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Tropical count of curves on abelian varieties

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We investigate the problem of counting tropical genus g curves in g-dimensional tropical abelian varieties. We do this by studying maps from principally polarized tropical abelian varieties into a fixed abelian variety. For $g = 2, 3$, we prove that the tropical count matches the count provided in [Göt98, BL99b, LS02] in the complex setting.

1. Introduction

One of the successes of tropical geometry has been the wide variety of so-called "correspondence theorems" that have been produced. These are typically theorems of the following form.

Theorem ("Correspondence Theorem"). *The tropical count of curves in setting* X^{tr} *matches the classical count in setting* X*, where* X^{tr} *is an appropriate tropicalization of* X*.*

Tropical curves are very well-suited to be studied combinatorially, and so with this type of theorem we are given the ability to study and solve many enumerative problems from a fresh perspective. In particular, we have the results of Boehm, Bringmann, Buchholz, and Markwig [BBBM14] which uses tropical geometry to prove Mirror Symmetry for the elliptic curve. There is the classic [Mik05] which counts tropical curves in toric surfaces. Furthermore, there has been extensive work on studying Hurwitz theory from the tropical perspective, see e.g. [CJM10, BCM13].

In fact, these correspondence theorems fit more broadly into the framework of the Gross-Siebert program [Gro11] which seeks to understand Mirror Symmetry through relating tropical, logarithmic, and complex algebraic geometry.

The goal of this paper is to prove the following correspondence theorem. Note that all of the definitions will follow afterwards.

Theorem 1.1 (Correspondence theorem for abelian varieties). *Let (A, L) be a g-dimensional complex abelian variety (for $g = 2, 3$), with L an ample line bundle inducing a polarization of type $(d_1, \ldots, d_g)$. Then the number of genus g curves of type c_L (respectively, in the linear system $|L|$) matches the tropical count.*

We will prove this in three separate steps. Our preliminary main result is the following.

Theorem 1.2. *Let A be a simple tropical torus of dimension g (where $g \geq 1$ is arbitrary) with a line bundle L inducing a polarization of type $(d_1, \ldots, d_g)$. Then the number of homomorphisms $f : (P, \theta) \to (A, c_L)$ (with P principally polarized by θ) such that $f^* c_L = (d_1 \cdots d_g)\theta$ is given by $\nu^\dagger(d_1, \ldots, d_g)$, where $\nu^\dagger(d_1, \ldots, d_g)$ is defined in Section 2.1.*

For $g \leq 3$, every simple principally polarized abelian variety is the Jacobian of a genus g curve, and so we can interpret this result in the context of embedded curves in abelian varieties. We then obtain the following theorem.

Theorem 1.3. *Let A be a tropical torus of dimension $g = 2, 3$ with a line bundle L such that (A, c_L) is a polarized tropical variety with polarization of type $(d_1, \ldots, d_g)$. Assume that A is simple and Torelli (Definition 2.28). Then the number of genus g curves of type c_L, up to translation in A, is $\nu^\dagger(d_1, \ldots, d_g)$.*

This is compared with [LS02] which provides the computation in the complex setting. Note that Lange and Sernesi provide a homological condition on their curve class, whereas we use one that is morally Poincaré dual (in the complex setting, this is exactly the case).

In the $g = 2$ case, we can now go even further by noting that curves are divisors, and so we can speak more refinedly of curves in a fixed linear system defined by L. We obtain then the following.

Theorem 1.4. *Let A be a simple tropical torus of dimension $g = 2$ with a line bundle L such that (A, c_L) is a polarized tropical surface with polarization of type (d_1, d_2). Then the number of genus g curves in the linear system $|L|$ is $(d_1 d_2)^2 \nu^\dagger(d_1, d_2)$.*

We can of course now specialize even further to the case that the polarization is primitive (i.e. of type $(1, n)$) to recover Theorem 3.2 of [Göt98].

Theorem 1.5. *Let A be a tropical torus with a line bundle L such that (A, c_L) is a polarized tropical abelian surface with primitive polarization of degree n. Assume that A contains no tropical elliptic curves. Then the number of genus 2 curves in the linear system $|L|$ is $n^2 \sigma_1(n)$.*

As is usual, performing a tropical count is a somewhat subtle task. One has to understand the combinatorics of the tropical objects, as well as the appropriate multiplicities, and then combine these together. This is related to the fact that the count in Theorem 1.3 is, in a certain sense, really a count of curves on a polarized abelian variety defined over a non-archimedean valued field $\mathbb{K}$.

As we will explain in Section 3.2, a tropical abelian variety can be seen as the image of an abelian $\mathbb{K}$-variety under a so-called *tropicalization* map. This map is essentially induced by the valuation map $\mathrm{val}\colon \mathbb{K}^* \to \mathbb{R}$, and allows us to relate the 'algebraic' count over $\mathbb{K}$ to the count of the underlying tropical objects, which is of a more combinatorial nature. However, it is important to note that information usually gets lost when passing from the algebraic to the tropical setting; this is analogous to the elementary fact that distinct elements of $\mathbb{K}^*$ can take the same value in $\mathbb{R}$ under the valuation map. The above mentioned multiplicities appear naturally as numerical factors keeping track of this defect.

In order to translate between algebraic and tropical geometry, we make use of some standard facts and constructions from rigid analytic geometry. We have included a summary of the results we need in Section 2.4.

1.1. Acknowledgements

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2. Definitions

2.1. Notation

We need the following definitions (see [Deb99]). For a finite abelian group G, let $G^* = \mathrm{Hom}(G, \mathbb{K}^\times)$ be its dual group of characters (where $\mathbb{K}$ is an algebraically closed field of characteristic zero; more will be said about this later). Furthermore, if $G \cong \Lambda/c(X)$ for lattices Λ, X and some full-rank homomorphism $c : \Lambda \to X$, then we define $G^\dagger = X/c^\dagger(\Lambda)$, where $c^\dagger$ is the unique map $c^\dagger : \Lambda \to X$ such that $c^\dagger \circ c = (\det c) Id_X$ and $c \circ c^\dagger = (\det c) Id_\Lambda$). This map is called the *adjugate map*[1].

Definition 2.1. For a finite abelian group G, we define

$$\nu(G) = \sum_{H \le G} \#\,\mathrm{Hom}^{\mathrm{sym}}(H, H^*)$$

where $\mathrm{Hom}^{\mathrm{sym}}$ refers to those homomorphisms $H \to \mathrm{Hom}(H, \mathbb{K}^\times)$ that are symmetric when viewed as bilinear functions $H \times H \to \mathbb{K}^\times$.

In the case that we have $G \cong \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_g\mathbb{Z}$ with $d_1 \mid \cdots \mid d_g$, then we define

$$\nu(d_1, \ldots, d_g) = \nu(G).$$

Finally, we define $\nu^\dagger(d_1, \ldots, d_g) = \nu(G^\dagger)$ as above.

Remark 2.2. Note that if we let $n = d_1 \cdots d_g$, then this can be written as

$$\nu^\dagger(d_1, \ldots, d_g) = \nu\left(\tfrac{n}{d_g}, \ldots, \tfrac{n}{d_1}\right).$$

This function satisfies the following property.

[1]Note that for a map $c : \Lambda \to X$ of full rank, we define $\det(c) = [X : c(\Lambda)]$, which is consistent with the case where c is an integral linear map $c : \mathbb{R}^N \to \mathbb{R}^N$.

Proposition 2.3. *Let G, G' be finite abelian groups such that* $\gcd(|G|, |G'|) = 1$. *Then*

$$\nu(G \times G') = \nu(G)\nu(G').$$

Equivalently, if $(d_1, \ldots, d_g)$ *and* $(d'_1, \ldots, d'_g)$ *are such that* $d_i \mid d_{i+1}$, $d'_i \mid d'_{i+1}$, *and* $\gcd(d_g, d'_g) = 1$, *then*

$$\nu(d_1 d'_1, \ldots, d_g d'_g) = \nu(d_1, \ldots, d_g)\nu(d'_1, \ldots, d'_g).$$

Proof. This follows due to the fact that if H, H' are coprime order, then

$$\mathrm{Hom}\left(H \times H', (H \times H')^*\right) = \mathrm{Hom}(H, H^*) \times \mathrm{Hom}\left(H', (H')^*\right)$$

(which itself follows due to the fact that the product and coproduct in the category of finite abelian groups coincide), and the fact that subgroups of $G \times G'$ are of the form $H \times H'$ for subgroups $H \leq G$ and $H' \leq G'$. □

We will also use the notation $\sigma_k(n) = \sum_{d|n} d^k$. Furthermore, unless otherwise stated, the notation $\mathrm{Hom}(-, -)$ will always refer to the Hom-sets in the category of abelian groups.

2.2. Tropical Tori/Abelian Varieties

A tropical variety can be defined in many ways; as a variety over the min-plus semi-ring, as a certain degeneration of an algebraic variety, or even a variety which locally has integer-affine structure (and whose transition functions preserve that). In any case, all of this simplifies greatly for the case of tropical tori, which have a remarkably simple definition.

In our definition, and in the discussion below, we follow [MZ08] with notation inspired by [Kat].

Definition 2.4. Let X be a rank g free abelian group. A g-dimensional tropical torus is given by the quotient

$$T = \mathrm{Hom}(X, \mathbb{R})/\Lambda$$

where $\Lambda \hookrightarrow \mathrm{Hom}(X, \mathbb{R})$ is a full rank sublattice. Note that the integral structure is given as $\mathrm{Hom}(X, \mathbb{Z}) \subset \mathrm{Hom}(X, \mathbb{R})$.

Remark 2.5. One main advantage of this definition is that it is basis invariant and provides a more natural definition of the dual torus.

Line bundles on tropical tori are defined in an analogous way to line bundles in the algebraic setting: they can be defined as continuous projections $\pi : L \to T$ whose fibres are the tropical line. Using this definition, they are classified cohomologically as in the algebraic setting as follows.

First, for a tropical torus T, denote by $\mathscr{T}_{\mathbb{Z}}^*$ the sheaf of 1-forms which take integer values on integer tangent vectors. Next, in the tropical setting, the sheaf $\mathscr{O}^\times$ of invertible regular functions is replaced by $\mathrm{Aff}_{\mathbb{Z}}$, the sheaf of affine-linear functions with integral slope. This fits into an exact sequence of sheaves defined on T given by

$$0 \to \mathbb{R} \to \mathrm{Aff}_{\mathbb{Z}} \to \mathscr{T}_{\mathbb{Z}}^* \to 0$$

(where the map $\mathbb{R} \to \mathrm{Aff}_{\mathbb{Z}}$ is the inclusion of constant functions, and $\mathrm{Aff}_{\mathbb{Z}} \to \mathscr{T}_{\mathbb{Z}}^*$ is given by $f \mapsto df$, which is an element of $\mathscr{T}_{\mathbb{Z}}^*$ since f is locally of the form $f(\vec{x}) = \langle df, \vec{x} \rangle + b$). In particular we obtain a long-exact sequence that is in part given by

$$\cdots \to H^0(\mathscr{T}_{\mathbb{Z}}^*) \to H^1(\mathbb{R}) \to H^1(\mathrm{Aff}_{\mathbb{Z}}) \xrightarrow{c} H^1(\mathscr{T}_{\mathbb{Z}}^*) \to H^2(\mathbb{R}) \to \cdots \tag{1}$$

We have as expected a bijection between line bundles and $H^1(\mathrm{Aff}_{\mathbb{Z}})$ (see [MZ08]).

Consider now $c_L := c(L)$ (where $c : H^1(\mathrm{Aff}_{\mathbb{Z}}) \to H^1(\mathscr{T}_{\mathbb{Z}}^*) \cong \mathrm{Hom}(\Lambda, X)$). That is, c_L is a map $c_L : \Lambda \to X$. This naturally induces a pairing $\Lambda \otimes \Lambda \to \mathbb{R}$ given by the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{c_L} & X \\ {\scriptstyle j}\downarrow & & \\ \mathrm{Hom}(X, \mathbb{R}) & & \end{array}$$

i.e. we define $\langle \lambda_1, \lambda_2 \rangle := j(\lambda_1)\big(c_L(\lambda_2)\big)$.

Lemma 2.6. *The pairing $\langle \, , \rangle$ is symmetric.*

Proof. As in the case of abelian varieties defined over $\mathbb{C}$, (see e.g. [BL04, Appendix B]), we can view elements of $H^1(\mathrm{Aff}_{\mathbb{Z}})$ as elements of $H^1(\Lambda, H^0(\widetilde{T}, \mathrm{Aff}_{\mathbb{Z}}))$ i.e. 1-cocycles on Λ with values in $H^0(\widetilde{T}, \mathrm{Aff}_{\mathbb{Z}})$, where $\widetilde{T} = \mathrm{Hom}(X, \mathbb{R})$ is the universal cover of T. These are functions $\phi : \Lambda \to$

$H^0(\widetilde{T}, \mathrm{Aff}_{\mathbb{Z}})$ which satisfy (for any $f \in \widetilde{T} = \mathrm{Hom}(X, \mathbb{R})$)

$$\phi(\lambda_1 + \lambda_2)(f) = \phi(\lambda_1)(\lambda_2 + f) + \phi(\lambda_2)(f). \tag{2}$$

Given that an element $\phi(\lambda)$ of $H^0(\widetilde{T}, \mathrm{Aff}_{\mathbb{Z}})$ is a globally-defined affine-linear function on $\mathrm{Hom}(X, \mathbb{R})$, we can write such an element as $\phi(\lambda)(f) = a_\lambda + f\big(c(\lambda)\big)$ with $c : \Lambda \to X$ (where the last term is of this form since we are considering affine-linear functions with *integer* slope). If we then examine the cocycle condition (2), we see that the elements a_λ satisfy

$$a_{\lambda_1+\lambda_2} = a_{\lambda_1} + a_{\lambda_2} + j(\lambda_1)\big(c(\lambda_2)\big).$$

Since the left-hand side of this equation is symmetric, it follows that $j(\lambda_1)\big(c_L(\lambda_2)\big) = j(\lambda_2)\big(c_L(\lambda_1)\big)$ as desired. □

We will assume from here onwards that c_L induces a positive-definite bilinear form on Λ. In particular, it is necessary that the image of Λ in X is of full rank; this is the tropical version of ampleness of a line bundle.

Definition 2.7. Let L be a line bundle on a tropical torus $T = \mathrm{Hom}(X, \mathbb{R})/\Lambda$. Note that the quotient $X/c_L(\Lambda)$ is a finite abelian group, and is hence isomorphic to $\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_g\mathbb{Z}$ for some integers $d_1 \mid \cdots \mid d_g$. We define the *type* of L to be the tuple $(d_1, \ldots, d_g)$. We define the *degree* of L to be the index $[X : c_L(\Lambda)] = d_1 \cdots d_g$. Finally, we say that the polarization is *primitive* if $\gcd(d_1, \ldots, d_g) = 1$.

Remark 2.8. If c_L is a polarization of type $(d_1, \ldots, d_g)$, then we will write $n = d_1 \cdots d_g$.

We will now define a (polarized) tropical abelian variety.

Definition 2.9. Let A be a tropical torus together with a line bundle L such that c_L induces a positive definite bilinear form. We call the pair (A, c_L) a polarized tropical abelian variety with polarization c_L. The degree of the polarization is the degree of c_L. If c_L has degree 1, then we call the polarization principal.

Remark 2.10. Note that if $A = \mathrm{Hom}(X, \mathbb{R})/\Lambda$ is principally polarized by c_L, then we have an isomorphism $c_L : \Lambda \to X$. Consequently, we can write any principally polarized abelian variety as $\mathrm{Hom}(\Lambda, \mathbb{R})/\Lambda$.

We will next investigate maps between tropical tori. The key is that such maps must preserve the integral structure of the tori, which greatly restricts their form.

We will start with defining those morphisms that preserve the identity of the tori.

Definition 2.11. Let $T_1 = \mathrm{Hom}(X_1, \mathbb{R})/\Lambda_1$ and $T_2 = \mathrm{Hom}(X_2, \mathbb{R})/\Lambda_2$ be tropical tori. A homomorphism $f : T_1 \to T_2$ consists of a pair of morphisms (g, h) where

$$g : \Lambda_1 \to \Lambda_2 \qquad h : X_2 \to X_1$$

such that the following diagram commutes.

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\;g\;} & \Lambda_2 \\ \downarrow & & \downarrow \\ \mathrm{Hom}(X_1, \mathbb{R}) & \xrightarrow[h^*]{} & \mathrm{Hom}(X_2, \mathbb{R}). \end{array}$$

Note of course that this is a necessary condition[2] for this to yield a map on the level of topological spaces $T_1 \to T_2$. It is also sufficient when we consider that the map must preserve the underlying integral structure.

Given such a map, we define the *topological degree* d_t to be the index $[\Lambda_2 : g(\Lambda_1)]$, and we define the *metric degree* d_m to be the index $[X_1 : h(X_2)]$. We define the *tropical degree* of the map to be the product $d_t d_m$.

Remark 2.12. Our definition of topological and metric degrees is defined so as to be an analog of the definition for maps between tropical curves. In that case, the topological degree is the number of pre-images of a point, whereas the metric degree is the degree to which the metric is scaled under the map.

In general, a morphism $T_1 \to T_2$ is the composition of a homomorphism as defined above together with a translation in $a \in T_2$. These are given by post-composing a homomorphism $f : T_1 \to T_2$ with the morphism

$$t_a : T_2 \to T_2 \qquad x \mapsto x + a.$$

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[2]This isn't quite correct; any map $h_{\mathbb{R}} : X_2 \otimes \mathbb{R} \to X_1 \otimes \mathbb{R}$ would actually suffice, but this is not relevant in our case.

Remark 2.13. We will use the term *homomorphism* to specifically refer to a morphism which preserves the identity, whereas a morphism may include a translation.

We next investigate how polarizations are affected by morphisms. So let A_1, A_2 be tropical tori, $A_i = \mathrm{Hom}(X_i, \mathbb{R})/\Lambda_i$. Translations do not affect polarizations, so we may assume that a morphism is in fact a homomorphism; let $f = (g, h)$ be a such morphism between them. Let c_L be a polarization on A_2. Then the induced polarization on A_1 is given by

$$\Lambda_1 \xrightarrow{g} \Lambda_2 \xrightarrow{c_L} X_2 \xrightarrow{h} X_1 , \qquad f^* c_L : \Lambda_1 \to X_1 .$$

Lastly, let us define the dual torus/abelian variety.

Definition 2.14. Let $A = \mathrm{Hom}(X, \mathbb{R})/\Lambda$ be a tropical torus. We define the dual torus to be $\widehat{A} = \mathrm{Hom}(\Lambda, \mathbb{R})/X$, where the inclusion $X \hookrightarrow \mathrm{Hom}(\Lambda, \mathbb{R})$ is given by

$$\mathrm{Hom}\big(\Lambda, \mathrm{Hom}(X, \mathbb{R})\big) \equiv \mathrm{Hom}(\Lambda \otimes X, \mathbb{R}) \equiv \mathrm{Hom}\big(X, \mathrm{Hom}(\Lambda, \mathbb{R})\big).$$

Moreover, if A is polarized by a degree n polarization c_L, then the dual polarization is given by $c_{\widehat{L}} = c_L^{\dagger}$ (the adjugate of c_L), and satisfies

$$c_L \circ c_{\widehat{L}} = n \cdot id_X \qquad\qquad c_{\widehat{L}} \circ c_L = n \cdot id_\Lambda .$$

Note that the dual polarization is of type $\left(\frac{n}{d_g}, \ldots, \frac{n}{d_1}\right)$. See by analogy over $\mathbb{C}$ [BL99a, Proposition 2.7].

Note that unlike the definition of dual polarization provided in [LS02], we do *not* have that $\widehat{\widehat{L}} \cong L$. The definition we provide (which is the same as in [BL99a]) is more natural in our context, and the count of curves we obtain is entirely equivalent.

This can be explained as follows. If we let L' be the dual line bundle as defined in [LS02], then we have the relation that $(L')^{\otimes k} = \widehat{L}$ for some $k \geq 1$. However, the count of curves of type c_L in the abelian variety A (all defined over $\mathbb{C}$) is obtained by looking at certain groups defined by the line bundle $(L')^{\otimes k}$, and so this is exactly the same line bundle that we use.

Next, given a polarized tropical abelian variety (A, c_L), we obtain a natural homomorphism $A \to \widehat{A}$ given by the pair (c_L, c_L). That is, we have

$$\begin{array}{ccc} \Lambda & \xrightarrow{c_L} & X \\ \downarrow & & \downarrow \\ \mathrm{Hom}(X, \mathbb{R}) & \xrightarrow[c_L^*]{} & \mathrm{Hom}(\Lambda, \mathbb{R}) \end{array}$$

which has tropical degree n^2 (each of the topological and metric degrees are n, respectively). Moreover, the compositions $A \to \widehat{A} \to A$ and $\widehat{A} \to A \to \widehat{A}$ are multiplication by n.

Remark 2.15 (See [MZ08]). One can equivalently define $\widehat{A} = \mathrm{Pic}^0(A)$, which by the long-exact-sequence (1) is isomorphic to our dual. Moreover, the map $A \to \widehat{A}$ is in this case given by $a \mapsto L^{-1} \otimes t_a^* L$ as in the complex case.

Note that we have a natural isomorphism $P \to \widehat{P}$ if P is principally polarized, which allows us to identify the two tori.

We will provide one last definition which is important for our case.

Definition 2.16. Let T be a tropical torus of dimension g. Then we say that T is *simple* if it does not admit any subtori of dimension $0 < h < g$.

Remark 2.17. As in the case over $\mathbb{C}$, T is simple if and only if it is not reducible. That is, T is simple if and only if there do not exist tori T_1, T_2 and an isogeny $T \to T_1 \times T_2$.

It follows that if A is a tropical abelian variety of dimension $g = 2, 3$, then it is simple if and only if there are no non-constant maps from elliptic curves to A, or from genus 2 curves in the $g = 3$ case.

2.3. Tropical Curves

We will work in a simplified setting which avoids a possible weighting function on vertices (which corresponds to certain degenerations of tropical curves). This section will be particularly brief; for more detail, see [MZ08].

Definition 2.18. Let Γ be a connected graph such that $h^1(\Gamma) = g$ together with a function $\ell : e(\Gamma) \to \mathbb{R}_{>0}$ (a so-called *metric* graph). Furthermore, we assume that Γ has no 2-valent vertices. We call such a pair a genus g abstract tropical curve.

Remark 2.19. Note that the integral structure comes from viewing an edge as $[0, \ell(e)] \subset \mathbb{R}$.

Definition 2.20. Let Γ be the graph underlying a tropical curve. We say that Γ is *m-edge connected* if the removal of any $k < m$ points in the interior of distinct edges leaves a connected graph.

Maps between tropical varieties are in general difficult to define. When the source is a curve and the target an abelian variety, then this is simpler.

Definition 2.21. A morphism from a tropical curve Γ to a tropical abelian variety A is

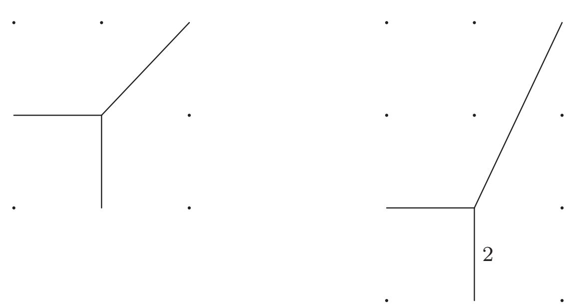
- A continuous map $f : \Gamma \to A$ that is locally affine-linear on the edges
- A weighting of the edges (i.e. a function $w : E(\Gamma) \to \mathbb{Z}_{\geq 0}$)

such that for each vertex of Γ with outgoing primitive tangent vectors $\{v_i\}$ (which are uniquely determined since each edge obtains an integral structure from its description as $[0, \ell(e)]$) we have $f_* v_i = w_i \xi_i$ for some primitive integral vectors $\{\xi_i\}$, and which satisfy

$$\sum_i w_i \xi_i = 0$$

(the *balancing condition*).

Example 2.22. Some common examples of local models are



where the weight w_e of an edge (if greater than 1) is written next to it.

Definition 2.23. The Jacobian of a tropical curve is defined in an analogous way to the Jacobian of a curve defined over $\mathbb{C}$. We have a notion of the space of 1-forms on Γ given by $\Omega(\Gamma)$. There is a map $H_1(\Gamma, \mathbb{Z}) \hookrightarrow \Omega(\Gamma)^\vee$ into the dual of $\Omega(\Gamma)$ given by integrating over 1-cycles. From this, we define

$$J(\Gamma) = \Omega(\Gamma)^\vee / H_1(\Gamma, \mathbb{Z}).$$

There is of course a map (depending on a choice of basepoint p_0) $\Gamma \to J(\Gamma)$ which is given by integrating along partial paths; that is, for $p \in \Gamma$, choose any path $\gamma : p_0 \to p$, which yields a map

$$\omega \mapsto \int_{p_0}^{p} \omega$$

which is well defined up to $H_1(\Gamma, \mathbb{Z})$, and so we have a map $\Gamma \to J(\Gamma)$ as claimed.

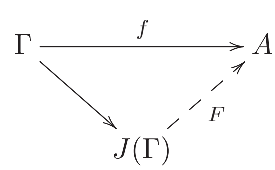
Remark 2.24. $J(\Gamma)$ is a polarized tropical abelian variety, where the polarization θ_Γ is the natural bilinear form associated to the quadratic form on q defined on $H_1(\Gamma)$) as

$$q\left(\sum_i k_i e_i\right) = \sum_i \ell(e_i) k_i^2$$

(which is called the *length pairing*). Note that as $J(\Gamma)$ is principally polarized, there is a unique (up to translation) line bundle Θ_Γ which induces this polarization θ_Γ (See [MZ08, Section 5.2]).

Most importantly for our purposes, the Jacobian satisfies the following universal property whose proof is nearly verbatim to the one over $\mathbb{C}$.

Proposition 2.25. *Let A be a tropical abelian variety, let Γ be a tropical curve, let $p_0 \in \Gamma$, and let $f : \Gamma \to A$ be a map such that $f(p_0) = 0 \in A$. Then there is a unique factorization*



through $J(\Gamma)$ (with F a homomorphism).

Using the universal property we can now make the following central definition. Note that in [Göt98] (and in [BL99b] as well) we restrict ourselves to curves in a fixed linear system $|L|$. In the case of abelian surfaces defined over $\mathbb{C}$, this is simply the statement that $\mathscr{O}\big(f(C)\big) = L$. This is equivalent to the fact that $F^*L = n\Theta_C$, which we will use as the basis for our definition as it suits us better.

Let $f : \Gamma \to (A, c_L)$ be a morphism from a tropical curve of genus g to a simple polarized tropical abelian variety of dimension g. This induces a morphism of polarized abelian varieties $F : (J(\Gamma), \theta_\Gamma) \to (A, c_L)$ (where, as above, the polarization on $J(\Gamma)$ is obtained by the length pairing on Γ).

Definition 2.26. Let $f : \Gamma \to (A, c_L)$ be a morphism as above, and let n be the degree of the polarization.

(i) We say that the image of Γ in A is of type c_L if

$$F^*c_L \cong n\theta_\Gamma. \tag{3}$$

(ii) Suppose now that A is an abelian surface. We say that the image of Γ in A is in the linear system $|L|$ if

$$F^*L \cong n\Theta_\Gamma.$$

Remark 2.27. An equivalent condition to (3) is that $\widehat{F}^*\theta_\Gamma \cong \widehat{c_L}$, where $\widehat{F} : \widehat{A} \to J(C)$ is the dual morphism and $\widehat{c_L}$ is the dual polarization.

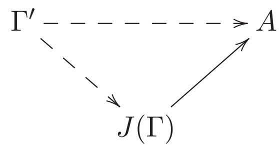
The second of these two conditions implies the first. However, if you compose a morphism together with a translation in A (which does not affect the polarization, as discussed earlier), then while you preserve the polarization, you will change the linear system unless you translate by an element of the kernel of the map $A \to \widehat{A}$. In particular, we note that it is a stronger condition than the former.

Finally, we introduce a genericity condition that we will assume holds for A.

Definition 2.28. Let A be a tropical abelian variety. We will say that A is *Torelli 3-connected* or more simply *Torelli* if it is not isogenous to the Jacobian of a curve which is not 3-connected.

The purpose of this condition is the following. As per [Viv13, Corollary 4.1.16], if we stay away from curves which are not 3-connected, then the tropical Torelli map $\mathcal{M}_g^{\mathrm{tr}} \to \mathcal{A}_g^{\mathrm{tr}}$ (which maps $\Gamma \mapsto J(\Gamma)$) is injective and in particular, the Jacobian determines the curve. This implies that for a g-dimensional A which is simple and Torelli, then for any non-constant map

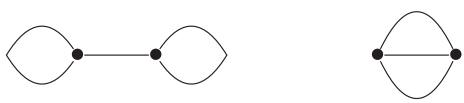
$\Gamma \to A$ (with Γ of genus g), we must have that Γ is 3-connected. In particular, the lifting problem



has a unique solution $\Gamma' = \Gamma$.

One can verify (by the construction of the 3-edge connectivization of a curve) that the abelian varieties which do not satisfy this condition (i.e. are isogenous to Jacobians of non-3-connected curves) form a positive codimension locus in the moduli space of tropical abelian varieties, and so a suitably generic abelian variety will always be Torelli.

Note that for abelian surfaces, A is Torelli if and only if it is simple. This follows since there are only two topological types of the graph underlying the tropical curve; the so-called barbell graph and theta graphs.



Since the Jacobian of the barbell graph (which is not 3-connected) is a product of two elliptic curves, the claim follows.

Finally, we note the following important property which is used later. Suppose that A is a simple g-dimensional abelian variety, and that $f : \Gamma \to A$ is a morphism from a genus g curve. Then the map is generically injective. That is, it does not factor through a map $\Gamma \to \Gamma'$ with $g(\Gamma') < g(\Gamma)$.

2.4. Rigid Analytic Varieties

Throughout this subsection, $\mathbb{K}$ denotes a non-archimedean field with absolute value $|\cdot|$ and valuation ring R. We shall assume that $\mathbb{K}$ is algebraically closed and of characteristic zero (though this is not essential for many of the statements below).

2.4.1. As we already indicated in Section 1, Theorems 1.2, 1.3 and 1.4 should be seen as statements concerning the counts of maps from curves, or from principally polarized abelian varieties, to an abelian $\mathbb{K}$-variety A, subject to conditions imposed by an appropriate choice of an ample line bundle on A. (These conditions can be formulated like in Definition 2.26.)

In order to translate statements between algebraic $\mathbb{K}$-varieties and tropical varieties, we shall pass through the category of *rigid analytic* varieties. In particular, this enables us to formulate the necessary geometric conditions on A for which this count matches the complex count. Moreover, this will allow us to use the powerful technique of rigid analytic uniformization of abelian varieties.

For an introduction to rigid analytic geometry, which is well suited for questions concerning curves and abelian varieties, we refer to the excellent textbook [Lüt16]. In particular, the reader will there find a thorough discussion of the various standard facts we recall in the paragraphs below.

2.4.2. Analytification. The rigid analytification functor, denoted $(\cdot)^{\mathrm{an}}$, takes a (proper) $\mathbb{K}$-variety X to its analytification X^{an}, which is a (proper) rigid-analytic space over $\mathbb{K}$. We shall use the following well-known properties of this functor:

- The functor $(\cdot)^{\mathrm{an}}$ is fully faithful on the category of *proper* $\mathbb{K}$-varieties.
- If X is a proper $\mathbb{K}$-variety, there is also an analytification functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ on $\mathcal{O}_X$-modules, which yields an equivalence from the category of coherent sheaves on X to the category of coherent sheaves on X^{an}.

2.4.3. Uniformization of abelian varieties. Any abelian $\mathbb{K}$-variety A allows a uniformization in the rigid analytic category. Here, by a uniformization of A, we mean the following data:

- A semi-abelian variety E, which is an extension

$$0 \to T \to E \to B \to 0$$

  of a torus T by an abelian variety B having *good reduction* over R.
- A lattice $M \subset E$ of rank equal to $\dim(T)$ and an exact sequence

$$0 \to M^{\mathrm{an}} \to E^{\mathrm{an}} \to A^{\mathrm{an}} \to 0.$$

Be aware that the morphism $E^{\mathrm{an}} \to A^{\mathrm{an}}$ only exists analytically, even though A and E are algebraic.

When the abelian part B of E is zero, we shall say that A is *uniformized by a torus*. Geometrically, this corresponds to the statement that A has maximally degenerate reduction over the valuation ring R. A basic example to keep in mind (which in fact started the whole theory) is Tate's uniformization of elliptic curves. This classic result asserts that if A is an elliptic curve with split multiplicative reduction over R, i.e., if A degenerates to a cycle of smooth rational curves, then A^{an} is isomorphic to a quotient $\mathbb{G}_{m,\mathbb{K}}/q^{\mathbb{Z}}$. Here q denotes an element such that $|q| < 1$. Tate's construction was generalized by Mumford to higher dimensional abelian varieties (with maximally degenerate reduction) in [Mum72].

2.4.4. Analytification of curves and abelian varieties. A smooth proper connected rigid $\mathbb{K}$-group carrying an *ample* line bundle is the analytification of an abelian variety. Thus, we shall speak about abelian varieties also in the rigid setting. As usual, an ample line bundle determines a *polarization.*

For later use, we include some details concerning polarizations of abelian varieties uniformized by tori. Let A be an abelian variety uniformized by a torus $T = \operatorname{Spec} \mathbb{K}[\widehat{M}]$ modulo a lattice $M \subset T$. Then a polarization of A is determined by an injective linear map $c_L\colon M \to \widehat{M}$ for which the induced bilinear form $\langle m_1, c_L(m_2)\rangle$ is symmetric and $|\langle m, c_L(m)\rangle| < 1$ for all $m \neq 0$. The degree of c_L is the number $[\widehat{M} : c_L(M)]$. The polarization induces a morphism from A to its dual $\widehat{A}$, which is uniformized by $\widehat{T} = \operatorname{Spec} \mathbb{K}[M]$ modulo the lattice $\widehat{M} \subset \widehat{T}$.

We also record the fact that if C is a smooth projective $\mathbb{K}$-curve, then the formation of the Jacobian variety commutes with analytification, i.e., $J(C^{\mathrm{an}}) = J(C)^{\mathrm{an}}$. Moreover, the analytic Jacobian again carries a principal polarization.

2.4.5. From algebraic to analytic. Let A be a simple abelian variety over $\mathbb{K}$ and assume that A is uniformized by a torus. By the properties of the analytification functor, the count of morphisms $g\colon P \to A$, say, with P principally polarized and g an isogeny, can be performed instead in the analytic category. In particular, this applies to the case where P is the Jacobian of a curve C of genus equal to the dimension of A, and g is induced by a non-constant map $C \to A$.

The advantage of working in the analytic category is that we can now use uniformization techniques; this in fact reduces our problem to a certain count of lattices. To be precise, we first need to establish the following lemma, which asserts that, in the situation outlined above, the abelian variety P is also uniformized by a torus.

Lemma 2.29. *Let P be a principally polarized abelian variety and let $g\colon P \to A$ be an isogeny. Then P is uniformized by a torus as well.*

Proof. Let P be uniformized by the data (E, N), where E is a semi-abelian variety and N is a lattice. As P is principally polarized, we can identify P

with its dual $\widehat{P}$. We denote by $\widehat{g}$ be the dual isogeny of g. The composition $g \circ \widehat{g}$ yields an isogeny $\widehat{A} \to A$. Analytically, this isogeny corresponds to an *injective* homomorphism $\widehat{M} \to M$ which has to factor through the lattice N (cf. e.g. [Lüt16, Prop. 6.4.1]). This is only possible if $\mathrm{rank}(N) = \dim(P)$, i.e., if the abelian part of E is zero. Thus also P is uniformized by a torus. □

3. Proof of main theorem

We are now prepared to give the proofs of the main theorems, Theorems 1.2–1.5. We will break this up into two main steps; first we will look at the combinatorics of the tropical maps which we will count without any reference to multiplicities. Second, we will compute the multiplicity to obtain the result.

3.1. Naïve Count of Maps/Curves

We begin by studying maps from principally polarized tropical abelian varieties. In particular, we will prove the following proposition.

Proposition 3.1. *Let $A = \mathrm{Hom}(X, \mathbb{R})/\Lambda$ be a simple tropical torus of dimension g with a line bundle L such that (A, c_L) is a polarized tropical abelian variety with polarization of type $(d_1, \dots, d_g)$. Then the number of homomorphisms $F : (P, \theta) \to (A, c_L)$ such that $F^*c = (d_1 \cdots d_g)\theta$ is given by $\#\{H \leq \Lambda/c_L^\dagger(X)\}$.*

The proof of Proposition 3.1 breaks up into two steps.

Proposition 3.2. *Let S_1, S_2 be defined as*

$$S_1 = \{F : (P, \theta) \to (A, c_L) \mid \theta \textit{ is principal}, F^*c_L = n \cdot \theta, F(0_P) = 0_A\}$$
$$S_2 = \{X \xrightarrow{f_1} I \xrightarrow{f_2} \Lambda \mid I \textit{ is a lattice with } \mathrm{rank}(I) = g, f_2 \circ f_1 = c_L^\dagger\}.$$

Then there is a bijection $S_1 \iff S_2$.

Proposition 3.3. *We have that*

$$|S_2| = \#\{H \leq \Lambda/c_L^\dagger(X)\}.$$

Proof of Proposition 3.2. Let $F : P \to A$ be an element in S_1, with $P = \mathrm{Hom}(I, \mathbb{R})/I$. We have a dual morphism $\widehat{F} : \widehat{A} \to P$ which yields a diagram

$$\begin{array}{ccccc} \multicolumn{5}{c}{\overset{c_L^{\dagger}}{\frown}} \\ X & \xrightarrow[f_1]{} & I & \xrightarrow[f_2]{} & \Lambda \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(\Lambda, \mathbb{R}) & \xrightarrow[f_2{}^*]{} & \mathrm{Hom}(I, \mathbb{R}) & \xrightarrow[f_1^*]{} & \mathrm{Hom}(X, \mathbb{R}) \end{array}$$

for some pair of homomorphisms (f_1, f_2). This yields an element of S_2.

Conversely, suppose we have a factorization $X \xrightarrow{f_1} I \xrightarrow{f_2} \Lambda$. Our goal is to produce a map $P \to A$ from a principally polarized abelian variety $P = \mathrm{Hom}(I, \mathbb{R})/I$ (which will satisfy the conditions outlined in the definition of S_1). The given factorization yields a diagram

$$\begin{array}{ccccc} X & \xrightarrow[f_1]{} & I & \xrightarrow[f_2]{} & \Lambda \\ \downarrow & & \vdots & & \downarrow \\ \mathrm{Hom}(\Lambda, \mathbb{R}) & \xrightarrow[f_2^*]{} & \mathrm{Hom}(I, \mathbb{R}) & \xrightarrow[f_1^*]{} & \mathrm{Hom}(X, \mathbb{R}) \end{array}$$

and we claim that we can fill in the dashed arrow to yield a principally polarized (tropical) abelian variety $P = \mathrm{Hom}(I, \mathbb{R})/I$ which will then necessarily satisfy the defining conditions of S_1. This map is constructed as follows: if we let $j : X \to \mathrm{Hom}(\Lambda, \mathbb{R})$ be the inclusion of the lattice, and let d be the index $[I : f_1(X)]$, then we can simply define the map $k : I \to \mathrm{Hom}(I, \mathbb{R})$ as

$$k = \frac{1}{d} f_2^* \circ j \circ f_1^{\dagger}.$$

One can easily check that this makes the diagram commute, and moreover, the resulting tropical abelian variety is by definition principally polarized. This yields our inverse map $S_2 \to S_1$. □

Proof of Proposition 3.3. The set S_2 consists of factorizations

$$\begin{array}{ccccc} X & \xrightarrow{f_1} & I & \xrightarrow{f_2} & \Lambda \\ \multicolumn{5}{c}{\underset{c_L^{\dagger}}{\smile}} \end{array}$$

with both f_1 and f_2 injective (this follows since c_L and $c_L^\dagger$ are). If we quotient out each term by X we end up with

$$0 \to I/f_1(X) \to \Lambda/c_L^\dagger(X).$$

We see that such a factorization is equivalent to a subgroup of $\Lambda/c_L^\dagger(X)$ as claimed. □

Next, we investigate the special cases $g = 2, 3$ in order to prove the naïve (i.e. non-multiplicity) versions of Theorems 1.3, 1.4.

Proposition 3.4. *Let (A, c) be a simple and Torelli polarized abelian variety of dimension $g = 2, 3$. Define the set*

$$S_3 = \{f : \Gamma \to A \mid f(\Gamma) \text{ is of type } c, g(\Gamma) = g\}$$

There is then a bijection

$$S_3 \iff S_1.$$

Finally, we specialize to genus 2.

Proposition 3.5. *Let (A, c_L) be a simple polarized abelian surface, where L is a line bundle on A. Define the set*

$$S_3' = \{f : \Gamma \to A \mid f(\Gamma) \in |L|, g(\Gamma) = 2\}$$

There is then a bijection

$$S_3' \iff \ker(A \to \widehat{A}) \times S_1.$$

We will prove both of these together.

Proofs of Propositions 3.4, 3.5. This proof is roughly the same as in the proof of Theorem 3.2 of [Göt98]. We begin by noting that the universal property of Jacobians (which is still valid in our case) yields a map $S_3 \to S_1$.

So consider an element $F : P \to A$ of S_1 where P is a principally polarized tropical abelian variety of dimension $g = 2, 3$. By the surjectivity of the Torelli map (see [BMV11, Remark 5.2.5]), it follows that $P = J(\Gamma)$ for some genus g tropical curve Γ. The composition $\Gamma \to P \to A$ yields the map $f : \Gamma \to A$ that we desire[3]. Moreover, since we are assuming that A is

[3] Note that this satisfies the conditions of definition 2.21 since the map $\Gamma \to J(\Gamma)$ does.

Torelli (recall the comments following definition 2.28), it follows that the maps $S_3 \to S_1$ and $S_1 \to S_3$ are inverse to each other.

Finally, all of this is up to translation in A. For $g = 2$, we additionally need to look at those elements $a \in A$ such that $t_a^* L \cong L$. However, by Remark 2.15, we see that this is the case if and only if $a \in \ker(A \to \widehat{A})$, whence the claim. □

3.2. Multiplicity Computation

3.2.1. Tropicalization of abelian varieties. We will first explain what we mean by *tropicalization* of abelian varieties over $\mathbb{K}$. Our discussion follows closely [BR15], see also [Gub07] for closely related results. To be precise, we allow arbitrary (non-degenerate) polarizations, but we restrict ourselves to abelian varieties uniformized by tori, as this is the only case we need. We continue to use the notation and terminology introduced in Section 2.4.3.

Let A be an abelian $\mathbb{K}$-variety of dimension g, which is uniformized by the data $M \subset T$. The points $T(\mathbb{K})$ can be naturally identified with the group $\mathrm{Hom}(\widehat{M}, \mathbb{K}^*)$. Via the group homomorphism

$$-\log\colon \mathrm{Hom}(\widehat{M}, \mathbb{K}^*) \cong T(\mathbb{K}) \to \mathbb{R}^g$$

defined by $(t_1, \ldots, t_g) \mapsto -(\log|t_1|, \ldots, \log|t_g|)$, we can identify $M(\mathbb{K})$ with a full rank lattice $\Lambda = -\log(M)$ in $\mathbb{R}^g$.

One easily checks that a polarization c_L yields, by composition with $-\log$, a polarization of the tropical torus $A^{\mathrm{tr}} = \mathbb{R}^g/\Lambda$, which we will continue to denote c_L. Thus, a pair (A, C_L) tropicalizes to a tropical polarized abelian variety in the sense of Definition 2.9. In conclusion, we obtain a functor

$$(A, c_L) \mapsto (A^{\mathrm{tr}}, c_L).$$

We shall need one additional, crucial, fact from [BR15]. Let C be a smooth projective and connected $\mathbb{K}$-curve of genus g, and let $J(C)$ denote its Jacobian variety. Let moreover C^{tr} denote the tropicalization of C, i.e., its *minimal skeleton* in the sense of non-archimedean geometry. Then one has a canonical isomorphism

$$J(C^{\mathrm{tr}}) \cong J(C)^{\mathrm{tr}}$$

as principally polarized tropical abelian varieties (see [BR15, 2.9, 2.10]).

Remark 3.6. The reader will find a closely related construction of tropicalization of abelian varieties in [Viv13], where the author works over a complete non-archimedean field with a discrete valuation, which allows the use of Néron models (instead of uniformization).

3.2.2. Analysis of multiplicities. We can understand multiplicity computations broadly as follows. Let $\mathscr{D}^{\mathrm{tr}}$ be a diagram of tropically defined objects which arises as the tropicalization of a diagram $\mathscr{D}$ defined over $\mathbb{K}$. Then the multiplicity associated to the diagram is the number of distinct diagrams $\mathscr{D}'$ such that $(\mathscr{D}')^{\mathrm{tr}} = \mathscr{D}^{\mathrm{tr}}$.

As an example, let A be an abelian variety defined over $\mathbb{K}$ and let A^{tr} be its tropicalization. Let $f : \Gamma \to A^{\mathrm{tr}}$ be a tropical morphism. The multiplicity is then the number of curves C which tropicalize to Γ and which fit into the diagram

$$\begin{array}{ccc} C & \dashrightarrow & A \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & A^{\mathrm{tr}}. \end{array}$$

That is, how many ways can we fill in the dashed corner?

In order to prove Theorem 1.2, all that remains is to compute the analogous multiplicity. We can see that this consists of counting all dashed lifts in the following diagram.

$$\begin{array}{ccc} P & \dashrightarrow & A \\ \downarrow & & \downarrow \\ P^{\mathrm{tr}} & \longrightarrow & A^{\mathrm{tr}} \end{array}$$

Recall now that if $A = \mathrm{Hom}(X, \mathbb{K}^\times)/\Lambda$, then homomorphisms $P^{\mathrm{tr}} \to A^{\mathrm{tr}}$ are in bijection with elements of the set S_2 defined above (by Proposition 3.2). That is, factorizations $X \xrightarrow{f_1} I \xrightarrow{f_2} \Lambda$.

Proposition 3.7. *Let $(f_1, f_2) \in S_2$, where $f_2 \circ f_1 = c_L^\dagger$. Let $H = I/f_1(X)$. Then the number of lifts of $P^{\mathrm{tr}} \to A^{\mathrm{tr}}$ is*

$$\# \mathrm{Hom}^{\mathrm{sym}}(H, H^*).$$

Proof. Our goal is to determine the number of dashed arrows that fit into the following diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{f_1} & I & \xrightarrow{f_2} & \Lambda \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(\Lambda, \mathbb{K}^\times) & \xrightarrow{f_2^*} & \mathrm{Hom}(I, \mathbb{K}^\times) & \xrightarrow{f_1^*} & \mathrm{Hom}(X, \mathbb{K}^\times) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(\Lambda, \mathbb{R}) & \xrightarrow{f_2^*} & \mathrm{Hom}(I, \mathbb{R}) & \xrightarrow{f_1^*} & \mathrm{Hom}(X, \mathbb{R}).
\end{array}
\tag{4}
$$

For this proof, we will regard a map $I \to \mathrm{Hom}(I, \mathbb{K}^\times)$ as a bilinear map $I \times I \to \mathbb{K}^\times$. In particular, we are thus trying to determine the number of (symmetric) bilinear maps $I \times I \to \mathbb{K}^\times$ (which make the diagram (4) commute, when viewed as a map $I \to \mathrm{Hom}(I, \mathbb{K}^\times)$). Let B be the set of such maps, and let $b_0 \in B$ be fixed.

We claim that there is a bijection $B \to \mathrm{Hom}^{\mathrm{sym}}(H, H^*)$. There is a map $\mathrm{Hom}^{\mathrm{sym}}(H, H^*) \to B$ given as follows. Let $\varphi : H \times H \to \mathbb{K}^\times$. Then we map φ to

$$(i_1, i_2) \mapsto b_0(i_1, i_2)\varphi([i_1], [i_2])$$

which is an element of B.

To construct the inverse, choose $b \in B$ arbitrary, and define a map $\varphi_b : H \times H \to \mathbb{K}^\times$ as

$$(g_1, g_2) \mapsto \frac{b(\overline{g_1}, \overline{g_2})}{b_0(\overline{g_1}, \overline{g_2})}$$

where $\overline{g_i}$ is any lift of g_i to I. As any pair of lifts only varies by an element of X; since the functions in B agree on elements of X, it follows that this is well defined. Finally, since the functions in B are symmetric, it follows that so are the resulting functions on H. □

We can now prove Theorem 1.2.

Proof of Theorem 1.2. From Proposition 3.2, the set of maps $P \to A$ which satisfy our condition are in bijection with the set S_2. From the above proposition, each contributes $\#\,\mathrm{Hom}^{\mathrm{sym}}(H, H^*)$ as multiplicity, and so the total count is

$$\sum_{H \leq \Lambda / c^\dagger(X)} \#\,\mathrm{Hom}^{\mathrm{sym}}(H, H^*) = \nu^\dagger(d_1, \ldots, d_g)$$

as claimed. □

We now specialize to the lower-dimensional cases. For $g = 2, 3$, all simple principally polarized abelian varieties are Jacobians, and so the new detail is that we need to examine what happens as we lift from Jacobians of tropical curves to the curves themselves (as well as lifting the above to $\mathbb{K}$).

Naïvely, this may pose a problem. One of the more fascinating aspects of tropical geometry is the notion of *superabundance*. In particular, for a certain class of curves you end up with positive-dimensional families all of whom have the same Jacobian. When we try to study the lifting problem

$$\begin{array}{ccc} C & \dashrightarrow & P \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & J(\Gamma) \end{array}$$

then we can possibly end up with infinitely many such lifts. This would, needless to say, make it difficult to count curves.

Let us now resolve this while proving Theorem 1.3.

Proof of Theorem 1.3. Since the count of 2- and 3-dimensional simple principally polarized tropical abelian varieties mapping $P \to A$ is already equal to the count of genus 2 (resp. genus 3) curves, we just need to ensure that there is no extra multiplicity when we lift from the tropical world to rigid analytic one. As stated above, this is something that could in principle occur.

So suppose that we have performed the first lift and produced the following diagram, and that we are trying to determine how many final dashed lifts can occur.

$$\begin{array}{ccccc} C & \dashrightarrow & P & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma & \longrightarrow & J(\Gamma) & \longrightarrow & A^{\mathrm{tr}} \end{array}$$

Since P is 2- (resp. 3-) dimensional, we know that $P = J(C_{\mathbb{K}})$ for some unique genus 2 (resp. genus 3) curve $C_{\mathbb{K}}$ defined over $\mathbb{K}$. Furthermore, $P^{\mathrm{tr}} = J(C_{\mathbb{K}})^{\mathrm{tr}} = J(C_{\mathbb{K}}^{\mathrm{tr}})$. Moreover, since A^{tr} is Torelli, we know that $C_{\mathbb{K}}^{\mathrm{tr}}$ must be 3-connected. In particular, it follows that $C_{\mathbb{K}}^{\mathrm{tr}} = \Gamma$. In sum, it follows that the lift exists and is unique.

It follows then that the count of genus g curves of type c_L is $\nu^{\dagger}(d_1, \ldots, d_g)$ as claimed. □

Lastly, we need to understand how multiplicity arises in the context of curves in linear systems.

Proof of Theorem 1.4. We will modify the notation in this proof so that A^{tr} is our tropical torus with a tropical line bundle L^{tr}, and we will let A be a fixed abelian variety defined over $\mathbb{K}$ which tropicalizes to A^{tr}. First, recall that L^{tr} induces a polarization on A^{tr} of type (d_1, d_2), and let $n = d_1 d_2$.

Recall next from Proposition 3.5 that the set of curves in a fixed linear system on A^{tr} is in bijection with the set $\ker(A^{\mathrm{tr}} \to \widehat{A^{\mathrm{tr}}}) \times S_1$. We understand how S_1 lifts to $\mathbb{K}$, and so we only need to determine how $\ker(A^{\mathrm{tr}} \to \widehat{A^{\mathrm{tr}}})$ does.

Finally, note that $K = \ker(A \to \widehat{A})$ and $K^{\mathrm{tr}} = \ker(A^{\mathrm{tr}} \to \widehat{A^{\mathrm{tr}}})$ fit into the diagram (via the Snake Lemma)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(c_L^*) & \longrightarrow & K & \longrightarrow & K^{\mathrm{tr}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(X, \mu_n) & \longrightarrow & A[n] & \longrightarrow & A^{\mathrm{tr}}[n] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c_L^*} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(\Lambda, \mu_n) & \longrightarrow & \widehat{A}[n] & \longrightarrow & \widehat{A^{\mathrm{tr}}}[n] & \longrightarrow & 0.
\end{array}$$

Since the bottom two rows are split exact, it follows that $K \to K^{\mathrm{tr}}$ is surjective, and since $\ker(c_L^*)$ has n elements, it follows that translations contribute a factor of n. From all of these theorems together, it follows that the count of curves arising from our lifting to $\mathbb{K}$, including multiplicity, is $n^2 \nu^\dagger(d_1, d_2)$ as claimed. $\square$

4. Some specific computations

We will now compare the enumerative counts resulting from Theorem 1.3 to other known results, as well as provide a few additional numerical computations.

Theorem (Theorem 1.5). *Let (A, c_L) be a simple polarized abelian surface with polarization of type $(1, n)$. Then the number of genus 2 curves in $|L|$ is $n^2 \sigma_1(n)$.*

Proof. It suffices to show that $\nu(1, n) = \sigma_1(n)$. But this is clear: the number of symmetric morphisms from $\mathbb{Z}/d\mathbb{Z}$ to $(\mathbb{Z}/d\mathbb{Z})^*$ is d, since such homomorphisms are trivially symmetric. Thus

$$\sum_{H \leq \mathbb{Z}/n\mathbb{Z}} \# \,\mathrm{Hom}^{\mathrm{sym}}(H, H^*) = \sum_{d|n} d = \sigma_1(n)$$

as claimed. □

We also have the following case.

Proposition 4.1. *Let p be a prime. Then*

$$\nu(p, pn) = \sigma_1(p^2 n) + p^3 \sigma_1(n)$$

Proof. We separate the proof into two parts. Let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/(pn)\mathbb{Z}$. First, we note that the cyclic subgroups satisfy

$$\sum_{\substack{H \leq G \\ H \text{ is cyclic}}} \# \operatorname{Hom}(H, H^*) = \sigma_1(p^2 n) \tag{5}$$

Since both sides are multiplicative, this can be reduced to the case $n = p^\ell$. Noting that for all non-identity g in an abelian p-group, we have that $|g|/\varphi(|g|) = p/(p-1)$, we have that

$$\begin{aligned}
\sum_{\substack{H \leq G \\ H \text{ is cyclic}}} \# \operatorname{Hom}(H, H^*) &= \sum_{g \in G} \frac{|g|}{\varphi(|g|)} \\
&= 1 + \sum_{0 \neq g \in G} \frac{p}{p-1} \\
&= 1 + (|G| - 1)\frac{p}{p-1} \\
&= 1 + p + \cdots + p^{\ell+2} = \sigma_1(p^{\ell+2})
\end{aligned}$$

as claimed.

Next, we have the non cyclic subgroups (which are all of the form $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/(pd)\mathbb{Z}$ for some $d \mid n$). To proceed, we must examine in a little more detail what the symmetry condition entails. Consider some abelian group $A = A_1 \times A_2$. We know that $\operatorname{Hom}(A, A^*)$ can be decomposed as

$$\operatorname{Hom}(A, A^*) = \operatorname{Hom}(A_1, A_1^*) \times \operatorname{Hom}(A_1, A_2^*) \times \operatorname{Hom}(A_2, A_1^*) \times \operatorname{Hom}(A_2, A_2^*)$$

and we seek to determine what is the subset of these that is symmetric. We have a natural bijection

$$\operatorname{Hom}(A_1, A_2^*) \iff \operatorname{Hom}(A_1 \times A_2, \mathbb{K}^\times) \iff \operatorname{Hom}(A_2, A_1^*)$$

and so the symmetric functions in $\operatorname{Hom}(A, A^*)$ are those whose projection to $\operatorname{Hom}(A_1, A_2^*) \times \operatorname{Hom}(A_2, A_1^*)$ lie in the diagonal. Furthermore, if $A_1 = \mathbb{Z}/p\mathbb{Z}$ and $A_2 = \mathbb{Z}/(pd)\mathbb{Z}$, we see that this set has order p.

From this discussion, it follows that these then satisfy

$$\begin{aligned}&\mathrm{Hom}^{\mathrm{sym}}\left(\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/(pd)\mathbb{Z},(\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/(pd)\mathbb{Z})^*\right)\\ &=\mathrm{Hom}\left(\mathbb{Z}/p\mathbb{Z},(\mathbb{Z}/p\mathbb{Z})^*\right)\times\mathrm{Hom}\left(\mathbb{Z}/p\mathbb{Z},(\mathbb{Z}/(pd)\mathbb{Z})^*\right)\\ &\times\mathrm{Hom}\left(\mathbb{Z}/(pd)\mathbb{Z},(\mathbb{Z}/(pd)\mathbb{Z})^*\right)\end{aligned}$$

which has order $p^3 d$. Summing over all of these yields $p^3\sigma_1(n)$ as claimed. □

Remark 4.2. We have the more general fact that the analog of equation (5) holds for *all* finite abelian groups.

It is immediate from Proposition 4.1 that the generating function $\sum_{n=1}^{\infty}\nu(p,pn)q^n$ can be written as

$$\sum_{n=1}^{\infty}\nu(p,pn)q^n = p^3E_2(\varepsilon^{p^2}) + E_2(\varepsilon) + \frac{p^3+1}{24}$$

where $q=\varepsilon^{p^2}$, and where $E_2(q) = -\frac{1}{24}+\sum_{n=1}^{\infty}\sigma_1(n)q^n$ is the Eisenstein series of weight 2. In particular, these generating functions are quasimodular forms (if not of pure weight) for a non-trivial congruence subgroup of $SL_2\mathbb{Z}$. Moreover, one suspects that there is a consistent definition of $\nu(p,0)$ which would allow us to account for the constant term; this could possibly be further explored using the language of tropical Gromov-Witten theory.

For other cases, we produce the following table of values (which have been computed in a custom SAGE program). Note that this does *not* agree with the table in [LS02], which has some numerical errors (which were also noted in [BOPY15]). Furthermore, unlike the values $\nu(1,d)=\sigma_1(d)$, these do not form a recognizable sequence as, say, the coefficients of a modular form.

d	$\nu(d,d)$	$\nu(d,d,d)$	(d_1,d_2)	$\nu(d_1,d_2)$
2	15	135	(2, 4)	39
3	40	1120	(2, 6)	60
4	151	11,287	(2, 8)	87
5	156	19,656	(2, 10)	90
6	600	151,200	(2, 12)	156
7	400	137,600	(3, 6)	120
8	1335	810,135	(3, 9)	148
9	1201	915,853	(3, 12)	280
10	2340	2,653,560	(4, 8)	375
11	1464	1,950,048	(4, 12)	604
12	6040	12,641,440	(4, 16)	823
13	2380	5,231,240	(5, 10)	468
14	6000	18,576,000	(5, 15)	624
15	6240	22,014,720	(6, 12)	1560
16	11,191	54,681,751	(6, 18)	2220

5. Conclusion and further work

The goal of this paper was to work towards understanding how the count of curves in abelian varieties adapts to the tropical setting. In the complex setting, this has been studied e.g. in [BL99b], [LS02], and [Ros14].

In future work we intend to look at extending this to higher genera in a number of potential ways. One could naturally look at developing a tropical analogue of [BL99b], which would presumably require a version of reduced tropical Gromov-Witten theory.

One could instead focus on hyperelliptic curves, using the work of [Cha13] on tropical hyperelliptic curves. One should expect some interesting results by trying to produce an analogous argument to the one provided in this paper, combined with an understanding of the Jacobians of tropical hyperelliptic curves.

One could furthermore follow in the vein of [Ros14] and look at trying to understand tropical genus 0 curves in the associated tropical Kummer surface to A. This would require an understanding of the relationship between the count of such curves and the count of hyperelliptic curves in A; in [Ros14] this was provided by using orbifold Gromov-Witten theory, which to the best of our knowledge has not been developed in the tropical setting.

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