continuation of $E(s,\kappa)$ at $s=0$ is given by

(4.28)
$$E(0,\kappa) = q^{-\kappa} + 12\,\sigma(\kappa) + 2\pi\sum_{n>0}\sqrt{\frac{\kappa}{n}}\,\sum_{c>0}\frac{S(n,-\kappa;c)}{c}\,I_1\left(\frac{4\pi}{c}\sqrt{\kappa n}\right)\,q^n.$$

For $\kappa=1$, this is recognized as the Petersson–Rademacher formula [39, 40] for the Klein modular invariant $j(\tau)=1/q+196884\,q+\cdots$, up to a suitable

[10] For negative modular weight, analytic continuation can lead to holomorphic anomalies but, as we shall see, these do not occur for $w=0$. We thank J. Manschot for discussions on this issue.

additive constant,

$$E(0,1) = j(\tau) + 12. \tag{4.29}$$

For $\kappa > 1$, the r.h.s. of (4.28) is obtained from its value at $\kappa = 1$ by acting with the the Hecke operator

$$(T_\kappa \cdot \Phi)(\tau) = \sum_{\substack{a,d>0 \\ ad=\kappa}} \sum_{b \bmod d} \Phi\left(\frac{a\tau + b}{d}\right), \tag{4.30}$$

which acts in the space of weak holomorphic modular functions and maps $\Phi = 1/q + \mathcal{O}(q)$ to $T_\kappa \cdot \Phi = 1/q^\kappa + \mathcal{O}(q)$. This can be checked using Selberg's identity (A.2) for the Kloosterman sum and the fact that T_κ acts on the Fourier coefficients as

$$(T_\kappa \cdot \tilde{E})_n = \kappa \sum_{d|(n,\kappa)} d^{-1}\, \tilde{E}_{n\kappa/d^2}. \tag{4.31}$$

Thus, the analytic continuation of $E(s,\kappa)$ at $s = 0$ is, in general, the weak holomorphic modular function

$$E(0,\kappa) = T_\kappa \cdot j(\tau) + 12\,\sigma(\kappa). \tag{4.32}$$

Indeed, a numerical evaluation of (4.28) reproduces the known coefficients of the j-function and its images under T_κ.

4.3. Shifted constrained Epstein zeta series

In this subsection we shall evaluate the modular integral (4.8) for an arbitrary modular function $F(\tau_1, \tau_2)$ of moderate growth. In the interest of simplicity, and with a view towards our main goal (4.1), we assume that

$$F_0(\tau_2) = \tau_2^{d/2}, \tag{4.33}$$

and we require that there exists an $a > 0$ such that

$$F_\kappa(\tau_2)\, \mathrm{e}^{2\pi\kappa\tau_2} \sim \mathrm{e}^{-a\tau_2} \tag{4.34}$$

as $\tau_2 \to \infty$. Here $F_\kappa(\tau_2) = \int_{-1/2}^{1/2} F(\tau_1, \tau_2)\, \mathrm{e}^{-2\pi \mathrm{i}\kappa\tau_1}\, d\tau_1$ is the κth Fourier coefficient of F. Note that these conditions are satisfied by the lattice partition function $\Gamma_{(d,d)}$ away from points of extended gauge symmetry.

By the same reasoning as in Equation (2.18) to (2.20), for large enough $\Re(s)$ we can unfold the sum over (c,d) in the non-holomorphic Poincaré series (4.7), arriving at

$$R_{\mathcal{T}}(F,s,\kappa) = \int_0^{\mathcal{T}} d\tau_2\, \tau_2^{s-2} e^{2\pi\kappa\tau_2}\, F_\kappa(\tau_2) - \int_{\mathcal{F}-\mathcal{F}_{\mathcal{T}}} d\mu\, F(\tau_1,\tau_2)\, \left(E(s,\kappa) - \tau_2^s\, e^{-2\pi i\kappa\tau}\right). \tag{4.35}$$

By our assumption (4.33), the term on the first line is finite as $\mathcal{T}\to\infty$. The term on the second line can be decomposed into

$$-\int_{\mathcal{F}-\mathcal{F}_{\mathcal{T}}} d\mu\, \left(F(\tau_1,\tau_2) - \tau_2^{d/2}\right)\, \left(E(s,\kappa) - \tau_2^s\, e^{-2\pi i\kappa\tau}\right) - \sum_{m=0}^{\infty} e_m(s,\kappa) \int_{\mathcal{T}}^{\infty} d\tau_2\, \tau_2^{-1-s-m+\frac{d}{2}}, \tag{4.36}$$

where

$$e_m(s,\kappa) = 2^{2(1-s)}\, \pi\, \frac{(\pi\kappa)^m\, \sigma_{1-2s-2m}(\kappa)\, \Gamma(2s+m-1)}{m!\, \Gamma(s+m)\, \Gamma(s)\, \zeta(2s+2m)}. \tag{4.37}$$

The first term in this expression is exponentially suppressed in the limit $\mathcal{T}\to\infty$, thus, up to exponentially suppressed terms, the modular integral (4.8) is given by

$$R_{\mathcal{T}}(F,s,\kappa) = \int_0^{\infty} d\tau_2\, \tau_2^{s-2} e^{2\pi\kappa\tau_2}\, F_\kappa(\tau_2) + \sum_{m=0}^{\infty} \frac{\mathcal{T}^{\frac{d}{2}-s-m}\, e_m(s,\kappa)}{\frac{d}{2}-s-m} + \cdots. \tag{4.38}$$

For $\Re(s) > d/2$, the second term in (4.38) is suppressed as $\mathcal{T}\to\infty$, and the cut-off integral (4.8) converges to the first term in (4.38) as the cut-off is removed. We define the renormalized integral as the analytic continuation of this result to all s, namely by the Mellin transform

$$\text{R.N.} \int_{\mathcal{F}} F(\tau_1,\tau_2)\, E(s,\kappa)\, d\mu \equiv \int_0^{\infty} d\tau_2\, \tau_2^{s-2} e^{2\pi\kappa\tau_2}\, F_\kappa(\tau_2). \tag{4.39}$$

As in the case considered in Section 2, the analytic structure of (4.39) can be read off from the growth of $e^{2\pi\kappa\tau_2} F_\kappa(\tau_2)$ near $\tau_2\to\infty$. Unlike the case studied in Section 2, the integral (4.39) is not expected to obey a simple functional equation under $s\to 1-s$.

Let us now specialise to the case $F(\tau_1,\tau_2)=\Gamma_{(d,d)}$. The κ-th Fourier coefficient is given by

$$F_\kappa(\tau)=\tau_2^{d/2}\sum_{p_L,p_R} \mathrm{e}^{-\pi\tau_2(p_L^2+p_R^2)}\delta\big(p_L^2-p_R^2-2\kappa\big), \tag{4.40}$$

where the sum is restricted to lattice vectors satisfying the usual level-matching constraint $p_L^2-p_R^2=2\kappa$. For such vectors, $p_L^2+p_R^2=2(p_R^2+\kappa)$, and the condition (4.34) is therefore obeyed away from loci on the symmetric space $\mathrm{O}(d,d,\mathbb{R})/\mathrm{O}(d)\times\mathrm{O}(d)$ where p_R^2 vanishes for some lattice vector satisfying the constraint above. Physically, these loci correspond to points of enhanced gauge symmetry, where additional massless states occur, leading to new infrared divergences. We shall always work away from such points.

Under this assumption, the Mellin transform (4.39) can be computed explicitly by integrating term by term, as in (3.14). The result is a 'shifted constrained Epstein zeta series'

$$\text{R.N.}\int_{\mathcal{F}} d\mu\, \Gamma_{(d,d)}(g,B)\, E(s,\kappa)=\frac{\Gamma(s+\frac{d}{2}-1)}{\pi^{s+\frac{d}{2}-1}}\,\mathcal{E}_V^d(g,B;s+\tfrac12 d-1,\kappa), \tag{4.41}$$

where

$$\mathcal{E}_V^d(g,B;s,\kappa)\equiv\sum_{p_L,p_R}\frac{\delta\big(p_L^2-p_R^2-2\kappa\big)}{(p_L^2+p_R^2-2\,\kappa)^s}. \tag{4.42}$$

This series is absolutely convergent when $\Re(s)>d$. Unlike the 'unshifted' Epstein zeta series (3.15), which we henceforth denote by $\mathcal{E}_V^d(g,B;s,0)$, the shifted series (4.42) diverges as $(2p_R^2)^{-s}$ at the points of enhanced gauge symmetry where the norm p_R^2 of some lattice vector satisfying the constraint vanishes. Moreover, using the differential Equation (3.10a), satisfied by the lattice partition function, together with the differential equation (4.11) satisfied by the non-holomorphic Poincaré series, we find that the shifted constrained Epstein zeta series satisfies

$$\begin{aligned}&\big[\Delta_{SO(d,d)}-s(s-1)+\tfrac14\, d(d-2)\big]\ \mathcal{E}_V^d(g,B;s+\tfrac12\, d-1,\kappa)\\ &\quad=2\pi\,\kappa\, s\,\mathcal{E}_V^d(g,B;s+\tfrac12\, d,\kappa).\end{aligned} \tag{4.43}$$

In the limit $s\to 0$, the r.h.s. vanishes and therefore $\mathcal{E}_V^d(s+\frac{d}{2}-1,\kappa)$ is an eigenmode of the Laplacian on $\mathrm{O}(d,d,\mathbb{R})/\mathrm{O}(d)\times\mathrm{O}(d)$ with eigenvalue $d(2-d)/4$. This argument assumes that $\mathcal{E}_V^d(s+\frac{d}{2},\kappa)$ is finite at $s=0$, otherwise the r.h.s. of (4.43) may be non-vanishing.

4.4. Holomorphic limit of the integral

Let us now investigate the limit of the integral (4.41) at $s = 0$. As discussed below Equation (4.7), the non-holomorphic Poincaré series becomes holomorphic in this limit, and one may hope to recover a modular integral of the form (4.1). Of course, infrared divergences require special care.

Returning to (4.38) at finite infrared cut-off $\mathcal{T}$, we see that all power-like terms vanish in the limit $s \to 0$, apart from those corresponding to $m = 1$ (whose coefficient goes to a constant $e_1(s,\kappa) = 12\sigma(\kappa)$, c.f. (4.29)) and to $m = d/2$ (when d is even), whose coefficient e_m vanishes linearly in s but which are divided by a vanishing number. Considering first the case where d is odd and greater than 3, we thus have

$$\lim_{s\to 0}\left(\int_{\mathcal{F}_\mathcal{T}} d\mu\, E(s,\kappa)\,\Gamma_{(d,d)}\right) = \frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}-1}}\,\mathcal{E}^d_V(g,B;\tfrac{1}{2}\,d-1,\kappa) + 12\sigma(\kappa)\,\frac{\mathcal{T}^{\frac{d}{2}-1}}{\frac{d}{2}-1}. \tag{4.44}$$

On the other hand, in the limit $s \to 0$, $E(s,\kappa)$ reduces to the holomorphic modular forms $T_\kappa \cdot j(\tau) + 12\,\sigma(\kappa)$, cf. (4.32). The constant term in this expression is responsible for the divergent term in (4.44). After subtracting this divergence, we conclude that

$$\text{R.N.}\int_{\mathcal{F}} d\mu\,\left[T_\kappa\cdot j(\tau) + 12\,\sigma(\kappa)\right]\,\Gamma_{(d,d)} = \frac{\Gamma(\frac{d}{2}-1)}{\pi^{d/2-1}}\,\mathcal{E}^d_V(g,B;\tfrac{1}{2}\,d-1,\kappa), \tag{4.45}$$

or, using (3.14),

$$\text{R.N.}\int_{\mathcal{F}} d\mu\, T_\kappa\cdot j(\tau)\,\Gamma_{(d,d)} = \frac{\Gamma(\frac{d}{2}-1)}{\pi^{d/2-1}}\left[\mathcal{E}^d_V(g,B;\tfrac{1}{2}\,d-1,\kappa) - 12\,\sigma(\kappa)\,\mathcal{E}^d_V(g,B;\tfrac{1}{2}\,d-1,0)\right]. \tag{4.46}$$

If d is even and $d \geq 4$, there is an additional constant term on the r.h.s. of (4.44), coming from $m = d/2$:

$$\lim_{s\to 0}\,(\dots) = \frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}-1}}\,\mathcal{E}^d_V(g,B;\tfrac{1}{2}\,d-1,\kappa) + 12\,\sigma(\kappa)\,\frac{\mathcal{T}^{\frac{d}{2}-1}}{\frac{d}{2}-1} + \frac{8\pi(\pi\kappa)^{d/2}\,\sigma_{1-d}(\kappa)}{(d/2)!\,(d-2)\,\zeta(d)}. \tag{4.47}$$

After subtracting the divergent term, we conclude that for $d \geq 4$ even,

$$\begin{aligned}(4.48)\quad \text{R.N.} \int_{\mathcal{F}} d\mu\, T_\kappa \cdot j(\tau) \Gamma_{(d,d)} &= \frac{\Gamma(\frac{d}{2}-1)}{\pi^{d/2-1}} \Big[\mathcal{E}_V^d(g,B;\tfrac{1}{2}\,d-1,\kappa) \, - 12\,\sigma(\kappa)\,\mathcal{E}_V^d(g,B;\tfrac{1}{2}\,d-1,0) \Big] \\ &\quad + \frac{8\pi(\pi\kappa)^{d/2}\,\sigma_{1-d}(\kappa)}{(d/2)!\,(d-2)\,\zeta(d)}.\end{aligned}$$

In the remainder of this subsection we deal with the low dimensional cases.

For $d = 0$, both $m = 0$ and $m = 1$ contribute, leading to

$$(4.49)\qquad \int_{\mathcal{F}_\mathcal{T}} d\mu\ [T_\kappa \cdot j(\tau) + 12\sigma(\kappa)] = -12\sigma(\kappa)\mathcal{T}^{-1} - 4\pi\sigma(\kappa) + \cdots.$$

Using $\int_{\mathcal{F}_\mathcal{T}} d\mu = \frac{\pi}{3} - \frac{1}{\mathcal{T}}$, we arrive at

$$(4.50)\qquad \int_{\mathcal{F}_\mathcal{T}} d\mu\, T_\kappa \cdot j(\tau) = -8\pi\,\sigma(\kappa),$$

which agrees with the result of Stokes' theorem (4.4). Note that this result is independent of the cutoff $\mathcal{T}$, since the only contribution to the modular integral comes from the region $\tau_2 \leq 1$ of $\mathcal{F}$.

For $d = 1$, only $m = 1$ contributes, leading to

$$\begin{aligned}(4.51)\qquad \int_{\mathcal{F}_\mathcal{T}} d\mu\ [T_\kappa \cdot j(\tau) + 12\sigma(\kappa)]\ \Gamma_{(1,1)}(R) \\ = -24\,\sigma(\kappa)\mathcal{T}^{-1/2} - 4\pi \sum_{\substack{m,n>0\\ mn=\kappa}} |m/R - nR|\end{aligned}$$

and therefore

$$(4.52)\qquad \int_{\mathcal{F}} d\mu\, T_\kappa \cdot j(\tau)\, \Gamma_{(1,1)}(R) = -4\pi \sum_{\substack{m,n>0\\ mn=\kappa}} |m/R - nR| - 4\pi\,\sigma(\kappa)(R + 1/R).$$

For $R > \sqrt{\kappa}$, Equation (4.52) reduces to $-8\pi\sigma(\kappa)R$, which would be the result of naively applying the unfolding trick to the partition sum $\Gamma_{(1,1)}(R)$, as in [9]. Similarly, for $R < 1/\sqrt{\kappa}$, Equation (4.52) reduces to $-8\pi\sigma(\kappa)/R$, as required by T-duality. For $1/\sqrt{\kappa} < R < \sqrt{\kappa}$, however, the naive unfolding of $\Gamma_{(1,1)}(R)$ fails, while our method still applies. We note that Equation (4.52) for $\kappa > 1$ can be derived from the $\kappa = 1$ result by observing that the lattice partition function satisfies[11]

$$T_\kappa \cdot \varGamma_{(1,1)}(R;\tau) = \sqrt{\kappa} \sum_{\substack{m,n>0 \\ mn=\kappa}} \varGamma_{(1,1)}(R\sqrt{m/n};\tau), \tag{4.53}$$

and invoking the Hermiticity of the Hecke operator T_κ with respect to the Petersson product. Equation (4.53) may be viewed as a p-adic analogue (for $\kappa = p$ prime) of (3.10a).

Finally, we consider the most complicated case $d = 2$. In this case the coefficient e_1 behaves as

$$e_1 = \alpha + \beta s + \mathcal{O}(s^2), \tag{4.54}$$

with

$$\alpha = 12\sigma(\kappa), \qquad \beta = -24\,\sigma(\kappa)\,[\gamma - 12\log\,A + \log(4\pi)] - 24\,\kappa\sigma'_{-1}(\kappa), \tag{4.55}$$

where A is the Glaisher constant. After integrating over τ_2, we arrive at

$$\int_{\mathcal{F}_\mathcal{T}} d\mu\, E(s,\kappa)\,\varGamma_{(2,2)} = \frac{\varGamma(s)}{\pi^s}\,\mathcal{E}^2_V(g,B;s,\kappa) - \frac{\alpha}{s} + \alpha\log\mathcal{T} - \beta + \mathcal{O}(s). \tag{4.56}$$

The pole and logarithmic divergences on the r.h.s. originate from the constant term $12\,\sigma(\kappa)\,\tau_2^s$ in the limit $s \to 0$ of $E(s,\kappa)$. After subtracting these terms, we conclude that

$$\int_{\mathcal{F}} d\mu\, T_\kappa \cdot j(\tau)\,\varGamma_{(2,2)} = \lim_{s\to 0}\left[\frac{\varGamma(s)}{\pi^s}\,\mathcal{E}^2_V(g,B;s,\kappa)\right] - \beta - 12\,\sigma(\kappa)\,I_2. \tag{4.57}$$

In particular, we conclude the Epstein zeta series $\mathcal{E}^2_V(g,B;s,\kappa)$ has a zero at $s = 0$.

On the other hand, the integral (4.57) may be determined using harmonicity and the singularity structure near points of enhanced symmetry. For example, for $\kappa = 1$, states with momenta $m_1 = n_1 = 0$, $m_2 = -n_2 = \pm 1$

[11] This identity is easily established for κ a prime number, and can be extended to the general case using the Hecke algebra $T_\kappa\,T_{\kappa'} = \sum_{d|(\kappa,\kappa')} d\,T_{\kappa\kappa'/d^2}$ (note the non-standard normalization of the Hecke operators in (4.30)).

become massless[12] at $T=U$, and induce a singularity of the form

$$-4\log|T-U|. \tag{4.58}$$

More generally, massless states arise whenever $j(T)=j(U)$, as required by invariance under the T-duality group $\mathrm{SL}(2,\mathbb{Z})_T\times\mathrm{SL}(2,\mathbb{Z})_U\ltimes\mathbb{Z}_2$. Combining this singularity structure with harmonicity, one can uniquely determine

$$\int_{\mathcal{F}} d\mu\; j(\tau)\,\Gamma_{(2,2)}(T,U) = -\log|j(T)-j(U)|^4 + \text{const}. \tag{4.59}$$

This result was confirmed in [8] by using the unfolding trick and of Borcherds' product formula for $j(T)-j(U)$.

Similarly to the one-dimensional case, the integral (4.57) can be evaluated for $\kappa>1$ by observing that the lattice partition function satisfies the p-adic analogue of (3.39)

$$T_\kappa^{(\tau)}\,\Gamma_{(2,2)} = T_\kappa^{(T)}\,\Gamma_{(2,2)} = T_\kappa^{(U)}\,\Gamma_{(2,2)}, \tag{4.60}$$

where $T_\kappa^{(T,U)}$ are the analogues of the Hecke operator (4.30), now acting on the moduli T and U. Invoking the Hermiticity of T_κ with respect to the Petersson product, one finds that (4.59) generalizes into

$$\int_{\mathcal{F}} d\mu\, T_\kappa\cdot j(\tau)\,\Gamma_{(2,2)}(T,U) = -\tfrac{1}{2}(T_\kappa^{(T)}+T_\kappa^{(U)})\log\,|j(T)-j(U)|^4 + \text{const}. \tag{4.61}$$

This result is consistent with the fact that additional massless states arise at $T=\kappa\,U$, together with the images of this locus under T-duality.

Comparing (4.61) with (4.57), we arrive at interesting identities between the constrained Epstein zeta series $\mathcal{E}_V^2(g,B;s,\kappa)$ at $s=0$ and the j function. It would be interesting to compute the constrained Epstein zeta series directly, perhaps along the lines of Section 3.2, and derive these identities independently.

More generally, it will be interesting to extend our method to modular integrals of non-symmetric lattice partition functions $\Gamma_{d+k,d}$. While the generalization of the non-holomorphic Poincaré series (4.7) to negative weight is obvious, its analytic continuation to the relevant value of s where the summand becomes holomorphic is subtle, and involves an interesting

[12] Note that four additional massless states appear at the special values $T=U=\mathrm{i}$ and $T=U=\rho$, where $\rho=\mathrm{e}^{\mathrm{i}\pi/3}$.

interplay of holomorphic and modular anomalies [37]. We hope to discuss this problem in future work[13].

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Appendix A. Properties of Kloosterman sums

Kloosterman sums play an central rôle in Number Theory. The classical Kloosterman sums for the modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ are defined by

$$S(a,b;c) = \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} \exp\left[\frac{2\pi\mathrm{i}}{c}(a\,d + b\,d^{-1})\right], \tag{A.1}$$

where a, b and c are integers, and d^{-1} is the inverse of d mod c, and enter in the explicit expression of the Fourier coefficients of modular forms. $S(a,b;c)$ is clearly symmetric under the exchange of a and b. Less evidently, it satisfies the Selberg identity

$$S(a,b;c) = \sum_{d|\gcd(a,b,c)} d\,S(ab/d^2, 1; c/d). \tag{A.2}$$

[13]Note added in proof: In a subsequent work [41], we proposed a method for evaluating modular integrals of non-symmetric lattice partition functions based on a different type of Poincaré series which does not require analytic continuation.

In the special case $a \neq 0$, $b = 0$, the Kloosterman sum reduces to the Ramanujan sum

$$\text{(A.3)} \quad S(a,0;c) = S(0,a;c) = \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} \exp\left(\frac{2\pi \mathrm{i}}{c} a\, d\right) = \sum_{d | \gcd(c,a)} d\, \mu(c/d),$$

with $\mu(n)$ the Möbius function. For $a = b = 0$, $S(a,b;c)$ reduces instead to the Euler totient function $\phi(c)$.

We now turn to the Kloosterman-Selberg zeta function (4.16). Using the trivial bound $|S(a,b;c)| < c$, one sees immediately that the sum over c converges absolutely when $\Re(s) > 1$. The Weil bound $|S(a,b;c)| < 2^{\nu(c)} \sqrt{c\, gcd(a,b,c)}$, where $\nu(n)$ is the number of divisors of n, shows that $Z(a,b;s)$ is in fact analytic when $\Re(s) > 3/4$. When one or both of the arguments vanish, it can be expressed in terms of the Riemann zeta function via

$$\text{(A.4)} \qquad Z(0,0;s) = \frac{\zeta(2s-1)}{\zeta(2s)}, \quad Z(0,\pm\kappa;s) = \frac{\sigma_{1-2s}(\kappa)}{\zeta(2s)} \quad (\kappa \neq 0),$$

where $\sigma_s(n)$ is the divisor function

$$\text{(A.5)} \qquad \sigma_s(n) = \sum_{d|n} d^s.$$

The usual notation $\sigma(n) \equiv \sigma_1(n)$ is used throughout the paper.

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