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Refined node polynomials via long edge graphs

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The generating functions of the Severi degrees for sufficiently ample line bundles on algebraic surfaces are multiplicative in the topological invariants of the surface and the line bundle. Recently new proofs of this fact were given for toric surfaces by Block, Colley, Kennedy and Liu, Osserman, using tropical geometry and in particular the combinatorial tool of long-edged graphs. In the first part of this paper these results are for $\mathbb{P}^2$ and rational ruled surfaces generalised to refined Severi degrees. In the second part of the paper we give a number of mostly conjectural generalisations of this result to singular surfaces, and curves with prescribed multiple points. The formulas involve modular forms and theta functions.

1. Introduction

The Severi degree $n^{d,\delta}$ is the number of δ-nodal degree d curves in the projective plane $\mathbb{P}^2$ through $d(d+3)/2-\delta$ general points. More generally for a pair (S,L) of a complex projective surface and a line bundle on S, the Severi degree $n^{(S,L),\delta}$ counts the number of δ-nodal curves in the linear system $|L|$ passing through $\dim|L|-\delta$ general points. In [DFI] it was conjectured that there are polynomials $n_\delta(d)$ in d, called *node polynomials*, such that $n^{d,\delta}=n_\delta(d)$, for d sufficiently large with respect to δ. In [Göt] it was conjectured that there are universal polynomials $t_\delta(x,y,z,w)$, such that for L sufficiently ample with respect to δ, $n^{(S,L),\delta}$ is obtained by substituting the intersection numbers L^2, LK_S, K_S^2, $\chi(\mathcal{O}_S)$: writing $n_\delta(S,L):=t_\delta(L^2,LK_S,K_S^2,\chi(\mathcal{O}_S))$ we should have $n_\delta(S,L)=n^{(S,L),\delta}$. The conjecture of [Göt] furthermore expresses the generating functions

$$n(d;t):=\sum_{\delta\geq 0}n_\delta(d)t^\delta,\qquad n((S,L);t):=\sum_{\delta\geq 0}n_\delta(S,L)t^\delta$$

in terms of some universal power series. The conjecture says that $n((S,L);t)$ is multiplicative in the parameters, i.e. we can write

$$n((S,L);t)=a_1(t)^{L^2}a_2(t)^{LK_S}a_3(t)^{K_S^2}a_4(t)^{\chi(\mathcal{O}_S)},\tag{1.1}$$

for some power series $a_i(t)\in\mathbb{Q}[[t]]$, and thus, with $b_2(t)=a_1(t)$, $b_1(t)=a_2(t)^{-3}$ and $b_0(t)=a_3(t)^9a_4(t)$ one gets

$$n(d;t)=b_1(t)b_1(t)^db_2(t)^{d^2}\tag{1.2}$$

Furthermore the conjecture gives explicit formulas for $a_1(t)$ and $a_4(t)$ in terms of modular forms.

We will call (1.1) and (1.2) the *multiplicativity* of $n((S,L);t)$ and $n(d;t)$. The Severi degrees of $\mathbb{P}^2$ and toric surfaces can be computed via tropical geometry, by the Mikhalkin correspondence theorem [Mik]. This was used in [FM] to prove the existence of the node polynomials $n_\delta(d)$, using Floor diagrams, which are combinatorial devices for encoding tropical curves. The conjecture of [Göt] was proven in [Tze], [KST], using the methods of complex geometry.

In [BCK] and [L] an alternative proof is given for the multiplicativity of the generating function $n(d;t)$ for the Severi degrees of $\mathbb{P}^2$. The starting point of their approach is the following elementary observation. Let

$$Q(d;t):=\log(n(d;t))=\sum_{\delta\geq 1}Q_\delta(d)t^\delta,$$
$$Q((S,L);t):=\log(n((S,L);t)=\sum_{\delta\geq 1}Q_\delta(S,L)t^\delta$$

be the formal logarithms of $n(d;t)$ and $n((S,L);t)$.

Remark 1. (1) If $Q_\delta(d)$ is a polynomial of degree 2 in d for all $\delta \geq 0$, then one can write $Q(d;t) = q_0(t) + q_1(t)d + q_2(t)d^2$, and, putting $B_i(t) = \exp(q_i(t))$, this gives

$$n(d;t) = \exp(Q(d;t)) = B_0(t)B_1(t)^d B_2(t)^{d^2}.$$

i.e. the multiplicativity (1.2).

(2) By the same argument, if, in dependence of (S, L), each $Q_\delta(S, L)$ is a linear combination of L^2, LK_S, K_S^2, $\chi(\mathcal{O}_S)$, then the multiplicativity (1.1) follows.

In [BCK] and [L] this is used to show the multiplicativity by showing that indeed all $Q_\delta(d)$ are polynomials of degree 2 in d. For this they introduce and employ long edge graphs, a modification of floor diagrams. In [LO], using again long edge graphs and part (2) of Remark 1, this result is extended to a large class of toric surfaces, and a generalisation is given to toric surfaces with rational singularities.

In [GS] and [BG] refined Severi degrees $N^{d,\delta}(y)$, and $N^{(S,L),\delta}(y)$ for (possibly singular) toric surfaces are introduced via tropical geometry. These are symmetric Laurent polynomials in a variable y, and the refined Severi degrees interpolate between the Severi degrees and the Welschinger numbers, i.e. $N^{(S,L),\delta}(1) = n^{(S,L),\delta}$, $N^{(S,L),\delta}(-1) = W^{(S,L),\delta}$. The Welschinger numbers $W^{d,\delta}$ count δ-nodal degree d real curves in $\mathbb{P}^2$ through $d(d+3)/2 - \delta$ real points with suitable signs, and $W^{(S,L),\delta}$ counts real δ-nodal curves in the linear system $|L|$ on a real algebraic surface S through a configuration of $\dim|L| - \delta$ real points. They are closely related to the Welschinger invariants, deformation invariants defined in genus 0. The Welschinger numbers depend in general on the point configuration, but in [Mik] it is shown that, for a so called *subtropical* configuration of points, they coincide with the tropical Welschinger invariants $W_{d,\delta}^{\text{trop}}$, $W_{(S,L),\delta}^{\text{trop}}$, defined via tropical geometry (and these are independent of the tropical configuration of points). In future we will assume that we are dealing with a subtropical configuration of points.

This note applies the methods of [BCK], [L] and [LO] to partially extend their results about Severi degrees to the refined Severi degrees and thus also to the Welschinger numbers.

In [GS] analogues of the conjectures of [Göt] are formulated for the refined Severi degrees.

Conjecture 2. *([BG],[GS]) There are polynomials* $t_\delta(x,y,z,q;y) \in \mathbb{Q}[x,y,z,w,y^{\pm 1}]$ *such that, for a pair* (S,L) *of a smooth toric surface and a* δ*-very ample toric line bundle, we have* $N^{(S,L),\delta}(y) = t_\delta(L^2, LK_S, K_S^2, \chi(\mathcal{O}_S))$.

We denote $N_\delta((S,L);y) := t_\delta(L^2, LK_S, K_S^2, \chi(\mathcal{O}_S))$, and $N_\delta(d;y) := N_\delta((\mathbb{P}^2, dH);y)$. We call the $N_\delta(d;y)$, $N_\delta((S,L);y)$ the refined node polynomials of $\mathbb{P}^2$, respectively S. In the case of $\mathbb{P}^2$, $\mathbb{P}(1,1,m)$ or a Hirzebruch surface Σ_m, a weak form of Conjecture 2 is proven in [BG, Thm. 4.2]. We introduce generating functions for the refined node polynomials. Let

$$N(d;y,t) := \sum_{\delta\geq 0} N_\delta(d;y)t^\delta, \qquad N((S,L);y,t) := \sum_{\delta\geq 0} N_\delta((S,L);y)t^\delta.$$

In [GS] it is again conjectured that $N((S,L);y,t)$ is multiplicative.

Conjecture 3. *([GS]) There exist power series* $A_i \in \mathbb{Q}[y^{\pm 1}][[t]]$, $i = 1,2,3,4$, *such that for all pairs* (S,L) *of a smooth toric surface and a toric line bundle we have*

$$\begin{aligned} N((S,L);y,t) &= A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)}, \\ N(d;y,t) &= A_1^{d^2} A_2^{-3d} A_3^{9} A_4. \end{aligned} \tag{1.3}$$

In [GS] so called refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ are introduced more generally for pairs (S,L) of a smooth projective surface and a line bundle. There is it conjectured that,

(1) if (S,L) is a pair of a smooth toric surface and a toric line bundle, then $\widetilde{N}^{(S,L),\delta}(y) = N^{(S,L),\delta}(y)$ (Conjecture 29),
(2) the $\widetilde{N}^{(S,L),\delta}(y)$ have an explicit multiplicative generating function (Conjecture 32).

Together these two conjectures imply a more precise version of Conjecture 3. They give a more explicit description of the A_i (see Conjecture 32, Remark 33 below). In particular two of these power series are expressed in terms of Jacobi forms.

In the first part of this note we adapt the method of long edge graphs and the proofs of [BCK], [L], [LO] to refined Severi degrees, to prove the multiplicativity also for the $N(d;y,t)$ and a weaker version of multiplicativity for rational ruled surfaces (see Theorem 26, Corollary 27). We combine this with computer calculations of the refined Severi degrees and the Welschinger numbers of $\mathbb{P}^2$ and rational ruled surfaces. This allows to determine the refined node polynomials of $\mathbb{P}^2$ and rational ruled surfaces for low values of δ, confirming the predictions of Conjecture 32, (see Corollary 36), and extending the results of [BG]. Together with our results on multiplicativity of the node polynomials, this can be seen as strong evidence for Conjecture 29 and Conjecture 32.

We then extend the results and conjectures to surfaces with singularities and to curves passing through (smooth or singular) points of S with possibly higher multiplicities. This in particular includes a conjectural generalization (Conjecture 42) of the results of [LO] for surfaces with rational double points to the refined Severi degrees, a remarkably simple conjectural formula (Conjecture 44) for the refined count of curves passing with higher multiplicities through A_1 singularities, and conjectural formulas (Conjecture 50) in terms of theta functions for counting curves with multiple points at smooth points of S.

Using the Caporaso-Harris recursion Theorem 10, we check these conjectures for the projective plane $\mathbb{P}^2$, the weighted projective plane $\mathbb{P}(1,1,m)$ and rational ruled surfaces Σ_m for low values of δ.

These conjectures give rise to the following general conjectural principle (see Remark 35 below): There are universal power series $B_1(y,q)$, $B_2(y,q)$, $B_3(y,q) \in Q[y^{\pm 1}][[q]]$ and to any condition c imposed on the curves at a point of S corresponds a power series $D_c(y,q) \in \mathbb{Q}[y^{\pm 1}][[q]]$, such that the refined count of curves in a sufficiently ample line bundle $|L|$ on S which satisfy conditions $c_1, \ldots, c_s$ is

$$\operatorname*{Coeff}_{q^{L(L-K_S)/2}} \left[B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} B_3(y,q)^{\chi(\mathcal{O}_S)} \prod_{i=1}^{s} D_{c_i}(y,q) \right]. \tag{1.4}$$

Conjecture 3 can be reformulated as a special case of this, with the condition for a curve to pass through a general point corresponding to the Jacobi form $\widetilde{DG}_2(y,q)$ (see Section 4).

2. Refined Severi degrees and long edge graphs

2.1. Refined Severi degrees and Floor diagrams

In [GS], [BG] refined Severi degrees were introduced. We will briefly recall some of the results and definitions.

A lattice polygon $\Delta \subset \mathbb{R}^2$ is a polygon with vertices of integer coordiates. The lattice length of an edge e of Δ is $\#(e \cap \mathbb{Z}^2) - 1$. We denote by $int(\Delta)$, $\partial(\Delta)$ its interior and its boundary. To a convex lattice polygon Δ one can associate a pair $S(\Delta)$, $L(\Delta)$ of a toric surface and a toric line bundle on $S(\Delta)$. The toric surface is defined by the fan given by the outer normal vectors of Δ. We have $\dim H^0(S(\Delta), L(\Delta)) = \#(\Delta \cap \mathbb{Z}^2)$. The arithmetic genus of a curve in $|L(\Delta)|$ is $g(\Delta) = \#(int(\Delta) \cap \mathbb{Z}^2)$. In [BG, Def. 3.8] refined Severi degrees $N^{\Delta,\delta}(y)$ are defined for any convex lattice polygon Δ. They are a count of tropical curves in $\mathbb{R}^2$ satisfying suitable point conditions with multiplicities which are Laurent polynomials in y. We also write $N^{S(\Delta),L(\Delta),\delta}(y) := N^{\Delta,\delta}(y)$. The $N^{\Delta,\delta}(y)$ interpolate between the Severi degrees (at $y = 1$) and the tropical Welschinger numbers (at $y = -1$).

Example 4. In the following we will be concerned only with the following lattice polygons $\Delta_{c,m,d} = \{(x, y) \in (\mathbb{R}_{\geq 0})^2 \mid y \leq d;\ x + my \leq md + c\}$, for $d \geq 0, m \geq 0, c \geq 0$. These are so called h-transversal lattice polygons, i.e. all the slopes of the outer normal vectors of Δ are integers or $\pm\infty$. This covers three different cases:

(1) $d \geq 0$, $m = 1$, $c = 0$. In this case $S(\Delta_{0,1,d}) = \mathbb{P}^2$, $L(\Delta_{0,1,d}) = dH$, with H the hyperplane bundle on $\mathbb{P}^2$.

(2) $d \geq 0$, $m \geq 1$, $c = 0$. In this case $S(\Delta_{0,m,d}) = \mathbb{P}(1, 1, m)$, $L(\Delta_{0,m,d}) = dH$, with H the hyperplane bundle on $\mathbb{P}(1, 1, m)$ with self intersection m.

(3) $d \geq 0$, $m \geq 0, c \geq 0$. In this case $S(\Delta_{c,m,d})$ is the m-th rational ruled surface $\Sigma_m := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(m))$. Let F be the class of the fibre of the ruling and let E be the class of a section with $E^2 = -m$. We denote $H := E + mF$. Then $L(\Delta_{c,m,d}) = cF + dH$.

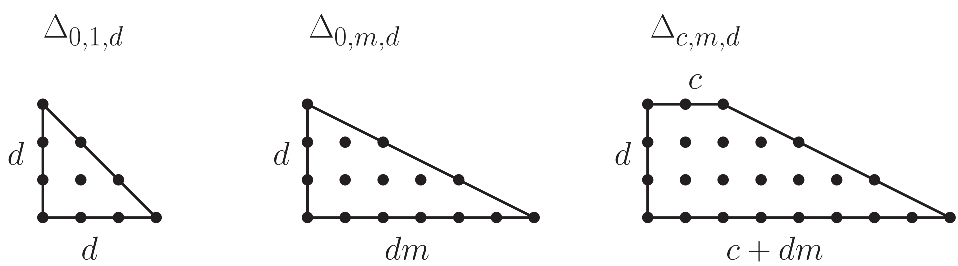

Figure 1: Lattice Polygons for $\mathbb{P}^2, \mathbb{P}(1, 1, m)$ and Σ_m.

Note that in some cases the same lattice polygon corresponds to different pairs of a surface and a line bundle, but by the above the refined Severi degree only depends on Δ.

In [BG] it was also shown that the refined Severi degrees can for h-transversal lattice polygons be computed in terms of Floor diagrams. Here we will not recall the definition of the refined Severi degrees as a count of tropical curves, but directly review them in terms of Floor diagrams which are very closely related to long-edge graphs. We will also restrict our attention to the lattice polygons $\Delta_{c,m,d}$ of Example 4, and thus to $\mathbb{P}^2$, $\mathbb{P}(1,1,m)$ and Σ_m. In the following we fix d, m, c and write $\Delta = \Delta_{c,m,d}$.

Definition 5. A Δ-*floor diagram* $\mathcal{D}$ consists of:

(1) A graph on a vertex set $\{1, \ldots, d\}$, possibly with multiple edges, with edges directed $i \to j$ if $i < j$. Edges e carry a weight $w(e) \in \mathbb{Z}_{>0}$.

(2) A sequence $(s_1, \ldots, s_d)$ of non-negative integers such that $s_1 + \cdots + s_d = c$.

(3) (Divergence Condition) For each vertex j of $\mathcal{D}$, we have

$$\operatorname{div}(j) \stackrel{\text{def}}{=} \sum_{\substack{\text{edges } e \\ j \stackrel{e}{\to} k}} w(e) - \sum_{\substack{\text{edges } e \\ i \stackrel{e}{\to} j}} w(e) \leq m + s_j.$$

We illustrate this and the following definitions with a Δ-floor diagram with Δ corresponding to $(\Sigma_1, 2H + 2F)$.

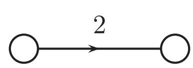

Figure 2: A $\Delta_{2,1,2}$-floor diagram with $(s_1, s_2) = (1, 1)$.

Notation 6. For an integer n we introduce the quantum number $[n]_y$ by

$$[n]_y = \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}} = y^{n-1/2} + y^{n-3/2} + \cdots + y^{-n+3/2} + y^{-n+1/2}.$$

Definition 7. We define the *refined multiplicity* $\mathrm{mult}(\mathcal{D}, y)$ of a floor diagram $\mathcal{D}$ as

$$\mathrm{mult}(\mathcal{D}, y) = \prod_{\text{edges } e} \left([w(e)]_y\right)^2 .$$

By definition $\mathrm{mult}(\mathcal{D}, y)$ is a Laurent polynomial in y with positive integral coefficients.

Definition 8. A *marking* of a floor diagram $\mathcal{D}$ is defined by the following four step process, which we illustrate using the floor diagram of Figure 2:

Step 1: For each vertex j of $\mathcal{D}$ create s_j new indistinguishable vertices and connect them to j with new edges directed towards j.

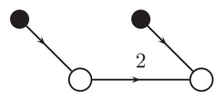

Figure 3: Step 1 applied to the floor diagram of Figure 2.

Step 2: For each vertex j of $\mathcal{D}$ create $m + s_j - div(j)$ new indistinguishable vertices and connect them to j with new edges directed away from j. This makes the divergence of vertex j equal to m.

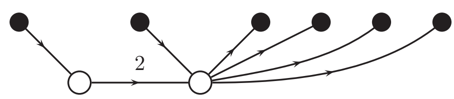

Figure 4: Step 2.

Step 3: Subdivide each edge of the original floor diagram $\mathcal{D}$ into two directed edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Denote the resulting graph $\widetilde{\mathcal{D}}$.

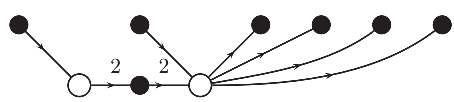

Figure 5: Step 3.

Step 4: Linearly order the vertices of $\widetilde{\mathcal{D}}$ extending the order of the vertices of the original floor diagram $\mathcal{D}$ such that, as before, each edge is directed from a smaller vertex to a larger vertex.

The extended graph $\widetilde{\mathcal{D}}$ together with the linear order on its vertices is called a marked floor diagram or *marking* of the floor diagram $\mathcal{D}$.

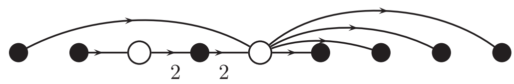

Figure 6: An ordering of the vertices of Figure 5.

The *cogenus* of a marked floor diagram $\widetilde{\mathcal{D}}$ is $\delta(\widetilde{\mathcal{D}}) := \#(\Delta \cap \mathbb{Z}^2) - 1 - k$, where k is the total number of vertices of $\widetilde{\mathcal{D}}$ (this coincides with the cogenus of the tropical curve corresponding to $\widetilde{\mathcal{D}}$, see e.g. [BG2, Def. 4.2]). We count marked floor diagrams up to equivalence. Two markings $\widetilde{\mathcal{D}}_1$, $\widetilde{\mathcal{D}}_2$ of a floor diagram $\mathcal{D}$ are *equivalent* if there exists an automorphism of weighted graphs which preserves the vertices of $\mathcal{D}$ and maps $\widetilde{\mathcal{D}}_1$ to $\widetilde{\mathcal{D}}_2$. We denote $\nu(\mathcal{D})$ the number of markings $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ up to equivalence. Denote by $\mathbf{FD}(\Delta, \delta)$ the set of Δ-floor diagrams $\mathcal{D}$ with cogenus δ.

Theorem 9. *([BG, Thm. 5.7]) For $\Delta = \Delta_{c,m,d}$ as in Example 4 and $\delta \geq 0$, we have*

$$N^{\Delta,\delta}(y) = \sum_{\mathcal{D}\in\mathbf{FD}(\Delta,\delta)} mult(\mathcal{D}; y) \cdot \nu(\mathcal{D}).$$

2.2. Caporaso-Harris type recursion

In [BG] also a Caporaso-Harris type recursion is proven for the refined Severi degrees of $\mathbb{P}^2$, $\mathbb{P}(1,1,m)$ or Σ_m, thus showing that they coincide with the refined Severi degrees as defined in [GS]. This recursion can be easily programmed in Maple, and has been extensively used in the course of this paper to find conjectural generating functions for the refined Severi degrees. In this section let S be $\mathbb{P}^2$, $\mathbb{P}(1,1,m)$ or Σ_m. We first recall the notations.

By a *sequence* we mean a collection $\alpha = (\alpha_1, \alpha_2, \ldots)$ of nonnegative integers, almost all of which are zero. For two sequences α, β we define $|\alpha| = \sum_i \alpha_i$, $I\alpha = \sum_i i\alpha_i$, $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$, and $\binom{\alpha}{\beta} = \prod_i \binom{\alpha_i}{\beta_i}$. We write $\alpha \leq \beta$ to mean $\alpha_i \leq \beta_i$ for all i. We write e_k for the sequence whose k-th element is 1 and all other ones 0. We usually omit trailing zeros. For sequences α, β, and $\delta \geq 0$, let $\gamma(L, \beta, \delta) = \dim |L| - HL + |\beta| - \delta$.

The relative refined Severi degrees $N^{(S,L),\delta}(\alpha, \beta)(y)$ are defined in [BG, Def. 7.2]. Here $N^{(S,L),\delta}(\alpha, \beta)(1)$ is the relative Severi degree, i.e. the number of δ-nodal curves in $|L|$ not containing H, through $\gamma(L, \beta, \delta)$ general points, and with α_k given points of contact of order k with H, and β_k arbitrary points of contact of order k with H. By definition the relative refined Severi degrees contain the refined Severi degrees as special case: $N^{(S,L),\delta}(0,(LH))(y) = N^{(S,L),\delta}(y)$.

Theorem 10. *([BG, Thm. 7.5]) Let L be a line bundle on S and let α, β be sequences with $I\alpha + I\beta = HL$, and let $\delta \geq 0$ be an integer. If $\gamma(L,\beta,\delta) > 0$, then*

$$\begin{aligned}
&N^{(S,L),\delta}(\alpha,\beta)(y)\\
&= \sum_{k:\beta_k>0} [k]_y \cdot N^{(S,L),\delta}(\alpha+e_k,\beta-e_k)(y)\\
&+ \sum_{\alpha',\beta',\delta'} \left(\prod_i [i]_y^{\beta_i'-\beta_i}\right) \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} N^{(S,L-H),\delta'}(\alpha',\beta')(y).
\end{aligned} \tag{2.1}$$

Here the second sum runs through all α',β',δ' satisfying the condition

$$\begin{aligned}
&\alpha' \leq \alpha,\ \ \beta' \geq \beta,\ \ I\alpha' + I\beta' = H(L-H),\\
&\delta' = \delta + g(L-H) - g(L) + |\beta'-\beta| - 1 = \delta - H(L-H) + |\beta'-\beta|.
\end{aligned} \tag{2.2}$$

Initial conditions: *if $\gamma(L,\beta,\delta) = 0$ we have $N^{(S,L),\delta}(\alpha,\beta)(y) = 0$, except for*

$$\begin{aligned}
N^{(\mathbb{P}^2,H),0}((1),(0))(y) &= 1,\\
N^{(\mathbb{P}(1,1,m),H),0}((1),(0))(y) &= 1\\
N^{(\Sigma_m,kF),0}((k),(0))(y) &= 1,
\end{aligned}$$

for all $k \geq 0$.

2.3. Long edge graphs

We review long edge graphs from [BCK], [L], [LO], working in the context of refined invariants. They are very closely related to Floor diagrams. We follow the presentation in [L], [LO]. The arguments used are similar to those of [L], [LO].

Definition 11. A *long edge graph G* is a graph (V,E) with a weight function $w : E \to \mathbb{Z}_{>0}$ satisfying the following.

(1) The vertex set is $V = \mathbb{Z}_{\geq 0}$, the edge set E is finite.

(2) G can have multiple edges, but no loops.

(3) G has no short edges, i.e. no edges connecting i and $i+1$ of weight 1.

An edge connecting i and j with $i < j$ will be denoted $(i \to j)$ (note that there can be more than one such edge). The *length* of an edge $e = (i \to j)$ is $\ell(e) := j - i$.

In the following figures the labelling of the vertices is suppressed, it always is consecutive from left to right starting with 0.

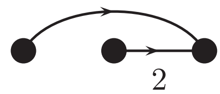

Figure 7: Long Edge Graph.

Definition 12. Given a long edge graph $G = (V, E, w)$, the *refined multiplicity* of G is

$$M(G)(y) := \prod_{e \in E} ([w(e)]_y)^2.$$

The *Severi multiplicity* $m(G)$ and the *Welschinger multiplicity* of G are

$$m(G) := M(G)(1) = \prod_{e \in E} w(e)^2,$$

$$r(G) := M(G)(-1) = \begin{cases} 1 & \text{all } w(e) \text{ are odd,} \\ 0 & \text{otherwise.} \end{cases}$$

The *cogenus* of G is $\delta(G) := \sum_{e \in E}(\ell(e)w(e) - 1)$.

We denote $\operatorname{minv}(G)$ (resp. $\operatorname{maxv}(G)$) the smallest (resp. largest) vertex i of G adjacent to an edge. The *length* of G is $l(G) := \operatorname{maxv}(G) - \operatorname{minv}(G)$.

We denote $G_{(k)}$ the graph obtained by shifting all edges of G to the right by k.

Definition 13. Let G be a long edge graph. For any $j \in \mathbb{Z}_{\geq 0}$ let $\lambda_j(G) := \sum_e w(e)$, for e running through the edges $(i \to k)$ with $i < j \leq k$.

For $\beta = (\beta_0, \ldots, \beta_M)$ a sequence of nonnegative integers, G is called *β-allowable* if $\operatorname{maxv}(G) \leq M+1$ and $\beta_{j-1} \geq \lambda_j(G)$ for all $j = 1, \ldots, M+1$. G is called *strictly β-allowable* if it is β-allowable and furthermore all edges incident to 0 or $M+1$ have weight 1. Also write $\overline{\lambda}_j(G) := \lambda_j(G) - \#\{\text{edges } (j-1 \to j)\}$. G is called *β-semiallowable* if $\operatorname{maxv}(G) \leq M+1$ and $\beta_{j-1} \geq \overline{\lambda}_j(G)$ for all j.

Definition 14. A long edge graph Γ is a *template* if for any vertex $1 \leq i \leq \ell(\Gamma) - 1$ there exists at least one edge $(j \to k)$ with $j < i < k$. A long edge graph G is called a *shifted template* if $G = \Gamma_{(k)}$ for some template $k \in \mathbb{Z}_{\geq 0}$.

Definition 15. Let G be β-allowable for $\beta = (\beta_0, \ldots, \beta_M)$. Define a new graph $\mathrm{ext}_\beta(G)$ by adding $\beta_{j-1} - \lambda_j(G)$ edges of weight 1 connecting $j-1$ and j for all $j = 1, \ldots, M+1$.

A *β-extended ordering* of G is a total ordering on the union of the vertices and edges of $\mathrm{ext}_\beta(G)$, such that

(1) it extends the natural ordering of the vertices $0, 1, 2, \ldots$,

(2) if an edge e connects vertices i and j, then e is between i and j.

Two extended orderings o, o' of G are considered equivalent if there is an automorphism of the edges, permuting only edges connecting the same vertices and of the same weight which sends o to o'.

Example 16. Let $\beta = (\beta_0, \beta_1, \beta_2) = (2, 3, 4)$. The long edge graph G in Figure 7 is strictly β-allowable. The corresponding $\mathrm{ext}_\beta(G)$ is depicted in Figure 8.

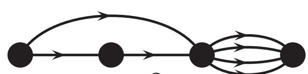

Figure 8: $\mathrm{ext}_\beta(G)$.

Definition 17. For a long edge graph let $P_\beta(G)$ be the number of β-extended orderings of G up to equivalence. Here $P_\beta(G)$ is defined to be 0, if G is not β-allowable. Furthermore let

$$P^s_\beta(G) := \begin{cases} P_\beta(G) & G \text{ strictly } \beta\text{-allowable}, \\ 0 & \text{otherwise}. \end{cases}$$

Definition 18. Given $\beta \in \mathbb{Z}^{M+1}_{\geq 0}$, define

$$N^\delta_\beta(y) := \sum_G M(G) P^s_\beta(G), \quad n^\delta_\beta := \sum_G m(G) P^s_\beta(G),$$
$$W^\delta_\beta := \sum_G r(G) P^s_\beta(G),$$

where the summation is over all long edge graphs G of cogenus δ.

In this paper we will mostly consider the following sequences.

Notation 19. Let $c, d, m \in \mathbb{Z}_{\geq 0}$. We put $s(c, m, d) := (e_0, \dots, e_d)$ with $e_i = c + mi$.

The connection to the refined Severi and tropical Welschinger numbers is given by

Theorem 20. *(1) For the refined Severi degrees of $\mathbb{P}^2$, $\mathbb{P}(1,1,m)$ and Σ_m we have $N^{d,\delta}(y) = N^{\delta}_{s(0,1,d)}(y)$, $N^{(\mathbb{P}(1,1,m),dH),\delta}(y) = N^{\delta}_{s(0,m,d)}(y)$, $N^{(\Sigma_m,cF+dH),\delta}(y) = N^{\delta}_{s(c,m,d)}(y)$.*

(2) For the Severi degrees we have $n^{d,\delta} = n^{\delta}_{s(0,1,d)}$, $n^{(\mathbb{P}(1,1,m),dH),\delta} = n^{\delta}_{s(0,m,d)}$, $n^{(\Sigma_m,cF+dH),\delta} = n^{\delta}_{s(c,m,d)}$.

(3) For the Welschinger numbers we have $W^{d,\delta} = W^{\delta}_{s(0,1,d)}$, $W^{(\mathbb{P}(1,1,m),dH),\delta} = W^{\delta}_{s(0,m,d)}$, $W^{(\Sigma_m,cF+dH),\delta} = W^{\delta}_{s(c,m,d)}$.

Proof. The proof is similar to that of [BCK, Thm. 2.7], we include it for completeness. It is enough to prove (1), because by Definition 18 and Definition 12 we have $n^{\delta}_{\beta} = N^{\delta}_{\beta}(1)$ and $W^{\delta}_{\beta} = N^{\delta}_{\beta}(-1)$, and we know $N^{(S,L),\delta}(1) = n^{(S,L),\delta}$, $N^{(S,L),\delta}(-1) = W^{(S,L),\delta}$ for any pair (S, L) of toric surface and toric line bundle. Furthermore it is enough to prove (1) in case $S = \Sigma_m$, because by Theorem 9 we have $N^{(\mathbb{P}(1,1,m),dH),\delta}(y) = N^{(\Sigma_m,dH),\delta}(y)$.

Let $\Delta = \Delta_{c,m,d}$ for $c, m, d \in \mathbb{Z}_{\geq 0}$. Let $\beta := s(c, m, d)$. We will show that N^{δ}_{β} is equal to the right hand side of Theorem 9, thus finishing the proof. First we show that there is a bijection between Δ-floor diagrams and strictly β-allowable long-edge graphs which respects the cogenus, by showing that both are in bijection to another set of graphs, which for the moment we will call β-graphs. A β-graph is defined precisely like a long edge graph, except that we also allow for short edges $(i \to i+1)$ of weight 1, and we require $\beta_{j-1} = \lambda_j(G)$ for $j = 1, \dots, d+1$, where as before $\lambda_j(G) = \sum_e w(e)$, with e running through the edges $(i \to k)$ with $i < j \leq k$. By definition it is clear that the map $G \mapsto \mathrm{ext}_{\beta}(G)$ defines a bijection from the strictly β-allowable long-edge graphs to the β-graphs, and the inverse is given by removing all short edges $(i \to i+1)$ of weight 1 from a β-graph. We define the cogenus of a β-graph by $\delta(G) = \sum_e (l(e)w(e) - 1)$, with e running over all edges of G. It is obvious that $\delta(G) = \delta(\mathrm{ext}_{\beta}(G))$.

If $\mathcal{D}$ is a Δ-floor diagram, we first perform steps (1) and (2) in Definition 8. Then we identify all vertices we have created in step (1) to a vertex 0, and we identify all vertices we have created in step (2) to a vertex $d+1$, in addition we add vertices $\mathbb{Z}_{\geq d+2}$ to the graph obtained this way. It is easy to see that in this way we get a β-graph $G(\mathcal{D})$. Clearly the map $\mathcal{D} \mapsto G(\mathcal{D})$ is injective, as all the steps are injective, and by definition is is also clear that it is surjective. If $\widetilde{\mathcal{D}}$ is a marking of $\mathcal{D}$, then we see that the total number of vertices of $\widetilde{\mathcal{D}}$ is equal to $d + \#E$ where E is the set of edges of $G(\mathcal{D})$. Defining $M(F) := \prod_e [w(e)]_y^2$ with e running through the edges of the β-graph F, Definitions 12 and 7 imply $\mathrm{mult}(\mathcal{D}) = M(G(\mathcal{D}))$ for a floor diagram $\mathcal{D}$ and $M(G) = M(\mathrm{ext}_\beta(G))$ for a long edge graph G. From the definitions we also see that

$$\begin{aligned}\delta(G(\mathcal{D})) &= \sum_{e\in E} w(e)l(e) - \#E = \sum_{i=1}^{d} \lambda_i(G(\mathcal{D})) - \#E\\ &= \#(\Delta \cap \mathbb{Z}^2) - d - \#E - 1 = \delta(\widetilde{\mathcal{D}}).\end{aligned}$$

Note that the markings of the Δ-floor diagram $\mathcal{D}$ are in bijection with the number of diagrams obtained by putting one vertex on every edge of $G(\mathcal{D})$ and ordering all the vertices of the new diagram, preserving the order of the vertices of $G(\mathcal{D})$, and such that the vertex introduced on an edge $(i \to j)$ lies between i and j. But this number clearly is the same as the number of linear orders on the union of the vertices and edges of $G(\mathcal{D})$, again preserving the order of the vertices and and such that the edge $(i \to j)$ lies between i and j. By definition this is just the number of β-extended orderings of the long edge graph corresponding to $\mathcal{D}$. □

Remark 21. More generally the methods of [BG] will show (using also the notations from [LO]) the following refined version of [LO, Thm. 2.12] (see [BG, Rem. 5.8]).

(1) For any $\delta \geq 0$, any h-transversal lattice polygon the refined Severi degree is

$$N^{\Delta,\delta}(y) = \sum_{(\mathbf{l},\mathbf{r})} N^{\delta-\delta(\mathbf{l},\mathbf{r})}_{\beta(d^t,\mathbf{r}-\mathbf{l})}(y).$$

Here the summation is over all reorderings $\mathbf{l}$ and $\mathbf{r}$ of the multisets of left and right directions of Δ, satisfying $\delta(\mathbf{l},\mathbf{r}) \leq \delta$, $\beta(d^t,\mathbf{r}-\mathbf{l}) \in \mathbb{Z}^{M+1}_{\geq 0}$.

(2) With the same index of summation we have

$$n^{\Delta,\delta} = \sum_{(\mathbf{l},\mathbf{r})} n^{\delta-\delta(\mathbf{l},\mathbf{r})}_{\beta(d^t,\mathbf{r}-\mathbf{l})}, \qquad W^{\Delta,\delta} = \sum_{(\mathbf{l},\mathbf{r})} W^{\delta-\delta(\mathbf{l},\mathbf{r})}_{\beta(d^t,\mathbf{r}-\mathbf{l})},$$

Following [L],[LO], we consider logarithmic versions of $P_\beta(G)$ and $P^s_\beta(G)$,

Definition 22. A *partition* of a long edge graph $G=(V,E,w)$ is a tuple $(G_1,\ldots,G_n)$ of nonempty long edge graphs such that the disjoint union of the (weighted) edge sets of $G_1,\ldots,G_n$ is the (weighted) edge set of G.

For any long edge graph define

$$\Phi_\beta(G) := \sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\sum_{G_1,\ldots,G_n}\prod_{j=1}^{n}P_\beta(G_j),$$
$$\Phi^s_\beta(G) := \sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\sum_{G_1,\ldots,G_n}\prod_{j=1}^{n}P^s_\beta(G_j),$$

where both summations are over the partitions of G.

Let

$$\begin{aligned}\mathcal{N}(\beta,y,t) &:= 1+\sum_{\delta>0}N^\delta_\beta(y)t^\delta,\\ \mathcal{Q}(\beta,y,t) &:= \log(\mathcal{N}(\beta,y,t)) = \sum_{\delta>0}Q^\delta_\beta(y)t^\delta.\end{aligned} \tag{2.3}$$

Then the same arguments as in [LO] show that

$$Q^\delta_\beta(y) = \sum_G M(G)\Phi^s_\beta(G), \tag{2.4}$$

where the summation is again over all long-edge graphs of cogenus δ.

Let $(S,L)=(S(\Delta),L(\Delta))$ be a pair of a toric surface and a toric line bundle, associated to a lattice polygon. Then analogously to (2.3) we set

$$\mathcal{N}((S,L);y,t) := 1+\sum_{\delta>0}N^{(S,L),\delta}(y)t^\delta,$$
$$\mathcal{Q}((S,L);y,t) := \log(\mathcal{N}((S,L);y,t)) = \sum_{\delta>0}Q^{(S,L),\delta}(y)t^\delta.$$

Clearly by Theorem 20 we have

$$Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^\delta_{s(0,m,d)}(y), \qquad Q^{(\Sigma_m,cF+dH),\delta}(y) = Q^\delta_{s(c,m,d)}(y).$$

Definition 23. Let G be a long edge graph. Let $\epsilon_0(G):=1$, if all edges adjacent to $\mathrm{minv}(G)$ have weight 1, and $\epsilon_0(G):=0$ otherwise. Similarly let

$\epsilon_1(G) := 1$, if all edges adjacent to $\mathrm{maxv}(G)$ have weight 1, and $\epsilon_1(G) := 0$ otherwise.

By [L, Lem. 2.15] we have $\Phi^s_\beta(G) = 0$, if G is not a shifted template. On the other hand [L, Cor. 3.5] says that for a template Γ we have

$$\Phi^s_\beta(\Gamma_{(k)}) = \begin{cases} \Phi_\beta(\Gamma_{(k)}) & 1 - \epsilon_0(\Gamma) \le k \le M + \epsilon_1(\Gamma) - \ell(\Gamma) \\ 0 & \text{otherwise.} \end{cases}$$

Together with (2.4), this gives the following refined version of [LO, Cor. 3.6].

Corollary 24. *Let $\beta = (\beta_0, \ldots, \beta_M) \in \mathbb{Z}^{M+1}_{\ge 0}$. Then*

$$Q^\delta_\beta(y) = \sum_\Gamma M(\Gamma) \sum_{k=1-\epsilon_0(\Gamma)}^{M-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_\beta(\Gamma_{(k)}),$$

where the first sum runs over all templates Γ of cogenus δ.

Theorem 25. *([LO, Thm. 3.8]) Let G be a long edge graph. There exists a linear multivariate function $\Phi(G, \beta)$ in β, such that for any β such that G is β-semiallowable, we have $\Phi_\beta(G) = \Phi(G, \beta)$. Furthermore writing $\beta = (\beta_0, \ldots, \beta_M) \in \mathbb{Z}^{M+1}_{\ge 0}$, the linear function $\Phi(G, \beta)$ is a linear combination of the β_i with* $\mathrm{minv}(G) \le i \le \mathrm{maxv}(G)$.

3. Multiplicativity theorems

In this section we will show that the generating functions for the refined Severi degrees on weighted projective spaces and rational ruled surfaces are multiplicative. Following [L] and [BCK], this is done by showing linearity of the logarithm of the generating function, and using Remark 1.

Theorem 26.

(1) Let $c \ge \delta$ and $d \ge \delta$, then $Q^{(\Sigma_m, cF+dH),\delta}(y)$ is a $\mathbb{Q}[y^{\pm 1}]$-linear combination of 1, c, d, cd, m, md, md^2.

(2) In particular if $c \ge \delta$, $d \ge \delta$, then $Q^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}(y)$ is a $\mathbb{Q}[y^{\pm 1}]$-linear combination of 1, $c+d$, cd.

(3) Fix $m \ge 1$, $c \ge 0$. If $d \ge \delta$ then $Q^{(\Sigma_m, dH+cF),\delta}(y)$ is a polynomial of degree 2 in d.

(4) *Fix $m \geq 1$. If $d \geq \delta$, then $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y)$ is a polynomial of degree 2 in d. In particular for $d \geq \delta$, $Q^{d,\delta}(y)$ is a polynomial of degree 2 in d.*

(5) *If $d, m \geq \delta$, then $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y)$ is a $\mathbb{Q}[y^{\pm 1}]$-linear combination of 1, m, d, dm, d^2m.*

Proof. (1) By Corollary 24 and Theorem 20, we have

$$Q^{(\Sigma_m,cF+dH),\delta}(y) = Q^{\delta}_{s(c,m,d)}(y) = \sum_{\Gamma} M(\Gamma) \sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c,m,d)}(\Gamma_{(k)}), \tag{3.1}$$

with Γ running through all templates of cogenus δ.

Let Γ now be a template of cogenus δ, and let k be an integer in $[1-\epsilon_0(\Gamma), d-\ell(\Gamma)+\epsilon_1(\Gamma)]$. Then by definition we get $\Phi_{s(c,m,d)}(\Gamma_{(k)}) = \Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$. On the other hand by [LO, Lem. 4.2] we have $\overline{\lambda}_i(\Gamma) \leq \delta$ for all i. By our assumption we have $c \geq \delta \geq \overline{\lambda}_i(\Gamma)$, thus Γ is $s(c+km, m, \ell(\Gamma)-1)$-semiallowable. Therefore $\Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$ is a linear function in the $c+lm$, $k \leq l \leq k+\ell(\Gamma)-1$, thus it is linear function in c, m and km of the form $\alpha + \beta(c+km) + \gamma m$, with $\alpha, \beta, \gamma \in \mathbb{Q}$.

Let $M_1 := d-\ell(\Gamma)+\epsilon_1(\Gamma)+\epsilon_0(\Gamma)$, $M_2 := d-\ell(\Gamma)+\epsilon_1(\Gamma)-\epsilon_0(\Gamma)+1$. It is easy to see (and was already used in [L]) that for a template Γ of cogenus δ we have $\ell(\Gamma)-\epsilon_1(\Gamma) \leq \delta$, so, by our assumption $d \geq \delta$, we have $M_1 \geq 0$. Recall that for integers $b \geq a-1$ we have the trivial identity

$$\sum_{k=a}^{b} k = \frac{(a+b)(b-a+1)}{2}.$$

Thus we get

$$\sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c,m,d)}(\Gamma_{(k)}) = \sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \big(\alpha + \beta(c+km) + \gamma m\big) = M_1(\alpha + \beta c + \gamma m) + \frac{M_1 M_2}{2} \beta m,$$

which is a $\mathbb{Q}$-linear combination of $1, c, d, cd, m, md, md^2$. Thus the claim follows by (3.1).

(2) By (1) $Q^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}(y)$ is a linear combination of 1, c, d, cd. It is clearly symmetric under exchange of c and d, and thus a linear combination of 1, $c+d$, cd.

(3) By Corollary 24 and Theorem 20,

$$Q^{(\Sigma_m,cF+dH),\delta}(y) = Q^{\delta}_{s(c,m,d)}(y) = \sum_{\Gamma} M(\Gamma) \sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c,m,d)}(\Gamma_{(k)}), \tag{3.2}$$

with Γ running through all templates of cogenus δ.

Let Γ be a template of cogenus δ, and let k be an integer in $[1-\epsilon_0(\Gamma), d-\ell(\Gamma)+\epsilon_1(\Gamma)]$. Then by definition we get $\Phi_{s(c,m,d)}(\Gamma_{(k)}) = \Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$. For a rational number a we denote by $\lceil a \rceil$ the smallest integer bigger or equal to a. We put

$$k_{min} := \max\left(1, \max\left(\left\lceil \frac{\overline{\lambda}_i(\Gamma)}{m}\right\rceil - i + 1 \,\middle|\, i = 1, \ldots, \ell(\Gamma)\right)\right).$$

For $k \geq k_{min}$ we have that $(k+i-1)m + c \geq \overline{\lambda}_i(\Gamma)$ for all i, thus Γ is $s(c+km, m, \ell(\Gamma)-1)$-semiallowable. Thus for $k \geq k_{min}$, we have that $\Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$ is a linear function in the lm, $k \leq l \leq k+\ell(\Gamma)-1$, thus it is a linear function $\alpha + \beta km + \gamma m$, with $\alpha, \beta, \gamma \in \mathbb{Q}$.

By [LO, Lem. 4.2], we have $\overline{\lambda}_i(\Gamma) \leq \delta - \ell(\Gamma) + i + \epsilon_1(\Gamma)$. As $\overline{\lambda}_i(\Gamma) \geq 0$, this implies

$$\left\lceil \frac{\overline{\lambda}_i(\Gamma)}{m}\right\rceil - i + 1 \leq \delta + \epsilon_1(\Gamma) - \ell(\Gamma) + 1$$

for all i. By the inequality $\ell(\Gamma) - \epsilon_1(\Gamma) \leq \delta$, already used in part (1), this implies $k_{min} \leq \delta + \epsilon_1(\Gamma) - \ell(\Gamma) + 1$. By our assumption $d \geq \delta$, we have $d - \ell(\Gamma) + \epsilon_1(\Gamma) - k_{min} + 1 \geq 0$. Therefore the same argument as in (1) shows that the sum

$$\sigma(\Gamma, k_{min}) := \sum_{k=k_{min}}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c,m,d)}(\Gamma_{(k)})$$

is a $\mathbb{Q}$-linear combination of 1, d, m, md, md^2. If we fix m, it is a linear combination of 1, d, d^2. But

$$\sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c+km,m,l(\Gamma)-1)}(\Gamma) = \sigma(\Gamma, k_{min}) + \sum_{k=1-\epsilon_0(\Gamma)}^{k_{min}-1} \Phi_{s(c+km,m,l(\Gamma)-1)}(\Gamma).$$

The second sum is for fixed m just a finite number, thus the claim follows.

(4) As $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{(\Sigma_m,dH),\delta}(y)$, (4) is a special case of (3).

(5) By Corollary 24 and Theorem 20,

(3.3)

$$Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{\delta}_{s(0,m,d)}(y) = \sum_{\Gamma} M(\Gamma) \sum_{k=1}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(0,m,d)}(\Gamma_{(k)}),$$

with Γ running through all templates of cogenus δ. According to Corollary 24, the inner sum starts at $k = 1 - \epsilon_0(\Gamma)$. But Γ is a template and therefore not $s(0,m,d)$-semiallowable. Thus (in case $\epsilon_0(\Gamma) = 1$), the contribution for $k = 0$ vanishes.

We have $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{\delta}_{s(0,m,d)}(y)$, which is computed by the case $c = 0$ of (3.3). If $m \geq \delta$, then $k_{min} = 1$ for all templates Γ of cogenus δ, thus

$$Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{\delta}_{s(0,m,d)}(y) = \sum_{\Gamma} M(\Gamma)\sigma(\Gamma, 1),$$

with Γ again running through the templates of cogenus δ. By (3) this is a $Q[y^{\pm 1}]$-linear combination of 1, d, m, md, md^2. □

By Remark 1 this result easily translates into multiplicativity results for the refined node polynomials of $\mathbb{P}^2$, Σ_m and $\mathbb{P}(1,1,m)$.

Corollary 27. *(1) There are power series $S_0, \ldots, S_6 \in \mathbb{Q}[y^{\pm 1}][[t]]$, such that*

$$N((\Sigma_m, cF + dH); y, t) = S_0 S_1^c S_2^d S_3^{cd} S_4^m S_5^{md} S_6^{md^2}$$

(2) There are power series $P_{m,0}, P_{m,1}, P_{m,2} \in \mathbb{Q}[y^{\pm 1}][[t]]$, such that for all $m \geq 1$

$$N((\mathbb{P}(1,1,m), dH); y, t) = P_{m,0} P_{m,1}^d P_{m,2}^{d^2}.$$

In particular $N(d; y, t) = P_{1,0} P_{1,1}^d P_{1,2}^{d^2}$.

Remark 28. This shows (1.3) for $\mathbb{P}^2$, but falls short of a proof for rational ruled surfaces. We have $(dH)^2 = d^2$, $dHK_{\mathbb{P}^2} = -3d$, on $\mathbb{P}^2$ and $(dH + cF)^2 = md^2 + 2dc$ and $(dH + cF)K_{\Sigma_m} = -md - 2c - 2d$ on Σ_m. Thus, for proving (1.3) for both $\mathbb{P}^2$ and rational ruled surfaces, we would need to show in addition $P_{1,2} = S_3^2 = S_6$, and $P_{1,1}^3 = S_5 = S_1^2 = S_2^2$.

4. Relation to the conjectural generating functions of the refined invariants

In this section we want to state the explicit version of Conjecture 3 from [GS], and prove some partial results towards this conjecture for $\mathbb{P}^2$ and rational ruled surfaces.

In [GS] refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ of pairs (S,L) of a smooth projective surface and a line bundle on S were introduced using complex geometry, using relative Hilbert schemes of points. We write $\widetilde{N}^{d,\delta}(y) := \widetilde{N}^{(\mathbb{P}^2,dH),\delta}(y)$. The $\widetilde{N}^{(S,L),\delta}(y)$ are symmetric Laurent polynomials in a variable y, whose coefficients can be expressed universally (independent of S and L) as polynomials in the four intersection numbers L^2, LK_S K_S^2 and $c_2(S)$ on the surface. For toric surfaces S and sufficiently ample line bundles L the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ and refined Severi degrees $N^{(S,L),\delta}(y)$ are conjectured to agree ([GS, Conj. 80]).

Conjecture 29. *Let (S,L) be a pair of a smooth toric surface and a line bundle on L.*

(1) If L is δ-very ample on S, then $\widetilde{N}^{(S,L),\delta}(y) = N^{(S,L),\delta}(y)$.

(2) $\widetilde{N}^{d,\delta}(y) = N^{d,\delta}(y)$ for $\delta \le 2d-2$.

(3) $\widetilde{N}^{(\mathbb{P}^1\times\mathbb{P}^1,dH+cF),\delta}(y) = N^{(\mathbb{P}^1\times\mathbb{P}^1,dH+cF),\delta}(y)$ for $\delta \le \min(2d,2c)$.

(4) $\widetilde{N}^{(\Sigma_m,dH+cF),\delta}(y) = N^{(\Sigma_m,dH+cF),\delta}(y)$ for $\delta \le \min(2d,c)$.

This conjecture was proven in [BG, Thm. 4.3], for $\mathbb{P}^2$ for $\delta \le 10$, for $\mathbb{P}^1\times\mathbb{P}^1$ for $\delta \le 6$ and for Σ_m for $\delta \le 2$. Corollary 38 below improves these bounds.

Remark 30. Note that by definition, Conjecture 29 implies Conjecture 2. In fact, assuming Conjecture 29, we obtain $N_\delta((S,L);y) = \widetilde{N}^{(S,L),\delta}(y)$ for any pair of a toric surface and a toric line bundle (S,L). In particular $N_\delta(d;y) = \widetilde{N}^{d,\delta}(y)$ for all d,δ, and $N_\delta((\Sigma_m, cF+dH);y) = \widetilde{N}^{(\Sigma_m,dH+cF),\delta}(y)$ for all m,d,c,δ.

In [GS, Conj. 67] also a multiplicative generating function for the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ is conjectured. Together with Conjecture 29 it gives a conjectural generating function for the $N_\delta((S,L);y)$, which we now state.

Notation 31. We start by introducing some notations about quasimodular forms and theta functions, and reviewing some standard facts, which we will use throughout the paper. Modular forms depend on a variable τ in the complex upper half plane, and have a Fourier development in terms of $q := e^{2\pi i\tau}$. We will write them as functions $f(q)$, because we are only interested in the coefficients of their Fourier development. Similarly theta functions will be written as functions $g(y, q)$, for $y = e^{2\pi iz}$, with $z \in \mathbb{C}$ and $q = e^{2\pi i\tau}$. The Eisenstein series

$$G_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n>0}\sum_{d|n} d^{2k-1}q^k$$

are for $2k \geq 4$ modular forms of weight $2k$ on $SL_2(\mathbb{Z})$, whereas $G_2(q)$ is only a quasimodular form of weight 2 on $SL_2(\mathbb{Z})$. The Dirichlet η-function and the discriminant $\Delta(q)$ are

$$\eta(q) := q^{1/24}\prod_{n>0}(1-q^n), \qquad \Delta(q) = \eta(q)^{24} = q\prod_{n>0}(1-q^n)^{24}.$$

The discriminant is a cusp form of weight 12 on $SL_2(\mathbb{Z})$. The operator $D := q\frac{\partial}{\partial q}$ sends (quasi)modular forms of weight $2k$ to quasimodular forms of weight $2k+2$. We denote two of the standard theta functions by

$$\begin{aligned}\theta(y) = \theta(y,q) &:= \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} y^{n+\frac{1}{2}} \\ &= q^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}})\prod_{n>0}(1-q^n)(1-q^n y)(1-q^n/y), \\ \theta_2(y,q) &:= \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2} y^n,\end{aligned}$$

and the theta zero value $\theta_2(q^2) := \theta_2(0, q^2) = \sum_{n\in\mathbb{Z}}(-1)^m q^{n^2} = \frac{\eta(q)^2}{\eta(q^2)}$. Let

$$\begin{aligned}\widetilde{\Delta}(y,q) &:= \frac{\eta(q)^{18}\theta(y)^2}{y-2+y^{-1}} = q\prod_{n=1}^{\infty}(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2, \\ \widetilde{DG}_2(y,q) &:= \sum_{m=1}^{\infty}\sum_{d|m}\frac{m}{d}[d]_y^2 q^m = q + (y+4+y^{-1})q^2 \\ &\qquad + (y^2+2y+6+2y^{-1}+y^{-2})q^3 + O(q^4), \\ D\widetilde{DG}_2(y,q) &= \sum_{m=1}^{\infty}\sum_{d|m}\frac{m^2}{d}[d]_y^2 q^m = q + (2y+8+2y^{-1})q^2 + O(q^3).\end{aligned}$$

Conjecture 32. *([GS]) There exist universal power series $B_1(y,q)$, $B_2(y,q)$ in $\mathbb{Q}[y,y^{-1}][\![q]\!]$, such that for all pairs (S,L) of a smooth projective surface S and a line bundle L on S, we have*

$$\sum_{\delta\geq 0} N_\delta((S,L);y)(\widetilde{DG}_2)^\delta = \frac{(\widetilde{DG}_2/q)^{\chi(L)} B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q)/q^2)^{\chi(\mathcal{O}_S)/2}} \tag{4.1}$$

Conjecture 32 has been proven for surfaces S with numerically trivial canonical bundle in [GS2, Cor. 9].

We give two equivalent reformulations. $\widetilde{DG}_2$ as a power series in q starts with q, let

$$\begin{aligned} g(t) := g(y,t) = t &+ ((-y^2-4y-1)/y)t^2 \\ &+ ((y^4+14y^3+30y^2+14y+1)/y^2)t^3 + O(t^4) \end{aligned}$$

be its compositional inverse. Write $g'(t) := \frac{\partial g}{\partial t}$.

Remark 33. Let $R \in \mathbb{Q}[y^{\pm 1}][[q]]$ be a formal power series. For polynomials $M_\delta((S,L);y) \in \mathbb{Q}[y^{\pm 1}]$ the following three formulas are equivalent:

(1) $$\sum_{\delta\geq 0} M_\delta((S,L);y)(\widetilde{DG}_2)^\delta = \frac{(\widetilde{DG}_2/q)^{\chi(L)} B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q)/q^2)^{\chi(\mathcal{O}_S)/2}} R(y,q)$$

(2) $$\sum_{\delta\geq 0} M_\delta((S,L);y)t^\delta = \frac{(t/g(t))^{\chi(L)} B_1(y,g(t))^{K_S^2}}{B_2(y,g(t))^{-LK_S}} \left(\frac{g(t)g'(t)}{\widetilde{\Delta}(y,g(t))}\right)^{\chi(\mathcal{O}_S)/2} R(y,g(t)),$$

(3) For all $\delta \geq 0$

$$\begin{aligned} &M_\delta((S,L);y) \\ &= \underset{q^{(L^2-LK_S)/2}}{\mathrm{Coeff}} \left[\widetilde{DG}_2(y,q)^{\chi(L)-1-\delta} \frac{B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} D\widetilde{DG}_2(y,q)}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q))^{\chi(\mathcal{O}_S)/2}} R(y,q)\right]. \end{aligned}$$

Proof. (2) is equivalent to (1) by noting that $D\widetilde{DG}_2(y,g(t)) = \frac{g(t)}{g'(t)} \frac{\partial \widetilde{DG}_2(y,g(t))}{\partial t}$ $= \frac{g(t)}{g'(t)}$.

Let A be a commutative ring, and let $f \in A[[q]]$, $h \in q + q^2A[[q]]$. Then we get by the residue formula that

$$f(q) = \sum_{l=0}^{\infty} h(q)^l \operatorname*{Coeff}_{q^0}\left[\frac{f(q)Dh(q)}{h(q)^{l+1}}\right].$$

Applying this with $h(q) = \widetilde{DG}_2$, and using the equality $\chi(L) = \frac{1}{2}(L^2 - LK_S) + \chi(\mathcal{O}_S)$, shows that (1) is equivalent to (3). $\square$

Remark 34. Now let (S, L) be a pair of a smooth toric surface and a toric line bundle. Note that by Remark 30, the conjectures Conjecture 29 Conjecture 32 imply (4.1) with on the left hand side the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ replaced by the refined node polynomials $N_\delta((S, L); y)$. In particular part (2) of Remark 33 shows that Conjecture 32 gives a more explicit version of Conjecture 3: using $\chi(L) = L(L - K_X)/2 + \chi(\mathcal{O}_X)$ it gives

$$A_1 = \left(\frac{t}{g(t)}\right)^{1/2}, \qquad A_4 = \left(\frac{t^2 g'(t)}{g(t)\widetilde{\Delta}(y, g(t))}\right)^{1/2}.$$

We can therefore view our multiplicativity results (Corollary 27) as evidence for Conjecture 32. Conversely, while the results of [GS2] do not apply to toric surfaces, it might be possible to prove Conjecture 32 using e.g. the geometry of Hilbert schemes, giving an alternative approach to multiplicativity for the node polynomials.

Remark 35. We will in the future mostly use the formula (3) of Remark 33. Note that this also has the following interpretation. Write

$$A^{(S,L)}(y, q) := \frac{B_1(y, q)^{K_S^2} B_2(y, q)^{LK_S} D\widetilde{DG}_2(y, q)}{(\widetilde{\Delta}(y, q) \cdot D\widetilde{DG}_2(y, q))^{\chi(\mathcal{O}_S)/2}}.$$

Then, for L sufficiently ample, the refined count of curves in $|L|$ with only nodes as singularities satisfying k general point conditions is

$$\operatorname*{Coeff}_{q^{L(L-K_S)/2}}[\widetilde{DG}_2(y, q)^k A^{(S,L)}(y, q)].$$

Thus it seems natural to expect the following general principle: To each condition c that we can impose at points of S to curves C in $|L|$ (e.g. C passing through a point with given multiplicity), or just to points in S, (e.g. S having a singular point) there corresponds a power series $L_c \in \mathbb{Q}[y^{\pm 1}][[q]]$, such that, for L sufficiently ample, the refined count of curves in $|L|$ on S satisfying conditions $c_1, \ldots, c_n$ is $\mathrm{Coeff}_{q^{L(L-K_S)/2}}[A^{(S,L)}(y,q)\prod_{i=1}^n L_{c_i}]$. According to this principle the power series corresponding to passing through a point of S would be $\widetilde{DG}_2$. In the second half of this paper we will give a number of instances of this principle.

By Remark 30 for $\mathbb{P}^2$ and rational ruled surfaces the conjecture says in particular

$$(4.2)\quad N_\delta(d;y) = \underset{q^{(d^2+3d)/2}}{\mathrm{Coeff}}\left[\widetilde{DG}_2(y,q)^{d(d+3)/2-\delta}\frac{B_1(y,q)^9}{B_2(y,q)^{3d}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\right]$$

$$(4.3)\quad N_\delta((\Sigma_m, cF+dH);y) =$$
$$\underset{q^{(d+1)(c+1+md/2)-1}}{\mathrm{Coeff}}\left[\frac{\widetilde{DG}_2(y,q)^{(d+1)(c+1+md/2)-1-\delta}B_1(y,q)^8}{B_2(y,q)^{2c+(m+2)d}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\right]$$

With the the power series $B_1(y,q)$, $B_2(y,q)$ given in [GS, Conj. 67] modulo q^{11} and in the arXiv version of this paper modulo q^{18}, we have the following corollary.

Corollary 36. *(1) The formula* (4.2) *and Conjecture 29(2) are true for* $\delta \le 17$.

(2) In case $m=0$ *the formula* (4.3) *and Conjecture 29(2) is true for* $\delta \le 12$.

(3) The formula (4.3) *and Conjecture 29(3) are true for all* m *and* $\delta \le 8$.

Proof. (1). Using the Caporaso-Harris recursion, we computed the $N^{d,\delta}(y)$ for $d \le 19$, $\delta \le 19$. This also computes the $Q^{d,\delta}$ for $d \le 19$, $\delta \le 19$. Part (4) of Theorem 26 gives $Q^{d,\delta} = Q_\delta(d)$ for $d \ge \delta$. As $Q_\delta(d;y)$ is a polynomial of degree 2 in d, the computation above determines $Q_\delta(d;y)$ and thus the $N_\delta(y;d)$ for $\delta \le 17$, giving the claim.

(2) and (3). Using again the Caporaso-Harris recursion we computed the $N^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}(y)$ for $c,d \le 13$, $\delta \le 13$. Again this gives the $Q^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}$ for $c,d \le 13$, $\delta \le 13$. By part (2) of Theorem 26 we have that $Q^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta} = Q_\delta((\mathbb{P}^1\times\mathbb{P}^1, cF+dH);y)$ for $c,d \ge \delta$. As $Q_\delta((\mathbb{P}^1\times\mathbb{P}^1, cF+dH);y)$ is a polynomial of bidegree $(1,1)$ in c,d, the computation above determines $Q_\delta((\mathbb{P}^1\times\mathbb{P}^1, cF+dH);y)$ and thus the $N_\delta((\Sigma_0, cF+dH);y)$ for $\delta \le 12$. As

$Q_\delta((\Sigma_m, cF+dH); y)$ is a linear combination of 1, c, cd, m, md, md^2, in order to prove (2) we only need to determine the coefficients of m, md, md^2. For this we can restrict to the case $m=1$, We computed $N^{(\Sigma_1, cF+dH),\delta}(y)$ for $c \le 9$, $d \le 10$. This determines the coefficients of m, md, md^2 of $Q_\delta((\Sigma_m, cF+dH); y)$ for $\delta \le 8$, giving the claim. □

As noted above, the refined Severi degrees $N^{(S,L),\delta}(y)$ specialize at $y=-1$ to the tropical Welschinger numbers $W^{(S,L),\delta}$. We specialize the above conjectures of [GS] to the tropical Welschinger numbers. As the Caporaso-Harris recursion for the tropical Welschinger numbers is computationally much more efficient than that for the refined Severi degrees, the conjectures for the tropical Welschinger numbers can be proven for much higher δ. We denote $W_\delta((S,L)) := N_\delta((S,L); -1)$, $W_\delta(d) := N_\delta(d; -1)$.

Let $\eta(q) := q^{1/24}\prod_{n>0}(1-q^n)$ the Dirichlet eta function, $G_2(q) := -\frac{1}{24} + \sum_{n>0}\sum_{d|n} dq^n$ be the Eisenstein series, and write

$$\overline{G}_2(q) := \widetilde{DG}_2(-1,q) = G_2(q) - G_2(q^2) = \sum_{n>0}\left(\sum_{d|n,\ d \text{ odd}} \frac{n}{d}\right) q^n.$$

We note that $\widetilde{DG_2}(-1,q) = \overline{G}_2(q)$, and $\widetilde{\Delta}(-1,q) = \eta(q)^{16}\eta(q^2)^4$. We write $\overline{B}_1(q) := B_1(-1,q)$, $\overline{B}_2(q) := B_2(-1,q)$. Conjecture 29 specializes to the following (see also [GS]).

Conjecture 37. *For the stable Welschinger numbers we have*

$$W_\delta(d) = \operatorname*{Coeff}_{q^{(d^2+3d)/2}}\left[\overline{G}_2(q)^{d(d+3)/2-\delta}\frac{\overline{B}_1(q)^9(D\overline{G}_2(q))^{1/2}}{\overline{B}_2(q)^{3d}\eta(q)^8\eta(q^2)^2}\right], \tag{4.4}$$

$$\begin{aligned} &W_\delta((\Sigma_m, cF+dH)) \\ &= \operatorname*{Coeff}_{q^{((d+1)(c+1+md/2)-1}}\left[\frac{\overline{G}_2(q)^{(d+1)(c+1+md/2)-1-\delta}\overline{B}_1(q)^8(D\overline{G}_2(q))^{1/2}}{\overline{B}_2(q)^{2c+(m+2)d}\eta(q)^8\eta(q^2)^2}\right]\end{aligned} \tag{4.5}$$

With $\overline{B}_1(q)$, $\overline{B}_2(q)$ given below modulo q^{31} we have the following corollary.

Corollary 38. *(1) The formula* (4.4) *is true for* $\delta \le 30$. *Furthermore for* $\delta \le 30$ *and* $d \ge \delta/3+1$ *we have* $W^{d,\delta} = W_\delta(d)$.

(2) On $\mathbb{P}^1 \times \mathbb{P}^1$ *the formula* (4.5) *is true for* $\delta \le 20$. *Furthermore for* $\delta \le 20$ *and* $\delta \le \min(20, 3c, 3d)$, *we have* $W^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta} = W_\delta(\mathbb{P}_1 \times \mathbb{P}_1, cF+dH)$.

(3) *For $m > 0$, the formula* (4.5) *is true for $\delta \le 11$. Furthermore for $\delta \le \min(11, 3d, c)$ we have $W^{(\Sigma_m, cF+dH),\delta} = W_\delta(\Sigma_m, cF + dH)$.*

Proof. (1) Using the Caporaso-Harris recursion, we computed to the $W^{d,\delta}$ for $d \le 32$, $\delta \le 33$. This also computes the $Q^{d,\delta}(-1)$ for $d \le 32$, $\delta \le 33$. The same argument as in the proof of Corollary 36 shows (1). Using again the Caporaso-Harris recursion we computed the $W^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}$ for $c, d \le 21, \delta \le 22$, and computed $W^{(\Sigma_1, cF+dH),\delta}(y)$ for $c, d, \delta \le 13$. The same argument as in the proof of Corollary 36 gives (2) and (3). $\square$

$$
\begin{aligned}
\overline{B}_1(q) = {} & 1 - q - q^2 - q^3 + 3q^4 + q^5 - 22q^6 + 67q^7 - 42q^8 - 319q^9 \\
& + 1207q^{10} - 1409q^{11} - 3916q^{12} + 20871q^{13} - 34984q^{14} - 37195q^{15} \\
& + 343984q^{16} - 760804q^{17} - 81881q^{18} + 5390386q^{19} - 15355174q^{20} \\
& + 8697631q^{21} + 79048885q^{22} - 293748773q^{23} + 329255395q^{24} \\
& + 1041894580q^{25} - 5367429980q^{26} + 8780479642q^{27} \\
& + 10991380947q^{28} - 93690763368q^{29} + 203324385877q^{30} + O(q^{31}), \\
\overline{B}_2(q) = {} & 1 + q + 2q^2 - q^3 + 4q^4 + 2q^5 - 11q^6 + 24q^7 + 4q^8 - 122q^9 \\
& + 313q^{10} - 162q^{11} - 1314q^{12} + 4532q^{13} - 4746q^{14} - 13943q^{15} \\
& + 68000q^{16} - 105786q^{17} - 124968q^{18} + 1025182q^{19} - 2139668q^{20} \\
& - 443505q^{21} + 15157596q^{22} - 41007212q^{23} + 19514894q^{24} \\
& + 214218876q^{25} - 755331892q^{26} + 780656576q^{27} + 2776494907q^{28} \\
& - 13420432234q^{29} + 20749875130q^{30} + O(q^{31}).
\end{aligned}
$$

5. Correction term for singularities

In this section we want to extend the above results and conjectures to surfaces with singularities. This section is partially motivated by the paper [LO], where this question is studied for the non-refined invariants for toric surfaces with rational double points. We have conjectured above and given evidence that there exist generating functions for the refined node polynomials on smooth toric surfaces S, of the form $A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)}$ for universal power series $A_i \in \mathbb{Q}[y^{\pm1}][[q]]$. It seems natural to conjecture that this extends to singular surfaces in the following form: for every analytic type of singularities c there is a universal power series $F_c(y, q)$ and the generating function for a singular surface S is $A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)} \prod_c F_c^{n_c}$, where n_c is the number of singularities of S of type c. For the case of toric surfaces given by h-transversal lattice polygons with only rational double points this problem has been solved in [LO] for the (non-refined) Severi degrees.

We start out by formulating a conjecture for general singular toric surfaces, and then give more precise results for specific singularities. For rational double points we conjecture that somewhat surprisingly the power series $F_c(y, q)$ is independent of y. In particular this says that the correction factor for A_n-singularities, determined in [LO] for the Severi degrees, is the same for the Severi degrees and the tropical Welschinger invariants.

Now let S be a normal toric surface. We want to formulate a conjecture about the refined Severi degrees $N^{(S,L),\delta}(y)$. Note that the curves counted in $N^{(S,L),\delta}(y)$ are not required to pass through any of the singular points of S. One can also reformulate the same conjecture in terms of the minimal resolution of S, i.e. a resolution $\pi : \widehat{S} \to S$, which contains no (-1) curves in the fibres of π.

Conjecture 39. *For every analytic type of singularities c there are formal power series $F_c \in \mathbb{Q}[y^{\pm 1}][[q]]$, $\widehat{F}_c \in \mathbb{Q}[y^{\pm 1}][[q]]$ such that the following hold. Let (S, L) be a pair of a projective toric surface and a toric line bundle on S. Let $\widehat{S}$ be a minimal toric resolution of S and denote by L also the pullback of L to $\widehat{S}$. Define $N^{(\widehat{S},L),\delta}(y) := N^{(S,L),\delta}(y)$. If L is δ-very ample on S, then*

$$(5.1)\quad N^{(S,L),\delta}(y) = \operatorname*{Coeff}_{q^{L(L-K_S)/2}}\left[\frac{\widetilde{DG}_2(y,q)^{\chi(L)-1-\delta}B_1(y,q)^{K_S^2}}{B_2(y,q)^{-LK_S}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\prod_c F_c(y,q)^{n_c}\right],$$

$$(5.2)\quad N^{(\widehat{S},L),\delta}(y) = \operatorname*{Coeff}_{q^{L(L-K_{\widehat{S}})/2}}\left[\frac{\widetilde{DG}_2(y,q)^{\chi(L)-1-\delta}B_1(y,q)^{K_{\widehat{S}}^2}}{B_2(y,q)^{-LK_{\widehat{S}}}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\prod_c \widehat{F}_c(y,q)^{n_c}\right].$$

Here c runs through the analytic types of singularities of S, and n_c is the number of singularities of S of type c.

We can see that the two formulas formulas (5.1), (5.2) are equivalent. Note that $LK_S = LK_{\widehat{S}}$. On the other hand it is easy to see that $K_{\widehat{S}}^2 = K_S^2 - \sum_c n_c e_c$ where e_c is a rational number depending only on the singularity type c. Thus the two formulas are equivalent, via the identification

$$\widehat{F}_c(y,q) = F_c(y,q)B_1(y,q)^{e_c}.$$

It turns out that the power series $\widehat{F}_c(y,q)$ are usually simpler, so we will restrict our attention to them. Note that for a rational double point c we have $e_c = 0$ and thus $F_c = \widehat{F}_c$.

We give a slightly more precise version of the conjecture for a weighted projective space $\mathbb{P}(1,1,m)$ and its minimal resolution Σ_m, and prove some special cases of it. In this case the exceptional divisor is the section E with self intersection $-m$. The weighted projective space $\mathbb{P}(1,1,m)$ has one singularity of type $\frac{1}{m}(1,1)$, i.e. the cyclic quotient of $\mathbb{C}^2$ by the m-th roots of unity μ_m acting by $\epsilon(x,y) = (\epsilon x, \epsilon y)$. We write c_m for this singularity. It is elementary to see that

$$\begin{aligned}
&K_{\Sigma_m} = -2H + (m-2)F = -\frac{m+2}{m}H - \frac{m-2}{m}E,\\
&K_{\mathbb{P}(1,1,m)} = -\frac{m+2}{m}H, \quad e_{c_m} = \frac{(m-2)^2}{m}, \quad K^2_{\Sigma_m} = 8,\\
&-dHK_{\Sigma_m} = d(m+2), \quad \chi(\Sigma_m, dH) = (md+2)(d+1)/2.
\end{aligned}$$

Conjecture 40. *If $\delta \leq 2d-1$, then*

$$N^{(\Sigma_m,dH),\delta}(y) = \operatorname*{Coeff}_{q^{\frac{m}{2}d^2+(\frac{m}{2}+1)d}}\left[\frac{\widetilde{DG}_2(y,q)^{\frac{m}{2}d^2+(\frac{m}{2}+1)d-\delta}B_1(y,q)^8}{B_2(y,q)^{d(m+2)}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\widehat{F}_{c_m}(y,q)\right]. \tag{5.3}$$

Furthermore we have for $m \geq 2$

$$\begin{aligned}
\widehat{F}_{c_m} = 1 - mq &+ ((m-2)y + (m^2/2 + 3m/2 - 5) + (m-2)y^{-1})q^2\\
&- \big((m^2+5m-14)y + (m^3+9m^2+44m-132)/6\\
&+ (m^2+5m-14)y^{-1}\big)q^3 + O(q^4),
\end{aligned}$$

and

$$\begin{aligned}
\widehat{F}_{c_2} &= \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} = 1 - 2q + 2q^4 - 2q^9 + \cdots,\\
\widehat{F}_{c_3} &= 1 - 3q + (y + 4 + y^{-1})q^2 - (10y + 18 + 10y^{-1})q^3\\
&\quad + ((6y^2 + 70y + 115 + 70y^{-1} + 6y^{-2})q^4\\
&\quad - ((y^3 + 94y^2 + 473y + 721y + 473y^{-1} + 94y^{-2} + y^{-3})q^5 + O(q^6)\\
\widehat{F}_{c_4} &= 1 - 4q + (2y + 9 + 2y^{-1})q^2 - (22y + 42 + 22y^{-1})q^3\\
&\quad + ((14y^2 + 164y + 273 + 164y^{-1} + 14y^{-2})q^4 + O(q^5).
\end{aligned}$$

Proposition 41. *Let $\delta_2 = 8$, $\delta_3 = 5$, $\delta_4 = 4$, $\delta_m = 3$ for $m \geq 5$. Then* (5.3) *is correct for $m \geq 2$ and $\delta \leq \min(\delta_m, d)$.*

Proof. Using the Caporaso-Harris recursion we computed $N^{(\Sigma_m, dH),\delta}$ for $2 \leq m \leq 4$, $\delta \leq \delta_m$ and $d \leq d_m$ with $d_2 = 10$, $d_3 = 7$, $d_4 = 6$. We find that in this range (5.3) holds for $\delta \leq \min(2d-1, \delta_m)$. By part (3) of Theorem 26 we have that $Q^{(\Sigma_m, dH),\delta}$ is a polynomial of degree 2 in d for $d \geq \delta$. By the computation we know this polynomial in the following cases: $(m = 2,\ \delta \leq 8)$, $(m = 3,\ \delta \leq 5)$, $(m = 4,\ \delta \leq 4)$. This shows the result for $m = 2, 3, 4$. Finally by part (5) of Theorem 26 we have that $Q^{(\Sigma_m, dH),\delta}(y)$ is for $d, m \geq \delta$ a polynomial in d and m of degree 2 in d and 1 in m. By the above we know this polynomial as a polynomial in d for $\delta = 0, 1, 2, 3$ and $m = 3, 4$. This determines it and thus also $Q^{(\Sigma_m, dH),\delta}(y)$ and therefore also $N^{(\Sigma_m, dH),\delta}(y)$, for $\delta = 0, 1, 2, 3$ and $d, m \geq \delta$. The result follows. □

The non-refined Severi degrees for toric surfaces with only rational double points given by h transversal lattice polygons have been studied in [LO]. The only rational double points which can occur in this case are A_n singularities. For such surfaces they prove the analogue of Conjecture 39 for $y = 1$ with precise bounds. Furthermore they show

$$F_{a_n}(1, q) = \frac{\eta(q)^{n+1}}{\eta(q^{n+1})} = \prod_{k>0} \frac{(1-q^k)^{n+1}}{1-q^{(n+1)k}},$$

where we denote $F_{a_n}(y, q)$ the power series $F_c(y, q)$ for c an A_n singularity. We conjecture that the same result holds also for the refined Severi degrees with the $F_{a_n}(y, q)$ independent of y.

Conjecture 42. *Let S be projective normal toric surface with only rational double points, more precisely with n_k singularities of type A_k for all k (with n_k only nonzero for finitely many k). If L is δ-very ample on S, then*

$$N^{(S,L),\delta}(y) = \underset{q^{L(L-K_S)/2}}{\operatorname{Coeff}} \Bigg[\widetilde{DG}_2(y, q)^{\chi(L)-1-\delta} \frac{B_1(y, q)^{K_{\widehat{S}}^2}}{B_2(y, q)^{-LK_{\widehat{S}}}} \left(\frac{D\widetilde{DG}_2(y, q)}{\widetilde{\Delta}(y, q)} \right)^{1/2} \prod_k \left(\frac{\eta(q)^{k+1}}{\eta(q^{k+1})} \right)^{n_k} \Bigg].$$

Remark 43. (1) $\mathbb{P}(1, 1, 2)$ has an A_1 singularity, and as we saw Σ_2 is a resolution of $\mathbb{P}(1, 1, 2)$. It is standard that $\theta_2(2\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)}$. Thus, for $\mathbb{P}(1,1,2)$, Conjecture 42 is a special case of Conjecture 40, and Proposition 41 gives evidence for it.

(2) We also used a version of the Caporaso-Harris recursion for $\mathbb{P}(1,2,3)$. With the line bundle dH with d small for H the hyperplane bundle. $\mathbb{P}(1,2,3)$ has one A_1 and one A_2 singularity, also in this case Conjecture 42 is confirmed in the realm considered.

(3) Note that the conjecture that the $F_{a_n}(y,q)$ are independent of y says in particular that the correction factor for the A_n singularities is the same for Severi degrees and tropical Welschinger invariants.

We want to generalise this conjecture in another direction. Let S be a singular toric surface with singular points $p_1,\ldots,p_r$ and a minimal toric resolution $\widehat{S}$ with exceptional divisors $E_1,\ldots,E_r$. Let L be a toric line bundle on S. We have seen that $N^{(\widehat{S},L),\delta}(y)=N^{(S,L),\delta}(y)$ is a refined count of δ-nodal curves on S, which are not required to pass through the singular locus of S. In a similar way we can interpret $N^{(\widehat{S},L-k_1E_1-\cdots-k_rE_r),\delta}(y)$ as a refined count of curves in $|L|$ on S which pass through the singular points p_i with multiplicity $-k_iE_i^2$. This even makes sense if L is only a class of Weil divisors on S, the k_i are not necessarily integral but $L-k_1E_1-\cdots-k_rE_r$ is a Cartier divisor on $\widehat{S}$. In this case the curves we count on S are Weil divisors.

Here we will consider this question only in the case that S has only A_1 singularities. Denote $\eta(q)=q^{1/24}\prod_{n>0}(1-q^n)$ the Dirichlet eta function. Let $\theta_2(q):=\sum_{n\in\mathbb{Z}}(-1)^nq^{n^2/2}$ be one of the standard theta functions. Recall the Jacobi triple product formula

$$\eta(q^2)^3=q^{1/4}\sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)}.$$

We define functions $f_l(q)$, for $l\in\mathbb{Z}_{\geq 0}$ by

$$\begin{aligned}f_{2k}(q)&=\frac{(-1)^k}{(2k)!}\sum_{n\in\mathbb{Z}}(-1)^n\left(\prod_{i=0}^{k-1}(n^2-i^2)\right)q^{n^2}\\&=\frac{(-1)^k}{(2k)!}\left(\prod_{i=0}^{k-1}(D-i^2)\right)\theta_2(q^2)\end{aligned}\tag{5.4}$$

$$f_{2k+1}(q) = \frac{(-1)^k}{(2k+1)!}\sum_{n\in\mathbb{Z}}(-1)^n(n+1/2)$$
$$\left(\prod_{i=0}^{k-1}((n+1/2)^2-(i+1/2)^2)\right)q^{(n+1/2)^2}$$
$$= \frac{(-1)^k}{(2k+1)!}\left(\prod_{i=0}^{k-1}(D-(i+1/2)^2)\right)\eta(q^2)^3.$$

Here as before we denote $D = q\frac{d}{dq}$. In particular we have

$$f_0(q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}, \quad f_1(q) = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n+1/2)^2},$$
$$f_2(q) = \sum_{n>0}(-1)^{n-1}n^2q^{n^2}.$$

We write $N^{(S,L),\delta}_{[k_1,\ldots,k_r]} := N^{(\widehat{S},L-k_1E_1-\cdots-k_rE_r),\delta}(y)$, to stress that we view it as a count of curves on S with prescribed multiplicities at the A_1-singularities.

Conjecture 44. *Let S be a toric surface with only A_1 singularities $p_1,\ldots,p_r$. Fix $k_1,\ldots,k_r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Let $\delta\geq 0$. Let L be a Weil divisor on S, such that $L-\sum_i k_iE_i$ is a Cartier divisor on $\widehat{S}$, which is δ-very ample on any irreducible curve in $\widehat{S}$ not contained in $E_1\cup\cdots\cup E_r$. Then*

$$N^{(S,L),\delta}_{[k_1,\ldots,k_r]}(y) = \operatorname*{Coeff}_{q^{L(L-K_S)/2}}\left[\frac{\widetilde{DG}_2(y,q)^{L(L-K_S)/2-\sum_i k_i^2-\delta}B_1(y,q)^{K_S^2}}{B_2(y,q)^{LK_S}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\prod_{i=1}^r f_{2k_i}(q)\right]. \tag{5.5}$$

Thus we claim that the correction factors for points of multiplicity k at A_1 singularities of S are given by the quasimodular forms $f_k(q)$.

Equivalently we can look at the same question on the blowup $\widehat{S}$. Write $\widehat{L} := L-k_1E_1-\cdots-k_rE_r$ and

$$\overline{f}_k(q) = \frac{f_k(q)}{q^{k^2/4}}, \quad k\in\mathbb{Z}_{\geq 0},$$

then (with the same assumptions) (5.5) is clearly equivalent to

$$N^{(\widehat{S},\widehat{L}),\delta}(y) = \operatorname*{Coeff}_{q^{\widehat{L}(\widehat{L}-K_{\widehat{S}})/2}} \left[\frac{\widetilde{DG}_2(y,q)^{\chi(\widehat{L})-1-\delta} B_1(y,q)^{K_{\widehat{S}}^2}}{B_2(y,q)^{\widehat{L}K_{\widehat{S}}}} \left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)} \right)^{1/2} \prod_{i=1}^{r} \overline{f}_{2k_i}(q) \right]. \tag{5.6}$$

In other words, the correction factors for $\widehat{L}$ not being sufficiently ample on $\widehat{S}$ are the $\overline{f}_l(q)$.

Remark 45. Under the assumptions of the conjecture, if the k_i are sufficiently large with respect to δ, then $\widehat{L}$ will be δ-very ample on $\widehat{S}$. This means by Conjecture 32 that for large l the correction factor $\overline{f}_l(q)$ should be 1 modulo some high power of q. In fact we find the following.

For $l \in \mathbb{Z}_{>0}$ we can rewrite

$$\overline{f}_l(q) = \sum_{m\geq 0} (-1)^m \frac{2m+l}{m+l} \binom{m+l}{l} q^{m(m+l)}.$$

In particular $\overline{f}_l(q) \equiv 1 \mod q^{l+1}$.

Proof. First we deal with the case l even. Note that

$$\prod_{i=0}^{k-1} (n^2 - i^2) = n \prod_{i=-k+1}^{k-1} (n+i).$$

Thus we get for $k > 0$

$$\begin{aligned}
\overline{f}_{2k}(q) &= \frac{(-1)^k}{(2k)!} \sum_{n\in\mathbb{Z}} (-1)^n \prod_{i=0}^{k-1} (n^2 - i^2) q^{n^2-k^2} \\
&= \sum_{n\geq k} (-1)^{n-k} \frac{2n}{2k} \binom{n+k-1}{2k-1} q^{n^2-k^2},
\end{aligned}$$

where we also have used that $\binom{n+k-1}{2k-1} = 0$ for $n < k$. Finally put $m = n - k$, so that $\frac{2n}{2k}\binom{n+k-1}{2k-1} = \frac{2m+2k}{m+2k}\binom{m+2k}{2k}$ and $n^2 - k^2 = m(m+2k)$.

The case l odd is similar. Note that

$$\prod_{i=0}^{k-1}((n+1/2)^2-(i+1/2)^2)=\prod_{i=-k+1}^{k}(n+i).$$

Thus we get

$$\begin{aligned}
&\overline{f}_{2k+1}(q)\\
&=\frac{(-1)^k}{(2k+1)!}\\
&\quad\sum_{n\geq 0}(-1)^n(2n+1)\left(\prod_{i=0}^{k-1}((n+1/2)^2-(i+1/2)^2)\right)q^{(n+1/2)^2-(k+1/2)^2}\\
&=\sum_{n\geq 0}(-1)^{n-k}\frac{2n+1}{2k+1}\binom{n+k}{2k}q^{(n+1/2)^2-(k+1/2)^2},
\end{aligned}$$

and put again $m:=n-k$. $\square$

Remark 46. It is again remarkable that the correction factors $f_k(q)$ are independent of the variable y. In particular this means again that the correction factor is the same for the Severi degrees and for the tropical Welschinger number.

We specialise the conjecture to case that S is the weighted projective space $\mathbb{P}(1,1,2)$ with the resolution Σ_2 with more precise bounds for the validity. Note that

$$\begin{aligned}
&\chi(\Sigma_2,dH-kE)=(d+1)^2-k^2,\\
&(dH-kE)K_{\Sigma_2}=(dH-kE)(-2H)=-4d,\quad K_{\Sigma_2}^2=8.
\end{aligned}$$

Conjecture 47. *Let $d,k\in\frac{1}{2}\mathbb{Z}$ with $d-k\in\mathbb{Z}$. Then for $\delta\leq 2(d-k)+1$, we have*

$$\begin{aligned}
N^{(\Sigma_2,dH-kE),\delta}(y)=\operatorname*{Coeff}_{q^{d^2+2d-k^2}}\Bigg[&\frac{\widetilde{DG}_2(y,q)^{d^2+2d-k^2-\delta}B_1(y,q)^8}{B_2(y,q)^{4d}}\\
&\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\overline{f}_{2k}(q)\Bigg].
\end{aligned}\tag{5.7}$$

Proposition 48. *(1) Conjecture 47 is true for all d, all $k\leq 5$ and $\delta\leq 4$.*

(2) The equation (5.7) *holds for all* $d, k \geq 0$ *with* $\delta \leq d - k$ *and* $\delta \leq 4$.

Proof. We compute $N^{(\Sigma_2,dH+cF),\delta}(y) = N^{(\Sigma_2,(d+c/2)H-c/2E),\delta}(y)$ for $\delta \leq 8$, $d \leq 6$ and $c \leq 5$, using the Caporaso-Harris recursion. We find in this realm that $N^{(\Sigma_2,(dH-kE),\delta}(y)$ is equal to the right hand side of Conjecture 47 for $\delta \leq 2(d-k)+1$. By Theorem 26 $Q^{(\Sigma_2,dH+cF),\delta}(y)$ is for fixed $c \geq 0$ and for $d \geq \delta$ a polynomial of degree 2 in d. Thus the above computations determine this polynomial for $\delta \leq 4$, and $c \leq 5$. On the other hand in dependence of c and d we have that $Q^{(\Sigma_2,dH+cF),\delta}(y)$ is for $c, d \geq \delta$ a polynomial in c and d of degree 2 in d and 1 in c. By the above we know this polynomial as a polynomial in d for $c = 4$ and $c = 5$. Thus it is determined and the claim follows. □

6. Counting curves with prescribed multiple points

Let S be a smooth projective surface, let $p_1, \ldots, p_r$ be general points on S, and let $\widehat{S}$ be the blowup of S in the p_i with exceptional divisors E_i. Let $n_1, \ldots, n_r \in \mathbb{Z}_{\geq 1}$. Let L be a sufficiently ample line bundle on S, and denote by the same letter its pullback to $\widehat{S}$. Note that $N^{(\widehat{S},L-\sum_i n_iE_i),\delta}(1)$ counts the complex curves on S in $|L|$ with points of multiplicity n_i in p_i which have in addition δ nodes and pass through $\dim(|L - \sum_i n_iE_i)|) - \delta$ general points of S. If L is sufficiently ample, then the multiple points at the p_i impose $\sum_i \binom{n_i+1}{2}$ independent conditions on curves in $|L|$. Furthermore we see that

$$\chi\left(L - \sum_i n_iE_i\right) = \chi(L) - \sum_i \binom{n_i+1}{2}.$$

Now assume that S is a smooth projective toric surface. Let the $p_i \in S$ be fixed points of the torus action, so that $\widehat{S}$ is again a toric surface and the exceptional divisors E_i are torus-invariant divisors. Then by the above we can view $N^{(\widehat{S},L-\sum_i n_iE_i),\delta}(y)$ as a refined count of curves in $|L|$ on S with points of multiplicity n_i at p_i for all i and in addition δ nodes which pass through

$$\dim(|L|) - \delta - \sum_i \binom{n_i+1}{2}$$

general points on S.

Notation 49. We denote $N^{(S,L),\delta}_{n_1,\ldots,n_r}(y) := N^{(\widehat{S},L-\sum_i n_iE_i),\delta}(y)$.

For an Eisenstein series $G_{2k}(q)$, we denote

$$\overline{G}_k(q) := G_k(q) - G_k(q^2) = \sum_{n>0} \sum_{\substack{d|n \\ \frac{n}{d} \text{ odd}}} d^{2k-1} q^n.$$

We write again $D := q\frac{\partial}{\partial q}$. Note that $D^l G_{2k}(q)$ and $D^l \overline{G}_{2k}(q)$ are quasimodular forms of weight $2k + 2l$.

Conjecture 50. *For each $i \geq 1$ there exists a universal power series $H_i \in \mathbb{Q}[y^{\pm 1}][[q]]$, such that, whenever L is sufficiently ample with respect to δ, r and $n_1, \dots, n_r$, we have*

$$\begin{aligned}(6.1)\qquad N^{(S,L),\delta}_{n_1,\dots,n_r}(y) = \operatorname*{Coeff}_{q^{(L^2-LK_S)/2}} \Bigg[& \widetilde{DG}_2(y,q)^{\chi(L)-1-\delta-\sum_i \binom{n_i+1}{2}} \\ & \frac{B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} D\widetilde{DG}_2(y,q)}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q))^{\chi(\mathcal{O}_S)/2}} \prod_{i=1}^{r} H_{n_i}(y,q) \Bigg].\end{aligned}$$

Furthermore we conjecture for all $m > 0$ the following:

1. *$H_m(y,q)$ can be expressed in terms of Jacobi theta functions and quasimodular forms.*
2. *$H_m(1,q)$ is a (usually non-homogeneous) polynomial in the $D^l G_{2k}(q)$ of weight $\leq 4k$.*
3. *$H_m(-1,q)$ is a (usually non-homogeneous) polynomial in the $D^l G_{2k}(q)$, $D^l \overline{G}_{2k}(q)$ of weight $\leq 2k$.*

For small m we explicitly conjecture the following formulas:

(1) For $m \leq 2$ we conjecture

$$\begin{aligned} H_1(y,q) &= \widetilde{DG}_2(y,q), \\ H_2(y,q) &= \frac{F_1(y,q)}{(y^{1/2}-y^{-1/2})^4} + \frac{F_2(y,q)}{(y^{1/2}-y^{-1/2})^2(y-y^{-1})}, \end{aligned}$$

with

$$F_1(y,q)=\sum_{n>0}\sum_{d|n}\frac{1}{2}\left(-\frac{n^3}{d^3}+\frac{n^2}{d}-\frac{n}{d}\right)(y^{d/2}-y^{-d/2})^2q^n$$
$$F_2(y,q)=\sum_{n>0}\sum_{d|n}\left(\frac{n^2}{d^2}-\frac{n}{2}\right)(y^d-y^{-d})q^n.$$

(2) For the specialisation at $y=1$ we conjecture the following (dropping the q from the notation).

$$\begin{aligned}
H_1(1)&=DG_2,\\
H_2(1)&=-\frac{1}{24}DG_2+\frac{1}{6}D^2G_2-\frac{1}{8}DG_4-\frac{1}{24}D^3G_2+\frac{1}{24}D^2G_4\\
H_3(1)&=\frac{DG_2}{90}-\frac{D^2G_2}{18}+\frac{DG_4}{24}+\frac{13D^3G_2}{288}-\frac{73D^2G_4}{1440}+\frac{DG_6}{120}-\frac{D^4G_2}{144}\\
&\quad+\frac{13D^3G_4}{1440}-\frac{D^2G_6}{480}+\frac{D^5G_2}{2880}-\frac{D^4G_4}{2016}+\frac{D^3G_6}{6912}+\frac{\Delta}{241920}\\
H_4(1)&=-\frac{9DG_2}{1120}+\frac{7D^2G_2}{160}-\frac{21DG_4}{640}-\frac{1063D^3G_2}{23040}+\frac{1207D^2G_4}{23040}-\frac{3DG_6}{320}\\
&\quad+\frac{79D^4G_2}{5760}-\frac{43D^3G_4}{2304}+\frac{149D^2G_6}{26880}-\frac{DG_8}{2688}-\frac{91D^5G_2}{69120}+\frac{95D^4G_4}{48384}\\
&\quad-\frac{461D^3G_6}{645120}+\frac{101D^2G_8}{1451520}-\frac{11\Delta}{5806080}+\frac{D^6G_2}{17280}-\frac{89D^5G_4}{967680}+\frac{D^4G_6}{25920}\\
&\quad-\frac{D^3G_8}{207360}+\frac{D\Delta}{2903040}-\frac{D^7G_2}{967680}+\frac{D^6G_4}{580608}-\frac{D^5G_6}{1244160}+\frac{D^4G_4}{8211456}\\
&\quad-\frac{D^2\Delta}{84913920}+\frac{\Delta G_4}{864864}
\end{aligned}$$

(3) At $y=-1$ we conjecture

$$\begin{aligned}
H_1(-1)&=\overline{G}_2(q),\\
H_2(-1)&=\frac{1}{8}\big(\overline{G}_2-D\overline{G}_2+\overline{G}_4-DG_2\big),\\
H_3(-1)&=\frac{1}{24}\overline{G}_2-\frac{1}{24}DG_2+\frac{7}{96}\overline{G}_4-\frac{7}{96}D\overline{G}_2+\frac{1}{2}\overline{G}_2^3-\frac{1}{192}D\overline{G}_4\\
&\qquad-\frac{5}{64}G_4\overline{G}_2+\frac{1}{96}D^2G_2-\frac{5}{1024}DG_4,
\end{aligned}$$

$$\begin{aligned}H_4(-1) = &\frac{3\overline{G}_2}{128} - \frac{5DG_2}{192} - \frac{67D\overline{G}_2}{1536} + \frac{67\overline{G}_4}{1536} + \frac{35D^2G_2}{2304} - \frac{247DG_4}{24576}\\ &+ \frac{55\overline{G}_2^3}{144} - \frac{55G_4\overline{G}_2}{1536} - \frac{11D\overline{G}_4}{4608} + \frac{D^3G_2}{192} + \frac{25D^2G_4}{6144} - \frac{7DG_6}{8192}\\ &+ \frac{11\overline{G}_2^4}{8} - \frac{13\overline{G}_2D^2G_2}{192} + \frac{35\overline{G}_2DG_4}{512} - \frac{21G_6\overline{G}_2}{1024} + \frac{D^2\overline{G}_4}{512}.\end{aligned}$$

Remark 51. Part (1) of Conjecture 50 is not formulated in a very precise way. We want to illustrate the statement for $H_1(y,q)$ and $H_2(y,q)$, which we have conjecturally determined. In addition to $D := q\frac{\partial}{\partial q}$, we also consider $' = y\frac{\partial}{\partial y}$. Writing $\widetilde{DG}_2(y,q) = \frac{F_0(y,q)}{y-2+y^{-1}}$ we have

$$\begin{aligned}F_0(y,q) &= -\frac{D\theta(y)}{\theta(y)} - 3G_2,\\ F_1(y,q) &= \frac{1}{2}\frac{(D\theta(y))^2}{\theta(y)^2} + 3\frac{D\theta(y)}{\theta(y)}G_2 + \frac{1}{2}\frac{D\theta(y)}{\theta(y)} + \frac{15}{8}G_4 - \frac{9}{4}DG_2 + \frac{3}{2}G_2,\\ F_2(y,q) &= -\frac{1}{2}\frac{D\theta(y)\theta'(y)}{\theta(y)^2} - \frac{1}{6}\frac{D\theta'(y)}{\theta(y)} - 2G_2\frac{\theta'(y)}{\theta(y)}.\end{aligned}$$

Proof. A similar computation has been done in [GS2, Rem 1.4]. By definition we have

$$F_0(y,q) = \sum_{m>0}\sum_{d>0} m(y^d - 2 + y^{-d})q^{md} = \sum_{md>0} my^dq^{md} - 2G_2(q) + \frac{1}{12}.$$

In [Z, page 456, compare (iii) and (vii)] it is proved that

$$\frac{\theta'(0)\theta(wy)}{\theta(w)\theta(y)} = \frac{wy-1}{(w-1)(y-1)} - \sum_{nd>0}\operatorname{sgn}(d)w^ny^dq^{nd}. \tag{6.2}$$

Write $w = e^x$ and take the coefficient of x on both sides of (6.2). By the identity [Z, eq. (7)] we have

$$\frac{x\theta'(0)}{\theta(w)} = \exp\left(2\sum_{k\geq 2} G_k(q)\frac{z_1^k}{k!}\right).$$

This gives

$$\operatorname*{Coeff}_{x}\left[\frac{\theta'(0)\theta(wy)}{\theta(w)\theta(y)}\right] = \operatorname*{Coeff}_{x^2}\left[\frac{\theta(wy)}{\theta(y)}\right] + G_2(\tau)$$
$$= \frac{1}{2}\frac{\theta''(y)}{\theta(y)} + G_2(\tau) = \frac{D\theta(y)}{\theta(y)} + G_2(\tau),$$

where the last step is by the heat equation $\frac{1}{2}\theta''(y) = D\theta(y)$. On the other hand we compute

$$\operatorname*{Coeff}_{z_1}\left[\frac{wy-1}{(w-1)(y-1)} - \sum_{nd>0} \operatorname{sgn}(d) w^n y^d q^{nd}\right] = \frac{1}{12} - \sum_{nd>0} n y^d q^{nd}.$$

This proves the formula for F_0.

We have

$$F_2(y,q) = \sum_{md>0} \operatorname{sgn}(d)(m^2 - md/2)y^d)q^{md}.$$

In [GS2, Rem. 1.4] it is shown (the statement there contains a misprint) that

$$\sum_{md>0} \operatorname{sgn}(d) m^2 y^d q^{md} = -\frac{1}{\theta(y)}\left(\frac{2}{3}D\theta'(y) + 2G_2(q)\theta'(y)\right).$$

We see by (6.2) that

$$\sum_{md>0} \operatorname{sgn}(d)(-md/2)y^d)q^{md} = \frac{1}{2}D\left(\frac{\theta'(0)\theta(wy)}{\theta(w)\theta(y)}\Big|_{w=1}\right) = \frac{1}{2}D\left(\frac{\theta'(y)}{\theta(y)}\right).$$

This shows the formula for F_2.

A similar but slightly more tedious computation shows the formula for F_1. □

The conjectural formulas of Conjecture 50 were found by doing computations for $\mathbb{P}^2$ and its blowup Σ_1 with exceptional divisor E. We use the Caporaso-Harris recursion formula to compute $N^{(\Sigma_1, dH+mF),\delta}(y) = N^{(\Sigma_1,(d+m)H-mE,\delta}$ for $d \leq 11$, $m \leq 4$ and $\delta \leq 22$, in this realm the following conjecture is true.

Conjecture 52. *There are power series $H_m(y,q) \in \mathbb{Q}[y^{\pm 1}][[q]]$, such that the following holds. For $d > 0$, and $0 \leq m \leq 4$ and $\delta \leq 2d + 1 - m(m+1)/2$*

we have

$$N_m^{(\mathbb{P}^2,dH),\delta}(y) =$$
$$\operatorname*{Coeff}_{q^{(d(d+3)/2}}\left[\widetilde{DG}_2(y,q)^{d(d+3)/2-m(m+1)/2-\delta}\frac{B_1(y,q)^9(D\widetilde{DG}_2(y,q))^{1/2}}{B_2(y,q)^{-3d}\widetilde{\Delta}(y,q)^{1/2}}H_m(y,q)\right].$$

Furthermore $H_1(y,q)$, $H_2(y,q)$ coincide with the functions with the same name from Conjecture 50, and $H_i(1,q)$, $H_i(-1,q)$ coincide for $i=1,2,3,4$ with the $H_i(1)$, $H_i(-1)$ from Conjecture 50.

Proposition 53. *Conjecture 52 is true from $m \leq 4$ and $\delta \leq 9$.*

Proof. The argument is the same as in several proofs before. By Theorem 26 we get that $Q^{(\Sigma_1,dH+mF),\delta}$ is for $\delta \leq d$ a polynomial of degree 2 in d, which we know for $9 \leq d \leq 11$. The result follows. □

Let S be a toric surface and $\widehat{S}$ be the blowup of S in torus fixed point. Given δ, if m is sufficiently large and L is sufficiently ample on S, then $L-mE$ will be sufficiently ample on $\widehat{S}$, so that Conjecture 32 will apply to the pair $(\widehat{S}, L-mE)$, giving

$$\begin{aligned}N_m^{(S,L),\delta}(y) &= N^{(\widehat{S},L-mE),\delta}(y)\\ &= \operatorname*{Coeff}_{q^{(L^2-LK_S)/2-\binom{m+1}{2}}}\Bigg[\widetilde{DG}_2(y,q)^{\chi(L)-1-\delta-\binom{m+1}{2}}\\ &\qquad\frac{B_1(y,q)^{K_S^2-1}B_2(y,q)^{LK_S+m}D\widetilde{DG}_2(y,q)}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q))^{\chi(\mathcal{O}_S)/2}}\Bigg].\end{aligned}$$

Combined with Conjecture 50 this leads to the following conjecture.

Conjecture 54. *We have*

$$\frac{H_m(y,q)}{q^{\binom{m+1}{2}}} \equiv \frac{B_2(y,q)^m}{B_1(y,q)} \mod q^{m+1}.$$

Thus, if eventually one would find a way to explicitly determine the functions $H_m(y,q)$ for all m, this could give the unknown power series $B_1(y,q)$, $B_2(y,q)$ and thus complete the conjectural formulas of [Göt],[GS].

It is natural to assume that the specialisation of Conjecture 50 and also of the previous conjectures Conjecture 39, Conjecture 44 to $y=1$ hold for the usual Severi degrees $n^{(S,L),\delta}$ for projective algebraic surfaces, not just for toric surfaces. Thus we get in particular the following generalisation of the original conjecture of [Göt].

Let S be a projective algebraic surface with A_1-singularties $q_1,\dots,q_s$. Let $p_1,\dots,p_r$ be distinct smooth points on S. Let $m_1,\dots,m_r\in\mathbb{Z}_{>0}$, $n_1,\dots,n_s\in\frac{1}{2}\mathbb{Z}_{\geq 0}$. Let $\widehat{S}$ be the blowup of S in $p_1,\dots,p_r,q_1,\dots,q_s$ and denote E_i, F_j the exceptional divisors over p_i, q_j respectively. Let L be a $\mathbb{Q}$-Cartier Weil divisor on S, such that $\widehat{L}:=L-\sum_{i=1}^r m_iE_i-\sum_{j=1}^s n_jF_j$ is a Cartier divisor on $\widehat{S}$, which is δ-very ample on all irreducible curves in $\widehat{S}$ not contained in $E_1\cup\cdots\cup E_r\cup F_1\cup\cdots\cup F_s$. Denote $n^{(S,L),\delta}_{(m_1,\dots,m_r),(n_1,\dots,n_s)}:=n^{(\widehat{S},\widehat{L}),\delta}$, which we could informally interpret as the number of curves in $|L|$ which have multiplicity m_i in p_i and n_j in q_j for all i,j and pass in addition through

$$\chi(L)-1-\sum_{i=1}^r\binom{m_i+1}{2}-\sum_{j=1}^s n_j^2$$

general points on S, and have δ nodes as other singularities, where we write $\chi(L)=L(L-K_S)/2+\chi(\mathcal{O}_S)$.

Conjecture 55. *Under the above assumptions we have*

$$\begin{aligned}(6.3)\quad n^{(S,L),\delta}_{(m_1,\dots,m_r),(n_1,\dots,n_s)}=\operatorname*{Coeff}_{q^{(L^2-LK_S)/2}}\Bigg[&DG_2(q)^{\chi(L)-\sum_i\binom{m_i+1}{2}-\sum_j n_j^2-1}\\ &\frac{B_1(q)^{K_S^2}B_2(q)^{LK_S}D^2G_2(q)}{(\Delta(q)\cdot D^2G_2(q))^{\chi(\mathcal{O}_S)/2}}\\ &\left(\prod_{i=1}^r H_{m_i}(1,q)\right)\left(\prod_{j=1}^s f_{2n_j}(q)\right)\Bigg].\end{aligned}$$

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