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Some remarks on K-lattices and the Adelic Heisenberg group for CM curves

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We define an adelic version of a CM elliptic curve E which is equipped with an action of the profinite completion of the endomorphism ring of E. The adelic elliptic curve so obtained is provided with a natural embedding into the adelic Heisenberg group. We embed into the adelic Heisenberg group the set of commensurability classes of arithmetic 1-dimensional $\mathbb{K}$-lattices (here and subsequently, $\mathbb{K}$ denotes a quadratic imaginary number field) and define theta functions on it. We also embed the groupoid of commensurability modulo dilations into the union of adelic Heisenberg groups relative to a set of representatives of elliptic curves with R-multiplication (R is the ring of algebraic integers of $\mathbb{K}$). We thus get adelic theta functions on the set of 1-dimensional $\mathbb{K}$-lattices and on the groupoid of commensurability modulo dilations. Adelic theta functions turn out to be acted by the adelic Heisenberg group and behave nicely under complex automorphisms (Theorems 6.12 and 6.14).

1. Introduction

The aim of this paper is to connect the natural action of the Heisenberg group on adelic theta functions with the adelic action stemming from the main theorem of complex multiplication for elliptic curves. We are also interested in defining an embedding of the moduli spaces of arithmetic 1-dimensional $\mathbb{K}$-lattices (here and subsequently, $\mathbb{K}$ denotes a quadratic imaginary number field) into the adelic Heisenberg group, in order to define on them theta functions with a nice behavior under complex automorphisms (Theorems 6.12 and 6.14).

After the seminal paper [2], many efforts have been devoted in recent years to the construction of quantum systems incorporating explicit class field theory for an imaginary quadratic number field $\mathbb{K}$ ([6], [10], [8], [4], [7]).

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More specifically, in [6] it is exhibited a quantum statistical mechanical system fully incorporating the explicit class field theory. The main ingredients of [6] was given in terms of commensurability of 1-dimensional $\mathbb{K}$-lattices. The connection between class field theory and quantum statistical mechanics is provided by a C^*-dynamical system containing an *arithmetic subalgebra* $\mathcal{A}_{\mathbb{Q}}$ with symmetry group isomorphic to $Gal(\mathbb{K}^{\mathrm{ab}}, \mathbb{K})$ and a set of *fabulous states* sending $\mathcal{A}_{\mathbb{Q}}$ to $\mathbb{K}^{\mathrm{ab}}$. The symmetry group action turns out to be compatible with Galois' one. The arithmetic subalgebra is defined by means of the *modular field*, namely by the field of modular function defined over $\mathbb{Q}^{\mathrm{ab}}$.

This work was intended as an attempt to define canonical *adelic theta functions* on commensurability classes of (arithmetic) 1-dimensional $\mathbb{K}$-lattices and on the groupoid of commensurability modulo dilations. The main ingredient of our construction is provided by the *adelic Heisenberg group* ([15]). Adelic theta functions have a nice behavior under the Galois action which incorporates all the properties stated in the *Main Theorem of Complex Multiplication* (see §§ 5 and 6).

Motivated by a purely algebraic definition of adelic theta function over an abelian variety and their deformations, David Mumford introduced, in a celebrated series of papers of the sixties ([14]), the *finite Heisenberg group* acting on the sections of an ample line bundle defined on the abelian variety. This led him to an adelic version of any abelian variety, defined as an extension of the set of its torsion points by the Barsotti-Tate group ([15], Ch. 3). It turned out that sections of the pull back of some line bundle on the *tower of isogenies* ([15], Definition 4.26), the so called *adelic theta functions*, are acted on by an adelic version of the Heisenberg group ([15], Ch.4).

In this work we apply Mumford's constructions to elliptic curves with complex multiplication (CM curves for short) and compare it with the moduli spaces of 1-dimensional $\mathbb{K}$-lattices introduced in [6]. We define an adelic version of a CM elliptic curve E which is equipped with an action of the profinite completion of the endomorphism ring of E. This is also provided with a natural embedding into the adelic Heisenberg group. This allows us to incorporate the endomorphism ring of the CM curve into the definitions of Heisenberg group and theta functions. We thus get an interpretation by means of Class Field Theory (Theorems 6.9, 6.11, 6.12 and 6.14) of the usual nice behavior of theta functions under automorphisms fixing the Hilbert field of $\mathbb{K}$ ([15], Proposition 5.6). One of our main results is an extension of the *Main Theorem of Complex Multiplication* ([18] II, Theorem 8.2) involving the Heisenberg group, which allows us to give a complete description of the behavior of theta functions under complex automorphisms (Theorem 6.9, Theorem 6.11).

Aimed at defining theta functions on them, we embed the set of commensurability classes of arithmetic K-lattices into the adelic Heisenberg group (Notations 3.16). We also embed the groupoid of commensurability modulo dilations into the union of finitely many adelic Heisenberg groups (indeed into the union of adelic Heisenberg groups corresponding to a set of representatives of elliptic curves with R-multiplication, modulo isomorphisms). We obtain theta functions on the set of 1-dimensional $\mathbb{K}$-lattices and on the groupoid of commensurability modulo dilations which are equipped with an action of the Heisenberg group and exhibiting a nice behavior under complex automorphisms (Theorems 6.12 and 6.14, Notations 6.13).

The paper is organized as follows. In Section 2 we collect some basic facts about *adèles* and *adelic elliptic curves* that will be needed in the following. In Section 3 we recall the spaces of 1-dimensional $\mathbb{K}$-lattices introduced in [6] and compare them with adelic CM curves in order to obtain natural morphisms into the Heisenberg group. Furthermore, we embed into the adelic Heisenberg group the set of commensurability classes of arithmetic 1-dimensional $\mathbb{K}$-lattices, and we also embed the groupoid of commensurability modulo dilations into the union of adelic Heisenberg groups relative to a set of representatives of elliptic curves with R-multiplication (Notations 3.16). In Section 4 we introduce the Adelic Heisenberg group of a CM curve and embed the adelic CM curve into it by means of a symmetric line bundle. We also describe the action of the complex automorphisms fixing the Hilbert field of $\mathbb{K}$ on the Heisenberg group. In Section 5 we state and prove a version of the *Main Theorem of Complex Multiplication* ([18] II, Theorem 8.2) concerning Heisenberg groups (Theorems 5.3 and 5.6). In a nutshell, what such theorems say is that if two different CM curves are mapped into each other by a complex automorphism then the embeddings into their Heisenberg groups can be made coherent with such a map. Finally, in Section 6 we introduce theta functions and study their behavior under complex automorphisms.

2. Notations

In this section we collect some basic facts about *adèles* and *adelic elliptic curves*, mainly to fix our notations.

2.1. Completions

Here and subsequently,, $\mathbb{K}$ denotes a quadratic imaginary number field and R denotes the ring of algebraic integers of $\mathbb{K}$. We denote by $\mathcal{I} = \mathcal{I}_R$ the set of (integral) ideals of R, by $\mathcal{J} = \mathcal{J}_R$ the group of fractional ideals of R, freely generated by the primes ([13], p. 91), and by $Cl(R)$ the *ideal class group* of R.

As usual

$$\hat{R} = \varprojlim_{I \in \mathcal{I}} \frac{R}{I} \subset \mathbb{A}$$

will be the completion of R, $\hat{R}^*$ the group of invertible elements of $\hat{R}$ and $\mathbb{A} = \mathbb{A}_{\mathbb{K},f}$ the ring of finite adèles of $\mathbb{K}$. Recall that

$$I^{-1} = \{x \in \mathbb{K} \mid xI \subset R\} \in \mathcal{J}$$

is a fractional ideal s.t. $I \cdot I^{-1} = R$. If $\Lambda \in \mathcal{J}$ is a fractional ideal of $\mathbb{K}$ then $\frac{I^{-1}\Lambda}{\Lambda}$ can be identified with a submodule of $\frac{\mathbb{K}}{\Lambda}$:

$$\frac{I^{-1}\Lambda}{\Lambda} = \left\{ x \in \frac{\mathbb{K}}{\Lambda} \simeq \frac{\mathbb{A}}{\hat{\Lambda}} \;\middle|\; xI = 0 \right\}, \quad \hat{\Lambda} := \Lambda \cdot \hat{R} \subset \mathbb{A}.$$

Remark 2.1. It is well known (see e.g. [18], II Proposition 1.4) that $\frac{I^{-1}\Lambda}{\Lambda}$ is a free $\frac{R}{I}$-module of rank 1. It is standard to deduce from this fact that, even though Λ is a *projective but usually non-free* R-module, the completion

$$\hat{\Lambda} \simeq \varprojlim_{I \in \mathcal{I}} \frac{I^{-1}\Lambda}{\Lambda} \tag{2.1}$$

is a free $\hat{R}$-module of rank 1. We denote by $\hat{\Lambda}^* \subset \hat{\Lambda}$ the set of $\hat{R}$*-module generators* of $\hat{\Lambda}$ which is obviously an $\hat{R}^*$*-torsor*.

Essentially by the same reason as above, for any pair of fractional ideals $\Lambda, \Gamma \in \mathcal{J}$ one also gets

$$\begin{aligned} \mathrm{Hom}_R\left(\frac{\mathbb{K}}{\Lambda}, \frac{\mathbb{K}}{\Gamma}\right) &= \varprojlim_{I \in \mathcal{I}} \mathrm{Hom}_{\frac{R}{I}}\left(\frac{I^{-1}\Lambda}{\Lambda}, \frac{I^{-1}\Gamma}{\Gamma}\right) \\ &\simeq \varprojlim_{I \in \mathcal{I}} \mathrm{Hom}_{\frac{R}{I}}\left(\frac{R}{I}, \frac{R}{I}\right) = \hat{R}, \end{aligned} \tag{2.2}$$

where the isomorphism can be made explicit through the choice of any pair of $\hat{R}$-module generators: $\lambda \in \hat{\Lambda}^*$, $\gamma \in \hat{\Gamma}^*$.

2.2. Adelic elliptic curves with complex multiplication

As above let $\mathbb{K}$ be a quadratic imaginary number field and let $\Lambda \subset \mathbb{K}$ be a fractional ideal of $\mathbb{K}$. Then $E = E_\Lambda := \frac{\mathbb{C}}{\Lambda}$ is an elliptic curve with $\mathrm{End}(E) =$

R ([18], p. 99). Following [18] p. 102, for $a \in R$ we denote by $E[a]$ the (a)-*torsion points* of E:

$$E[a] := \{x \in E \mid \; ax = 0\} \simeq \frac{(a)^{-1}\Lambda}{\Lambda}$$

(compare with [18], Proposition 1.4). It is a standard fact that the *Barsotti-Tate module*

$$T(E) := \varprojlim E[a]$$

is a free $\hat{R}$-module of rank 1. Similarly, we denote by $T(E)^*$ the $\hat{R}^*$-torsor of $\hat{R}$-module generators of $T(E)$.

Now we imitate [15] Definition 4.1 to define an *adelic version* of E taking into account its complex multiplication structure.

Definition 2.2. We define the *adelic elliptic curve* associated to E as

$$V(E) := \left\{ (x_a)_{a \in R} \;\middle|\; \frac{a}{b} x_a = x_b, \text{ if } b \mid a, \; x_1 \in E_{tor} \right\}.$$

This is equipped with projections

$$\nu_b : V(E) \to E_{tor}, \quad (x_a)_{a \in R} \to x_b.$$

By [15], pp. 48-49, we have

$$V(E) \simeq T(E) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{2.3}$$

and

$$0 \longrightarrow T(E) \longrightarrow V(E) \xrightarrow{\nu_1} E_{tor} \longrightarrow 0. \tag{2.4}$$

Remark 2.3.

1) If we fix an $\hat{R}$-module generator $u \in T(E)^*$ then (2.2) implies that any morphism $\phi \in \mathrm{Hom}_R\left(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\Lambda}\right)$ can be represented by an element $\psi \in \hat{R}$. Then the multiplication by ψ gives rise by (2.3) and (2.4) to a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{R} & \longrightarrow & \mathbb{A} & \longrightarrow & \frac{\mathbb{K}}{R} & \longrightarrow & 0 \\
 & & \downarrow \psi & & \downarrow \psi & & \downarrow \phi & & \\
0 & \longrightarrow & T(E) & \longrightarrow & V(E) & \longrightarrow & \frac{\mathbb{K}}{\Lambda} & \longrightarrow & 0
\end{array}$$

By the snake lemma ϕ is an isomorphism iff ψ belongs to $T(E)^*$.

2) In particular, any choice of a $\hat{R}$-module generator $u \in T(E)^*$ gives rise to a commutative diagram of $\hat{R}$-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{R} & \longrightarrow & \mathbb{A} & \longrightarrow & \frac{\mathbb{K}}{R} & \longrightarrow & 0 \\ & & \updownarrow & & \updownarrow & & \updownarrow & & \\ 0 & \longrightarrow & T(E) & \longrightarrow & V(E) & \xrightarrow{\nu_1} & E_{tor} & \longrightarrow & 0 \end{array}$$

3. The moduli space of $\mathbb{K}$-lattices.

Let us recall the following (see [6], Def. 4.1):

Definition 3.1.

1) A 1-dimensional $\mathbb{K}$ lattice (Λ, ϕ) is a finitely generated R-submodule $\Lambda \subset \mathbb{C}$ s.t. $\Lambda \otimes_R \mathbb{K} \simeq \mathbb{K}$, together with an R-morphism $\phi : \frac{\mathbb{K}}{R} \to \frac{\mathbb{K}\Lambda}{\Lambda}$.
2) An invertible 1-dimensional $\mathbb{K}$ lattice (Λ, ϕ) is a finitely generated R-submodule $\Lambda \subset \mathbb{C}$ s.t. $\Lambda \otimes_R \mathbb{K} \simeq \mathbb{K}$, together with an R-isomorphism $\phi : \frac{\mathbb{K}}{R} \to \frac{\mathbb{K}\Lambda}{\Lambda}$.

Since for any finitely generated R-submodule $\Lambda \subset \mathbb{C}$ there exists $k \in \mathbb{C}$ s.t. $k\Lambda \subset \mathbb{K}$ ([5], Lemma 3.111), we also give the following:

Definition 3.2. An arithmetic (invertible) 1-dimensional $\mathbb{K}$ lattice (Λ, ϕ) is a finitely generated R-submodule $\Lambda \subset \mathbb{K}$ s.t. $\Lambda \otimes_R \mathbb{K} \simeq \mathbb{K}$, together with a R-morphism (isomorphism) $\phi : \frac{\mathbb{K}}{R} \to \frac{\mathbb{K}}{\Lambda}$.

The following result is an immediate consequence of Remark 2.3.

Theorem 3.3. *Any morphism $\phi \in \mathrm{Hom}_R\big(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\Lambda}\big)$ gives rise to map of short exact sequences:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{R} & \longrightarrow & \mathbb{A} & \longrightarrow & \frac{\mathbb{K}}{R} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T(E) & \longrightarrow & V(E) & \longrightarrow & \frac{\mathbb{K}}{\Lambda} & \longrightarrow & 0 \end{array}$$

where $E := \frac{\mathbb{C}}{\Lambda}$. The corresponding $\mathbb{K}$-lattice is invertible iff the vertical maps are isomorphisms. In particular any $\phi \in \mathrm{Hom}_R\big(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\Lambda}\big)$ corresponds to a point *of* $x_\phi \in T(E)$ *(the image of* $1 \in \hat{R}$*) and the corresponding* $\mathbb{K}$*-lattice is invertible iff* x_ϕ *belongs to* $T(E)^*$.

Following [18], pag 98, we denote by $\mathcal{ELL}(R)$ the moduli space of elliptic curves with $\mathrm{End}(E) \simeq R$. Then ([18], II Proposition 1.2) $\mathcal{ELL}(R)$ is a $Cl(R)$-torsor.

Definition 3.4.

1) Let $\rho \in \mathbb{A}^*$ be an *idèle*. Recall that the *ideal of* ρ ([18] p. 119) is the fractional ideal
$$(\rho) := \prod_{\mathfrak{p}} \mathfrak{p}^{ord_{\mathfrak{p}} \rho_{\mathfrak{p}}} \in \mathcal{J}$$
(where the product is over all the primes of R) and the *multiplication by* ρ map is defined as
$$\rho : \mathcal{J} \longrightarrow \mathcal{J}, \quad \Lambda \longrightarrow (\rho)\Lambda.$$
By an abuse of notations, we use the same symbol for the corresponding map on $Cl(R)$
$$\rho : Cl(R) \longrightarrow Cl(R), \quad [\Lambda] \longrightarrow \rho[\Lambda] := [(\rho)\Lambda],$$
and on $\mathcal{ELL}(R)$
$$\rho : \mathcal{ELL}(R) \longrightarrow \mathcal{ELL}(R), \quad E_{[\Lambda]} \longrightarrow E_{\rho[\Lambda]}.$$

2) We denote by $\mathcal{A}$ ($\mathcal{A}^*$) the set of (invertible) arithmetic $\mathbb{K}$-lattices and by $\mathcal{L}$ ($\mathcal{L}^*$) the set of (invertible) $\mathbb{K}$-lattices modulo dilations. Fix an idèle $\rho \in \mathbb{A}^*$ and a fractional ideal $\Lambda \in \mathcal{J}$. The *multiplication by* ρ *map on* $\frac{\mathbb{K}}{R}$ is defined by the commutativity of the following diagram (compare with [18] p. 159)
$$\begin{array}{ccc} \frac{\mathbb{K}}{R} & \xrightarrow{\quad \rho \quad} & \frac{\mathbb{K}}{\rho R} \\ \downarrow & & \downarrow \\ \bigoplus_{\mathfrak{p}} \frac{\mathbb{K}_{\mathfrak{p}}}{R_{\mathfrak{p}}} & \xrightarrow{\oplus_{\mathfrak{p}} \rho_{\mathfrak{p}} \cdot} & \bigoplus_{\mathfrak{p}} \frac{\mathbb{K}_{\mathfrak{p}}}{\rho_{\mathfrak{p}} R_{\mathfrak{p}}} \end{array}$$
and it can be viewed as an element of $\mathrm{Hom}_R\left(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\rho R}\right) \simeq \hat{R}$ (compare with (2.2)), hence as a lattice $\Phi_\rho \in \mathcal{A}$ (we also define $\Psi_\rho := [\phi_\rho] \in \mathcal{L}$).

Obviously such a lattice is invertible as the inverse of $\rho : \frac{\mathbb{K}}{R} \to \frac{\mathbb{K}}{\rho R}$ is provided by $\rho^{-1} \in \mathbb{A}^*$, thus Φ_ρ belongs to $\mathcal{A}^*$ ($\Psi_\rho \in \mathcal{L}^*$).

Theorem 3.5.

1) *The map* Φ *just defined*

$$\Phi : \mathbb{A}^* \longrightarrow \mathcal{A}^*, \quad \rho \longrightarrow \Phi_\rho$$

is bijective.

2) $\mathbb{A}^* \simeq \hat{R} \times_{\hat{R}^*} \mathcal{A}^* \simeq \hat{R} \times_{\hat{R}^*} \mathbb{A}^*$.

Proof. (1) Surjectivity: the multiplication map

$$\rho : \mathbb{A}^* \longrightarrow \mathcal{J}, \quad \rho \longrightarrow (\rho)R$$

is obviously surjective. Furthermore, any two invertible lattices in $\mathrm{Hom}_R\left(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\rho R}\right)$ differ (by Theorem 3.3) by an element of $T(E)^*$, which is an $\hat{R}^*$-torsor (compare with Section 2.2).

Injectivity: if $\Phi_\rho = \Phi_{\rho'}$ then $\rho R = \rho' R := \Lambda$ and $\frac{\rho}{\rho'} = id \in \mathrm{Hom}_R\left(\frac{\mathbb{K}}{\Lambda}, \frac{\mathbb{K}}{\Lambda}\right)$, hence $\frac{\rho}{\rho'} = 1 \in \mathbb{A}^*$ because of Theorem 3.3.

(2) Just combine (1) with Proposition 4.6 of [6]. □

Taking quotients by $\mathbb{K}^*$ in the theorem above we also have

Corollary 3.6.

1) *The map*

$$\psi : \mathbb{A}^*/\mathbb{K}^* \longrightarrow \mathcal{L}^*, \quad [\rho] \longrightarrow \psi_\rho,$$

defined on the idèle class group $\mathbb{A}^*/\mathbb{K}^*$ *([11] p. 142), is bijective and the projection* $\mathcal{L}^* \to Cl(R)$ *coincides with the usual Class Field map ([11], p. 224).*

2) $\mathcal{L} \simeq \hat{R} \times_{\hat{R}^*} \mathcal{L}^* \simeq \hat{R} \times_{\hat{R}^*} (\mathbb{A}^*/\mathbb{K}^*)$.

Lemma 3.7. *Consider an arithmetic 1-dimensional* $\mathbb{K}$ *lattice* (Λ, ϕ)*. Any 1-dimensional* $\mathbb{K}$ *lattice which is commensurable to* (Λ, ϕ) *is arithmetic.*

Proof. Assume that (Λ', ψ) is commensurable to (Λ, ϕ). Then both the natural projections $\Lambda \to \Lambda \cap \Lambda'$, $\Lambda' \to \Lambda \cap \Lambda'$ have finite index hence the same happens for $\Lambda + \Lambda' \to \Lambda \cap \Lambda'$. So there exists an integer n s.t. $n(\Lambda + \Lambda') \subset \Lambda \cap \Lambda'$. The thesis follows since $n\Lambda' \subset n(\Lambda + \Lambda') \subset \Lambda \cap \Lambda' \subset \mathbb{K}$. □

Notations 3.8. We denote by S the set of *non-archimedean places of* $\mathbb{K}$ ([3], p. 189). For any $v \in S$ we denote by $\mathfrak{p}_v \subset R$ the prime ideal corresponding to v and we choose a *prime element* $\pi_v \in \mathfrak{p}_v$ ([3], p. 42).

Lemma 3.9. *Consider two fractional ideals* Λ, Λ', *assume* $\Lambda \subset \Lambda'$ *and set* $\Lambda = \prod_{v\in S} \mathfrak{p}_v^{e(v)} \cdot \Lambda'$. *For any* $\gamma \in T(\frac{\mathbb{C}}{\Lambda'})^*$ *there exists* $\delta \in T(\frac{\mathbb{C}}{\Lambda})^*$ *s.t. the natural map* $\mathrm{Hom}_R(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\Lambda}) \to \mathrm{Hom}_R(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\Lambda'})$ *induced by projection is represented, via isomorphisms stated in* (2.2), *by the multiplication by* $\prod_{v\in S} \pi_v^{e(v)}$.

Proof. Fix $\delta' \in T(\frac{\mathbb{C}}{\Lambda})^*$ and assume that the map in $\mathrm{Hom}_R(\frac{\mathbb{K}}{\Lambda}, \frac{\mathbb{K}}{\Lambda'})$ induced by projection is represented by $\alpha \in \hat{R}$ via (2.2). Since $Ker(\alpha) \simeq \frac{\Lambda'}{\Lambda} \simeq \prod_{v\in S} \mathfrak{p}_v^{e(v)}$, we may assume $\alpha = \prod_{v\in S} \pi_v^{e(v)} \alpha'$, with $\alpha' \in \hat{R}^*$. In order to conclude it suffices to choose $\delta = \delta' \cdot \alpha'$. □

We recall the following fact observed in the proof of Proposition 4.5 of [6]:

Lemma 3.10. *Two invertible 1-dimensional* $\mathbb{K}$ *lattices are commensurable iff they coincide.*

Furthermore, we get

Lemma 3.11. *A non-invertible arithmetic 1-dimensional* $\mathbb{K}$ *lattice is commensurable to an invertible one.*

Proof. Any arithmetic lattice is represented by means of (2.2) by an element $\rho \in \hat{R}$ which can be written as $\rho = \alpha \cdot \rho'$, where α is a suitable product of prime elements and $\rho' \in \hat{R}^*$. Lemma 3.9 implies that $\rho' \in \hat{R}^*$ can be interpreted as an invertible lattice for some $\Lambda' \subset \Lambda$. □

Corollary 3.12. *The set* $\mathcal{F}$ *of arithmetic 1-dimensional* $\mathbb{K}$ *lattices modulo commensurability can be identified with* $\bigcup_\Lambda T(E_\Lambda)^*$.

Proof. Just combine Lemmas 3.10 and 3.11. □

Definition 3.13. Consider two fractional ideals Λ and Λ' and set $\Gamma = \Lambda + \Lambda'$. Denote by $p : \mathrm{Hom}_R(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\Lambda}) \to \mathrm{Hom}_R(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\Gamma})$ and by $q : \mathrm{Hom}_R(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\Lambda'}) \to \mathrm{Hom}_R(\frac{\mathbb{K}}{R}, \frac{\mathbb{K}}{\Gamma})$ the natural projection and consider the corresponding ρ and ρ' obtained by means of Lemma 3.9. Then the lattices (Λ, ϕ) and (Λ', ϕ') are commensurable iff $p(\phi) = p'(\phi')$ so that the set of such commensurable lattices can be identified with the fiber product $\mathcal{F}_{\Lambda,\Lambda'} \subset T(E_\Lambda) \times T(E_{\Lambda'})$ arising from the Cartesian square:

$$\begin{array}{ccc} \mathcal{F}_{\Lambda,\Lambda'} & \longrightarrow & T(E_\Lambda) \\ q\downarrow & & \downarrow p \\ T(E_{\Lambda'}) & \longrightarrow & T(E_\Gamma) \end{array}$$

In the notations of Definition 3.13, Corollary 3.12 implies the following:

Theorem 3.14. *Fix a set of representatives Λ_i of $Cl(R)$, $1 \leq i \leq \sharp Cl(R)$. Set $\mathcal{E} = \{1, 2, \ldots, \sharp Cl(R)\} \times \mathcal{J}$. Then the groupoid of commensurability modulo dilations can be identified with:*

$$\bigcup_{(\Lambda_i,\Lambda)\in\mathcal{E}} \mathcal{F}_{\Lambda_i,\Lambda} \subset \bigcup_{(\Lambda_i,\Lambda)\in\mathcal{E}} T(E_{\Lambda_i}) \times T(E_\Lambda).$$

Proof. If Λ and Λ' are commensurable, Lemma 3.7 implies that there exists $k \in \mathbb{C}$ s.t. $k\Lambda = \Lambda_i$ and $k\Lambda' \in Cl(R)$. Then we may conclude by means of Corollary 3.12 and Definition 3.13. □

We recall the following (compare with [15], Definition 4.21 and Lemma 4.22 and remind that we are working with fields of characteristic 0):

Definition 3.15. Consider two elliptic curves E and E'. A $\mathbb{Q}$-*isogeny* from E to E' is a triple (Z, f_1, f_2), where Z is an elliptic curve and $f_1 : Z \to E$, $f_2 : Z \to E'$ are isogenies. Two $\mathbb{Q}$-isogenies (Z, f_1, f_2), (W, g_1, g_2) are *equivalent* if there is an elliptic curve C and isogenies $a : C \to Z$, $b : C \to W$ so that $f_i \circ a = g_i \circ b$, $i = 1, 2$. Any $\mathbb{Q}$-isogeny $\alpha = (Z, f_1, f_2)$ induces an isomorphism $V(\alpha) : V(E) \to V(E')$ and equivalent $\mathbb{Q}$-isogenies induce the same map.

Equivalent classes of $\mathbb{Q}$-isogenies from E to E' for a fixed E are in 1-1 correspondence with open subgroups of $V(E)$ (the open subgroup $V(\alpha)^{-1}T(E')$ $\subset V(E)$ corresponding to the $\mathbb{Q}$-isogeny $\alpha = (Z, f_1, f_2)$).

Notations 3.16.

1) Consider two fractional ideals Λ, Λ' of R, then there is an obvious $\mathbb{Q}$-*isogeny* $\alpha_{\Lambda,\Lambda'} := (E_{\Lambda\cap\Lambda'}, p_1, p_2)$ from E_Λ to $E_{\Lambda'}$, where p_1 and p_2 are the natural projections. Recalling Definition 3.15, we have a natural morphism

$$V(\alpha_{\Lambda,\Lambda'})^{-1} : T(E_{\Lambda'})^* \to V(E_\Lambda).$$

2) Gluing the morphisms $V(\alpha_{\Lambda,\Lambda'})^{-1}$ just defined we get a natural map from the set of arithmetic 1-dimensional $\mathbb{K}$ lattices modulo commensurability, $\mathcal{F}$ (compare with Corollary 3.12), to $V(E_\Lambda)$:

$$\rho_\Lambda : \mathcal{F} = \bigcup_{\Lambda'} T(E_{\Lambda'})^* \to V(E_\Lambda), \quad \rho_\Lambda\, |_{T(E_{\Lambda'})^*} := V(\alpha_{\Lambda,\Lambda'})^{-1}.$$

3) With notations as in Definition 3.13 and in Theorem 3.14 we can define a natural map

$$\xi : \bigcup_{(\Lambda_i,\Lambda)\in\mathcal{E}} T(E_{\Lambda_i}) \times T(E_\Lambda) \longrightarrow \bigcup_{\Lambda_i} V(E_{\Lambda_i}),$$

acting on each $T(E_{\Lambda_i}) \times T(E_\Lambda)$ as

$$\xi : T(E_{\Lambda_i}) \times T(E_\Lambda) \to V(E_{\Lambda_i}), \quad (x,y) \to V(\alpha_{\Lambda_i,\Lambda})^{-1}(y)\cdot x.$$

By Theorem 3.14, such a map restricts to a map from the groupoid of commensurability modulo dilations

$$\xi : \bigcup_{(\Lambda_i,\Lambda)\in\mathcal{E}} \mathcal{F}_{\Lambda_i,\Lambda} \longrightarrow \bigcup_{\Lambda_i} V(E_{\Lambda_i}), \quad \xi\,|_{\mathcal{F}_{\Lambda_i,\Lambda}} : (x,y) \to V(\alpha_{\Lambda_i,\Lambda})^{-1}(y)\cdot x.$$

4. The Adelic Heisenberg group.

Notations 4.1.

1) Let $L \in Pic(E)$ be a line bundle on an elliptic curve E with projection map $p_L : L \longrightarrow E$. We assume the curve E to be uniformized as

$$\pi_\Lambda : \mathbb{C} \longrightarrow \frac{\mathbb{C}}{\Lambda} \simeq E.$$

By [1], Proposition 2.1.6 and Lemma 2.1.7, the first Chern class of L, $c_1(L)$, can be identified with an alternating form

$$\mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{R}, \;\; (z,w) \longrightarrow r\mathrm{Im}\,(z\overline{w}),$$

s.t. $r\mathrm{Im}\,(\Lambda\overline{\Lambda}) \subset \mathbb{Z}$.

If we fix an integral basis of the lattice Λ: $\lambda := l_x + il_y$, $\mu := m_x + im_y$ s.t. $\mathrm{Im}\,(\frac{\lambda}{\mu}) > 0$ then, in order to have $r\mathrm{Im}\,(\Lambda\overline{\Lambda}) \subset \mathbb{Z}$, we must have

$$r = \frac{d}{l_x m_y - l_y m_x} \tag{4.1}$$

where d is the *degree* of L (compare with [9], §12.2).

2) As in [1], §2.4, we denote by t_x the *translation by* $x \in E$ and define:

$$K(L) := \{x \in E \mid t_x^* L \simeq L\}, \ \ \Lambda(L) := \pi_\Lambda^{-1}(K(L)).$$

By [1], §2.4,

$$\Lambda(L) = \{z \in \mathbb{C} \mid r\mathrm{Im}\,(z\overline{\Lambda}) \subset \mathbb{Z}\} \simeq \frac{1}{d}\Lambda, \tag{4.2}$$

hence we have:

$$K(L) \simeq \frac{1/d\Lambda}{\Lambda} \simeq E[d]. \tag{4.3}$$

Remark 4.2. Assume that $End(E) \simeq R$ and fix $a \in R$. Then $E \xrightarrow{a} E$ has degree $|a|^2$ ([17] §5, Proposition 2.3, [9] §12, Proposition 1.3) hence (4.3) implies:

$$K(a^* L) = E[|a|^2 d] \tag{4.4}$$

Definition 4.3. Let $x \in E$, then a biholomorphic map $\phi : L \longrightarrow L$ is called an *automorphism of* L *over* x, if the diagram

$$\begin{array}{ccc} L & \xrightarrow{\ \phi\ } & L \\ {\scriptstyle p_L}\downarrow & & \downarrow{\scriptstyle p_L} \\ Y & \xrightarrow{\ t_x\ } & X \end{array}$$

commutes ([1], 6.1). This of course forces x to belong to $K(L)$. We denote by $\mathcal{G}(L)_x$ the set of automorphisms of L over x and set

$$\mathcal{G}(L) := \bigcup_{x \in K(L)} \mathcal{G}(L)_x.$$

Then $\mathcal{G}(L)$ is a group via composition of automorphisms, the *Heisenberg group* of L, which is known to be a central extension of $K(L)$:

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(L) \xrightarrow{g_L} K(L) \longrightarrow 0$$

(compare with [15], §1 and with [1], §6). As for any central extension of an abelian group we can define a *commutator map*:

$$e^L : K(L) \times K(L) \longrightarrow \mathbb{C}^*$$

$$e^L(x,y) := \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1},$$

where $g_L(\tilde{x}) = x$ and $g_L(\tilde{y}) = y$. If z, $w \in \mathbb{C}$ are chosen in such a way that $\pi_\Lambda(z) = x$ and $\pi_\Lambda(w) = y$, then ([1], Proposition 6.3.1) we have

$$e^L(x,y) = \exp\{-2\pi i r \mathrm{Im}\,(z\overline{w})\} \in \boldsymbol{\mu}, \tag{4.5}$$

which obviously takes values in the *group of roots of unity* $\boldsymbol{\mu}$.

Remark 4.4. Combining (4.1), (4.3) and (4.5) one can recognize in e^L the inverse of the *Weil Pairing* on $E[d]$ ([9], §12):

$$e^L(x,y) = W(x,y)^{-1}$$

From now on we assume that Λ is a fractional ideal of $\mathbb{K}$ (so $End(E) \simeq R$).

Lemma 4.5. *Assume that $ay = x \in E$, then we have:*

1) *if $x \in K(L)$ then $y \in K(a^*L)$;*
2) *if $y \in K(a^*L)$ then*

$$x \in K(L) \Leftrightarrow e^{a^*L}\,|_{A_x} \equiv 1,$$

where $A_x := a^{-1}(x) \times a^{-1}(x)$.

Proof. (1) Combining (4.3) with Remark 4.2 we find

$$K(L) = E[d] \subset E[|a|^2 d] = K(a^*L).$$

(2) We have a commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\text{id}} & \mathbb{C} \\ {\scriptstyle \pi_\Lambda}\downarrow & & \downarrow{\scriptstyle \pi_{a\Lambda}} \\ E & \xrightarrow{\text{a}} & E \end{array}$$

showing that (recall (4.2))

$$\begin{aligned} \Lambda(L) &= \{z \in \mathbb{C} \mid r\mathrm{Im}\,(z\overline{\Lambda}) \subset \mathbb{Z}\} \\ &\subset \{z \in \mathbb{C} \mid r\mathrm{Im}\,(z\overline{a\Lambda}) \subset \mathbb{Z}\} = \Lambda(a^*L). \end{aligned} \tag{4.6}$$

Choose a set of representatives of $\frac{\Lambda}{a\Lambda}$: $v_i \in \Lambda$, $1 \le i \le |a|^2$.
If $z \in \Lambda(a^*L)$ then (4.6) shows that

$$z \in \Lambda(L) \;\Leftrightarrow\; r\mathrm{Im}\,(z\overline{v_i}) \in \mathbb{Z},\; 1 \le i \le |a|^2. \tag{4.7}$$

If $z \in \mathbb{C}$ projects on both x and y, and if $e^{a^*L}|_{A_x} \equiv 1$ then (4.5) implies

$$r\mathrm{Im}\,(z\overline{v_i}) = r\mathrm{Im}\,(z(\overline{z+v_i})) \in \mathbb{Z},\; 1 \le i \le |a|^2.$$

Conversely, if $r\mathrm{Im}\,(z\overline{v_i}) \in \mathbb{Z}$, $1 \le i \le |a|^2$, then we have

$$r\mathrm{Im}\,((z+v_i)(\overline{z+v_j})) \in \mathbb{Z},\; 1 \le i,j \le |a|^2$$

hence $e^{a^*L}|_{A_x} \equiv 1$. □

Consider now the pull-back commutative diagram

$$\begin{array}{ccc} a^*L & \xrightarrow{\tilde{a}} & L \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{a} & E \end{array}$$

where we put $q := p_{a^*L}$ and $p := p_L$ in order to ease the notations. The following Lemma relates the automorphisms of L over a point x to the automorphisms of a^*L over any point y, s.t. $ay = x$.

Lemma 4.6. *Assume that $ay = x \in E$ and fix $\phi \in \mathcal{G}(L)_x$. Then there exists a unique $\psi \in \mathcal{G}(a^*L)_y$ s.t. $\tilde{a} \circ \psi = \phi \circ \tilde{a}$.*

Proof. It is an easy application of the universal property of pull-back diagrams. Since $ay = x$, we also have $t_x \circ a = a \circ t_y$, and

$$a \circ t_y \circ q = t_x \circ a \circ q = t_x \circ p \circ \tilde{a} = p \circ \phi \circ \tilde{a}$$

hence $f := t_y \circ q : a^*L \longrightarrow E$ and $g := \phi \circ \tilde{a} : a^*L \longrightarrow L$ satisfy $a \circ f = p \circ g$. By the universal property of pull-back diagrams there exists a unique $\psi : a^*L \longrightarrow a^*L$ such that $f = q \circ \psi$ and $g = \tilde{a} \circ \psi$, which amounts to say $\psi \in \mathcal{G}(a^*L)_y$ and $\tilde{a} \circ \psi = \phi \circ \tilde{a}$. □

Definition 4.7. Following [15], Definition 4.5 we will say that the automorphism $\psi \in \mathcal{G}(a^*L)_y$ defined in Lemma 4.6 *covers* ϕ *over* t_y. We will also say that ϕ *is covered by* ψ *over* t_y.

Theorem 4.8. *Assume that* $ay = x \in E$*, and that* $y \in K(a^*L)$*, TFAE:*

1) $e^{a^*L} |_{A_x} \equiv 1$*;*
2) $x \in K(L)$*;*
3) *for any* $\psi \in \mathcal{G}(a^*L)_y$ *there exists* $\phi \in \mathcal{G}(L)_x$ *s.t.* $\tilde{a} \circ \psi = \phi \circ \tilde{a}$*;*
4) *there are* $\psi \in \mathcal{G}(a^*L)_y$ *and* $\phi \in \mathcal{G}(L)_x$ *s.t.* $\tilde{a} \circ \psi = \phi \circ \tilde{a}$*.*

Proof. (1) $\Leftrightarrow$ (2) follows by Lemma 4.5.
(2) $\Rightarrow$ (3) follows by Lemma 4.6.
(3) $\Rightarrow$ (4) is obvious.
(4) $\Rightarrow$ (2) follows because if $\phi \in \mathcal{G}(L)_x$ then $x \in K(L)$. □

Remark 4.9.

1) Under the hypotheses of either Lemma 4.6 or Theorem 4.8 we have a bijective map

$$\mathcal{G}(L)_x \leftrightarrow \mathcal{G}(a^*L)_y$$

defined by the relation

$$\phi \leftrightarrow \psi \quad \Leftrightarrow \quad \tilde{a} \circ \psi = \phi \circ \tilde{a}.$$

2) It is immediate to check that the bijective maps just defined preserve the composition of automorphisms, so they glue to give a group morphism

$$\delta_{a,L} : \mathcal{G}(a^*L) |_{a^{-1}(K(L))} \longrightarrow \mathcal{G}(L).$$

Definition 4.10. Let $x = (x_a)_{a\in R} \in V(E)$ so that $x_1 \in E_{tor}$. Set

$$I_x := \{a \in R \mid \; ax_1 \in K(L)\}$$

and

$$J_x := \{a \in R \mid \; x_a \in K(a^*L)\}.$$

Lemma 4.11. *Let $x = (x_a)_{a\in R} \in V(E)$ so that $x_1 \in E_{tor}$ and choose $k \in \mathbb{K}$ s.t. $\pi_\Lambda(k) = x_1$. Then we have*

$$I_x = \overline{J_x} \simeq \frac{1}{kd}\Lambda \cap R.$$

Proof. $I_x \simeq \frac{1}{kd}\Lambda \cap R :$ by (4.2), for any $a \in R$ we have:

$$ax_1 \in K(L) \;\Leftrightarrow\; ak \in \frac{1}{d}\Lambda, \;\Leftrightarrow\; a \in \frac{1}{kd}\Lambda \cap R.$$

$\overline{J_x} \simeq \frac{1}{kd}\Lambda \cap R :$ Choose $k_a \in \mathbb{K}$ s.t. $ak_a = k$ and $\pi_\Lambda(k_a) = x_a$. Combining (4.2) and (4.3) with Remark 4.2, for any $a \in R$ we have:

$$x_a \in K(a^*L) \;\Leftrightarrow\; k_a \in \frac{1}{a\overline{a}d}\Lambda, \;\Leftrightarrow\; k = ak_a \in \frac{1}{\overline{a}d}\Lambda, \;\Leftrightarrow\; \overline{a} \in \frac{1}{kd}\Lambda \cap R.$$

□

Remark 4.12. If $x \in V(E)$ and $a, b \in J_x$ then by Remark 4.9 we have canonical isomorphisms:

$$\mathcal{G}(a^*L)_{x_a} \simeq \mathcal{G}((ab)^*L)_{x_{ab}} \simeq \mathcal{G}(b^*L)_{x_b}$$

in the sense that any $\phi_a \in \mathcal{G}(a^*L)_{x_a}$ ($\phi_b \in \mathcal{G}(b^*L)_{x_b}$) is covered by a unique element $\phi_{ab} \in \mathcal{G}((ab)^*L)_{x_{ab}}$ over $t_{x_{ab}}$:

$$\tilde{b} \circ \phi_{ab} = \phi_a \circ \tilde{b}, \;\; \tilde{a} \circ \phi_{ab} = \phi_b \circ \tilde{a}.$$

By the previous remark we can give the following

Definition 4.13.

1) We denote by $\hat{\mathcal{G}}(L)$ the *adelic Heisenberg group* i.e. the group of $\alpha = (x, (\phi_a)_{a\in J_x})$ s.t. $x \in V(E)$, $\phi_a \in \mathcal{G}(a^*L)_{x_a}$ with ϕ_{ab} covering ϕ_a over $t_{x_{ab}}$, $\forall a \in J_x$. Any collection of elements $(\phi_a)_{a\in J_x}$, $\phi_a \in \mathcal{G}(a^*L)_{x_a}$ with ϕ_{ab} covering ϕ_a over $t_{x_{ab}}$ will be called a *coherent system of automorphisms* defined over x.

2) Using the (rather cumbersome) notations of Remark 4.9 (2), if $(\phi_a)_{a\in J_x}$ is a coherent system of automorphisms then

$$\delta_{b,a^*L}(\phi_{ab}) = \phi_a, \ \ \forall a \in J_x, \ \forall b \in R.$$

3) Consider the closed subgroup $\mathcal{V}_a := \nu_a^{-1}(K(a^*L)) \subset V(E)$. Obviously $a \in J_x \ \forall x \in \mathcal{V}_a$ so it is well defined a group morphism

$$\gamma_a : \hat{\mathcal{G}}(L) \mid_{\mathcal{V}_a} \longrightarrow \mathcal{G}(a^*L), \quad (x, (\phi_b)_{b\in J_x}) \longrightarrow \phi_a.$$

Theorem 4.14.

1) *We have an exact sequence:*

$$1 \to \mathbb{C}^* \to \hat{\mathcal{G}}(L) \xrightarrow{g} V(E) \to 0.$$

2)

$$\hat{\mathcal{G}}(L) \mid_{\mathcal{V}_a} \simeq \nu_a^* \mathcal{G}(a^*L).$$

3)

$$\hat{\mathcal{G}}(L) \mid_{T(E)} \simeq \nu_1^* \mathbb{C},$$

so there exists a morphism $\sigma^L : T(E) \to \hat{\mathcal{G}}(L)$ providing a section of the restriction to $T(E)$ of the sequence above ($\sigma(x)$ is defined as the unique element of $\hat{\mathcal{G}}(L)$ lifting the identity).

Proof. (1) By Lemma 4.11 the map $\hat{\mathcal{G}}(L) \to V(E)$ is surjective because $\frac{1}{kd}\Lambda \cap R = J_x \neq \emptyset$, $\forall x \in V(E)$, so there are coherent systems of automorphisms defined over any $x \in V(E)$. Furthermore, for any $x \in V(E)$, Remark 4.9 implies that any coherent system of automorphisms $(\phi_a)_{a\in J_x}$, $\phi_a \in \mathcal{G}(a^*L)_{x_a}$, is uniquely determined by any of its components, so we have:

$$\hat{\mathcal{G}}(L)_x \simeq \mathcal{G}(a^*L)_{x_a} \simeq \mathbb{C}^*, \ \ \forall a \in J_x.$$

(2) Consider the pull-back diagram

$$\begin{array}{ccc} \nu_a^*\mathcal{G}(a^*L) & \longrightarrow & \mathcal{G}(a^*L) \\ \downarrow & & \downarrow g_{a^*L} \\ \mathcal{V}_a & \xrightarrow{\nu_a} & K(a^*L) \end{array}$$

Obviously the morphisms $g_{a^*L} \circ \gamma_a$ and $\nu_a \circ g$ coincide on $\hat{\mathcal{G}}(L)\mid_{\mathcal{V}_a}$ so, by the universal property of pull-back, there is a unique morphism $\iota : \hat{\mathcal{G}}(L)\mid_{\mathcal{V}_a} \longrightarrow \nu_a^*\mathcal{G}(a^*L)$ s.t.

$$\begin{array}{ccc} \hat{\mathcal{G}}(L)\mid_{\mathcal{V}_a} & \xrightarrow{\ \iota\ } & \nu_a^*\mathcal{G}(a^*L) \\ \downarrow & & \downarrow \\ \mathcal{V}_a & \xrightarrow{\ \mathrm{id}\ } & \mathcal{V}_a \end{array}$$

commutes. Finally, ι is an isomorphism because it is bijective on the fibers (both isomorphic to $\mathbb{C}^*$) of the vertical maps of the last diagram.
(3) This is a particular case of (2). □

Lemma 4.15. *Let* $(x, (\phi_a)_{a \in J_x}), (y, (\phi_a)_{a \in J_y}) \in \hat{\mathcal{G}}(L)$ *and* $a, b \in J_x \cap J_y$*, then*

$$e^{a^*L}(x_a, y_a) = e^{b^*L}(x_b, y_b)$$

Proof. We argue as in the proof of Lemma 4.5. We have commutative diagrams

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\ \mathrm{id}\ } & \mathbb{C} \\ {\scriptstyle \pi_{b\Lambda}}\downarrow & & \downarrow{\scriptstyle \pi_{ab\Lambda}} \\ E & \xrightarrow{\ \mathrm{a}\ } & E \end{array}$$

and

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\ \mathrm{id}\ } & \mathbb{C} \\ {\scriptstyle \pi_{a\Lambda}}\downarrow & & \downarrow{\scriptstyle \pi_{ab\Lambda}} \\ E & \xrightarrow{\ \mathrm{b}\ } & E \end{array}$$

so if we choose $z, v \in \mathbb{C}$ s.t.

$$\pi_{ab\Lambda}(z) = x_{ab}, \ \ \pi_{ab\Lambda}(v) = y_{ab}$$

then

$$\pi_{a\Lambda}(z) = x_a, \ \pi_{a\Lambda}(v) = y_a, \ \pi_{b\Lambda}(z) = x_b, \ \ \pi_{b\Lambda}(v) = x_b.$$

Finally, (4.5) implies

$$e^{a^*L}(x_a, y_a) = \exp\{-2\pi i r \mathrm{Im}\,(z\overline{v})\} = e^{b^*L}(x_b, y_b).$$

□

Remark 4.16. Similarly as above (compare with Definition 4.3), we can define a *commutator map*:

$$\tilde{e}^L : V(E) \times V(E) \to \mathbb{C}^1, \qquad \tilde{e}^L(x, x') = yy'y^{-1}y'^{-1}$$

where $y, y' \in \hat{\mathcal{G}}(L)$ are chosen in such a way that $g(y) = x, g(y') = x'$. By Lemma 4.15 we have

$$\tilde{e}^L(x, x') = e^{a^*L}(x_a, x'_a) \in \boldsymbol{\mu}, \ \forall a \in J_x \cap J_{x'}. \tag{4.8}$$

with values in the group of the roots of unity.

Theorem 4.17. *Let $x_a \in E_{tor}$ and set*

$$\tilde{A}_{x_a} := \nu_a^{-1}(x_a) \times \nu_a^{-1}(x_a) \subset V(E) \times V(E).$$

Then the restriction $\tilde{e}^L(\cdot, \cdot) \mid_{\tilde{A}_{x_a}}$ is constant iff $x_a \in K(a^*L) = E[|a|^2 d]$.

Proof. If $x_a \in K(a^*L) = E[|a|^2 d]$ then 4.8 implies

$$\tilde{e}^L(x, x') = e^{a^*L}(x_a, x_a) = 1, \ \forall (x, x') \in \tilde{A}_{x_a}.$$

Conversely, assume that $\tilde{e}^L(\cdot, \cdot) \mid_{\tilde{A}_{x_a}} \equiv 1$ and fix $b \in R$ s.t. $\overline{b}x_a \in K(a^*L)$. By Lemma 4.11

$$x_{ab} \in K((ab)^*L), \ \forall x \in \nu_a^{-1}(x_a)$$

which means that

$$ab \in J_x, \ \ \forall x \in \nu_a^{-1}(x_a). \tag{4.9}$$

Consider $r, s \in b^{-1}(x_a)$ and choose $x, x' \in \nu_a^{-1}(x_a)$ s.t. $x_{ab} = r$ and $x'_{ab} = s$. Then $(x, x') \in \tilde{A}_{x_a}$ and combining 4.8 with the hypothesis $\tilde{e}^L(\cdot, \cdot) \mid_{\tilde{A}_{x_a}} \equiv 1$

we find

$$e^{(ab)^*L}(r,s) = \tilde{e}^L(x,x') = 1, \ \forall r,s \in b^{-1}(x_a).$$

We are done by means of Theorem 4.8. □

Remark 4.18. Despite the strong link with the Weil Pairing noticed in Remark 4.4, last Theorem shows that the commutator map $\tilde{e}^L(\cdot,\cdot)\mid_{\tilde{A}_{x_a}}$ cannot be constant if $x_a \notin K(a^*L)$. This proves that the bilinear form $\tilde{e}^L(\cdot,\cdot)$ on $V(E)$ *cannot be induced by any bilinear form on* E_{tor} (such as the Weil pairing) via projection

$$\nu_a \times \nu_a : V(E) \times V(E) \to E_{tor} \times E_{tor}.$$

Assume now that L is *symmetric*: $(-1)^*L \simeq L$. Correspondingly we have an involution $i : \mathcal{G}(L) \to \mathcal{G}(L)$. By an abuse of notations, we use the same symbol for the corresponding morphism of $\hat{\mathcal{G}}(L)$:

$$i : \hat{\mathcal{G}}(L) \to \hat{\mathcal{G}}(L).$$

We recall the following ([15], p. 58):

Definition 4.19. Fix $x \in V(E)$, $y \in \hat{\mathcal{G}}(L)$ s.t. $2g(y) = x$. Then $\tau(x) \in \hat{\mathcal{G}}(L)$ defined as $\tau(x) := yi(y)^{-1}$ does not depend of the choice of y, so we have a map:

$$\tau : V(E) \to \hat{\mathcal{G}}(L),$$

providing a section of the exact sequence of Theorem 4.14.

Proposition 4.20.

$$\tau(x)\tau(y) = \tilde{e}^L(\tfrac{x}{2}, y)\tau(x+y)$$

Proof. Observe that, for any pair $x' \in \hat{\mathcal{G}}(L)$, $y' \in \hat{\mathcal{G}}(L)$ s.t. $2g(x') = x$ and $2g(y') = y$ we have $x'y' = \tilde{e}^L(\frac{x}{2}, \frac{y}{2})y'x'$, $2g(i(x')^{-1}) = x$ and $2g(i(y')^{-1}) = y$ (and so $i(x')i(y') = \tilde{e}^L(\frac{x}{2}, \frac{y}{2})i(y')i(x')$). Then we get

$$\begin{aligned}\tau(x)\tau(y) &= x'i(x')^{-1}y'i(y')^{-1} = \tilde{e}^L(\tfrac{x}{2}, \tfrac{y}{2})x'y'i(x')^{-1}i(y')^{-1} \\ &= \tilde{e}^L(\tfrac{x}{2}, y)x'y'i(y')^{-1}i(x')^{-1} = \tilde{e}^L(\tfrac{x}{2}, y)x'y'i(x'y')^{-1} \\ &= \tilde{e}^L(\tfrac{x}{2}, y)\tau(x+y).\end{aligned}$$

□

5. The Main Theorem of Complex Multiplication for Adelic Curves

In this section we are going to study the behavior of the Adelic Heisenberg Group previously defined under the action of field automorphisms of $\mathbb{C}$. We begin with some technical Lemmas which will be needed in the following.

Lemma 5.1. *Consider two normalized ([18] §II, Proposition 1.1) elliptic curves* E, E' *s.t.* $End(E) \simeq End(E') \simeq R$, *equipped with line bundles* $L \to E$, $L' \to E'$. *Assume there are bijective maps* $\sigma : E \to E'$, $\tilde{\sigma} : L \to L'$ *making commutative the following diagram*

$$\begin{array}{ccc} L & \xrightarrow{\tilde{\sigma}} & L' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ E & \xrightarrow{\sigma} & E' \end{array}$$

with σ *commuting with any* $a \in R$. *Then for any* $a \in R$ *there exists a unique bijective* $\tilde{\sigma}_a : a^*L \to a^*L'$ *satisfying* $q_a \circ \tilde{\sigma}_a = \sigma \circ p_a$ *and* $\tilde{a} \circ \tilde{\sigma}_a = \tilde{\sigma} \circ \tilde{a}$ ($p_a : a^*L \to E$, $q_a : a^*L' \to E'$ *denote the natural projections). Furthermore, we have*

$$\tilde{a} \circ \tilde{\sigma}_{ab} = \tilde{\sigma}_b \circ \tilde{a} : (ab)^*L \to b^*L', \ \ \forall a, b \in R. \tag{5.1}$$

Proof. Consider the maps $\sigma \circ p_a : a^*L \to E'$, $\tilde{\sigma} \circ \tilde{a} : a^*L \to L'$. Since σ commutes with a, we have

$$a \circ \sigma \circ p_a = \sigma \circ a \circ p_a = \sigma \circ p \circ \tilde{a} = q \circ \tilde{\sigma} \circ \tilde{a},$$

and the Universal Property implies there exists a unique $\tilde{\sigma}_a : a^*L \to a^*L'$ s.t. $q_a \circ \tilde{\sigma}_a = \sigma \circ p_a$ and $\tilde{a} \circ \tilde{\sigma}_a = \tilde{\sigma} \circ \tilde{a}$.
In order to prove 5.1 it suffices show that

1) $\tilde{b} \circ \tilde{a} \circ \tilde{\sigma}_{ab} = \tilde{b} \circ \tilde{\sigma}_b \circ \tilde{a}$,
2) $q_b \circ \tilde{a} \circ \tilde{\sigma}_{ab} = q_b \circ \tilde{\sigma}_b \circ \tilde{a}$.

(1) from $\tilde{b} \circ \tilde{\sigma}_b = \tilde{\sigma} \circ \tilde{b}$ and $\widetilde{ab} \circ \tilde{\sigma}_{ab} = \tilde{\sigma} \circ \widetilde{ab}$ we have

$$\tilde{b} \circ \tilde{a} \circ \tilde{\sigma}_{ab} = \tilde{\sigma} \circ \tilde{b} \circ \tilde{a} = \tilde{b} \circ \tilde{\sigma}_b \circ \tilde{a}.$$

(2) from $q_b \circ \tilde{\sigma}_b = \sigma \circ p_b$ and $q_{ab} \circ \tilde{\sigma}_{ab} = \sigma \circ p_{ab}$ we have

$$q_b \circ \tilde{\sigma}_b \circ \tilde{a} = \sigma \circ p_b \circ \tilde{a} = \sigma \circ a \circ p_{ab} = a \circ \sigma \circ p_{ab} = a \circ q_{ab} \circ \tilde{\sigma}_{ab} = q_b \circ \tilde{a} \circ \tilde{\sigma}_{ab}.$$

□

Lemma 5.2. *Keep the hypothesis of Lemma 5.1 and assume additionally that L and L' are obtained via pull-back of the hyperplane bundle by means of embeddings $a : E \to \mathbb{P}(\mathbb{C}^n)$, $b : E' \to \mathbb{P}(\mathbb{C}^n)$ and that $\tilde{\sigma}$ is induced by a map $\tilde{\sigma} : (\mathbb{C}^n)^* \to (\mathbb{C}^n)^*$, obtained by acting on each coordinate with $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then we have an isomorphism*

$$\mathcal{G}(L) \leftrightarrow \mathcal{G}(L'), \;\; \phi \leftrightarrow \phi^\sigma := \tilde{\sigma} \circ \phi \circ \tilde{\sigma}^{-1}.$$

Proof. The proof is immediate since any $\phi \in \mathcal{G}(L)$ correspond by a matrix $U \in GL(\mathbb{C}^n)$ via *canonical representation* ([1], §6.4) and it is immediate to check that ϕ^σ is represented by $\sigma(U)$. □

First of all we would like to study the behavior of the Adelic Heisenberg Group under the action of field automorphisms of $\mathbb{C}$ fixing E.

Consider an elliptic curve E with complex multiplication by R, embedded in $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^3)$ by means of the Weierstrass model and assume it is defined over a field $\mathbb{L}$. Denote by σ a field automorphisms of $\mathbb{C}$ fixing $\mathbb{K} \cdot \mathbb{L}$. By [18] §II, Theorem 4.1, σ fix also $\mathbb{H}$, the *Hilbert field* of $\mathbb{K}$, and may be interpreted as an element $\sigma \in Gal(\overline{\mathbb{H}}, \mathbb{H})$. Consider also a normalization $E \simeq \frac{\mathbb{C}}{\Lambda}$ ([18], §II.1) with Λ a fractional ideal of $\mathbb{K}$ and denote by L the line bundle providing the embedding $\frac{\mathbb{C}}{\Lambda} \to \mathbb{P}^2$. Observe that L is symmetric so all the results of Section 4 can be applied.

Theorem 5.3. *The field automorphism σ acts as automorphism of $\hat{\mathcal{G}}(L)$ and we have: $\tau(\sigma(x)) = \sigma(\tau(x)) \in \hat{\mathcal{G}}(L)$, $\forall x \in V(E)$.*

Proof. Since σ fixes $\mathbb{L}$, it provides a bijection of E_{tor} which is induced (compare with Lemma 5.2), via embedding $E \subset \mathbb{P}^2$, by the bijective map $\tilde{\sigma} : \mathbb{C}^3 \leftrightarrow \mathbb{C}^3$ (acting as σ in any coordinate). Such a map pull-back to L via uniformization in such a way that we have a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\tilde{\sigma}} & L \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{\sigma} & E \end{array}$$

Moreover, by [18], Theorem 2.2 (b), the multiplication map by any $a \in R$ is still defined over $\mathbb{K} \cdot \mathbb{L}$ so $a \circ \sigma = \sigma \circ a$, $\forall a \in R$ and we may apply Lemma 5.1.
First of all we observe that σ extend to an automorphism

$$\sigma : V(E) \to V(E)$$

because, if $x = (x_a)_{a \in R}$ belongs to $V(E)$ then also $\sigma(x) := (\sigma(x_a))_{a \in R}$ stays in $V(E)$ since

$$a\sigma(x_{ab}) = \sigma(ax_{ab}) = \sigma(x_b).$$

Furthermore, we have

$$ay \in K(L) = E[d] \Leftrightarrow ady = 0 \Leftrightarrow \sigma(ady) = ad\sigma(y) = 0 \Leftrightarrow a\sigma(y) \in K(L),$$

so (compare with 4.11)

$$I_x = I_{\sigma(x)}, \;\; J_x = J_{\sigma(x)}, \;\; \forall x \in V(E).$$

If $\phi \in \mathcal{G}(a^*L)$ define $\phi^\sigma := \tilde{\sigma}_a \circ \phi \circ \tilde{\sigma}_a^{-1} \in \mathcal{G}(a^*L)$. Fix $\alpha = (x, (\phi_a)_{a \in J_x}) \in \hat{\mathcal{G}}(L)$ and set $\alpha^\sigma = (\sigma(x), (\phi_a^\sigma)_{a \in J_x})$. Then 5.1 implies

$$\begin{aligned} \tilde{a} \circ \phi_{ab}^\sigma &= \tilde{a} \circ \tilde{\sigma}_{ab} \circ \phi_{ab} \circ \tilde{\sigma}_{ab}^{-1} = \tilde{\sigma}_b \circ \tilde{a} \circ \phi_{ab} \circ \tilde{\sigma}_{ab}^{-1} = \tilde{\sigma}_b \circ \phi_b \circ \tilde{a} \circ \tilde{\sigma}_{ab}^{-1} \\ &= \tilde{\sigma}_b \circ \phi_b \circ \tilde{\sigma}_b^{-1} \circ \tilde{a} = \phi_b^\sigma \circ \tilde{a} \end{aligned}$$

so we have $\alpha^\sigma \in \hat{\mathcal{G}}(L)$ and we have the desired extension of σ to an automorphism of $\hat{\mathcal{G}}(L)$

$$\sigma : \hat{\mathcal{G}}(L) \to \hat{\mathcal{G}}(L).$$

In order to conclude the proof we have

$$\tau(\sigma(x)) = \sigma(y) i(\sigma(y))^{-1}, \;\; \forall y \in V(E) \; \mid \; 2g(y) = x,$$

since $2g(\sigma(y)) = \sigma(2g(y)) = \sigma(x)$ and $\tau(\sigma(x))$ does not depend on y' s.t. $2g(y') = \sigma(x)$. Finally, we have

$$\tau(\sigma(x)) = \sigma(y) i(\sigma(y))^{-1} = \sigma(y i(y)^{-1}) = \sigma(\tau(x)),$$

because $\sigma : \hat{\mathcal{G}}(L) \to \hat{\mathcal{G}}(L)$ is a morphism commuting with i. □

What we are going to do now is to study the behavior of the Adelic Heisenberg Group under the action of any field automorphisms of $\mathbb{C}$. Consider again an elliptic curve E with complex multiplication by R, embedded in $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^3)$ by means of the Weierstrass model and assume it is defined over a field $\mathbb{L}$. Recall the *Main Theorem of Complex Multiplication* ([18] II, Theorem 8.2, see also [19], Ch. 5 and [12], Ch. 10):

Theorem 5.4. *Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ fixing $\mathbb{K}$ and let s be an idèle of $\mathbb{K}$ corresponding to σ via Artin map. Fix a complex analytic isomorphism:*

$$f : \frac{\mathbb{C}}{\Lambda} \to E(\mathbb{C}),$$

where Λ is a fractional ideal. Then there exists a unique complex analytic isomorphism:

$$g : \frac{\mathbb{C}}{s^{-1}\Lambda} \to E^\sigma(\mathbb{C}),$$

so that the following diagram commutes:

$$\begin{array}{ccc} \frac{\mathbb{K}}{\Lambda} & \xrightarrow{s^{-1}} & \frac{\mathbb{K}}{s^{-1}\Lambda} \\ f\downarrow & & \downarrow g \\ E(\mathbb{C}) & \xrightarrow{\sigma} & E^\sigma(\mathbb{C}) \end{array}$$

The main purpose of the rest of this section is to lift such a commutative diagram to adelic Heisenberg groups. To the ease notations we put $E' = E^\sigma$. We may assume $E(\mathbb{C})$ and $E(\mathbb{C})'$ both embedded $\mathbb{P}^2(\mathbb{C}) = \mathbb{P}(\mathbb{C}^3)$ by means of Weierstrass models and we define $L := f^*\mathcal{O}_{\mathbb{P}^2}(1)$, $L' := g^*\mathcal{O}_{\mathbb{P}^2}(1)$. Like in Lemma 5.2, $\sigma : E \to E'$ is induced by $\tilde{\sigma} : (\mathbb{C}^3)^* \to (\mathbb{C}^3)^*$ pulling back to a map $L \to L'$ and providing a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\tilde{\sigma}} & L' \\ \mathrm{p}\downarrow & & \downarrow \mathrm{q} \\ E & \xrightarrow{\sigma} & E' \end{array}$$

Lemma 5.5.

1) *$a^\sigma = a$, $\forall a \in R$;*
2) *if $x, y \in E$ are s.t. $ay = x$ then $a\sigma(y) = \sigma(x)$;*
3) *σ extends to an isomorphism*

$$\sigma : V(E) \to V(E),\ x = (x_a)_{a\in R} \to \sigma(x) := (\sigma(x_a))_{a\in R} = s^{-1}x.$$

4)

$$I_x = I_{\sigma(x)}, \ \ J_x = J_{\sigma(x)}, \ \ \forall x \in V(E).$$

Proof. (1) First proof: since E and E' are normalized by the maps f and g of Theorem 5.4, the statement follows directly from [18] II, Theorem 2.2.

Second proof: even more directly, Theorem 5.4 implies

$$a^\sigma(x') = \sigma \circ a \circ \sigma^{-1}(x') = \sigma(asx') = ax', \ \ \forall x' \in E_{tor}.$$

(2) $ay = x \Rightarrow \sigma(ay) = \sigma(x) \Rightarrow a\sigma(y) = \sigma(x)$ thanks to (1).
(3) It follows just combining (2) with Theorem 5.4 since σ acts as the multiplication by s^{-1} on each component x_a.
(4) We get

$$\begin{aligned} ax_1 \in K(L) = E[d] \Leftrightarrow adx_1 = 0 \Leftrightarrow \sigma(adx_1) = ad\sigma(x_1) = 0 \\ \Leftrightarrow a\sigma(x_1) \in K(L'), \end{aligned}$$

hence $I_x = I_{\sigma(x)}$, $\forall x \in V(E)$ and we are done by Lemma 4.11. □

Theorem 5.6. *The isomorphism: $s^{-1} : V(E) \to V(E')$ lifts to an isomorphism of adelic Heisenberg groups*

$$\sigma : \hat{\mathcal{G}}(L) \to \hat{\mathcal{G}}(L')$$

commuting with the sections $\tau : V(E) \to \hat{\mathcal{G}}(L)$ and $\tau' : V(E') \to \hat{\mathcal{G}}(L')$:

$$\sigma(\tau(x)) = \tau'(\sigma(x)), \qquad \forall x \in V(E).$$

Proof. Combining Lemmas 5.1, 5.2 and 5.5 the proof is very similar to that of Theorem 5.3.

By Lemma 5.5, (1) we can apply Lemma 5.1 to L and L' so for any $a \in R$ there exists a unique bijective $\tilde{\sigma}_a : a^*L \to a^*L'$ satisfying $q_a \circ \tilde{\sigma}_a = \sigma \circ p_a$ and $\tilde{a} \circ \tilde{\sigma}_a = \tilde{\sigma} \circ \tilde{a}$ and

$$\tilde{a} \circ \tilde{\sigma}_{ab} = \tilde{\sigma}_b \circ \tilde{a} : (ab)^*L \to b^*L', \ \ \forall a, b \in R. \tag{5.2}$$

Furthermore, Lemma 5.2 implies that for any $a \in R$ we have an isomorphism

$$\mathcal{G}(a^*L) \leftrightarrow \mathcal{G}(a^*L) \ \ \phi \leftrightarrow \phi^\sigma := \tilde{\sigma}_a \circ \phi \circ \tilde{\sigma}_a^{-1}.$$

Since $I_x = I_{\sigma(x)}$, $\forall x \in V(E)$ (compare with Lemma 5.5, (4)), for any $\alpha = (x, (\phi_a)_{a\in J_x}) \in \hat{\mathcal{G}}(L)$ we define $\alpha^\sigma = (\sigma(x), (\phi_a^\sigma)_{a\in J_x})$. Then 5.2 implies

$$\begin{aligned}\tilde{a} \circ \phi_{ab}^\sigma &= \tilde{a} \circ \tilde{\sigma}_{ab} \circ \phi_{ab} \circ \tilde{\sigma}_{ab}^{-1} = \tilde{\sigma}_b \circ \tilde{a} \circ \phi_{ab} \circ \tilde{\sigma}_{ab}^{-1} = \tilde{\sigma}_b \circ \phi_b \circ \tilde{a} \circ \tilde{\sigma}_{ab}^{-1} \\ &= \tilde{\sigma}_b \circ \phi_b \circ \tilde{\sigma}_b^{-1} \circ \tilde{a} = \phi_b^\sigma \circ \tilde{a}\end{aligned}$$

so we have $\alpha^\sigma \in \hat{\mathcal{G}}(L)$ and we have the desired extension of σ to an automorphism of $\hat{\mathcal{G}}(L)$

$$\sigma : \hat{\mathcal{G}}(L) \to \hat{\mathcal{G}}(L).$$

In order to conclude the proof we have

$$\tau'(\sigma(x)) = \sigma(y)i(\sigma(y))^{-1}, \ \ \forall y \in V(E) \ \mid \ 2g(y) = x,$$

since $2g(\sigma(y)) = \sigma(2g(y)) = \sigma(x)$ and $\tau(\sigma(x))$ does not depend on y' s.t. $2g(y') = \sigma(x)$. Finally, we have

$$\tau'(\sigma(x)) = \sigma(y)i(\sigma(y))^{-1} = \sigma(yi(y)^{-1}) = \sigma(\tau(x)),$$

because $\sigma : \hat{\mathcal{G}}(L) \to \hat{\mathcal{G}}(L)$ is a morphism commuting with i. □

6. Adelic Thetas

What we are going to do in this section is to show that *canonical representations* $\tilde{\rho}_a$ defined in 6.1, fit for different $a \in R$ to give a representation U (Proposition 6.5) of the adelic Heisenberg group $\hat{\mathcal{G}}(L)$ into the direct limit (Definition 6.3)

$$\hat{H}^0(L) \simeq \varinjlim_{a\in R} H^0(a^*L).$$

This allows us to define *adelic theta functions* θ_s, for any $s \in \hat{H}^0(L)$ (Definition 6.6), on $V(E)$ by means of the lifting $\tau : V(E) \to \hat{\mathcal{G}}(L)$ defined in 4.19 (in all this section the line bundle L is assumed to be symmetric). We obtain in such a way (Proposition 6.8) a vector space of functions over which $\hat{\mathcal{G}}(L)$ is represented by means of *translations and characters* (as it usually happens in canonical representations).

Combining all that with the properties of the adelic action on $V(E)$ stemming from the main theorem of complex multiplication for elliptic curves (proved in §5) we find a nice intertwining between theta functions and $\mathbb{C}$-automorphisms (Theorems 6.9 and 6.11).

Last but not least, composing theta functions with the embeddings defined in Notations 3.16, we are going to define theta functions exhibiting a nice behavior under $\mathbb{C}$-automorphisms on commensurability classes of arithmetic 1-dimensional $\mathbb{K}$-lattices (Theorem 6.12) and on the groupoid of commensurability modulo dilations (Notations 6.13, Theorem 6.14).

Definition 6.1. By an abuse of notations, for any $\psi \in \mathcal{G}(a^*L)_x$ we still denote by ψ its image via *canonical representation* ([1], 6.4):

$$\tilde{\rho}_a : \mathcal{G}(a^*L) \to GL(H^0(a^*L)), \;\; \psi = \tilde{\rho}_a(\psi) : s \to \psi \circ s \circ t_{-x}.$$

If $s \in H^0(a^*L)$ and $b \in R$ then $b \circ id = b = id \circ b = p_a \circ s \circ b$ so, by universal property, there exists $\hat{b}(s) \in H^0((ab)^*L)$ s.t. $\tilde{b}(\hat{b}(s)) = s \circ b$. We find a linear map

$$\hat{b} : H^0(a^*L) \hookrightarrow H^0((ab)^*L).$$

Lemma 6.2. *Consider $\psi \in \mathcal{G}((ab)^*L)$ and $\phi \in \mathcal{G}(a^*L)$ s.t. $\tilde{b} \circ \psi = \phi \circ \tilde{b}$. Then we have a commutative square:*

$$\begin{array}{ccc} H^0(a^*L) & \xrightarrow{\hat{b}} & H^0((ab)^*L) \\ \phi \big\downarrow & & \big\downarrow \psi \\ H^0(a^*L) & \xrightarrow{\hat{b}} & H^0((ab)^*L) \end{array}$$

Proof. For any $s \in H^0(a^*L)$, $\hat{b}(s) \in H^0((ab)^*L)$ is characterized by $\tilde{b} \circ \hat{b}(s) = s \circ b$ so we are left to prove that $\tilde{b} \circ \psi(\hat{b}(s)) = \phi(s) \circ b$, $\forall s \in H^0(a^*L)$. We have

$$\begin{aligned} \tilde{b} \circ \psi(\hat{b}(s)) &= \tilde{b} \circ \psi \circ \hat{b}(s) \circ t_{-bx} = \phi \circ \tilde{b} \circ \hat{b}(s) \circ t_{-x} \\ &= \phi \circ s \circ b \circ t_{-x} = \phi \circ s \circ t_{-bx} \circ b = \phi(s) \circ b. \end{aligned}$$

□

Definition 6.3.

1) Set

$$\hat{H}^0(L) \simeq \varinjlim_{a \in R} H^0(a^*L), \;\; \iota_a : H^0(a^*L) \to \hat{H}^0(L),$$

with ι_a denoting the canonical inclusion. For any $\alpha = (x, (\phi_a)_{a \in J_x}) \in \hat{\mathcal{G}}(L)$, define

$$U_\alpha : \hat{H}^0(L) \to \hat{H}^0(L), \qquad U_\alpha \mid_{H^0(a^*L)} = \tilde{\rho}_a(\phi_a), \; \forall a \in J_x.$$

Such a U_α is well defined by Lemma 6.2.

2) For any $a \in R$ we denote by GL_a the group

$$GL_a := \{(\phi_r)_{r\in(a)} \mid \ \phi_b \in GL(H^0(b^*L)),\ \phi_{ab} \circ \hat{c} = \hat{c} \circ \phi_{abc}\ \forall b, c \in R\}.$$

3) If $a \mid b$ then $(b) \subset (a)$ and there an obvious injective group morphism $GL_a \hookrightarrow GL_b$. So can define the limit group

$$GL(\hat{H}^0(L)) := \varinjlim_{a\in R} GL_a.$$

Remark 6.4. Observe that Lemma 6.2 implies that U_α defined in 6.3, (1) belongs to $GL(\hat{H}^0(L))$. The following Proposition shows that the correspondence $\alpha \to U_\alpha$ is indeed a representation of $\hat{\mathcal{G}}(L)$ in $GL(\hat{H}^0(L))$.

Proposition 6.5. *Let L be very ample, choose a section $s \in H^0(L)$ and assume everything defined over some field $\mathbb{L}$. The map defined in 6.3, (1)*

$$U : \hat{\mathcal{G}}(L) \to GL(\hat{H}^0(L)), \quad \alpha \to U_\alpha$$

is a group morphism. Furthermore, we have:

$$U_{\tau(x)} \circ U_{\tau(y)}(s) = \tilde{e}(\tfrac{x}{2}, y) U_{\tau(x+y)}(s).$$

Proof. Fix $\alpha = (x, (\phi_a)_{a\in J_x}) \in \hat{\mathcal{G}}(L)$ and $\beta = (x, (\phi_a)_{a\in J_y}) \in \hat{\mathcal{G}}(L)$. If $a \in J_x \cap J_y$, then α and β belong to $\hat{\mathcal{G}}(L) \mid_{\mathcal{V}_a}$ (compare with Definition 4.13, (3)). What we are going to prove is that

$$U : \hat{\mathcal{G}}(L) \mid_{\mathcal{V}_a} \to GL(\hat{H}^0(L))$$

is a group morphism. Theorem 4.14, (2) implies

$$\hat{\mathcal{G}}(L) \mid_{\mathcal{V}_a} \simeq \nu_a^* \mathcal{G}(a^*L).$$

Moreover, $U : \nu_a^* \mathcal{G}(a^*L) \to GL(\hat{H}^0(L))$ obviously factorizes through GL_a

$$U : \nu_a^* \mathcal{G}(a^*L) \to GL_a \subset GL(\hat{H}^0(L))$$

and we have a commutative diagram (compare with Definition 6.1):

$$\begin{array}{ccc} \nu_a^*\mathcal{G}(a^*L) & \xrightarrow{\quad U \quad} & GL_a \\ \downarrow & & \downarrow \\ \mathcal{G}(a^*L) & \xrightarrow{\ \rho_a\ } & GL(H^0(a^*L)) \end{array}$$

i.e. the map $U \mid_{\hat{\mathcal{G}}(L)|_{\nu_a}}$ factorizes through ρ_a and must be a morphism since ρ_a it is. Finally,

$$U_{\tau(x)} \circ U_{\tau(y)}(s) = \tilde{e}(\tfrac{x}{2}, y) U_{\tau(x+y)}(s)$$

follows from Proposition 4.20. □

Recall [15], Definition 5.5:

Definition 6.6. Fix $x \in V(E)$, $s \in \iota_b(H^0(b^*L)) \subset \hat{H}^0(L)$ and assume $l \in L(0)^*$, also defined on $\mathbb{L}$. We define the *adelic theta function associated to* s:

$$\theta_s : V(E) \to \overline{\mathbb{L}},$$

in such a way that

$$\theta_s(x) = l\big(\phi_{ab}^{-1}\big(\hat{a}\big(s\big)\big(x_{ab}\big)\big),\ \forall a \mid ab \in J_x,$$

if $\tau(x) = \big(x, (\phi_c)_{c \in J_x}\big)$.

Such a theta function is well defined in view of the following:

Lemma 6.7. *If both ab and cb belong to J_x then*

$$\phi_{ab}^{-1}(\hat{a}(s)(x_{ab})) = \phi_{cb}^{-1}(\hat{c}(s)(x_{cb})) = \phi_{acb}^{-1}(\hat{ac}(s)(x_{acb})).$$

Proof. We prove

$$\phi_{ab}^{-1}(\hat{a}(s)(x_{ab})) = \phi_{acb}^{-1}(\hat{ac}(s)(x_{acb})).$$

Since $\phi_{acb}^{-1}(\hat{ac}(s)(x_{acb})) \in (acb)^*L \mid_0$ and any $\tilde{c}$ acts as the identity on the zero-fiber, we have

$$\begin{aligned} \phi_{acb}^{-1}(\hat{ac}(s)(x_{acb})) &= \tilde{c}(\phi_{acb}^{-1}(\hat{ac}(s)(x_{acb}))) \\ &= \phi_{ab}^{-1}(\tilde{c}(\hat{c} \circ \hat{a}(s)(x_{acb}))) = \phi_{ab}^{-1}(\hat{a}(s)(cx_{acb})) = \phi_{ab}^{-1}(\hat{a}(s)(x_{ab})), \end{aligned}$$

because $\tilde{c}(\hat{c}(s)) = s \circ c$, $\forall s \in H^0((ab)^*L)$ (compare with Definition 6.1). □

We recall some properties of adelic theta functions (see [15], Chap. 5):

Proposition 6.8.

1) $\theta_s(x) = l(U_{\tau(-x)}s)$
2) $\theta_{U_{\tau(y)}s}(x) = \tilde{e}(y, \frac{x}{2})\theta_s(x-y)$

Proof. (1) Assume that $\tau(x) = (x, (\phi_a)_{a\in J_x})$. By Proposition 6.5, we have

$$U_{\tau(x)} \circ U_{\tau(-x)} = \tilde{e}(\tfrac{x}{2}, -x)U_{\tau(0)} = id$$

so $U_{\tau(x)}^{-1} = U_{\tau(-x)}$ with $\tau(-x) = (-x, (\phi_a^{-1})_{a\in J_x})$. In order to ease the notations, if $s \in \iota_b(H^0(b^*L)) \subset \hat{H}^0(L)$ we set $s_{ab} = \hat{a}(s)$, if $ab \in J_x$, so we have:

$$U_{\tau(-x)}s = \phi_{ab}^{-1}s_{ab}(x_{ab}), \text{ and } \theta_s(x) = l(\phi_{ab}^{-1}s_{ab}(x_{ab})) = l(U_{\tau(-x)}s).$$

(2) It follows combining (1) with Proposition 6.5:

$$\theta_{U_{\tau(y)}s}(x) = l(U_{\tau(-x)\circ\tau(y)}s) = \tilde{e}(y, \tfrac{x}{2})l(U_{\tau(y-x)}s) = \tilde{e}(y, \tfrac{x}{2})\theta_s(x-y).$$

□

Theorem 6.9.

1) *As in Theorem 5.3, consider an elliptic curve E with complex multiplication by R, embedded in $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^3)$ by means of the Weierstrass model and assume it is defined over a field $\mathbb{L}$. Denote by σ a field automorphisms of $\mathbb{C}$ fixing $\mathbb{K}\cdot\mathbb{L}$. Consider also a normalization $E \simeq \frac{\mathbb{C}}{\Lambda}$ ([18], §II.1) with Λ a fractional ideal of $\mathbb{K}$ and denote by L the line bundle providing the embedding $\frac{\mathbb{C}}{\Lambda} \to \mathbb{P}^2$. Fix a section $s \in H^0(L)$ corresponding to a line of $\mathbb{P}^2$ also defined on $\mathbb{K}\cdot\mathbb{L}$. Then we have:*

$$\sigma(\theta_s(x)) = \theta_s(\sigma(x)) = \theta_s(l\cdot x),$$

where $l \in \mathbb{A}^*$ is the idèle of $\mathbb{K}$ corresponding to $\sigma : E_{tor} \to E_{tor}$ (according to §2).

2) *Keep notations as in Theorems 5.4 and 5.6, define $l \in \mathbb{A}^*$ as the inverse of the idèle corresponding to σ via Artin map, consider $E(\mathbb{C})$ and $E(\mathbb{C})'$ both embedded $\mathbb{P}^2(\mathbb{C})$ by means of Weierstrass models and define $L := f^*\mathcal{O}_{\mathbb{P}^2}(1)$, $L' := g^*\mathcal{O}_{\mathbb{P}^2}(1)$. Fix sections $s \in H^0(E, L)$ and $s' \in H^0(E', L')$ corresponding to the same line in $\mathbb{P}^2(\mathbb{K})$. Then we have:*

$$\sigma(\theta_s(x)) = \theta_{s'}(\sigma(x)) = \theta_{s'}(l\cdot x).$$

Proof. We are going to prove (2) as the proof of (1) runs the same way. By Theorem 5.6, $\tau'(\sigma(x)) = \sigma(\tau(x)) \in \hat{\mathcal{G}}(L')$, $\forall x \in V(E)$ so $\tau'(\sigma(x)) = (\sigma(x), (\phi_a^\sigma)_{a\in J_x})$. Then we have:

$$\theta_{s'}(\sigma(x)) = l((\phi_a^\sigma)^{-1}(s'_a(\sigma(x_a)))) = l((\phi_a^\sigma)^{-1}(\sigma(s_a(x_a)))),$$

by definition of s and s',

$$l((\phi_a^\sigma)^{-1}(\sigma(s(x_a)))) = l(\sigma((\phi_a)^{-1}(s(x_a))))$$

by definition of $\sigma : \mathcal{G}(a^*L) \to \mathcal{G}(a^*L')$, and finally

$$l(\sigma((\phi_a)^{-1}(s(x_a)))) = \sigma(l((\phi_a)^{-1}(s(x_a)))) = \sigma(\theta_s(x))$$

since l is defined over $\mathbb{K}$. □

We have furthermore the important Corollary (see [15], Proposition 5.6):

Definition 6.10. (Compare with [18], Theorem 8.2) Fix an automorphism of the complex numbers σ, and assume $\sigma\,|_{\mathbb{K}^{ab}} = [t, \mathbb{K}]$, $\sigma\,|_{\mathbb{Q}^{ab}} = [r, \mathbb{Q}]$ via Artin maps:

$$[\,\cdot\,, \mathbb{K}] : \mathbb{A}_{\mathbb{K}}^* \to Gal(\mathbb{K}^{ab}, \mathbb{K}), \quad [\,\cdot\,, \mathbb{Q}] : \mathbb{A}_{\mathbb{Q}}^* \to Gal(\mathbb{Q}^{ab}, \mathbb{Q}).$$

We define:

$$\chi_\sigma = V(E) \times V(E) \to \overline{\mathbb{Q}}^1, \qquad \chi_\sigma(x, y) = \frac{r^{-1}\tilde{e}(x, y)}{\tilde{e}(s^{-1}x, s^{-1}y)}.$$

We state our main result concerning the behaviour of adelic theta functions under automorphisms:

Theorem 6.11.

1) *With notations as in Theorem 6.9 (1), let t and r be ideles of $\mathbb{K}$ and $\mathbb{Q}$ corresponding to σ via Artin maps. Then we have:*

$$\sigma(\theta_{U_{\tau(y)}s}(x)) = \chi_\sigma(y, \tfrac{x}{2})\theta_{U_{\tau(t^{-1}y)}s}(t^{-1}x).$$

2) *With notations as in Theorem 6.9 (2), let t and r be ideles of $\mathbb{K}$ and $\mathbb{Q}$ corresponding to σ via Artin maps. Then we have:*

$$\sigma(\theta_{U_{\tau(y)}s}(x)) = \chi_\sigma(y, \tfrac{x}{2})\theta_{U_{\tau(t^{-1}y)}s'}(t^{-1}x).$$

Proof. We are going to prove (2) as the proof of (1) runs the same way. By Proposition 6.8, $\theta_{U_{\tau(y)}s}(x) = \tilde{e}(y, \frac{x}{2})\theta_s(x-y)$, so Theorem 6.9 implies:

$$\begin{aligned}\sigma(\theta_{U_{\tau(y)}s}(x)) &= \sigma(\tilde{e}(y, \tfrac{x}{2})\theta_s(x-y)) \\ &= r^{-1}\tilde{e}(y, \tfrac{x}{2})\theta_{s'}(\sigma(x-y)) = r^{-1}\tilde{e}(y, \tfrac{x}{2})\theta_{s'}(t^{-1}(x-y)),\end{aligned}$$

and we conclude by applying Proposition 6.8 once again. □

Finally, we may apply our results to the set of arithmetic 1-dimensional $\mathbb{K}$ lattices modulo commensurability and to the groupoid of commensurability modulo dilations.

Theorem 6.12. *Keep notations as in Theorem 6.9 (1) and let $l \in \mathbb{A}^*$ be an idèle of $\mathbb{K}$ corresponding to $\sigma : E_{tor} \to E_{tor}$ (according to §2). Recall the map*

$$\rho_\Lambda : \mathcal{F} \to V(E_\Lambda), \quad \rho_\Lambda \mid_{T(E_{\Lambda'})^*} := V(\alpha_{\Lambda,\Lambda'})^{-1},$$

defined in 3.16 and acting on the set of arithmetic 1-dimensional $\mathbb{K}$ lattices modulo commensurability $\mathcal{F}$. Set

$$\tilde{\theta}_s := \theta_s \circ \rho_\Lambda : \mathcal{F} \longrightarrow \bar{\mathbb{L}}. \tag{6.1}$$

Then we have

$$\sigma(\tilde{\theta}_s(x)) = \tilde{\theta}_s(l \cdot x).$$

Proof. Recall that $\mathcal{F} = \bigcup_{\Lambda'} T(E_{\Lambda'})^*$ by Corollary 3.12. By 3.16 (2) and 6.1, then

$$\tilde{\theta} \mid_{T(E_{\Lambda'})^*} = \theta_s \circ V(\alpha_{\Lambda,\Lambda'})^{-1}.$$

Then it follows from Theorem 6.9 that

$$\begin{aligned}\sigma(\tilde{\theta}_s(x)) &= \sigma(\theta_s(V(\alpha_{\Lambda,\Lambda'})^{-1}(x))) = \theta_s(l \cdot V(\alpha_{\Lambda,\Lambda'})^{-1}(x))) \\ &= \theta_s(V(\alpha_{\Lambda,\Lambda'})^{-1}(l \cdot x))) = \tilde{\theta}_s(l \cdot x), \ \ \forall x \in T(E_{\Lambda'})^*\end{aligned}$$

(the third equality depends on the fact that $V(\alpha_{\Lambda,\Lambda'})^{-1}$ is just the multiplication by an idèle once one has fixed an element in $T(E)^*$). We are done because of $\mathcal{F} = \bigcup_{\Lambda'} T(E_{\Lambda'})^*$. □

Notations 6.13. Like in Theorem 3.14 fix a set of representatives Λ_i of $Cl(R)$, $1 \leq i \leq \sharp Cl(R)$, and define $E_i := \frac{\mathbb{C}}{\Lambda_i}$. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ fixing $\mathbb{K}$ and let s be an idèle of $\mathbb{K}$ corresponding to σ via Artin map. Set moreover $E'_i := \sigma(E_i)$ like in Theorems 5.4 and 5.6, consider $E(\mathbb{C})$ and $E(\mathbb{C})'$ both embedded

$\mathbb{P}^2(\mathbb{C})$ by means of Weierstrass models and define $L_i := f_i^*\mathcal{O}_{\mathbb{P}^2}(1)$, $L'_i := g_i^*\mathcal{O}_{\mathbb{P}^2}(1)$. Fix sections $s_i \in H^0(E_i, L_i)$ and $s'_i \in H^0(E'_i, L'_i)$ corresponding to the same line in $\mathbb{P}^2(\mathbb{K})$. Recall (compare with 3.16) that we defined a map from the groupoid of commensurability modulo dilations $\mathcal{S}$ to $\bigcup_{\Lambda_i} V(E_{\Lambda_i})$:

$$\xi : \mathcal{S} = \bigcup_{(\Lambda_i,\Lambda)\in\mathcal{E}} \mathcal{F}_{\Lambda_i,\Lambda} \longrightarrow \bigcup_{\Lambda_i} V(E_{\Lambda_i}), \quad \xi\,|_{\mathcal{F}_{\Lambda_i,\Lambda}}: (x,y) \to V(\alpha_{\Lambda_i,\Lambda})^{-1}(y)\cdot x.$$

The following Theorem can be proved just as Theorem 6.12

Theorem 6.14. *Keeping notations as above, define*

$$\Theta, \Theta' : \mathcal{S} \to \overline{\mathbb{L}}$$

by

$$\Theta\,|_{\mathcal{F}_{\Lambda_i,\Lambda}} := \theta_{s_i} \circ \xi, \;\; \Theta'\,|_{\mathcal{F}_{\Lambda_i,\Lambda}} := \theta_{s'_i} \circ \xi.$$

Then we have

$$\sigma(\Theta(x,y)) = \Theta'(s^{-1}x, y).$$

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