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Asymptotics of the $D^8\mathcal{R}^4$ genus-two string invariant

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We continue our investigation of the modular graph functions and string invariants that arise at genus-two as coefficients of low energy effective interactions in Type II superstring theory. In previous work, the non-separating degeneration of a genus-two modular graph function of weight w was shown to be given by a Laurent polynomial in the degeneration parameter t of degree (w, w). The coefficients of this polynomial generalize genus-one modular graph functions, up to terms which are exponentially suppressed in t as $t \to \infty$. In this paper, we evaluate this expansion explicitly for the modular graph functions associated with the $D^8\mathcal{R}^4$ effective interaction for which the Laurent polynomial has degree $(2, 2)$. We also prove that the separating degeneration is given by a polynomial in the degeneration parameter $\ln(|v|)$ up to contributions which are power-behaved in v as $v \to 0$. We further extract the complete, or tropical, degeneration and compare it with the independent calculation of the integrand of the sum of Feynman diagrams that contributes to two-loop type II supergravity expanded to the same order in the low energy expansion. We find that the tropical limit of the string theory integrand reproduces the supergravity integrand as its leading term, but also includes sub-leading terms proportional to odd zeta values that are absent in supergravity and can be ascribed to higher-derivative stringy interactions.

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1. Introduction

The low energy dynamics of string theory may be described in terms of an effective action which encodes the influence of massive string states upon the massless sector. The effective action admits an expansion in powers of space-time derivatives or, equivalently, in powers of the momenta of the massless states. The expansion contains the supersymmetrized Einstein–Hilbert action as its leading term, plus an infinite series of higher derivative effective interactions. The coefficients of the effective interactions are functions of the scalar fields that are associated with geometrical data of the target-space and are referred to as target-space moduli. These coefficients exhibit a rich mathematical structure. While relatively little is known about their exact dependence on the target-space moduli, precise statements can be made order by order in a variety of further expansions at asymptotic values of the target-space moduli. String perturbation theory uses an expansion in powers of the string coupling g_s, which is related to the constant value of the dilaton field.

The low energy effective action may be extracted from superstring scattering amplitudes. In closed superstring perturbation theory an amplitude is given by an infinite power series in g_s where the coefficient of g_s^{-2+2h} for $h \geq 0$ is an integral over the moduli space of compact super-Riemann surfaces of genus h. Many features of superstring amplitudes have been established to all orders in g_s, most notably the absence of the ultraviolet divergence and anomaly problems that plague perturbative quantum field theories containing gravity. However, explicit formulas for the amplitudes have been obtained so far only at low orders in the g_s expansion. Our interest in this paper will be in the simplest non-trivial Type II amplitude, namely for the scattering of four gravitons, whose explicit form is known only for $h \leq 2$. The low energy expansion of the four-graviton amplitude is given by a sum over $k \geq 0$ of effective interactions which are schematically of the form $D^{2k}\mathcal{R}^4$, where D and $\mathcal{R}$ respectively stand for the covariant derivative and the Riemann tensor of the target-space, suitably contracted. The coefficients of the effective interactions will be described next.

The genus-zero ($h = 0$) term is the tree-level contribution. Its leading low energy expansion reproduces the amplitudes arising in classical Einstein gravity. The coefficients of the higher order effective interactions in the low energy expansion of the four-graviton amplitude are polynomials in odd Riemann zeta values with rational coefficients. More generally, the coefficients

in the expansion of the tree-level amplitude with more than four gravitons are single-valued multiple zeta values [1, 2].

The genus-one ($h = 1$) four-graviton amplitude involves an integral over the moduli space $\mathcal{H}_1/SL(2,\mathbb{Z})$ of complex tori Σ (where $\mathcal{H}_1$ is the Poincaré upper half plane) and an integral over four points on Σ, corresponding to the four gravitons [3]. The coefficients of the effective interactions in the low-energy expansion of the integral over the points only, without integrating over $\mathcal{H}_1/SL(2,\mathbb{Z})$, are $SL(2,\mathbb{Z})$-invariant *modular graph functions* on $\mathcal{H}_1$ that were recently studied in some detail in [4, 5, 6, 7]. Although their structure still remains to be fully elucidated, it is clear that modular graph functions (and their generalizations to modular forms) generalize the multiple-zeta values that arise in the tree-level expansion, and satisfy algebraic identities that generalize those satisfied by multiple zeta values [8, 9, 10] (see also [11, 12, 13, 14, 15] for further studies).

Much less is known about the coefficients of the low-energy expansion of higher-genus ($h \geq 2$) amplitudes in superstring perturbation theory. The genus-two four-graviton amplitude was evaluated explicitly in [16, 17, 18, 19] for Type II and Heterotic strings by projecting the moduli space of super Riemann surfaces to that of Riemann surfaces. The Type II amplitude was reproduced in the pure spinor formulation and extended to include fermions in [20, 21]. The structure of the genus-two four-graviton amplitude generalizes that of its genus-one counterpart. It is given by an integral over the moduli space $\mathcal{M}_2 \approx \mathcal{H}_2/Sp(4,\mathbb{Z})$ of genus two compact Riemann surfaces Σ (where $\mathcal{H}_2$ is the Siegel upper half space of rank two, parametrized by the period matrix Ω) of an integral over four points on Σ, again corresponding to the four gravitons. The low-energy expansion of the integral over the points only, without integrating over $\mathcal{M}_2$, now gives rise to $Sp(4,\mathbb{Z})$-invariant functions on $\mathcal{H}_2$ which, by analogy with the genus-one case, are referred to as *genus-two modular graph functions* [22].

In the low energy expansion of the genus-two four-graviton amplitude in Type II superstrings, the effective interactions $\mathcal{R}^4$ and $D^2\mathcal{R}^4$ have vanishing coefficients. The first non-zero term is $D^4\mathcal{R}^4$, whose coefficient is constant on $\mathcal{H}_2$ and matches the predictions of S-duality in Type IIB string theory [23]. The next order term is the effective interaction $D^6\mathcal{R}^4$. Its coefficient is a non-trivial $Sp(4,\mathbb{Z})$-invariant function φ on $\mathcal{H}_2$, which was shown in [24] to be proportional to the genus-two Kawazumi-Zhang invariant [25, 26] (see also [27, 28]). The invariant φ satisfies a Laplace eigenvalue equation on $\mathcal{H}_2$ [29], which was later used in combination with known leading asymptotics [30, 31] to establish its representation as a generalized Borcherds-type theta-lift [32]. The latter provides the full asymptotic expansion near the boundary of

moduli space $\mathcal{M}_2$, including all exponentially suppressed terms. The integral of $\varphi(\Omega)$ over $\mathcal{M}_2$ can be computed using the eigenvalue equation and also matches the S-duality prediction [29].

The genus-two contribution to higher order effective interactions, schematically of the form $D^{2k}\mathcal{R}^4$ for $k \geq 4$, may also be derived from the four-graviton amplitude and, as was pointed out in [24], produces further and novel genus-two $Sp(4,\mathbb{Z})$-invariant modular graph functions on $\mathcal{H}_2$. The goal of previous work in [22], of this paper, and of future work, is to gain understanding of these novel invariants, and of any algebraic and differential relations they may satisfy, at a level comparable to the one that has been achieved for the Kawazumi-Zhang invariant or for the genus-one case. One important step in this direction, which has proven to be invaluable also at genus one, is to obtain the behavior of the novel invariants under degenerations of the genus-two Riemann surface.

Powerful techniques were developed in [22] to analyze the behavior of general classes of modular graph functions at arbitrary genus near the non-separating degeneration of the Riemann surface. The non-separating degeneration of a genus-two surface Σ corresponds to letting a non-trivial homology cycle become infinitely long while keeping independent cycles finite so that a genus-two surface Σ degenerates to a torus Σ_1 with two punctures.

1.1. Summary of results

In this paper we will extend the techniques and results of [22] to obtain the expansions of the genus-two modular graph function $\mathcal{B}_{(2,0)}(\Omega)$ associated with the $D^8\mathcal{R}^4$ effective interaction around both the non-separating and the separating degeneration limits. We will also consider the further degeneration, known as the "tropical" limit, in which the two-dimensional surface reduces to a two-loop irreducible graph shown in Figure 12 on page 401. In the non-separating case this will be compared with the expression obtained from low energy expansion of two-loop supergravity. Our results may be summarized as follows:

1. With the help of the genus-two Arakelov Green function, the string invariant $\mathcal{B}_{(2,0)}(\Omega)$ may be decomposed into a sum of three non-trivial $Sp(4,\mathbb{Z})$-invariant genus-two modular graph functions $\mathcal{Z}_i(\Omega)$ defined in (2.12),

$$\mathcal{B}_{(2,0)}(\Omega) = \tfrac{1}{2}\mathcal{Z}_1(\Omega) - \mathcal{Z}_2(\Omega) + \tfrac{1}{2}\mathcal{Z}_3(\Omega) \tag{1.1}$$

Throughout, Ω will denote a genus-two period matrix, and Y will denote its imaginary part, whose components are given by,

$$\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \qquad Y = \operatorname{Im}\Omega = \begin{pmatrix} \tau_2 & v_2 \\ v_2 & \sigma_2 \end{pmatrix} \tag{1.2}$$

with $\tau = \tau_1 + i\tau_2, v = v_1 + iv_2, \sigma = \sigma_1 + i\sigma_2$, and $\tau_1, \tau_2, v_1, v_2, \sigma_1, \sigma_2 \in \mathbb{R}$. The matrix Ω takes values in the Siegel upper-half plane $\mathcal{H}_2$ so that the matrix Y is positive definite.

2. Near the *non-separating degeneration* $t \to \infty$, Theorem 3 of [22] states that each modular graph function $\mathcal{Z}_i(\Omega)$ is given by a Laurent polynomial in the degeneration parameter t, of degree (w, w) for $w = 2$, with exponentially small corrections,

$$\mathcal{Z}_i(\Omega) = \sum_{n=-w}^{w} (\pi t)^n \, \mathfrak{z}_i^{(n)}(v|\tau) + \mathcal{O}(e^{-2\pi t}) \tag{1.3}$$

The subgroup of the modular group $Sp(4, \mathbb{Z})$ which leaves the degeneration invariant is isomorphic to the Fourier-Jacobi group $SL(2, \mathbb{Z}) \ltimes (\mathbb{Z}^2 \ltimes \mathbb{Z})$. It was shown in [22] that the non-separating degeneration takes a strikingly simple form in terms of a special combination t of the moduli given by $t = \det(\operatorname{Im}\Omega)/\operatorname{Im}(\tau)$. The parameter t and the coefficients $\mathfrak{z}_i^{(n)}(v|\tau)$ are invariant under the modular subgroup $SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ acting on $v \in \Sigma_1$ and $\tau \in \mathcal{H}_1$. The coefficients $\mathfrak{z}_i^{(n)}(v|\tau)$ may be thought of equivalently as non-holomorphic elliptic functions, non-holomorphic Jacobi forms, or modular graph functions with $(\mathbb{R}/\mathbb{Z})^2$-character, and are evaluated explicitly in this paper, see Eq. (3.21).

3. Near the *separating degeneration* $v \to 0$, we show that each modular graph function $\mathcal{Z}_i(\Omega)$ is given by a polynomial in $(-\ln|\hat{v}|)$ of degree $w = 2$, up to corrections that are power behaved in $|v|^{\frac{1}{2}}$ (see Eq. (4.19)),

$$\mathcal{Z}_i(\Omega) = \sum_{n=0}^{w} (-\ln|\hat{v}|)^n \, \mathfrak{s}_i^{(n)}(\tau, \sigma) + \mathcal{O}(|\hat{v}|^{\frac{1}{2}}) \qquad \hat{v} = 2\pi v \, \eta(\tau)^2 \eta(\sigma)^2 \tag{1.4}$$

where η is the Dedekind eta-function. The degeneration parameter $\hat{v}$ and the coefficients $\mathfrak{s}_i^{(n)}(\tau, \sigma)$ are invariant under the residual modular

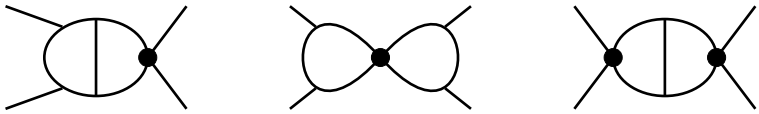
group $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})'$ of the separating degeneration as $v \to 0$ while keeping τ and σ fixed. This is in fact a special case of a result valid for a general class of genus two modular graph functions of degree w, as we show in Section 4.7.

4. Near the *tropical degeneration*, the matrix Y is uniformly scaled to ∞ keeping the ratios of its entries fixed, so that the parameter V defined by $V = (\det Y)^{-1/2}$ tends to zero. In this limit, each modular graph function $\mathcal{Z}_i(\Omega)$ is given by a Laurent polynomial with exponentially small corrections (see Eq. (5.12), (5.13), (5.16)),

(1.5)
$$\mathcal{Z}_i(\Omega) = \sum_{n=-w}^{2w} V^n \, \mathfrak{j}_i^{(n)}(S) + \mathcal{O}(e^{-1/V}) \qquad\qquad S = \frac{v_2}{\tau_2} + i\sqrt{\frac{t}{\tau_2}}$$

The coefficients $\mathfrak{j}_i^{(n)}(S)$ are *modular local Laurent polynomials*, which belong to a class of non-holomorphic modular functions first encountered in the study of two-loop supergravity amplitudes [33] and further developed in the mathematics literature [34, 35, 36]. In the vicinity of the cusp $S \to i\infty$, this degeneration is obtained by extracting the behavior of the genus-one functions $\mathfrak{z}_i^{(n)}(v|\tau)$ in the limit $\tau_2 \to \infty$ keeping v_2/τ_2 fixed in the range $0 < v_2/\tau_2 < 1$. The tropical degeneration near the separating degeneration is obtained by extracting the behavior of the coefficients $\mathfrak{s}_i^{(n)}(\tau,\sigma)$ as both $\tau_2, \sigma_2 \to \infty$, keeping their ratio fixed.

5. We compare the tropical limit of $\mathcal{B}_{(2,0)}$ with the coefficient, $\mathcal{B}_{(2,0)}^{(sg)}$, of the $D^8\mathcal{R}^4$ interaction in the low energy expansion of the two-loop contribution to Type II supergravity [38], which can be expressed as a sum of scalar field theory diagrams as shown in Figure 12. In order to make this comparison we use a world-line formalism that mimics the string theory world-sheet formalism and expresses $\mathcal{B}_{(2,0)}^{(sg)}$ as a linear sum of three contributions, which are built out of the world-line Arakelov Green function and are field theory analogues of $\mathcal{Z}_i(\Omega)$. The tropical degeneration $V \to 0$ is found to reproduce the known supergravity integrand at order $\mathcal{O}(1/V^2)$, but includes additional sub-leading terms proportional to odd zeta values such as $\zeta(3)V$, $\zeta(5)V^3$ and $\zeta(3)^2V^4$. The same phenomenon holds for the Kawazumi-Zhang invariant, whose tropical limit includes a single subleading term proportional to $\zeta(3)V^2$ [32]. In field theory language each subleading term can be interpreted



(a) (b) (c)

Figure 1: Examples of modifications of two-loop supergravity diagrams in which the black nodes indicate higher-derivative local interaction vertices, which arise in the tropical limit.

as a two-loop Feynman integrand where one of the supergravity interaction vertices is replaced by a higher derivative tree-level effective interaction, such as $\mathcal{R}^4$, $D^4\mathcal{R}^4$ or $D^6\mathcal{R}^4$ as is indicated in Figure 1. The effect of such higher derivative interactions implies a particular pattern of logarithmic divergences when the amplitude is considered in lower dimensions by compactification on a torus. We leave a detailed analysis of this phenomenon to a forthcoming publication [37].

These results should be important for elucidating further properties of higher genus modular graph functions. Among many outstanding issues still to be understood are algebraic and differential identities between the genus-two modular graph function and the possibility of generalised theta-lift representation for these functions analogous to the representation of the Kawazumi–Zhang invariant found in [32]. Moreover, these results are important in determining the genus-two contribution to the coefficient of the $D^8\mathcal{R}^4$ effective interaction, which is given by integration of $\mathcal{B}_{(2,0)}$ over the moduli space of genus-two Riemann surfaces, or equivalently over a fundamental domain $\mathcal{H}_2/Sp(4,\mathbb{Z})$ in the Siegel upper-half plane.

1.2. Organization

The remainder of this paper is organized as follows. In Section 2, we review the low energy expansion of the genus-two contribution to the four graviton amplitude. Appendix A reviews related material concerning some basic features of genus-one surfaces, including details of how various genus-two integrals reduce to integrals over genus-one surfaces with punctures. In Section 3 we use the methods developed in [22] to give a detailed evaluation of the expansion of $\mathcal{B}_{(2,0)}(\Omega)$, the $Sp(4,\mathbb{Z})$ invariant coefficient of $D^8\mathcal{R}^4$, around the non-separating degeneration. The determination of the expansion coefficients $\mathfrak{z}_i^{(n)}(v|\tau)$ in (1.3) involves a large number of steps, which are detailed

in Appendix B. In Section 4 we obtain a general result on the separating degeneration of genus-two modular graph functions, and apply this to obtain the explicit coefficients, $\mathfrak{s}_i^{(n)}$ (defined in (1.4)) of the expansion of $\mathcal{B}_{(2,0)}(\Omega)$. In Section 5 we take the further limit that gives the complete (or tropical) degeneration and evaluate the coefficients, $\mathfrak{j}_i^{(n)}$ (defined in 1.5), relevant to this degeneration. Technical details needed in deriving the coefficients in the tropical limit are given in Appendix C. In Section 6, we review and extend the computation of the coefficient of $D^8\mathcal{R}^4$ in the expansion of the two-loop supergravity amplitude using world-line techniques, and compare this with the tropical limit of the non-separating degeneration of the string amplitude.

2. Structure of genus-two string invariants

The genus-two contribution to the four-graviton scattering amplitude $\mathcal{A}^{(2)}(\epsilon_i, k_i)$ is proportional to an integral of a scalar function $\mathcal{B}^{(2)}(s_{ij}|\Omega)$ over the moduli space $\mathcal{M}_2$ of genus-two compact Riemann surfaces [19, 23],

$$\mathcal{A}^{(2)}(\epsilon_i, k_i) = \frac{\pi}{4}\kappa_{10}^2\, g_s^2\, \mathcal{R}^4 \int_{\mathcal{M}_2} \frac{|d^3\Omega|^2}{(\det Y)^3}\, \mathcal{B}^{(2)}(s_{ij}|\Omega) \tag{2.1}$$

The polarization tensors and momenta of the four gravitons are respectively denoted by ϵ_i and k_i with $i = 1, 2, 3, 4$. The kinematic invariants are defined by $s_{ij} = -\alpha' k_i \cdot k_j/2$ and satisfy the relations $s_{12} = s_{34}$, $s_{13} = s_{24}$, $s_{14} = s_{23}$, and $s_{12} + s_{13} + s_{14} = 0$ due to momentum conservation. The gravitational constant in ten-dimensional space-time is denoted by κ_{10}^2, while α' is the string scale. The quantity $\mathcal{R}^4$ represents a particular scalar contraction of four powers of the linearized Riemann curvature tensor whose detailed form can be found in [3] and is dictated by supersymmetry. Its explicit expression will not be needed here.

The period matrix $\Omega \in \mathcal{H}_2$ parametrizes the complex structure of a genus-two Riemann surface Σ. Given a choice of canonical homology cycles $\mathfrak{A}_I, \mathfrak{B}_I$ for $I = 1, 2$, and dual holomorphic one-forms ω_I on Σ, the period matrix is defined by,

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad\qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ} \tag{2.2}$$

The Riemann bilinear relations imply that Ω is a symmetric matrix, and that its imaginary part Y is positive definite, along with the following integral

relation,[1]

$$\frac{i}{2}\int_\Sigma \omega_I \wedge \overline{\omega}^J = \delta_I{}^J \qquad\qquad \overline{\omega}^J = (Y^{-1})^{JK}\,\overline{\omega_K} \tag{2.3}$$

where Y^{-1} denotes the inverse of the matrix Y. Further properties of the period matrix, including its behavior under modular transformations $Sp(4,\mathbb{Z})$, are well-known and are reviewed, for example, in subsection 2.2 of [22]. The measure factor in the integrand of (2.1) is the $Sp(4,\mathbb{Z})$-invariant volume form for the Siegel metric on $\mathcal{H}_2$, which will not be needed in the sequel. By construction, the function $\mathcal{B}^{(2)}(s_{ij}|\Omega)$ will be invariant under $Sp(4,\mathbb{Z})$ transformations of Ω, for arbitrary values of s_{ij}. Therefore, the integral in (2.1) is intrinsically defined, and we may represent $\mathcal{M}_2$ by a fundamental domain $\mathcal{H}_2/Sp(4,\mathbb{Z})$ for the action of the modular group $Sp(4,\mathbb{Z})$ on the Siegel upper half space $\mathcal{H}_2$.

2.1. Genus-two string invariants

The function $\mathcal{B}^{(2)}(s_{ij}|\Omega)$ is the starting point of our study, and may be viewed as a generating function in the parameters s_{ij} for genus-two modular graph functions derived from string theory. It is given by an integral over four points z_i, corresponding to the four gravitons, on a genus-two Riemann surface Σ with period matrix Ω,[2]

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \frac{1}{16}\int_{\Sigma^4} \frac{\mathcal{Y}\wedge\bar{\mathcal{Y}}}{(\det Y)^2}\exp\left\{\sum_{1\le i<j\le 4} s_{ij}\,\mathcal{G}(z_i,z_j|\Omega)\right\} \tag{2.4}$$

Here, $\mathcal{Y}$ is a holomorphic $(1,0)$-form in each point $z_i\in\Sigma$ and is linear in the s_{ij}. It was constructed in [19], and is given for example in eq. (2.29) of

[1]Throughout, we shall use the Einstein summation convention for the indices $I,J=1,2$ which implies summation over any repeated lower and upper index of the same name. When no confusion is expected to arise, we shall not exhibit the dependence on the moduli and the coordinates in differential forms, but we shall exhibit these dependences for functions.

[2]In the earlier paper [22], a Riemann surface of genus h was denoted Σ_h and we used the notations κ_h for the canonical Kähler form, $\mathcal{G}_h$ for the Arakelov Green function, and G_h for the string Green function. Since the present paper deals exclusively with genus-two surfaces and their degenerations, we shall drop the subscript "2" throughout, and use the notation $g=\mathcal{G}_1$ for the standard Green function on the torus.

[22], but its explicit form will be needed only for a special arrangement of the parameters s_{ij} which will be given below.

The Green function $\mathcal{G}(z_i, z_j|\Omega)$ is formally the inverse of the Laplace operator on Σ. Due to the zero mode of the Laplace operator, the inverse is not unique, and generally fails to be conformal invariant. However, momentum conservation relations between the parameters s_{ij} imply that the exponential in (2.4) is invariant under the following shift of $\mathcal{G}$,

$$\mathcal{G}(z_i, z_j|\Omega) \to \mathcal{G}(z_i, z_j|\Omega) + c(z_i) + c(z_j) \tag{2.5}$$

for an arbitrary function c. Since a conformal transformation on $\mathcal{G}$ produces a shift in $\mathcal{G}$ which is precisely of the form (2.5) (see for example [39]), the exponential in (2.4) and thus $\mathcal{B}^{(2)}(s_{ij}|\Omega)$ are conformal invariant, as is essential in a consistent string theory amplitude. Two convenient choices will be used below for $\mathcal{G}$, the first being the familiar string Green function [39], the other being the Arakelov Green function (see for example [40]), both of which will be discussed in subsection 2.5. We note that the behavior as $z_j \to z_i$ of either of these scalar Green functions is $\mathcal{G}(z_i, z_j|\Omega) \approx -\ln|z_i - z_j|^2$.

The integrals in (2.4) are absolutely convergent for $\mathrm{Re}\,(s_{ij}) < 1$, and admit a Taylor series in powers of s_{ij} with unit radius of convergence in s_{12}, s_{13} and s_{14}. Expanding the exponential in powers of s_{ij}, using the relation $s_{12} + s_{13} + s_{14} = 0$ and the fact that $\mathcal{B}^{(2)}(s_{ij}|\Omega)$ is symmetric in the variables s_{ij}, leads to an infinite series which may be organized as follows [4],

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \sum_{p,q=0}^{\infty} \mathcal{B}_{(p,q)}(\Omega)\, \frac{\sigma_2^p\, \sigma_3^q}{p!\, q!} \tag{2.6}$$

where σ_n are symmetric polynomials in s_{ij} defined by $\sigma_n = (s_{12})^n + (s_{13})^n + (s_{14})^n$. Since $\mathcal{B}^{(2)}(s_{ij}|\Omega)$ is $Sp(4,\mathbb{Z})$-invariant for any value of s_{ij}, each coefficient $\mathcal{B}_{(p,q)}(\Omega)$ is itself $Sp(4,\mathbb{Z})$-invariant and defines a genus-two modular graph function, in the sense of [22]. To identify the structure of the effective interaction to which each coefficient corresponds, it will be convenient to introduce the notion of *weight*, defined as the number of Green function $\mathcal{G}$ factors in the Taylor series expansion in powers of s_{ij}, which is given by,

$$w = 2p + 3q - 2 \tag{2.7}$$

The linearity of $\mathcal{Y}$ in s_{ij} implies that a modular graph function $\mathcal{B}_{(p,q)}(\Omega)$ of weight w corresponds to an effective interaction of the form $D^{2w+4}\mathcal{R}^4$. For

low weights, $w \leq 3$, there is a unique effective interaction for each w, but for $w \geq 4$ several independent effective interactions may correspond to the same weight.

2.2. Convergence of the integrals over $\mathcal{M}_2$

The integrals in (2.4) over the points $z_i \in \Sigma$ are absolutely convergent for any point Ω in the interior of moduli space as long as $\mathrm{Re}\,(s_{ij}) < 1$. However, this convergence is non-uniform as Ω moves to the boundary of $\mathcal{M}_2$ so that the summation in (2.6) and the integration in (2.1) cannot be legitimately interchanged. Mathematically, this is due to the fact that $\mathcal{G}$ grows linearly with Y, so that the domain of absolute convergence of the integral over $\mathcal{M}_2$ is restricted to $\mathrm{Re}\,(s_{ij}) = 0$. Away from this set, analytic continuation in s_{ij} is required, and may be carried out along similar lines as the construction of the genus-one amplitude in [41]. Physically, the divergences arise because non-analyticities, such as logarithmic branch cuts, in the variables s_{ij} are produced by this analytic continuation, and these functions cannot be expanded in a convergent Taylor series at $s_{ij} = 0$.

Even when the integrals of $\mathcal{B}_{(p,q)}(\Omega)$ over $\mathcal{M}_2$ are not absolutely convergent, due to the appearance of non-analyticities in s_{ij} as explained above, it is still possible to extract the strength of the corresponding effective interactions. However, this requires isolating the contribution to the integral from the boundary of $\mathcal{M}_2$ first, carrying out its analytic continuation, and then identifying the low energy expansion of the analytic remainder of the amplitude. Carrying out this procedure will be the subject of future work [37].

2.3. Low weights: the Kawazumi-Zhang invariant

In this subsection, we briefly review the results for the string invariants for weights $w \leq 1$. Since $\mathcal{Y}$ is linear in s_{ij}, we have $\mathcal{B}_{(0,0)}(\Omega) = 0$, reflecting the absence of genus-two corrections to the effective interaction $\mathcal{R}^4$. Weight zero corresponds to the effective interaction $D^4\mathcal{R}^4$ whose coefficient $\mathcal{B}_{(1,0)}(\Omega)$ is constant on $\mathcal{M}_2$ (equal to -2 in our conventions) and, upon integration over $\mathcal{M}_2$, provides an important consistency check with the implications of S-duality in Type IIB superstrings [23].

The weight $w = 1$ coefficient $\mathcal{B}_{(0,1)}(\Omega)$ of the effective interaction $D^6\mathcal{R}^4$ is proportional to the Kawazumi-Zhang invariant $\varphi(\Omega)$ for genus two, specifically $B_{(0,1)}(\Omega) = 4\,\varphi(\Omega)$. The contribution from the exponential in (2.4) is

linear in $\mathcal{G}$ and therefore one may integrate explicitly over two of the four points in (2.4), using (2.3), giving the following formula,

$$\varphi(\Omega) = -\frac{1}{8}\Big(2\,\delta_{J_1}{}^{I_2}\delta_{J_2}{}^{I_1} - \delta_{J_1}{}^{I_1}\delta_{J_2}{}^{I_2}\Big) \times \int_{\Sigma^2} \omega_{I_1}(z_1)\,\overline{\omega}^{J_1}(z_1)\,\omega_{I_2}(z_2)\,\overline{\omega}^{J_2}(z_2)\,\mathcal{G}(z_1,z_2|\Omega) \tag{2.8}$$

The complete asymptotics of $\varphi(\Omega)$ is known thanks to the theta-lift representation established in [32], based on the Laplace-Beltrami eigenvalue equation derived in [29] and on known leading asymptotics [30, 31]. Its integral over $\mathcal{M}_2$ can be computed using the eigenvalue equation and is also in agreement with S-duality [29].

2.4. The string invariant $\mathcal{B}_{(2,0)}$

At weight $w = 2$, there is a single effective interaction, of the form $D^8\mathcal{R}^4$ corresponding to $p = 2$ and $q = 0$ in (2.6) for a single kinematic invariant $(\sigma_2)^2$. The explicit form of the corresponding string invariant was given in [24, 22]. It may be obtained by expanding (2.4) to second order in s_{ij} (equivalently to second order in $\mathcal{G}$), and setting $s_{13} = 0$ so that the expression for $\mathcal{Y}$ simplifies. The function $\mathcal{B}_{(2,0)}(\Omega)$ is given as follows,

$$\mathcal{B}_{(2,0)}(\Omega) = \int_{\Sigma^4} \frac{|\Delta(z_1,z_3)\Delta(z_2,z_4)|^2}{64\,(\det Y)^2} \times \Big(\mathcal{G}(z_1,z_2|\Omega) + \mathcal{G}(z_3,z_4|\Omega) - (z_1 \leftrightarrow z_3)\Big)^2 \tag{2.9}$$

where $\Delta(z_i,z_j)$ is the holomorphic two-form on Σ^2 defined by,

$$\Delta(x,y) = \omega_1(x)\wedge\omega_2(y) - \omega_2(x)\wedge\omega_1(y) \tag{2.10}$$

The string invariant $\mathcal{B}_{(2,0)}(\Omega)$ will be the central object of our study in this paper.

In the sequel, we shall find it useful to decompose $\mathcal{B}_{(2,0)}(\Omega)$ into a sum of three terms by expanding the integrand in (2.9),

$$\mathcal{B}_{(2,0)}(\Omega) = \frac{1}{2}\mathcal{Z}_1(\Omega) - \mathcal{Z}_2(\Omega) + \frac{1}{2}\mathcal{Z}_3(\Omega) \tag{2.11}$$

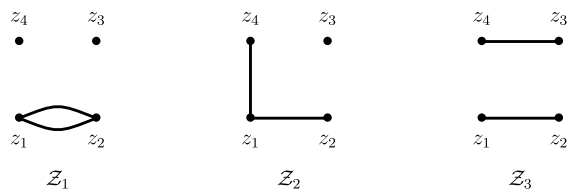


Figure 2: Graphs representing the three distinct contributions $\mathcal{Z}_i$ to $\mathcal{B}_{(2,0)}$ in (2.11), where each line represents a factor of the Green function $\mathcal{G}$.

where the 8-fold symmetry group generated by the permutations (3214), (1432) and (3412) allows us to reduce the terms bilinear in the Green functions as follows,

$$
\begin{aligned}
\mathcal{Z}_1(\Omega) &= \int_{\Sigma^4} \frac{|\Delta(z_1,z_3)\Delta(z_2,z_4)|^2}{8\,(\det Y)^2}\,\mathcal{G}(z_1,z_2|\Omega)^2 \\
\mathcal{Z}_2(\Omega) &= \int_{\Sigma^4} \frac{|\Delta(z_1,z_3)\Delta(z_2,z_4)|^2}{8\,(\det Y)^2}\,\mathcal{G}(z_1,z_2|\Omega)\,\mathcal{G}(z_1,z_4|\Omega) \\
\mathcal{Z}_3(\Omega) &= \int_{\Sigma^4} \frac{|\Delta(z_1,z_3)\Delta(z_2,z_4)|^2}{8\,(\det Y)^2}\,\mathcal{G}(z_1,z_2|\Omega)\,\mathcal{G}(z_3,z_4|\Omega)
\end{aligned}
\tag{2.12}
$$

The modular graph function $\mathcal{B}_{(2,0)}(\Omega)$ is invariant under shifts of $\mathcal{G}$ given in (2.5), just as the generating function $\mathcal{B}^{(2)}(s_{ij}|\Omega)$ was from which $\mathcal{B}_{(2,0)}(\Omega)$ is derived. Thus, $\mathcal{B}_{(2,0)}(\Omega)$ is independent of the type of Green function chosen to represent it, and is conformal invariant. However, once we split $\mathcal{B}_{(2,0)}(\Omega)$ into a sum of three terms, as in (2.11), each individual term $\mathcal{Z}_i(\Omega)$ will generally fail to be invariant under the shifts (2.5), and fail to be conformal invariant. This shortcoming may be remedied by using the conformal invariant Arakelov Green function in each term $\mathcal{Z}_i$. As a result, each $\mathcal{Z}_i$ will be a genuine genus-two modular graph function in the sense of [22], and may be represented graphically as in Figure 2.

The expressions for the modular graph functions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ may be simplified by integrating over the points z_3 and z_4 in $\mathcal{Z}_1$ and the point z_3 in $\mathcal{Z}_2$ using (2.3) and (2.10). The resulting expressions are as follows,

$$
\begin{aligned}
\mathcal{Z}_1(\Omega) &= 8\int_{\Sigma^2} \kappa(z_1)\kappa(z_2)\,\mathcal{G}(z_1,z_2|\Omega)^2 \\
\mathcal{Z}_2(\Omega) &= \int_{\Sigma^3} \kappa(z_1)\frac{|\Delta(z_2,z_4)|^2}{\det Y}\,\mathcal{G}(z_1,z_2|\Omega)\,\mathcal{G}(z_1,z_4|\Omega)
\end{aligned}
\tag{2.13}
$$

where $\kappa(z)$ is the canonical Kähler form defined in (2.14).

We close this subsection by noting two further motivations for splitting $\mathcal{B}_{(2,0)}$ into three individual modular graph functions $\mathcal{Z}_i$. One motivation will be to establish detailed, graph by graph agreement with the supergravity calculations – even though the original supergravity calculation for the complete $\mathcal{B}_{(2,0)}$ integrand [33] is much simpler than the one required for each of the three terms separately. A second motivation stems from the ultimate goal to obtain algebraic and differential equations satisfied by genus-two modular graph functions of weight $w \geq 1$. Experience with the corresponding problem at genus one has revealed that, for high enough weight, one has to deal with a system of equations involving several modular graph functions rather than a single equation for a single function [6]. Therefore, it may be useful to build a "library" of functions such as the individual modular graph functions $\mathcal{Z}_i$.

2.5. The Arakelov Green function

As discussed in the preceding subsection, the use of the Arakelov Green function is crucial for obtaining a decomposition of the integrand (2.11) into a sum of well-defined conformal invariant modular graph functions $\mathcal{Z}_i$. In this subsection, we review the salient features of the Arakelov Green function on a genus-two Riemann surface Σ, and evaluate it concretely. The starting point is the canonical Kähler form κ normalized to unit integral,

$$\kappa = \frac{i}{4}\,\omega_I \wedge \overline{\omega}^I = \frac{i}{4}(Y^{-1})^{IJ}\omega_I \wedge \overline{\omega}_J \qquad\qquad \int_\Sigma \kappa = 1 \tag{2.14}$$

The canonical Kähler form depends only on the holomorphic one-forms ω_I and their periods. It is conformal and modular invariant, and uniquely determined by its integral over Σ.

The Arakelov Green function $\mathcal{G}(z, y|\Omega)$ on a genus-two Riemann surface Σ is a real-valued symmetric function on $\Sigma \times \Sigma \times \mathcal{H}_2$, which provides an inverse to the scalar Laplace operator on Σ equipped with the canonical Kähler form κ. Expressing κ in local complex coordinates $\kappa = \frac{i}{2}\kappa_{z\bar{z}}(z)\,dz \wedge d\bar{z}$, the Arakelov Green function is defined by the following equations,[3]

$$\partial_{\bar{z}}\,\partial_z\,\mathcal{G}(z, y|\Omega) = -\pi\,\delta^{(2)}(z, y) + \pi\,\kappa_{z\bar{z}}(z) \qquad\qquad \int_\Sigma \kappa(z)\,\mathcal{G}(z, y|\Omega) = 0 \tag{2.15}$$

[3]Throughout, the "coordinate" Dirac δ-function is normalized by $\frac{i}{2}\int_\Sigma dz \wedge d\bar{z}\,\delta^{(2)}(z, y) = 1$.

An explicit expression for $\mathcal{G}$ may be obtained by relating it to another Green function G which is often used in string theory [39], and defined by,

(2.16)
$$G(x,y|\Omega) = -\ln|E(x,y|\Omega)|^2 + 2\pi\,\mathrm{Im}\left(\int_y^x \omega_I\right)(Y^{-1})^{IJ}\,\mathrm{Im}\left(\int_y^x \omega_J\right)$$

where $E(x,y|\Omega)$ is the prime form [42], which is a holomorphic form of weight $(-1/2,0)$ in each variable x,y on the covering space of Σ^2. As a result, the Green function $G(x,y|\Omega)$ is not a conformal scalar. Therefore, one should be careful to calculate with $G(x,y|\Omega)$ on a simply connected fundamental domain for Σ obtained by cutting the surface along suitably chosen curves $\mathfrak{A}_I,\mathfrak{B}_I$. The Green functions G and $\mathcal{G}$ are related as follows,

$$\mathcal{G}(x,y|\Omega) = G(x,y|\Omega) - \gamma(x|\Omega) - \gamma(y|\Omega) + \gamma_1(\Omega) \tag{2.17}$$

where

$$\gamma(x|\Omega) = \int_\Sigma \kappa(z)G(x,z|\Omega) \qquad\qquad \gamma_1(\Omega) = \int_\Sigma \kappa(z)\gamma(z|\Omega) \tag{2.18}$$

These relations ensure that $\mathcal{G}$ integrates to zero against the canonical Kähler form κ, as required by (2.15), and that the function $\mathcal{G}$ defined by (2.17) is invariant when G is shifted as in (2.5). As a result, the Arakelov Green function is conformal invariant, and each individual function $\mathcal{Z}_i$ defined in (2.12) will be a conformal invariant modular graph form, as promised earlier. Henceforth, $\mathcal{G}$ will denote the Arakelov Green function, and it will understood throughout that (2.12) is expressed in terms of the Arakelov Green function $\mathcal{G}$.

3. The non-separating degeneration

In this section we shall obtain the non-separating degeneration, in the form given in (1.3), of the genus-two modular graph functions $\mathcal{Z}_i$, which were expressed in terms of the Arakelov Green function $\mathcal{G}$ in (2.12). We begin with a review of the methods developed in [22] for parametrizing and calculating this degeneration, apply the method to reproduce the degeneration of $\mathcal{B}_{(0,1)}$ (proportional to the Kawazumi-Zhang invariant), and then calculate the degenerations of the string invariants $\mathcal{Z}_i$ and $\mathcal{B}_{(2,0)}$.

3.1. Funnel construction of the non-separating degeneration

In the non-separating degeneration, a compact genus-two surface Σ degenerates to a genus-one surface $\Sigma_1 \setminus \{p_a, p_b\}$, where Σ_1 is a compact genus-one surface and p_a, p_b are the two points which are the remnants of the degenerating funnel of the surface Σ. Our interest is in evaluating the expansion of the form (1.3) in a neighborhood of the non-separating node where a non-trivial homology cycle of the surface Σ becomes a long and skinny, but finite, funnel. The imaginary part of the period matrix Y, introduced in (1.2), and its inverse Y^{-1}, may be parametrized as follows,

(3.1)
$$Y = \begin{pmatrix} \tau_2 & \tau_2 u_2 \\ \tau_2 u_2 & t + \tau_2 u_2^2 \end{pmatrix} \qquad Y^{-1} = \begin{pmatrix} \tau_2^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{t} \begin{pmatrix} u_2^2 & -u_2 \\ -u_2 & 1 \end{pmatrix}$$

where $v_2 = \tau_2 u_2$ and $t = (\det Y)/\tau_2 = \sigma_2 - \tau_2 u_2^2$. The non-separating degeneration corresponds to letting t become large while keeping the other independent moduli finite. We stress that the above expression for Y^{-1} in terms of t is exact.

The methods developed in [22] are tailored to obtaining the expansion of (1.3), *exactly to all orders in powers of* t while neglecting any contributions that vanish exponentially in the large t limit. To carry out the construction, the genus-two surface Σ is parametrized in terms of t as well. This parametrization may be approached from two opposite directions which are intimately connected and equivalent to one another. The first approach starts from the genus-two surface, degenerates the period matrix according to (3.1) for large but finite t, and infers the degenerations of other functions and forms on Σ, such as the canonical Kähler form, the string Green function, and the Arakelov Green function. The second approach constructs the genus-two surface Σ, near a non-separating degeneration node, in terms of a compact genus-one surface Σ_1. As was shown in [22], the link between these two approaches is a family of Morse functions $f(z)$ which may be constructed from either approach.

For our purpose, it will be convenient to construct Σ starting from a compact torus Σ_1. We shall denote by $g(z, y|\tau) = g(z - y|\tau)$ the genus-one scalar Green function on Σ_1 which, by translation invariance, depends only on the difference $z - y$, and obeys,

(3.2)
$$\tau_2 \partial_{\bar z} \partial_z g(z|\tau) = -\pi \tau_2 \delta^{(2)}(z) + \pi \qquad \int_{\Sigma_1} dz \wedge d\bar z \, g(z|\tau) = 0$$

Its explicit expression in terms of ϑ-functions is given by,

$$g(z|\tau) = -\ln\left|\frac{\vartheta_1(x-y|\tau)}{\eta(\tau)}\right|^2 + \frac{2\pi}{\tau_2}\,(\mathrm{Im}\,z)^2 \tag{3.3}$$

We add two punctures p_a, p_b on Σ_1 to produce a punctured genus-one surface $\Sigma_1 \setminus \{p_a, p_b\}$. On this surface we introduce the Morse-type function f defined by,

$$f(z|\tau) = g(z, p_b|\tau) - g(z, p_a|\tau) \tag{3.4}$$

We observe that f is well-defined, single-valued, and harmonic on $\Sigma_1 \setminus \{p_a, p_b\}$, and tends to $-\infty$ as $z \to p_a$ and to $+\infty$ as $z \to p_b$. We define level sets $\mathfrak{C}_a$ and $\mathfrak{C}_b$ by,

$$\begin{aligned} \mathfrak{C}_a &= \{z \in \Sigma_1 \text{ such that } f(z) = -2\pi t\} \\ \mathfrak{C}_b &= \{z \in \Sigma_1 \text{ such that } f(z) = +2\pi t\} \end{aligned} \tag{3.5}$$

For sufficiently large values of t, each level set is connected and has the topology of a circle. (As t is decreased, each level set ultimately becomes disconnected and splits into two circles.) We define the genus-one surface Σ_{ab} with a boundary $\partial\Sigma_{ab} = \mathfrak{C}_a \cup \mathfrak{C}_b$ by cutting out of Σ_1 the two discs with boundaries $\mathfrak{C}_a$ and $\mathfrak{C}_b$, or equivalently,

$$\Sigma_{ab} = \{z \in \Sigma_1 \text{ such that } -2\pi t \le f(z) \le +2\pi t\} \tag{3.6}$$

The genus-two surface Σ is obtained from Σ_{ab} by gluing together its boundary curves $\mathfrak{C}_a$ and $\mathfrak{C}_b$. The full moduli space of Σ requires identifying $\mathfrak{C}_a$ and $\mathfrak{C}_b$ after a twist by the angle $\mathrm{Re}\,(\sigma)$, where σ is the bottom diagonal entry of Ω in (3.1). However, the Laurent polynomial part in the expansion of (1.3) is independent of $\mathrm{Re}\,(\sigma)$, since any dependence on $\mathrm{Re}\,(\sigma)$ of a modular invariant function must be exponential in σ in view of the periodicity $\sigma \to \sigma + 1$. Thus, the twist by $\mathrm{Re}\,(\sigma)$ is immaterial for our purposes, and may be ignored.

To complete the construction of Σ, we specify a canonical homology basis $\mathfrak{A}_I, \mathfrak{B}_I$ for $H_1(\Sigma, \mathbb{Z})$, and its dual basis of holomorphic one-forms ω_I for $I = 1, 2$. The cycles $\mathfrak{A}_1, \mathfrak{B}_1$ are chosen to be a canonical homology basis for $H_1(\Sigma_1, \mathbb{Z})$, the cycle $\mathfrak{A}_2$ is homologous to $\mathfrak{C}_a \approx \mathfrak{C}_b$, and the cycle $\mathfrak{B}_2$ consists of a curve which lies in Σ_{ab} and which connects $\mathfrak{C}_a$ to $\mathfrak{C}_b$, as shown in Figure 3.

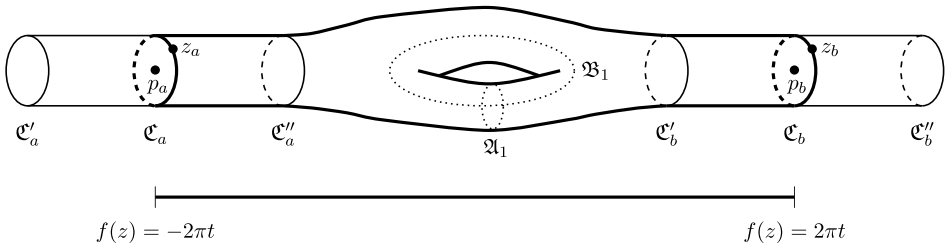


Figure 3: The funnel construction near the non-separating divisor of a genus-two surface Σ. The surface Σ_{ab} is obtained from the compact surface Σ_1 by removing the discs with boundaries $\mathfrak{C}_a$ and $\mathfrak{C}_b$ centered at the punctures p_a, p_b respectively. The surface Σ is obtained from Σ_{ab} by pairwise identifying the cycles $\mathfrak{C}_a \approx \mathfrak{C}_b$, (as well as identifying $\mathfrak{C}'_a \approx \mathfrak{C}'_b$ and $\mathfrak{C}''_a \approx \mathfrak{C}''_b$). A canonical homology basis for Σ is obtained by choosing the cycles $\mathfrak{A}_1, \mathfrak{B}_1$ of the surface Σ_1, along with a cycle $\mathfrak{A}_2$ homologous to the cycles $\mathfrak{C}_a, \mathfrak{C}'_a, \mathfrak{C}''_a, \mathfrak{C}_b, \mathfrak{C}'_b, \mathfrak{C}''_b$. The cycle $\mathfrak{B}_2$ is obtained by connecting z_a to z_b by a curve in Σ_{ab} and identifying the points $z_a \approx z_b$. The punctures p_a, p_b lie on Σ_1, but do not belong to either Σ_{ab} or Σ. The function $f(z)$ is constant on $\mathfrak{C}_a$ and $\mathfrak{C}_b$ and increases from $-2\pi t$ on $\mathfrak{C}_a$ to $2\pi t$ on $\mathfrak{C}_b$.

To specify the holomorphic one-forms on Σ, represented by Σ_{ab} with identified boundary components, we represent the torus by $\Sigma_1 = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and introduce a complex coordinate z subject to the identifications $z \approx z+1$ along $\mathfrak{A}_1$ and $z \approx z+\tau$ along $\mathfrak{B}_1$. The dual basis of holomorphic one-forms then consists of the normalized holomorphic one-form ω_1 on Σ_1 and the holomorphic one-form $\omega_2 = \omega_t + u_2\omega_1$ defined on Σ_{ab} by,

$$\omega_1 = dz \qquad\qquad \omega_t = \frac{i}{2\pi}\partial_z f(z)\, dz \ , \tag{3.7}$$

It follows from the above construction that ω_I is canonically normalized on the cycles $\mathfrak{A}_I$ as in (2.2), while on $\mathfrak{B}_I$ cycles we recover Ω of (3.1) with $v = p_b - p_a$ and t given by the construction above. Since we have not included the twist when identifying $\mathfrak{C}_a$ and $\mathfrak{C}_b$ we do not have access to the entry $\mathrm{Re}\,(\sigma)$ but, as argued earlier, we have no need for this variable here. The non-separating degeneration corresponds to $t \to \infty$ keeping τ and v fixed.

3.2. Degeneration of the Green functions

The key to the striking results for the non-separating degeneration obtained in [22] is the use not of the naive modulus σ_2 but rather instead of the special

parameter t which is invariant under the Jacobi group, $SL(2,\mathbb{Z}) \ltimes (\mathbb{Z}^2 \ltimes \mathbb{Z})$. It is in terms of t that the power series expansion terminates and becomes a Laurent polynomial of finite degree.

We begin by recalling the degeneration of the genus-two canonical Kähler form κ in terms of the normalized genus-one Kähler form κ_1,

$$\kappa = \frac{1}{2}\kappa_1 + \frac{i}{4t}\omega_t \wedge \bar{\omega}_t + \mathcal{O}(e^{-2\pi t}) \qquad \kappa_1 = \frac{i}{2\tau_2}\omega_1 \wedge \overline{\omega}_1 \tag{3.8}$$

The degeneration of the string Green function G is given by,[4]

$$G(x,y) = g(x,y) + \frac{1}{8\pi t}\big(f(x) - f(y)\big)^2 + \mathcal{O}(e^{-2\pi t}) \tag{3.9}$$

while the degeneration of the Arakelov Green function $\mathcal{G}(x,y)$ is given by,

$$\begin{aligned}\mathcal{G}(x,y) = \frac{\pi t}{12} &+ g(x,y) \\ &- \frac{1}{4}\Big(g(x,p_a) + g(x,p_b) + g(y,p_a) + g(y,p_b) - g(p_a,p_b)\Big) \\ &+ \frac{1}{16\pi t}\Big(f(x)^2 + f(y)^2 - 4f(x)f(y) - 2F_2(v)\Big) + \mathcal{O}(e^{-2\pi t})\end{aligned} \tag{3.10}$$

Here and below, we find it useful to introduce the following notation,

$$F_k(v) = \frac{1}{k!}\int_{\Sigma_1} \kappa_1(z) f(z)^k \tag{3.11}$$

Clearly, F_k vanishes when k is odd in view of translation and reflection symmetry of the genus-one Green function $g(z)$. For even k, we shall evaluate F_k in (3.31) and (3.32). The combination F_2 may be evaluated explicitly, using its definition, and we find,

$$F_2(v) = E_2 - g_2(v) \tag{3.12}$$

where E_2 is the genus-one non-holomorphic Eisenstein series, and g_k is defined to be the genus-one Green function for $k = 1$ and for higher values is defined recursively as follows,

$$g_{k+1}(z) = \int_{\Sigma_1} \kappa_1(x)\, g(z,x)\, g_k(x) \tag{3.13}$$

[4]Henceforth, when no confusion is expected to arise, we shall often suppress the dependence on the periods τ and Ω to simplify and shorten the notations.

The non-holomorphic Eisenstein series E_k is simply related to g_k by $E_k = g_k(0)$.

For the degeneration of both the Green functions G and $\mathcal{G}$, the asymptotic Laurent polynomial is at most of degree $(1,1)$. The functions $\gamma(x)$ and γ_1, which account for the difference between $G(x,y)$ and $\mathcal{G}(x,y)$ through (2.17), are given as follows,

$$\begin{aligned} \gamma(x) &= \frac{\pi t}{12} + \frac{1}{4}g(x,p_a) + \frac{1}{4}g(x,p_b) + \frac{f(x)^2}{16\pi t} + \frac{F_2(v)}{4\pi t} + \mathcal{O}(e^{-2\pi t}) \\ \gamma_1 &= \frac{\pi t}{4} + \frac{1}{4}g(v) + \frac{3F_2(v)}{8\pi t} + \mathcal{O}(e^{-2\pi t}) \end{aligned} \tag{3.14}$$

The invariants $\mathcal{B}_{(p,q)}$ that we wish to calculate involve integrals of powers of the Green function over the genus-two surface. In view of the number of terms in (3.10) compared to (3.9) it will prove convenient to first study the asymptotics of the integrals defined using the Green function $G(x,y)$ rather than the Arakelov Green function. We may then obtain the asymptotics of the individual modular graph invariants associated with the individual graphs, such as $\mathcal{Z}_i$ (in Figure 2) by a simple conversion formula involving $\gamma(x)$ and γ_1.

3.3. Degeneration of the Kawazumi–Zhang invariant

In terms of the Arakelov Green function $\mathcal{G}$, the Kawazumi-Zhang invariant is given by,

$$\varphi(\Omega) = -\frac{1}{4}\int_{\Sigma^2} \omega_I(z_1)\overline{\omega}^J(z_1)\omega_J(z_2)\overline{\omega}^I(z_2)\mathcal{G}(z_1,z_2) \tag{3.15}$$

Its degeneration limits were discussed in [32]. In the approach to the non-separating limit, $t \to \infty$, it has an expansion,

$$\varphi(\Omega) = \frac{1}{6}\pi t + \frac{1}{2}g(v) + \frac{5F_2(v)}{4\pi t} + \mathcal{O}(e^{-2\pi t}) \tag{3.16}$$

In accord with the general Theorem on the non-separating degeneration (1.3), the Kawazumi-Zhang invariant indeed produces a Laurent polynomial in t of degree $(1,1)$ with coefficients which are modular functions and their generalizations to include the dependence on v. The precise nature of the generalization this entails will be spelled out in subsection 3.6.

3.4. Degeneration of the string invariant $\mathcal{B}_{(2,0)}$

The modular graph function $\mathcal{B}_{(2,0)}(\Omega)$ may be expressed, via the Arakelov Green function and (2.11), as a sum of three modular graph functions $\mathcal{Z}_i(\Omega)$, which were defined in (2.12) with simplified expressions given in (2.13). To evaluate their non-separating degeneration, we observe in (3.9) and (3.10) that the degeneration formula for the string Green function G is simpler than the one for the Arakelov Green function $\mathcal{G}$. Therefore, we shall first calculate the degeneration of the analogues Z_i of $\mathcal{Z}_i$ in which $G(x,y)$ is replaced by $\mathcal{G}(x,y)$ in (2.12) and (2.13), which leads to the following definitions,

$$\begin{aligned} Z_1(\Omega) &= 8\int_{\Sigma^2}\kappa(z_1)\kappa(z_2)G(z_1,z_2)^2 \\ Z_2(\Omega) &= \int_{\Sigma^3}\kappa(z_1)\frac{\Delta(z_2,z_4)|^2}{\det Y}\,G(z_1,z_2)\,G(z_1,z_4) \\ Z_3(\Omega) &= \int_{\Sigma^4}\frac{|\Delta(z_1,z_3)\Delta(z_2,z_4)|^2}{8\,(\det Y)^2}\,G(z_1,z_2)\,G(z_3,z_4) \end{aligned} \tag{3.17}$$

Unlike the functions $\mathcal{Z}_i(\Omega)$, each individual function $Z_i(\Omega)$ fails to be conformally invariant, but the expression for $\mathcal{B}_{(2,0)}(\Omega)$, which is given by,

$$\mathcal{B}_{(2,0)} = \frac{1}{2}\mathcal{Z}_1 - \mathcal{Z}_2 + \frac{1}{2}\mathcal{Z}_3 = \frac{1}{2}Z_1 - Z_2 + \frac{1}{2}Z_3 \tag{3.18}$$

clearly remains conformal invariant. The relations between the individual functions $\mathcal{Z}_i(\Omega)$ and $Z_i(\Omega)$ may be recovered in terms of three functions $\gamma_i(\Omega)$,

$$\begin{aligned} \mathcal{Z}_1 &= Z_1 + 8\gamma_1^2 - 16\gamma_2 & \gamma_1 &= \int_\Sigma \kappa(x)\gamma(x) \\ \mathcal{Z}_2 &= Z_2 + 8\gamma_1^2 - 8\gamma_2 - \gamma_3 & \gamma_2 &= \int_\Sigma \kappa(x)\gamma(x)^2 \\ \mathcal{Z}_3 &= Z_3 + 8\gamma_1^2 - 2\gamma_3 & \gamma_3 &= \int_{\Sigma^2}\frac{|\Delta(x,y)|^2}{\det Y}\gamma(x)\gamma(y) \end{aligned} \tag{3.19}$$

The calculations required to extract the t-dependence of these integrals are complicated and have been relegated to Appendix B. The key steps involved are as follows.

1. The integrals of (3.17) and (3.19) over the compact genus-two Riemann surface Σ are expressed in terms of integrals over the genus-one surface

Σ_{ab} with boundary. The Green functions on Σ are expressed in terms of g and f using (3.9), while the integration measure is expressed in terms of the genus-one differentials ω_1 and ω_t using (3.8) and,

$$\Delta(z_i, z_j) = \omega_1(z_i) \wedge \omega_t(z_j) - \omega_t(z_i) \wedge \omega_1(z_j) \tag{3.20}$$

2. The remaining determinant factor is given by $\det Y = t\tau_2$.
3. The integrals over Σ_{ab} obtained in this manner are then analyzed and recast in the form of a Laurent polynomial in t with coefficients which can be expressed as convergent integrals over the compact genus-one Riemann surface Σ_1. The difficulty involved in this last step is strongly correlated with the structure of the associated Feynman graph and its renormalization properties, and will be given systematically in Appendix B.

The resulting expressions involve various modular functions and their generalizations which will be defined and discussed in subsection 3.6 and in Appendix B. The functions $Z_i(\Omega)$ and $\gamma_i(\Omega)$ will be computed in Appendix B, and give the following result for $\mathcal{Z}_i$,

(3.21)

$$\begin{aligned}
\mathcal{Z}_1(\Omega) = {} & \frac{13\pi^2 t^2}{90} + \frac{\pi t}{3} g(v) + 4E_2 + \frac{1}{2} g(v)^2 - \frac{1}{2} F_2(v) \\
& + \frac{1}{\pi t}\left(- D_3 - D_3^{(1)}(v) - \frac{1}{2} g F_2(v) + 2g_3(v) + 4\zeta(3) + \frac{1}{4\pi}\Delta_v F_4(v)\right) \\
& + \frac{1}{8\pi^2 t^2}\left(3F_2(v)^2 + 12F_4(v) + \mathcal{K}^c(v)\right) + \mathcal{O}(e^{-2\pi t}) \\
\mathcal{Z}_2(\Omega) = {} & - \frac{7\pi^2 t^2}{90} - \frac{\pi t}{3} g(v) - E_2 - \frac{1}{2} g(v)^2 + \frac{1}{2} F_2(v) \\
& + \frac{1}{\pi t}\Bigg(- 2D_3 + \frac{1}{2} g(v) F_2(v) + 2g_3(v) + 2\zeta(3) \\
& - \frac{1}{16\pi}\Delta_v \left(F_2(v)^2 + 2F_4(v)\right)\Bigg) - \frac{(\Delta_\tau + 5)F_4(v)}{4\pi^2 t^2} + \mathcal{O}(e^{-2\pi t}) \\
\mathcal{Z}_3(\Omega) = {} & \frac{(\pi t)^2}{18} + \frac{\pi t}{3} g(v) + \frac{1}{6} F_2(v) + \frac{1}{2} g(v)^2 \\
& + \frac{1}{\pi t}\left(-\frac{1}{2} g(v) F_2(v) + \frac{1}{8\pi}\Delta_v F_2(v)^2\right) \\
& + \frac{1}{8\pi^2 t^2}(\Delta_\tau + 5) F_2(v)^2 + \mathcal{O}(e^{-2\pi t})
\end{aligned}$$

The total string invariant $\mathcal{B}_{(2,0)}$ is then obtained from (2.11) and is given by,

(3.22)
$$\begin{aligned}\mathcal{B}_{(2,0)}(\Omega) = {} & \frac{8\pi^2 t^2}{45} + \frac{2\pi t}{3} g(v) + 3E_2 + g(v)^2 - \frac{2}{3} F_2(v) \\ & + \frac{1}{\pi t}\Bigg(\frac{3}{2} D_3 - \frac{1}{2} D_3^{(1)}(v) - g_3(v) - g(v) F_2(v) \\ & + \frac{1}{8\pi} \Delta_v \left(F_2(v)^2 + 2F_4(v)\right)\Bigg) \\ & + \frac{1}{16\pi^2 t^2} \left(\Delta_\tau + 8\right) \Big(F_2(v)^2 + 4F_4(v)\Big) + \frac{\mathcal{K}^c(v)}{16\pi^2 t^2} + \mathcal{O}(e^{-2\pi t})\end{aligned}$$

The definition of the various modular graph functions involved in these results will be given in the next subsections, while the corresponding derivations are relegated to Appendix B.

3.5. Modular graph functions occurring in $\mathcal{Z}_i$

All the genus-one modular graph functions and their generalizations occurring in the non-separating degeneration of the genus-two modular graph functions $\mathcal{Z}_i$ are built from the canonical volume form $\kappa_1(x)$ and the scalar Green function $g(x,y) = g(x-y)$ on the compact genus-one surface Σ_1. The most familiar such function is the non-holomorphic Eisenstein series E_k which may be defined by,[5]

$$E_k(\tau) = \prod_{i=1}^{k} \int_{\Sigma_1} \kappa_1(z_i)\, g(z_i - z_{i+1}|\tau) = {\sum_{m,n\in\mathbb{Z}}}' \frac{\tau_2^k}{\pi^k |m+\tau n|^{2k}} \tag{3.23}$$

where $z_{k+1} = z_1$, and the prime on the sum indicates that the term $m = n = 0$ is omitted. Another familiar family of modular graph functions is defined by,

$$D_k(\tau) = \int_{\Sigma_1} \kappa_1(z)\, g(z|\tau)^k = {\sum_{m_r,n_r\in\mathbb{Z}}}' \delta_{m,0}\delta_{n,0} \prod_{r=1}^{k} \frac{\tau_2}{\pi |m_r + \tau n_r|^2} \tag{3.24}$$

where $m = m_1 + \cdots + m_k$ and $n = n_1 + \cdots + n_k$. Both $E_k(\tau)$ and $D_k(\tau)$ are given by convergent integrals and sums for $k \geq 2$, are invariant under

[5]In this subsection and the next, we exhibit the dependence on moduli for added clarity.

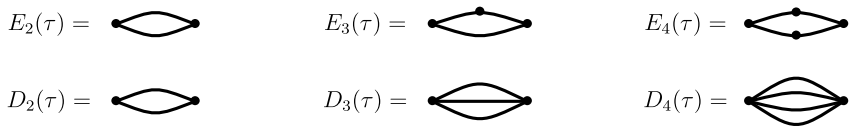


Figure 4: Modular graph functions $E_k(\tau)$ and $D_k(\tau)$.

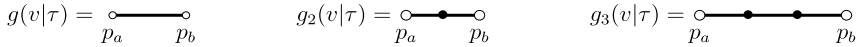


Figure 5: Modular graph functions $g_k(v|\tau)$.

$SL(2,\mathbb{Z})$ modular transformations of τ, and obey the following coincidence relations $D_2 = E_2$ and $D_3 = E_3 + \zeta(3)$. Their graphical representation is well known, and illustrated in Figure 4.

3.6. Generalized modular graph functions occurring in $\mathcal{Z}_i$

The remaining coefficients with non-trivial τ-dependence in $\mathcal{Z}_i$ also depend on the punctures through the combination $v = p_b - p_a$. The simplest of these is given by the Green function $g(v) = g(v|\tau)$ itself. Closely related are the iterated Green functions, $g_k(v) = g_k(v|\tau)$, which may defined recursively by (3.13), or in terms of a Kronecker-Eisenstein sum by,

$$g_k(v|\tau) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\tau_2^k\, e^{2\pi i(mu_2 - nu_1)}}{\pi^k |m+\tau n|^{2k}} \tag{3.25}$$

where u_1, u_2 are real and defined by $v = u_1 + \tau u_2$. For $k = 1$, we recover the Green function, $g_1(v|\tau) = g(v|\tau)$, while we also have $g_k(0|\tau) = E_k(\tau)$. The functions $g_k(v|\tau)$ are invariant under $SL(2,\mathbb{Z})$ transformations,

$$g_k(v'|\tau') = g_k(v|\tau) \qquad v' = \frac{v}{c\tau + d} \qquad \tau' = \frac{a\tau + b}{c\tau + d} \tag{3.26}$$

for $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The transformation induced on the real variables u_1, u_2 is linear and given by $u_1' = au_1 - bu_2$ and $u_2' = -cu_1 + du_2$. Thus, (u_1, u_2) provides a $(\mathbb{R}/\mathbb{Z})^2$-valued character and the generalization of modular graph functions provided by the functions $g_k(v|\tau)$ may be viewed as the result of introducing $\mathbb{R}/\mathbb{Z}$-*valued characters* in the Kronecker-Eisenstein sums. Their graphical representation is given in Figure 5.

Another interpretation of the generalization is to note the relation between $g_k(v|\tau)$ and the *single-valued elliptic polylogarithms* $D_{k,\ell}(v|\tau)$ introduced by Zagier [51],

$$D_{k,\ell}(v|\tau) = \frac{(2i\tau_2)^{k+\ell-1}}{2\pi i} \sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(mu_2-nu_1)}}{(m+n\tau)^k(m+n\bar{\tau})^\ell} \tag{3.27}$$

While $g_k(v|\tau)$ are modular functions satisfying (3.26), $D_{k,\ell}$ transforms as a modular form of weight $(1-\ell, 1-k)$. The functions $g_k(v|\tau)$ are special cases of Zagier's $D_{k,\ell}$-forms when $\ell = k$,

$$D_{k,k}(v|\tau) \quad = \quad (-4\pi\tau_2)^{k-1} g_k(v|\tau) \tag{3.28}$$

Further properties and interrelations satisfied by the forms $D_{k,\ell}$ and the functions g_k are provided in Appendix A.

The remaining coefficient functions are all generalized modular graph functions in the sense defined above, either as modular graph functions with character, or as single-valued elliptic polylogarithms. We give below the definitions of these functions, along with their graphical representations. We have the following infinite families,

$$D_\ell^{(k)}(v|\tau) = \int_{\Sigma_1} \kappa_1(z) g(v+z|\tau)^k g(z|\tau)^{\ell-k} \tag{3.29}$$

for $k, \ell \geq 0$ integers. They obey the symmetry relation $D_\ell^{(k)}(v|\tau) = D_\ell^{(\ell-k)}(v|\tau)$, and restrict to modular graph functions by the relation $D_\ell^{(k)}(0|\tau) = D_\ell(\tau)$, while for $k = 1$, they satisfy a simple differential relation,

$$\Delta_v D_\ell^{(1)}(v|\tau) = 4\pi D_{\ell-1}(\tau) - 4\pi g(v|\tau)^{\ell-1} \tag{3.30}$$

The graphical representation of these functions is illustrated in Figure 6.

The modular graph function $F_\ell(v|\tau)$, defined in (3.11), may be expressed as a linear combination of these functions,

$$F_\ell(v|\tau) = \sum_{k=0}^{\ell} \frac{(-)^{\ell-k}}{k!\,(\ell-k)!} D_\ell^{(k)}(v|\tau) \tag{3.31}$$

Figure 6: Modular graph functions $D_\ell^{(k)}(v|\tau)$.

For odd values of ℓ the sum vanishes in view of the symmetry relation of $D_\ell^{(k)}$, for $\ell = 2$ we have (3.12), while for F_2 and F_4, the formula reduces to,

$$\begin{aligned} F_2(v|\tau) &= E_2(\tau) - g_2(v|\tau) \\ F_4(v|\tau) &= \frac{1}{12} D_4(\tau) - \frac{1}{3} D_4^{(1)}(v|\tau) + \frac{1}{4} D_4^{(2)}(v|\tau) \end{aligned} \tag{3.32}$$

where we have made use of $D_2(\tau) = E_2(\tau)$ and $D_2^{(1)}(v|\tau) = g_2(v|\tau)$ on the first line.

3.7. Higher generalized modular graph functions

The degeneration of the modular graph function $\mathcal{Z}_1$ involves substantially more complicated genus-one modular graph functions than its Kawazumi-Zhang or $\mathcal{Z}_2$ and $\mathcal{Z}_3$ counterparts. The complication arises from their higher loop order, including three and four loops, and the need for subtractions in some of the graphs, as will be explained below. The main source of the complication is the integral (B.48) appearing in the degeneration of $Z_1^{(a)}$ defined in the first line of (B.39),

$$\mathcal{K} = \frac{\tau_2^2}{\pi^2} \int_{\Sigma_{ab}} \kappa_1(z) \int_{\Sigma_{ab}} \kappa_1(w) \, |\partial_z f(z) \partial_w f(w)|^2 \, g(z,w)^2 \tag{3.33}$$

As is shown in Appendices B.4 and B.5, the non-separating degeneration of $\mathcal{K}$ consists of a polynomial of degree four in t, plus terms which are exponentially suppressed in t. To extract the polynomial in t, we express f in terms of the genus-one Green function g, and expand the integrand into 16 terms, which may be regrouped in terms of 5 distinct building blocks,

$$\mathcal{K} = 2\mathcal{K}_{abab} + 2\mathcal{K}_{abba} + 2\mathcal{K}_{aabb} - 8\,\mathrm{Re}\,(\mathcal{K}_{aaab}) + 2\mathcal{K}_{aaaa} \tag{3.34}$$

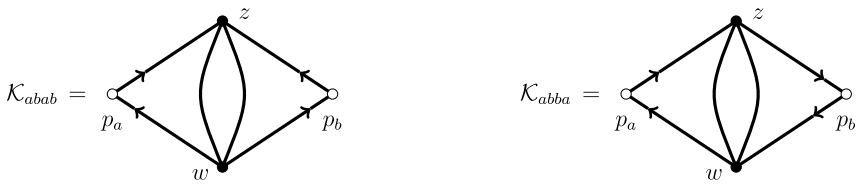


Figure 7: Modular graph functions $\mathcal{K}_{abab}$ and $\mathcal{K}_{abba}$. An arrow flowing into a vertex indicates a ∂-derivative with respect to the coordinate of the vertex, while an arrow flowing out of a vertex indicates a $\bar{\partial}$-derivative with respect to the coordinate of the vertex.

The graphical representation of the functions $\mathcal{K}_{abab}$ and $\mathcal{K}_{abba}$ is given in Figure 7. These functions are given by the following convergent integrals,

(3.35)
$$\begin{aligned}\mathcal{K}_{abab} &= \frac{\tau_2^2}{\pi^2}\int_{\Sigma_1}\kappa_1(z)\int_{\Sigma_1}\kappa_1(w)\,\partial_z g(z,p_a)\partial_{\bar{z}} g(z,p_b)\,g(z,w)^2\,\partial_w g(w,p_a)\\ &\quad\times\partial_{\bar{w}} g(w,p_b)\\ \mathcal{K}_{abba} &= \frac{\tau_2^2}{\pi^2}\int_{\Sigma_1}\kappa_1(z)\int_{\Sigma_1}\kappa_1(w)\,\partial_z g(z,p_a)\partial_{\bar{z}} g(z,p_b)\,g(z,w)^2\,\partial_w g(w,p_b)\\ &\quad\times\partial_{\bar{w}} g(w,p_a)\end{aligned}$$

and contribute a polynomial of degree zero in t, up to exponential corrections.

The remaining three functions $\mathcal{K}_{aabb}$, $\mathcal{K}_{aaab}$ and $\mathcal{K}_{aaaa}$ do have non-trivial polynomial t-dependence, and are represented schematically in Figures 8 and 9. We isolate this dependence by splitting the integrals as follows,

$$\begin{aligned}\mathcal{K}_{aabb} &= \mathcal{K}^0_{aabb} + \mathcal{K}^1_{aabb}\\ \mathcal{K}_{aaab} &= \mathcal{K}^0_{aaab} + \mathcal{K}^1_{aaab}\\ \mathcal{K}_{aaaa} &= \mathcal{K}^0_{aaaa} + \mathcal{K}^1_{aaaa}\end{aligned} \tag{3.36}$$

where the contributions $\mathcal{K}^0$ are constant in t, while the contributions $\mathcal{K}^1$ are polynomials in t with vanishing constant part, up to exponentially suppressed contributions. The contributions $\mathcal{K}^0_{aabb}$ and $\mathcal{K}^0_{aaab}$ are given by the following convergent integrals, while the t-dependent parts $\mathcal{K}^1_{aabb}, \mathcal{K}^1_{aaab}$ and

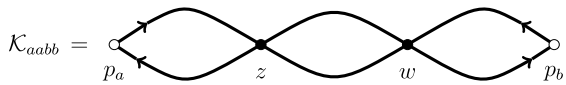


Figure 8: Schematic representation of the modular graph function $\mathcal{K}^0_{aabb}$.

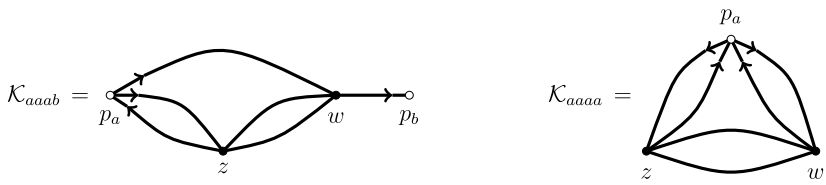


Figure 9: Schematic representation of $\mathcal{K}^0_{aaab}$ and $\mathcal{K}^0_{aaaa}$.

$\mathcal{K}^1_{aaaa}$ will be evaluated below.

(3.37)
$$\begin{aligned}\mathcal{K}^0_{aabb} &= \frac{\tau_2^2}{\pi^2}\int_{\Sigma_1}\kappa_1(z)\int_{\Sigma_1}\kappa_1(w)\,|\partial_z g(z,p_a)|^2|\partial_w g(w,p_b)|^2\\ &\qquad\times\Big(g(z,w)^2-g(p_a,w)^2-g(z,p_b)^2+g(p_a,p_b)^2\Big)\\ \mathcal{K}^0_{aaab} &= \frac{\tau_2^2}{\pi^2}\int_{\Sigma_1}\kappa_1(z)\int_{\Sigma_1}\kappa_1(w)|\partial_z g(z,p_a)|^2\partial_w g(w,p_a)\partial_{\bar w} g(w,p_b)\\ &\qquad\times\Big(g(z,w)^2-g(p_a,w)^2\Big)\end{aligned}$$

Finally, the modular graph function $\mathcal{K}^0_{aaaa}$ is given by the following convergent integral,

$$\begin{aligned}\mathcal{K}^0_{aaaa} &= \frac{\tau_2}{\pi}\int_{\Sigma_1}\kappa_1(z)|\partial_z g(z)|^2\Big(W(z)-4\zeta(3)\Big)\\ W(z) &= \frac{\tau_2}{\pi}\int_{\Sigma_1}\kappa_1(w)|\partial_w g(w)|^2\Big(g(z,w)-g(z)\Big)\Big(g(z,w)-g(w)\Big)\end{aligned}\tag{3.38}$$

The integral over z on the second line is convergent, but that the integral of each term in the parentheses separately is divergent due to the double pole of $|\partial_z g(z)|^2$ at $z=0$.

The functions $\mathcal{K}^1_{aabb}$, $\mathcal{K}^1_{aaab}$ and $\mathcal{K}^1_{aaaa}$ are polynomials in t whose coefficients are genus-one modular graph functions of the customary type. Their

contribution to $\mathcal{K}$ is given by,

$$\mathcal{K}^t = 2\mathcal{K}^1_{abab} + 2\mathcal{K}^1_{abba} + 2\mathcal{K}^1_{aabb} - 8\,\mathrm{Re}\,\left(\mathcal{K}^1_{aaab}\right) \tag{3.39}$$

and computed in Appendix B.4, using the variational method introduced in [22, §3.6]. The functions $\mathcal{K}^0_{aabb}$, $\mathcal{K}^0_{aaab}$ and $\mathcal{K}^0_{aaaa}$ are more exotic genus-one modular graph functions, which are schematically represented in Figures 8 and 9 (these figures, however, do not indicate the subtractions in the integrand). The remainder $\mathcal{K}^c = \lim_{t\to\infty}(\mathcal{K} - \mathcal{K}^t)$ is given by,

(3.40)

$$\begin{aligned}\mathcal{K}^c = {} & 2\mathcal{K}_{abab} + 2\mathcal{K}_{abba} + 2\mathcal{K}^0_{aabb} - 4\mathcal{K}^0_{aaab} - 4(\mathcal{K}^0_{aaab})^* + 2\mathcal{K}^0_{aaaa} \\ & + 4g(v)\left(D_3 + 2\zeta(3) - D_3^{(1)}(v) + \frac{\Delta_v F_4(v)}{2\pi}\right) - 3g(v)^4 - \frac{7}{4}E_2^2 + \frac{5}{4}D_4 \\ & + \frac{3}{2}E_4\end{aligned}$$

This concludes the explanation of the modular graph functions appearing in the minimal degenerations (3.21) of the string integrands $\mathcal{Z}_i$ and $\mathcal{B}_{(2,0)}$.

4. The separating degeneration

In the separating degeneration, a genus-two Riemann surface Σ tends to the union of two genus-one surfaces which intersect at a common puncture. Denoting the corresponding compact genus-one surfaces by Σ_1 and Σ_1' with respective moduli τ and σ, the punctured surfaces are $\Sigma_1 \setminus \{p\}$ and $\Sigma_1' \setminus \{p'\}$ respectively, where the punctures p and p' are identified with one another. We shall examine the degeneration of string invariants in a neighborhood of the separating degeneration, which we parametrize by the off-diagonal element v of the period matrix. We begin by presenting a review of the degeneration of the Abelian differentials, the canonical Kähler form, and the Arakelov Green function, and derive the degenerations of the Kawazumi-Zhang and $\mathcal{B}_{(2,0)}$ example of higher invariants at genus two.

4.1. Funnel construction of the separating degeneration

A convenient parametrization of the neighborhood of the separating divisor is provided by the funnel construction given in [42]. We shall carry out this construction here in the simplest case of a genus-two surface Σ because this

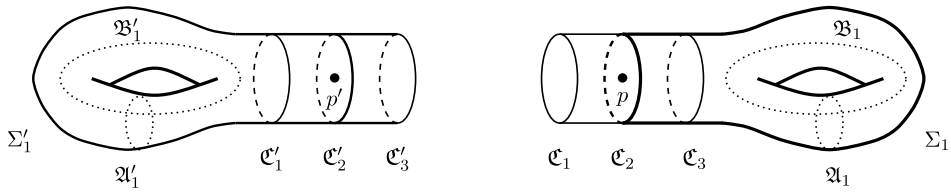


Figure 10: Funnel construction of a family of genus-two Riemann surfaces Σ near the separating divisor in terms of two compact genus-one surfaces Σ_1 and Σ'_1. The circles $\mathfrak{C}_i$ and $\mathfrak{C}'_i$ for $i = 1, 2, 3$ are centered respectively at the punctures p and p' and respectively bound the discs $\mathfrak{D}_i$ and $\mathfrak{D}'_i$. The surface Σ is constructed from the surfaces $\Sigma_1 \setminus \mathfrak{D}_1$ and $\Sigma'_1 \setminus \mathfrak{D}'_3$ by identifying the annuli $[\mathfrak{C}_1, \mathfrak{C}_3]$ and $[\mathfrak{C}'_1, \mathfrak{C}'_3]$.

is the focus of the present paper, but the construction is easily generalized to arbitrary genus.

For genus two, the starting point of the construction of Σ in [42] is provided by the compact genus-one surfaces Σ_1 and Σ'_1, to which we add punctures, respectively p and p'. Next, we introduce a system of local complex coordinates $(x, \bar{x})$ and $(x', \bar{x}')$ on each surface, and denote the coordinates of the punctures simply by p and p'. We specify (simply connected) discs $\mathfrak{D}_i$ centered at p with boundaries $\mathfrak{C}_i$ on Σ_1, and (simply connected) discs $\mathfrak{D}'_i$ centered at p' with boundaries $\mathfrak{C}'_i$ on Σ'_1 for $i = 1, 2, 3$, as shown in Figure 10.

The genus-two surface Σ is obtained by identifying the annulus $[\mathfrak{C}_1, \mathfrak{C}_3]$ with the annulus $[\mathfrak{C}'_1, \mathfrak{C}'_3]$ with respective local complex coordinates x and x' via the relation,

$$(x - p)(x' - p') = v_s \tag{4.1}$$

Here v_s is a complex parameter governing the separating degeneration (which is referred to as t in [42]) and is such that the separating degeneration corresponds to the limit $v_s \to 0$. Customarily, the curves $\mathfrak{C}_i$ and $\mathfrak{C}'_i$ are defined to be circles in the local complex coordinates on the surfaces but here instead we shall use a more intrinsic definition, which will be given below. Next, we shall construct the Abelian differentials and Green function on Σ in terms of its genus-one data along with v_s.

4.2. Global funnel construction

In analogy with the construction of the neighborhood of the non-separating node, we may also here provide a convenient intrinsic characterization of $\mathfrak{C}_i$

and $\mathfrak{C}'_i$ as level-curves of the scalar Green functions g and g' on the genus-one surfaces Σ_1 and Σ'_1,

$$\begin{aligned} \mathfrak{C}_i &= \{x \in \Sigma_1 \text{ such that } g(x-p|\tau) = t_i\} \\ \mathfrak{C}'_i &= \{x' \in \Sigma'_1 \text{ such that } g(x'-p'|\sigma) = t'_i\} \end{aligned} \tag{4.2}$$

for sufficiently large values of t_i, t'_i so that each level-set $\mathfrak{C}_i$ and $\mathfrak{C}'_i$ is connected. The curves are related by the following relation between their t_i-values, valid for $i = 1, 2, 3$,

$$t_i + t'_i = -\ln|2\pi v_s\, \eta(\tau)^2 \eta(\sigma)^2|^2 + \mathcal{O}(v_s^2) \tag{4.3}$$

Here, we have used the short-distance expansion of the scalar Green function on the torus, given by $g(z|\tau) = -\ln|2\pi z\, \eta(\tau)^2|^2 + \mathcal{O}(z^2)$ to convert (4.1) into the expression above.

When performing integrals over the genus-two surface Σ, it will be convenient to decompose the integral into a sum of the contribution from $\Sigma_1 \setminus \mathfrak{D}_2$ plus the contribution from $\Sigma'_1 \setminus \mathfrak{D}'_2$ where the curves $\mathfrak{C}_2$ and $\mathfrak{C}'_2$ are defined so that $t_2 = t'_2$. Under these conditions, the Abelian differentials ω_1 and ω_2 remain uniformly bounded throughout Σ by a constant of order $\mathcal{O}(v_s^0)$, with corrections which are suppressed by powers of v_s.

4.3. Degeneration of Abelian differentials

We choose canonical homology bases for the genus-one surfaces by $\mathfrak{A}_1, \mathfrak{B}_1 \subset \Sigma_1 \setminus \{p\}$ and $\mathfrak{A}'_1, \mathfrak{B}'_1 \subset \Sigma'_1 \setminus \{p'\}$, and extend those to a canonical homology basis $\mathfrak{A}_I, \mathfrak{B}_I$ for $I = 1, 2$ for Σ by setting $\mathfrak{A}_2 = \mathfrak{A}'_1, \mathfrak{B}_2 = \mathfrak{B}'_1$. The genus-one holomorphic Abelian differentials ω and ω' respectively on Σ_1 and Σ'_1 are normalized as follows,

$$\oint_{\mathfrak{A}_1} \omega = \oint_{\mathfrak{A}'_1} \omega' = 1 \qquad \oint_{\mathfrak{B}_1} \omega = \tau \qquad \oint_{\mathfrak{B}'_1} \omega' = \sigma \tag{4.4}$$

To construct holomorphic 1-forms on the genus-two surface with period matrix,

$$\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \tag{4.5}$$

we extend ω to a differential ω_1 on the genus-two surface Σ and ω' to a differential on the genus-two surface Σ by using the identification (4.1).

Choosing complex coordinates x, x' on Σ_1 and Σ_1' such that $\omega = dx$ and $\omega' = dx'$, we see that the differential dx extends to $-v_s/(x'-p')^2 dx'$ in Σ_1' while the differential dx' extends to $-v_s/(x-p)^2 dx$ in Σ_1. Thus, the extensions are governed by meromorphic 1-forms with a double pole. The meromorphic 1-forms $\varpi(x,y)$ and $\varpi'(x',y')$ respectively on Σ_1 and Σ_1' are normalized to have vanishing $\mathfrak{A}$-periods and a double pole of unit strength at $x = y$ and $x' = y'$. Their $\mathfrak{B}$-periods are given by the Riemann bilinear relations,

$$\oint_{\mathfrak{B}_1} \varpi(x,y) = 2\pi i \omega(y) \qquad\qquad \oint_{\mathfrak{B}_1'} \varpi(x',y') = 2\pi i \omega'(y') \tag{4.6}$$

The holomorphic 1-forms ω_1 and ω_2 on the genus-two surface Σ, canonically normalized on $\mathfrak{A}_1$ and $\mathfrak{A}_2$-cycles, are then given as follows,

$$\begin{aligned} \omega_1 &= \begin{cases} \omega(x) & x \in \Sigma_1 \setminus \mathfrak{D}_1 \\ v\,\varpi'(x',p')/(2\pi i \omega'(p')) & x' \in \Sigma_1' \setminus \mathfrak{D}_3' \end{cases} \\ \omega_2 &= \begin{cases} v\,\varpi(x,p)/(2\pi i \omega(p)) & x \in \Sigma_1 \setminus \mathfrak{D}_1 \\ \omega'(x') & x' \in \Sigma_1' \setminus \mathfrak{D}_3' \end{cases} \end{aligned} \tag{4.7}$$

The parameter v_s is related to the entry v of the genus-two period matrix by,

$$v = \oint_{\mathfrak{B}_1} \omega_2 = \oint_{\mathfrak{B}_2} \omega_1 = -2\pi i\, v_s\, \omega(p)\, \omega'(p') \tag{4.8}$$

The expressions in (4.7) are valid up to corrections of order $\mathcal{O}(v^2)$ which have been omitted.

4.4. Degeneration of the Green function

The degeneration of the string Green function G of (2.16) on the genus-two Riemann surface Σ was obtained in [24], and is given by,

$$G = \begin{cases} g(x-y|\tau) + 2\ln(2\pi|\eta(\tau)|^2) & x, y \in \Sigma_1 \setminus \mathfrak{D}_1 \\ g(x'-y'|\sigma) + 2\ln(2\pi|\eta(\sigma)|^2) & x', y' \in \Sigma_1' \setminus \mathfrak{D}_3' \\ g(x-p|\tau) + g(y'-p'|\sigma) + \ln\left((2\pi)^3 |v\eta(\tau)\eta(\sigma)|^4\right) & \\ \quad x \in \Sigma_1 \setminus \mathfrak{D}_1, y' \in \Sigma_1' \setminus \mathfrak{D}_3' & \end{cases} \tag{4.9}$$

up to terms of order $\mathcal{O}(v)$ which will be omitted in the sequel. The degeneration of the canonical Kähler form κ of the genus-two Riemann surface Σ is given as follows,

$$\kappa = \begin{cases} \frac{1}{2}\kappa_1(x) = \frac{i}{4\tau_2}\omega(x)\wedge\overline{\omega}(x) & x\in\Sigma_1\setminus\mathfrak{D}_1 \\ \frac{1}{2}\kappa_1'(x') = \frac{i}{4\sigma_2}\omega'(x')\wedge\overline{\omega}'(x') & x'\in\Sigma_1'\setminus\mathfrak{D}_3' \end{cases} \tag{4.10}$$

From these results we readily obtain the separating degeneration formulas for the Arakelov Green function $\mathcal{G}$ on the genus-two Riemann surface Σ, which are given as follows,

(4.11)

$$\mathcal{G} = \begin{cases} -\frac{1}{2}\ln|\hat{v}| + g(x-y|\tau) - \frac{1}{2}g(x-p|\tau) - \frac{1}{2}g(y-p|\tau) \quad x,y\in\Sigma_1\setminus\mathfrak{D}_1 \\ -\frac{1}{2}\ln|\hat{v}| + g(x'-y'|\sigma) - \frac{1}{2}g(x'-p'|\sigma) - \frac{1}{2}g(y'-p'|\sigma) \\ \quad x',y'\in\Sigma_1'\setminus\mathfrak{D}_3' \\ \frac{1}{2}\log|\hat{v}| + \frac{1}{2}g(x-p|\tau) + \frac{1}{2}g(y'-p'|\sigma)\ , \quad x\in\Sigma_1\setminus\mathfrak{D}_1, y'\in\Sigma_1'\setminus\mathfrak{D}_3' \end{cases}$$

where $\hat{v}$ is related to v by the Dedekind eta-function η,

$$\hat{v} = 2\pi\, v\, \eta(\tau)^2\eta(\sigma)^2 \tag{4.12}$$

In the vicinity of the separating degeneration the genus-two modular group $Sp(4,\mathbb{Z})$ restricts to its $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})'$ subgroup which acts by Möbius transformations on (τ,σ), and v by,

(4.13)

$$\tau\to\frac{a\tau+b}{c\tau+d} \qquad \sigma\to\frac{a'\sigma+b'}{c'\sigma+d'} \qquad v\to\frac{v}{(c\tau+d)(c'\sigma+d')}$$

with $a,b,c,d,a',b',c',d'\in\mathbb{Z}$ and $ad-bc = a'd'-b'c' = 1$. Since $\eta(\tau)^2$ transforms as a one-form under $SL(2,\mathbb{Z})$ acting on τ, the combination $\hat{v}$ is invariant under $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})'$, as well as under exchange of τ and σ.

4.5. Degeneration of the genus-two Kawazumi-Zhang invariant

As a warm-up, we consider the behavior of the genus-two KZ-invariant under separating degeneration. Instead of the expression (3.15) for the KZ-invariant, it will be more convenient to use the expression given in [24] and valid for any scalar Green function,

$$\varphi(\Omega) = -\frac{1}{8}\int_{\Sigma^2} P(x,y|\Omega)\,\mathcal{G}(x,y|\Omega) \tag{4.14}$$

The bi-form $P(x,y|\Omega)$ is symmetric in x,y and defined by,

$$P(x,y|\Omega) = \Big(2(Y^{-1})^{IL}(Y^{-1})^{JK} - (Y^{-1})^{IJ}(Y^{-1})^{KL}\Big) \times \omega_I(x)\,\overline{\omega}_J(x)\,\omega_K(y)\,\overline{\omega}_L(y) \tag{4.15}$$

Up to terms of order $\mathcal{O}(v)$, the form $P(x,y|\Omega)$ degenerates as follows,

$$P(x,y|\Omega) \to \begin{cases} -4\,\kappa_1(x)\kappa_1(y) & x,y \in \Sigma_1 \setminus \mathfrak{D}_1 \\ -4\,\kappa_1'(x')\kappa_1'(y') & x',y' \in \Sigma_1' \setminus \mathfrak{D}_3' \\ +4\,\kappa_1(x)\kappa_1'(y') & x \in \Sigma_1 \setminus \mathfrak{D}_1, y' \in \Sigma_1' \setminus \mathfrak{D}_3' \end{cases} \tag{4.16}$$

Combining the asymptotic behaviors to this order of the Arakelov Green function $\mathcal{G}$ in (4.11) and of the differential P in (4.7), we see that all the contributions of the genus-one Green functions in (4.11) integrate to zero against the degeneration of P, and only the contribution of the terms proportional to $\ln|\hat{v}|$ survive, giving,

$$\varphi = -\ln|\hat{v}| + \mathcal{O}(|\hat{v}|) \tag{4.17}$$

This result is consistent with part a) of the main Theorem in [30] for $(h_1,h_2) = (1,1)$, and is identical to the more precise asymptotics derived in [24].

4.6. Degeneration of the genus-two invariants $\mathcal{Z}_i$ and $\mathcal{B}_{(2,0)}$

Our starting point is the expression for the genus-two modular graph functions $\mathcal{Z}_i$ of (2.12), and its simplified form (2.13) after some of the trivial integrals over points on the surface have been performed, along with the relation (2.11). To evaluate the degenerations of these invariants, neglecting terms of order $\mathcal{O}(|\hat{v}|)$, we use the asymptotics of the Arakelov Green function in (4.11), of the canonical Kähler form κ in (4.10), and of the combination involving the bi-holomorphic form Δ,

$$\Delta = \begin{cases} 0 & x,y \in \Sigma_1 \setminus \mathfrak{D}_1 \text{ or } x',y' \in \Sigma_1' \setminus \mathfrak{D}_3' \\ +\omega(x) \wedge \omega'(y') & x \in \Sigma_1 \setminus \mathfrak{D}_1,\ y' \in \Sigma_1' \setminus \mathfrak{D}_3' \\ -\omega'(x') \wedge \omega(y) & x' \in \Sigma_1' \setminus \mathfrak{D}_3',\ y \in \Sigma_1 \setminus \mathfrak{D}_1 \end{cases} \tag{4.18}$$

The results are as follows,

$$\begin{aligned}\mathcal{Z}_1 &= 2\,\ln^2|\hat{v}| + 4E_2(\tau) + 4E_2(\sigma) + \mathcal{O}(|\hat{v}|)\\ \mathcal{Z}_2 &= -2\,\ln^2|\hat{v}| - E_2(\tau) - E_2(\sigma) + \mathcal{O}(|\hat{v}|)\\ \mathcal{Z}_3 &= 2\,\ln^2|\hat{v}| + \mathcal{O}(|\hat{v}|)\end{aligned} \tag{4.19}$$

Summing the contributions gives,

$$\mathcal{B}_{(2,0)} = 4\,\ln^2|\hat{v}| + 3E_2(\tau) + 3E_2(\sigma) + \mathcal{O}(|\hat{v}|) \tag{4.20}$$

Note in particular that $\mathcal{B}_{(2,0)} - 4\,\varphi^2$ is finite as $v \to 0$.

4.7. Degeneration of general genus-two modular graph functions

General classes of modular graph functions at genus two and beyond were constructed in subsection 2.8 of [22], and are given as follows,

$$\mathcal{C}[n_{ij}; c(\sigma)] = c^{I_1\cdots I_N}_{J_1\cdots J_N} \int_{\Sigma^N} \prod_{i=1}^{N} \omega_{I_i}(z_i)\,\overline{\omega}^{J_i}(z_i) \prod_{1\le i<j\le N} \mathcal{G}(z_i, z_j)^{n_{ij}} \tag{4.21}$$

Here, $n_{ij} \ge 0$ are integers, while $c^{I_1\cdots I_N}_{J_1\cdots J_N}$ is an invariant modular tensor built out of a linear combination of products of Kronecker δ-symbols. Given these properties, it may be expressed as follows,

$$c^{I_1\cdots I_N}_{J_1\cdots J_N} = \sum_{\sigma\in\mathfrak{S}_N} c(\sigma) \prod_{i=1}^{N} \delta_{J_{\sigma(i)}}{}^{I_i} \tag{4.22}$$

where $c(\sigma)$ are constants which depend on the permutation $\sigma \in \mathfrak{S}_N$. The weight w of the modular graph function $\mathcal{C}$ is given as follows,

$$w = \sum_{1\le i<j\le N} n_{ij} \tag{4.23}$$

We shall limit attention to the case of genus-two though the results extend to higher genus. The asymptotics of $\mathcal{C}$ under separating degeneration is given by the following theorem.

Theorem 1 *The behavior of the modular graph function $\mathcal{C}$ in a neighborhood of the separating degeneration node is given by a polynomial of degree*

w in $\ln|\hat{v}|$ plus terms that are suppressed by positive powers of $|\hat{v}|$, by the following expression,

$$\mathcal{C}[n_{ij}, c(\sigma)] = \sum_{k=0}^{w} \rho_k(\tau,\sigma)\big(-\ln|\hat{v}|\big)^k + \mathcal{O}(|\hat{v}|^{1-\varepsilon}) \tag{4.24}$$

for any $\varepsilon > 0$. The expansion parameter $\ln|\hat{v}|$ and coefficients $\rho_k(\tau,\sigma)$ are invariant under the residual group $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})'$ acting on τ, σ and v as in (4.13).

The proof of the theorem proceeds as follows. Up to corrections of order $\mathcal{O}(|\hat{v}|)$, the differential forms $\omega_{I_i}(z_i)\overline{\omega}^{J_i}(z_i)$ on the genus-two surface reduces to a linear combination of $\kappa_1(z_i)$ when $z_i \in \Sigma_1 \setminus \mathfrak{D}_1$ and $\kappa_1'(z_i)$ when $z_i \in \Sigma_1' \setminus \mathfrak{D}_3'$. Therefore, to order $\mathcal{O}(|\hat{v}|)$ the integral over Σ^N reduces to a sum of integrals over these genus-one components of products of powers of $\ln|\hat{v}|$, $g(z_i - p|\tau)$, $g'(z_i' - p'|\sigma)$, $g(z_i, z_j|\tau)$ and $g(z_i', z_j'|\sigma)$, all of which integrate to produce terms obeying the properties of the expansion announced in the Theorem. This part is straightforward.

The more delicate part of the proof consists in showing that the corrections of order $\mathcal{O}(|\hat{v}|)$ to the leading contributions in the separating degeneration of the holomorphic Abelian differentials is indeed suppressed as indicated by the theorem. This part is not straightforward in view of the fact that the coefficient of v is a meromorphic differential which has a double pole at the punctures, so that its contribution to various integrals near the punctures could overcome the $\mathcal{O}(|\hat{v}|)$ suppression factor.

To proceed, we shall represent the genus-two surface Σ by the union of the genus-one surfaces with boundary, given by $\Sigma_1 \setminus \mathfrak{D}_2$ and $\Sigma_1' \setminus \mathfrak{D}_2'$, where we choose the curves $\mathfrak{C}_2$ and $\mathfrak{C}_2'$ to be defined by,

$$t_2 = t_2' = -\ln|2\pi\, v\, \eta(\tau)^2\eta(\sigma)^2| + \mathcal{O}(|v|^2) \tag{4.25}$$

For sufficiently small v, the parameters t_2 and t_2' are large and the curves $\mathfrak{C}_2$ and $\mathfrak{C}_2'$ are approximately circles centered at the punctures with radii squared of order v,

$$\begin{aligned} \mathfrak{C}_2 &= \{z \in \Sigma_1 \text{ such that } |z-p| = |v|^{\frac{1}{2}} + \mathcal{O}(|v|^{\frac{3}{2}})\} \\ \mathfrak{C}_2' &= \{z' \in \Sigma_1' \text{ such that } |z'-p'| = |v|^{\frac{1}{2}} + \mathcal{O}(|v|^{\frac{3}{2}})\} \end{aligned} \tag{4.26}$$

The terms of fastest growth in the integrals required in Theorem 1 are as follows,

$$|v|^2 \int_{\Sigma_1 \setminus \mathfrak{D}_2} |\varpi(x,p)|^2 f(x) \tag{4.27}$$

The cases we need are when $f(x)$ is continuous throughout Σ_1 or behaves as a power of a logarithm near the puncture p. For f continuous near p, the integral is of the form,

$$|v|^2 \int_{|x-p|>|v|^{\frac{1}{2}}} d^2x \frac{f(x)}{|x-p|^4} = |v| \int_{|\tilde{x}|>1} d^2\tilde{x} \frac{f(p+|v|^{\frac{1}{2}}\tilde{x})}{|\tilde{x}|^4} \tag{4.28}$$

where the equality was obtained by changing variables locally by setting $x = p + |v|^{\frac{1}{2}}\tilde{x}$. Thus, the contribution to the integrals from the Abelian differential with double pole is suppressed by a power of $|v|$. The same scaling argument shows that upon multiplying f by a factor of $g(x-p|\tau)^n$, the suppression factor is instead $|v|(\ln|v|)^n$. An analogous argument goes through for multiple integrations, say over variables x, y, involving also powers of the Green function $g(x-y|\tau)^n$, as may be seen from the following double integral,

$$\begin{aligned} |v|^4 \int_{(\Sigma_1\setminus\mathfrak{D}_2)^2} & |\varpi(x,p)|^2|\varpi(y,p)|^2 g(x-y|\tau)^n \\ &\approx |v|^2 \int_{|\tilde{x}|,|\tilde{y}|>1} \frac{d^2\tilde{x}\, d^2\tilde{y}}{|\tilde{x}|^4|\tilde{y}|^4}\Big(-\ln|v||\tilde{x}-\tilde{y}|^2\Big)^n \end{aligned} \tag{4.29}$$

which is now suppressed by $|v|^2$ times powers of $\ln|v|$. For any n, the integrals are therefore bounded by $|v|^{1-\varepsilon}$ for any $\varepsilon > 0$, which concludes the proof of the Theorem. An alternative proof may be given using the variational method on $\ln|v|$, which is closer in spirit to the proof given for the non-separating degeneration in terms of t.

5. The tropical degeneration

The complete degeneration of a compact genus-two Riemann surface is obtained by letting the imaginary part of the period matrix Y scale to ∞ while keeping the ratios of its entries fixed. We shall refer to this limit as the *tropical degeneration* because maximal degenerations are generally described by tropical geometry [43]. The tropical degeneration provides a suitable framework for examining the relation between the integrands for amplitudes of superstring theory and those of the associated supergravity [44]. In this section, we shall review the geometry and symmetries of the tropical degeneration, and then obtain the corresponding asymptotic behavior of the string invariant $\mathcal{B}_{(2,0)}$. This study will prepare the ground for the comparison between string and supergravity amplitudes in Section 6.2.

5.1. Geometry and symmetry of the tropical degeneration

The geometry and symmetries of the tropical degeneration are most easily exposed by parametrizing the imaginary part of the period matrix Y, given in (1.2) and (3.1) in terms of a positive real variable V and a parameter $S = S_1 + iS_2$ in the Poincaré upper half-plane [45],

$$Y = \frac{1}{VS_2}\begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix} \tag{5.1}$$

The relation between the two systems of coordinates is given by,

$$V = \frac{1}{\sqrt{t\tau_2}} = (\det Y)^{-\frac{1}{2}} \qquad\qquad S = u_2 + i\sqrt{\frac{t}{\tau_2}} \tag{5.2}$$

where we recall that $v_2 = \tau_2 u_2$ and $t = \sigma_2 - \tau_2 u_2^2$. The tropical degeneration corresponds to letting $V \to 0$ while keeping S fixed. In terms of the original variables, it arises equivalently by taking $t\tau_2 \to \infty$ while keeping u_2 and t/τ_2 fixed and non-zero.

The subgroup of the genus-two modular group $Sp(4,\mathbb{Z})$ which leaves the tropical degeneration invariant acts on Ω by 2×2 matrices A, B with integer entries,

$$\Omega \to \Omega' = A(\Omega + B)A^t \tag{5.3}$$

where $A \in GL(2,\mathbb{Z})$ and B is symmetric. Parametrizing the matrix A by $a, b, c, d \in \mathbb{Z}$ as exhibited below, we find that V is invariant, while S transforms as follows,

$$A = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \qquad S \to \left.\frac{aS+b}{cS+d}\right|_{\det A=1} \qquad S \to \left.\frac{a\bar{S}+b}{c\bar{S}+d}\right|_{\det A=-1} \tag{5.4}$$

The modular subgroup of these transformations is $GL(2,\mathbb{Z})\ltimes\mathbb{Z}^3 \subset Sp(4,\mathbb{Z})$.[6]

We shall expand genus-two string invariants near the tropical degeneration within the approximation where all Laurent polynomial contributions in the components of Y are retained but exponential contributions are neglected. This asymptotic expansion is the analogue of the one used for the

[6]The parameterization of two-loop supergravity in terms of the coordinates S and V was introduced in the analysis of properties of two-loop maximal supergravity in [45, 46, 33], where the complex coordinate S was denoted by τ.

non-separating degeneration where all exponential contributions in t are neglected. Since periodicity forces the dependence on the moduli $\mathrm{Re}\,(\Omega)$ to be exponential, this dependence will vanish within the above approximation. Since the action of the transformation matrix B given in (5.3) affects only $\mathrm{Re}\,(\Omega)$ and not Y, the $\mathbb{Z}^3$ components of the residual modular group acts trivially. Similarly, the center of $GL(2,\mathbb{Z})$, which consists of the group of matrices $A = \pm I$ also acts trivially on Y. Therefore, the proper modular subgroup acting in the tropical degeneration will be $PGL(2,\mathbb{Z}) = GL(2,\mathbb{Z})/\{\pm I\}$. The corresponding fundamental domain may be chosen as follows,

$$\mathcal{F} = \left\{S \in \mathcal{H}_1,\, 0 < S_1 < \tfrac{1}{2},\, |S| > 1\right\} \tag{5.5}$$

It will be convenient to consider the six-fold covering space $\hat{\mathcal{F}}$ of $\mathcal{F}$ defined as follows,

$$\hat{\mathcal{F}} = \left\{S \in \mathcal{H}_1,\, 0 < S_1 < 1,\, |S - \tfrac{1}{2}| > \tfrac{1}{4}\right\}\ , \tag{5.6}$$

which happens to be a fundamental domain for the congruence subgroup $\Gamma_0(2)$ of matrices A in (5.4) with $\det A = 1$ and $c = 0$ modulo 2 (see Figure 11). The corresponding deck transformations $\mathfrak{S} = \{\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5\}$ act on S by,

$$\begin{array}{lll} \Pi_0(S) = S & \Pi_2(S) = 1 - S^{-1} & \Pi_4(S) = (1-S)^{-1} \\ \Pi_1(S) = 1 - \bar{S} & \Pi_3(S) = \bar{S}^{-1} & \Pi_5(S) = (1-\bar{S}^{-1})^{-1} \end{array} \tag{5.7}$$

and form a group $\mathfrak{S}$ which is isomorphic to the permutation group $\mathfrak{S}_3$, so that $\mathcal{F} = \hat{\mathcal{F}}/\mathfrak{S}$. The action of $PSL(2,\mathbb{Z}) \subset PGL(2,\mathbb{Z})$ has isolated fixed points at $S = i\infty, i$ and $e^{2\pi i/6}$ and their images under $SL(2,\mathbb{Z})$. The set of fixed points $\mathcal{S}$ of transformations in $PGL(2,\mathbb{Z})$ with $\det A = -1$ consists of the fixed line of the transformation $S \to -\bar{S}$ and its images under $SL(2,\mathbb{Z})$. The boundary components of $\mathcal{F}$, located at $S_1 = 0$, $S_1 = \frac{1}{2}$, and $|S| = 1$ are fixed lines respectively of $S \to -\bar{S}$, $S \to 1 - \bar{S}$, and $\bar{S} \to 1/\bar{S}$. The fundamental domain $\mathcal{F}$ has been defined as an open subset of $\mathcal{H}_1$ which excludes the points in $\mathcal{S}$.

A more physical interpretation of the fixed lines may be obtained by changing variables from V, S to real variables $L_1, L_2, L_3 > 0$ related to one another as follows [45],

$$Y = 2\pi \begin{pmatrix} L_1 + L_2 & L_1 \\ L_1 & L_1 + L_3 \end{pmatrix} \qquad V = \frac{2\pi}{(L_1L_2 + L_2L_3 + L_3L_1)^{\frac{1}{2}}} \tag{5.8}$$

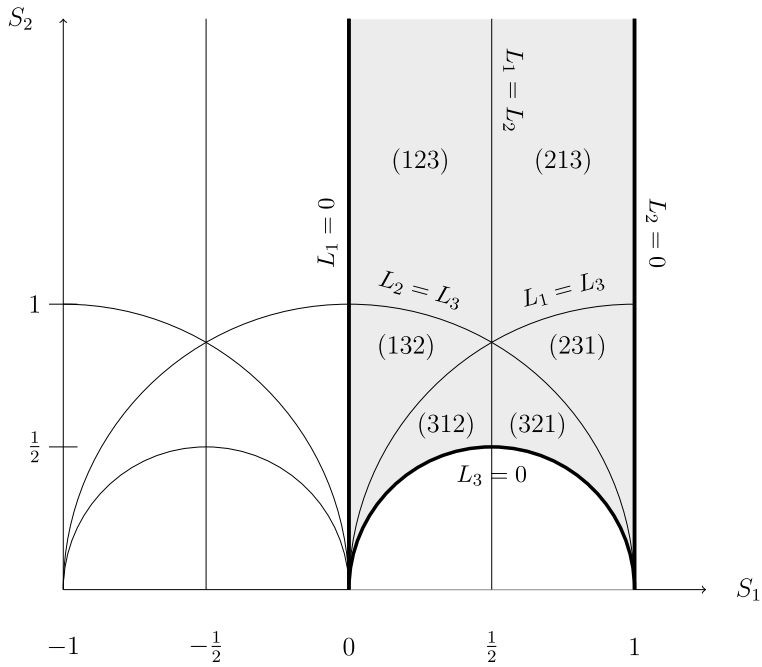


Figure 11: The extended fundamental domain $\hat{\mathcal{F}}$ (shaded in grey) is a six-fold cover of the fundamental domain $\mathcal{F}$ of $GL(2,\mathbb{Z})$ labeled by (123). Here the labels (ijk) denote the ordering $L_i < L_j < L_k$ of the Schwinger parameters.

The variable S takes the following form,

$$S_1 = \frac{L_1}{L_1 + L_2} \qquad S_2 = \frac{(L_1L_2 + L_2L_3 + L_3L_1)^{\frac{1}{2}}}{L_1 + L_2} \tag{5.9}$$

As will be further explained in the next section, the L_i's arise as Schwinger parameters in supergravity Feynman diagrams. The domain where all L_i's are positive coincides with the domain $\hat{\mathcal{F}}$ and the group $\mathfrak{S}$ of (5.7) acts by permuting the L_i's (namely, Π_1, Π_3, Π_5 exchange (L_1, L_2), (L_3, L_1), (L_2, L_3), respectively while Π_2, Π_4 act by circular permutations). The domain $\mathcal{F}$ corresponds to the particular choice of ordering $L_1 < L_2 < L_3$. The boundary components of $\hat{\mathcal{F}}$, namely $S_1 = 0$, $S_1 = 1$ and $|S - \frac{1}{2}| = \frac{1}{4}$ respectively correspond to the vanishing of L_1, L_2, and L_3. The intersection of the tropical degeneration with the non-separating degeneration ($t \to \infty$ for fixed τ and v), corresponds to $L_3/L_1, L_3/L_2 \to \infty$ while keeping L_1/L_2 fixed, or equiva-

lently $V \to 0$ and $S_2 \to \infty$ keeping VS_2 and S_1 fixed. The intersection of the tropical degeneration with the separating degeneration ($v \to 0$ for fixed τ and σ), corresponds to $V, L_1 \to 0$ while keeping L_3/L_2 fixed, or equivalently $V, S_1 \to 0$ keeping S_2 fixed.

5.2. Tropical limit of string invariants

The asymptotic behavior of genus-two modular graph functions near the tropical degeneration consists of a Laurent expansion in the variable V with coefficients which are functions only of S (and $\bar{S}$), plus exponentially suppressed contributions, which we neglect. Since the original modular graph function is invariant under $Sp(4,\mathbb{Z})$ and the expansion parameter V is invariant under its $PGL(2,\mathbb{Z})$ subgroup, it follows that the expansion coefficients of the Laurent polynomial in V must also be invariant under $PGL(2,\mathbb{Z})$. If the genus-two string invariant is real-analytic away from the separating degeneration locus $v = 0$ (and $Sp(4,\mathbb{Z})$ images thereof), then each expansion coefficient will be a real-analytic function of S away from the locus $\mathcal{S}$ given by the union of images of the line $S_1 = 0$ under the action of $GL(2,\mathbb{Z})$.The Laurent coefficients are real-analytic modular invariant functions on $\mathcal{H}_1 \setminus \mathcal{S}$, a class of functions known as *modular local polynomials*, first encountered in the study of two-loop supergravity amplitudes [33] and further developed in the mathematics literature [35, 36]. We postpone a general discussion of these functions to subsection 5.4, and concentrate here on the specific examples of the first two non-trivial genus-two string invariants, $\mathcal{B}_{(0,1)}$ and $\mathcal{B}_{(2,0)}$.

5.3. Tropical limit of non-separating degenerations

In terms of the variables t, τ_2, u_2 introduced in (3.1), the fundamental domain $\mathcal{F}$ covers the region $t > \tau_2(1-u_2^2)$, $0 < u_2 < \frac{1}{2}$ which includes the non-separating degeneration $t \to \infty$ for τ_2 fixed. We can therefore access the tropical limit $V \to 0$ for S_2 near the cusp of $\mathcal{F}$ by starting from the asymptotic series (1.3) in the non-separating degeneration limit $t \to \infty$. In taking this limit, we shall retain only terms which are power-behaved in τ_2, since exponentially suppressed terms will not contribute to the Laurent expansion around $V = 0$. Due to the modular graph nature of the coefficients in the large t expansion (1.3), it will turn out that in the limit $\tau_2 \to \infty$ keeping $u_2 = v_2/\tau_2 \in]0, 1/2[$ fixed, each of these coefficients reduces to a Laurent polynomial in τ_2, with coefficients given by Bernoulli polynomials in u_2. After transcribing these results in terms of V and S, we will be able to express

the Laurent coefficients around $V = 0$ in terms of a family of local modular functions $A_{i,j}(S)$ defined in the next subsection (see (5.32)). This process is rather involved and the derivations are relegated to Appendix (C), where the results are first expressed in terms of the variables t, τ_2 and u_2.

The tropical limit of the Kawazumi-Zhang invariant φ, first obtained at leading order in [29] and then extended to all orders in [32], is derived by letting $\tau_2 \to 0$ at fixed u_2 in the expansion (3.16), and retaining only power-like terms in τ_2. The result reads

$$\varphi^{(t)} = \frac{5\pi}{6V} A_{1,0} + \frac{5V^2}{4\pi^2}\zeta(3)\, A_{0,0} \tag{5.10}$$

where $A_{0,0} = 1$ and

$$A_{1,0}(S) \;=\; \frac{S_2}{5} + \frac{6}{5S_2}\, B_2(S_1) + \frac{1}{S_2^3}\left(\frac{1}{30} + B_4(S_1)\right) \;. \tag{5.11}$$

where $B_{2k}(S_1)$ are Bernoulli polynomials of even index. These expressions are valid in the extended fundamental domain $\hat{\mathcal{F}}$ only, and can be extended to continuous (but non differentiable) functions on the full upper half plane by $GL(2,\mathbb{Z})$ invariance. As we shall see in the next section, the leading Laurent coefficient in φ of order $1/V$ matches the two-loop supergravity integrand, while the sub-leading term, proportional to $\zeta(3)V$, can be traced to a two-loop diagram with a higher derivative $\mathcal{R}^4$ interaction on one vertex.

Similarly, the tropical limit of the genus-two modular graph functions $\mathcal{Z}_2, \mathcal{Z}_3$ is given by,

$$\begin{aligned}\mathcal{Z}_2^{(t)} = \frac{32\pi^2}{V^2}&\left[\frac{1}{504}A_{0,0} - \frac{1}{1008}A_{0,2} - \frac{5}{792}A_{1,1} - \frac{17}{960}A_{2,0}\right]\\ &- \frac{5V\zeta(3)}{2\pi}\, A_{1,0} - \frac{7V^3\zeta(5)}{4\pi^3}A_{0,1}\end{aligned} \tag{5.12}$$

$$\begin{aligned}\mathcal{Z}_3^{(t)} = \frac{32\pi^2}{V^2}&\left[-\frac{11}{7560}A_{0,0} + \frac{1}{1512}A_{0,2} + \frac{1}{792}A_{11} + \frac{17}{576}A_{2,0}\right]\\ &+ \frac{5V\zeta(3)}{6\pi}A_{1,0} + \frac{11V^4\zeta(3)^2}{8\pi^4}A_{0,0}\end{aligned} \tag{5.13}$$

where the functions $A_{i,j} = A_{i,j}(S)$ are given in the extended fundamental

domain $\hat{\mathcal{F}}$ by the following expressions,

$$
\begin{aligned}
A_{0,1}(S) &= S_2 + \frac{1}{S_2}\left(\frac{5}{6} + B_2\right) \\
A_{0,2}(S) &= S_2^2 + \left(\frac{2}{3} + 2B_2\right) + \frac{1}{S_2^2}\left(\frac{7}{10} + 2B_2 + B_4\right) \\
A_{1,1}(S) &= \frac{S_2^2}{7} + \left(\frac{1}{70} + \frac{9}{7}B_2\right) + \frac{1}{S_2^2}\left(\frac{9}{7}B_2 + \frac{15}{7}B_4\right) \\
&\quad + \frac{1}{S_2^4}\left(\frac{11}{420} + \frac{3}{2}B_4 + B_6\right) \\
A_{2,0}(S) &= \frac{S_2^2}{33} + \left(\frac{20}{693} + \frac{20}{33}B_2\right) + \frac{1}{S_2^2}\left(\frac{20}{33}B_2 + \frac{70}{33}B_4\right) \\
&\quad + \frac{1}{S_2^4}\left(\frac{20}{11}B_4 + \frac{28}{11}B_6\right) + \frac{1}{S_2^6}\left(\frac{1}{630} + \frac{4}{3}B_6 + B_8\right)
\end{aligned}
\tag{5.14}
$$

where for brevity we denote $B_{2n} = B_{2n}(S_1)$. After extending them to the full upper-half plane by modular invariance, the $A_{i,j}$'s become eigenfunctions of the Laplace-Beltrami operator $\Delta = S_2^2\left(\partial_{S_1}^2 + \partial_{S_2}^2\right)$ on $\mathcal{H}_1\backslash\mathcal{S}$,

$$
\Delta A_{i,j} - n(n+1)\, A_{i,j} = 0\ , \quad n = 3i + j \tag{5.15}
$$

up to a delta function source supported on the singular locus $\mathcal{S}$. They provide a basis for the class of functions encountered in the low energy expansion of the two-loop supergravity amplitude computed to high order in [33]. In particular, they are invariant under the group of permutations $\mathfrak{S}_3$ on L_1, L_2, L_3. This particular basis was constructed by Zagier [34] and will be reviewed in the next subsection.

For the remaining string invariant $\mathcal{Z}_1$, in principle we would need to compute the tropical limit of the integral $\mathcal{K}^0_{aaaa}$ defined in (3.38), which appears to be a project in its own right. By construction however, $\mathcal{K}^0_{aaaa}$ is only a function of τ, independent of the variable v, and therefore the tropical limit of $\mathcal{K}^0_{aaaa}/t^2$ cannot be written as a linear combination of $V^\alpha A_{i,j}$ without spoiling the coefficients of the higher powers of t. Fortunately, there are other offending terms coming from the tropical limit of $\mathcal{K}^c$ which are also independent of u_2. Collecting these terms together, we obtain

$$
\begin{aligned}
\mathcal{Z}_1^{(t)} &= \frac{32\pi^2}{V^2}\left[-\frac{1}{315}A_{0,0} + \frac{1}{252}A_{0,2} - \frac{1}{792}A_{1,1} + \frac{23}{960}A_{2,0}\right] \\
&\quad + \frac{V\zeta(3)}{\pi}\left[\frac{18}{5}A_{0,1} - \frac{1}{2}A_{1,0}\right] - \frac{V^3\zeta(5)}{2\pi^3}A_{0,1} + \frac{3\zeta(3)^2V^4}{8\pi^4}A_{0,0}
\end{aligned}
\tag{5.16}
$$

$$+\frac{1}{8\pi^2 t^2}\left[\mathcal{K}^0_{aaaa}-\frac{2y^4}{945}+\frac{8y\zeta(3)}{5}+\frac{145\zeta(5)}{6y}+\frac{3\zeta(7)}{4y^3}\right]$$

For consistency with the symmetries of the tropical limit, the bracket on the last line must be proportional to $V^4 A_{0,0}$ with no dependence on S. We conclude that the tropical limit of $\mathcal{K}^0_{aaaa}$ must be given by

(5.17)
$$\mathcal{K}^0_{aaaa}=\frac{2y^4}{945}-\frac{8y\zeta(3)}{5}-\frac{145\zeta(5)}{6y}-\frac{3\zeta(7)}{4y^3}+(\beta-3)\,\frac{\zeta(3)^2}{\pi^2 y^2}+\mathcal{O}(e^{-2y})$$

where the coefficient β is unknown at this stage. Note that the naive evaluation of the integral (3.38) by replacing $g(z)$ by its polynomial approximation $\mathfrak{g}_1(z)$ and ignoring the term proportional to $\zeta(3)$ correctly produces the leading term in (5.17).

We conclude that the tropical degeneration of the string invariant $\mathcal{B}_{(2,0)}$ is given by,

(5.18)
$$\begin{aligned}\mathcal{B}^{(t)}_{(2,0)}&=\frac{1}{2}\mathcal{Z}^{(t)}_1-\mathcal{Z}^{(t)}_2+\frac{1}{2}\mathcal{Z}^{(t)}_3\\&=\frac{32\pi^2}{V^2}\left[-\frac{13}{3024}A_{0,0}+\frac{5}{1512}A_{0,2}+\frac{5}{792}A_{1,1}+\frac{2}{45}A_{2,0}\right]\\&\quad+\frac{V}{\pi}\zeta(3)\,\left[\frac{9}{5}A_{0,1}+\frac{8}{3}A_{1,0}\right]+\frac{3V^3}{2\pi^3}\zeta(5)A_{0,1}+(\beta+11)\frac{\zeta(3)^2V^4}{16\pi^4}A_{0,0}\end{aligned}$$

where the coefficient β could in principle be determined by a full analysis of the tropical limit of the integral $\mathcal{K}^0_{aaaa}$ defined in (3.38), which we leave for future work.

5.4. Modular local polynomials

In this subsection we review the construction of the space of modular local polynomials, following [34] and expanding thereon. Our goal is to construct functions $A(S)$ on the Poincaré upper half-plane, which are invariant under the action (5.4), real-analytic away from the locus $\mathcal{S}$, and given in each connected domain of $\mathcal{H}_1\backslash\mathcal{S}$ by a Laurent polynomial in S_2, with coefficients which are polynomial in S_1. We further require that these functions satisfy the Laplace eigenvalue equation,

(5.19)
$$[\Delta-n(n+1)]\;A=0$$

away from $\mathcal{S}$, with $n \geq 0$ integer. Since the differential operator $D_k = \partial_S + \frac{k}{S-\bar{S}}$ satisfies $\Delta_{k+2} \cdot D_k - D_k \Delta_k = -kD_k$, where $\Delta_k = 4D_{k-2} \circ (S_2^2 \partial_{\bar{S}})$ is the Laplacian acting on weight k modular forms, it is clear that the function given locally by $A = D_{-2n}^{(n)} P$ where $D_{-2n}^{(n)}$ is the iterated derivative operator

$$D_{-2n}^{(n)} = \frac{(-2i)^n n!}{(2n)!} D_{-2} \circ D_{-4} \circ \cdots \circ D_{-2n+2} \circ D_{-2n} \tag{5.20}$$

will satisfy (5.19) whenever P is annihilated by the Laplacian Δ_{-2n}, in particular when P is a holomorphic function of S. Since the operator (5.20) can be written as

$$D_{-2n}^{(n)} = \frac{(-2i)^n n!}{(2n)!} \sum_{m=0}^{n} \binom{n}{m} \frac{(-n-m)_m}{(S-\bar{S})^m} \frac{\partial^{n-m}}{\partial S^{n-m}} \tag{5.21}$$

where $(k)_m = k(k+1)\ldots(k+m-1)$ is the ascending Pochhammer symbol, it is also clear that whenever P is a polynomial in S, the resulting function A will be a Laurent polynomial in S_2 with coefficients which are polynomial in S_1. In order for $A(S)$ to be invariant under the action (5.4), the polynomial P should transform according to,

(5.22)

$$P|_\gamma(S) = \begin{cases} (cS+d)^{-2n} P\left(\frac{aS+b}{cS+d}\right) & \text{if} \quad \det\gamma = +1 \\ (cS+d)^{-2n} \overline{P\left(\frac{a\bar{S}+b}{c\bar{S}+d}\right)} & \text{if} \quad \det\gamma = -1 \end{cases} \qquad \text{where} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

It is easy to check that this action preserves the space V_{2n} of polynomials of degree at most $2n$, while polynomials of higher degree are mapped into rational functions. Thus, the functions of interest are of the form $A = D_{-2n}^{(n)} P$ where $P(S)$ is a polynomial in S of degree at most $2n$.

Since the extended fundamental domain $\hat{\mathcal{F}}$ covers a single connected component of $\mathcal{H}_1 \backslash \mathcal{S}$, we must restrict to functions are invariant under the deck transformations (5.7). We claim that this amounts to requiring that $P(S)$ is a sum of monomials $\sum_i C_i u^i v^{n-3i}$, where C_i's are real constants and we set,

$$u = S^2(1-S)^2 \ , \qquad v = S^2 - S + 1 \tag{5.23}$$

This result may be established by considering the generating function $(S+t)^{2n}$ for the space of polynomials in V_{2n} parametrized by $t \in \mathbb{R}$. The operator

(5.20) acts on this function by,

$$D^{(n)}_{-2n}\,(S+t)^{2n} = |S+t|^{2n} S_2^{-n} \tag{5.24}$$

The deck transformations $\Pi_i \in \mathfrak{S}$ acting on the functions $|S+t|^{2n}S_2^{-n}$ lift to purely holomorphic transformations $\tilde{\Pi}_i$ acting on the functions $(S+t)^{2n}$,

$$\Pi_i\left(|S+t|^{2n}S_2^{-n}\right) = D^{(n)}_{-2n}\left\{\tilde{\Pi}_i\left((S+t)^{2n}\right)\right\} \tag{5.25}$$

where

$$\begin{aligned} &\tilde{\Pi}_0(S+t)^{2n} = (S+t)^{2n} && \tilde{\Pi}_1(S+t)^{2n} = (S-t-1)^{2n} \\ &\tilde{\Pi}_2(S+t)^{2n} = (S+tS-1)^{2n} && \tilde{\Pi}_3(S+t)^{2n} = (tS+1)^{2n} \\ &\tilde{\Pi}_4(S+t)^{2n} = (tS-t-1)^{2n} && \tilde{\Pi}_5(S+t)^{2n} = (S+tS-t)^{2n} \end{aligned} \tag{5.26}$$

The projection of the generating function $(S+t)^{2n}$ onto the space $\mathcal{P}_n^{\mathfrak{S}}$ of weight $2n$ $\mathfrak{S}$-invariant polynomials is given by summing over all images,

$$P_t(S) = \frac{1}{6}\sum_{i=0}^{5}\tilde{\Pi}_i(S+t)^{2n} \tag{5.27}$$

Invariance under $S \to 1-S$ trivially implies that $P_t(S)$ is a polynomial in $S(1-S)$, and therefore a polynomial in u, v. It remains to show that the only allowed monomials are those of the form $u^i v^j$ with $n = 3i+j$.

For this purpose, we linearize the action of $\mathfrak{G}$ by introducing a set of three complex variables, z_1, z_2, z_3 in terms of which u and v are given by symmetric polynomials,

$$\begin{aligned} z_1 + z_2 + z_3 &= 0 \\ z_1^2 + z_2^2 + z_3^2 &= 2v \\ z_1^3 + z_2^3 + z_3^3 &= 3\sqrt{u} \end{aligned} \tag{5.28}$$

The projected generating function $P_t(S)$ can then be obtained as

$$z_2^{2n} P_t(-z_1/z_2) = F_t(z_1, z_2, z_3) \tag{5.29}$$

where F_t is the following polynomial, which is homogeneous of degree $2n$ in the variables z_i, and invariant under permutations of the z_i's,

$$F_t(z_1,z_2,z_3) = (tz_1-z_2)^{2n} + (tz_1-z_3)^{2n} + (tz_2-z_1)^{2n} + (tz_2-z_3)^{2n} + (tz_3-z_1)^{2n} + (tz_3-z_2)^{2n} \tag{5.30}$$

Since F_t is a symmetric polynomial in z_i, it may be expressed as a polynomial in v and $\sqrt{u}$. Under the parity transformation $z_i \to -z_i$, the polynomial F_t is invariant, while v is even but $\sqrt{u}$ is odd. Since the polynomials u, v, and F_t have respective homogeneity degree weights 2, 6, and $2n$, we have the decomposition,

$$F_t(z_1,z_2,z_3) = \sum_{i,j\geq 0;\, 3i+j=n} C_i(t)\, u^i v^j \tag{5.31}$$

for some polynomials $C_i(t)$ in t with real coefficients, thus proving the announced result.

We are now ready to define the family of functions $A_{i,j}$ whose first few members appeared in the previous subsection: they are simply the descendents of the monomials

$$A_{i,j}(S) = D^{(n)}_{-2n}(u^i v^j) \qquad n = 3i+j \quad \text{with} \quad i,j \geq 0 \tag{5.32}$$

In the fundamental domain $\hat{\mathcal{F}}$, the modular function $A_{i,j}(S)$ takes the following form,

$$A_{i,j}(S) = \sum_{k=0}^{2i+j} A^{(k)}_{i,j}(S_1)\, S_2^{i+j-2k} \tag{5.33}$$

where $A^{(k)}_{i,j}(S_1)$ is a polynomial of degree k in $S_1(1-S_1)$, and thus of degree $2k$ in S_1. Since it is invariant under $S_1 \mapsto 1-S_1$, it may be expressed as a linear combination of Bernoulli polynomials $B_{2k}(S_1)$ of even index. After expressing S_1, S_2 in terms of L_1, L_2, L_3 using (5.9), the function $A_{i,j}$ is then by construction a homogenous function of the L_i's, invariant under permutations. Multiplying $A_{i,j}$ by a power V^α and expressing it in terms of the variables t, τ_2, u_2, we see that $V^\alpha A_{i,j}$ has a Laurent expansion near $t=\infty$ with powers ranging from $\frac{1}{2}(i+j-\alpha)$ to $-\frac{1}{2}(3i+j+\alpha)$. This Laurent expansion is compatible with that of a genus-two modular graph function with weight w only when

$$i+j \leq 2w+\alpha\ , \quad 3i+j \leq 2w-\alpha\ , \quad |\alpha| \leq 2w\ , \quad i+j-\alpha \text{ even} \tag{5.34}$$

For $w=2$ and $\alpha=-2$, this constraint singles out the functions $A_{0,0}, A_{0,2}, A_{1,1}, A_{2,0}$ appearing in the leading term in (C.90). The subleading terms in

the same equation also satisfy the requirement (5.34), but it is worth mentioning that terms with $\alpha = -4, -3, 0, 2$ are in principle allowed, although they do not occur in practice. In particular, agreement with supergravity requires $\alpha \geq -w$. Finally, since $A_{i,j}$ satisfies (5.19) with $n = 3i + j$, it easily follows that $V^\alpha A_{i,j}$ is an eigenmode of the Laplacian $\Delta_{\mathcal{H}_2}$ on the Siegel upper half plane with eigenvalue $\frac{1}{2}[n(n+1) + \alpha(\alpha+3)]$, away from the separating degeneration locus.

6. Low energy expansion in two-loop supergravity

The amplitudes of closed superstring theory are related at energy scales $\ll (\alpha')^{-1/2}$ to amplitudes in maximal supergravity. At tree level this connection is easy to demonstrate, but at loop level the connection to higher genus string amplitudes is more subtle due to ultraviolet divergences occurring in supergravity loop amplitudes. Still, the maximal degeneration of the *integrand* of the genus-h superstring amplitude is expected to be related to the sum of integrands of the corresponding supergravity amplitude, and in fact provides an efficient reorganisation of the sum over Feynman diagrams [47, 48, 49]. In this section we will compare the low energy expansion of the integrands in maximal supergravity with the genus-two string theory results of the preceding sections. Our discussion will highlight the fact that the integrands of the Feynman diagrams do not capture the full content of the tropical limit that was analyzed in the last section – the terms proportional to odd zeta values do not arise from the field theory expression.

The Feynman diagrams contributing to the two-loop four-graviton amplitude in maximal supergravity were expressed in an efficient manner in [38], where it was demonstrated that they could be reduced to a sum of diagrams of the form shown in Figure 12. As indicated in that figure, each diagram has the structure of a graph in ϕ^3 quantum field theory multiplied by a kinematic factor of $s^2\,\mathcal{R}^4$. The full amplitude, which is symmetric in the external states is obtained by summing over the diagrams with the three inequivalent permutations of the external particles, which involve kinematic factors of $t^2\,\mathcal{R}^4$ and $u^2\,\mathcal{R}^4$ in addition to the $s^2\,\mathcal{R}^4$ term shown in Figure 12. Note that graphs in which more than two vertices are attached to a single line are absent.

After integrating over the loop momenta the expression for each Feynman integral involves integration over seven "Schwinger" parameters. These may be interpreted as the positions of the four vertices, t_i $(i = 1, 2, 3, 4)$, and the parameters, L_1, L_2 and L_3, which label the lengths of the lines, and which take values in the range $0 \leq L_i \leq \infty$. These real parameters of the

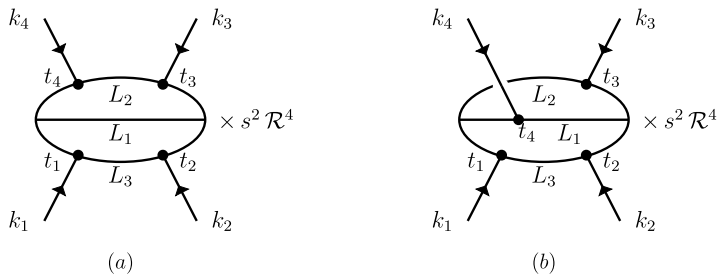


Figure 12: (a) A "planar" Feynman diagram with a pair of external states connected to two different lines of a two-loop vacuum diagram. (b) A "non-planar" Feynman diagram in which one pair of external states is attached to a single line and the other states are each attached to separate lines.

Feynman integrand can be understood as the moduli of tropical Riemann surfaces [44] and are analogous to the seven complex parameters that enter into the integrand of the genus-two superstring amplitude, (2.1) and (2.4), which label the complex positions of the four vertex operators and the three complex moduli of the compact genus-two surface.

Expanding the sum of Feynman diagrams in powers of s, t and u and integrating each term over t_i leads to a power series of the form (2.6) with the string coefficient functions $\mathcal{B}_{(p,q)}(\Omega)$ replaced by their supergravity counterparts, $\mathcal{B}^{(sg)}_{(p,q)}(L_1, L_2, L_3)$[7]. These functions can be obtained reasonably straightforwardly up to any given order in the low energy expansion by expanding the Feynman diagrams. In [33] these coefficients were explicitly evaluated up to terms with $2p + 3q = 6$ (terms of order s^6), and it is straightforward to generate them to much higher order. The expressions for $\mathcal{B}^{(sg)}_{(p,q)}(L_1, L_2, L_3)$ are sums of the local modular functions $A_{i,j}(S)$ (that were defined in the last section) with rational coefficients, multiplied by a factor of V^{-w} (where w was defined in (2.7)).

We are here interested in studying the detailed correspondence between the tropical limit of the genus-two string amplitude and the supergravity expression. For this purpose we would like to express the Feynman diagrams in terms of sums over word-lines in a manner that mimics the expression of string theory amplitudes as sums over world-sheets. Such a world-line procedure was described in the context of scalar field theory in [48, 49, 50] and in the context of the two-loop four-graviton amplitude in maximal supergravity

[7]The superscript $^{(sg)}$ will be used to label the supergravity versions of the various quantities in the following equations.

in [33]. In the latter reference it was shown that the low energy expansion of the sum of supergravity Feynman diagrams is reproduced by the sum of word-line diagrams. In other words, the coefficient $\mathcal{B}^{(sg)}_{(2,0)}$ of the term at order σ_2^2 in the low energy expansion is given by a world-line expression analogous to the world-sheet expression in (2.9).

Such a world-line formulation will allow us to evaluate quantities $\mathcal{Z}_1^{(sg)}$, $\mathcal{Z}_2^{(sg)}, \mathcal{Z}_3^{(sg)}$ that are the supergravity analogues of the integrals of bilinears in the world-sheet Green function that were defined by (2.11) and (2.12). We will then compare them with the tropical limit of the string invariants, $\mathcal{Z}_1^{(t)}, \mathcal{Z}_2^{(t)}, \mathcal{Z}_3^{(t)}$ that were computed in Section 5. The form of the supergravity expressions, when expressed in the world-line formalism, is given by,

$$\mathcal{Z}_1^{(sg)} = \int_{\Gamma^4} \frac{\Delta^{(sg)}(1,3)\Delta^{(sg)}(2,4)}{8\,(\det Y^{(sg)})^2}\mathcal{G}^{(sg)}(1,2)^2 \tag{6.1}$$

$$\mathcal{Z}_2^{(sg)} = \int_{\Gamma^4} \frac{\Delta^{(sg)}(1,3)\Delta^{(sg)}(2,4)}{8\,(\det Y^{(sg)})^2}\mathcal{G}^{(sg)}(1,2)\,\mathcal{G}^{(sg)}(1,4) \tag{6.2}$$

$$\mathcal{Z}_3^{(sg)} = \int_{\Gamma^4} \frac{\Delta^{(sg)}(1,3)\Delta^{(sg)}(2,4)}{8\,(\det Y^{(sg)})^2}\mathcal{G}^{(sg)}(1,2)\,\mathcal{G}^{(sg)}(3,4) \tag{6.3}$$

where $\mathcal{G}^{(sg)}(i,j) = \mathcal{G}^{(sg)}(t_i,t_j)$ now denotes the Arakelov Green function on the two-loop graph Γ with three edges of length L_1, L_2, L_3 and t_i, t_j ($i = 1,2,3,4$) label the positions of the vertex operators that are to be integrated over the network of world-lines in the graph (as shown in Figure 12). We normalize the (real) period matrix of the graph, $Y^{(sg)}$, so that it agrees with the imaginary part of the period matrix of the Riemann surface Σ in the maximal degeneration limit (where we have made a particular choice for the arbitrary overall normalisation)

$$Y^{(sg)} = 2\pi \begin{pmatrix} L_1 + L_2 & L_1 \\ L_1 & L_1 + L_3 \end{pmatrix} \tag{6.4}$$

The measure in (6.1)–(6.3) involves factors of $\Delta^{(sg)}(i,j)$, each of which is a two-form on $\Gamma \times \Gamma$, and is the limit of the corresponding factor $|\Delta(z_i,z_j)|^2$ in the string measure defined in (2.12). It reduces to $\pm 4\,dt_i\,dt_j$ if the points i,j are on different edges, and zero otherwise.

As in the string computation, the expression for the total coefficient

$$\mathcal{B}^{(sg)}_{(p,q)} = \frac{1}{2}\mathcal{Z}_1^{(sg)} - \mathcal{Z}_2^{(sg)} + \frac{1}{2}\mathcal{Z}_3^{(sg)} = \frac{1}{2}Z_1^{(sg)} - Z_2^{(sg)} + \frac{1}{2}Z_3^{(sg)} \tag{6.5}$$

is independent of whether one uses the world-line Green function, $G^{(sg)}(t_i, t_j)$ (as in [50, 33]) or the Arakelov Green function, $\mathcal{G}^{(sg)}(t_i, t_j)$, but the individual contributions $\mathcal{Z}_i$ and Z_i differ in the two cases. In order to compare these with the tropical limit of the string calculation, it is crucial that we use the Arakelov Green function in the following. As stressed earlier, the use of the Arakelov Green function guarantees the conformal invariance of each individual component $\mathcal{Z}_i$.

However, just as in the string case, it is far more convenient to first compute the diagrams with the world-line Green function $G^{(sg)}(t_i, t_j)$, giving rise to contributions $Z_i^{(sg)}$, which may then be transcribed into $\mathcal{Z}_i^{(sg)}$ by using the relation between the Green functions. We will see that the supergravity results $\mathcal{Z}_i^{(sg)}(L_1, L_2, L_3)$ reproduces the leading term in the tropical limit $\mathcal{Z}_i^{(t)}(\Omega)$ of the string invariant, upon identifying the graph period matrix (6.4) (up to an overall scale factor) with the imaginary part of the period matrix Ω. However, the tropical limit of the string amplitude also contains subleading terms which do not arise in the supergravity calculations, but can nevertheless be understood as two-loop amplitudes with higher-derivative vertices.

6.1. Green functions on graphs

We first recall the general definition of the world-line propagator from [50], and its relation to the Arakelov-Green function. We consider a graph Γ with h loops, and denote the edges by e_i. We choose a basis a_I, $I = 1 \dots h$ of homology cycles in $H^1(\Gamma, \mathbb{Z})$. The dual one-forms $\omega_I \in H_1(\Gamma, \mathbb{Z})$ are given by $\pm \mathrm{d}t$ on the edge e_j if the edge e_j lies on the cycle a_I, with a sign depending on the orientation of e_j along a_I, or zero of e_j does not belong to a_I. The period matrix of the graph is defined by $Y_{IJ}^{(sg)} = 2\pi \int_{a_I} \omega_J$ and is a symmetric positive real matrix. The world-line Green function on Γ is given by[8]

$$G^{(sg)}(t, t') = -s(p(t, t')) + 2\pi Y^{(sg)IJ} \int_{p(t,t')} \omega_I \int_{p(t,t')} \omega_J \tag{6.6}$$

where $s(t, t')$ is the length of the path $p(t, t')$ and $Y^{(sg)IJ}$ is the inverse of $Y_{IJ}^{(sg)}$. Note that $G^{(sg)}(t, t')$ depends on the choice of homology basis and the

[8]We use a somewhat unusual normalization of the Green function such that it agrees with the string Green function (2.16) in the tropical limit.

choice of path, which we fix by cutting the graph along h edges such that it becomes simply connected. Moreover it satisfies,

$$\partial_t^2 G^{(sg)}(t,t') = -2\delta(t-t') + 2h\,\kappa^{(sg)}(t)\ , \tag{6.7}$$

where $\kappa^{(sg)}(t)$ is the Arakelov one-form on Γ, given on the edge e_i by

$$\kappa_i^{(sg)} = \frac{2\pi}{h} s_{IJ} Y^{(sg)\,IJ} \mathrm{d}t \tag{6.8}$$

where $s_{IJ} = \pm 1$ if the edge e_i belongs to both a_I and a_J, or $s_{IJ} = 0$ if it does not. Alternatively, $\kappa_i^{(sg)} = \mathrm{d}t/(L_i + r_i)$, where L_i is the length of the edge e_i and r_i is the effective resistance between the two endpoints of e_i once the edge is removed from Γ. By Foster's theorem from electric network theory, $\sum_i \frac{L_i}{L_i+r_i} = h$ so $\int_\Gamma \kappa^{(sg)} = 1$. Note that unless $h = 1$, the r.h.s. of (6.7) does not integrate to zero. The Arakelov Green function $\mathcal{G}^{(sg)}(t,t')$ is obtained from $G^{(sg)}(t,t')$ using the relation,

$$\mathcal{G}^{(sg)}(t,t') = G^{(sg)}(t,t') - \gamma^{(sg)}(t) - \gamma^{(sg)}(t') + \gamma_1^{(sg)} \tag{6.9}$$

where

$$\gamma^{(sg)}(t) = \frac{1}{2}\int_\Gamma G^{(sg)}(t,t')\,\kappa^{(sg)}(t')\ , \quad \gamma_1^{(sg)} = \int_\Gamma \gamma^{(sg)}(t)\,\kappa^{(sg)}(t) \tag{6.10}$$

The Arakelov Green function satisfies,

$$\partial_t^2 \mathcal{G}^{(sg)}(t,t') = -2\delta(t-t') + \kappa^{(sg)}(t)\ , \quad \int_\Gamma \mathcal{G}^{(sg)}(t,t')\,\kappa^{(sg)}(t') = 0 \tag{6.11}$$

so, unlike the world-line Green function, its integral using the Arakelov measure, vanishes.

6.2. World-line evaluation of two-loop supergravity invariants

We now apply the previous formulae to the diagrams of Figure 12 with three edges of e_i length L_i, $i = 1,2,3$. We choose a basis of $H_1(\Gamma)$ such that the loop $B_1 = e_1 - e_2$, and $B_2 = e_2 - e_3$. Then, on the edge e_1, the Abelian differentials (ω_1, ω_2) reduce to $(\mathrm{d}t, 0)$; on e_2 to $(\mathrm{d}t, -\mathrm{d}t)$; and on e_3 to $(0, \mathrm{d}t)$. The period matrix of the graph is given in (6.4). The canonical volume form is then

$$\kappa^{(sg)}(t) = \frac{(L_j + L_k)\,\mathrm{d}t}{2\Delta_L} \quad (t \in e_i) \tag{6.12}$$

where $\Delta_L = L_1L_2 + L_2L_3 + L_3L_1 = V^{-2}$ and $\{i,j,k\} = \{1,2,3\}$. It is straightforward to check that $\int_\Gamma \kappa^{(sg)} = 1$. The world-line Green function is given by [33]

(6.13)
$$G^{(sg)}(t,t') = \begin{cases} -\frac{1}{2}|t-t'| + \frac{L_j+L_k}{2\Delta_L}(t-t')^2 & t,t' \in e_i \\ -\frac{1}{2}(t+t') + \frac{(L_j+L_k)t'^2}{2\Delta_L} + \frac{(L_i+L_k)t^2}{2\Delta_L} + \frac{2L_k tt'}{2\Delta_L} & t \in e_i, t' \in e_j \end{cases}$$

From this it follows that

(6.14)
$$\begin{aligned} \gamma^{(sg)}(t) &= \frac{L_j+L_k}{4\Delta_L} t(t-L_i) \\ &\quad - \frac{4L_1L_2L_3 + L_i^2(L_j+L_k) + L_i(L_j^2+L_k^2) + L_jL_k(L_j+L_k)}{24\Delta_L} \\ \gamma_1^{(sg)} &= -\frac{(L_1+L_2)(L_2+L_3)(L_3+L_1)}{16\Delta_L} \end{aligned}$$

where as before $t \in e_i$ and $\{i,j,k\} = \{1,2,3\}$, from which it is easy to obtain $\mathcal{G}^{(sg)}(t,t')$.

Using (6.9) it is easy to show that the supergravity invariants (6.1)–(6.3) are related to their counterparts $Z_i^{(sg)}$ defined using the world-line Green function (6.13) via the analogue of (3.19),

$$\begin{aligned} \mathcal{Z}_1^{(sg)} &= Z_1^{(sg)} + 32(\gamma_1^{(sg)})^2 - 64\gamma_2^{(sg)} \\ \mathcal{Z}_2^{(sg)} &= Z_2^{(sg)} - 32\gamma_3^{(sg)} - 32\gamma_2^{(sg)} + 32(\gamma_1^{(sg)})^2 \\ \mathcal{Z}_3^{(sg)} &= Z_3^{(sg)} - 64\gamma_3^{(sg)} + 32(\gamma_1^{(sg)})^2 \end{aligned} \tag{6.15}$$

where

$$\begin{aligned} \gamma_2^{(sg)} = \int_\Gamma \gamma^{(sg)}(t)^2\kappa^{(sg)}(t) &= \frac{1}{2880\Delta_L^2}\Bigg[13\sum_{i\neq j} L_i^4L_j^2 + 20\sum_{i<j} L_i^3L_j^3 \\ &\quad + L_1L_2L_3\left(26\sum_i L_i^3 + 67\sum_{i\neq j} L_i^2L_j\right) + 106L_1^2L_2^2L_3^2\Bigg] \end{aligned} \tag{6.16}$$

$$\gamma_3^{(sg)} = \int_{\Gamma^2} \gamma^{(sg)}(t)\gamma^{(sg)}(t')\,\frac{\Delta^{(sg)}(t,t')}{(\det Y^{(sg)})^2} \tag{6.17}$$

$$= \frac{1}{576\Delta_L}\left[2\sum_{i\neq j} L_i^3 L_j + 5\sum_{i<j} L_i^2 L_j^2 + 7L_1L_2L_3\sum_i L_i^3\right]$$

The integrals $Z_i^{(sg)}$ and $\mathcal{Z}_i^{(sg)}$ may now be evaluated using the above procedure. In the case of (6.1), if both t_1, t_2 are on the same edge e_1, then the integral over t_3, t_4 along the edges e_2, e_3 produces $2(L_2+L_3)^2/\Delta_L^2 = 8\kappa^{(sg)}(t_1)\kappa^{(sg)}(t_2)$. If t_1, t_2 are instead on distinct edges e_1, e_2, then the integral over t_3, t_4 along the edges e_2, e_3 produces $2(L_1+L_3)(L_2+L_3)/\Delta_L^2 = 8\kappa^{(sg)}(t_1)\kappa^{(sg)}(t_2)$.

In the interest of brevity we will only display the results for $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$ which are based on the Arakelov Green function, suppressing the intermediate results for the Z_i's, which are based on the string Green function. We shall express the result both in terms of the Schwinger parameters L_i, and in terms of the local modular functions $A_{i,j}$ introduced in Section 5.4, in order to facilitate comparison with the tropical limit of the string integrand.

• For the supergravity integrand associated with the graph 1 in Figure 2, we find,

(6.18)
$$\begin{aligned}\mathcal{Z}_1^{(sg)} &= \frac{2\pi^2}{9}\Bigg[-\frac{4}{5}\Delta_L + \frac{13}{20}(L_1+L_2+L_3)^2 - \frac{17}{10}(L_1+L_2+L_3)\frac{L_1L_2L_3}{\Delta_L} \\ &\quad + \frac{69}{20}\left(\frac{L_1L_2L_3}{\Delta_L}\right)^2\Bigg] \\ &= \frac{32\pi^2}{V^2}\left[-\frac{1}{315}A_{0,0} + \frac{1}{252}A_{0,2} - \frac{1}{792}A_{1,1} + \frac{23}{960}A_{2,0}\right]\end{aligned}$$

This result precisely reproduces the leading term in the tropical limit $\mathcal{Z}_1^{(t)}$ in (C.89) of the string invariant, up to subleading terms proportional to odd zeta values, namely

(6.19)
$$\mathcal{Z}_1^{(t)} = \mathcal{Z}_1^{(sg)} + \zeta(3)\,V\left[\frac{18}{5\pi}A_{01} - \frac{1}{2\pi}A_{01}\right] - \zeta(5)\,\frac{V^3}{2\pi^3}A_{01} + \beta\frac{\zeta(3)^2V^4}{\pi^4}$$

• For the supergravity integrand associated with the graph 2 in Figure 2,

we get instead,

(6.20)
$$\begin{aligned}\mathcal{Z}_2^{(sg)} &= \frac{2\pi^2}{9}\Big[\frac{1}{5}\Delta_L - \frac{7}{20}(L_1+L_2+L_3)^2 + \frac{23}{10}(L_1+L_2+L_3)\frac{L_1L_2L_3}{\Delta_L} \\ &\quad - \frac{51}{20}\left(\frac{L_1L_2L_3}{\Delta_L}\right)^2\Big] \\ &= \frac{32\pi^2}{V^2}\left[\frac{1}{504}A_{0,0} - \frac{1}{1008}A_{0,2} - \frac{5}{792}A_{1,1} - \frac{17}{960}A_{2,0}\right]\end{aligned}$$

Again, this result precisely matches the tropical limit $\mathcal{Z}_2^{(t)}$ in (C.86) of the string invariant, up to subleading terms proportional to odd zeta values,

$$\mathcal{Z}_2^{(t)} = \mathcal{Z}_2^{(sg)} - \frac{5}{2\pi}\zeta(3)\,V\,A_{10} - \frac{7}{4\pi^3}\zeta(5)V^3A_{01} \tag{6.21}$$

• Finally, for the supergravity integrand associated with the graph 3 in Figure 2 we find,

$$\begin{aligned}\mathcal{Z}_3^{(sg)} &= \frac{2\pi^2}{9}\Big[\frac{1}{4}(L_1+L_2+L_3)^2 - \frac{5}{2}(L_1+L_2+L_3)\frac{L_1L_2L_3}{\Delta_L} \\ &\quad + \frac{17}{4}\left(\frac{L_1L_2L_3}{\Delta_L}\right)^2\Big] \\ &= \frac{32\pi^2}{V^2}\left[-\frac{11}{7560}A_{0,0} + \frac{1}{1512}A_{0,2} + \frac{1}{792}A_{11} + \frac{17}{576}A_{2,0}\right]\end{aligned} \tag{6.22}$$

Comparing with the tropical limit of $\mathcal{Z}_3$ in (C.87), we have

$$\mathcal{Z}_3^{(t)} = \mathcal{Z}_3^{(sg)} + \frac{5}{6\pi}\zeta(3)VA_{10} + \frac{11\zeta(3)^2}{8\pi^2}V^4 \tag{6.23}$$

Combining these results, we find that the total supergravity invariant which determines the $D^8\mathcal{R}^4$ coupling is given by

(6.24)
$$\begin{aligned}\mathcal{B}_{(2,0)}^{(sg)} &= \frac{1}{2}(\mathcal{Z}_1^{(sg)} - 2\mathcal{Z}_2^{(sg)} + \mathcal{Z}_3^{(sg)}) \\ &= \frac{2\pi^2}{9}\Big[-\frac{3}{5}\Delta_L + \frac{4}{5}(L_1+L_2+L_3)^2 - \frac{22}{5}(L_1+L_2+L_3)\frac{L_1L_2L_3}{\Delta_L}\end{aligned}$$

$$
\begin{aligned}
&+\frac{32}{5}\left(\frac{L_1L_2L_3}{\Delta_L}\right)^2\Bigg]\\
&=\frac{32\pi^2}{V^2}\left[-\frac{13}{1512}A_{0,0}+\frac{5}{756}A_{0,2}+\frac{5}{396}A_{1,1}+\frac{4}{45}A_{2,0}\right]
\end{aligned}
$$

which agrees (up to an overall normalization convention) with the result in [33]. Comparing with the tropical limit of the string invariant given in (C.90), we find

$$
\begin{aligned}
\mathcal{B}^{(t)}_{(2,0)}=\mathcal{B}^{(sg)}_{(2,0)}+V\,\zeta(3)\,\left[\frac{18}{5\pi}A_{01}+\frac{16}{3\pi}A_{10}\right]+\frac{3}{\pi^3}\zeta(5)V^3A_{01}\\
+(\beta+11)\,\frac{\zeta(3)^2V^4}{16\pi^4}
\end{aligned}
\tag{6.25}
$$

The leading term proportional to $1/V^2$ in the string integrand is therefore exactly reproduced by the supergravity computation. The subleading terms proportional to V, V^3 and V^4 are stringy corrections which can be interpreted as two-loop Feynman diagrams with one gravity vertex (or two vertices) replaced by higher derivative couplings.

Appendix A. Genus-one basics and integration formulas

In this appendix, we summarize various definitions and results for functions and forms on a compact genus-one surface Σ_1, including the volume form κ_1, the Green function g and its successive convolutes g_n, the non-holomorphic Eisenstein series E_a and its associated modular forms $D_{a,b}$. We shall also evaluate various integrals on the genus-one surface with two boundary components Σ_{ab} and reduce them to integrals on Σ_1 which have a smooth limit as $t\to\infty$ and Σ_{ab} tends to Σ_1 with punctures.

A.1. Genus-one differentials and scalar Green function

We parametrize a genus-one surface $\Sigma_1=\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with modulus $\tau\in\mathcal{H}_1$ by a complex coordinate $z=\alpha+\beta\tau$ where $\alpha,\beta\in\mathbb{R}/\mathbb{Z}$. We choose canonical $\mathfrak{A}_1$ and $\mathfrak{B}_1$ homology cycles along the identifications $z\approx z+1$ and $z\approx z+\tau$ respectively, and denote the holomorphic Abelian differential dual to the $\mathfrak{A}_1$-cycle by $\omega_1(z)=dz$. The volume form κ_1 of unit area, and the corresponding "coordinate" Dirac δ-function are as follows,

$$
\kappa_1(z)=\frac{i}{2\tau_2}dz\wedge d\bar z=d\alpha\wedge d\beta \qquad\qquad \delta^{(2)}(z)=\frac{1}{\tau_2}\delta(\alpha)\delta(\beta)
\tag{A.1}
$$

The derivatives with respect to z are related to those with respect to α, β by,

$$\partial_z = \frac{1}{2i\tau_2}(\partial_\beta - \bar\tau\partial_\alpha) \qquad \partial_{\bar z} = -\frac{1}{2i\tau_2}(\partial_\beta - \tau\partial_\alpha) \tag{A.2}$$

while the (negative) Laplace operator in z is given by,

$$\Delta_z = 4\tau_2\partial_{\bar z}\partial_z = \frac{1}{\tau_2}(\partial_\beta - \tau\partial_\alpha)(\partial_\beta - \bar\tau\partial_\alpha) \tag{A.3}$$

The scalar Green function g is defined by,

$$\Delta_z g(z|\tau) = -4\pi\tau_2\delta^{(2)}(z) + 4\pi \qquad \int_{\Sigma_1}\kappa_1(z)g(z|\tau) = 0 \tag{A.4}$$

It may be expressed as a double Fourier series (which converges provided $z \notin \mathbb{Z} + \mathbb{Z}\tau$),

$$g(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|m+n\tau|^2}\, e^{2\pi i(m\beta - n\alpha)} \tag{A.5}$$

or in terms of the Jacobi ϑ-function, and the Dedekind η-function,

$$g(z|\tau) = -\ln\left|\frac{\vartheta_1(z|\tau)}{\eta(\tau)}\right|^2 + \frac{2\pi}{\tau_2}\,(\mathrm{Im}\, z)^2 \tag{A.6}$$

The Green function $g(z|\tau)$ is doubly periodic in z with periods $\mathbb{Z} + \mathbb{Z}\tau$ and is invariant under $SL(2,\mathbb{Z})$ modular transformations, as given in (3.26).

A.2. Kronecker-Eisenstein series and elliptic polylogarithms

Iterated integrals of the scalar Green function and its derivatives give non-holomorphic Eisenstein series and the elliptic polylogarithm functions $D_{a,b}(z|\tau)$ introduced in [51]. In this subsection, we shall provide the precise normalizations of these integrals, and often replace $D_{a,a}$ by a more transparent modular function g_a. These functions are defined as follows using the notation $z = \alpha + \tau\beta$, for $\alpha, \beta \in \mathbb{R}$, and for $b - a \in \mathbb{Z}$,

$$\begin{aligned} g_a(z|\tau) &= \sum_{(m,n)\neq(0,0)} \frac{\tau_2^a\, e^{2\pi i(m\beta - n\alpha)}}{\pi^a|m+n\tau|^{2a}} \\ D_{a,b}(z|\tau) &= \frac{(2i\tau_2)^{a+b-1}}{2\pi i} \sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(m\beta - n\alpha)}}{(m+n\tau)^a(m+n\bar\tau)^b} \end{aligned} \tag{A.7}$$

Clearly, we have $g_1(z|\tau) = g(z|\tau)$ and $g_a(0|\tau) = E_a(\tau)$, and

$$D_{a,a}(z|\tau) = (-4\pi\tau_2)^{a-1} g_a(z|\tau) \tag{A.8}$$

The functions $g_a(z|\tau)$ satisfy the modular transformation law of g in (3.26), while $D_{a,b}$ for $a \neq b$ transforms as a modular form. They satisfy the following integral relations,

$$\begin{aligned} g_{a_1+a_2}(z|\tau) &= \int_{\Sigma_1} \kappa_1(w)\, g_{a_1}(z-w|\tau)\, g_{a_2}(w|\tau) \\ D_{a_1+a_2,b_1+b_2}(z|\tau) &= -4\pi\tau_2 \int_{\Sigma_1} \kappa_1(w)\, D_{a_1,b_1}(z-w|\tau) D_{a_2,b_2}(w|\tau) \end{aligned} \tag{A.9}$$

The functions $g_a(z|\tau)$ and $D_{a,b}(z|\tau)$ satisfy the following differential equations,

$$\begin{aligned} \partial_z^n g_a(z|\tau) &= (2\pi i)^n (-4\pi\tau_2)^{1-a} D_{a,a-n}(z|\tau) \\ \Delta_z g_a(z|\tau) &= -4\pi g_{a-1}(z|\tau) \\ \partial_z D_{a,b}(z|\tau) &= +2\pi i D_{a,b-1}(z|\tau) \\ \partial_{\bar z} D_{a,b}(z|\tau) &= -2\pi i D_{a-1,b}(z|\tau) \\ \Delta_z D_{a,a}(z|\tau) &= 16\pi^2\tau_2 D_{a-1,a-1}(z|\tau) \end{aligned} \tag{A.10}$$

The differential relations given thus far were with respect to the parameter z. Actually, several differential relations with respect to the modulus τ will also be useful in the sequel and will be derived here. The basic differential equation for $D_{a,b}$ in the modulus, from which all others may be deduced, is given by,

$$2i\tau_2 \partial_\tau D_{a,b}(v|\tau) = a D_{a+1,b-1}(v|\tau) + (b-1) D_{a,b}(v|\tau) \tag{A.11}$$

We record the standard normalization the Laplace operator on the upper half plane,

$$\Delta_\tau = 4\tau_2^2 \partial_\tau \partial_{\bar\tau} \tag{A.12}$$

For given α, β the function $g_a(\alpha + \tau\beta|\tau)$ is an eigenfunction of Δ_τ with eigenvalue $a(a-1)$.

A.3. Reducing integrals on Σ_{ab} to integrals on Σ_1

Extracting the power behavior in t of integrals of various products of Green functions is achieved by recasting integrals over the genus-one surface Σ_{ab}

with boundary to a sum over integrals over the compact genus-one surface Σ_1 without boundary. The key result, obtained in Section 3.5 of [22], states the following relation between integrals for $I, J \in \{1, t\}$,

$$\int_{\Sigma_{ab}} \omega_I \wedge \overline{\omega}_J \, \psi = \int_{\Sigma_1} \omega_I \wedge \overline{\omega}_J \, \psi + \mathcal{O}(e^{-2\pi t}) \tag{A.13}$$

provided ψ is smooth near the punctures p_a, p_b and $(I, J) \neq (t, t)$. The relation also holds when $I = J = t$ provided ψ vanishes at both punctures. Several of the integrals below were derived in Section 4.4 of [22]. In the remainder of this appendix, we shall no longer indicate the exponentially suppressed terms, which will always be understood.

A.4. Integrals involving two punctures

We refer to integrals involving two punctures as those whose integration measure is singular at both punctures. The following integrals [22] are valid for any integer $n \geq 0$,

$$\int_{\Sigma_{ab}} \omega_t \wedge \overline{\omega}_1 \, f^n = \int_{\Sigma_{ab}} \omega_t \wedge \overline{\omega_t} \, f^{2n+1} = 0 \tag{A.14}$$

and

$$\int_{\Sigma_{ab}} \omega_t \wedge \overline{\omega_t} \; f^{2n} = -\frac{2i}{2n+1} (2\pi)^{2n} t^{2n+1} \tag{A.15}$$

Throughout, it will be convenient to use the following notation,

$$f_n(z) = g_n(z - p_b) - g_n(z - p_a) \tag{A.16}$$

where for $n = 1$ we recover $f_1(z) = f(z)$. We have the following integrals,

$$\int_{\Sigma_{ab}} \omega_t(z) \wedge \overline{\omega}_1(z) \, g_n(z - w) = \frac{\tau_2}{\pi} \, \partial_w f_{n+1}(w) \tag{A.17}$$

For any function $\psi(z)$ which is smooth on Σ_{ab}, and whose Laplacian $\partial_z \partial_{\bar{z}} \psi(z)$ is smooth on Σ_{ab}, but which does not need to extend to a smooth function at the punctures $z = p_a, p_b$, we have the following integral formula,

(A.18)
$$\int_{\Sigma_{ab}} \omega_t \wedge \overline{\omega_t} \, f^n \psi = - \frac{i \, (2\pi t)^{n+1}}{4\pi^2 (n+1)} \int_0^{2\pi} d\theta \Big(\psi \left(p_b^\theta \right) + (-)^n \psi \left(p_a^\theta \right) \Big)$$

$$- \frac{i\tau_2}{2\pi^2(n+1)(n+2)} \int_{\Sigma_1} \kappa_1(z)\, f(z)^{n+2}\, \partial_z \partial_{\bar z} \psi(z)$$

where p_a^θ, p_b^θ are defined in terms of the variable R as follows,

$$\begin{aligned} -2\ln R &= 2\pi t + \lambda(\tau) + g(v|\tau) \\ p_a^\theta &= p_a + R\, e^{i\theta} \\ p_b^\theta &= p_b + R\, e^{i\theta} \end{aligned} \tag{A.19}$$

and λ is given by,

$$\begin{aligned} g(z|\tau) &= -\ln|z|^2 - \lambda(\tau) + \mathcal{O}(z) \\ \lambda(\tau) &= \ln\left|2\pi\eta(\tau)^2\right|^2 \end{aligned} \tag{A.20}$$

In particular, the following special cases will be used in the sequel,

$$\begin{aligned} &\int_{\Sigma_{ab}} \omega_t(z) \wedge \overline{\omega}_t(z)\, f(z)\, g(z,w) = -i\pi t^2 f(w) + \frac{i}{12\pi} f(w)^3 \\ &\int_{\Sigma_{ab}} \omega_t(z) \wedge \overline{\omega}_t(z)\, f(z)^{2n}\, g(z,w) \\ &\quad = -\frac{i\,(2\pi t)^{2n+1}}{4\pi^2(2n+1)} \int_0^{2\pi} d\theta \Big(g(w,p_a^\theta) + g(w,p_b^\theta) \Big) \\ &\qquad - \frac{i\,(2n)!}{2\pi} F_{2n+2} + \frac{i\, f(w)^{2n+2}}{4\pi(2n+1)(n+1)} \end{aligned} \tag{A.21}$$

When ψ extends to a smooth function at the punctures, the function ψ inside the θ-integral in (A.18) is constant up to exponentially suppressed corrections, and simplifies as follows,

$$\begin{aligned} \int_{\Sigma_{ab}} \omega_t \wedge \overline{\omega}_t\, f^n \psi = &- i\, \frac{(2\pi)^n t^{n+1}}{n+1} \Big(\psi(p_b) + (-)^n \psi(p_a) \Big) \\ &- \frac{i\tau_2}{2\pi^2(n+1)(n+2)} \int_{\Sigma_1} \kappa_1(z)\, f(z)^{n+2}\, \partial_z \partial_{\bar z} \psi(z) \end{aligned} \tag{A.22}$$

A.5. Integrals involving at most one puncture

We refer to integrals involving at most one puncture as those whose integrand is singular at most at only one puncture. The following integrals are

valid for any integer $n \geq 0$,

$$\frac{\tau_2}{\pi}\int_{\Sigma_{ab}} \kappa_1(z)|\partial_z g(z,p_a)|^2\, g(z,p_a)^{n-1} = \frac{1}{n}\Big(T^n - D_n\Big) \tag{A.23}$$

where we have used the parameter T, defined by,

$$T = -2\ln R - \lambda = 2\pi t + g(v) \tag{A.24}$$

We also use the following integrals,

(A.25)

$$\frac{\tau_2}{\pi}\int_{\Sigma_1} \kappa_1(z)|\partial_z g(z,p_a)|^2\Big(g(z,p_b)^2 - g(p_a,p_b)^2\Big) = -D_3^{(1)}(v) - 2D_4^{(a)}(v)$$

$$\frac{\tau_2}{\pi}\int_{\Sigma_{ab}} \kappa_1(z)\partial_{\bar z} g(z,p_a)\partial_z g(z,p_b) g(z,p_a)^{n-1} = \frac{1}{n}\Big(g(v)^n - D_n\Big)$$

where the function $D_n^{(1)}(v)$ was defined in (3.29) and $D_4^{(a)}(v)$ is defined by,

$$D_4^{(a)}(z|\tau) = \frac{\tau_2}{\pi}\int_{\Sigma_1} \kappa_1(x) g(z+x)\partial_x g(z+x) g(x)\partial_{\bar x} g(x) \tag{A.26}$$

It may be expressed as the Laplacian in the variable z of the function $D_4^{(2)}$ as follows,

$$\Delta_z D_4^{(2)}(z|\tau) = -16\pi D_4^{(a)}(z|\tau) \tag{A.27}$$

Alternatively, it may also be expressed in terms of the Laplacian in z of F_4, using the above identity, (3.32), and (3.30), and we find,

$$D_4^{(a)}(v) = \frac{1}{3}g_1^3 - \frac{1}{3}D_3 - \frac{1}{4\pi}\Delta_v F_4(v) \tag{A.28}$$

It is in this form that we shall present the final results involving $D_4^{(a)}$.

Furthermore, we have the following integrals for $n \geq 1$ which involve the Green function $g(z,w)$ at a generic point w on Σ_{ab},

$$\begin{aligned}\frac{\tau_2}{\pi}\int_{\Sigma_{ab}} &\kappa_1(z)|\partial_z g(z,p_a)|^2 g(z,p_a)^{n-1} g(z,w) \\ &= \frac{1}{n(n+1)}\Big(D_{n+1} - g(w,p_a)^{n+1}\Big) - \frac{1}{n}D_{n+1}^{(1)}(w-p_a)\end{aligned} \tag{A.29}$$

$$+\frac{T^n}{2\pi n}\int_0^{2\pi} d\theta\, g(w,p_a^\theta)$$

In the special case $n=1$, we have $D_2=E_2$ and $D_2^{(1)}(w-p_a)=g_2(w-p_a)$.

Appendix B. Non-separating degeneration of Z_i and $\mathcal{Z}_i$

In this appendix we will present some of the core calculations of this paper and compute the three contributions to $\mathcal{B}_{(2,0)}$ defined in (2.11), (2.12) in terms of the functions $\mathcal{Z}_i$ and in (3.17) and (3.18) in terms of the functions Z_i. In the process, we shall evaluate the intermediate functions of (3.19) as well. The first ingredient in this evaluation is the relation (3.9) between the Green function G on the genus-two Riemann surface Σ, in terms of which the integrals in Z_i are expressed, and its representation in terms of the genus-one surface Σ_{ab}, which we repeat here for convenience,

$$G(x,y)=g(x,y)+\frac{1}{8\pi t}\Big(f(x)-f(y)\Big)^2+\mathcal{O}(e^{-2\pi t}) \tag{B.1}$$

The second ingredient is the analogous expression for the integration measure, which may be decomposed in terms of the following factors,

$$|\Delta(z_i,z_j)|^2=|\omega_1(z_i)\wedge\omega_t(z_j)-\omega_t(z_i)\wedge\omega_1(z_j)|^2=\nu_{ij}^- - \nu_{ij}^+ \tag{B.2}$$

where the forms $\nu_{ij}^\pm$ have been separated according to their parity in the form ω_t and its complex conjugate at each point. These forms are given explicitly by,[9]

$$\begin{aligned}\nu_{ij}^+&=\omega_1(i)\wedge\overline{\omega}_1(i)\wedge\omega_t(j)\wedge\overline{\omega}_t(j)+\omega_t(i)\wedge\overline{\omega}_t(i)\wedge\omega_1(j)\wedge\overline{\omega}_1(j)\\ \nu_{ij}^-&=\omega_1(i)\wedge\overline{\omega}_t(i)\wedge\omega_t(j)\wedge\overline{\omega}_1(j)+\omega_t(i)\wedge\overline{\omega}_1(i)\wedge\omega_1(j)\wedge\overline{\omega}_t(j)\end{aligned} \tag{B.3}$$

The third ingredient is the representation of the genus-two Arakelov Kähler form κ in terms of data on Σ_{ab}, given by (3.8) which we repeat here for convenience,

$$\kappa=\frac{1}{2}\kappa_1+\frac{i}{4t}\omega_t\wedge\bar{\omega}_t+\mathcal{O}(e^{-2\pi t}) \qquad \kappa_1=\frac{i}{2\tau_2}\omega_1\wedge\overline{\omega}_1 \tag{B.4}$$

[9]For the sake of brevity, we shall often abbreviate the points z_i by i in the arguments of functions and forms, and we shall omit the wedge in the product of forms.

The determinant is given by $\det Y = t\tau_2$. Finally, we shall extract the t-power dependence of the integrals over Σ_{ab} and cast the result in terms of a Laurent polynomial in t with coefficients given by convergent integrals over Σ_1. A very useful tool will be Stokes theorem on the surface Σ_{ab} for a $(1,0)$ form $\omega = \omega_z(z)dz$, formulated as follows,

(B.5)
$$\int_{\Sigma_{ab}} \kappa_1(z)\,\partial_{\bar z}\omega_z(z) = -\frac{i}{2\tau_2}\oint_{\partial\Sigma_{ab}} dz\,\omega_z(z) = \frac{i}{2\tau_2}\left(\oint_{\mathfrak{C}_a} + \oint_{\mathfrak{C}_b}\right)\omega_z(z)dz$$

We shall carry out these procedures in increasing order of difficulty and complexity. We begin with Z_3, then Z_2 and finally compute Z_1.

B.1. Degeneration of Z_3

We start with the simplest modular graph function Z_3 defined in (3.17), which corresponds to the disconnected diagram on the right of Figure 2. Using (B.1), it will be convenient to expand Z_3, in terms of the number of f functions, into a sum of 3 terms,

(B.6)
$$\begin{aligned}
Z_3^{(a)} &= \frac{1}{8t^2\tau_2^2}\int_{\Sigma^4_{ab}} |\Delta(1,3)\Delta(2,4)|^2\, g(1,2)\, g(3,4)\\
Z_3^{(b)} &= \frac{1}{32\pi t^3\tau_2^2}\int_{\Sigma^4_{ab}} |\Delta(1,3)\Delta(2,4)|^2\, g(1,2)\,\big(f(3)-f(4)\big)^2\\
Z_3^{(c)} &= \frac{1}{512\pi^2 t^4\tau_2^2}\int_{\Sigma^4_{ab}} |\Delta(1,3)\Delta(2,4)|^2\,\big(f(1)-f(2)\big)^2\,\big(f(3)-f(4)\big)^2
\end{aligned}$$

where it is understood that Δ is expressed in terms of ω_1 and ω_t using (B.2). We will start with the last of these integrals since it is the simplest.

B.1.1. Evaluating $Z_3^{(c)}$ Thanks to the property (A.14), the contributions of ν_{13}^- and ν_{24}^- in (B.2) integrate to zero. Further using symmetry and the property that terms linear in f integrate to zero against either $|\omega_1|^2$ or $|\omega_t|^2$ (see (A.14) and (A.15)), we can replace $(f(1)-f(2))^2(f(3)-f(4))^2$ appearing in $Z_3^{(c)}$ by $2(f(1)^2+f(2)^2)f(3)^2$ to obtain,

(B.7)
$$Z_3^{(c)} = \frac{1}{256\pi^2 t^4\tau_2^2}\int_{\Sigma^4_{ab}} \nu_{13}^+\,\nu_{24}^+\Big(f(1)^2+f(2)^2\Big)f(3)^2$$

The integral over the point 4 can be computed using (A.14) and (A.15),

$$\int_{\Sigma^{(4)}_{ab}} \nu^+_{24} = -2it|\omega_1(2)|^2 - 2i\tau_2|\omega_t(2)|^2 = -8t\tau_2\kappa(2) \tag{B.8}$$

Using the function F_ℓ of (3.11) and (3.12) as well as equation (A.15) to compute the remaining integrals successively, we arrive at,

$$Z^{(c)}_3 = \frac{\pi^2 t^2}{9} + F_2 + \frac{F_2^2}{4\pi^2 t^2} + \mathcal{O}(e^{-2\pi t}) \tag{B.9}$$

where $F_2(v) = E_2 - g_2(v)$ as is familiar by now.

B.1.2. Evaluating $Z^{(b)}_3$ In $Z^{(b)}_3$, the contributions of ν^-_{13} and ν^-_{24} similarly integrate to zero. Using symmetry again and the fact that terms linear in $f(3)$ and $f(4)$ integrate to zero to replace $(f(3) - f(4))^2$ by $2f(3)^2$. In this way we find,

$$Z^{(b)}_3 = \frac{1}{16\pi t^3 \tau_2^2} \int_{\Sigma^4_{ab}} \nu^+_{13}\, \nu^+_{24}\, f(3)^2\, g(1,2) \tag{B.10}$$

Integrating over point z_4 using (B.8), we have,

$$Z^{(b)}_3 = -\frac{1}{2\pi t^2 \tau_2} \int_{\Sigma^3_{ab}} \nu^+_{13}\, \kappa(2)\, f(3)^2\, g(1,2) \tag{B.11}$$

The part proportional to $\kappa_1(1)$ in ν^+_{13} and the part proportional to $\kappa_1(2)$ in $\kappa(2)$ integrate to zero. Taking these simplifications into account, the measure in point 3 is proportional to $\kappa_1(3)$ and the integral over this point may be performed, simplifying the result to give,

$$Z^{(b)}_3 = -\frac{F_2}{2\pi t^3} \int_{\Sigma^2_{ab}} \omega_t(z)\, \overline{\omega}_t(z)\, \omega_t(w)\, \overline{\omega}_t(w)\, g(z,w) \tag{B.12}$$

To evaluate the integrals, we use (A.15) and (A.21) to arrive at,

$$Z^{(b)}_3 = \frac{2}{3}F_2 + \frac{2gF_2}{\pi t} + \frac{F_2^2}{\pi^2 t^2} + \mathcal{O}(e^{-2\pi t}) \tag{B.13}$$

where g is shorthand for $g = g(v)$.

B.1.3. Evaluating $Z_3^{(a)}$ The computation of $Z_3^{(a)}$ is slightly more complicated. In contrast with the previous two cases, it is now the contributions from ν_{13}^+ and ν_{24}^+ that integrate to zero against the Green functions $g(1,2)g(3,4)$. The remaining contribution is given as follows,

$$Z_3^{(a)} = \frac{1}{8t^2\tau_2^2}\int_{\Sigma_{ab}^4} \nu_{13}^-\,\nu_{24}^-\,g(1,2)\,g(3,4) \tag{B.14}$$

This integral is manifestly convergent when extended to the punctures, so that Σ_{ab} may be replaced by Σ_1, up to exponentially suppressed corrections which we neglect. We carry out the integrals over the points 2 and 4 using the following relation (A.17) and its complex conjugate, for the special case $n = 1$. The contributions arising from the two terms in ν_{24}^- are pairwise equal, and we may simplify the result as follows,

$$Z_3^{(a)} = -\frac{1}{4\pi^2t^2}\int_{\Sigma_{ab}^2} \nu_{zw}^-\,\partial_{\bar z}f_2(z)\,\partial_w f_2(w) \tag{B.15}$$

where z and w respectively stand for the point z_1 and z_3, and f_n was defined in (A.16). To evaluate the remaining integrations over the points 1 and 3, we use (A.17), and we find,

$$Z_3^{(a)} = \frac{F_2^2}{\pi^2t^2} + \frac{\tau_2^2}{\pi^4t^2}\left|\partial_w^2 f_3(w)\right|^2\Big|_{w=p_a} \tag{B.16}$$

Expressing the result in terms of $D_{3,1}$ using (A.10), we have,

$$Z_3^{(a)} = \frac{F_2^2}{\pi^2t^2} + \frac{1}{16\pi^4t^2\tau_2^2}\Big|D_{3,1}(v|\tau) - D_{3,1}(0|\tau)\Big|^2 \tag{B.17}$$

Using the differential relation (A.11) for $a = b = 2$, and a suitable rearrangement formula,

$$\begin{aligned} D_{3,1}(v|\tau) - D_{3,1}(0|\tau) &= 4\pi i\tau_2^2\partial_\tau F_2(v|\tau) \\ \Delta_\tau F_2^2 - 4F_2^2 &= 8\tau_2^2|\partial_\tau F_2|^2 \end{aligned} \tag{B.18}$$

we simplify the final expression for $Z_3^{(a)}$ as follows,

$$Z_3^{(a)} = \frac{4F_2^2 + \Delta_\tau F_2^2}{8\pi^2t^2} \tag{B.19}$$

Collecting the three contributions $Z_3^{(a,b,c)}$, we arrive at our final formula,

$$Z_3 = \frac{\pi^2 t^2}{9} + \frac{5F_2}{3} + \frac{2gF_2}{\pi t} + \frac{\Delta_\tau F_2^2 + 14F_2^2}{8\pi^2 t^2} + \mathcal{O}(e^{-2\pi t}) \tag{B.20}$$

B.2. Degeneration of Z_2

The modular graph function Z_2 is defined in (3.17) and corresponds to the L-shape diagram of Figure 2. Using (3.9), one decomposes Z_2 into a sum of 3 terms,

(B.21)

$$\begin{aligned}
Z_2^{(a)} &= \frac{1}{8t^2\tau_2^2}\int_{\Sigma_{ab}^4} |\Delta(1,3)\Delta(2,4)|^2\, g(1,2)\, g(1,4) \\
Z_2^{(b)} &= \frac{1}{32\pi t^3\tau_2^2}\int_{\Sigma_{ab}^4} |\Delta(1,3)\Delta(2,4)|^2\, g(1,2)\, \big(f(1)-f(4)\big)^2 \\
Z_2^{(c)} &= \frac{1}{512\pi^2 t^4\tau_2^2}\int_{\Sigma_{ab}^4} |\Delta(1,3)\Delta(2,4)|^2\, \big(f(1)-f(2)\big)^2\, \big(f(1)-f(4)\big)^2
\end{aligned}$$

We proceed to evaluating these integrals again in order of increasing difficulty.

B.2.1. Evaluating $Z_2^{(c)}$ The contributions from ν_{13}^- and ν_{24}^- vanish upon integrating with respect to points 3 and 4. As a result, and using the symmetries of the integrand, the integral reduces to,

$$Z_2^{(c)} = \frac{1}{8^3\pi^2 t^4\tau_2^2}\int_{\Sigma_{ab}^4} \nu_{13}^+\,\nu_{24}^+\Big(f(1)^4 + 2f(1)^2 f(4)^2 + f(2)^2 f(4)^2\Big) \tag{B.22}$$

The integral over point 3 may be carried out with the help of (B.8) while the integral over point 4 can be computed using (A.15). Performing also the integrals over the remaining points 1 and 2, we find,

$$Z_2^{(c)} = \frac{14\pi^2 t^2}{45} + \frac{2}{3}F_2 + \frac{6F_4 + F_2^2}{4\pi^2 t^2} + \mathcal{O}(e^{-2\pi t}) \tag{B.23}$$

where F_4 was defined in (3.11) and given explicitly in (3.32).

B.2.2. Evaluating $Z_2^{(b)}$ The contributions from ν_{13}^- and ν_{24}^- similarly integrate to zero in $Z_2^{(b)}$. Using (A.15) we find,

$$Z_2^{(b)} = \frac{1}{32\pi t^3\tau_2^2}\int_{\Sigma_{ab}^4} \nu_{13}^+\,\nu_{24}^+\left(f(1)^2+f(4)^2\right)g(1,2) \tag{B.24}$$

The integral over point 3 may be performed using (B.8), while the one over point 4 may be performed using (A.15), and we find,

(B.25)
$$Z_2^{(b)} = \frac{2}{\pi t}\int_{\Sigma_{ab}^2}\kappa(1)\kappa(2)f(1)^2g(1,2) + \frac{iF_2}{\pi t^2}\int_{\Sigma_{ab}^2}\kappa(1)\omega_t(2)\overline{\omega}_t(2)\,g(1,2)$$

The contribution of κ_1 in κ cancels out for point 2 in the first integral and point 1 in the second integral. The remaining integrals may be evaluated using (A.21) for both the integrals in points 1 and 2, and we find,

(B.26)
$$Z_2^{(b)} = \frac{3\pi^2t^2}{10} + \frac{2}{3}\pi tg + \frac{F_2}{2} + \frac{D_3 - D_3^{(1)} + 2gF_2}{2\pi t} + \frac{3F_2^2 - 5F_4}{4\pi^2t^2} + \mathcal{O}(e^{-2\pi t})$$

The function $D_3^{(1)}$ was defined in (3.29), while F_2 and F_4 were given in (3.11).

B.2.3. Evaluating $Z_2^{(a)}$ The contribution from this integral is simplified by integrating over the point 3, and we have,

$$Z_2^{(a)} = \frac{i}{4t^2\tau_2^2}\int_{\Sigma_{ab}^3}\left(t\,|\omega_1(1)|^2 + \tau_2\,|\omega_t(1)|^2\right)\left(\nu_{24}^- - \nu_{24}^+\right)g(1,2)g(1,4) \tag{B.27}$$

We begin by carrying out the integrals over points 2 and 4. The terms proportional to $|\omega_1(2)|^2$ and $|\omega_1(4)|^2$ in ν_{24}^+ integrate to zero in view of the normalization of g, leaving only the contribution from ν_{24}^-. To evaluate the integrals over the points 2 and 4, we use the relation derived earlier in (A.17) for $n=1$, and its complex conjugate, and we find,

$$\begin{aligned} Z_2^{(a)} = &-\frac{i}{2\pi^2 t}\int_{\Sigma_{ab}}\omega_1(w)\overline{\omega}_1(w)|\partial_w f_2(w)|^2 \\ &-\frac{i\tau_2}{2\pi^2t^2}\int_{\Sigma_{ab}}\omega_t(w)\overline{\omega_t(w)}|\partial_w f_2(w)|^2 \end{aligned} \tag{B.28}$$

where $w = z_1$ represents the remaining integration point 1. The first integral is evaluated by integrating by parts in w, and using the following Laplacian relation,

$$\tau_2 \partial_{\bar{w}} \partial_w f_2(w) = -\pi f(w) \tag{B.29}$$

The remaining integral is carried out using the definitions of E_3 and g_3, and we find,

$$Z_2^{(a)} = -\frac{2}{\pi t}\Big(E_3(\tau) - g_3(v|\tau)\Big) - \frac{i\tau_2}{2\pi^2 t^2}\int_{\Sigma_{ab}} \omega_t(w)\,\overline{\omega}_t(w)\,|\partial_w f_2(w)|^2 \tag{B.30}$$

To evaluate the remaining integral we use (A.18) for $n = 0$ and $\psi(w) = |\partial_w f_2(w)|^2$. Since ψ is regular at the punctures, we may use the simplified formula (A.22). We evaluate $|\partial_w f_2(w)|^2$ at the punctures in terms of the function $D_{2,1}$, by using the first line of (A.10). Taking into account the fact that $D_{2,1}(0|\tau) = 0$, we find,

$$\begin{aligned} &\int_{\Sigma_{ab}} \omega_t(w)\,\overline{\omega}_t(w)\,|\partial_w f_2(w)|^2 \\ &\quad = -2it|\partial_v g_2(v)|^2 - \frac{i\tau_2}{4\pi^2}\int_{\Sigma_{ab}} \kappa_1(w) f(w)^2 \partial_w \partial_{\bar{w}} |\partial_w f_2|^2 \end{aligned} \tag{B.31}$$

The Laplacian of ψ may be simplified with the help of (B.29) and is given by,

$$\begin{aligned} \partial_w \partial_{\bar{w}} \psi(w) = \frac{\pi^2}{\tau_2^2} f(w)^2 + |\partial_w^2 f_2(w)|^2 - \frac{\pi}{\tau_2}\partial_w f_2(w)\,\partial_{\bar{w}} f(w) \\ - \frac{\pi}{\tau_2}\partial_{\bar{w}} f_2(w)\,\partial_w f(w) \end{aligned} \tag{B.32}$$

Integrating by parts in the last two terms above so as to regroup in terms of $\partial_w \partial_{\bar{w}} f_2(w)$ and then using (B.29), the integral in (B.31) takes the following form,

$$\int_{\Sigma_{ab}} \kappa_1(w) f(w)^2 \partial_w \partial_{\bar{w}} |\partial_w f_2|^2 = \frac{8\pi^2}{\tau_2^2} F_4 + \int_{\Sigma_{ab}} \kappa_1(w) f(w)^2 |\partial_w^2 f_2|^2 \tag{B.33}$$

Combining the derivative relations in $\partial_w^2 g_2$ and $\partial_\tau g$ in (A.10), we obtain,

$$\partial_w^2 f_2(w) = 2\pi i \partial_\tau f(w) \tag{B.34}$$

Using furthermore the relation $\partial_{\bar\tau}\partial_\tau f(w)=0$, we may rearrange the integral as follows,

$$\int_{\Sigma_{ab}} \kappa_1(w) f(w)^2 \partial_w\partial_{\bar w}|\partial_w f_2|^2 = \frac{8\pi^2}{\tau_2^2}F_4 + \frac{2\pi^2}{\tau_2^2}\Delta_\tau F_4 \tag{B.35}$$

Using also the relation,

$$8\tau_2|\partial_v g_2(v)|^2 = \Delta_v F_2(v)^2 - 8\pi g(v)F_2(v) \tag{B.36}$$

we obtain the following expression,

$$Z_2^{(a)} = \frac{1}{\pi t}(2g_3 - 2E_3 + gF_2) - \frac{\Delta_v F_2^2}{8\pi^2 t} - \frac{(\Delta_\tau+4)F_4}{4\pi^2 t^2} + \mathcal{O}(e^{-2\pi t}) \tag{B.37}$$

Collecting the three contributions $Z_2^{(a,b,c)}$, we arrive at our final result,

$$\begin{aligned} Z_2 = {} & \frac{11\pi^2 t^2}{18} + \frac{2g\pi t}{3} + \frac{7F_2}{6} \\ & + \frac{1}{2\pi t}\left(D_3 - D_3^{(1)} + 4gF_2 + 4g_3 - 4E_3 - \frac{\Delta_v F_2^2}{4\pi}\right) \\ & + \frac{4F_2^2 - (\Delta_\tau+3)F_4}{4\pi^2 t^2} + \mathcal{O}(e^{-2\pi t}) \end{aligned} \tag{B.38}$$

B.3. Degeneration of Z_1

The modular graph function Z_1 was defined in (3.17) and corresponds to the one-loop graph on the left of Figure 2. Using (3.9), one finds that Z_1 decomposes into a sum of 3 terms,

$$\begin{aligned} Z_1^{(a)} &= 8\int_{\Sigma_{ab}^2} \kappa(1)\,\kappa(2)\,g(1,2)^2 \\ Z_1^{(b)} &= \frac{2}{\pi t}\int_{\Sigma_{ab}^2} \kappa(1)\,\kappa(2)\,g(1,2)\,\big(f(1)-f(2)\big)^2 \\ Z_1^{(c)} &= \frac{1}{8\pi^2 t^2}\int_{\Sigma_{ab}^2} \kappa(1)\,\kappa(2)\,\big(f(1)-f(2)\big)^4 \end{aligned} \tag{B.39}$$

The contributions $Z_1^{(c)}$ and $Z_1^{(b)}$ are routine, but $Z_1^{(a)}$ will involve some rather serious analysis.

B.3.1. Evaluating $Z_1^{(c)}$ Substituting κ by its expression on Σ_{ab}, given in (3.8), the integrals in (B.39) can be evaluated successively using (A.15), and we find,

$$Z_1^{(c)} = \frac{11\pi^2 t^2}{15} + F_2 + \frac{12F_4 + 3F_2^2}{4\pi^2 t^2} + \mathcal{O}(e^{-2\pi t}) \tag{B.40}$$

B.3.2. Evaluating $Z_1^{(b)}$ Using symmetry under the exchange of the points 1 and 2, $Z_1^{(b)}$ may be decomposed into a sum of two terms,

$$Z_1^{(b,1)} = \frac{4}{\pi t} \int_{\Sigma_{ab}^2} \kappa(1)\,\kappa(2) g(1,2)\, f(1)^2 \tag{B.41}$$

$$Z_1^{(b,2)} = -\frac{4}{\pi t} \int_{\Sigma_{ab}^2} \kappa(1)\,\kappa(2)\, g(1,2)\, f(1) f(2) \tag{B.42}$$

The integral over point 2 in $Z_1^{(b,1)}$ may be computed using (A.21),

$$\begin{aligned} Z_1^{(b,1)} = \frac{1}{2\pi t^2} \int_{\Sigma_{ab}} \kappa(z) f(z)^2 \\ \times \left(F_2(v) - \frac{1}{2} f(z)^2 + \frac{t}{\pi} \int_0^{2\pi} d\theta \Big(g(z, p_a^\theta) + g(z, p_b^\theta) \Big) \right) \end{aligned} \tag{B.43}$$

where the quantities p_a^θ and p_b^θ were introduced in (A.19). The remaining integrals in z may be computed using (A.21) again, and we find,

$$Z_1^{(b,1)} = \frac{3\pi^2 t^2}{5} + \frac{4\pi t}{3} g + \frac{F_2}{3} + \frac{D_3 - D_3^{(1)}}{\pi t} + \frac{F_2^2 - 5F_4}{2\pi^2 t^2} \tag{B.44}$$

The second integral $Z_1^{(b,2)}$ is a sum of three terms obtained by decomposing each κ into its κ_1 and its $\omega_t \wedge \overline{\omega}_t$ parts. The resulting integrals may be carried out using (A.21),

$$Z_1^{(b,2)} = -\frac{8}{15}\pi^2 t^2 - 2F_2 + \frac{2}{\pi t}(g_3 - E_3) + \frac{2F_4}{\pi^2 t^2} \tag{B.45}$$

Combining these results with the contribution (B.44) from $Z_1^{(b,2)}$, we find,

$$Z_1^{(b)} = \frac{\pi^2 t^2}{15} + \frac{4\pi t}{3} g - \frac{5}{3} F_2 + \frac{D_3 - D_3^{(1)} - 2E_3 + 2g_3}{\pi t} + \frac{F_2^2 - F_4}{2\pi^2 t^2} \tag{B.46}$$

B.3.3. Evaluating $Z_1^{(a)}$ We finally come to the most difficult part of the computation, namely the evaluation of $Z_1^{(a)}$. Substituting the volume form κ by (B.4), we decompose $Z_1^{(a)}$ as follows,

$$Z_1^{(a)} = 2\int_{\Sigma_{ab}^2} \kappa_1(1)\, g(1,2)^2 \left(\kappa_1(2) + \frac{i}{t}\omega_t(2)\,\overline{\omega_t(2)}\right) + \frac{\mathcal{K}}{8\pi^2 t^2} \tag{B.47}$$

where $\mathcal{K}$ is given by,

$$\mathcal{K} = -4\pi^2 \int_{\Sigma_{ab}^2} \omega_t(1)\overline{\omega_t(1)}\omega_t(2)\overline{\omega_t(2)}\, g(1,2)^2 \tag{B.48}$$

The factor of $8\pi^2 t^2$ has been extracted for later convenience. Carrying out the integral over point 1 in the integral in the first term of (B.47) gives a result that is independent of point 2, namely E_2. Performing the remaining integral, we find,

$$Z_1^{(a)} = 6E_2 + \frac{\mathcal{K}}{8\pi^2 t^2} \tag{B.49}$$

The degeneration of $\mathcal{K}$ is complicated, but the calculation of its variation in t up to exponentially suppressed corrections may be obtained using the variational method developed in Section 3.6 of [22] and is relatively simple. Therefore, we shall split the function $\mathcal{K}$ into a sum of two contributions,

$$\mathcal{K} = \mathcal{K}^c + \mathcal{K}^t + \mathcal{O}(e^{-2\pi t}) \tag{B.50}$$

where $\mathcal{K}^c$ is independent of t, and $\mathcal{K}^t$ is a polynomial in t of degree four, with vanishing constant term. The variational method will allow us to compute $\mathcal{K}^t$ completely in the next subsection but does not give us access to $\mathcal{K}^c$, which will have to be computed by other methods in the subsequent subsection. The result for $\mathcal{K}^t$ will be found to be,

$$\frac{\mathcal{K}^t}{8\pi^2 t^2} = \frac{2\pi^2 t^2}{3} + \frac{4\pi t}{3} g + 2g^2 + \frac{1}{\pi t}\left(D_3 - D_3^{(1)} + 2\zeta(3) + \frac{\Delta_v F_4}{2\pi}\right) \tag{B.51}$$

Collecting the three contributions $Z_1^{(a)}, Z_1^{(b)}, Z_1^{(c)}$, we arrive at the final formula,

$$Z_1 = \frac{22\pi^2 t^2}{15} + \frac{8\pi t}{3} g + 6E_2 - \frac{2}{3}F_2 + 2g^2 \tag{B.52}$$

$$+\frac{2}{\pi t}\left(g_3 - D_3^{(1)} + 2\zeta(3) + \frac{\Delta_v F_4}{2\pi}\right)$$
$$+\frac{\mathcal{K}^c + 20F_4 + 10F_2^2}{8\pi^2 t^2} + \mathcal{O}(e^{-2\pi t})$$

where $\mathcal{K}^c$ is independent of t. Next, we proceed to the calculation of $\mathcal{K}^t$ using the variational method the next subsection and then of the constant contribution $\mathcal{K}^c$ in subsection B.5.

B.4. Variational calculation of $\mathcal{K}^t$

In this subsection, we shall calculate the variation $\delta\mathcal{K}^t = \delta\mathcal{K}$ under an infinitesimal variation δt holding all other moduli fixed.[10] We begin by recasting the defining formula (B.48) for $\mathcal{K}$ in the following form,

$$\mathcal{K} = \frac{\tau_2^2}{\pi^2}\int_{\Sigma_{ab}} \kappa_1(w) \int_{\Sigma_{ab}} \kappa_1(z)\, |\partial_z f(z)|^2\, |\partial_w f(w)|^2\, g(w,z)^2 \tag{B.53}$$

The integrand is independent of t, so that all t-dependence arises from the dependence on t of the integration domain, $\Sigma_{ab} = \{z \in \Sigma_1,\, |f(z)| \leq 2\pi t\}$. As a result, $\delta\mathcal{K}$ is given entirely by the effects of varying the integration regions for both z and w (which contribute equally) with t, and we have,

$$\delta\mathcal{K} = \frac{2\tau_2^2}{\pi^2}\int_{\Sigma_{ab}} \kappa_1(z)|\partial_z f|^2 \left[\int_{\delta\mathfrak{D}_a \cup \delta\mathfrak{D}_b} \kappa_1(w)|\partial_w f|^2 g(z,w)^2\right] \tag{B.54}$$

The infinitesimal integration domains $\delta\mathfrak{D}_a, \delta\mathfrak{D}_b$ are defined as follows,

$$\begin{aligned} \delta\mathfrak{D}_a &= \{w \in \Sigma_1,\, -2\pi(t+\delta t) \leq f(w) \leq -2\pi t\} \\ \delta\mathfrak{D}_b &= \{w \in \Sigma_1,\, +2\pi t \leq f(w) \leq 2\pi(t+\delta t)\} \end{aligned} \tag{B.55}$$

The w-integrals may be simplified as follows. We begin with the contribution from $\delta\mathfrak{D}_b$, the one from $\delta\mathfrak{D}_a$ being analogous. We parametrize w in $\delta\mathfrak{D}_b$ as follows,

$$2\pi t \leq g(w,p_b) - g(w,p_a) \leq 2\pi(t+\delta t) \tag{B.56}$$

[10] Throughout, we shall neglect all contributions which are exponentially suppressed in t.

Up to exponential corrections, which we neglect, $g(w,p_a)$ equals $g(p_b,p_a) = g(v)$ for $w \in \delta\mathfrak{D}_b$. Furthermore, the Green function in the funnel is given by (A.20),

$$g(w,p_b) = -\ln|w-p_b|^2 - \lambda + \mathcal{O}(w-p_b) \tag{B.57}$$

where λ was defined as well. In terms of the variable R introduced in (A.19), the domain $\delta\mathfrak{D}_b$ consists of the points w restricted by,

$$R\,e^{-\pi\delta t} \leq |w-p_b| \leq R \tag{B.58}$$

and may be parametrized by two real coordinates x, y,

$$w = p_b + R\,e^{-x-i\theta} \qquad 0 \leq x \leq \pi\delta t \qquad 0 \leq \theta \leq 2\pi \tag{B.59}$$

With this parametrization, the integral in x may be evaluated at $x=0$, so that we find the simplified formulas,

$$\tau_2 \int_{\delta\mathfrak{D}_{a,b}} \kappa_1(w)|\partial_w f|^2 g(z,w)^2 = \pi\delta t \int_0^{2\pi} d\theta\, g(z,p^\theta_{a,b})^2 \tag{B.60}$$

To evaluate the z-integrals, we split up the calculation of $\delta\mathcal{K}$ into three parts,

$$\delta\mathcal{K} = \delta\mathcal{K}_{(m)} + \delta\mathcal{K}_{(a)} + \delta\mathcal{K}_{(b)} \tag{B.61}$$

where

$$\begin{aligned} \delta\mathcal{K}_{(m)} &= 4\tau_2\,\delta t \int_{\Sigma_{ab}} \kappa_1(z)|\partial_z f|^2 \Big(g(z,p_a)^2 + g(z,p_b)^2\Big) \\ \delta\mathcal{K}_{(a,b)} &= \frac{2\tau_2}{\pi}\,\delta t \int_{\Sigma_{ab}} \kappa_1(z)|\partial_z f|^2 \int_0^{2\pi} d\theta \Big(g(z,p^\theta_{a,b})^2 - g(z,p_{a,b})^2\Big) \end{aligned} \tag{B.62}$$

The purpose of this rearrangement is to simplify the integrand for the most complicated part of the calculation, namely in $\delta\mathcal{K}_{(m)}$, and be left with $\delta\mathcal{K}_{(a)}, \delta\mathcal{K}_{(b)}$ which receive contributions only from the funnel parts. We shall now evaluate each part in turn.

B.4.1. Calculating $\delta\mathcal{K}_{(m)}$ We begin by expanding the factor $|\partial_z f|^2$,

$$\delta\mathcal{K}_{(m)} = 4\tau_2\,\delta t \int_{\Sigma_{ab}} \kappa_1(z)|\partial_z g(z,p_a)|^2 \Big(g(z,p_a)^2 + g(z,p_b)^2\Big) \tag{B.63}$$

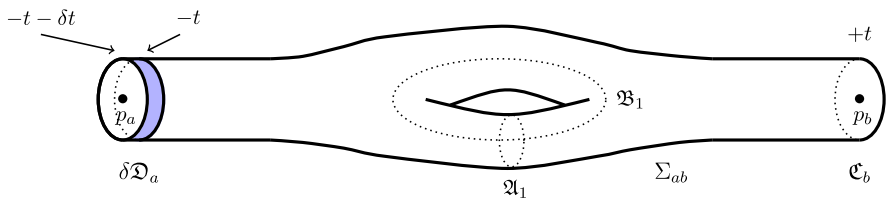


Figure 13: The variational method evaluates the contribution from varying the boundary cycles of Σ_{ab} through a variation of t, here represented for the variation of the cycle $\mathfrak{C}_a$.

$$
\begin{aligned}
&+4\tau_2\,\delta t\int_{\Sigma_{ab}}\kappa_1(z)|\partial_z g(z,p_b)|^2\Big(g(z,p_a)^2+g(z,p_b)^2\Big)\\
&-4\tau_2\,\delta t\int_{\Sigma_{ab}}\kappa_1(z)\,\partial_{\bar z}g(z,p_a)\partial_z g(z,p_b)\,\Big(g(z,p_a)^2+g(z,p_b)^2\Big)+\text{c.c}
\end{aligned}
$$

where addition of the complex conjugate applies only to the last line. To evaluate the first two lines, we use the integral (A.23) in terms of the parameter T introduced in (A.24). Using also the integrals of (A.25), and putting all together, we have,

$$
\delta\mathcal{K}_{(m)}=4\delta T\left(\frac{1}{3}T^3+Tg(v)^2+\frac{1}{3}D_3-\frac{2}{3}g(v)^3-D_3^{(1)}-2D_4^{(a)}\right) \tag{B.64}
$$

Integrating the above equation, we obtain,

$$
\begin{aligned}
\mathcal{K}_{(m)}=&\frac{T^4}{3}+2T^2g(v)^2+\frac{4}{3}TD_3-\frac{8}{3}Tg(v)^3-4TD_3^{(1)}-8TD_4^{(a)}\\
&+\mathcal{O}(t^0)
\end{aligned} \tag{B.65}
$$

B.4.2. Calculation of $\delta\mathcal{K}_{(a)}$ and $\delta\mathcal{K}_{(b)}$ The bulk contributions to $\mathcal{K}_{(a)}$ and $\mathcal{K}_{(b)}$ are exponentially suppressed, as is manifest by Taylor expanding the Green function. To evaluate the contributions from the funnel, we may approximate all functions by their form strictly in the funnel, and extend their functional form arbitrarily beyond the funnel.

To evaluate $\mathcal{K}_{(b)}$, we extend the range of z near p_b by requiring only the condition $f(z)\leq 2\pi t$ and dropping the lower minimum condition on $f(z)$. Furthermore, we use the approximations suitable for the funnel for the

Green functions $g(z,w)$ and $g(w,p_b)$,

$$\begin{aligned} g(z,w) &= -\ln|z-w|^2 - \lambda \\ g(w,p_b) &= -\ln|w-p_b|^2 - \lambda \end{aligned} \tag{B.66}$$

Within this approximation, the integration domain for z then becomes $R \leq |z-p_b|$, and may be parametrized by two real coordinates α, β,

$$z = p_b + R\, e^{\alpha+i\theta} \qquad 0 \leq \alpha \qquad 0 \leq \theta \leq 2\pi \tag{B.67}$$

The integral over y combines with the integral over θ, and we find after some simplifications,

(B.68)
$$\delta\mathcal{K}_{(b)} = 4\delta t \int_0^\infty d\alpha \int_0^{2\pi} d\theta \left(\left(2\alpha - T + \ln|1-e^{-\alpha-i\theta}|^2\right)^2 - (2\alpha - T)^2 \right)$$

Using the vanishing for $\alpha > 0$ of the integral over θ of a single power of $\ln|1-e^{-\alpha-i\theta}|^2$, we are left with performing the integral of the square of the logarithm, which may be done by Taylor expanding each logarithm, performing the integrals over θ and then performing the integrals over α. The result is given by $\delta\mathcal{K}_{(b)} = 8\pi\zeta(3)\,\delta t$. Upon integration in T, we find,

$$\mathcal{K}_{(a)} + \mathcal{K}_{(b)} = 8\zeta(3)\,T + \mathcal{O}(t^0) \tag{B.69}$$

Putting all of this together, we find,

$$\begin{aligned} \mathcal{K} = \frac{T^4}{3} + 2T^2 g(v)^2 + \frac{4}{3}TD_3 - \frac{8}{3}Tg(v)^3 + 8\zeta(3)T - 4TD_3^{(1)} \\ - 8TD_4^{(a)} + \mathcal{O}(t^0) \end{aligned} \tag{B.70}$$

Using the definition of $T = 2\pi t + g(v)$, and retaining only the t-dependent terms in the above formula for $\mathcal{K}$ we readily obtain (B.51).

B.5. Calculation of $\mathcal{K}^c$

Having calculated the non-constant t-dependence $\mathcal{K}^t$ of $\mathcal{K}$ in the preceding subsection, we define $\mathcal{K}^c$ by the following limit,

$$\mathcal{K}^c = \lim_{t\to\infty} \left(\mathcal{K} - \mathcal{K}^t\right) \tag{B.71}$$

We shall compute all contributions to $\mathcal{K}$, cancel the ones with non-constant dependence on t, and then extract the remainder in the limit. Although this procedure duplicates the variational method to some extent, the confirmation of the validity of the t-dependent terms will be of value in this rather tricky calculation.

B.5.1. Partitioning the integral We recall the starting formula for $\mathcal{K}$, with the function f expanded into its two contributions,

$$\mathcal{K} = \frac{\tau_2^2}{\pi^2} \int_{\Sigma_{ab}} \kappa_1(z) \int_{\Sigma_{ab}} \kappa_1(w) \Big| \partial_z g(z,p_a) - \partial_z g(z,p_b) \Big|^2 \times \Big| \partial_w g(w,p_a) - \partial_w g(w,p_b) \Big|^2 g(z,w)^2 \tag{B.72}$$

We decompose $\mathcal{K}$ into a sum over sixteen contributions,

$$\mathcal{K} = \sum_{\alpha,\bar\alpha,\beta,\bar\beta \in \{a,b\}} (-)^{\#(a)} \mathcal{K}_{\alpha\bar\alpha\beta\bar\beta} \tag{B.73}$$

obtained by expanding both absolute-value-squared factors in (B.72) into the following basis,

$$\mathcal{K}_{\alpha\bar\alpha\beta\bar\beta} = \frac{\tau_2^2}{\pi^2} \int_{\Sigma_{ab}} \kappa_1(z) \int_{\Sigma_{ab}} \kappa_1(w)\, \partial_z g(z,p_\alpha)\, \partial_{\bar z} g(z,p_{\bar\alpha})\, \partial_w g(w,p_\beta) \times \partial_{\bar w} g(w,p_{\bar\beta})\, g(z,w)^2 \tag{B.74}$$

Here, $\#(a)$ is the number of a-labels amongst $\alpha, \bar\alpha, \beta, \bar\beta$ (which equals the number of b-labels modulo two). Swapping z with w and complex conjugating produce the following relations,

$$\mathcal{K}_{\alpha\bar\alpha\beta\bar\beta} = \mathcal{K}_{\beta\bar\beta\alpha\bar\alpha} = \big(\mathcal{K}_{\bar\alpha\alpha\bar\beta\beta}\big)^* \tag{B.75}$$

Taking also into account the symmetry under swapping p_a and p_b, the sum over 16 terms reduces to a sum over 5 irreducible terms,

$$\mathcal{K} = 2\mathcal{K}_{abab} + 2\mathcal{K}_{abba} + 2\mathcal{K}_{aabb} - 4\mathcal{K}_{aaab} - 4\mathcal{K}^*_{aaab} + 2\mathcal{K}_{aaaa} \tag{B.76}$$

Two of these integrals are finite in the limit $t \to \infty$, and may thus be extended to finite integrals over the compact torus Σ_1, up to exponential

corrections which we neglect,

(B.77)
$$\begin{aligned}\mathcal{K}_{abab} &= \frac{\tau_2^2}{\pi^2}\int_{\Sigma_1}\kappa_1(z)\int_{\Sigma_1}\kappa_1(w)\,\partial_z g(z,p_a)\partial_{\bar z}g(z,p_b)\,g(z,w)^2\,\partial_w g(w,p_a)\\ &\quad\times\partial_{\bar w}g(w,p_b)\\ \mathcal{K}_{abba} &= \frac{\tau_2^2}{\pi^2}\int_{\Sigma_1}\kappa_1(z)\int_{\Sigma_1}\kappa_1(w)\,\partial_z g(z,p_a)\partial_{\bar z}g(z,p_b)\,g(z,w)^2\,\partial_w g(w,p_b)\\ &\quad\times\partial_{\bar w}g(w,p_a)\end{aligned}$$

They are both three-loop Feynman diagrams represented in Figure 7 on page 379. The remaining integrals do have non-trivial polynomial t-dependence.

To express the remaining contributions, $\mathcal{K}_{aabb}$, $\mathcal{K}_{aaab}$ and $\mathcal{K}_{aaaa}$, in terms of a polynomial in t whose coefficients are convergent integrals over the compact surface Σ_1, we proceed as follows. We split the integrals into a part $\mathcal{K}^0$ which is given by a convergent integral as $t\to\infty$, and a part $\mathcal{K}^1$ which has non-trivial polynomial t dependence and which is easier to evaluate than the original integral,

$$\begin{aligned}\mathcal{K}_{aabb} &= \mathcal{K}^0_{aabb}+\mathcal{K}^1_{aabb}\\ \mathcal{K}_{aaab} &= \mathcal{K}^0_{aaab}+\mathcal{K}^1_{aaab}\\ \mathcal{K}_{aaaa} &= \mathcal{K}^0_{aaaa}+\mathcal{K}^1_{aaaa}\end{aligned}\tag{B.78}$$

We shall begin by discussing the first two functions above, and then proceed to the most intricate case of the last function.

B.5.2. Decomposing $\mathcal{K}_{aabb}$ and $\mathcal{K}_{aaab}$ The functions $\mathcal{K}_{aabb}$ and $\mathcal{K}_{aaab}$ are schematically represented by three-loop Feynman diagrams in Figures 8 and 9 respectively. The functions $\mathcal{K}^0_{aabb}$ and $\mathcal{K}^0_{aaab}$ are defined by,

$$\begin{aligned}\mathcal{K}^0_{aaab} &= \frac{\tau_2^2}{\pi^2}\int_{\Sigma_{ab}}\kappa_1(z)\int_{\Sigma_{ab}}\kappa_1(w)|\partial_z g(z,p_a)|^2\partial_w g(w,p_a)\partial_{\bar w}g(w,p_b)\\ &\quad\times\Big(g(z,w)^2-g(p_a,w)^2\Big)\\ \mathcal{K}^0_{aabb} &= \frac{\tau_2^2}{\pi^2}\int_{\Sigma_{ab}}\kappa_1(z)\int_{\Sigma_{ab}}\kappa_1(w)\,|\partial_z g(z,p_a)|^2|\partial_w g(w,p_b)|^2\\ &\quad\times\Big(g(z,w)^2-g(p_a,w)^2-g(z,p_b)^2+g(p_a,p_b)^2\Big)\end{aligned}\tag{B.79}$$

They do have finite limits as $t \to \infty$, so that the integration domains may be smoothly extended from Σ_{ab} to Σ_1, and the resulting integrals evaluate to generalized modular graph functions. The finiteness of the integrals in $\mathcal{K}^0_{aaab}$ and $\mathcal{K}^0_{aabb}$ as $t \to \infty$ may be proven conveniently by using the variational method, but we shall not present these proofs here.

The functions $\mathcal{K}^1_{aabb}$ and $\mathcal{K}^1_{aaab}$ are defined by,

(B.80)
$$\begin{aligned}\mathcal{K}^1_{aabb} &= \frac{\tau_2^2}{\pi^2}\int_{\Sigma_{ab}}\kappa_1(z)\int_{\Sigma_{ab}}\kappa_1(w)|\partial_z g(z,p_a)|^2|\partial_w g(w,p_b)|^2 \\ &\quad\times\Big(g(p_a,w)^2+g(z,p_b)^2-g(p_a,p_b)^2\Big)\\ \mathcal{K}^1_{aaab} &= \frac{\tau_2^2}{\pi^2}\int_{\Sigma_{ab}}\kappa_1(z)\int_{\Sigma_{ab}}\kappa_1(w)\,|\partial_z g(z,p_a)|^2\partial_w g(w,p_a)\partial_{\bar w} g(w,p_b)\,g(p_a,w)^2\end{aligned}$$

The integrals $\mathcal{K}^1_{aabb}$ and $\mathcal{K}^1_{aaab}$ do have non-trivial dependence on t which may, however, be easily evaluated. To compute $\mathcal{K}^1_{aaab}$ we note that its integral over z may be performed using (A.23), while to compute $\mathcal{K}^1_{aabb}$ we use the fact that the z-integral may be similarly computed for the first and third term in the parentheses of the integrand, while for the second term it is the w-integral that may be readily computed. The results are as follows,

(B.81)
$$\begin{aligned}\mathcal{K}^1_{aabb} &= \frac{\tau_2 T}{\pi}\int_{\Sigma_{ab}}\kappa_1(w)|\partial_w g(w,p_b)|^2\Big(2g(w,p_a)^2-g(p_a,p_b)^2\Big)\\ \mathcal{K}^1_{aaab} &= \frac{\tau_2 T}{\pi}\int_{\Sigma_{ab}}\kappa_1(w)\,\partial_w g(w,p_a)\partial_{\bar w} g(w,p_b)\,g(w,p_a)^2\end{aligned}$$

The remaining w-integrals are readily evaluated, and we obtain,

(B.82)
$$\begin{aligned}\mathcal{K}^1_{aabb} &= T^2\,g(v)^2-2T\,D_3^{(1)}-4T\,D_4^{(a)}+\mathcal{O}(e^{-2\pi t})\\ \mathcal{K}^1_{aaab} &= \frac{T}{3}(g(v)^3-D_3)+\mathcal{O}(e^{-2\pi t})\end{aligned}$$

B.5.3. Decomposing $\mathcal{K}_{aaaa}$ The remaining integral $\mathcal{K}_{aaaa}$ is given by the four-loop Feynman diagram in Figure 9,

(B.83)
$$\mathcal{K}_{aaaa} = \frac{\tau_2^2}{\pi^2}\int_{\Sigma_{ab}}\kappa_1(z)\int_{\Sigma_{ab}}\kappa_1(w)\,|\partial_z g(z,p_a)|^2|\partial_w g(w,p_a)|^2\,g(z,w)^2$$

We rearrange $\mathcal{K}_{aaaa}$ as the sum of three integrals,

(B.84)
$$\mathcal{K}_{aaaa} = K_1+K_2+K_3$$

where each part is defined as follows,

(B.85)

$$\begin{aligned}
K_1 =& \frac{\tau_2^2}{\pi^2}\int_{\Sigma_{ab}}\kappa_1(z)\int_{\Sigma_{ab}}\kappa_1(w)|\partial_z g(z,p_a)|^2|\partial_w g(w,p_a)|^2 \\
&\times\Big(g(z,w)-g(z,p_a)\Big)\Big(g(z,w)-g(p_a,w)\Big) \\
K_2 =& \frac{2\tau_2^2}{\pi^2}\int_{\Sigma_{ab}}\kappa_1(z)\int_{\Sigma_{ab}}\kappa_1(w)|\partial_z g(z,p_a)|^2|\partial_w g(w,p_a)|^2 g(z,w)g(z,p_a) \\
K_3 =& -\frac{\tau_2^2}{\pi^2}\int_{\Sigma_{ab}}\kappa_1(z)\int_{\Sigma_{ab}}\kappa_1(w)|\partial_z g(z,p_a)|^2|\partial_w g(w,p_a)|^2 g(z,p_a)g(w,p_a)
\end{aligned}$$

The purpose of the rearrangement is to expose the last two factors in the integrand of K_1 which vanish at $z=p_a$ and $w=p_a$, and decrease the orders of the poles at these points. For fixed z away from p_a, the w-integral is absolutely convergent. However, the multiple integration over both z,w will not, in fact, be convergent yet. In quantum field theory language, K_1 has no sub-divergences, but it has a primitive divergence, with which we shall deal shortly. The remaining integrals K_2 and K_3 are simpler and can be evaluated exactly. It is straightforward to evaluate K_3 using (A.23) and we obtain,

$$K_3 = -\frac{1}{4}(T^2-E_2)^2+\mathcal{O}(e^{-2\pi t}) \tag{B.86}$$

To evaluate K_2, we use a formula analogous to (A.21) to carry out the integral over w,

$$\begin{aligned}
&\frac{2\tau_2}{\pi}\int_{\Sigma_{ab}}\kappa_1(w)|\partial_w g(w,p_a)|^2 g(z,w) \\
&\quad = E_2 - 2g_2(z,p_a) - g(z,p_a)^2 + \frac{T}{\pi}\int d\theta\, g(z,p_a^\theta)
\end{aligned} \tag{B.87}$$

where p_a^θ was defined in (A.19). We find,

(B.88)

$$\begin{aligned}
K_2 =& \frac{\tau_2}{\pi}\int_{\Sigma_{ab}}\kappa_1(z)|\partial_z g(z,p_a)|^2 g(z,p_a)\left(-E_2-g(z,p_a)^2+2Tg(z,p_a)\right) \\
&+\frac{2\tau_2}{\pi}\int_{\Sigma_{ab}}\kappa_1(z)|\partial_z g(z,p_a)|^2 g(z,p_a)\left(E_2-g_2(z,p_a)\right)
\end{aligned}$$

$$+\frac{\tau_2 T}{\pi^2}\int_{\Sigma_{ab}}\kappa_1(z)|\partial_z g(z,p_a)|^2 g(z,p_a)\int_0^{2\pi}d\theta\Big(g(z,p_a^\theta)-g(z,p_a)\Big)$$

The first line is evaluated with the help of (A.15). To evaluate the second line, we integrate by parts twice and use the Laplace equations for g and g_2, taking into account that any $\delta(z,p_a)$ vanishes since the surface Σ_{ab} does not contain the point p_a. The integral on the last line receives contributions only from the region where w is in the funnel. Evaluating these contributions by using for g the Green function on the plane, we find that the contribution vanishes. Adding all up, we find,

$$K_2+K_3=\frac{T^4}{6}-\frac{2T}{3}D_3-\frac{7}{8}E_2^2+\frac{5}{8}D_4+\frac{3}{4}E_4+\mathcal{O}(e^{-2\pi t}) \tag{B.89}$$

It remains to analyze K_1. We do this by recasting it in the following way,

(B.90)

$$K_1=\frac{\tau_2}{\pi}\int_{\Sigma_{ab}}\kappa_1(z)|\partial_z g(z,p_a)|^2 W_{ab}(z)$$

$$W_{ab}(z)=\frac{\tau_2}{\pi}\int_{\Sigma_{ab}}\kappa_1(w)|\partial_w g(w,p_a)|^2\Big(g(z,w)-g(z,p_a)\Big)\Big(g(z,w)-g(p_a,w)\Big)$$

and first studying the function $W_{ab}(z)$. When z is in the funnel, the entire contribution of the integral in w arises for w in the funnel as well, in view of the second factor in parentheses in the integrand. Thus, we use the following parametrization,

(B.91)

$$z=p_a+R\,e^{x+iy}\qquad\qquad w=p_a+R\,e^{\alpha+i\beta}\qquad\qquad 0\le y,\beta\le 2\pi$$

where x,y are data determined by z, and α,β are to be integrated over. The condition $w\in\Sigma_{ab}$ restricts α to be positive and bounded above by a quantity of order $-\ln R$. The integral inside the funnel will rapidly converge as α becomes large and may be extended to ∞. For $z\in\Sigma_{ab}$ we require $x>0$. We shall also be interested in continuing the point z to the interior of the disc $\mathfrak{D}_a$ in Σ_1 outside of Σ_{ab} where $x<0$. Under these assumptions, and using for g the expression suitable for the funnel, the integral reduces to,

$$W_{ab}(z)=\frac{1}{\pi}\int_0^\infty d\alpha\int_0^{2\pi}d\beta\left(-\ln\left|1-e^{\alpha-x+i\beta-iy}\right|^2\right) \tag{B.92}$$

$$\times \left(-\ln \left| 1 - e^{x-\alpha-i\beta+iy} \right|^2 \right)$$

The integral is independent of y by translation invariance in β. For $x > 0$, we split the integration region for α into two regions, $0 \leq \alpha \leq x$ and $x \leq \alpha$. Expanding the logarithms into absolutely convergent Taylor series, the integrals over β and α may be carried out,

$$W_{ab}(z) = 4\zeta(3)\theta(x) - 2\varepsilon(x) \sum_{m=1}^{\infty} \frac{e^{-2m|x|}}{m^3} \tag{B.93}$$

where $\theta(x)$ is the Heaviside step function and $\varepsilon(x) = \theta(x) - \theta(-x)$ is the sign function. The function $W_{ab}(z)$ is continuous and differentiable once at $x = 0$, and vanishes as $z \to p_a$ as expected from its defining integral in (B.90).

Using this result to evaluate the contribution to K_1 from the region where z is in the funnel, we see that the $\zeta(3)$ term produces $4T\zeta(3)$ upon integrating over $z \in \Sigma_{ab}$. The z-integral of the second term above is localized in the funnel thanks to the exponential suppression. The sum of its contributions from the funnel and insider the disc $\mathfrak{D}_a$ cancel. Therefore, we may extend the domains of integration for $W(z)$ and K_1 from Σ_{ab} to Σ_1,

$$\begin{aligned} \mathcal{K}^0_{aaaa} &= K_1 - 4T\zeta(3) = \frac{\tau_2}{\pi} \int_{\Sigma_1} \kappa_1(z) |\partial_z g(z)|^2 \Big(W(z) - 4\zeta(3) \Big) \\ W(z) &= \frac{\tau_2}{\pi} \int_{\Sigma_1} \kappa_1(w) |\partial_w g(w)|^2 \Big(g(z,w) - g(z) \Big) \Big(g(z,w) - g(w) \Big) \end{aligned} \tag{B.94}$$

Here, we have set $p_a = 0$ by translation invariance on Σ_1. While the integrand of $W_{ab}(z)$ vanishes at $z = 0$ for w away from $w = 0$, a careful analysis shows that the integral over $w \in \Sigma_1$ evaluates as follows $\lim_{z\to 0} W_{ab}(z) = 4\zeta(3)$. This limit arises from the original integral over Σ_{ab} by taking the limit as $t \to 0$ of $W(z) = W_{ab}(z) + \mathcal{O}(e^{-2\pi t})$ for $z \neq 0$. In terms of $\mathcal{K}^0_{aaaa}$ we have,

$$\mathcal{K}_{aaaa} = \frac{T^4}{6} - \frac{2T}{3} D_3 + 4T\zeta(3) - \frac{7}{8} E_2^2 + \frac{5}{8} D_4 + \frac{3}{4} E_4 + \mathcal{K}^0_{aaaa} + \mathcal{O}(e^{-2\pi t}) \tag{B.95}$$

This concludes the decomposition of $\mathcal{K}_{aaaa}$.

B.5.4. Summary of results for $\mathcal{K}$ Putting all together, we find,

(B.96)
$$\begin{aligned}\mathcal{K} =& \frac{T^4}{3} + 2T^2 g(v)^2 + \frac{4T}{3} D_3 - \frac{8T}{3} g(v)^3 + 8T\zeta(3) - 4T D_3^{(1)}(v) - 8T D_4^{(a)}(v) \\ & - \frac{7}{4} E_2^2 + \frac{5}{4} D_4 + \frac{3}{2} E_4 + 2\mathcal{K}_{abab} + 2\mathcal{K}_{abba} + 2\mathcal{K}^0_{aabb} - 4\mathcal{K}^0_{aaab} - 4(\mathcal{K}^0_{aaab})^* \\ & + 2\mathcal{K}^0_{aaaa} + \mathcal{O}(e^{-2\pi t})\end{aligned}$$

Splitting the contributions into $\mathcal{K}^t$ and $\mathcal{K}^c$ we recover precisely $\mathcal{K}^t$ of (B.51), a fact which provides a double check on the calculations since (B.51) was obtained by the variational method, and allows us to compute the constant part,

(B.97)
$$\begin{aligned}\mathcal{K}^c =& 2\mathcal{K}_{abab} + 2\mathcal{K}_{abba} + 2\mathcal{K}^0_{aabb} - 4\mathcal{K}^0_{aaab} - 4(\mathcal{K}^0_{aaab})^* + 2\mathcal{K}^0_{aaaa} \\ & + 4g(v)\left(D_3 + 2\zeta(3) - D_3^{(1)} + \frac{\Delta_v F_4}{2\pi}\right) - 3g(v)^4 - \frac{7}{4} E_2^2 + \frac{5}{4} D_4 \\ & + \frac{3}{2} E_4\end{aligned}$$

This result completes the calculation of the functions $Z_i(\Omega)$.

B.6. Calculation of the functions γ_i and $\mathcal{Z}_i$

The conversion of the functions $Z_i(\Omega)$ to the genuine modular graph functions $\mathcal{Z}_i(\Omega)$ is achieved with the help of the formulas (3.19). The functions $\gamma(x)$ and γ_1 govern the conversion of the standard string Green function G into the Arakelov Green function $\mathcal{G}$ and were given in (3.14). The function $\gamma(x)$ is readily obtained from its definition and, In the non-separating degeneration limit, is obtained by substituting (3.8) and (3.9) in (2.18),

$$\begin{aligned}\gamma(x) =& \int_{\Sigma_{ab}} \left(\frac{1}{2}\kappa_1(y) + \frac{i}{4t}\omega_t \wedge \overline{\omega}_t(y)\right)\left(g(x,y) + \frac{(f(x)-f(y))^2}{8\pi t}\right) \\ & + \mathcal{O}(e^{-2\pi t})\end{aligned} \tag{B.98}$$

The integral of κ_1 against g vanishes while against the term in f^2 it may be evaluated in terms of F_2. The remaining integrals are evaluated using (A.15) and the second equation in (A.21) for $n = 0$ and generic $w = y$, so that we

recover the result for $\gamma(x)$,

$$\gamma(x) = \frac{\pi t}{12} + \frac{1}{4}g(x,p_a) + \frac{1}{4}g(x,p_b) + \frac{f(x)^2}{16\pi t} + \frac{F_2(v)}{4\pi t} + \mathcal{O}(e^{-2\pi t}) \tag{B.99}$$

announced in (3.14), and repeated here for convenience. To obtain also γ_1 simply requires a further integration using the formulas of (A.15) and (A.21) for $n = 0$. Using these results, we shall now also evaluate the remaining functions γ_2 and γ_3,

$$\begin{aligned} \gamma_2 &= \int_{\Sigma_{ab}} \left(\frac{1}{2}\kappa_1(x) + \frac{i}{4t}\omega_t \wedge \overline{\omega}_t(x)\right)\gamma(x)^2 \\ \gamma_3 &= \int_{\Sigma^2_{ab}} \left(\nu^-_{xy} - \nu^+_{xy}\right)\gamma(x)\gamma(y) \end{aligned} \tag{B.100}$$

where $\gamma(x)$ is given in (3.14), and $\nu^{\pm}_{xy}$ was defined in (B.3). The calculation of γ_2 may be performed using the following integrals, in addition to those of (A.15) and (A.21),

(B.101)

$$\begin{aligned} \frac{\tau_2}{\pi}\int_{\Sigma_{ab}} \kappa_1(z)|\partial_z f(z)|^2 g(z,p_a) &= \frac{T^2}{2} + Tg(v) - \frac{3}{2}g(v)^2 + F_2(v) \\ \frac{\tau_2}{\pi}\int_{\Sigma_{ab}} \kappa_1(z)|\partial_z f(z)|^2 g(z,p_a)g(z,p_b) &= T^2 g(v) + \frac{1}{3}D_3 - \frac{1}{3}g(v)^3 - D_3^{(1)}(v) \\ &\quad - 2D_4^{(a)}(v) \end{aligned}$$

as well as the rearrangement $(g(x,p_a)+g(x,p_b))^2 = f(x)^2+4g(x,p_a,)g(x,p_b)$. The calculation of γ_3 may be performed using the following results,

$$\begin{aligned} \int_{\Sigma_{ab}} \omega_t \wedge \overline{\omega}_1(x)\gamma(x) &= -\frac{\tau_2}{2\pi}\partial_v g_2(v) \\ \int_{\Sigma_{ab}} \omega_1 \wedge \overline{\omega}_1(x)\gamma(x) &= -2i\tau_2\left(\frac{\pi t}{12} + \frac{3F_2}{8\pi t}\right) \\ \int_{\Sigma_{ab}} \omega_t \wedge \overline{\omega}_t(x)\gamma(x) &= -\frac{5\pi i}{6}t^2 - itg(v) - \frac{3iF_2(v)}{4\pi} \end{aligned} \tag{B.102}$$

Using the rearrangement formula (B.36), we arrive at the following results,

(B.103)

$$\gamma_1 = \frac{\pi t}{4} + \frac{1}{4}g + \frac{3gF_2}{8\pi t} + \mathcal{O}(e^{-2\pi t})$$

$$\gamma_2 = \frac{41\pi^2 t^2}{360} + \frac{5\pi t}{24} g + \frac{1}{24}(5E_2 - 2g_2 + 3g^2)$$
$$+ \frac{1}{16\pi t}\left(D_3 - D_3^{(1)} + 2gF_2 + \frac{\Delta_v F_4}{4\pi}\right) + \frac{F_4 + 2F_2^2}{16\pi^2 t^2} + \mathcal{O}(e^{-2\pi t})$$
$$\gamma_3 = \frac{5\pi^2 t^2}{18} + \frac{1}{3}\pi g t + \frac{3}{2}F_2 + \frac{2gF_2}{\pi t} - \frac{\Delta_v F_2^2}{16\pi^2 t} + \frac{9F_2^2}{8\pi^2 t^2} + \mathcal{O}(e^{-2\pi t})$$

It is now a simple matter of algebraic substitution to use formulas (3.19) with Z_1 given in (B.52), Z_2 given in (B.38), and Z_3 given in (B.52) and the above expressions for $\gamma_1, \gamma_2, \gamma_3$ in order to obtain the expressions for $\mathcal{Z}_1, \mathcal{Z}_2$ and $\mathcal{Z}_3$ given in the body of the paper in (3.21).

Appendix C. Tropical limits of modular graph functions

In this appendix, we shall derive the limit as $\tau \to i\infty$ of the various modular graph functions which appear as coefficients in the non-separating degeneration of the genus-two string invariants φ, $\mathcal{Z}_i$ and $\mathcal{B}_{(2,0)}$. The results give the behavior of these functions in their tropical limit near the non-separating degeneration node of Section 5.

In the first two subsections we review, without derivation, the Bernoulli polynomials and the limits of standard modular graph functions and elliptic polylogarithms. In the third subsection, we derive the limits of the functions $D_3^{(1)}$, $D_4^{(1)}$, $D_4^{(a)}$, F_2 and F_4. In the fourth subsection, we present the limit for the one-loop self-energy graph, and related graphs. In the four subsequent subsections, we obtain the limits of the modular graph functions $\mathcal{K}_{abab}, \mathcal{K}_{abba}, \mathcal{K}^0_{aabb}, \mathcal{K}^0_{aaab}$ and $\mathcal{K}^0_{aaaa}$.

C.1. Bernoulli polynomials

The Bernoulli polynomials $B_k(x)$ are defined for all integers $k \geq 0$ by the Taylor series,

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} B_k(x) = \frac{z\, e^{xz}}{e^z - 1} \tag{C.1}$$

for $x \in \mathbb{C}$, and $z \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$. From this definition, we have the following relations,

$$\begin{aligned} B_k(1-x) &= (-)^k B_k(x) \\ B_k'(x) &= k B_{k-1}(x) \end{aligned} \tag{C.2}$$

The Bernoulli polynomials sum up the following Fourier series when $0 \leq x \leq 1$ and $k \geq 2$,

$$\sum_{n\neq 0} \frac{e^{2\pi i n x}}{n^k} = -\frac{(2\pi i)^k}{k!} B_k(x) \tag{C.3}$$

The function defined by $B_k(x)$ in the interval $x \in [0,1]$ for $k \geq 2$ takes the same values at $x = 1$ and at $x = 0$, and extends to a *continuous periodic function* on $\mathbb{R}$ by translation of the interval by $\mathbb{Z}$. Its successive derivatives, however, are not continuous. The validity of the formula may be extended to all $x \in \mathbb{R}$,

$$\sum_{n\neq 0} \frac{e^{2\pi i n x}}{n^k} = -\frac{(2\pi i)^k}{k!} B_k(\{x\}) \tag{C.4}$$

where the fractional part $\{x\}$ of $x \in \mathbb{R}$, defined so that $0 \leq \{x\} < 1$ and $x - \{x\} \in \mathbb{Z}$. The Bernoulli polynomials with even index k for $k \leq 8$ are given explicitly by $B_0(x) = 1$ and

$$\begin{aligned} B_2(x) &= \frac{1}{6} - x + x^2 \\ B_4(x) &= -\frac{1}{30} + x^2 - 2x^3 + x^4 \\ B_6(x) &= \frac{1}{42} - \frac{1}{2}x^2 + \frac{5}{2}x^4 - 3x^5 + x^6 \\ B_8(x) &= -\frac{1}{30} + \frac{2}{3}x^2 - \frac{7}{3}x^4 + \frac{14}{3}x^6 - 4x^7 + x^8 \,. \end{aligned} \tag{C.5}$$

Note that the Bernoulli numbers B_k are related to the Bernoulli polynomials by $B_k = B_k(0)$.

C.2. Eisenstein series and standard modular graph functions

The limit $\tau_2 \to \infty$ of the non-holomorphic Eisenstein series $E_n(\tau)$ and of the elliptic polylogarithms $g_n(v|\tau)$ and $D_{a,b}(v|\tau)$ for $u_2 = v_2/\tau_2$ fixed and $0 < u_2 < \frac{1}{2}$ is well known,

$$\begin{aligned} E_n(\tau) &= \frac{2\zeta(2n)}{\pi^{2n}} y^n + 2\,\frac{\Gamma(n-\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n)} \zeta(2n+1) y^{1-n} + \mathcal{O}(e^{-2y}) \\ g_n(v|\tau) &= -\frac{(-4y)^n}{(2n)!}\, B_{2n}(u_2) + \mathcal{O}(e^{-2yu_2}) \\ D_{a,b}(v|\tau) &= \frac{(4y)^{a+b-1}}{(a+b)!} B_{a+b}(u_2) + \mathcal{O}(e^{-2yu_2}) \end{aligned} \tag{C.6}$$

where $y = \pi\tau_2$. The asymptotics of the modular graph functions D_ℓ defined in (3.24) was studied in [6]. For the values $\ell = 3, 4$ relevant to this paper we have,

$$D_3(\tau) = \frac{2}{945}\, y^3 + \zeta(3) + \frac{3\zeta(5)}{4\, y^2} + \mathcal{O}(e^{-2y}) \tag{C.7}$$

$$D_4(\tau) = \frac{y^4}{945} + \frac{2\zeta(3)}{3}\, y + \frac{10\zeta(5)}{y} - \frac{3\zeta(3)^2}{y^2} + \frac{9\zeta(7)}{4y^3} + \mathcal{O}(e^{-2y}) \tag{C.8}$$

We note the relations $D_3 = E_3 + \zeta(3)$ and $D_4 = 24C_{2,1,1} + 3E_2^2 - 18E_4$ established in [6].

C.3. Degeneration of $D_\ell^{(1)}$, $D_4^{(2)}$, $D_4^{(a)}$, F_2 and F_4

The method developed in [7] for computing the asymptotics of $D_\ell(\tau)$ defined in (3.29) applies just as well to the generalized modular graph function $D_\ell^{(1)}(v|\tau)$. Consider the decomposition of the genus-one Green function into $g(z|\tau) = \mathfrak{g}_1(z|\tau) + \mathfrak{g}_2(z|\tau) + \mathfrak{g}_3(z|\tau)$, where we use the parametrization $z = \alpha + \beta\tau$ for $\alpha \in \mathbb{R}/\mathbb{Z}$ and $-1 < \beta < 1$,

$$\begin{aligned}
\mathfrak{g}_1(z) &= 2y\, B_2(|\beta|) \\
\mathfrak{g}_2(z) &= \sum_{m\neq 0} \frac{1}{|m|} e^{2\pi \mathrm{i} m(-\alpha+\tau_1\beta)-2y|m\beta|} \\
\mathfrak{g}_3(z) &= \sum_{m\neq 0} \frac{1}{|m|} \sum_{k\neq 0} e^{2\pi \mathrm{i} m[-\alpha+\tau_1(\beta+k)]-2y|m(\beta+k)|}
\end{aligned} \tag{C.9}$$

Given the range for β, with strict inequalities, the term $\mathfrak{g}_3(z|\tau)$ contributes to $D_\ell^{(1)}$ terms which are exponentially suppressed by $\mathcal{O}(e^{-2\pi\tau_2 u_2})$ and can therefore be omitted. Similarly terms linear in $\mathfrak{g}_2(z)$ integrate to zero. In this way we find,

(C.10)

$$\begin{aligned}
D_\ell^{(1)}(v|\tau) = (2y)^\ell \int_{-1/2}^{1/2} d\beta\, B_2(|\beta|)^{\ell-1}\, B_2(|u_2-\beta|) \\
+ \sum_{\substack{\ell-1=\ell_1+\ell_2 \\ \ell_1\geq 0, \ell_2\geq 2}} \frac{(\ell_1+\ell_2)!}{\ell_1!\,\ell_2!} \sum_{n=1}^{2\ell_1+3} P(\ell_1, n; u_2)\, S(\ell_2, n)\, (2y)^{\ell_1-n+1} \\
+ \mathcal{O}(e^{-2yu_2})
\end{aligned}$$

where $P(\ell_1, n; u_2)$ are quadratic polynomials in u_2, defined by

$$\int_{-\infty}^{\infty} d\beta\, B_2(|\beta|)^{\ell_1} B_2(u_2-\beta)\, e^{-y|\beta|} = \sum_{n=1}^{2\ell_1+3} P(\ell_1, n; u_2)\, y^{-n} \tag{C.11}$$

and $S(\ell_2, n)$ is defined by the multiple sum,

$$S(\ell_2, n) = \sum_{m_1,\cdots,m_{\ell_2}\neq 0} \frac{\delta(\sum_{i=1}^{\ell_2} m_i)}{|m_1\cdots m_{\ell_2}|(|m_1|+\cdots+|m_{\ell_2}|)} \tag{C.12}$$

Here, we have replaced $B_2(|u_2-\beta|)$ by $B_2(u_2-\beta)$, since for $y\to\infty$ the integral is dominated by contributions from the region $|\beta| \ll 1$ so that effectively $\beta < u_2$, up to corrections of order $\mathcal{O}(e^{-2yu_2})$, which we neglect. In particular, from (C.10), we have,

$$\begin{aligned} D_3^{(1)}(v|\tau) &= -\frac{8y^3}{15}B_6 - \frac{4y^3}{9}B_4 + 2\zeta(3)\,B_2 + \frac{\zeta(5)}{4\,y^2} + \mathcal{O}(e^{-2yu_2}) \\ D_4^{(1)}(v|\tau) &= y^4\left(\frac{4}{7}B_8 + \frac{16}{15}B_6 + \frac{2}{9}B_4\right) + 2\zeta(3)B_2\,y \\ &\quad + \frac{3\zeta(5)}{2\,y}\left(B_2+\tfrac{1}{6}\right) - \frac{3\zeta(3)^2}{4\,y^2} + \frac{9\zeta(7)}{8\,y^3} + \mathcal{O}(e^{-2yu_2}) \end{aligned} \tag{C.13}$$

where here and henceforth, we omit the argument in $B_{2n}(|u_2|)$. One may check that these results are consistent with the differential equation (3.30). We note that, upon setting $u_2 = 0$, the polynomial part of $D_\ell^{(1)}(v|\tau)$ does not reduce to the polynomial part of $D_\ell(\tau)$, since terms of order $\mathcal{O}(e^{-2yu_2})$ cannot be neglected in this limit.

C.3.1. Degeneration of $D_4^{(2)}(v|\tau)$ The same method readily applies to the calculation of $D_4^{(2)}(v|\tau)$. Replacing g by $\mathfrak{g}_1+\mathfrak{g}_2$ and using the fact that terms linear in $\mathfrak{g}_2$ integrate to zero, we get,

$$\begin{aligned} D_4^{(2)} = &\int_{\Sigma_1} \kappa_1(z)\Big(\mathfrak{g}_1^2(z)\,\mathfrak{g}_1^2(z-v) + 2\mathfrak{g}_1^2(z-v)\,\mathfrak{g}_2^2(z) + \mathfrak{g}_2(z)^2\,\mathfrak{g}_2(z-v)^2\Big) \\ &+ \mathcal{O}(e^{-2yu_2}) \end{aligned} \tag{C.14}$$

The first term can be evaluated directly and produces a linear combination of Bernoulli polynomials $B_{2k}(u_2)$ for $k\leq 4$. The last term is exponentially

suppressed as $y \to \infty$. The second term in (C.14) gives,

$$8\pi^2\tau_2^2 \sum_{m\neq 0} \frac{1}{m^2} \int_{-1/2}^{1/2} d\beta\, B_2(|u_2-\beta|)^2\, e^{-4\pi\tau_2|m\beta|} \tag{C.15}$$

which can be evaluated by extending the integral to the full real axis. In total, we find,

$$\begin{aligned} D_4^{(2)}(v|\tau) = &-y^4\left(\frac{8}{35}B_8+\frac{32}{45}B_6+\frac{8}{27}B_4-\frac{1}{2025}\right) \\ &+y\,\zeta(3)\left(8B_4+\frac{8}{3}B_2+\frac{4}{45}\right) \\ &+\frac{\zeta(5)}{y}\left(6B_2+\frac{1}{3}\right)+\frac{3\zeta(7)}{4\,y^3}+\mathcal{O}(e^{-2yu_2}) \end{aligned} \tag{C.16}$$

C.3.2. Degeneration of $D_4^{(a)}(v|\tau)$ Although the function $D_4^{(a)}(v|\tau)$ does not enter into the final expressions for the degeneration of the genus-two string invariants considered in this paper, it will be useful to have at intermediate stages. It was defined in (A.26) and related to the function $D_4^{(2)}(v|\tau)$ in (A.27). Thus, its degeneration may be obtained directly from that of $D_4^{(2)}$ by differentiation in u_2,

$$D_4^{(a)}(v|\tau) = -\frac{1}{16\pi\tau_2}\partial_{u_2}^2 D_4^{(2)}(v|\tau) \tag{C.17}$$

which is readily computed using the differentiation rule for Bernoulli polynomials,

$$\begin{aligned} D_4^{(a)}(v|\tau) = &\, y^3\left(\frac{4B_6}{5}+\frac{4B_4}{3}+\frac{2B_2}{9}\right) \\ &-\zeta(3)\left(6B_2+\frac{1}{3}\right)-\frac{3\zeta(5)}{4\,y^2}+\mathcal{O}(e^{-2yu_2}) \end{aligned} \tag{C.18}$$

C.3.3. Degeneration of F_2, F_2^2 and F_4 The degeneration of F_2, F_2^2, and F_4 are obtained from the definitions of F_2 and F_4 in (3.32), and the relation to the modular graph functions D_4, $D_4^{(1)}$ and $D_2^{(2)}$ which have already

been calculated above, and we find,

$$
\begin{aligned}
F_2(v|\tau) &= \frac{y^2}{45} + \frac{\zeta(3)}{y} + \frac{2}{3}y^2 B_4 + \mathcal{O}(e^{-2yu_2}) \\
F_2(v|\tau)^2 &= y^4\left(\frac{2}{2835} + \frac{16}{27}B_6 + \frac{4}{9}B_8\right) \\
&\quad + y\zeta(3)\left(\frac{2}{45} + \frac{4}{3}B_4\right) + \frac{\zeta(3)^2}{y^2} + \mathcal{O}(e^{-2yu_2}) \\
F_4(v|\tau) &= \frac{2y^4}{15}\left(\frac{1}{630} + \frac{4}{15}B_6 + B_8\right) + 2y\,\zeta(3)\left(\frac{1}{30} + B_4\right) \\
&\quad + \frac{1}{y}\zeta(5)\left(\frac{5}{6} + B_2\right) + \mathcal{O}(e^{-2yu_2})
\end{aligned}
\tag{C.19}
$$

Note that every term in the combination $10F_4 - 3F_2^2$ involves either $\zeta(3)$ or $\zeta(5)$.

C.4. Self-energy and related graphs

For the evaluation of the remaining modular graph functions, we shall make use of one-loop graphs with two Green function factors, possibly with various derivatives. Some of these graphs were already studied in [6] to which we refer for their complete derivation and degeneration, while other graphs appear for the first time, and for which we shall give a complete derivation.

The Fourier transform $\mathcal{T}(M,N)$ of the square of the Green function (which is often referred to as the self-energy graph in quantum field theory), is given by,

$$
g(z)^2 = \sum_{M,N\in\mathbb{Z}} \mathcal{T}(M,N)\, e^{2\pi i(Mz_2 - Nz_1)} \tag{C.20}
$$

where $z = z_1 + \tau z_2$ with $z_1, z_2 \in \mathbb{R}$, and the Fourier coefficients $\mathcal{T}(M,N)$ are given as follows,

$$
\mathcal{T}(M,N) = \sum_{(m,n)\neq(0,0),(M,N)} \frac{\tau_2^2}{\pi^2\left|m+n\tau\right|^2\left|m-M+(n-N)\tau\right|^2} \tag{C.21}
$$

We have $\mathcal{T}(0,0) = E_2$. Closely related is the following integral,

$$
\mathcal{F}(M,N) = \frac{\tau_2}{\pi}\int_{\Sigma_1} \kappa_1(z)|\partial_z g(z)|^2\Big(e^{2\pi i(Mz_2 - Nz_1)} - 1\Big) \tag{C.22}
$$

which clearly satisfies $\mathcal{F}(0,0)=0$. For $(M,N)\neq(0,0)$, both $\mathcal{T}(M,N)$ and $\mathcal{F}(M,N)$ are invariant under $(M,N)\to(-M,-N)$. The function $\mathcal{F}(M,N)$ may be evaluated by integrating by parts successively in z and $\bar{z}$, using the fact that $\delta(z)$ cancels against the expression in the parentheses and expressing the remaining integral in terms of $\mathcal{T}(M,N)$,

$$\mathcal{F}(M,N)=-\frac{\tau_2}{\pi|M+N\tau|^2}-\frac{\pi}{2\tau_2}|M+N\tau|^2\mathcal{T}(M,N) \tag{C.23}$$

The degeneration of $\mathcal{T}(M,N)$ for $(M,N)\neq(0,0)$ was evaluated in the Appendix of [6], while that of $\mathcal{F}(M,N)$ may be deduced from the above relation between the two quantities. For $M\neq 0$, we have,

$$\begin{aligned}\mathcal{T}(M,0)&=\frac{2\tau_2^2}{3M^2}-\frac{6\tau_2^2}{\pi^2M^4}+\frac{8\tau_2^2}{M^2}\mathfrak{J}(M)+\mathcal{O}(e^{-2\pi\tau_2})\\ \mathcal{F}(M,0)&=\frac{2\tau_2}{\pi M^2}-\frac{\pi\tau_2}{3}-4\pi\tau_2\,\mathfrak{J}(M)+\mathcal{O}(e^{-2\pi\tau_2})\end{aligned} \tag{C.24}$$

where the function $\mathfrak{J}(M)$ is conveniently given by the following integral representation,

$$\mathfrak{J}(M)=\int_0^\infty dt\,\frac{2-e^{2\pi iMt}-e^{-2\pi iMt}}{e^{4yt}-1} \tag{C.25}$$

and we continue to use the notation $y=\pi\tau_2$. Evaluating the integral, we find,

$$\mathfrak{J}(M)=\frac{1}{2y}\left(\gamma_E+\Psi\left(\frac{i\pi M}{2y}\right)\right) \tag{C.26}$$

where $\Psi(z)=d\ln\Gamma(z)$ and γ_E is the Euler-Mascheroni constant. Thus $\mathfrak{J}(M)$ grows logarithmically with M.

C.5. Sums involving powers of $\mathfrak{J}(M)$

We shall now present a general procedure to evaluate sums over M involving powers of $\mathfrak{J}(M)$, in an expansion wherewe omit exponential corrections $\mathcal{O}(e^{-2yu_2})$. As usual, we assume $0<u_2<\frac{1}{2}$. The sums of interest may be expressed as follows,

$$\sum_{M\neq 0}e^{2\pi iMu_2}\frac{\mathfrak{J}(M)^p}{\pi^nM^n}=-\frac{(2i)^n}{n!}\prod_{i=1}^p\int_0^\infty\frac{dt_i}{e^{4yt_i}-1}\,\mathfrak{B}_n^{(p)}(u_2;t) \tag{C.27}$$

where the double-index family of functions $\mathfrak{B}_n^{(p)}(u_2,t)$ is defined by,

$$\mathfrak{B}_n^{(p)}(u_2;t) = -\frac{n!}{(2\pi i)^n}\sum_{M\neq 0}\frac{e^{2\pi i M u_2}}{M^n}\prod_{i=1}^{p}(2-e^{2\pi i M t_i}-e^{-2\pi i M t_i}) \tag{C.28}$$

and where t stands for the array $t=(t_1,\cdots,t_p)$. The function $\mathfrak{B}_n^{(p)}(u_2;t)$ is the sum of 3^p terms, obtained by summing the Bernoulli polynomial B_n in the variable u_2 shifted by the various combinations of $\pm t_i$, $\pm t_i \pm t_j$ and so on for i,j mutually distinct[11] The normalization factor has been included so that each term contributes a Bernoulli polynomial with its natural multiplicity. For example, we have,

(C.29)

$$\begin{aligned}
\mathfrak{B}_n^{(1)}(u_2;t) &= 2B_n(u_2) - B_n(u_2+t) - B_n(u_2-t)\\
\mathfrak{B}_n^{(2)}(u_2;t) &= 4B_n(u_2) - 2B_n(u_2+t_1) - 2B_n(u_2-t_1)\\
&\quad - 2B_n(u_2+t_2) - 2B_n(u_2-t_2) + B_n(u_2+t_1+t_2)\\
&\quad + B_n(u_2+t_1-t_2) + B_n(u_2-t_1+t_2) + B_n(u_2-t_1-t_2)
\end{aligned}$$

The function $\mathfrak{B}_n^{(p)}(u_2;t)$ is a polynomial in u_2 and the variables t_i of overall degree at most n; it is a symmetric function of the t_i; it is even in each t_i separately and vanishes whenever $t_i=0$ for at least one value of i. With those properties in mind, it becomes straightforward to compute and simplify these functions, and for $p=1$ we find,

$$\mathfrak{B}_n^{(1)}(u_2;t) = -2\sum_{k=1}^{[n/2]}\binom{n}{2k}t^{2k}B_{n-2k}(u_2) \tag{C.30}$$

To the orders needed here, we shall also make use of the following results,

$$\begin{aligned}
&\mathfrak{B}_2^{(2)}(u_2,t)=0 && \mathfrak{B}_2^{(3)}(u_2,t)=0\\
&\mathfrak{B}_3^{(2)}(u_2,t)=0 && \mathfrak{B}_4^{(2)}(u_2,t)=24\,t_1^2t_2^2
\end{aligned} \tag{C.31}$$

[11] We assume that the t_i's are small enough so that the shifted $\tilde{u}_2$ remains in the interval $0<\tilde{u}_2<1/2$. This is indeed the region which dominates the integral in the limit $y\to\infty$.

The integrals for the remaining terms may be evaluated using the following formula,

$$\int_0^\infty dt\, \frac{t^n}{e^{4yt}-1} = \frac{n!\,\zeta(n+1)}{(4y)^{n+1}} \tag{C.32}$$

Applying the integrals to the formula in (C.30), we find,

$$\sum_{M\neq 0} e^{2\pi i M u_2}\, \frac{\mathfrak{I}(M)}{\pi^n M^n} = 2(2i)^n \sum_{k=1}^{[n/2]} \frac{B_{n-2k}(u_2)}{(n-2k)!}\, \frac{\zeta(2k+1)}{(4y)^{2k+1}} \tag{C.33}$$

Applying the integrals to the formula in (C.31), we find,

$$\begin{aligned} &\sum_{M\neq 0} e^{2\pi i M u_2}\, \frac{\mathfrak{I}(M)^2}{\pi^n M^n} = 0 \qquad\qquad n=2,3 \\ &\sum_{M\neq 0} e^{2\pi i M u_2}\, \frac{\mathfrak{I}(M)^2}{\pi^4 M^4} = -\frac{\zeta(3)^2}{64y^6} \end{aligned} \tag{C.34}$$

When $u_2 = 0$, we use instead the identity, valid up to $\mathcal{O}(e^{2y})$ terms

$$\sum_{M\neq 0} \frac{\mathfrak{I}(M)}{\pi^n M^n} = \frac{2(2\mathrm{i})^n}{n!} \int_0^\infty \mathrm{d}t \frac{B_n(t)-B_n(0)}{e^{4yt}-1} \tag{C.35}$$

and evaluate the integral using (C.32).

C.6. Degeneration of $\mathcal{K}^0_{aabb}$

This function was defined in (B.79), and we shall use translation invariance to shift z and w by p_a, so that the integral may be expressed directly in terms of v,

(C.36)

$$\begin{aligned} \mathcal{K}^0_{aabb} = \frac{\tau_2^2}{\pi^2} \int_{\Sigma_1} \kappa_1(z) \int_{\Sigma_1} \kappa_1(w) |\partial_z g(z)|^2 |\partial_w g(w-v)|^2 \\ \times \Big(g(z-w)^2 - g(w)^2 - g(z-v)^2 + g(v)^2 \Big) \end{aligned}$$

The Fourier transform of the combination in the parentheses of the integrand may be evaluated with the help of (C.20) and its dependence on z and w

may be factored using the following formula, and its analogue for $w \to v$,

(C.37)
$$g(z-w)^2 - g(w)^2 = \sum_{M,N\in\mathbb{Z}} \mathcal{T}(M,N)\, e^{-2\pi i(Mw_2 - Nw_1)}\Big(e^{2\pi i(Mz_2-Nz_1)} - 1\Big)$$

where we continue to use the notation $v = u_1 + \tau u_2$, $z = z_1 + \tau z_2$ and $w = w_1 + \tau w_2$. In terms of the Fourier coefficients $\mathcal{T}$ and $\mathcal{F}$, the function $\mathcal{K}^0_{aabb}$ takes on the following form,

$$\mathcal{K}^0_{aabb} = \sum_{M,N\in\mathbb{Z}} \mathcal{T}(M,N)|\mathcal{F}(M,N)|^2\, e^{2\pi i(Mu_2 - Nu_1)} \tag{C.38}$$

Retaining only the constant Fourier mode in u_1 in the limit where we omit exponential dependence on u_2, we are led to keep only the contribution from $N = 0$, and we find,

$$\mathcal{K}^0_{aabb} = \sum_{M\neq 0} \mathcal{T}(M,0)|\mathcal{F}(M,0)|^2\, e^{2\pi i M u_2} + \mathcal{O}(e^{-2yu_2}) \tag{C.39}$$

Note that $F(0,0) = 0$ so that only $M \neq 0$ contributes. Expressing both $\mathcal{T}$ and $\mathcal{F}$ in terms of the function $\mathfrak{J}$ using (C.24), we find,

$$\begin{aligned}\mathcal{K}^0_{aabb} = 128y^4 \sum_{M\neq 0} \frac{e^{2\pi i M u_2}}{\pi^2 M^2}\left(\frac{1}{12} - \frac{3}{4\pi^2 M^2} + \mathfrak{I}(M)\right)\\ \times\left(\frac{1}{12} - \frac{1}{2\pi^2 M^2} + \mathfrak{I}(M)\right)^2\end{aligned} \tag{C.40}$$

As a result of the right most equation in (C.31), the contribution from the term in $\mathfrak{I}^3$ sums to zero. The integrals for the remaining terms may be evaluated using the remaining formulas in (C.30), (C.31), and (C.33) and we find,

$$\begin{aligned}\mathcal{K}^0_{aabb} = \frac{4y^4}{135}\left(5B_2 + 35B_4 + 32B_6 + \frac{36}{7}B_8\right)\\ - \frac{y}{3}\zeta(3)\Big(1 + 28B_2 + 32B_4\Big) - \frac{\zeta(5)}{6y}\Big(7 + 48B_2\Big)\\ + \frac{7\zeta(3)^2}{2y^2} - \frac{\zeta(7)}{y^3} + \mathcal{O}(e^{-2yu_2})\end{aligned} \tag{C.41}$$

The leading term in fact agrees with the naive evaluation of the integral (C.36) by replacing $g(z)$ by its polynomial approximation $\mathfrak{g}_1(z)$.

C.7. Degeneration of $\mathcal{K}_{abab}$ and $\mathcal{K}_{abba}$

The starting point is pair of integrals defined in (C.42). By translation invariance and reflection symmetry, we may shift z, w by p_a and express the result in terms of v,

(C.42)
$$\mathcal{K}_{abab} = \frac{\tau_2^2}{\pi^2} \int_{\Sigma_1} \kappa_1(z) \int_{\Sigma_1} \kappa_1(w) \partial_z g(z-v) \partial_{\bar{z}} g(z) g(z-w)^2 \partial_w g(w-v) \partial_{\bar{w}} g(w)$$

(C.43)
$$\mathcal{K}_{abba} = \frac{\tau_2^2}{\pi^2} \int_{\Sigma_1} \kappa_1(z) \int_{\Sigma_1} \kappa_1(w) \partial_z g(z-v) \partial_{\bar{z}} g(z) g(z-w)^2 \partial_w g(w) \partial_{\bar{w}} g(w-v)$$

As we shall neglect exponential corrections we are interested only in the zero mode in u_1, where $v = u_1 + \tau u_2$. To extract it projection, we define the following integrals,

$$\begin{aligned} L^+(z,w;u_2) &= \int_0^1 du_1\, \partial_z g(z-v)\, \partial_w g(w-v) \\ L^-(z,w;u_2) &= \int_0^1 du_1\, \partial_z g(z-v)\, \partial_{\bar{w}} g(w-v) \end{aligned} \tag{C.44}$$

The dependence of $L^\pm$ on the modulus τ is understood throughout. It will be convenient to parametrize $z = z_1 + \tau z_2$ and $w = w_1 + \tau w_2$ with $z_1, z_2, w_1, w_2 \in \mathbb{R}$ and with ranges $0 \le z_1, w_1 \le 1$ and $-\frac{1}{2} \le z_2, w_2 \le \frac{1}{2}$. Since $L^\pm(z,w;u_2)$ depends only on the combination $z_1 - w_1$, it is clear that the remaining integration over w_1 has the net effect of projecting onto the constant Fourier mode of the complex conjugate combinations as well, and we have,

(C.45)
$$\begin{aligned} \mathcal{K}_{abab} &= \frac{\tau_2^2}{\pi^2} \int_0^1 dz_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} dz_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} dw_2\, g(z-w)^2 L^+(z,w;u_2) \overline{L^+(z,w;0)} \\ &\quad + \mathcal{O}(e^{-2\pi\tau_2 u_2}) \\ \mathcal{K}_{abba} &= \frac{\tau_2^2}{\pi^2} \int_0^1 dz_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} dz_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} dw_2\, g(z-w)^2 L^-(z,w;u_2) \overline{L^-(z,w;0)} \\ &\quad + \mathcal{O}(e^{-2\pi\tau_2 u_2}) \end{aligned}$$

We shall compute $L^\pm$ for $-\frac{1}{2} \leq u_2 < \frac{1}{2}$ using the following formula,

$$\partial_z g(z) = -i\pi \frac{e^{2\pi i(z_1+\tau z_2)}+1}{e^{2\pi i(z_1+\tau z_2)}-1} - 2\pi i z_2 + \mathcal{O}(e^{-\pi\tau_2}) \tag{C.46}$$

It will be convenient to organize the result in terms of the Fourier components in $z_1 - w_1$,

$$L^\pm(z,w;u_2) = 4\pi^2 \sum_{n=-\infty}^{\infty} \ell_n^\pm(z_2,w_2;u_2)\, e^{2\pi i n(z_1-w_1)} \tag{C.47}$$

where the Fourier coefficients $\ell_n^\pm$ are given by,

(C.48)

$$\begin{aligned}
\ell_0^\pm(z_2,w_2;u_2) &= \mp\Big(z_2-u_2-\frac{1}{2}\varepsilon(z_2-u_2)\Big)\Big(w_2-u_2-\frac{1}{2}\varepsilon(w_2-u_2)\Big)\\
\ell_n^+(z_2,w_2;u_2) &= e^{2\pi i n\tau(z_2-w_2)}\,\tilde{\ell}_n^+(z_2,w_2;u_2)\\
\ell_n^-(z_2,w_2;u_2) &= e^{2\pi i n\left(\tau(z_2-u_2)-\bar{\tau}(w_2-u_2)\right)}\,\tilde{\ell}_n^-(z_2,w_2;u_2)
\end{aligned}$$

and the functions $\tilde{\ell}_n^\pm$ are given by (using the convention $\theta(0)=0$),

(C.49)

$$\begin{aligned}
\tilde{\ell}_n^+(z_2,w_w;u_2) &= \theta(n)\theta(z_2-u_2)\theta(u_2-w_2)+\theta(-n)\theta(u_2-z_2)\theta(w_2-u_2)\\
\tilde{\ell}_n^-(z_2,w_w;u_2) &= \theta(n)\theta(z_2-u_2)\theta(w_2-u_2)+\theta(-n)\theta(u_2-z_2)\theta(u_2-w_2)
\end{aligned}$$

We shall also use the Fourier expansion of $g(z-w)^2$, given in (C.20). The integrals over z_1 may now be performed (where we abbreviate $z=z_2, w=w_2$),

(C.50)

$$\begin{aligned}
\mathcal{K}_{abab} &= 16y^2 \sum_{M,N,n} T(M,N) \int_{-\frac{1}{2}}^{\frac{1}{2}} dz \int_{-\frac{1}{2}}^{\frac{1}{2}} dw\, e^{2\pi i M(z-w)}\, \ell_n^+(z,w;u_2)\\
&\quad\times \overline{\ell_{n-N}^+(z,w;0)}\\
\mathcal{K}_{abba} &= 16y^2 \sum_{M,N,n} T(M,N) \int_{-\frac{1}{2}}^{\frac{1}{2}} dz \int_{-\frac{1}{2}}^{\frac{1}{2}} dw\, e^{2\pi i M(z-w)}\, \ell_n^-(z,w;u_2)\\
&\quad\times \overline{\ell_{n-N}^-(z,w;0)}
\end{aligned}$$

up to exponential corrections. There are five cases to be distinguished,

(C.51)

(1)	$n \neq 0,$	$n \neq N$	
(2)	$n = 0,$	$n \neq N$	requiring $N \neq 0$
(3)	$n \neq 0,$	$n = N$	requiring $N \neq 0$
(4)	$n = 0,$	$n = N, M \neq 0$	requiring $N = 0$
(5)	$n = 0,$	$n = N, M = 0$	requiring $N = 0$

We designate the contributions to $\mathcal{K}_{abab}$ and $\mathcal{K}_{abba}$ of each sum range above respectively by $\mathcal{K}^{(i)}_{abab}$ and $\mathcal{K}^{(i)}_{abba}$ for $i = 1, 2, 3, 4, 5$. The following contributions are pairwise equal,

$$\begin{aligned} \mathcal{K}^{(1)}_{abab} &= \mathcal{K}^{(1)}_{abba} = \mathcal{O}(e^{-2yu_2}) \\ \mathcal{K}^{(5)}_{abab} &= \mathcal{K}^{(5)}_{abba} = 4y^2 E_2\, B_2^2 + \mathcal{O}(e^{-2yu_2}) \end{aligned} \tag{C.52}$$

as well as,

(C.53)

$$\begin{aligned} \mathcal{K}^{(4)}_{abab} = \mathcal{K}^{(4)}_{abba} &= y^2 \sum_{M\neq 0} T(M,0) e^{2\pi i M u_2} \left(\frac{1}{\pi^4 M^4} - \frac{4B_2 + \frac{1}{3}}{\pi^2 M^2} - \frac{4iB_1}{\pi^3 M^3} \right) + \text{c.c.} \\ &\quad + y^2 \sum_{M\neq 0} T(M,0) \left(\frac{2}{\pi^4 M^4} + \frac{8B_2 + \frac{2}{3}}{\pi^2 M^2} \right) + \mathcal{O}(e^{-2yu_2}) \\ \mathcal{K}^{(2)}_{abab} = \overline{\mathcal{K}^{(3)}_{abab}} &= y^2 \sum_{M} \sum_{N\neq 0} T(M,N) \left(\frac{1}{\pi^4 (M + \bar\tau N)^4} + \frac{4B_2 + \frac{1}{3}}{\pi^2 (M + \bar\tau N)^2} \right) \\ \mathcal{K}^{(2)}_{abba} = \mathcal{K}^{(3)}_{abba} &= y^2 \sum_{M} \sum_{N\neq 0} T(M,N) \left(\frac{1 + 4yNB_1(u_2)}{\pi^4 |M + \tau N|^4} + \frac{4B_2 + \frac{1}{3}}{\pi^2 |M + \tau N|^2} \right) \end{aligned}$$

We shall denote the contribution of the first and second line of (C.53) by $\mathcal{K}^{(4')}_{abab}$ and $\mathcal{K}^{(4'')}_{abab}$, respectively.

The term proportional to $B_1(u_2)$ cancels since the summand that multiplies it is odd in $(M, N) \to (-M, -N)$ while $T(M, N)$ is even. The asymptotics of the first line of (C.53) can be computed using the techniques de-

veloped earlier,

(C.54)
$$\begin{aligned}\mathcal{K}^{(4')}_{abab} = \mathcal{K}^{(4')}_{abba} &= 2y^4\left(\frac{164}{105}B_8 + \frac{416}{135}B_6 + \frac{92}{135}B_4 - \frac{8}{14175}\right)\\ &\quad - 2y\zeta(3)\left(\frac{10}{3}B_4 + 2B_2 + \frac{4}{45}\right) + \frac{\zeta(5)}{y}\left(B_2 + \frac{1}{6}\right) - \frac{\zeta(7)}{8y^3}\\ &\quad + \mathcal{O}(e^{-2yu_2})\end{aligned}$$

For the remaining terms, we recognize the contributions with $(M,N) \neq 0$ as arising from 2-loop modular graph forms, whose definition and normalization we recall here,

$$\mathcal{C}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \left(\frac{\tau_2}{\pi}\right)^{\frac{1}{2}\sum_i (a_i+b_i)} \sum_{p_1,p_2,p_3\in\Lambda'} \frac{\delta(p_1+p_2+p_3)}{p_1^{a_1}p_2^{a_2}p_3^{a_3}\bar{p}_1^{b_1}\bar{p}_2^{b_2}\bar{p}_3^{b_3}} \tag{C.55}$$

for $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ and $\Lambda' = \Lambda \setminus \{0\}$. When $b_i = a_i$ for $i = 1,2,3$, we shall use the simplified notation $\mathcal{C}_{a_1,a_2,a_3}$ instead. In terms of these functions, we have

$$\begin{aligned}\mathcal{K}^{(3)}_{abab} + \frac{1}{2}\mathcal{K}^{(4'')}_{abab} &= \mathcal{C}\begin{bmatrix} 4 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + 4y\left(B_2 + \frac{1}{12}\right)\mathcal{C}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\\ \mathcal{K}^{(2)}_{abba} + \frac{1}{2}\mathcal{K}^{(4'')}_{abba} &= \mathcal{C}_{2,1,1} + 4y\left(B_2 + \frac{1}{12}\right)\mathcal{C}_{1,1,1}\end{aligned} \tag{C.56}$$

We shall need the asymptotics of $\mathcal{C}_{1,1,1}$ and $\mathcal{C}_{2,1,1}$, as well as of D_4,

$$\mathcal{C}_{1,1,1} = \frac{2y^3}{945} + \zeta(3) + \frac{3\zeta(5)}{4y^2} + \mathcal{O}(e^{-2y}) \tag{C.57}$$
$$\mathcal{C}_{2,1,1} = \frac{2y^4}{14175} + \frac{y\zeta(3)}{45} + \frac{5\zeta(5)}{12y} - \frac{\zeta(3)^2}{4y^2} + \frac{9\zeta(7)}{16y^3} + \mathcal{O}(e^{-2y})$$
$$D_4 = \frac{y^4}{945} + \frac{2}{3}y\zeta(3) + \frac{10\zeta(5)}{y} - \frac{3\zeta(3)^2}{y^2} + \frac{9\zeta(7)}{4y^3} + \mathcal{O}(e^{-2y}) \tag{C.58}$$

The asymptotics of $\mathcal{C}\left[\begin{smallmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right]$ and $\mathcal{C}\left[\begin{smallmatrix} 4 & 1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right]$ can be obtained by taking successive derivatives with respect to $\nabla = 2i\tau_2^2\partial_\tau$ (which reduces to $\tau_2^2\partial_{\tau_2}$ when acting on functions independent of τ_1):

$$\mathcal{C}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{3\tau_2}\nabla E_3 = \frac{2y^3}{945} - \frac{\zeta(5)}{2y^2} + \mathcal{O}(e^{-2y}) \tag{C.59}$$

while we also have (see eq (4.26) of [9]),

$$\mathcal{C}\begin{bmatrix}4 & 1 & 1\\ 0 & 1 & 1\end{bmatrix} = \frac{1}{24\tau_2^2}\nabla^2 D_4 - \frac{1}{4\tau_2^2}E_2\,\nabla^2 E_2 \tag{C.60}$$

As a result we obtain the following asymptotics,

$$\mathcal{C}\begin{bmatrix}4 & 1 & 1\\ 0 & 1 & 1\end{bmatrix} = \frac{2y^4}{14175} + \frac{y\zeta(3)}{45} - \frac{\zeta(3)^2}{4y^2} + \frac{9\zeta(7)}{16y^3} + \mathcal{O}(e^{-2y}) \tag{C.61}$$

Combining these results, we find

(C.62)

$$\begin{aligned}\mathcal{K}_{abab} = {} & \frac{y^4}{945}\left(1 + 44B_2 + 1372B_4 + 5824B_6 + 2952B_8\right)\\ & - \frac{y\zeta(3)}{3}\left(8B_4 + 8B_2 + \frac{1}{3}\right) - \frac{\zeta(5)}{y}\left(3B_2 + \frac{1}{6}\right) - \frac{\zeta(3)^2}{2y^2} + \frac{\zeta(7)}{y^3}\\ & + \mathcal{O}(e^{-2yu_2})\end{aligned}$$

(C.63)

$$\begin{aligned}\mathcal{K}_{abba} = {} & \frac{y^4}{945}\left(1 + 44B_2 + 1372B_4 + 5824B_6 + 2952B_8\right)\\ & - \frac{y\zeta(3)}{3}\left(8B_4 - 16B_2 - \frac{5}{3}\right) + \frac{\zeta(5)}{y}\left(7B_2 + \frac{3}{2}\right) - \frac{\zeta(3)^2}{2y^2} + \frac{\zeta(7)}{y^3}\\ & + \mathcal{O}(e^{-2yu_2})\end{aligned}$$

The leading term in each expression agrees with the result obtained by evaluating the integrals (C.42), (C.43) after replacing $g(z)$ by its polynomial approximation $\mathfrak{g}_1(z)$.

C.8. Degeneration of $\mathcal{K}^0_{aaab}$

The modular graph function $\mathcal{K}^0_{aaab}$ was defined in (B.79). Using translation invariance, we may shift z and w by p_a and express the result solely in terms of v,

$$\begin{aligned}\mathcal{K}^0_{aaab} = {} & \frac{\tau_2^2}{\pi^2}\int_{\Sigma_1}\kappa_1(z)\int_{\Sigma_1}\kappa_1(w)|\partial_z g(z)|^2\partial_w g(w)\partial_{\bar{w}} g(w-v)\\ & \times\left(g(z,w)^2 - g(w)^2\right)\end{aligned} \tag{C.64}$$

We use the relation (C.37) to factorize the dependence of the integrand on z and w. The result may be expressed as follows,

$$\mathcal{K}^0_{aaab} = \sum_{(M,N)\neq(0,0)} \mathcal{T}(M,N)\,\mathcal{F}(M,N)\,\mathcal{G}(M,N) \tag{C.65}$$

where $\mathcal{T}$ was defined in (C.21) and $\mathcal{F}$ was calculated in (C.22) in terms of $\mathcal{T}$. The coefficients $\mathcal{G}(M,N)$ are defined as follows,

$$\mathcal{G}(M,N) = \frac{\tau_2}{\pi}\int_{\Sigma_1} \kappa_1(w)\partial_w g(w)\partial_{\bar{w}} g(w-v)\, e^{-2\pi i(Mw_2 - Nw_1)} \tag{C.66}$$

Since we neglect exponentially suppressed contributions, we retain only the zero mode in u_1, which is calculated by integrating over u_1,

$$\int_0^1 du_1 g(w-v) = 2yB_2(|w_2-u_2|) \tag{C.67}$$

Substituting this result into the definition of $\mathcal{G}(M,N)$, we find,

(C.68)

$$\begin{aligned}\mathcal{G}(M,N) = i\tau_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} dw_2\, \partial_{u_2} B_2(|w_2-u_2|) e^{-2\pi i M w_2} \int_0^1 dw_1 \partial_w g(w)\, e^{2\pi i N w_1} \\ + \mathcal{O}(e^{-2yu_2})\end{aligned}$$

Within this approximation, the w_1-integral may be carried out using (C.46), and we find,

(C.69)

$$\begin{aligned}\mathcal{G}(M,0) &= -\frac{2iyB_1(u_2)}{\pi M} + \frac{y}{\pi^2 M^2} + \left(\frac{2iyB_1(u_2)}{\pi M} + \frac{y}{\pi^2 M^2}\right) e^{-2\pi i M u_2} \\ \mathcal{G}(M,N) &= -\frac{2iyB_1(u_2)}{\pi(M+N\tau)} + \frac{y}{\pi^2(M+N\tau)^2}\end{aligned}$$

where on the first line $M \neq 0$ and on the second line $N \neq 0$. Since we have retained only the zero mode the above formulas are valid up to exponentially suppressed contributions.

We split the sum in (C.65) according to whether $N = 0$ or not, and further split the $N = 0$ part into parts with and without exponential u_2-dependence in $\mathcal{G}(M,0)$,

$$\mathcal{K}^0_{aaab} = \mathcal{K}_A + \mathcal{K}_B + \mathcal{K}_C \tag{C.70}$$

where

$$
\begin{aligned}
\mathcal{K}_A &= y \sum_{M\neq 0} \mathcal{T}(M,0)\,\mathcal{F}(M,0)\left(\frac{2iB_1(u_2)}{\pi M}+\frac{1}{\pi^2 M^2}\right) e^{-2\pi i M u_2} \\
\mathcal{K}_B &= y \sum_{M\neq 0} \mathcal{T}(M,0)\,\mathcal{F}(M,0)\,\frac{1}{\pi^2 M^2} \\
\mathcal{K}_C &= y \sum_{N\neq 0}\sum_{M} \mathcal{T}(M,N)\,\mathcal{F}(M,N)\,\frac{1}{\pi^2 (M+N\tau)^2}
\end{aligned}
\tag{C.71}
$$

We have simplified these expressions by using the fact that the terms which are proportional to $B_1(u_2)$ in the non-exponential terms in $\mathcal{G}$ are odd under $(M,N)\to(-M,-N)$ and sum to zero since both $\mathcal{T}$ and $\mathcal{F}$ are even.

C.8.1. Calculating $\mathcal{K}_B+\mathcal{K}_C$ As a result, $\mathcal{K}_B$ and $\mathcal{K}_C$ are independent of u_2. Their sum is an ordinary genus-one modular graph function,

$$
\mathcal{K}_B+\mathcal{K}_C = \sum_{(M,N)\neq(0,0)} \mathcal{T}(M,N)\,\mathcal{F}(M,N)\,\frac{\tau_2}{\pi(M+N\tau)^2}
\tag{C.72}
$$

Using the explicit expression for $\mathcal{F}$ of (C.23), we find more explicitly,

$$
\begin{aligned}
\mathcal{K}_B+\mathcal{K}_C = -\sum_{(M,N)\neq(0,0)} &\left(\frac{\tau_2^2\,\mathcal{T}(M,N)}{\pi^2(M+N\tau)^3(M+N\bar\tau)}\right. \\
&\left.+\frac{1}{2}\mathcal{T}(M,N)^2\,\frac{M+N\bar\tau}{M+N\tau}\right)
\end{aligned}
\tag{C.73}
$$

This is recognized as the sum of the following modular graph functions [9],

$$
\mathcal{K}_B+\mathcal{K}_C = -\mathcal{C}\begin{bmatrix}3&1&1\\1&1&1\end{bmatrix} - \frac{1}{2}\,\mathcal{C}\left[\begin{array}{cc|cc|c}1&1&1&1&1\\1&1&1&1&-1\end{array}\right]
\tag{C.74}
$$

We may simplify the trihedral modular graph function using the rules of [9, §7],

$$
\mathcal{C}\left[\begin{array}{cc|cc|c}1&1&1&1&1\\1&1&1&1&-1\end{array}\right] = 2\,\mathcal{C}\left[\begin{array}{cc|cc|c}1&1&1&1&1\\1&1&1&0&0\end{array}\right] = 2\,\mathcal{C}\left[\begin{array}{cc|cc|c}1&1&2&1&0\\1&1&1&0&0\end{array}\right] - 2\,\mathcal{C}\left[\begin{array}{cc|cc|c}1&1&2&0&1\\1&1&1&0&0\end{array}\right]
\tag{C.75}
$$

Using now also the algebraic reduction formulas of [9], we find,

$$\mathcal{C}\left[\begin{array}{cc|cc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{array}\right] = \frac{1}{2\tau_2}\nabla E_2^2 - \frac{1}{6\tau_2}\nabla D_4 + 2\,\mathcal{C}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{C.76}$$

Hence we have,

$$\mathcal{K}_B + \mathcal{K}_C = -\frac{1}{4\tau_2}\nabla E_2^2 + \frac{1}{12\tau_2}\nabla D_4 - 2\,\mathcal{C}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{C.77}$$

To evaluate the last term is more involved. We begin with the observation that this modular graph form satisfies the following differential equation (for the rules of differentiation and further manipulation of modular graph forms, see [9]),

$$\nabla\left(\tau_2\,\mathcal{C}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = 3\tau_2^2\mathcal{C}\begin{bmatrix} 4 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \frac{1}{4}(\nabla E_2)^2 - \frac{1}{20}\nabla^2 E_4 \tag{C.78}$$

Since the Laurent expansion on the right side is known, we obtain the following Laurent expansion by integrating the above differential equation

$$\mathcal{C}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{2y^4}{14175} + \frac{\zeta(3)y}{45} + \frac{c}{y} + \frac{\zeta(3)^2}{2y^2} - \frac{3\zeta(7)}{4y^3} + \mathcal{O}(e^{-2y}) \tag{C.79}$$

where c is the integration constant which is left undetermined by the above calculation. It may be evaluated by direct summation, and one finds $c = -5\zeta(5)/12$. Using these asymptotics we obtain,

$$\mathcal{K}_B + \mathcal{K}_C = -\frac{2y^4}{4725} + \frac{15\zeta(7)}{16y^3} + \mathcal{O}(e^{-2y}) \tag{C.80}$$

C.8.2. Calculating $\mathcal{K}_A$ It remains to evaluate $\mathcal{K}_A$, which we write out explicitly as follows,

(C.81)

$$\begin{aligned}\mathcal{K}_A = &-32y^4\sum_{M\neq 0}\left(\frac{1}{12} - \frac{3}{4\pi^2M^2} + \mathfrak{J}(M)\right)\left(\frac{1}{12} - \frac{1}{2\pi^2M^2} + \mathfrak{J}(M)\right) \\ &\times\left(\frac{2iB_1}{\pi^3M^3} + \frac{1}{\pi^4M^4}\right)e^{-2\pi iMu_2}\end{aligned}$$

The contributions to $\mathcal{K}_A = \mathcal{K}_A^+ + \mathcal{K}_A^-$ respectively with an even or odd power of M multiplying the exponential, are given by,

(C.82)
$$\begin{aligned}\mathcal{K}_A^+ = & -y^4 \sum_{M\neq 0} \frac{e^{2\pi i M u_2}}{\pi^4 M^4}\Bigg(32\mathfrak{J}(M)^2 - \frac{40\mathfrak{J}(M)}{\pi^2 M^2} + \frac{16\mathfrak{J}(M)}{3} + \frac{12}{\pi^4 M^4} \\ & - \frac{10}{3\pi^2 M^2} + \frac{2}{9}\Bigg) \\ \mathcal{K}_A^- = & -2iB_1 y^4 \sum_{M\neq 0} \frac{e^{2\pi i M u_2}}{\pi^3 M^3}\Bigg(32\mathfrak{J}(M)^2 - \frac{40\mathfrak{J}(M)}{\pi^2 M^2} + \frac{16\mathfrak{J}(M)}{3} + \frac{12}{\pi^4 M^4} \\ & - \frac{10}{3\pi^2 M^2} + \frac{2}{9}\Bigg)\end{aligned}$$

We use (C.27), (C.33), (C.29), (C.30) to evaluate the remaining integrals, and we find,

(C.83)
$$\begin{aligned}\mathcal{K}_A^+ = & \, y^4\left(\frac{8}{105}B_8 + \frac{8}{27}B_6 + \frac{4}{27}B_4\right) - \frac{y}{3}\zeta(3)(10B_4 + 4B_2) \\ & - \frac{\zeta(5)}{6y}(15B_2 + 1) + \frac{\zeta(3)^2}{2y^2} - \frac{5\zeta(7)}{16y^3} + \mathcal{O}(e^{-2yu_2}) \\ \mathcal{K}_A^- = & \, y^4\left(-\frac{64}{105}B_8 - \frac{32}{15}B_6 - \frac{52}{45}B_4 - \frac{16}{315}B_2 + \frac{2}{4725}\right) \\ & + \frac{y}{3}\zeta(3)\left(40B_4 + 18B_2 + \frac{1}{3}\right) + \frac{5\zeta(5)}{y}\left(B_2 + \frac{1}{12}\right) + \mathcal{O}(e^{-2yu_2})\end{aligned}$$

Collecting all terms together we get

(C.84)
$$\begin{aligned}\mathcal{K}_{aaab}^0 = & -\frac{y^4}{945}(48B_2 + 952B_4 + 1736B_6 + 504B_8) \\ & + y\zeta(3)\left(10B_4 + \frac{14}{3}B_2 + \frac{1}{9}\right) + \frac{\zeta(5)}{4y}(10B_2 + 1) + \frac{\zeta(3)^2}{2y^2} + \frac{5\zeta(7)}{8y^3} \\ & + \mathcal{O}(e^{-2yu_2})\end{aligned}$$

The leading term agrees with the result obtained by evaluating the integral (C.64) after replacing $g(z)$ by its polynomial approximation $\mathfrak{g}_1(z)$.

C.9. Summary of the tropical degeneration

Collecting the various contributions computed in the previous subsections we can now state the tropical limits of the string invariants considered in Section 5.2. For the Kawazumi-Zhang invariant, we recover the result obtained earlier in [32]:

$$\varphi^{(t)} = \frac{\pi t}{6} + y\, B_2 + \frac{5y^2}{6\pi t}\left(B_4 + \frac{1}{30}\right) + \frac{5\zeta(3)}{4\pi y t} \tag{C.85}$$

where we recall that we denote $B_{2n} = B_{2n}(|u_2|)$, and assume that $|u_2| < 1$.

For the invariants $\mathcal{Z}_2$ and $\mathcal{Z}_3$ defined in (3.21), we find

(C.86)

$$\begin{aligned}\mathcal{Z}_2^{(t)} = &-\frac{7(\pi t)^2}{90} - \frac{2\pi t y}{3}B_2 - y^2\left(\frac{5}{3}B_4 + \frac{2}{3}B_2 + \frac{1}{45}\right)\\ &-\frac{y^3}{\pi t}\left(\frac{74}{45}B_6 + \frac{4}{3}B_4 + \frac{1}{189}\right) - 17\frac{y^4}{(\pi t)^2}\left(\frac{1}{30}B_8 + \frac{2}{45}B_6 + \frac{1}{18900}\right)\\ &-\frac{\zeta(3)}{2}\left(\frac{1}{y} + \frac{6B_2}{\pi t} + \frac{(5B_4 + \frac{1}{6})y}{(\pi t)^2}\right) - \frac{7\zeta(5)}{4}\left(\frac{1}{y^2\pi t} + \frac{(B_2 + \frac{5}{6})y}{(\pi t)^2}\right)\end{aligned}$$

(C.87)

$$\begin{aligned}\mathcal{Z}_3^{(t)} = &\frac{(\pi t)^2}{18} + \frac{2\pi t y}{3}B_2 + y\left(\frac{19}{9}B_4 + \frac{2}{3}B_2 + \frac{2}{135}\right)\\ &+\frac{y^3}{\pi t}\left(\frac{22}{9}B_6 + \frac{16}{9}B_4 + \frac{1}{945}\right) + 17\frac{y^4}{(\pi t)^2}\left(\frac{1}{18}B_8 + \frac{2}{27}B_6 + \frac{1}{11340}\right)\\ &+\frac{\zeta(3)}{6}\left(\frac{1}{y} + \frac{6B_2}{\pi t} + \frac{(5B_4 + \frac{1}{6})y}{(\pi t)^2}\right) + \frac{11}{8(\pi y t)^2}\zeta(3)^2\end{aligned}$$

For the invariant $\mathcal{Z}_1$ defined in (3.21), we require the tropical limit of the term $\mathcal{K}^c$ defined in (3.40). Using the results of Sections C.6-§C.8 we find

(C.88)

$$\begin{aligned}\mathcal{K}^c = &\, y^4\left(\frac{16}{5}B_8 + \frac{64}{15}B_6 + \frac{2}{675}\right) - 32y\zeta(3)\left(B_4 + \frac{1}{60}\right)\\ &-\frac{\zeta(5)}{y}\left(16B_2 - \frac{65}{6}\right) - \frac{9\,\zeta(3)^2}{2y^2} + \frac{3\zeta(7)}{4y^3} + \mathcal{K}^0_{aaaa} + \mathcal{O}(e^{-2yu_2})\end{aligned}$$

where $\mathcal{K}^0_{aaaa}$ is the integral (3.38). We have not analyzed the tropical limit of this integral (which is a function of τ but not of v), but we shall be able to infer it indirectly, up to an undetermined term proportional to $1/y^2$ (see (5.17)). It follows from (C.88) and earlier results in this section that the tropical limit of $\mathcal{Z}_1$ is given by

(C.89)

$$\begin{aligned}\mathcal{Z}_1^{(t)} = {} & \frac{13(\pi t)^2}{90} + \frac{2\pi t y}{3} B_2 + y^2 \left(\frac{5}{3} B_4 + \frac{2}{3} B_2 + \frac{4}{45} \right) \\ & + \frac{y^3}{\pi t} \left(\frac{86}{45} B_6 + \frac{4}{3} B_4 - \frac{1}{945} \right) \\ & + \frac{y^4}{(\pi t)^2} \left(\frac{23}{30} B_8 + \frac{46}{45} B_6 + \frac{1}{1050} \right) \\ & + \zeta(3) \left(\frac{7}{2y} + \frac{3B_2 + 3}{\pi t} - \frac{y(B_4(u_2) - \frac{11}{30})}{2(\pi t)^2} \right) \\ & + \frac{\zeta(5)}{2} \left(-\frac{1}{2\pi t\, y^2} - \frac{B_2(u_2) - \frac{125}{24}}{2(\pi t)^2\, y} \right) + \frac{3\zeta(7)}{32\pi^2 y^3 t^2} - \frac{3\zeta(3)^2}{(4\pi y t)^2} + \frac{\mathcal{K}^0_{aaaa}}{8\pi^2 t^2}\end{aligned}$$

The tropical limit of the complete string invariant $\mathcal{B}^{(t)}_{(2,0)} = \frac{1}{2}\big(\mathcal{Z}_1^{(t)} - 2\mathcal{Z}_2^{(t)} + \mathcal{Z}_3^{(t)}\big)$ is then given by

(C.90)

$$\begin{aligned}\mathcal{B}^{(t)}_{(2,0)} = {} & \frac{8(\pi t)^2}{45} + \frac{4\pi t y}{3} B_2 + y^2 \left(\frac{32}{9} B_4 + \frac{4}{3} B_2 + \frac{2}{27} \right) \\ & + \frac{y^3}{\pi t} \left(\frac{172}{45} B_6 + \frac{26}{9} B_4 + \frac{1}{189} \right) \\ & + \frac{y^4}{(\pi t)^2} \left(\frac{64}{45} B_8 + \frac{256}{135} B_6 + \frac{241}{113400} \right) \\ & + \zeta(3) \left(\frac{7}{3y} + \frac{5B_2 + \frac{3}{2}}{\pi t} + \frac{(8B_4 + \frac{17}{30})\, y}{3(\pi t)^2} \right) \\ & + \zeta(5) \left(\frac{3}{2y^2 \pi t} + \frac{(3B_2 + \frac{265}{48})}{2y\,(\pi t)^2} \right) + \frac{3\zeta(7)}{64\pi^2 y^3 t^2} + \frac{19\,\zeta(3)^2}{32(\pi y t)^2} + \frac{\mathcal{K}^0_{aaaa}}{16\pi^2 t^2}\end{aligned}$$

Upon changing variables from $(t, y = \pi\tau_2, u_2)$ to (V, S_1, S_2) using (5.9), and making use of the functions $A_{i,j}$ defined in (5.11), (5.14), we recover the results announced in Section 5.2.

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