

# Feynman integrals, L-series and Kloosterman moments

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This work lies at an intersection of three subjects: quantum field theory, algebraic geometry and number theory, in a situation where dialogue between practitioners has revealed rich structure. It contains a theorem and 7 conjectures, tested deeply by 3 optimized algorithms, on relations between Feynman integrals and L-series defined by products, over the primes, of data determined by moments of Kloosterman sums in finite fields. There is an extended introduction, for readers who may not be familiar with all three of these subjects. Notable new results include conjectural evaluations of non-critical L-series of modular forms of weights 3, 4 and 6, by determinants of Feynman integrals, an evaluation for the weight 5 problem, at a critical integer, and formulas for determinants of arbitrary size, tested up to 30 loops. It is shown that the functional equation for Kloosterman moments determines much but not all of the structure of the L-series. In particular, for problems with odd numbers of Bessel functions, it misses a crucial feature captured in this work by novel and intensively tested conjectures. For the 9-Bessel problem, these lead to an astounding compression of data at the primes.

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## 1. Introduction

As indicated in the abstract, this article concerns a fruitful intersection of three fascinating subjects: quantum field theory [12], algebraic geometry [5] and number theory [30], with a focus on Feynman integrals that evaluate to L-series defined by moments of Kloosterman sums [39].

Feynman diagrams with internal edges associated to massive particles invariably define *integrals of products of Bessel functions*. Simple cases were studied in [3]. *Kloosterman moments* are mere integers, resulting from finite sums in finite fields. Bridges between these seemingly different constructs are provided by *L-series*. These are defined by products over all the primes, in the manner that Euler obtained zeta values by considering

$$(1) \quad \zeta(s) = \prod_p \frac{1}{1 - p^{-s}} = \sum_{n>0} \frac{1}{n^s}.$$

Key data comes from *functional equations* [28] for Kloosterman moments, at powers of primes. These yield functional equations for L-series, similar to that used by Riemann to study the analytic continuation of  $\zeta(s)$  inside a critical strip and most spectacularly on a critical line, with the real part of  $s$  set to  $\frac{1}{2}$ . Our strips will be wider, containing *critical integers*, on the real line. At these, *mirabile dictu*, L-series evaluate to Feynman integrals [12, 15].

The remainder of this introduction provides further orientation, for those who wish it. A reader who finds fun in formulas may fast-forward, taking note that each of the 24 signs  $\stackrel{?}{=}$  in Section 7 indicates an empirical result that is, to my knowledge, unproven, yet has been tested numerically to a precision of at least 500 decimal digits. These leave ample opportunity for a fortunate reader with a talent for proof.

### 1.1. Arena

*Quantum field theorists* predict probabilities for outcomes of interactions of particles, as at the large hadron collider, by making expansions in coupling constants that are hopefully small enough to permit accurate comparison

with experimental data, as is the case for the standard model of particle physics, at high energy.

They do this by evaluating integrals defined by Feynman diagrams, whose external edges specify the process under study, while the internal edges represent the possibility of so-called virtual propagation, in which the energy  $E$  and momentum  $p$  are no longer related by Einstein’s mass-shell condition  $E^2 = (mc^2)^2 + (pc)^2$ , for a particle of mass  $m$ , with  $c$  the speed of light. In a spacetime with even dimension  $D$ , like our own with  $D = 4$ , the propagator, between spacetime events at adjacent vertices of the diagram, involves a *Bessel function*, when  $m > 0$ . Integrations must be performed over all spacetime separations of the vertices.

Thus, whenever the diagram has at least two internal massive edges, it yields an integral of a product of Bessel functions. That may be disguised, by Fourier transformation, or by use of parametric integration, yet it is always the case. I have therefore devoted considerable effort to the study of integrals of Bessel functions, notably in work with David Bailey, Jon Borwein and Larry Glasser [3].

*Algebraic geometers* may take interest in a Feynman integral when shown its equivalent representation as a projective integral over Schwinger parameters, with an integrand whose denominator is a polynomial in the parameters associated to the edges, with coefficients determined by the external data and the internal masses. From this it may be possible to determine, by what Spencer Bloch likes to call *pure thought*, the type of number that the Feynman diagram will yield and in particular whether the result will be a simple polylogarithm or something more refined, such as the elliptic dilogarithms in [1, 6] or the elliptic trilogarithm found by Bloch, Kerr and Vanhove in [5] for a sunrise diagram, with off-shell external data, in two spacetime dimensions. This has 5 Bessel functions, at 3 loops, giving

$$(2) \quad \mathcal{S}_{5,3}(w) := 2^3 \int_0^\infty I_0(wt) K_0^4(t) t dt.$$

In 2007, I had conjectured [11] that

$$(3) \quad \mathcal{S}_{5,3} := \mathcal{S}_{5,3}(1) = \frac{\pi^3}{2} \left(1 - \frac{1}{\sqrt{5}}\right) \left(\sum_{n=-\infty}^\infty e^{-n^2\pi\sqrt{15}}\right)^4$$

in the on-shell case, with  $w = 1$ . This is now proven [5, 43].

*Number theorists* may take interest in a Feynman integral when shown its evaluation, as in (3). If the coefficients of the polynomial of Schwinger

parameters are rational numbers the integral is, by definition, a *period* [35]. This arises when there is a single large energy scale in the process, allowing effective neglect of internal masses. Then the period may be a multiple zeta value (MZV) as in my work with Dirk Kreimer [13]. MZVs hold lively interest for mathematicians [29, 32] and I have devoted considerable effort to their study, notably in work with Johannes Blümlein and Jos Vermaseren [7]. Yet Francis Brown and Oliver Schnetz have shown that even single-scale massless diagrams outgrow the kindergarten of polylogarithms, such as MZVs, when the loop-number (which mathematicians call the first Betti number) of the diagram reaches 7, for massless diagrams with two external particles [17].

When the scale is set by a single mass, results may be given by L-series of modular forms, as in (3), where the 3-loop evaluation

$$(4) \quad \frac{S_{5,3}}{\zeta(2)} = \frac{48}{5} L_5(1)$$

comes from combining work in [5] with work by Mathew Rogers, James Wan and John Zucker [40] on an L-series,  $L_5(s)$ , defined by a modular form [38] of weight 3 and level 15, with a functional equation derived from Kloosterman moments at  $n = 5$ . This is the subject of Sections 5.1 and 7.1.

My purpose is to address the *simplest* cases in which interests of quantum field theorists, algebraic geometers and number theorists intersect and the Feynman integrals are not mere polylogarithms.

Accordingly, let  $D = 2$ , since in two spacetime dimensions I need not instruct the reader in Freeman Dyson's miracle of renormalization [23], required to render finite the Feynman integrals of the real world, with  $D = 4$ .

Let all particles be massive and spin-less, as is the Higgs boson. Crucially, let them have the same mass,  $m$ , and work in a system of units where  $m = c = 1$  and Planck's constant is  $h = 2\pi$ . Then an internal edge has a propagator  $K_0(t)$ , where  $t$  is the separation, in proper time, of the vertices that the edge connects. The Bessel function  $K_0(t)$  falls off exponentially fast at large  $t$ , so there will be no infra-red problems. At  $t = 0$  it has a logarithmic singularity, benign enough to avoid ultra-violet problems.

Finally, and most drastically, let each diagram have precisely two vertices. Then, as illustrated in Section 7, there are two cases: vacuum diagrams, with no external data, or so-called sunrise diagrams, where energy  $E$  and momentum  $p$  enter at one vertex and leave at the other. For the latter, work on the external mass-shell, with  $w^2 := E^2 - p^2 = 1$ . Then the external data will contribute  $I_0(t)$  to the integrand. The Bessel functions  $I_0(t)$  and  $K_0(t)$  are born of the same second order differential equation, but  $I_0(t)$  is quite

contrary to its sibling, being well behaved at  $t = 0$ , where it is regular, and poorly behaved at large  $t$ , where it increases exponentially.

Thus we shall be studying single integrals of products of Bessel functions. This tiny part of the vast arena of quantum field theory holds remarkable surprises, illuminated by L-series associated with Kloosterman sums.

### 1.2. *Précis* of Section 2, on Kloosterman sums

Here, I define *Kloosterman sums* in finite fields,  $\mathbf{F}_q$ , with  $q = p^k$ , for prime  $p \geq 2$  and integer exponent  $k \geq 1$ . These sums bring us as close as possible to emulating Bessel functions, while rigourously eschewing all branches of analysis. From them I shall obtain what Phillippe Robba (1941–1988) called *symmetric powers of the  $p$ -adic Bessel equation* [39]. These provide integers,  $c_n(q)$ , for integer powers  $n \geq 1$ . I shall refer to them as *Bessel moments* in finite fields. They may seem remote from the hard analysis required for Feynman diagrams with masses, which involves integration over products of Bessel functions that are themselves defined as integrals or infinite sums. Yet  $c_n(q)$  brings us as close as possible to capturing the features of an integral of the product of  $n$  Bessel functions while allowing ourselves only finite summations that yield integers. The key fact is that  $c_n(p^k)$  is *predictable* for  $k > d(p, n)$ , where  $d(p, n)$  is a vital integer, associated with a functional equation for Robba's  $p$ -adic problem. This has deep consequences for the evaluation of the Feynman integrals.

### 1.3. *Précis* of Section 3, with Algorithm 1

This is devoted to determining data for a generating function that gives important information about all moments  $c_n(p^k)$ , for each prime  $p$ . I have tried to keep this section simple and self-contained, avoiding profound mathematical concepts that may be unfamiliar to physicists, like myself. Yet it is still a delicate task, since one must carefully distinguish the cases of odd and even  $n$ , with an important sign,  $s(p, n)$ , prominent for odd  $n$ , where the residue of  $p$  modulo 8 is also important. Moreover, the prime  $p = 2$  is a very special case that was only recently understood, thanks to fine work by Zhiwei Yun [46], whose crucial contribution I have condensed to a single line, in (37), where the even prime, 2, talks deeply with its neighbour, 3.

A result for the sign, from Lei Fu and Daqing Wan [28], has been condensed to a simple recursion in (13), from which I derive Algorithm 1, enabling efficient computation of a generating function for  $d(p, n)$ . Fu and Wan also corrected a faulty conjecture by Robba on the case  $p \equiv 3 \pmod{4}$  in (32).

#### 1.4. *Précis of Section 4, with a theorem and corollaries*

A reader more concerned with results than with derivations may wish to skim the details of the previous section and concentrate instead on this section, where Theorem 1 encapsulates all the hard-won findings on  $d(p, n)$  and leads to Corollary 1, with a remark on an uncanny parallel to my work with Kreimer [13], which connects enumerations of irreducible MZVs to enumeration of modular forms. Corollary 2 is included for those who prefer floors to generating functions, with a remark that favours the latter.

#### 1.5. *Précis of Section 5, with Conjectures 1, 2 and 3*

Here, I propose Conjecture 1, which constrains Bessel moments, with odd  $n$ , even more tightly than the functional equation. It is proven at  $n \leq 5$  and agrees well with Ronald Evans' inspired discoveries [19, 24] at  $n = 7$ . I set store on it to illuminate uncharted territory at  $n \geq 9$ , where it has survived tenacious testing. Moreover, I refine it by Conjectures 2 and 3, which distinguish the cases  $n \equiv 1 \pmod{4}$  and  $n \equiv 3 \pmod{4}$ , where they are even more predictive. They too survive deep testing.

#### 1.6. *Précis of Section 6, with Algorithms 2 and 3*

Here, I hone the computational challenges to their ineluctable essentials. This was a necessary step, since the innermost declarations of Algorithm 2 were executed about  $10^{13}$  times, in tests of the conjectures. Algorithm 3 ensures that one uses Algorithm 2 as sparingly as possible.

#### 1.7. *Précis of Section 7, with Conjectures 4, 5, 6 and 7*

Here, at last, we get to study the Feynman integrals, finding wonderfully simple results for them as critical values of L-series defined by infinite products over data at the primes, from Kloosterman moments. Some of these results were reported in [12, 15]. Since then there has been dramatic progress, thanks to work with Anton Mellit [14]. We have been able to achieve results at  $n = 7$ , where the functional equation was found to be significantly different. By using *determinants* of Feynman integrals, including those from vacuum diagrams, we reached non-critical values that are the subject of conjectures by Fernando Rodriguez Villegas, on logarithmic *Mahler measures* [45]. Moreover we reach a place that was inaccessible to him: the value of a L-series with modular weight 6 at  $s = 7$ . This we claim to evaluate in

Conjecture 6 of Section 7.8, using a  $2 \times 2$  determinant that includes 7-loop Feynman integrals with 8 Bessel functions.

The route to Conjecture 6 was via the striking Conjecture 5 of Section 7.7, which is dedicated to the memory of Richard Crandall, in regret that he did not live to see it. Further determinants yield powers of  $\pi$  and, from diligent study of these, I advance Conjectures 4 and 7, with examples in (133) and (166), by way of *amuse bouche*. These have been checked numerically by evaluating hundreds of Feynman integrals, with up to 30 loops, at 500-digit precision.

### 1.8. *Précis* of Section 8, on harder problems

*What proverb more common, what proverb more true, than that after pride comes a fall?*

Mindful of Kingsley’s [33] question, I report attempts with  $n > 8$  Bessel functions. At  $n = 9$ , Conjecture 2 places hitherto unexplored constraints on the data at the primes. In a sense that I make precise,  $c_9(p)$  determines  $c_9(p^2)$  *more often than not*, for  $p \leq 631$ . Yet that has not produced, so far, a result for Feynman integrals with 9 Bessel functions. As Francis Brown is fond of reminding me, all good things come to an end. I respond with a question from his almost namesake, Browning [18].

*Ah, but a man’s reach should exceed his grasp, or what’s a heaven for?*

## 2. Functional equation for Kloosterman moments

Kloosterman sums,  $K(a)$ , with  $a \in \mathbf{F}_q$  and  $q = p^k$ , are defined by

$$(5) \quad K(a) := \sum_{x \in \mathbf{F}_q^*} \exp\left(\frac{2\pi i}{p} \operatorname{trace}\left(x + \frac{a}{x}\right)\right)$$

with a trace of Frobenius, whose details I postpone to Section 6, where they are truly needed. The key information that we need for Feynman integrals is given by the Bessel moments  $c_n(q)$ , defined by

$$(6) \quad c_n(q) := -\frac{1 + S_n(q)}{q^2}, \quad S_n(q) := \sum_{a \in \mathbf{F}_q^*} \sum_{j=0}^n [g(a)]^j [h(a)]^{n-j}$$

with  $K(a) = -g(a) - h(a)$  and  $g(a)h(a) = q$ . These moments are integers that bring us as close as possible to emulating Feynman integrals, while

performing only finite summations. Hence I have devoted considerable effort to refining algorithms for their efficient computation.

Notwithstanding the appearance of an infinite sum in the exponential of

$$(7) \quad Z_n(p, T) := \exp \left( - \sum_{k>0} \frac{c_n(p^k)}{k} T^k \right)$$

the result is a polynomial in  $T$ , whose degree,  $r(p, n)$ , I shall call the *raw* degree. In essence, (7) tells us that  $c_n(p^k)$  is predictable for  $k > r(p, n)$ .

For  $n \leq 4$  the situation is very simple, with  $Z_1(p, T) = Z_2(p, T) = 1$ ,

$$(8) \quad Z_3(p, T) = 1 - \left(\frac{p}{3}\right) T, \quad Z_4(p, T) = \begin{cases} 1 & \text{if } p = 2 \\ 1 - T & \text{if } p > 2 \end{cases}$$

and a Legendre symbol  $\left(\frac{p}{3}\right) = 0$  or  $\pm 1$ , for  $p \equiv 0$  or  $\pm 1 \pmod{3}$ .

It is often possible to improve predictability by extracting factors from (7). Let  $y = p^{k-1}T$ , for odd  $n = 2k + 1$ , and  $y = p^{k-2}T$ , for even  $n = 2k$ . Then there are exponents  $a(p, n)$  and  $b(p, n)$  such that

$$(9) \quad Z_n(p, T) = (1 - y)^{a(p,n)} (1 + y)^{b(p,n)} M_n(p, T)$$

where  $M_n(p, T)$  has even degree  $2d(p, n) = r(p, n) - a(p, n) - b(p, n)$ , which I shall call the *net* degree. Most importantly, there is a functional equation

$$(10) \quad M_n(p, T) = (p^{n-3}T^2)^{d(p,n)} M_n \left( p, \frac{1}{p^{n-3}T} \right)$$

from which it follows that  $Z_n(p, T)$  may be computed from moments of Kloosterman sums in finite fields  $\mathbf{F}_q$  with characteristic  $p$  and  $q \leq p^{d(p,n)}$ , provided that one knows the raw degree of (7) and the exponents in (9). It is proven that  $d(p, n) \leq 1$  for  $n \leq 8$ . For  $5 \leq n \leq 8$ , we may therefore determine  $c_n(p^k)$  for all  $k > 1$  knowing only  $c_n(p)$ . This circumstance is associated with the existence of automorphic forms of modular weight  $n - 2$  for  $5 \leq n \leq 8$ , where the L-series associated with  $c_n(p)$  provide evaluations of Feynman integrals in quantum field theory.

### 3. Case by case study of moments

We must take care to distinguish cases by the parities of  $n$  and  $p$ , as follows.

### 3.1. Moments with odd $n$ and $p > 2$

Let  $p > 2$  be an odd prime and  $n$  be an odd integer. Then the raw degree is given recursively by

$$(11) \quad r(p, n) = r(p, n - 2) + \begin{cases} 0 & \text{if } p|n \\ 1 & \text{otherwise} \end{cases}$$

with  $r(p, 1) = 0$ . Using *floor* delimiters in  $\lfloor x \rfloor$  to denote the largest integer not exceeding  $x$ , we obtain the solution [39]

$$(12) \quad r(p, n) = \frac{n - 1}{2} - \left\lfloor \frac{n + p}{2p} \right\rfloor.$$

The sign  $s(p, n)$  of Theorem 4.2.4 in [46] satisfies the recursion

$$(13) \quad \frac{s(p, n)}{s(p, n - 2)} = t(p, n) := \begin{cases} \left(\frac{-2}{p}\right) & \text{if } p|n \\ \left(\frac{p}{n}\right) & \text{otherwise} \end{cases}$$

with  $s(p, 1) = 1$  and the Legendre symbol

$$(14) \quad \left(\frac{-2}{p}\right) = (-1)^{(p-1)(p-3)/8} = \begin{cases} +1 & \text{if } p \equiv 1 \text{ or } 3 \pmod{8} \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod{8} \end{cases}$$

occurring when  $p$  divides  $n$ . Otherwise we encounter the Jacobi symbol

$$(15) \quad \begin{aligned} \left(\frac{p}{n}\right) &= \left(\frac{p}{-n}\right) = \left(\frac{n}{p}\right) \left(\frac{-1}{p}\right)^{(n-1)/2} \\ &= \left(\frac{p}{n + 2p}\right) \left(\frac{-1}{p}\right) = \left(\frac{p}{n + 4p}\right). \end{aligned}$$

In both cases,  $t(p, n) = t(p, -n) = t(p, n + 4p)$  and we easily evaluate

$$(16) \quad \frac{s(p, n)}{s(p, -2 - n)} = \prod_{k=0}^{(n-1)/2} t(p, n - 2k)t(p, 2k - n) = 1.$$

More care must be taken to determine the sign of

$$(17) \quad \frac{s(p, n + 2p)}{s(p, -2 - n)} = t(p, p) \prod_{k=1}^{(n+p)/2} t(p, p + 2k)t(p, p - 2k) \\ = \left(\frac{-2}{p}\right) \left(\frac{-1}{p}\right)^{u(p,n)}$$

with an exponent

$$(18) \quad u(p, n) := \frac{n + p}{2} - \left\lfloor \frac{n + p}{2p} \right\rfloor = \frac{p + 1}{2} + r(p, n).$$

Noting that  $\left(\frac{-1}{p}\right)^{(p+1)/2} = (-1)^{(p^2-1)/4} = 1$ , we obtain, from (16,17),

$$(19) \quad s(p, 2p - 2 - n) = s(p, n + 2p) = \left(\frac{-2}{p}\right) \left(\frac{-1}{p}\right)^{r(p,n)} s(p, n).$$

Moreover,  $s(p, n) = s(p, n + 4p)$ , since  $r(p, n + 2p) - r(p, n) = p - 1$  is even. It follows from (19) that the signs  $s(p, n)$  with  $1 < n < p$  suffice to determine all other signs.

The net degree  $2d(p, n) = r(p, n) - a(p, n) - b(p, n)$  is obtained by removing from Fu and Wan’s polynomial (9) a factor  $(1 - y)^{a(p,n)}(1 + y)^{b(p,n)}$ , with  $y = p^{(n-3)/2}T$  and exponents

$$(20) \quad a(p, n) = \frac{1 - s(p, n)(-1)^{r(p,n)}}{2}, \quad b(p, n) = \frac{1 - s(p, n)}{2}$$

that are either 0 or 1. It follows that

$$(21) \quad \varepsilon(p, n) := \left\lfloor \frac{r(p, n)}{2} \right\rfloor - d(p, n) = \frac{1 - s(p, n)}{2} \frac{1 + (-1)^{r(p,n)}}{2}$$

vanishes when  $r(p, n)$  is odd or  $s(p, n) = 1$ . Sufficient information is encoded by the economical polynomial

$$(22) \quad g_p(x) := \sum_{p > 2k+1 > 0} \varepsilon(p, 2k + 1)x^{2k+1}$$

for which  $\left\lfloor \frac{p-3}{4} \right\rfloor$  Legendre symbols suffice, in the following algorithm.

**Algorithm 1.** [Compute  $g_p(x)$  for prime  $p$ .] Set  $s = n = 1$  and  $g = 0$ . While  $n + 4 < p$ , add 4 to  $n$ , change the sign of  $s$  if  $n(2 - n)$  is not a square modulo  $p$ , add  $x^n$  to  $g$  if  $s$  is negative. Return  $g_p(x) = g$ .

For convenience, here are the results for odd primes  $p \leq 31$ :

$$(23) \quad g_3 = g_5 = 0, \quad g_7 = g_{13} = x^5, \quad g_{11} = g_{29} = x^5 + x^9, \quad g_{17} = x^9,$$

$$(24) \quad g_{19} = x^9 + x^{13} + x^{17}, \quad g_{23} = x^{17}, \quad g_{31} = x^9 + x^{17} + x^{21} + x^{25}.$$

To determine  $\varepsilon(p, n)$  for odd  $n \geq p$ , we use (19), which gives

$$(25) \quad \varepsilon(p, 2p - 2 - n) = \varepsilon(p, n + 2p) = \begin{cases} \varepsilon(p, n) & \text{if } \left(\frac{-2}{p}\right) = +1 \\ 1 - \varepsilon(p, n) & \text{if } \left(\frac{-2}{p}\right) = -1. \end{cases}$$

Defining the polynomial

$$(26) \quad h_p(x) := \left(1 + \left(\frac{-2}{p}\right)x^{2p}\right) \left(g_p(x) + \left(\frac{-2}{p}\right)x^{2p-2}g_p(1/x)\right)$$

we obtain, from (22,25) and the periodicity of  $\varepsilon(p, n) = \varepsilon(p, n + 4p)$ ,

$$(27) \quad (1 - x^{4p}) \sum_{k \geq 0} \varepsilon(p, 2k + 1)x^{2k+1} = \begin{cases} h_p(x) & \text{if } \left(\frac{-2}{p}\right) = +1 \\ e_p(x) + h_p(x) & \text{if } \left(\frac{-2}{p}\right) = -1 \end{cases}$$

with an extra polynomial

$$(28) \quad e_p(x) := \sum_{p \leq 4k+3 < 3p} x^{4k+3} = \begin{cases} (x^{p+2} - x^{3p})/(1 - x^4) & \text{if } p \equiv 5 \pmod{8} \\ (x^p - x^{3p+2})/(1 - x^4) & \text{if } p \equiv 7 \pmod{8} \end{cases}$$

to take account of the cases with  $\left(\frac{-2}{p}\right) = -1$  in (25).

To determine a generating function  $\sum_{k \geq 0} d(p, 2k + 1)x^{2k+1}$  from (21,27), it suffices to evaluate the polynomial

$$(29) \quad f_p(x) := (1 - x^2)(1 - x^{4p}) \sum_{k \geq 0} \left\lfloor \frac{r(p, 2k + 1)}{2} \right\rfloor x^{2k+1}$$

which depends on the sign  $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$ , as follows:

$$(30) \quad f_p(x) = x^5 \left(\frac{1 - x^{4p}}{1 - x^4}\right) - \begin{cases} x^p & \text{if } p \equiv 1 \pmod{4} \\ x^{p+2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Before assembling these results for odd  $n$ , we consider the even moments.

**3.2. Moments for even  $n$  and  $p > 2$**

Let  $p > 2$  be an odd prime and  $n = 2k$  be an even positive integer. Then the raw degree is given recursively by (11) with  $r(p, 2) = 0$ . The solution is  $r(p, 2k) = k - 1 - \lfloor \frac{k}{p} \rfloor$ . The net degree  $2d(p, n) = r(p, n) - a(p, n) - b(p, n)$  is obtained by removing a factor  $(1 - y)^{a(p,n)}(1 + y)^{b(p,n)}$ , with  $y = p^{(n-4)/2}T$  and exponents determined by [26–28]

$$(31) \quad \sum_{k>0} (a(p, 2k) + b(p, 2k)) x^{2k} = \frac{x^4}{1 - x^4} + \frac{x^{2p}}{(1 - x^2)(1 - x^{2p})}$$

$$(32) \quad (1 - x^2)(1 - x^{4p}) \sum_{k>0} b(p, 2k) x^{2k} = \begin{cases} 0 & \text{if } p \equiv 1 \pmod{4} \\ x^{2p} & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

with a distinction of cases in (32) that does not affect the outcome for [39]

$$(33) \quad d(p, 2k) = \left\lfloor \frac{k - 1}{2} \right\rfloor - \left\lfloor \frac{k}{p} \right\rfloor.$$

**3.3. Moments for odd  $n$  at  $p = 2$**

Here one uses the recursions (11,13), where there is now no need to consider the case  $p|n$ . Hence we obtain a raw degree  $r(2, 2k + 1) = k$  and an easily evaluated Jacobi symbol

$$(34) \quad \frac{s(2, n)}{s(2, n - 2)} = \left( \frac{2}{n} \right) = (-1)^{(n^2-1)/8} = \begin{cases} +1 & \text{if } n \equiv 1 \text{ or } 7 \pmod{8} \\ -1 & \text{if } n \equiv 3 \text{ or } 5 \pmod{8} \end{cases}$$

in the recursion for the sign. Then, by induction, we prove that

$$(35) \quad s(2, n) = (-1)^{(n+3)(n^2-1)/48} = \begin{cases} +1 & \text{if } n \equiv 1, 5 \text{ or } 7 \pmod{8} \\ -1 & \text{if } n \equiv 3 \pmod{8} \end{cases}$$

with  $p = 2$  singling out the residue 3 modulo 8 as a special case. From (20), we obtain

$$(36) \quad \sum_{k \geq 0} a(2, 2k + 1) x^{2k+1} = \frac{x^7}{1 - x^8}, \quad \sum_{k \geq 0} b(2, 2k + 1) x^{2k+1} = \frac{x^3}{1 - x^8}.$$

### 3.4. Moments for even $n$ at $p = 2$

The subtlest question has been left to last. At  $p = 2$ , what is the pattern of exponents in (9) for even moments? The raw degree is  $r(2, 2k) = \lfloor \frac{k-1}{2} \rfloor$ , but to obtain the net degree  $2d(2, 2k) = r(2, 2k) - a(2, 2k) - b(2, 2k)$  we need a pair of recursions recently proven by Yun [46]:

$$(37) \quad a(2, 2k + 24) - a(2, 2k) = 1 = b(2, 2k + 24) - b(2, 2k)$$

with 24 signally a dialogue between the neighbours  $p = 2$  and  $p = 3$ .

To determine  $a(2, 2k)$  and  $b(2, 2k)$  for  $2k \leq 24$ , it suffices to evaluate  $Z_{2k}(2, T)$  from moments of Kloosterman sums in  $\mathbf{F}_{2^N}$  with  $N \leq 5$ , which is easily accomplished by Algorithm 2. We obtain  $Z_2(2, T) = Z_4(2, T) = 1$ ,  $Z_6(2, T) = 1 + y$  and  $Z_8(2, T) = 1 - y$  merely from working in  $\mathbf{F}_2$ . Work in  $\mathbf{F}_4$  quickly determines  $Z_{10}(2, T) = 1 + 6T + 2^7T^2$  and  $Z_{12}(2, T) = 1 - y^2$ , with  $y = 2^{k-2}T$  for  $Z_{2k}(2, T)$ . Thereafter

$$(38) \quad Z_{14}(2, T) = (1 + y)(1 - 54T + 2^{11}T^2)$$

$$(39) \quad Z_{16}(2, T) = (1 - y)(1 + 132T + 2^{13}T^2)$$

$$(40) \quad Z_{18}(2, T) = (1 - y^2)(1 - 114T + 2^{15}T^2)$$

$$(41) \quad Z_{20}(2, T) = (1 - y^2)(1 + 72T + 2^{17}T^2)$$

reveal that  $d(2, 2k) \leq 1$  for  $2k \leq 20$ . Moments in  $\mathbf{F}_{32}$  complete the picture:

$$(42) \quad Z_{22}(2, T) = (1 + y)(1 - 270(T + 2^{19}T^3) - 230720T^2 + 2^{38}T^4)$$

$$(43) \quad Z_{24}(2, T) = (1 - y)(1 - y^2)(1 + 12T + 2^{21}T^2).$$

The values  $a(2, 2k)$  and  $b(2, 2k)$ , with  $k = 1$  to 12, thus form the sequences

$$(44) \quad \begin{array}{cccccccccccc} 0, & 0, & 0, & 1, & 0, & 1, & 0, & 1, & 1, & 1, & 0, & 2 \\ 0, & 0, & 1, & 0, & 0, & 1, & 1, & 0, & 1, & 1, & 1, & 1 \end{array}$$

respectively. Then the recursions in (37) prove that

$$(45) \quad \sum_{k>0} a(2, 2k)x^{2k} = \frac{x^8}{1 - x^8} \left( 1 + \frac{x^4}{1 - x^6} \right)$$

$$(46) \quad \sum_{k>0} b(2, 2k)x^{2k} = \frac{x^6}{(1 - x^6)(1 - x^8)}.$$

Hence we obtain, from (36,45,46), the generating function

$$(47) \quad \sum_{n>0} d(2, n)x^n = \frac{x^5}{(1-x^2)(1-x^4)} + \frac{x^{10}}{(1-x^4)(1-x^6)}.$$

#### 4. Generating functions for all cases

I now assemble the results for  $d(p, n)$ . For prime  $p > 2$ , let

$$(48) \quad d_p(x) := \begin{cases} x^p(1+x^{2p}) & \text{if } p \equiv 1 \pmod 8 \\ x^{p+2}(1+x^{2p}) & \text{if } p \equiv 3 \pmod 8 \\ x^p(1+x^2) & \text{if } p \equiv 5 \text{ or } 7 \pmod 8 \end{cases}$$

$$(49) \quad C_p(x) := \frac{x^5}{(1-x)(1-x^4)} - \frac{x^{2p}}{(1-x^2)(1-x^{2p})}$$

$$(50) \quad D_p(x) := C_p(x) - \frac{d_p(x)}{(1-x^4)(1-x^{4p})} - \frac{g_p(x) + \left(\frac{-2}{p}\right)x^{2p-2}g_p(1/x)}{1 - \left(\frac{-2}{p}\right)x^{2p}}$$

with  $g_p(x)$  determined by  $\left[\frac{p-3}{4}\right]$  Legendre symbols, via Algorithm 1.

**Theorem 1.** *The net degree  $2d(p, n)$  is generated by*

$$(51) \quad \sum_{n>0} d(p, n)x^n = \begin{cases} C_3(x) & \text{if } p = 2 \\ D_p(x) & \text{if } p > 2. \end{cases}$$

*Proof.* For odd  $n$ , (30) gives  $d_p(x) = x^p(1+x^{2p})$  for  $p \equiv 1 \pmod 8$  and  $d_p(x) = x^{p+2}(1+x^{2p})$  for  $p \equiv 3 \pmod 8$ . For  $p \equiv 5 \pmod 8$ , the extra term in (28) gives  $d_p(x) = x^p(1+x^{2p}) + x^{p+2} - x^{3p} = x^p(1+x^2)$ . For  $p \equiv 7 \pmod 8$ , it gives  $d_p(x) = x^{p+2}(1+x^{2p}) + x^p - x^{3p+2} = x^p(1+x^2)$ . Since  $d_p(x)$  and  $g_p(x)$  are odd functions, only  $C_p(x)$  contributes for even  $n$  and  $p > 2$ , giving (33). Setting  $p = 3$  in (49), we obtain the result for  $p = 2$  in (47). □

**Corollary 1.** *Let  $M(w)$  be the dimension of the space of cusp forms of weight  $w$  for the fundamental modular group. Then*

$$(52) \quad d(2, 2k) = d(3, 2k) = M(2k - 2) = \left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k}{3} \right\rfloor.$$

*Proof.* The cusp forms are generated by products of the unique cusp form with  $w = 12$  and powers of Eisenstein series with weight 4 or 6. Hence

$$(53) \quad \sum_{w>0} M(w)x^w = \frac{x^{12}}{(1-x^4)(1-x^6)}.$$

The even term in (47) proves that  $M(2k - 2) = d(2, 2k)$ . Theorem 1 gives  $d(2, 2k) = d(3, 2k)$ . Then (33), at  $p = 3$ , completes the proof of (52). □

**Remark 1.** The Broadhurst-Kreimer conjectures [10, 13], for the embedding of the two-letter alphabet of MZVs in the three-letter alphabet [7] of nested alternating sums, also use the generating function (53) for cusp forms. I was reluctant to regard this as anything more than coincidence, since it was hard for me to see how MZVs, at genus zero, talk to Eisenstein series, at genus 1. Yet Don Zagier [29, 32] and Francis Brown [16] seem to believe this to be no accident. In the present case, the coincidences of Corollary 1 are proven.

**Corollary 2.**  $d(3, 2k + 1) = \lfloor \frac{k}{3} \rfloor$  and  $d(5, 2k + 1) = \lfloor \frac{k-2}{2} \rfloor - \lfloor \frac{k-2}{10} \rfloor$ .

*Proof.* Noting that  $g_3(x) = g_5(x) = 0$ , we obtain

$$(54) \quad \sum_{k \geq 0} d(3, 2k + 1)x^{2k+1} = \frac{x^7}{(1-x^2)(1-x^6)}$$

$$(55) \quad \sum_{k \geq 0} d(5, 2k + 1)x^{2k+1} = \frac{x^9}{(1-x^2)(1-x^4)} - \frac{x^{25}}{(1-x^2)(1-x^{20})}$$

from (50). Conversion to the stated floors follows immediately. □

**Remark 2.** For  $p > 5$ , floors may be obtained by collecting terms, as here:

$$(56) \quad \sum_{k \geq 0} d(7, 2k + 1)x^{2k+1} = \frac{x^7 + x^{13} + x^{17} + x^{19} - x^{21} + x^{23} + x^{25} + x^{29}}{(1-x^2)(1-x^{28})}$$

$$(57) \quad \sum_{k \geq 0} d(17, 2k+1)x^{2k+1} = \frac{x^5 + x^{11} + x^{13} + x^{19} + x^{25} + x^{27} + x^{31} + x^{35}}{(1-x^2)(1-x^{34})}$$

with each monomial in the numerator yielding a floor. In these examples, we obtain 8 terms, while there is only one in  $g_7(x) = x^5$ , or  $g_{17}(x) = x^9$ . In

general, the number of monomials in the numerator is given by

$$(58) \quad n_p := \begin{cases} \frac{p-1}{2} & \text{if } p \equiv 1 \text{ or } 3 \pmod{8} \\ p-1 & \text{if } p \equiv 5 \pmod{8} \\ p+1 & \text{if } p \equiv 7 \pmod{8} \end{cases}$$

while the number of monomials in  $g_p(x)$  is asymptotic to  $\frac{1}{8}p$ , providing a saving in data by factors of 4 or 8, at large  $p$ .

### 5. Cores from odd moments

The *core* of a non-zero integer  $N$  is the unique square-free integer  $d|N$  such that  $N/d$  is the square of an integer. For odd  $n$ , let

$$(59) \quad c_n^\pm(p) := p^{m(n)} Z_n \left( p, \frac{\pm 1}{p^{(n-3)/2}} \right)$$

$$(60) \quad m(n) := \begin{cases} (n-1)^2/16 & \text{if } n \equiv 1 \pmod{4} \\ (n-3)(n+1)/16 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

As helpfully remarked by a referee, the rational numbers in (59) are non-negative, since the zeroes of  $Z_n(p, T)$  lie on the circle with  $|T|^2 = p^{3-n}$ .

**Conjecture 1.** *For odd  $n \geq 1$  and prime  $p \geq 2$ , if  $c_n^\pm(p)$  is non-zero then it is an integer whose core has no prime divisor greater than  $n$ .*

**Remark 3.** Recalling that  $y := p^{(n-3)/2}T$ , for odd  $n$ , we see from (9) that  $c_n^+(p) = 0$  if  $a(p, n) > 0$  and  $c_n^-(p) = 0$  if  $b(p, n) > 0$ . Moreover  $c_1^\pm(p) = 1$  and  $c_3^\pm(p) = 1 \mp \left(\frac{p}{3}\right)$  satisfy Conjecture 1. The conjecture has highly non-trivial content for odd  $n \geq 5$ .

#### 5.1. A modular form at $n = 5$

At  $n = 5$ , with  $y = pT$ , we have  $Z_5(3, T) = 1 + y$ ,  $Z_5(5, T) = 1 - y$  and  $Z_5(p, T) = 1 - y^2$  for  $\left(\frac{p}{15}\right) = -1$ . Since  $c_5^\pm(p) := pZ_5(p, \pm 1/p)$  is evaluated at  $y = \pm 1$ , Conjecture 1 holds in all of these cases.

When  $\left(\frac{p}{15}\right) = 1$ , we have  $Z_5(p, T) = 1 - c_5(p)T + y^2$  and hence obtain  $c_5^\pm(p) = 2p \mp c_5(p)$ . It follows that  $4p = c_5^+(p) + c_5^-(p)$  and  $-2c_5(p) =$

$c_5^+(p) - c_5^-(p)$ . There are *unique* positive integers,  $u$  and  $v$ , such that

$$(61) \quad 4p = \begin{cases} 15u^2 + v^2 & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = +1 \\ 3u^2 + 5v^2 & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = -1 \end{cases}$$

and it follows from work by Peters, Top and van der Vlugt [38] that the pair  $(u, v)$  determines

$$(62) \quad -2c_5(p) = \begin{cases} 15u^2 - v^2 & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = +1 \\ 3u^2 - 5v^2 & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = -1. \end{cases}$$

Hence Conjecture 1 holds at  $n = 5$  and the cores of  $(c_5^+(p), c_5^-(p))$  are  $(15, 1)$  for  $p \equiv 1$  or  $4 \pmod{15}$ , or  $(3, 5)$  for  $p \equiv 2$  or  $8 \pmod{15}$ . Underlying this great simplicity is a modular form of weight 3 and level 15, with a Fourier series, in the upper half of the complex  $\tau$ -plane,

$$(63) \quad f_{3,15}(\tau) = (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3 = \sum_{n>0} A_5(n)q^n$$

where  $q := \exp(2\pi i\tau)$ ,  $\eta_n := \eta(q^n)$ ,  $\eta(q) := q^{1/24} \prod_{k>0} (1 - q^k)$ ,  $A_5(1) = 1$  and, for prime  $p$ ,  $A_5(p) = c_5(p)$ .

### 5.2. A Hecke eigenform at $n = 7$

At  $n = 7$ , with  $y = p^2T$ , we have  $Z_7(3, T) = 1 - 10T + y^2$ ,  $Z_7(5, T) = 1 - y^2$ ,  $Z_7(7, T) = 1 - 70T + y^2$  and  $Z_7(11, T) = (1 - y)^2(1 + y)$ . Thus  $c_7^\pm(3) = 2(9 \mp 5)$ ,  $c_7^\pm(5) = 0$ ,  $c_7^\pm(7) = 14(7 \mp 5)$  and  $c_7^\pm(11) = 0$  all conform to Conjecture 1, since they are non-negative integers and when non-zero have cores that divide  $2 \times 3 \times 7 = 42$ .

For  $\left(\frac{p}{105}\right) = \pm 1$  we have  $c_7^\pm(p) = 0$ . For  $p = 2$  and for  $11 < p < 10^5$ ,  $c_7^\pm(p)/2 = p^2 \pm c_7(p)$  vanishes only at  $p = 48281$ . Otherwise it is a positive integer whose core divides 42. Hence Conjecture 1 holds at  $n = 7$  for all primes  $p < 10^5$ .

The answer 42, for the least common multiple of cores, comes from the existence of a weight-3 newform on  $\Gamma_0(525)$ , with quartic nebentypus and Hecke eigenvalues in  $\mathbf{Q}(\sqrt{-1}, \sqrt{6}, \sqrt{14})$ . For prime  $p$  coprime to 105, its  $p$ -th eigenvalue,  $\lambda_p$ , determines

$$(64) \quad c_7(p) = \left(\frac{p}{105}\right) (|\lambda_p|^2 - p^2)$$

as conjectured [24] by Evans and Stein and proven [46] by Yun and Vincent.

For  $\left(\frac{p}{105}\right) = \pm 1$ , we have  $c_7^\mp(p)/2 = |\lambda_p|^2$ , which vanishes at  $p = 11$  and at  $p = 48281$ . If  $\lambda_p$  is non-zero, then the core  $\mu$  of  $|\lambda_p|^2$  is determined by a signature composed of the signs  $\left(\frac{p}{q}\right)$ , for  $q = 3, 5$  and  $7$ , as follows:

$$(65) \quad \mu(+, +, +) = 1, \mu(-, -, +) = 3, \mu(+, -, -) = 7, \mu(-, +, -) = 21,$$

$$(66) \quad \mu(-, -, -) = 2, \mu(+, +, -) = 6, \mu(-, +, +) = 14, \mu(+, -, +) = 42.$$

This map from signatures to cores is multiplicative: if primes  $p_1, p_2$  and  $p_3$  give cores  $\mu_1, \mu_2$  and  $\mu_3$ , and  $\left(\frac{p_3}{q}\right) = \left(\frac{p_1 p_2}{q}\right)$ , for  $q = 3, 5$  and  $7$ , then  $\mu_3$  is the core of  $\mu_1 \mu_2$ . Hence 3 values of  $\mu$ , at the primes 2, 13 and 17, with signatures  $(-, -+), (+, -, -)$  and  $(-, -, -)$ , determine all 8 cores in (65,66).

### 5.3. Cores at $n = 9$ and $n = 11$

For  $n = 9$ , with  $y = p^3 T$ , we have  $Z_9(3, T) = 1 + 6T + y^2$ ,  $Z_9(5, T) = (1 + y)(1 - 106T + y^2)$ , and  $Z_9(7, T) = (1 - y)(1 + 238T + y^2)$ , for  $p|105$ . In these cases,  $c_7^\pm(p)$  either vanishes or is positive, with a core dividing 210, in accord with Conjecture 1. For  $\left(\frac{p}{105}\right) = -1$ , we have  $Z_9(p, T) = (1 - y^2)(1 - c_9(p)T + y^2)$  and hence  $c_7^\pm(p) = 0$ . The interesting case is  $\left(\frac{p}{105}\right) = 1$  for which the irreducible quartic polynomial

$$(67) \quad Z_9(p, T) = 1 - c_9(p)(T + p^6 T^3) + \frac{1}{2}(c_9^2(p) - c_9(p^2))T^2 + p^{12} T^4$$

requires computations in both  $\mathbf{F}_p$  and  $\mathbf{F}_{p^2}$ . At  $p = 2$  we find that  $c_9^+(2) = 5 \times 3^2$  and  $c_9^-(2) = 21$ , while at  $p = 13$  we have  $c_9^+(13) = 105 \times 24^2$  and  $c_9^-(13) = 80^2$ . There is a regular pattern:

$$(68) \quad (c_9^+(p), c_9^-(p)) = \begin{cases} (105u^2, v^2) & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{35}\right) = +1 \\ (5u^2, 21v^2) & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{35}\right) = -1 \end{cases}$$

with positive integers  $(u, v)$  determined by  $c_9(p)$  and  $c_9(p^2)$ , for all  $p \leq 631$  having the signatures in (68). Since  $c_9(p) = O(p^3)$  and  $c_9(p^2) = O(p^6)$ , much economy is achieved by the pair  $(u, v)$ , with  $u = O(v) = O(p^2)$ . In Section 8, I shall show how to achieve even greater economy.

For  $n = 11$ , with  $y = p^4 T$ , we have  $Z_{11}(3, T) = (1 - y)(1 + 150T + y^2)$  and  $Z_{11}(p, T)$  vanishing at  $y = \pm 1$ , for  $p = 5, 7$  and  $11$ . Thus we need compute only  $c_{11}^-(3) = 6^3$  to check Conjecture 1 for  $p|1155$ . For  $\left(\frac{p}{1155}\right) = \pm 1$ , we have  $c_{11}^\pm(p) = 0$  and  $c^\mp(p)$  an even positive integer for  $p \leq 631$ , with the

exception of the case  $p = 17$ , where

$$(69) \quad Z_{11}(17, T) = (1 - y)(1 + y)^2(1 - 27290T + y^2)$$

and hence  $c_{11}^+(17) = c_{11}^-(17) = 0$ . When  $c_{11}^\mp(p)/2$  is non-zero, its core,  $\mu$ , is determined by the signature  $\left(\frac{p}{q}\right)$ , for  $q = 3, 5, 7$  and  $11$ , according to a multiplicative map generated by  $\mu(+, +, -, -) = 2, \mu(+, -, -, +) = 3, \mu(+, -, +, +) = 5$  and  $\mu(+, -, -, -) = 11$ . Thus all the cores divide  $330$ .

### 5.4. Doctrine of signatures

For odd positive integer  $n$ , let  $n!! := \prod_{0 < 2k < n} (n - 2k)$  be the product of all odd positive integers not exceeding  $n$ . Let  $\omega(n!!)$  be the number of distinct primes  $q|n!!$ . Let the signature of a prime  $p$  coprime to  $n!!$  be the list of  $\omega(n!!)$  signs  $\left(\frac{p}{q}\right)$  for primes  $q|n!!$ .

**Conjecture 2.** *For  $n \equiv 1 \pmod 4, n \geq 5$  and  $\left(\frac{p}{n!!}\right) = 1, (c_n^+(p), c_n^-(p))$  is a pair of positive integers whose cores  $(\mu^+, \mu^-)$  are determined by a multiplicative map, from signatures to pairs of cores. Moreover, for each pair, the core of  $\mu^+\mu^-$  coincides with the core of  $n!!$ .*

**Conjecture 3.** *For  $n \equiv 3 \pmod 4, n \geq 7$  and  $\left(\frac{p}{n!!}\right) = \pm 1, c_n^\mp(p)/2$  is a non-negative integer. If this integer is non-zero, its core  $\mu$  is determined by a multiplicative map, from signatures to cores.*

These agree with the previous findings. At  $n = 5$  the pairs of cores are  $(15, 1)$  and  $(3, 5)$ ; at  $n = 9$  they are  $(105, 1)$  and  $(5, 21)$ . At  $n = 7$ , the cores are the 8 divisors of 42; at  $n = 11$  they are the 16 divisors of 330.

### 5.5. Tests at $n \geq 13$

At  $n = 13$ , the primes dividing  $13!!$  are 3, 5, 7, 11 and 13, with product 15015, which is also the core of  $13!!$ . At these 5 primes, Conjecture 1 holds, with non-zero values for  $c_{13}^+(3)$  and for all 5 cases of  $c_{13}^-(p)$  with  $p|15015$ . Conjecture 2 holds for  $\left(\frac{p}{15105}\right) = 1$  and  $p \leq 109$ , with a map generated by  $\mu^-(-, +, -, +, +) = 2, \mu^-(-, -, +, +, +) = 5, \mu^-(-, +, +, -, +) = 11$  and  $\mu^-(+, +, +, -, -) = 13$ . In all cases the core of  $\mu^+\mu^-$  is 15015. Hence  $\mu^-$  is a divisor of  $2 \times 5 \times 11 \times 13 = 1430$  and  $\mu^+$  is a divisor of 30030. The map was determined by data at 4 of the primes with  $\left(\frac{p}{15105}\right) = 1$  and tested at the remaining 12 cases with  $p \leq 109$ .

At  $n = 15$ , Conjecture 1 holds at the primes  $p|15015$ , where  $c_{15}^+(3)$ ,  $c_{15}^-(3)$  and  $c_{15}^+(5)$  are non-zero. Conjecture 2 holds for  $\left(\frac{p}{15015}\right) = \pm 1$  and  $p \leq 109$ , with a map generated by  $\mu(+, +, +, -, +) = 3$ ,  $\mu(-, -, +, -, +) = 5$ ,  $\mu(+, +, +, -, -) = 7$ ,  $\mu(+, -, -, -, -) = 11$  and  $\mu(+, +, -, -, +) = 13$ . Hence  $\mu$  is a divisor of 15015. The map was determined by data at 5 of the primes with  $\left(\frac{p}{15105}\right) = \pm 1$  and tested at 17 others cases with  $p \leq 109$ .

At  $n = 17$  and  $n = 19$ , we require data on  $c_n(p^k)$  with  $k \leq 4$  to test the conjectures. Algorithm 2 thus requires us to access a table of traces more than  $2p^8$  times. This becomes a daunting task at quite modest  $p$ . Tests of the conjectures were conducted for  $p \leq 31$ , at no small cost.

At  $n = 17$ , Conjecture 1 holds for the odd primes up to 17, with 7 non-zero values of  $c_{17}^\pm(p)$ , of which the largest is  $c_{17}^-(17) = 715 \times 135085536^2$ . The two primes  $p \leq 31$  with  $\left(\frac{p}{17!}\right) = 1$  give  $(c_{17}^+(2), c_{17}^-(2)) = (1105 \times 3^4, 385 \times 3^2)$  and  $(c_{17}^+(29), c_{17}^-(29)) = (35 \times 172876746240^2, 12155 \times 1827678336^2)$ . In both cases the core of  $\mu^+\mu^-$  is 17017, which is indeed the core 17!!.

At  $n = 19$ , Conjecture 1 holds for the odd primes up to 19, with 11 non-zero values of  $c_{19}^\pm(p)$ , of which the largest two are  $c_{19}^+(17) = 285 \times 167446803456^2$  and  $c_{19}^-(17) = 42 \times 1239609466368^2$ . Primes  $p \leq 31$  with  $\left(\frac{p}{19!}\right) = \pm 1$  yield

$$(70) \quad \mu(-, -, +, -, -, +, -) = 105, \quad \mu(-, -, +, +, +, -, +) = 19 \times 105,$$

$$(71) \quad \mu(-, +, +, -, +, -, -) = 13 \times 105, \quad \mu(+, +, -, +, -, -, -) = 7 \times 19,$$

with  $c_{19}^-(31)/2 = 7 \times 19 \times 3787940863744^2$  obtained by accessing a table of traces in  $\mathbf{F}_{31^4}$  more than  $1.7 \times 10^{12}$  times, by efficient algorithms, below.

### 6. Efficient computation of $c_n(p^k)$ and $Z_n(p, T)$

The trace in the definition

$$(72) \quad K(a) := \sum_{x \in \mathbf{F}_q^*} \exp\left(\frac{2\pi i}{p} \operatorname{trace}\left(x + \frac{a}{x}\right)\right)$$

of a Kloosterman sum is a trace of Frobenius in  $\mathbf{F}_q$  over  $\mathbf{F}_p$ . For  $q = p^k$ , it maps an element  $z$  of  $\mathbf{F}_q$ , in this case  $z = x + a/x$ , to an element

$$(73) \quad \operatorname{trace}(z) := \sum_{j=0}^{k-1} z^{p^j} = z + z^p + z^{p^2} + \dots + z^{q/p}$$

of  $\mathbf{F}_p$ , which we may take to be an integer in  $[0, p - 1]$ , modulo  $p$ , giving a  $p$ -th root of unity in the summand of (72). We need  $q - 1$  traces to evaluate the sum in (72) over all elements of  $\mathbf{F}_q$  except 0. We need  $K(a)$  for  $q - 1$  elements of  $\mathbf{F}_q$ , to evaluate the sum over symmetric powers in

$$(74) \quad c_n(q) := -\frac{1 + S_n(q)}{q^2}, \quad S_n(q) := \sum_{a \in \mathbf{F}_q^*} \sum_{j=0}^n [g(a)]^j [h(a)]^{n-j}$$

with  $K(a) = -g(a) - h(a)$  and  $g(a)h(a) = q$ . As written, (72,74) appear to ask for  $(q - 1)^2$  traces, each with  $k$  terms in (73). With  $q = 31^4$ , we do not want to evaluate  $4(31^4 - 1)^2 > 3 \times 10^{12}$  powers of elements of  $\mathbf{F}_q$ . The following algorithm shows how to avoid this.

### 6.1. Algorithm for $c_n(p^k)$

**Algorithm 2.** [Evaluation of  $c_n(q)$ , for  $q = p^k$  and all  $n \in [1, N]$ .]

- 1) Find a monic polynomial,  $f$ , of degree  $k$ , that is irreducible over  $\mathbf{F}_p[x]$ .  
[For  $q = 31^4$ , we may use  $f \equiv x^4 + x^3 + 2x^2 - 4x + 3 \pmod{31}$ .]
- 2) Find a polynomial,  $g$ , that is, modulo  $f$ , a primitive root for  $\mathbf{F}_q^*$ .  
[In the example, we may use  $g \equiv 7x^2 + 9x + 14 \pmod{f}$ , since  $g^{(q-1)/d} - 1$  is non-zero for all divisors  $d|(q - 1)$  with  $d > 1$ .]
- 3) For  $m \in [0, p - 1]$ , store in  $C[m]$  the value of  $\cos(2\pi m/p)$ , at suitable numerical precision. [The sum in (72) is real, so cosines suffice.]
- 4) For  $n \in [1, N]$  and  $m \in [1, N]$ , store in  $U[n, m]$  the integer coefficient of  $x^n y^m$  in the expansion of  $1/(1 + xy + x^2q)$ . [This vectorizes (74).]
- 5) For  $m \in [0, k - 1]$ , store in  $X[m]$  the trace of  $z^m$ , with  $z \equiv x \pmod{f(x)}$ .  
[In the example, we obtain, from (73),  $X \equiv [4, 30, 28, 17] \pmod{31}$ .]
- 6) For  $r \in [1, q - 1]$ , store in  $T[r]$  the trace of  $g^r$ , using the data in  $X$ , as follows. Set  $r = 0, t = 1$ . While  $r < q - 1$ , add 1 to  $r$ , multiply  $t$  by  $g$ , set  $T[r] = \sum_m t_m X[m]$ , where  $t_m$  is the coefficient of  $x^m$  in  $t$ .  
[In the example,  $g$  gives  $T[1] = 7X[2] + 9X[1] + 14X[0] \equiv 26 \pmod{31}$ , then  $g^2 = 15x^3 + 24x^2 + 14x + 18$  gives  $T[2] \equiv 28 \pmod{31}$ , and so on.]
- 7) For  $a \in [1, q - 1]$ , store  $K[a] = \sum_{0 < r < q} C[m(a, r)]$ , with  $m(a, r) \in [0, p - 1]$  given by  $m(a, r) \equiv T[r] + T[s(a, r)] \pmod{p}$ , where  $s(a, r) \in [1, q - 1]$  is given by  $s(a, r) \equiv (a - r) \pmod{q - 1}$ .
- 8) For  $m \in [1, N]$ , store  $V[m] = \sum_{0 < a < q} (K[a])^m$ .

- 9) For  $n \in [1, N]$ , compute  $S[n] = \sum_{0 < m \leq N} U[n, m]V[m]$  and return  $c_n(q)$  as the integer nearest to  $-(1 + S[n])/q^2$ .

For large  $q$ , most time is spent in Step 7, where  $T$  is accessed  $2(q - 1)^2$  times. The entries in  $T$  are determined in Step 6 by  $(q - 1)$  multiplications in  $\mathbf{F}_q$  and  $O(kq)$  accesses to  $X$ . The  $k$  entries in  $X$  are determined in Step 5 by binary exponentiations requiring  $O(k^2 \log(q)/\log(2))$  multiplications.

### 6.2. Algorithm for $Z_n(p, T)$

**Algorithm 3.** [Determination of  $Z_n(p, T)$  for all  $n \in [1, N]$ .]

- 1) Use Theorem 1 and Algorithm 1 to determine the largest  $d(p, n)$  for  $n \in [1, N]$  and record this as  $D$ . If  $D > 0$ , then use Algorithm 2, for  $k \in [1, D]$ , and store  $C[n, k] = c_n(p^k)$ , for  $n \in [1, N]$ .
- 2) For  $n \in [1, N]$ , set  $d = d(p, n)$ ,  $y = p^{\lfloor (n-3)/2 \rfloor} T$ ,  $z = p^{n-3} T^2$ . Find  $F(T) = (1 - y)^{a(p,n)}(1 + y)^{b(p,n)}$ , using (20), or (31,32), or (36), or (45,46), according as the parities of  $n$  and  $p$ . Develop the expansion

$$\exp\left(-\sum_{0 < k \leq d} C[n, k] T^k / k\right) / F(T) = \sum_{0 \leq k \leq d} m_k T^k + O(T^{d+1}).$$

Return  $Z_n(p, T) = F(T)(m_d T^d + \sum_{0 \leq k < d} m_k (1 + z^{d-k}) T^k)$ .

This minimizes the number,  $D$ , of uses of Algorithm 2 and maximally exploits the functional equation (10). To obtain only  $Z_{2k}(p, T)$ , set  $D$  to the largest  $d(p, 2k)$  for  $2k \leq N$ , which is often smaller than the  $D$  above.

### 6.3. Features of the database

The database compiled with these algorithms contains determinations of  $Z_n(p, T)$  for  $p < 10^5$  at  $n \leq 8$ ; for  $p \leq 631$  at  $n \leq 12$ ; for  $p \leq 109$  at  $n \leq 16$ ; for  $p \leq 31$  at  $n \leq 20$ .

At small fixed  $p$ , Algorithm 3 was used to extend the range of  $n$  as follows. At  $p = 2$ , determinations were made for even  $n \leq 200$  and odd  $n \leq 65$ ; at  $p = 3$  for even  $n \leq 116$  and odd  $n \leq 59$ ; at  $p = 5$  for even  $n \leq 44$  and odd  $n \leq 35$ ; at  $p = 7$  and  $p = 11$  for all  $n \leq 27$ .

The results are available in files, on serious request to the author. They conform with Conjectures 1, 2 and 3, at all listed  $p$  and odd  $n$ .

For even  $n$ ,  $M_n(p, T)$  was always found to be irreducible, with  $d(p, n)$  pairs of complex roots at  $T = \exp(\pm i\theta)p^{(3-n)/2}$  and  $0 < \theta < \pi$ . For odd  $n$ , only  $M_7(11, T) = (1 - y)^2$ ,  $M_7(48281, T) = (1 - y)^2$  and  $M_{11}(17, T) = (1 + y)^2(1 - 27290T + y^2)$  were found to be reducible.

For odd  $n < p$ , the choice of exponent in (60) is almost always minimal. There is only one case in the database with non-zero  $c_n^\pm(p)$  divisible by  $p$ , for odd  $n < p$ , namely at  $n = 13$  and  $p = 109$ , where

$$(75) \quad c_{13}^+(109) = 3 \times 5 \times 7 \times 11 \times 109^2 \times 515328^2$$

$$(76) \quad c_{13}^-(109) = 13 \times 109^2 \times 2399960^2$$

are in full accord with Conjectures 1 and 2, but also divisible by  $p^2$ .

From Corollary 1, we obtain a bound  $d(2, 2k) = d(3, 2k) \leq \lfloor \frac{k+1}{6} \rfloor$ . This relatively slow growth enables one to determine very large values of  $c_n(q)$  when  $n$  is even and  $q$  has characteristic 2 or 3. Here are examples yielding probable primes, of sizes ranging from 3238 to 4366 decimal digits:

$$(77) \quad p_{3238} = -2^{-157}c_{142}(2^{157}), \quad p_{3449} = 2^{-632}c_{156}(2^{158}),$$

$$(78) \quad p_{3614} = 2^{-2}3^{-279}c_{116}(3^{139}), \quad p_{3638} = -c_{82}(3^{193}),$$

$$(79) \quad p_{3903} = 2^{-174}c_{154}(2^{174}), \quad p_{4366} = -2^{-200}c_{150}(2^{200}).$$

### 7. L-series and Feynman integrals

For  $n < 8$  and  $s$  with a suitably large real part, I define the L-series

$$(80) \quad L_n(s) := \prod_{p \geq 2} \frac{1}{Z_n(p, p^{-s})} = \sum_{m > 0} \frac{A_n(m)}{m^s}$$

with  $A_n(1) = 1$ ,  $A_n(m_1 m_2) = A_n(m_1)A_n(m_2)$ , for  $\text{gcd}(m_1, m_2) = 1$ , and  $A_n(p) = c_n(p)$  at prime  $p$ . Then  $L_1(s) = L_2(s) = 1$  and from (8) we obtain

$$(81) \quad L_3(s) = \sum_{m > 0} \left(\frac{m}{3}\right) m^{-s} = \sum_{k \geq 0} \left( \frac{1}{(3k+1)^s} - \frac{1}{(3k+2)^s} \right)$$

$$(82) \quad L_4(s) = (1 - 2^{-s})\zeta(s) = \sum_{k \geq 0} \frac{1}{(2k+1)^s}$$

with a functional equation

$$(83) \quad \Lambda_3(s) := \left(\frac{3}{\pi}\right)^{s/2} \Gamma\left(\frac{s+1}{2}\right) L_3(s) = \Lambda_3(1-s)$$

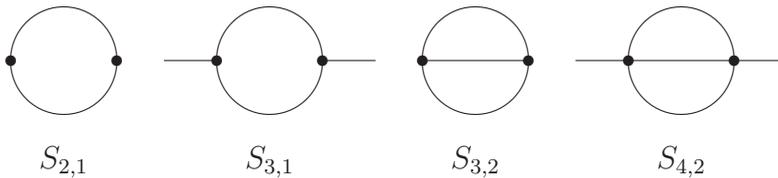
for the Dirichlet L-series (81), giving analytic continuation to  $L_3(0) = \sqrt{3}L_3(1)/\pi = \frac{1}{3}$ , while at  $n = 4$  Riemann and Euler give us  $L_4(0) = 0$  and  $L_4(2) = \pi^2/8$ .

Now consider Feynman diagrams, like those illustrated below, evaluated, in two spacetime dimensions, by integrals of the form

$$(84) \quad S_{n,s} := 2^s \int_0^\infty [I_0(t)]^{n-s-1} [K_0(t)]^{s+1} t dt$$

with  $n$  Bessel functions and loop-number (i.e. first Betti number)  $s$  satisfying  $s < n \leq 2s + 2$  and  $s > 1$  for  $n = 2s + 2$ , to ensure convergence.

The internal scalar particles have unit mass and account for the Bessel function  $K_0(t)$  in the integrand. Hence the two-loop vacuum integral  $S_{3,2}$  has a propagator  $K_0(t)$  associated with each of its three internal edges.  $S_{4,2} := 2^2 \int_0^\infty I_0(t)K_0^3(t) t dt$  is a two-loop on-shell sunrise diagram, with the Bessel function  $I_0(t)$  coming from external half-edges, whose momenta are on the unit mass shell. The one-loop diagram  $S_{3,1}$  is obtained from  $S_{3,2}$  by cutting an internal edge. Removing the external half-edges from  $S_{3,1}$ , we obtain the one-loop vacuum diagram  $S_{2,1}$ . If we join up the half-edges in  $S_{4,2}$ , we obtain a three-loop vacuum diagram,  $S_{4,3}$ .



For  $n \leq 4$ , the  $s$ -loop integral  $S_{n,s}$  is an integer multiple [3] of  $L_n(s)$ :

$$(85) \quad S_{1,0} = L_1(0) = 1$$

$$(86) \quad S_{2,1} = L_2(1) = 1$$

$$(87) \quad S_{3,1} = 2L_3(1) = \frac{2\pi}{\sqrt{3^3}}$$

$$(88) \quad S_{3,2} = 3L_3(2) = 3 \sum_{k \geq 0} \left( \frac{1}{(3k+1)^2} - \frac{1}{(3k+2)^2} \right)$$

$$(89) \quad S_{4,2} = 2L_4(2) = \frac{\pi^2}{4}$$

$$(90) \quad S_{4,3} = 8L_4(3) = 7\zeta(3).$$

### 7.1. Proofs for 5 Bessel functions

In a conference talk, *Reciprocal PSLQ and the tiny nome of Bologna*, given in June 2007 at the Zentrum für interdisziplinäre Forschung in Bielefeld, I presented the empirical evaluation [11]

$$(91) \quad C := \frac{S_{5,3}}{8\pi^2} = \frac{\pi}{16} \left(1 - \frac{1}{\sqrt{5}}\right) \left(\sum_{n=-\infty}^{\infty} e^{-n^2\pi\sqrt{15}}\right)^4$$

for the on-shell 3-loop sunrise diagram  $S_{5,3}$  in two spacetime dimensions. This implies a neat evaluation as a product of values of the gamma function [36]

$$(92) \quad S_{5,3} = \frac{1}{30\sqrt{5}} \prod_{k=0}^3 \Gamma\left(\frac{2^k}{15}\right)$$

by applying the Chowla-Selberg theorem to elliptic integrals at the 15th singular value [3]. Intense work in 2007 with Jon Borwein, in Halifax, Nova Scotia, showed that (92) is equivalent to

$$(93) \quad S_{5,3} = \frac{4\pi}{\sqrt{15}} S_{5,2}.$$

Discussions with Spencer Bloch and Francis Brown, at a summer school in Les Houches, organized by Dirk Kreimer in 2010, pointed to a connection with the modular form  $f_{3,15} = (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3$  of Section 5.1. This came from the representation of

$$(94) \quad S_{5,3} = \int_0^\infty \int_0^\infty \int_0^\infty \frac{da \, db \, dc}{(abc + ab + bc + ca)(a + b + c) + ab + bc + ca}$$

as an integral over Schwinger parameters. Then counts of the zeros of the denominator of the integrand, in finite fields  $\mathbf{F}_p$ , implicated the Fourier coefficients  $A_5(p) = c_5(p)$  of  $f_{3,15}$ , at small primes.

Thus I was led to investigate the L-series obtained by setting  $n = 5$  in (80). This has a functional equation

$$(95) \quad \Lambda_5(s) := \left(\frac{15}{\pi^2}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_5(s) = \Lambda_5(3-s)$$

and analytic continuation yields a convergent expansion for

$$(96) \quad \frac{2\pi}{\sqrt{15}} L_5(1) = L_5(2) = \sum_{n>0} \frac{A_5(n)}{n^2} \left(1 + \frac{4\pi n}{\sqrt{15}}\right) \exp\left(-\frac{2\pi n}{\sqrt{15}}\right).$$

Numerical comparison with (92) then gave the evaluations

$$(97) \quad S_{5,2} = 3L_5(2)$$

$$(98) \quad S_{5,3} = \frac{48}{5} \zeta(2) L_5(1).$$

The neat result (97) was proven by combining my work with Bailey, Borwein and Glasser [3], on  $S_{5,2}$ , with work by Rogers, Wan and Zucker [40] on  $L_5(2)$ . The proof of (98) follows from work by Bloch, Kerr and Vanhove [7] on  $S_{5,3}$ . These authors evaluated the 3-loop sunrise diagram off the external mass shell, in terms of an elliptic trilogarithm. Delicate work on the on-shell limit, elucidated by Detchat Samart [43], then gave a reduction to gamma values. Hence all the equations (91) to (98) now have proofs, after 8 years of hard work.

## 7.2. Conjectures for 5 Bessel functions

The 4-loop vacuum integral with 5 Bessel functions has a representation

$$(99) \quad S_{5,4} = 2 \int_0^\infty \int_0^\infty \int_0^\infty \frac{\log(a+b+c+1) da db dc}{(abc+ab+bc+ca)(a+b+c)+ab+bc+ca}$$

with the same denominator as for  $S_{5,3}$  in (94), but a logarithmic numerator, resulting from integration over an extra Schwinger parameter. This resisted 8 years of effort to find a relation to the L-series  $L_5(s)$  until Anton Mellit and I met at the Mainz Institute for Theoretical Physics in 2015 and experimented with determinants of matrices of Bessel moments. On the basis of numerical investigation, we arrived at the conjectures

$$(100) \quad \det \int_0^\infty I_0(t) K_0^3(t) \begin{bmatrix} K_0(t) & t^2 K_0(t) \\ I_0(t) & t^2 I_0(t) \end{bmatrix} t dt \stackrel{?}{=} \frac{2\pi^3}{\sqrt{3^3 5^3}}$$

$$(101) \quad \det \int_0^\infty K_0^3(t) \begin{bmatrix} K_0^2(t) & t^2 K_0^2(t) \\ I_0^2(t) & t^2 I_0^2(t) \end{bmatrix} t dt \stackrel{?}{=} \frac{45}{8\pi^2} L_5(4)$$

with question marks indicating that these evaluations are as yet unproven.

Our rationale for these constructions was as follows. In 2007, reciprocal PSLQ for the matrix whose determinant is taken in (100) gave

$$(102) \quad \begin{bmatrix} \pi^2 C & \pi^2 \left(\frac{2}{15}\right)^2 \left(13C - \frac{1}{10C}\right) \\ \frac{\sqrt{15}\pi}{2} C & \frac{\sqrt{15}\pi}{2} \left(\frac{2}{15}\right)^2 \left(13C + \frac{1}{10C}\right) \end{bmatrix}$$

and since then the first column and second row have been proven. The entry in row 1 and column 2 is conjectural, though checked to 1000-digit precision. It has precisely the form to make the determinant independent of  $C$ . In (101), we chose a matrix whose second row is identical to that in (100), but with vacuum integrals appearing in the first row, and were rewarded by a result for  $L_5(4)$ , outside the critical strip. Conjectures (100,101) may be combined with proven results to obtain the striking evaluation

$$(103) \quad \frac{L_5(4)}{L_5(2)\zeta(2)} \stackrel{?}{=} \frac{4}{5} \int_0^\infty (R - t^2) K_0^5(t) t dt$$

$$(104) \quad R := 13 \left(\frac{2}{15}\right)^2 + 32 \prod_{k=0}^3 \frac{\Gamma(1 - 2^k/15)}{\Gamma(2^k/15)}$$

with our lucky integer 13 inferred from (102). It would have been hard to arrive at (103,104) without taking determinants in (100,101).

### 7.3. Conjectures for 6 Bessel functions

At  $n = 6$ , I found, with help from Francis Brown at Les Houches in 2010, a modular form of weight 4 and level 6

$$(105) \quad f_{4,6}(\tau) = (\eta_1 \eta_2 \eta_3 \eta_6)^2 = \sum_{n>0} A_6(n) q^n$$

with  $A_6(p)$  at the primes agreeing with counts in  $\mathbf{F}_p$  of zeros of the denominator of the Feynman integrand for the 4-loop sunrise diagram  $S_{6,4}$ , with 6 Bessel functions. Work by Hulek, Spandaw, van Geemen and van Straten [31] showed that (105) solves the Kloosterman problem at  $n = 6$ , giving  $A_6(p) = c_6(p)$  at the primes. Then the functional equation

$$(106) \quad \Lambda_6(s) := \left(\frac{6}{\pi^2}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_6(s) = \Lambda_6(4-s)$$

yielded convergent expansions for

$$(107) \quad L_6(2) = \sum_{n>0} \frac{A_6(n)}{n^2} \left( 2 + \frac{4\pi n}{\sqrt{6}} \right) \exp \left( -\frac{2\pi n}{\sqrt{6}} \right)$$

$$(108) \quad 2\zeta(2)L_6(1) = L_6(3) = \sum_{n>0} \frac{A_6(n)}{n^3} \left( 1 + \frac{2\pi n}{\sqrt{6}} + \frac{2\pi^2 n^2}{3} \right) \exp \left( -\frac{2\pi n}{\sqrt{6}} \right)$$

at the three integers inside the critical strip, with  $0 < s < 4$ , and numerical investigation led to the conjectures

$$(109) \quad S_{6,2} \stackrel{?}{=} 6L_6(2)$$

$$(110) \quad S_{6,3} \stackrel{?}{=} 12L_6(3) = 24\zeta(2)L_6(1)$$

$$(111) \quad S_{6,4} \stackrel{?}{=} 48\zeta(2)L_6(2)$$

which have been checked at 1000-digit precision. It is notable that  $S_{6,4}$  is pulled down to  $L_6(2)$ , by a multiple of  $\zeta(2)$ . Hence conjectures (109,111) imply the sum rule [3]

$$(112) \quad \int_0^\infty I_0(t)K_0^3(t) (\pi^2 I_0^2(t) - 3K_0^2(t)) t dt \stackrel{?}{=} 0.$$

It was harder to relate Feynman integrals to  $L_6(5)$ , outside the critical strip. This problem was cracked by using determinants in conjectures

$$(113) \quad \det \int_0^\infty I_0(t)K_0^4(t) \begin{bmatrix} K_0(t) & t^2 K_0(t) \\ I_0(t) & t^2 I_0(t) \end{bmatrix} t dt \stackrel{?}{=} \frac{5}{32} \zeta(4)$$

$$(114) \quad \det \int_0^\infty K_0^4(t) \begin{bmatrix} K_0^2(t) & t^2 K_0^2(t) \\ I_0^2(t) & t^2 I_0^2(t) \end{bmatrix} t dt \stackrel{?}{=} \frac{27}{4\pi^2} L_6(5)$$

that neatly follow the pattern discovered at  $n = 5$ , in (100,101).

### 7.4. Mahler measures and vacuum diagrams

A Laurent polynomial  $P(x_1, \dots, x_n)$  has a logarithmic Mahler measure

$$(115) \quad m(P) := \int_0^1 dt_1 \cdots \int_0^1 dt_n \log (|P(e^{2\pi i t_1}, \dots, e^{2\pi i t_n})|)$$

that may sometimes evaluate to an L-series [41]. For example Christopher Deninger [21] conjectured that

$$(116) \quad m \left( 1 + x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} \right) = \frac{15}{4\pi^2} L_{2,15}(2)$$

where the L-series comes from the modular form  $\eta_1\eta_3\eta_5\eta_{15}$ , with weight 2 and level 15. This was proven by Mathew Rogers and Wadim Zudilin [42]. David Boyd conjectured [9] and Anton Mellit proved [37] that

$$(117) \quad m \left( 1 + x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} + x_1x_2 + \frac{1}{x_1x_2} \right) = \frac{7}{2\pi^2} L_{2,14}(2)$$

where the L-series comes from  $\eta_1\eta_2\eta_7\eta_{14}$ , with weight 2 and level 14.

It is instructive to note that

$$(118) \quad m(1 + x_1 + x_2) = \frac{\sqrt{3}}{4\pi} S_{3,2} = \frac{\sqrt{3^3}}{4\pi} L_3(2)$$

$$(119) \quad m(1 + x_1 + x_2 + x_3) = \frac{1}{2\pi^2} S_{4,3} = \frac{7}{2\pi^2} \zeta(3) = \frac{4}{\pi^2} L_4(3)$$

evaluate in terms of vacuum diagrams  $S_{n,n-1} := 2^{n-1} \int_0^\infty K_0^n(t) t dt$ , with  $n$  Bessel functions at  $n - 1$  loops. One might therefore expect the L-series for the 5 and 6-Bessel modular forms,  $(\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3$  and  $(\eta_1\eta_2\eta_3\eta_6)^2$ , to determine Mahler measures, outside their critical strips, where (101,114) give conjectural evaluations of  $L_5(4)$  and  $L_6(5)$  in terms of determinants that include  $S_{5,4}$  and  $S_{6,5}$ , respectively. Indeed they do, via relations

$$(120) \quad m(1 + x_1 + x_2 + x_3 + x_4) \stackrel{?}{=} 6 \left( \frac{\sqrt{15}}{2\pi} \right)^5 L_5(4)$$

$$(121) \quad m(1 + x_1 + x_2 + x_3 + x_4 + x_5) \stackrel{?}{=} 3 \left( \frac{\sqrt{6}}{\pi} \right)^6 L_6(5)$$

conjectured by Rodriguez Villegas [45], listed in [25] and tested in [2], at 1000-digit precision, using a Bessel formula [8]

$$(122) \quad m(1 + x_1 + \dots + x_{n-1}) = -\log(2) - \gamma - \int_0^\infty \frac{dJ_0^n(t)}{dt} \log(t) dt$$

that I had derived from Kluyver’s work in 1905 on  $n$ -step walks [34]. Here  $\gamma$  is Euler’s constant and the oscillatory Bessel function  $J_0(t) := I_0(it)$  leads to a demanding quadrature, at high precision, as in the case of off-shell

sunrise diagrams with spacelike external data,  $w^2 < 0$ . As far as I know, no-one has related logarithmic Mahler measures to non-critical L-series from modular forms with weight greater than 4. Nevertheless, the story of the relation between vacuum diagrams and non-critical L-series, is not finished, as I shall show at  $n = 8$ , with a modular form of weight 6.

### 7.5. Conjectures for 7 Bessel functions

Until last year, no relation between L-series and Feynman integrals with 7 Bessel functions had been discovered. Undaunted by previous failure, Anton Mellit proposed that we use my determinations of local factors, from Kloosterman sums in  $\mathbf{F}_q$ , in the L-series

$$(123) \quad L_7(s) := \prod_{p \geq 2} \frac{1}{Z_7(p, p^{-s})} = \sum_{n > 0} \frac{A_7(n)}{n^s}$$

and try to discover a functional relation that would allow analytic continuation to critical values with  $0 < s < 5$ . Crucial in this endeavour were my determinations of the local factors

$$(124) \quad Z_7(2, 2^{-s}) = \left(1 - \frac{1}{2^{s-2}}\right) \left(1 + \frac{5}{2^{s-2}} + \frac{1}{2^{2s-4}}\right)$$

$$(125) \quad Z_7(3, 3^{-s}) = 1 - \frac{10}{3^{s-2}} + \frac{1}{3^{2s-4}}$$

$$(126) \quad Z_7(5, 5^{-s}) = 1 - \frac{1}{5^{2s-4}}$$

$$(127) \quad Z_7(7, 7^{-s}) = 1 - \frac{10}{7^{s-3}} + \frac{1}{7^{2s-4}}$$

at  $p \leq 7$ , where Evans had been silent [24]. Thereafter, it was sufficient to use **Sage**, with commands kindly provided by William Stein, to determine local factors from Hecke eigenvalues,  $\lambda_p \in \mathbf{Q}(\sqrt{-1}, \sqrt{6}, \sqrt{14})$ , at prime  $p$ , of the newform on  $\Gamma_0(525)$ , with weight 3 and quartic nebentypus, that gives  $|\lambda_p|^2 = p^2 \pm A_7(p)$ , for  $\left(\frac{p}{105}\right) = \pm 1$ . Using computer power kindly provided at the Humboldt Universität zu Berlin, I ran **Sage** for several days and verified that for  $7 < p < 10^5$  the cores of  $|\lambda_p|^2$  are divisors of 42, determined from signatures by the multiplicative map (65,66), except in the cases  $p = 11$  and  $p = 48281$ , with signature  $(-, +, +)$  and  $A_7(p) = p^2$ , where  $\lambda_p$  vanishes.

Having determined  $A_7(n)$  for  $n \leq 100000$ , I used Tim Dokchitser’s code [22] `computel`, in `Pari-GP`, to determine the viability of a functional equation

$$(128) \quad \Lambda_7(s) := \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_7(s) \stackrel{?}{=} \Lambda_7(5-s)$$

with the crucial factor  $\Gamma((s-1)/2)$  found empirically. Then `computel` professed itself ready to compute  $L_7(s)$  and was used at low precision to determine a conjectural integer 20 in

$$(129) \quad S_{7,4} := 2^4 \int_0^\infty I_0^2(t) K_0^5(t) t dt \stackrel{?}{=} 20\zeta(2)L_7(2)$$

which was then confirmed to 1000-digit precision, in less than 7 hours.

The 4-loop Feynman integral  $S_{7,4}$  is the sole integral with 7 Bessel functions that I have been able to relate to  $L_7(s)$ . It is not hard to see why there is only one. The factor  $\Gamma((s-1)/2)$  in the functional equation (128) seems to render  $s = 1$  and hence  $s = 5 - 1 = 4$  inaccessible. Inside the critical strip, with  $0 < s < 5$ , that leaves only  $s = 2$ , which is equivalent to  $s = 5 - 2 = 3$ , by the functional equation.

No determinant was found to permit an excursion to  $s = 6$ , outside the critical strip. However a  $3 \times 3$  matrix of moments of 7 Bessel functions,

$$(130) \quad M_3 := \int_0^\infty I_0(t) K_0^4(t) \begin{bmatrix} K_0^2(t) & t^2 K_0^2(t) & t^4 K_0^2(t) \\ I_0(t) K_0(t) & t^2 I_0(t) K_0(t) & t^4 I_0(t) K_0(t) \\ I_0^2(t) & t^2 I_0^2(t) & t^4 I_0^2(t) \end{bmatrix} t dt$$

gave the intriguing numerical result

$$(131) \quad \det M_3 \stackrel{?}{=} \frac{2^4 \pi^6}{\sqrt{3^3 5^5 7^7}}$$

with the square root of  $3^3 5^5 7^7$  resonating with the square root of  $3^3 5^5$  in (100). Defining  $M_k$  to be the  $k \times k$  matrix with elements

$$(132) \quad (M_k)_{a,b} := \int_0^\infty [I_0(t)]^a [K_0(t)]^{2k+1-a} t^{2b-1} dt$$

that are moments of  $n = 2k + 1$  Bessel functions, I found a similar pattern, working up to a  $15 \times 15$  matrix at  $n = 31$ , where the striking evaluation

$$(133) \quad \det M_{15} \stackrel{?}{=} \frac{2^{182} \pi^{120}}{3^{33} 5^{20} 7^5 \sqrt{11^3 13^9 17^{17} 19^{19} 23^{23} 29^{29} 31^{31}}}$$

was found and checked at 500-digit precision.

**Conjecture 4.** *The  $k \times k$  matrix with elements (132) has determinant*

$$(134) \quad \det M_k = \prod_{j=1}^k \frac{(2j)^{k-j} \pi^j}{\sqrt{(2j+1)^{2j+1}}}.$$

### 7.6. Conjectures for 8 Bessel functions

At  $n = 8$ , we need to modify the definition (80), which served well for  $n < 8$ . I discovered that the modular form

$$(135) \quad f_{6,6}(\tau) = \left( \frac{\eta_2^3 \eta_3^3}{\eta_1 \eta_6} \right)^3 + \left( \frac{\eta_1^3 \eta_6^3}{\eta_2 \eta_3} \right)^3 = \sum_{n>0} A_8(n) q^n$$

with weight 6 and level 6, gives  $A_8(p) \equiv c_8(p) \pmod p$ , at the primes. However, we do not have equality between  $A_n(p)$  and  $c_n(p)$  for  $n = 8$  and prime  $p > 2$ . Instead I found that

$$(136) \quad c_8(p) = \begin{cases} A_8(p) & \text{if } p = 2 \\ p^4 + A_8(p) & \text{if } p > 2 \end{cases}$$

and hence, from (8), that

$$(137) \quad L_8(s) := \prod_{p \geq 2} \frac{Z_4(p, p^{4-s})}{Z_8(p, p^{-s})} = \sum_{n>0} \frac{A_8(n)}{n^s}$$

is the L-series corresponding to the modular form (135), as was recently proven by Yun [46]. Then the functional equation

$$(138) \quad \Lambda_8(s) := \left( \frac{6}{\pi^2} \right)^{s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_8(s) = \Lambda_8(6-s)$$

enables analytic continuation inside the critical strip,  $0 < s < 6$ , where

$$(139) \quad L_8(3) = \sum_{n>0} \frac{A_8(n)}{n^3} \left( 2 + \frac{4\pi n}{\sqrt{6}} + \frac{2\pi^2 n^2}{3} \right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right)$$

$$(140) \quad L_8(4) = \sum_{n>0} \frac{A_8(n)}{n^4} \left( 1 + \frac{2\pi n}{\sqrt{6}} + \frac{4\pi^2 n^2}{9} + \frac{4\pi^3 n^3}{9\sqrt{6}} \right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right)$$

are given by rapidly convergent series. At  $s = 5$  we obtain

$$(141) \quad L_8(5) = \sum_{n>0} \frac{A_8(n)}{n^5} \left( 1 + \frac{2\pi n}{\sqrt{6}} + \frac{\pi^2 n^2}{3} + \frac{2\pi^3 n^3}{9\sqrt{6}} + \frac{\pi^4 n^4}{27} \right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right)$$

which should be a rational multiple of  $\zeta(2)L_8(3)$ , according to experts [30] in Eichler-Shimura-Manin theory. I verified, at 1000-digit precision, that

$$(142) \quad \frac{L_8(5)}{\zeta(2)L_8(3)} \stackrel{?}{=} \frac{4}{7}$$

and expect equality to be soon proven, by at least one expert.

Evaluating (139,140) numerically and using (142), I found that

$$(143) \quad S_{8,3} \stackrel{?}{=} 8L_8(3)$$

$$(144) \quad S_{8,4} \stackrel{?}{=} 36L_8(4)$$

$$(145) \quad S_{8,5} \stackrel{?}{=} 216L_8(5)$$

$$(146) \quad S_{8,6} \stackrel{?}{=} 864\zeta(2)L_8(4)$$

and have checked these evaluations at 1000-digit precision. Since these four Feynman integrals have evaluations in terms of only two independent critical values of  $L_8(s)$ , we obtain two conjectural sum rules, namely

$$(147) \quad \int_0^\infty (9\pi^2 I_0^2(t) - 14K_0^2(t)) I_0^2(t) K_0^4(t) t dt \stackrel{?}{=} 0$$

$$(148) \quad \int_0^\infty (\pi^2 I_0^2(t) - K_0^2(t)) I_0(t) K_0^5(t) t dt \stackrel{?}{=} 0$$

of which the second is merely the tip of a remarkable empirical iceberg.

### 7.7. A conjecture dedicated to Richard Crandall

Richard Crandall (1947–2012) was proudly part Cherokee, a physicist and a computational number theorist. He studied under Richard Feynman at Caltech, under Viki Weisskopf at MIT, and worked in the physics department at Reed College, Oregon, from 1978 until his untimely death from acute myeloid leukemia. With Carl Pomerance, he wrote a particularly fine book, *Prime numbers, a computational perspective* [20], whose well-thumbed first edition has been a source of delight to me for 15 years. His work on *Integrals of the Ising class* [4] led him to high-precision evaluation of Bessel moments.

In Richard's memory, I offer a striking conjecture that generalizes (148), to an astounding degree, and has resulted in the discovery of a 204433-digit probable prime divisor of a Bessel moment.

**Conjecture 5.** For positive integer  $n$ , let

$$(149) \quad A(n) := \left(\frac{2}{\pi}\right)^4 \int_0^\infty (\pi^2 I_0^2(t) - K_0^2(t)) I_0(t) K_0^5(t) (2t)^{2n-1} dt.$$

Then  $A(n)$  is a non-negative integer and for  $1 < n < x$  the number of cases for which precisely one prime  $p \geq n/2$  divides  $A(n)$  is asymptotic to

$$(150) \quad N(x) := \frac{\exp(\gamma)[\log(x/2)]^2}{4 + 8 \log(2)}.$$

**Remark 4.** From numerical integration, at high precision, I found that, very probably,  $A(n)$  gives a sequence of integers, beginning with 0, 1, 2, 15, 302, 12559, 900288, 98986140, 15459635718 for  $n = 1$  to 9, with  $A(1) = 0$  corresponding to sum rule (148). Then I used the recursion

$$(151) \quad \sum_{k=0}^4 (-1)^k (k+1) P_k(2n+k) A(n+k) = 0$$

with polynomial coefficients  $P_0(x) = x^9$ ,  $P_1(x) = 15x^7 + 63x^5 + 45x^3 + 5x$ ,  $P_2(x) = 91x^5 + 365x^3 + 188x$ ,  $P_3(x) = 205x^3 + 371x$  and  $P_4(x) = \frac{576}{5}x$ , determined by the differential equation satisfied by both  $I_0(t)$  and  $K_0(t)$ , discovering that the (presumed) integers  $A(n)$  are rich in odd prime factors  $p < n/2$ . For example,

$$(152) \quad A(33) = 2 \times 3^{23} \times 5^{11} \times 7^6 \times 11^2 \times 13^2 \times p_{56}$$

$$(153) \quad A(36) = 3^{25} \times 5^{13} \times 7^8 \times 11^4 \times 13^2 \times 17^2 \times p_{59}$$

$$(154) \quad A(49) = 2^3 \times 3^{39} \times 5^{16} \times 7^{10} \times 11^6 \times 13^4 \times 17^2 \times 19^3 \times 23^2 \times p_{86}$$

where  $p_{56}$  is the 56-digit prime

$$57992474894877287439798522082574263282518819530344295461.$$

Similarly,  $p_{59}$  and  $p_{86}$  are primes with 59 and 86 decimal digits.

Now consider the sequence of integers  $n > 1$  for which  $A(n)$  has precisely one prime divisor  $p \geq n/2$ . It begins with 3, 5, 7, 9, 11, 33, 36,

49 and contains 5 more integers with  $n \leq 2000$ , namely 453, 727, 1560, 1569, 1627, for which proofs of primality were relatively easy. There are 8 cases with  $2000 < n < 10^5$  for which  $A(n)$  has only one probable prime divisor  $p \geq n/2$ . They are 5078, 6605, 17663, 27281, 29298, 29708, 39509, 98653. Here I relied on strong Lucas and Fermat tests of probable prime divisors with up to 204433 decimal digits.

**Heuristics.** Asymptotic expansion of the Bessel functions in (149) gives

$$(155) \quad A(n) = \frac{8(2n-5)!}{\pi} \left( 1 - \frac{1}{4n} - \frac{3}{32n^2} + \frac{141}{128n^3} + \frac{14019}{2048n^4} + O(1/n^5) \right)$$

which grows factorially, with  $A(60000)$  a 557365-digit integer.

For  $n > 1$ , let  $A(n) = B(n)C(n)$ , where no prime  $p \geq n/2$  divides the integer  $B(n)$  and no prime  $p < n/2$  divides the integer  $C(n) > 0$ , which grows more slowly, with  $C(60000)$  a 124405-digit integer. Empirically, it seems that

$$(156) \quad \log(C(n)) = n \log(K) + O(\sqrt{n})$$

with  $K \approx 118$  and a fluctuating term suppressed by a square root.

Here I shall argue that

$$(157) \quad K = (4e)^2 = 118.22489758 \dots$$

For odd prime  $p < n/2$  let  $k$  be the largest integer such that  $p^{2k} | B(n)$ . Then experiment reveals that  $k \geq \lceil n/p \rceil - 2$  and that this lower bound is rarely exceeded for  $p > \sqrt{n}$ . For  $\sqrt{n} \geq p > 2$ , the empirical estimate  $k = (n + O(\sqrt{n})) / (p - 1)$  is adequate for the present purpose. Combining these observations, I obtain (157) using the following sums over primes:

$$(158) \quad \sum_{p < x} \frac{\log(p)}{p-1} = \log(x) - \gamma + o(1)$$

$$(159) \quad \sum_{\sqrt{x} < p < x} \left( \left\lfloor \frac{x}{p} \right\rfloor - \frac{x}{p} \right) \frac{\log(p)}{x} = \gamma + o(1).$$

Mertens' third theorem then gives the probability of  $C(n)$  being prime, heuristically, as asymptotic to  $\exp(\gamma) \log(n/2) / \log(K^n)$  and the number of prime values of  $C(n)$  with  $1 < n < x$  as asymptotic to  $N(x)$ , as defined in (150), which gives  $N(60000) \approx 19.8$ . This is comfortably close to the number 20 of probable primes that I discovered with  $n \in [2, 60000]$ .

Work by Armin Straub confirmed my findings up to  $n = 17663$ . Then Paul Zimmermann and Bruno Salvy confirmed them up to  $n = 39509$ . I encountered an unexpectedly long gap until the next case, at  $n = 98653$ , where the 204433-digit probable prime  $C(98653)$  and was found on 15 November 2015, after considerable work with `OpenPFGW`. Neil Sloane has recorded 98653 as entry 21 in sequence A265079 of *The on-line encyclopedia of integer sequences* [44].

### 7.8. Evaluation of an L-series of weight 6 outside its critical strip

The rationality of the moments  $A(n)$ , asserted in Conjecture 5, was a source of joy, since it implies rich rational substructure in matrices of moments of 8 Bessel functions. We had hoped for a  $3 \times 3$  matrix, involving 7-loop Feynman integrals, with a determinant that might yield  $L_8(7)$ , outside the critical strip, where a Mahler measure had failed to do the job. Thanks to the conjectures that  $A(1)$ ,  $A(2)$  and  $A(3)$  evaluate to the integers 0, 1 and 2, respectively, we found something even better.

**Conjecture 6.** *The determinant of the  $2 \times 2$  matrix*

$$(160) \quad \mathcal{M}_2 := \int_0^\infty K_0^6(t) \begin{bmatrix} K_0^2(t) & t^2(1-2t^2)K_0^2(t) \\ I_0^2(t) & t^2(1-2t^2)I_0^2(t) \end{bmatrix} t dt$$

*with 8-Bessel moments up to 7 loops, evaluates the L-series (137) for the weight-6 modular form (135), outside its critical strip, as follows:*

$$(161) \quad L_8(7) = \frac{128\pi^2}{6075} \det \mathcal{M}_2.$$

### 7.9. Further evaluations of determinants of Feynman integrals

Now consider the  $3 \times 3$  matrix

$$(162) \quad N_3 := \int_0^\infty I_0(t)K_0^5(t) \begin{bmatrix} K_0^2(t) & t^2K_0^2(t) & t^4K_0^2(t) \\ I_0(t)K_0(t) & t^2I_0(t)K_0(t) & t^4I_0(t)K_0(t) \\ I_0^2(t) & t^2I_0^2(t) & t^4I_0^2(t) \end{bmatrix} t dt$$

obtained by adding an extra  $K_0(t)$  to the integrand in (130). The elements of its first column are evaluated by  $S_{8,6}$ , in (146), by  $S_{8,5}$ , in (145), and by  $S_{8,4}$ , in (144). The elements in its first row are related to those in its third row, by Conjecture 5. Thus we have 5 relations constraining the 9 elements.

There is a 6th empirical constraint:

$$(163) \quad \det N_3 \stackrel{?}{=} \frac{5}{3} \frac{\pi^8}{2^{19}}$$

which has likewise been checked at high precision and is the tip of another empirical iceberg.

Let  $N_k$  be the  $k \times k$  matrix with elements

$$(164) \quad (N_k)_{a,b} := \int_0^\infty [I_0(t)]^a [K_0(t)]^{2k+2-a} t^{2b-1} dt,$$

which are moments of  $2k + 2$  Bessel functions. For integer  $m > 0$ , let

$$(165) \quad D_m := \frac{2\pi^{m^2/2}}{\Gamma(m/2)} \prod_{j=1}^m \frac{(2j-1)^{m-j}}{(2j)^j},$$

which is a rational multiple of an integer power of  $\pi^2$ .

**Conjecture 7.** *For every integer  $k > 0$ , the determinant of  $N_k$  is  $D_{k+1}$ .*

**Examples.** The constant  $D_1 = 1$  is not a subject of the conjecture; one may think of it as referring to the empty matrix.  $D_2 = (\pi/4)^2$  correctly evaluates the two-loop integral  $\int_0^\infty I_0(t)K_0^3(t) t dt = S_{4,2}/4$  defined by (164) at  $a = b = k = 1$ .  $D_3 = \frac{5}{32}\zeta(4)$  gives conjecture (113), for 6 Bessel functions, and  $D_4 = \frac{5}{3}\pi^8/2^{19}$  gives conjecture (163), for 8 Bessel functions. I have checked Conjecture 7 up to  $k = 15$ , where a matrix of Feynman integrals with up to 30 loops is predicted to have determinant

$$(166) \quad D_{16} = 17^7 \left(\frac{19}{7}\right)^6 \left(\frac{23}{65}\right)^4 \frac{7}{3} \frac{29}{11} \frac{\pi^{128}}{2^{291}}$$

in agreement with 225 numerical quadratures at 500-digit precision.

### 8. Harder problems at $n > 8$

The bad news is that, despite strong effort, I have not yet evaluated an L-series for  $n > 8$  Bessel functions in terms of Feynman integrals. Yet I feel that it may still be possible. My intuition is based on the following remarkably tight structure, imposed by Conjecture 2 on the case  $n = 9$ .

For the 52 primes in the set  $\mathcal{S} := \{2 < p \leq 631, (\frac{p}{105}) = 1\}$ , I found that

$$(167) \quad \frac{p c_9(p)}{4} = \begin{cases} b_p^2 - 945a_p^2 & \text{if } p \equiv 1 \pmod{3} \\ 21b_p^2 - 45a_p^2 & \text{if } p \equiv 2 \pmod{3} \end{cases}$$

$$(168) \quad \frac{4p^6 + c_9^2(p) - c_9(p^2)}{16p^2} = \begin{cases} b_p^2 + 945a_p^2 & \text{if } p \equiv 1 \pmod{3} \\ 21b_p^2 + 45a_p^2 & \text{if } p \equiv 2 \pmod{3} \end{cases}$$

where  $(a_p, b_p)$  is a pair of positive integers, determined by  $c_9(p)$  and  $c_9(p^2)$ . For each of these 52 primes, I define an integer measure,  $\alpha_p$ , of how close  $c_9(p)$  comes to determining  $c_9(p^2)$ , as follows.

For  $p \in \mathcal{S}$ , let  $\alpha_p$  be the number of positive integers  $m < a_p$  such that  $p c_9(p)/4 + 945m^2$  is a square, for  $p \equiv 1 \pmod{3}$ , or  $(p c_9(p)/4 + 45m^2)/21$  is a square, for  $p \equiv 2 \pmod{3}$ . If  $\alpha_p = 0$ , I say that  $c_9(p)$  determines  $c_9(p^2)$ , since we may take  $a_p$  as the smallest possible positive integer consistent with (167), which then tells us the positive integer  $b_p$  and hence  $c_9(p^2)$ , from (168). If  $\alpha_p$  is positive, I say that  $c_9(p)$  fails to determine  $c_9(p^2)$ , via (167,168), and take the value of  $\alpha_p$  as a measure of how far it falls short. In either case,  $\alpha_p$  and  $c_9(p)$  determine  $c_9(p^2)$ .

Here are the 52 values of  $\alpha_p \leq 4$ ,

$$(169) \quad 0000100131100001000002040000010000101020200000000111$$

concatenated as a string, with increasing  $p$ , in order to save space. To obtain these, I used Algorithm 2, with  $k = 1$ , for each of the 52 primes  $p \in \mathcal{S}$ , to determine  $c_9(p)$ , which was quickly done. At  $k = 2$

$$(170) \quad 2 \sum_{p \in \mathcal{S}} (p^2 - 1)^2 = 3019245508224 > 3 \times 10^{12}$$

accesses to tables of traces of Frobenius were made to determine  $c_9(p^2)$ , which took far longer, by a factor of more than  $2 \times 10^5$ . Yet the results from all that long work on  $c_9(p^2)$  are encoded in one line, by the 52 integers  $\alpha_p \leq 4$  listed in (169), of which 36 vanish, showing that, by my definition,  $c_9(p)$  determines  $c_9(p^2)$  *more often than not*, for  $p \leq 631$ . It was the astounding data compression in (169) that led me to Conjectures 1, 2 and 3. I remark that there is no evidence that this phenomenon is limited to small primes. On the contrary, the average value of  $\alpha_p$  for the 26 smaller primes in  $\mathcal{S}$  is about 0.54, while for the 26 larger primes it is about 0.38.

**Example 1.** At  $p = 577$ , we have  $c_9(577)/4 = -16930160$  and  $\alpha_{577} = 0$ . The *smallest* case with  $-577 \times 16930160 + 945m^2$  a square is  $m = 3216$ . Then  $(a_{577}, b_{577}) = (3216, 2260)$  give  $c_9(577^2) = 100104812100156676$ .

**Example 2.** At  $p = 617$ ,  $c_9(617)/4 = -5181267$  and  $\alpha_{617} = 1$  determine  $m = 11872$  as the *second* case with  $(-617 \times 5181267 + 45m^2)/21$  a square. Then  $(a_{617}, b_{617}) = (11872, 12239)$  give  $c_9(617^2) = -92449542374608124$ .

These show how a single line, in (169), spares any future worker 99.9995% of the effort that I have expended at  $n = 9$ . The functional equation (10) is very fine, but still lacking in the structure implied by my Conjectures 1, 2 and 3, tested for  $n < 20$ .

I conclude with questions [18, 33] posed in the introduction.

- 1) *What proverb more common, what proverb more true, than that after pride comes a fall?*
- 2) *Ah, but a man's reach should exceed his grasp, or what's a heaven for?*

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