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Amplitude recursions with an extra marked point*

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The recursive calculation of Selberg integrals by Aomoto and Terasoma using the Knizhnik–Zamolodchikov equation and the Drinfeld associator makes use of an auxiliary point and facilitates the recursive evaluation of string amplitudes at genus zero: open-string N-point amplitudes can be obtained from those at $N-1$ points.

We establish a similar formalism at genus one, which allows the recursive calculation of genus-one Selberg integrals using an extra marked point in a differential equation of Knizhnik–Zamolodchikov–Bernard type. Hereby genus-one Selberg integrals are related to genus-zero Selberg integrals. Accordingly, N-point open-string amplitudes at one loop can be obtained from $(N+2)$-point open-string amplitudes at tree level. The construction is related to and in accordance with various recent results in intersection theory and string theory.

Keywords and phrases: Selberg integrals, string scattering, KZ equation, KZB equation.

1. Introduction

1.1. Recursion for open-string amplitudes at genus zero

Scattering amplitudes in open superstring theories at tree level are correlation functions of vertex operators inserted on the boundary of a disk. When evaluating those conformal correlators, the properties of the particular string theory in question can be straightforwardly incorporated in the so-called polarization part. What remains is the evaluation of the so-called

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configuration-space integrals such as the four-point Veneziano amplitude [Ven68]

$$\int_0^1 dx_3\, x_3^{s_{13}}(x_3-1)^{s_{23}}\,\frac{s_{13}}{x_3} = \frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})}\,. \tag{1}$$

The complex parameters

$$s_{i_1\dots i_r} = \alpha'(k_{i_1}+\ldots+k_{i_r})^2 \tag{2}$$

are Mandelstam variables built from the momenta k_{i_p} of the external particles. Throughout this article, these variables are assumed to meet a sufficient condition for the convergence of the integrals to be considered, such as $\Re(s_{i_1\dots i_r})>0$ for consecutive insertion points $x_{i_1}<\cdots<x_{i_r}$ [Man74, BD19].

In the case of N-point interactions and for appropriately fixed s_{ij}, the integrands of the configuration-space integrals are defined on the configuration space[1] $\mathcal{F}_{N,3}$ of $N-3$ insertion points x_i on $\mathbb{R}\setminus\{0,1\}$: these are the unfixed insertion points on the real line, which parametrises the boundary of the disk, formed by the tree-level worldsheet, embedded into the Riemann sphere. The SL(2)-symmetry of the Riemann sphere is used to fix three of the N punctures at zero, one and infinity. The configuration-space integrals are obtained from iteratively integrating these integrands over the $N-3$ variables of $\mathcal{F}_{N,3}$, i.e. the unfixed insertion points, on the unit interval. Finally, α' serves as counting parameter and will be identified with the inverse string tension, when considering actual string scattering amplitudes.

The N-point configuration-space integrals in genus-zero open-string amplitudes are examples of Selberg integrals [Sel44], which we denote and construct as follows: consider the $(L+1)$-punctured Riemann sphere with

$$(x_1,x_2,x_{L+1}) = (0,1,\infty) \tag{3}$$

fixed by the SL(2)-symmetry of the Riemann sphere. Writing

$$x_{ij} = x_{i,j} = x_i - x_j\,, \tag{4}$$

the corresponding integrals of Selberg type are iteratively defined by

(5)
$$\mathrm{S}[i_{k+1},\dots,i_L](x_1,\dots,x_k) = \int_0^{x_k}\frac{dx_{k+1}}{x_{k+1,i_{k+1}}}\,\mathrm{S}[i_{k+2},\dots,i_L](x_1,\dots,x_{k+1})\,,$$

[1] This is the real moduli space $\mathcal{M}_{0,N}=\mathcal{M}_{0,N}(\mathbb{R})=\mathcal{F}_{N,3}$. Below, we will introduce more general configuration spaces $\mathcal{F}_{L+1,k+1}$, which is why we rather use the notation $\mathcal{F}_{N,3}$ than $\mathcal{M}_{0,N}(\mathbb{R})$.

and the empty Selberg integral (or Selberg seed) is defined as[2]

$$(6)\qquad \mathrm{S}[](x_1,\ldots,x_L)=\prod_{0\le x_i<x_j\le 1} x_{ji}^{s_{ij}}\,.$$

The definition (5) presumes that the so-called admissibility condition

$$(7)\qquad 1\le i_p<p \qquad \forall p\in\{k+1,\ldots,L\}$$

is met. The integral in eq. (5) is said to be of type $(k, L{+}1)$ and is, for fixed s_{ij}, a function on $\mathcal{F}_{k+1,3}$.

Aomoto [Aom87] and Terasoma [Ter02] showed that Selberg integrals of type $(2, L)$ can be obtained algebraically from those of type $(2, L{-}1)$: one starts from a basis vector $\mathbf{S}(x_3)$ for Selberg integrals of type $(3, L{+}1)$, which contain an auxiliary point x_3 in contrast to the integrals of type $(2, L)$ and $(2, L{-}1)$, respectively. Taking the derivative with respect to x_3 leads to an equation of Knizhnik–Zamolodchikov (KZ) type [KZ84]

$$(8)\qquad \frac{d}{dx_3}\,\mathbf{S}(x_3)=\Big(\frac{e_0}{x_3}+\frac{e_1}{x_3-1}\Big)\,\mathbf{S}(x_3)\,,$$

where the (braid) matrices e_0 and e_1 have entries which are homogeneous polynomials of degree one in the parameters s_{ij}. The regularized boundary values

$$(9)\qquad \mathbf{C}_0=\lim_{x_3\to 0} x^{-e_0}\,\mathbf{S}(x_3)\ ,\quad \mathbf{C}_1=\lim_{x_3\to 1}(1-x_3)^{-e_1}\,\mathbf{S}(x_3)\,,$$

of the differential equation (46) can be shown to be related by the Drinfeld associator [Dri89, Dri91]

$$(10)\qquad \mathbf{C}_1=\mathbf{\Phi}(e_0,e_1)\ \mathbf{C}_0\,.$$

What makes this construction useful for physicists is the fact that the $(N{-}1)$-point and the N-point configuration-space integrals at genus zero can be identified (upon proper assignment of the Mandelstam variables) as linear combinations of the components of $\mathbf{C}_0$ and $\mathbf{C}_1$ respectively, where

[2]We use the notation $\prod_{x_a\le x_i<x_j\le x_b}=\prod_{i,j\in\{1,2,\ldots,L\}:\,x_a\le x_i<x_j\le x_b}$. This notation ensures that all the differences x_{ji} appearing in the product from eq. (6) are positive and real. In agreement with the standard notation in string theory, we will include the absolute values in the definitions of the Koba–Nielsen factors and propagators, and usually write the Mandelstam variables $s_{ij}=s_{ji}$ as s_{ij} with $i<j$.

$N{=}L$. This relationship has been used to derive a recursive construction for all configuration-space integrals on genus zero: it provides an analogue of the Parke–Taylor formula [PT86] for string theory [BSST14].

1.2. Open-string scattering at genus one

For a long time physicists have tried to find a similar recursive algorithm at genus one. In this article, we are going to establish such a construction. One-loop open-string amplitudes are calculated on an annulus: again, there is a polarization part and configuration-space integrals. For simplicity we are going to stick to those configuration-space integrals where points are inserted on one boundary exclusively. Upon embedding the annulus into a torus, the relevant boundary is identified with the A-cycle and parametrised by the unit interval. In the two-point case, the open-string one-loop configuration-space integral is of the form

$$\int_0^1 dz_2\, e^{s_{12}\,\tilde{\Gamma}_{21}} g_{21}^{(0)}\,. \tag{11}$$

The functions $g_{ij}^{(0)}$ and $\tilde{\Gamma}_{ij} = \int^{z_{ij}} dz\, g^{(1)}(z,\tau)$ (cf. eq. (85)) are defined by an infinite class of functions $g_{ij}^{(n)} = g^{(n)}(z_{ij},\tau)$ and integrals thereof, where n is a non-negative integer, $z_1 = 0$ and $z_{ij} = z_i - z_j$ is the difference of insertion points on the A-cycle of the torus. These functions are suitable genus-one analogues of the fractions in $\frac{dx_{k+1}}{x_{k+1,i_{k+1}}}$ from eq. (5) and the genus-zero propagator $\log x_{ji}$ appearing in eq. (6) in exponentiated form, respectively. They are defined by the Eisenstein–Kronecker series $F(z,\eta,\tau)$ [Kro81, BL11]

$$F(z,\eta,\tau) = \frac{\theta_1'(0,\tau)\theta_1(z+\eta,\tau)}{\theta_1(z,\tau)\theta_1(\eta,\tau)}\,, \tag{12}$$

where θ_1 is the odd Jacobi function and $'$ denotes a derivative with respect to the first argument: expanding in the second complex argument η, the function $g^{(n)}$ is the coefficient of η^{n-1}, i.e.

$$\eta F(z,\eta,\tau) = \sum_{n=0}^{\infty} g^{(n)}(z,\tau)\eta^n\,. \tag{13}$$

Various properties of these functions will be discussed thoroughly in subsection 3.1.

Considering the similarity between genus-zero configuration-space integrals such as the four-point example in eq. (1) and genus-zero Selberg integrals (5), it is very natural to define a suitable genus-one analogue of Selberg integrals:

Definition 1. Let $L \geq 2$, $0 = z_1 < z_L < ... < z_2 < 1$ and τ the modular parameter of the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Let the empty genus-one Selberg integral (or genus-one Selberg seed) be

$$\mathrm{S}^\tau = \mathrm{S}^\tau\Big[\;\Big](z_1,\ldots,z_L) = \prod_{0=z_1\leq z_i<z_j\leq z_2} \exp\left(s_{ij}\,\tilde{\Gamma}_{ji}\right) . \tag{14}$$

Genus-one Selberg integrals are then defined recursively by

$$\begin{aligned} &\mathrm{S}^\tau\left[\begin{smallmatrix} n_{k+1}, & \ldots, & n_L \\ i_{k+1}, & \ldots, & i_L \end{smallmatrix}\right](z_1,\ldots,z_k) \\ &\qquad = \int_0^{z_k} dz_{k+1}\, g^{(n_{k+1})}_{k+1,i_{k+1}}\, \mathrm{S}^\tau\left[\begin{smallmatrix} n_{k+2}, & \ldots, & n_L \\ i_{k+2}, & \ldots, & i_L \end{smallmatrix}\right](z_1,\ldots,z_{k+1})\,, \end{aligned} \tag{15}$$

where $1 \leq i_p < p$ for $k+1 \leq p \leq L$ and $n_{k+1},\ldots,n_L$ are non-negative integers.

The successful concept to calculate open-string configuration-space integrals from Selberg integrals, which worked for genus zero, will be extended here: starting from genus-one Selberg integrals of type $(1, L-1)$, which contain the genus-one open-string configuration-space integrals, one can introduce an auxiliary point z_2 leading to genus-one Selberg integrals of type $(2, L)$. Given a class of type-$(2, L)$ Selberg integrals of fixed weight $w = \sum_{i=k+1}^{L} n_i$, one can find a vector $\mathbf{S}^\tau_w(z_2)$ of basis elements with respect to Fay identities (a genus-one generalization of partial fractioning) and integration by parts. Concatenating all those basis vectors into an infinitely long vector $\mathbf{S}^\tau(z_2)$, one has constructed the genus-one analogue of the (finite-length) vector $\mathbf{S}(x_3)$ from above. This article is devoted to the construction of these integrals and to proving the following theorem:

Theorem 2 (Elliptic KZB-system)**.** *Let $\mathbf{S}^\tau(z_2)$ be the vector of genus-one Selberg integrals of type $(2, L)$ with auxiliary point z_2. The derivative with respect to the auxiliary point z_2 can be written in the form*

$$\frac{\partial}{\partial z_2}\,\mathbf{S}^\tau(z_2) = \sum_{n\geq 0} g^{(n)}_{21}\, x^{(n)}\,\mathbf{S}^\tau(z_2)\,, \tag{16}$$

which is a system of elliptic KZB-type. The non-vanishing entries of the matrices $x^{(n)}$ are $\mathbb{Z}$-linear combinations of the parameters s_{ij}.

While showing the closure of the above system of differential equations is elaborate, two regularized boundary values can be easily associated to each other following the statements in the next proposition.

Proposition 3. *The regularized boundary values*

(17)
$$\mathbf{C}_1^\tau = \lim_{z_2\to 1}(-2\pi i(1-z_2))^{-x^{(1)}}\,\mathbf{S}^\tau(z_2) \;\textit{and}\; \mathbf{C}_0^\tau \;= \lim_{z_2\to 0}(-2\pi i z_2)^{-x^{(1)}}\,\mathbf{S}^\tau(z_2)$$

are related by the A-cycle component $\Phi(x^{(0)}, x^{(1)}, x^{(2)}, \ldots)$ *of the KZB associator via*

$$\mathbf{C}_1^\tau = \Phi(x^{(0)}, x^{(1)}, x^{(2)}, \ldots)\,\mathbf{C}_0^\tau\;. \tag{18}$$

The regularized boundary value $\mathbf{C}_1^\tau$ *contains* $(L-1)$*-point configuration-space integrals at genus one whereas* $\mathbf{C}_0^\tau$ *contains* $(L+1)$*-point configuration-space integrals at genus zero.*

Therefore, the N-point configuration-space integrals appearing in open-string amplitudes at genus one can be calculated from the $(N+2)$-point integrals at genus zero via eq. (18), with $N{=}L{-}1$. As examples we will consider the construction suitable for two-, three- and four-point configuration-space integrals at genus one, which allow the determination of the (planar) two-, three- and four-point one-loop scattering amplitudes in open string theory.

1.3. Contents

In section 2 we are going to review the recursive evaluation of Selberg integrals at genus zero. We will apply the technique to genus-zero open-string amplitudes in a way equivalent to the approach in ref. [BSST14]. We are going to develop the genus-one formalism in section 3, prove the main theorem there and discuss the relation between genus-one objects and those at genus zero. Three examples are provided in section 4. In section 5 we conclude and point out several open questions.

2. Genus zero (tree level)

In this section we are going to review the recursive evaluation of genus-zero Selberg integrals of Aomoto and Terasoma [Aom90, Ter02] and relate it to the formalism for calculating open-string tree-level configuration-space integrals put forward in ref. [BSST14].

2.1. Singularities, iterated integrals and multiple zeta values

Configuration-space integrals for open-string tree-level amplitudes are defined on the boundary of a disk, on which the integration parameters x_i are located. Integrating over all configurations of the parameters while keeping their ordering along the boundary intact leads to iterated integrals.

Whithin the context of open-string tree-level configuration-space integrals, all integrations can either be performed trivially or can be traced back to iterated integrals of the following differential form on the Riemann sphere with a simple pole at the fixed insertion points $a_j \in \{x_1, x_2\} = \{0, 1\}$:

$$\frac{dx_i}{x_i - a_j}\,. \tag{19}$$

Accordingly, the canonical form of iterated integrals appearing in the α'-expansion of genus-zero configuration-space integrals are multiple polylogarithms

$$G(a_1, a_2, \dots, a_r; x) = \int_0^x dx' \, \frac{1}{x' - a_1} G(a_2, \dots, a_r; x'), \quad G(;x) = 1\,, \tag{20}$$

with $a_i \in \{0, 1\}$ and $a_r \neq 0$. Below, it will be useful to write this subclass of Goncharov polylogarithms [Gon95, Gon01] indexed by words of the form

$$w = e_0^{n_r - 1} e_1 \dots e_0^{n_1 - 1} e_1\,, \tag{21}$$

where $n_i \geq 1$:

$$G_w(x) = G(\underbrace{0, \dots, 0}_{n_r - 1}, 1, \dots, \underbrace{0, \dots, 0}_{n_1 - 1}, 1; x)\,. \tag{22}$$

Evaluating the above multiple polylogarithms with $n_r > 1$ at $x = 1$ leads to multiple zeta values

$$\zeta_w = (-1)^r G_w(1) = \sum_{1 \leq k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}}\,, \tag{23}$$

which are the transcendental numbers appearing in the α'-expansion of open-string configuration-space integrals at genus zero.

In the above definitions of multiple polylogarithms and multiple zeta values divergent situations have been excluded. However, using the definitions

$$
\begin{aligned}
G_{e_0}(x) &= G(0;x) = \log(x) \\
\zeta_{e_1} &= \zeta_{e_0} = 0
\end{aligned}
\tag{24}
$$

and shuffle relations between iterated integrals extended to all multiple polylogarithms and multiple zeta values

$$
G_{w'}(x)G_{w''}(x) = G_{w' \sqcup\!\sqcup\, w''}(x),\ \zeta_{w'}\zeta_{w''} = \zeta_{w' \sqcup\!\sqcup\, w''} \quad w', w'' \in \{e_0, e_1\}^{\times}\,,
\tag{25}
$$

one can extend the definition to all words from $\{e_0, e_1\}^{\times}$. This regularization scheme is referred to as *shuffle regularization* (or *tangential basepoint regularization*), see e.g. refs. [Del89, Bro14].

2.2. Selberg integrals

In comparison to the iterated integrals defined in eq. (20) above, genus-zero configuration-space integrals have one more ingredient: the empty integral to be iteratively integrated with integration kernels $1/(x-a_i)$ is not one, but the so-called Koba–Nielsen factor. It contains the open-string tree-level Green function, also known as *propagator*, weighted by Mandelstam variables s_{ij} defined in eq. (2). The Green function is the integral over the differential form from eq. (19): $\log|x_{ij}| = G(0, |x_{ij}|) = \int_1^{|x_{ij}|} \frac{dx}{x}$, where $x_{ij} = x_i - x_j$ is the difference of two insertion points.

The class of integrals accommodating the above features are *Selberg integrals* [Sel44, Aom87, Ter02]. Consider L points on the unit interval with the ordering

$$
0 = x_1 < x_L < x_{L-1} < \cdots < x_3 < x_2 = 1
\tag{26}
$$

and define the empty Selberg integral or *Selberg seed*

$$
\mathrm{S} = \mathrm{S}[](x_1, \ldots, x_L) = \prod_{0 \le x_i < x_j \le 1} \exp\left(s_{ij} \log x_{ji}\right) = \prod_{0 \le x_i < x_j \le 1} x_{ji}^{s_{ij}}\,,
\tag{27}
$$

with[3] complex parameters s_{ij}. The Selberg seed is integrated over various

[3]The empty integral S[] is called the *Koba–Nielsen factor* if the parameters s_{ij}

integration kernels of the form $1/x_{ij}$ which lead to functions denoted by

(28)
$$\mathrm{S}[i_{k+1},\dots,i_L](x_1,\dots,x_k)=\int_0^{x_k}\frac{dx_{k+1}}{x_{k+1,i_{k+1}}}\,\mathrm{S}[i_{k+2},\dots,i_L](x_1,\dots,x_{k+1})\,,$$

where

$$1\leq i_p<p\qquad\forall p\in\{k+1,\dots,L\}\,. \tag{29}$$

The above *admissibility condition* motivates the definition of *admissible* iterated integrals: the integration kernel $1/x_{k+1,i_{k+1}}$ in eq. (28) can not depend on variables which have already been integrated out. In accordance with ref. [Aom87], this property is called *admissibility* and an integral with an integrand of the form $S\prod_k 1/x_{k,i_k}$ satisfying eq. (29) *admissible.* As argued in subsection 2.4 and subsection 2.5, Selberg integrals of length $L-3$

$$\begin{aligned}\mathrm{S}[i_4,\dots,i_L](x_1,x_2,x_3)&=\int_0^{x_3}\frac{dx_4}{x_{4,i_4}}\,\mathrm{S}[i_5,\dots,i_L](x_1,\dots,x_4)\\&=\int_{\mathcal{C}(x_3)}\prod_{i=4}^{L}dx_i\;\mathrm{S}\prod_{k=4}^{L}\frac{1}{x_{k,i_k}}\,,\end{aligned} \tag{30}$$

where $\mathcal{C}(x_3)$ is the region of integration denoted by

$$\mathcal{C}(x_i)=\{0=x_1<x_L<x_{L-1}<\cdots<x_i\} \tag{31}$$

for $x_i\leq x_2=1$, include in the limit of merging punctures $x_3\to 1=x_2$ all integrals appearing in the calculation of L-point open-string tree-level scattering amplitudes.

For appropriately fixed s_{ij} and fixed unintegrated insertion points $x_1=0,x_2=1,x_3,\dots,x_k$ and $x_{L+1}=\infty$, the integrands in the Selberg integrals $\mathrm{S}[i_{k+1},\dots,i_L](x_1,\dots,x_k)$ are functions defined on the configuration space of the $L-k$ insertion points $x_{k+1},\dots,x_L$ on the k-punctured real line $\mathbb{R}\setminus\{x_1,\dots,x_k\}$:

$$\begin{aligned}&\mathcal{F}_{L+1,k+1}=\\&\quad\{(x_{k+1},x_{k+2},\dots,x_L)\in(\mathbb{R}\setminus\{x_1,\dots,x_k\})^{L-k}|\forall i\neq j:x_i\neq x_j\}\,.\end{aligned} \tag{32}$$

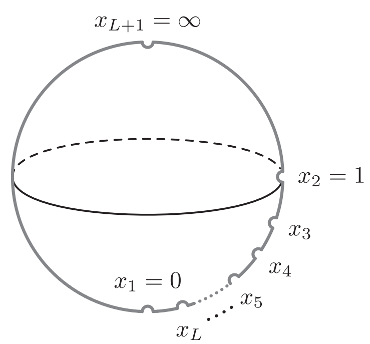
are identified with the momenta of the corresponding external states of a scattering amplitude. Note that according to eq. (26) all the differences x_{ji} in definition (27) are positive and real. This does apply to the genus-zero propagator mentioned above as well.

The differential forms

$$\bigwedge_{p=k+1}^{L} \frac{dx_p}{x_{p,i_p}}\,, \tag{33}$$

where $1 \leq i_p < p$, appearing in the integrands of the Selberg integrals in eq. (28), represent elements of a basis of the twisted de Rham cohomology of the $(L+1)$-punctured Riemann sphere with $k+1$ fixed coordinates: their twisted cohomology classes defined by the connection $d+d\log S$ pulled back to $\mathcal{F}_{L+1,k+1}$ appear in such a basis [Aom87], see also ref. [Miz19].

The Selberg integrals $\mathrm{S}[i_{k+1},\dots,i_L](x_1,\dots,x_k)$ in turn, depend on the unintegrated variables $x_1 = 0, x_2 = 1, x_3,\dots,x_k$ and $x_{L+1} = \infty$ with $x_i \neq x_j$, and are therefore functions defined on the configuration space $\mathcal{F}_{k+1,3}$. In particular, the Selberg seed from eq. (27) is defined on $\mathcal{F}_{L+1,3}$. A configuration of the form (26) in $\mathcal{F}_{L+1,3}$ can be depicted on the real line plus infinity embedded into a circle on the Riemann sphere as follows:



(34) .

2.3. KZ equation for an auxiliary point

The open-string configuration-space integrals at genus zero are recovered from the integrals

$$\mathrm{S}[i_4,\dots,i_L](x_1 = 0, x_2 = 1, x_3) \tag{35}$$

defined in eq. (30) in the following two regularized limits:

- in the limit $x_3 \to x_2 = 1$, it is merged with the point x_2 and one fixed puncture is removed. The integrands of the Selberg integrals defined on $\mathcal{F}_{L+1,4}$ degenerate to integrands on $\mathcal{F}_{L,3}$ of the integrals $\mathrm{S}[i_4,\dots,i_L](x_1{=}0, x_2{=}1, x_3{=}x_2)$. The space $\mathcal{F}_{L,3}$ is the configuration space known from open-string calculations with three fixed coordinates on which L-point tree-level amplitudes are defined. Indeed, as shown below, they will be recovered in this limit from the Selberg integrals.

- the merging of $x_3 \to x_1 = 0$ is slightly more involved and will lead to the $(L-1)$-point integrals in a certain soft limit, which leads to an additional degeneration of the integrands to functions defined on the configuration space $\mathcal{F}_{L-1,3}$ relevant for $L-1$-point tree-level amplitudes.

Thus, for $x_3 \in (0,1)$ the puncture interpolates between the L- and $(L-1)$-point open-string configuration-space integrals. These two boundary values can be related by a recursive procedure involving matrix operations [Aom87, Ter02], which leads to the genus-zero string recursion in ref. [BSST14]. The main idea hereby is the use of x_3 as an *auxiliary* insertion point, such that differentiating with respect to x_3 leads to a KZ equation (8), whose regularised boundary values in the above limits can be related via the Drinfeld associator according to eq. (10). In the remainder of this subsection, we will review differential equations for the Selberg integrals, while limits/boundary values of the differential equation are discussed in subsection 2.4 below. Attached to the point x_3 there is an auxiliary external momentum k_3 with associated Mandelstam variables s_{3i}, $i \in \{1,2,4,5,\dots L\}$. For the moment we are not imposing any conditions like the momentum conservation and consider the variables $s_{ij} = s_{ji}$ as independent parameters whose interpretation as Mandelstam variables in a scattering amplitude context will become clear when considering the limits $x_3 \to 0$ and $x_3 \to 1$ below.

Therefore, let us explore differential equations with respect to the auxiliary point x_3 acting on the Selberg integrals (35):

$$\frac{d}{dx_3}\,\mathrm{S}[i_4, i_5, \dots, i_L](0,1,x_3) = \frac{d}{dx_3}\int_{\mathcal{C}(x_3)} \prod_{i=4}^{L} dx_i \;\mathrm{S} \prod_{k=4}^{L} \frac{1}{x_{k,i_k}}\,. \tag{36}$$

Noting that the Selberg seed (27) vanishes for $x_i = x_j$ and $\Re(s_{ij}) > 0$

$$\mathrm{S}|_{x_i = x_j} = 0\,, \tag{37}$$

it follows that the derivative in eq. (36) only acts non-trivially on the integrand and not on the integration domain. The identity

$$\frac{\partial}{\partial x_i}\frac{1}{x_{ij}} = -\frac{\partial}{\partial x_j}\frac{1}{x_{ij}} \tag{38}$$

and integration by parts may be used to let partial derivatives act on the Selberg seed only:

$$\frac{d}{dx_3}\,\mathrm{S}[i_4,i_5,\dots,i_L](0,1,x_3)=\int_{\mathcal{C}(x_3)}\prod_{i=4}^{L}dx_i\left(\sum_{j\in U_3}\frac{\partial}{\partial x_j}\,\mathrm{S}\right)\prod_{k=4}^{L}\frac{1}{x_{ki_k}}\,. \tag{39}$$

The set U_3 in the previous equation can be stated explicitly:

$$\begin{aligned}U_3=\Bigg\{&j\in\{3,4,\dots,L\}\,\Bigg|\,j=3\text{ or there exist labels }3=j_1,j_2,\dots,j_m=j\\ &\text{such that }\prod_{i=1}^{m-1}\frac{1}{x_{j_{i+1},j_i}}\text{ is a factor of }\prod_{k=4}^{L}\frac{1}{x_{ki_k}}\Bigg\}.\end{aligned} \tag{40}$$

Partial derivatives of the Selberg seed yield factors of s_{jl}/x_{jl}

$$\frac{\partial}{\partial x_j}\,\mathrm{S}=\sum_{l\neq j}\frac{s_{jl}}{x_{jl}}\,\mathrm{S}\,, \tag{41}$$

such that

$$\frac{d}{dx_3}\,\mathrm{S}[i_4,i_5,\dots,i_L](0,1,x_3)=\int_{\mathcal{C}(x_3)}\prod_{i=4}^{L}dx_i\;\mathrm{S}\sum_{j\in U_3}\sum_{l\notin U_3}\frac{s_{jl}}{x_{jl}}\prod_{k=4}^{L}\frac{1}{x_{ki_k}}\,. \tag{42}$$

Admissibility of $\mathrm{S}[i_4,i_5,\dots,i_L](0,1,x_3)$ implies that upon consecutive applications of partial fractioning

$$\frac{1}{x_{k,l}}\frac{1}{x_{k,m}}=\left(\frac{1}{x_{k,l}}-\frac{1}{x_{k,m}}\right)\frac{1}{x_{l,m}}\,, \tag{43}$$

where $k>l>m$, we will again find (admissible) Selberg integrals, however, with different labels i_k.

All integrals on the right-hand side of eq. (36) will contain a prefactor of the form

$$\frac{s_{ij}}{x_{31}}=\frac{s_{ij}}{x_3}\quad\text{or}\quad\frac{s_{ij}}{x_{32}}=\frac{s_{ij}}{x_3-1}\,, \tag{44}$$

since the indices in x_{31} and x_{32} can no longer be reduced by partial fractioning. Accordingly, if we consider the vector of all admissible integrals

$$\mathbf{S}(x_3) = \big(\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3)\big)_{1\leq i_k<k}\,, \tag{45}$$

its derivative with respect to x_3 can be phrased in terms of a vector equation

$$\frac{d}{dx_3}\,\mathbf{S}(x_3) = \Big(\frac{e_0}{x_3} + \frac{e_1}{x_3-1}\Big)\,\mathbf{S}(x_3)\,, \tag{46}$$

where the entries of the $(L{-}1)!/2\times(L{-}1)!/2$ matrices e_0 and e_1 either vanish or are $\mathbb{Z}$-linear combinations of the parameters s_{ij}. In an amplitude context later, this implies (cf. eq. (2)) that e_0 and e_1 are proportional to α'.

The fact that the derivative of $\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3)$ is expressible as a linear combination of iterated integrals $\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3)$ originates in the property mentioned below eq. (33) of the differential forms appearing in the integrand in eq. (36): they are the building blocks for the so-called fibration basis of the twisted de Rham cohomology of the $(L{+}1)$-punctured Riemann sphere with four fixed coordinates [Aom87, Miz19]. Note that for each $4\leq k\leq L$, one can get rid of one particular index $1\leq i'_k<k$ by partial fractioning and integration by parts. Thus, one can identify a suitable basis of the iterated integrals $\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3)$ as

$$\mathcal{B}_{i'_4,i'_5,\dots,i'_L} = \{\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3)|1\leq i_k<k, i_k\neq i'_k\}\,, \tag{47}$$

that is, the ticked indices do *not* appear as labels. Accordingly, one can reduce the vector of all admissible integrals to

$$\mathbf{S}(x_3)|_{\mathcal{B}_{i'_4,i'_5,\dots,i'_L}} \tag{48}$$

for which one finds the differential equation

$$\frac{d}{dx_3}\,\mathbf{S}(x_3)|_{\mathcal{B}_{i'_4,i'_5,\dots,i'_L}} = \Big(\frac{e_0}{x_3} + \frac{e_1}{x_3-1}\Big)\,\mathbf{S}(x_3)|_{\mathcal{B}_{i'_4,i'_5,\dots,i'_L}}\,. \tag{49}$$

While the matrices e_0 and e_1 contain entries from the same class as for the matrices in eq. (46), they are now of dimension $(L-2)!\times(L-2)!$. These matrices turn out to be braid matrices, that is, representations of the braid group of $L{+}1$ distinguishable strands with three strands held fixed. It is well known how to obtain these matrices recursively [Aom87, Ter02, Miz19].

Of course, the choice of the basis is a priori arbitrary. However, depending on the intended use, certain choices turn out to be much more beneficial than others in practice. For example, the recursive definition of the matrices in e_0 and e_1 in ref. [Miz19] are constructed for the choice $\mathcal{B}_{1,1,\dots,1}$, i.e. $2 \leq i_k < k$. On the other hand, the limits considered in subsection 2.4 will conveniently be formulated in the basis $\mathcal{B}_{2,2,\dots,2}$.

Example. Let us consider the simplest example $L = 4$ and show the above calculational steps explicitly for the basis $\mathcal{B}_2 = \{\mathrm{S}[1](0,1,x_3), \mathrm{S}[3](0,1,x_3)\}$, where

$$\mathrm{S}[i_4](0,1,x_3) = \int_0^{x_3} dx_4 \, \mathrm{S}\,\frac{1}{x_{4,i_4}}\,, \qquad \mathrm{S} = x_{41}^{s_{14}} x_{31}^{s_{13}} x_{34}^{s_{34}} x_{21}^{s_{12}} x_{24}^{s_{24}} x_{23}^{s_{23}}\,. \tag{50}$$

The integrands are functions defined on $\mathcal{F}_{5,4} = \{x_4 \in \mathbb{R} | x_4 \neq x_1, x_2, x_3, x_5\}$ and we consider the following order of the punctures:

$$0 = x_1 < x_4 < x_3 < x_2 = 1 < x_5 = \infty\,. \tag{51}$$

The set $\mathcal{B}_2$ is indeed a basis, since the only remaining Selberg integral $\mathrm{S}[2](0,1,x_3)$ can be expressed in terms of elements in $\mathcal{B}_2$ using

$$s_{14}\,\mathrm{S}[1](0,1,x_3) + s_{24}\,\mathrm{S}[2](0,1,x_3) + s_{34}\,\mathrm{S}[3](0,1,x_3) = 0\,. \tag{52}$$

Now, let us calculate the derivatives of $\mathbf{S}(x_3)|_{\mathcal{B}_2}$ in order to recover the KZ equation (49): for the first basis element, we find $U_3(\mathrm{S}[1](0,1,x_3)) = \{3\}$, such that according to eq. (42)

$$\begin{aligned}\frac{d}{dx_3}\mathrm{S}[1](0,1,x_3) &= \int_0^{x_3} dx_4\, \mathrm{S}\left(\frac{s_{13}}{x_{31}} + \frac{s_{34}}{x_{34}} + \frac{s_{23}}{x_{32}}\right)\frac{1}{x_{41}}\\ &= \frac{s_{13}}{x_3}\mathrm{S}[1](0,1,x_3) + \frac{s_{23}}{x_3-1}\mathrm{S}[1](0,1,x_3)\\ &\quad + \frac{s_{34}}{x_3}\left(\mathrm{S}[1](0,1,x_3) - \mathrm{S}[3](0,1,x_3)\right)\end{aligned} \tag{53}$$

where we have used the partial fractioning identity (43) for the third equality. Similarly, for the second basis element we find $U_3(\mathrm{S}[3](0,1,x_3)) = \{3,4\}$, such that

$$\begin{aligned}\frac{d}{dx_3}\mathrm{S}[3](0,1,x_3) &= \int_0^{x_3} dx_4\, \mathrm{S}\left(\frac{s_{13}}{x_{31}} + \frac{s_{23}}{x_{32}} + \frac{s_{14}}{x_{41}} + \frac{s_{24}}{x_{42}}\right)\frac{1}{x_{43}}\\ &= \frac{s_{13}}{x_3}\mathrm{S}[3](0,1,x_3) + \frac{s_{23}}{x_3-1}\mathrm{S}[3](0,1,x_3)\\ &\quad + \frac{s_{14}}{x_3}\left(\mathrm{S}[3](0,1,x_3) - \mathrm{S}[1](0,1,x_3)\right)\end{aligned}$$

$$+\frac{1}{x_3-1}\left((s_{24}+s_{34})\,\mathrm{S}[3](0,1,x_3)+s_{14}\,\mathrm{S}[1](0,1,x_3)\right)\,, \tag{54}$$

where we have again used partial fractioning (43) and integration by parts (52) for the fourth equality. Overall we find the differential equation

$$\frac{d}{dx_3}\,\mathbf{S}(x_3)|_{\mathcal{B}_2}=\left(\frac{e_0}{x_3}+\frac{e_1}{x_3-1}\right)\mathbf{S}(x_3)|_{\mathcal{B}_2}\,, \tag{55}$$

which is indeed of the form of the KZ equation (49) with the matrices

$$e_0=\begin{pmatrix} s_{13}+s_{34} & -s_{34} \\ -s_{14} & s_{13}+s_{14}\end{pmatrix}\,,\qquad e_1=\begin{pmatrix} s_{23} & 0 \\ s_{14} & s_{234}\end{pmatrix} \tag{56}$$

given by the braid matrices from ref. [Miz19].

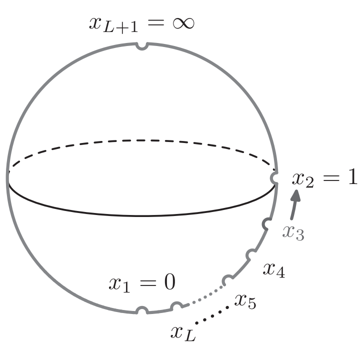
2.4. Boundary values for the KZ equation

Equation (46) is of KZ type [KZ84]. Well known from refs. [Dri89, Dri91], we provide a brief summary of the relation between the two boundary values $z_3 \to 0, 1$ in appendix A.

Let us consider the regularized limits (9) when taking the auxiliary point x_3 to either zero or one in (46)[4]:

$$\mathbf{C}_0=\lim_{x_3\to 0} x_3^{-e_0}\,\mathbf{S}(x_3)\,,\quad \mathbf{C}_1=\lim_{x_3\to 1}(1-x_3)^{-e_1}\,\mathbf{S}(x_3)\,. \tag{57}$$

Boundary value $\mathbf{C}_1^\tau$: Let us start by considering the limit $x_3 \to x_2 = 1$, which is depicted in the following figure:

(58) 

.

[4]The following paragraph is closely related to the original analysis of Selberg integrals in ref. [Ter02], which serves as the prime reference for our investigation and led to the formulation of the amplitude recursion in ref. [BSST14].

The relevant integrals in the amplitude recursion in this limit turn out to be the Selberg integrals in $\mathcal{B}_{2,2,\ldots,2} \cap \mathcal{B}_{3,3,\ldots,3}$ with integrands defined on the configuration space $\mathcal{F}_{L+1,4}$ with $1 \leq i_k < k$ and $i_k \neq 2, 3$. For these integrals, the action of the prefactor $(1-x_3)^{-e_1}$ is particularly simple: on the one hand, the set U_3 in eq. (42) is simply $U_3 = \{3\}$. On the other hand, the only appearance of the insertion point x_2 in the integral $\mathrm{S}[i_4, i_5, \ldots, i_L](x_1 = 0, x_2 = 1, x_3)$ with $i_k \neq 2$ is in the Selberg seed. Therefore using partial fractioning to obtain the KZ form from eq. (42) does not introduce any factor of $1/x_{32}$ other than s_{23}/x_{32} obtained from differentiating the Selberg seed. Thus, for the basis $\mathcal{B}_{2,2,\ldots,2}$, the representation e_1 in the KZ equation (49) is of the form

$$e_1 = \begin{pmatrix} s_{23}\mathbb{I}_{(L-3)!\times(L-3)!} & 0_{(L-3)\times(L-3)!} \\ A_{(L-3)!\times(L-3)} & B_{(L-3)\times(L-3)} \end{pmatrix}, \tag{59}$$

where the upper left block proportional to the identity corresponds to the integrals in $\mathcal{B}_{2,2,\ldots,2} \cap \mathcal{B}_{3,3,\ldots,3}$, (cf. example (56)). For this subclass of integrals, the regularization factor $(1-x_3)^{-e_1}$ only contributes with the scalar $(1-x_3)^{-s_{23}} = x_{23}^{-s_{23}}$ and the corresponding entries of the regularized limit $\mathbf{C}_1$ can be calculated as

$$\begin{aligned} &\lim_{x_3\to x_2} x_{23}^{-s_{23}}\, \mathrm{S}[i_4, i_5, \ldots, i_L](0, 1, x_3) \\ &\quad = \int_{\mathcal{C}(x_3\to x_2)} \prod_{i=4}^{L} dx_i \prod_{0\leq x_j < x_l < x_3} x_{lj}^{s_{jl}} \prod_{0\leq x_n < x_3} x_{2n}^{s_{2n}+s_{3n}} \prod_{k=4}^{L} \frac{1}{x_{ki_k}} \\ &\quad = \mathrm{S}[i_4, i_5, \ldots, i_L](0, 1, x_3 = x_2)\big|_{s_{23}=0}^{\tilde{s}_{ij}=s_{ij}+\delta_{i2}s_{3j}}\,. \end{aligned} \tag{60}$$

Thus the regularization $x_{23}^{-s_{23}}$ cancels the factor $x_{23}^{s_{23}}$ in the Selberg seed S, which would otherwise render the integral vanishing. Moreover, the punctures x_2 and x_3 have merged, such that the associated Mandelstam variables, and hence, the momenta of the external states, are added to yield effective Mandelstam variables $\tilde{s}_{ij} = s_{ij} + \delta_{i2}s_{3j}$ for $i, j \in \{1, \ldots, L\} \setminus \{3\}$, $i < j$.

The resulting differential form and integration domain in the integral (60) are known from the type-$(L, 3)$ Selberg integrals defined on the configuration space $\mathcal{F}_{L,3}$. Thus, in this limit, the forms in $\mathbf{S}(x_3)$ can be expressed as linear combinations of the Parke–Taylor forms of L-point string amplitudes, which are discussed in the next subsection. In terms of the disk picture 58, we are modifying the relative distances on the boundary by taking the limit

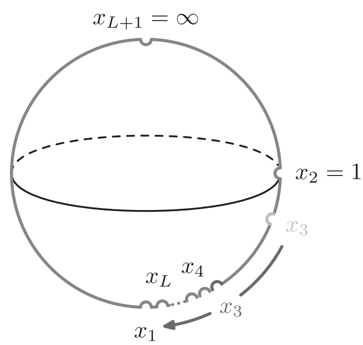
$x_3 \to x_2 = 1$. Upon identification of the points x_2 and x_3 we find the transition

$$\mathcal{F}_{L+1,4} \to \mathcal{F}_{L,3} \tag{61}$$

with the L insertion points $x_1 = 0, x_2 = 1, x_4, x_5, \ldots, x_L$ and $x_{L+1} = \infty$, which is the setup suitable for describing L-point amplitudes.

Boundary value $\mathbf{C}_0^\tau$: For the limit $x_3 \to 0$, we are facing the following situation

(62)



This limit can be described in the basis $\mathcal{B}_{2,2,\ldots,2}$, since for this choice, the maximum eigenvalue of e_0 is given by

$$s_{\max} = s_{1,3,4,\ldots,L}\,. \tag{63}$$

This can be seen by repeating the observation that led to eq. (59) for e_1: deriving eq. (49) for $\mathcal{B}_{2,2,\ldots,2}$, assembles all the s_{ij} with $i, j \neq 2, L+1$ in the matrix e_0. Therefore, the regularization factor $z_0^{-e_0}$ in $\mathbf{C}_0$ can at most contribute with a factor $x_3^{-s_{\max}}$ to each integral.

The behavior of these entries for $x_3 \to x_1 = 0$ may be determined using the change of variables $x_i = x_3 w_i$ for $0 = x_1 \leq x_i < x_2 = 1$, such that in particular $w_1 = 0$ and $w_3 = 1$. This yields for $i_k \neq 2$

$$\begin{aligned}
&\lim_{x_3\to 0} x_3^{-s_{\max}}\, \mathrm{S}[i_4,\ldots,i_L](0,1,x_3)\\
&= \lim_{x_3\to 0} x_3^{-s_{\max}} \int_{\mathcal{C}(x_3)} \prod_{i=4}^{L} dx_i \prod_{0\leq x_j<x_l<x_3} x_{lj}^{s_{jl}} \prod_{0\leq x_m<x_3} x_{3m}^{s_{3m}} \prod_{0\leq x_n<x_2} x_{2n}^{s_{2n}} \prod_{k=4}^{L} \frac{1}{x_{ki_k}}\\
&= \lim_{x_3\to 0} \int_{0=w_1<w_i<w_3=1} \prod_{i=4}^{L} dw_i \prod_{0\leq w_j<w_l<x_3} w_{lj}^{s_{jl}} \prod_{0\leq w_m<w_3} w_{3m}^{s_{3m}} \prod_{0\leq x_n<x_2} (1-x_3 w_n)^{s_{2n}} \prod_{k=4}^{L} \frac{1}{w_{ki_k}}
\end{aligned}$$

$$= \int_{0=w_1<w_i<w_3=1} \prod_{i=4}^{L} dw_i \prod_{0\leq w_j<w_l<x_3} w_{lj}^{s_{jl}} \prod_{0\leq w_m<w_3} w_{3m}^{s_{3m}} \prod_{k=4}^{L} \frac{1}{w_{ki_k}}$$

(64)

$$= \mathrm{S}[i_4, i_5, \ldots, i_L](0, 1, w_3 = 1)|_{s_{2j}=0}\,,$$

which is, as for the $x_3 \to 1$ limit, an integral with integrand defined on $\mathcal{F}_{L,3}$.

Note that if we would not restrict to the basis $\mathcal{B}_{2,2,\ldots,2}$ and there were r indices $k_j \in \{4, 5 \ldots, L\}$ such that $i_{k_j} = 2$, then the change of variables would leave r factors of x_3 in the quotient of the measure and the denominator

$$\prod_{k=4}^{L} \frac{dx_k}{x_{ki_k}} = x_3^r \prod_{k=4, k\notin\{k_j\}}^{L} \frac{dw_k}{w_{ki_k}} \prod_{j=1}^{r} \frac{dw_{k_j}}{x_3 w_{k_j} - 1}\,, \tag{65}$$

which vanishes for $x_3 \to 0$. Therefore, the entries of $\mathbf{C}_0$ are linear combinations of integrals

$$\lim_{x_3\to 0} x_3^{-s_{\max}}\, \mathrm{S}[i_4, \ldots, i_L](0, 1, x_3)$$

(66)

$$= \begin{cases} \mathrm{S}[i_4, i_5, \ldots, i_L](0, 1, w_3 = 1)|_{s_{2j}=0} & \text{if } \mathrm{S}[i_4, \ldots, i_L](0, 1, x_3) \in \mathcal{B}_{2,2,\ldots,2}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Mandelstam variables: According to eq. (60), the Mandelstam variables s_{3j} associated to the momentum of the auxiliary insertion point x_3 are redundant in $\mathbf{C}_1$: they simply appear as a splitting of the effective momentum $\tilde{s}_{2j} = s_{2j} + s_{3j}$ associated to the insertion point at $x_2 = 1$ and thus, may be chosen to be set to zero. This choice is more subtle in the boundary value $\mathbf{C}_0$ with the non-vanishing entries being calculated according to eq. (64): here, the Mandelstam variables s_{3j} are not at all redundant, i.e. an artificial splitting of the momentum contribution, but encode the full momentum of the insertion point $w_3 = 1$. Thus, it may be expected that setting this momentum to zero effectively removes one external state, leaving an integrand defined on $\mathcal{F}_{L-1,3}$. This expectation is true for certain linear combinations of Selberg integrals, as argued in the next subsection.

Summary of subsection: The vector of type-$(2, L)$ Selberg integrals $\mathbf{S}(x_3)$ with integrands on $\mathcal{F}_{L+1,4}$ encodes for $N = L$ the N- and $(N-1)$-point amplitudes, in the regularized limits $\mathbf{C}_0$ and $\mathbf{C}_1$, which can be related to each other using the Drinfeld associator $\Phi(e_0, e_1)$, with e_0 and e_1 determined by the KZ eq. (46), as follows

$$\begin{tikzcd}
& \mathbf{C}_1 \text{ on } \mathcal{F}_{N,3} = \mathcal{M}_{0,N} \arrow[r, "s_{3j} \to 0"] & N\text{-point amplitudes on } \mathcal{M}_{0,N} \\
\mathbf{S}(x_3) \text{ on } \mathcal{F}_{N+1,4} \arrow[ur, "x_3 \to x_2 = 1"] \arrow[dr, "x_3 \to x_1 = 0"'] & & \\
& \mathbf{C}_0 \text{ on } \mathcal{F}_{N,3} = \mathcal{M}_{0,N} \arrow[uu, "\Phi(e_0, e_1)"'] \arrow[r, "s_{3j} \to 0"] & (N{-}1)\text{-point amplitudes on } \mathcal{M}_{0,N-1} \arrow[uu, "\Phi(e_0, e_1)|_{s_{3j}=0}"']
\end{tikzcd} \tag{67}$$

where the exact degeneracy to the amplitudes as $s_{3j} \to 0$ and the corresponding map $\Phi(e_0, e_1)|_{s_{3j}=0}$ will be explored in the next subsections.

2.5. Open string amplitudes at genus zero

Open-string tree-level amplitudes arise as correlators between vertex-operators inserted at the boundary of the disk worldsheet. Usually one makes use of the conformal symmetry of the worldsheet in order to place the boundary of the disk at the real line. Evaluating the correlators allows to frame the amplitude in the form [BSS13]:

$$A_{\text{open}}(1, L, L-1, ..., 2, L+1; \alpha') = \mathbf{Z}^T \, \mathbf{MK} \; \mathbf{A}_{\text{YM}} \, . \tag{68}$$

While the Yang–Mills tree-level amplitudes $\mathbf{A}_{\text{YM}}$ can be obtained (for example) from BCFW recursion relations [BCF05, BCFW05], the object $\mathbf{MK}$ is known as the momentum kernel and can be represented as a matrix of dimension $(L-2)! \times (L-2)!$. A recursive formula is known for any multiplicity [Sti09, BBDSV11]. The vector $\mathbf{Z}$ consists of $(L-2)!$ so-called Z-integrals [BSS13]

$$Z(q_1, q_2, \ldots, q_{L+1}) = \int_{\mathcal{C}(x_2=1)} \prod_{i=3}^{L} dx_i \; \text{KN} \, \frac{x_{1,L+1} x_{2,L+1} x_{12}}{x_{q_1 q_2} x_{q_2 q_3} \cdots x_{q_L q_{L+1}} x_{q_{L+1} q_1}} \, , \tag{69}$$

where the factor $x_{1,L+1} x_{2,L+1} x_{12}$ together with the fixing of the coordinates $(x_1, x_2, x_{L+1}) = (0, 1, \infty)$ (cf. eq. (3)) corresponds to dividing out the gauge volume $\mathcal{V}_{\text{CKG}}$ of the conformal Killing group $\text{SL}(2, \mathbb{C})$. The quotient together with the integration measure is called Parke–Taylor form, while KN is called

Koba–Nielsen factor[5] and defined by

$$\mathrm{KN} = \prod_{0=x_1\leq x_i<x_j\leq x_2=1} |x_{ij}|^{s_{ij}} = \prod_{0=x_1\leq x_i<x_j\leq x_2=1} \exp(s_{ij}\log|x_{ij}|)\,. \tag{70}$$

Note that we have defined the Selberg seed in eq. (27) in exactly the same way: it is constructed to equal the $(L+1)$-point Koba–Nielsen factor

$$\mathrm{S} = \mathrm{KN}\,. \tag{71}$$

Since $\log x_{ij}$ is (almost) the genus-zero string propagator, the Koba–Nielsen factor can easily be identified as a generating functional of graphs connecting the vertex operators, where each edge connecting vertex operators at positions x_i and x_j is weighted by the corresponding Mandelstam variable s_{ij}.

Iterated integrals in x_i over various derivatives of the Koba–Nielsen factor, in particular the Z-integrals defined in eq. (69), fall in the class of Selberg integrals [Sel44]. It is only those integrals, which need to be calculated in order to determine the full open-string tree-level amplitude at any multiplicity.

Concretely, in ref. [BSST14] a vector of iterated integrals has been constructed, which is related via a basis transformation to the vector of genus-zero Selberg integrals $\mathbf{S}(x_3)|_{\mathcal{B}_{1,1,\ldots,1}}$ in the basis $\mathcal{B}_{1,1,\ldots,1}$

$$\hat{\mathbf{F}}(x_3) = \boldsymbol{B}\,\mathbf{S}(x_3)|_{\mathcal{B}_{1,1,\ldots,1}}\,. \tag{72}$$

The transformation matrix (of the corresponding bases of twisted forms) $\boldsymbol{B}$ has been calculated in ref. [Kad20], its non-vanishing entries are polynomials over $\mathbb{Z}$ of degree $L-3$ in s_{ij}. According to the KZ eq. (49), the function $\hat{\mathbf{F}}(x_3)$ satisfies a KZ equation with matrices $\hat{\boldsymbol{e}}_0 = \boldsymbol{B}\boldsymbol{e}_0\boldsymbol{B}^{-1}$ and $\hat{\boldsymbol{e}}_1 = \boldsymbol{B}\boldsymbol{e}_1\boldsymbol{B}^{-1}$, where $\boldsymbol{e}_i$ are the matrices in the KZ equation satisfied by the Selberg integrals $\mathbf{S}(x_3)|_{\mathcal{B}_{1,1,\ldots,1}}$. Moreover, the first $(L-3)!$-entries of the regularized limit as $x_3\to 1$ are linear combinations of the integrals in eq. (60) and contain the L-point, tree-level amplitudes with effective Mandelstam variables $\tilde{s}_{ij} = s_{ij} + \delta_{i2}s_{3j}$ for $i,j\in\{1,\ldots,L\}\setminus\{3\}$, $i<j$, such that

$$\hat{\mathbf{C}}_1 = \lim_{x_3\to 1}(1-x_3)^{-\hat{e}_1}\,\hat{\mathbf{F}}(x_3) = \begin{pmatrix} \mathbf{Z}^T\,\mathbf{MK}|_{L\text{-point},\,\tilde{s}_{ij}=s_{ij}+\delta_{i2}s_{3j}} \\ \vdots \end{pmatrix}, \tag{73}$$

[5] Here, for consistency with other articles, we have written the absolute value, despite all x_{ji} are real and positive in our conventions.

where the (labels of the) Mandelstam variables $\tilde{s}_{ij}$ correspond to the insertion points $x_1 < x_L < x_{L-1} < \cdots < x_4 < x_2 = 1$. On the other hand, the only non-vanishing entries of the regularized boundary value

$$\hat{\mathbf{C}}_0 = \lim_{x_3 \to 0} x^{-\hat{e}_0} \, \hat{\mathbf{F}}(x_3) \tag{74}$$

degenerate in the soft limit $s_{3j} \to 0$ to the $(L-1)$-point, tree-level amplitudes

$$\lim_{s_{3j} \to 0} \hat{\mathbf{C}}_0 = \begin{pmatrix} \mathbf{Z}^T \, \mathbf{MK}|_{(L-1)-\text{point}, \, s_{ij}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{75}$$

where the Mandelstam variables s_{ij} are associated to the insertion points $w_1 = 0 < w_L < w_{L-1} < \cdots < w_4 = 1$. The limit $s_{3j} \to 0$ effectively leads to a merging of the punctures w_4 and w_3 in the integrals (66), since these integrals are linearly combined in $\hat{\mathbf{C}}_0$ such that the integrands are total derivatives with respect to w_4 in the limit $s_{3j} \to 0$. Thus, taking the limit $s_{3j} \to 0$ of the associator equation

$$\hat{\mathbf{C}}_1 = \mathbf{\Phi}(\hat{e}_0, \hat{e}_1) \hat{\mathbf{C}}_0 \tag{76}$$

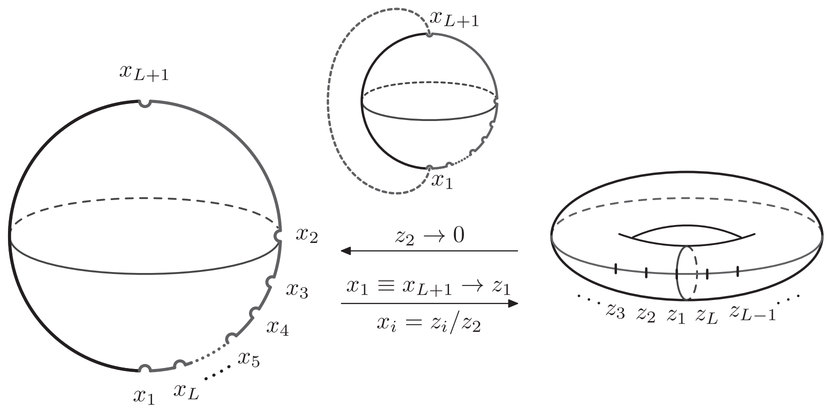
leads to a recursion relating the L-point to the $(L-1)$-point, genus-zero amplitudes

$$\begin{pmatrix} \mathbf{Z}^T \, \mathbf{MK}|_{L-\text{point}, \, s_{ij}} \\ \vdots \end{pmatrix} = \mathbf{\Phi}(\hat{e}_0, \hat{e}_1)|_{s_{3j}=0} \begin{pmatrix} \mathbf{Z}^T \, \mathbf{MK}|_{(L-1)-\text{point}, \, s_{ij}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{77}$$

3. Genus one (one-loop level)

In this section, we develop and explore the genus-one version of the concepts from section 2 and link the resulting formalism to the evaluation of open-string configuration-space integrals at one loop. The genus-one recursion is remarkably similar to the genus-zero recursion of ref. [BSST14] reviewed in subsection 2.5.

While the genus-zero recursion relates N-point configuration-space integrals to $(N-1)$-point versions thereof and is thus a recursion in the number of external legs, the genus-one mechanism relates N-point one-loop configuration-space integrals to $(N{+}2)$-point tree-level configuration-space integrals, thus linking objects[6] occurring at different genera:

(78)


.

In the genus-zero recursion, the Drinfeld associator effectively adds an additional puncture to an $(N{-}1)$-point interaction resulting in an N-point tree-level interaction. On the other hand, as shown below, the genus-one recursion amounts to two external states of the $(N{+}2)$-point tree-level interaction being glued together by the elliptic analogue of the Drinfeld associator, the elliptic Knizhnik–Zamolodchikov–Bernard (KZB) associator, to form a genus-one worldsheet of N external string states.

In the current section, we follow the structure of the previous section 2: in subsection 3.1 to subsection 3.5 we introduce elliptic iterated integrals, a genus-one version of Selberg integrals, the elliptic KZB associator and the KZB equation for an auxiliary marked point. In the subsequent subsection 3.6, the relation to open-string configuration-space integrals is drawn, where we also discuss some practicalities. In section 4, the first orders in α' of the two-, three- and four-point one-loop configuration-space integrals are calculated using the genus-one associator mechanism and are shown to match known results.

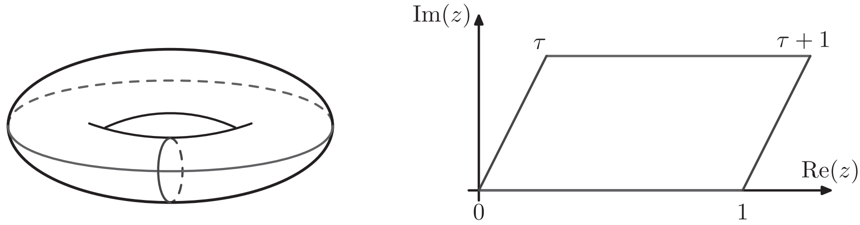
3.1. Singularities, iterated integrals and elliptic multiple zeta values

In the following, we will consider the annulus formed by open-string worldsheets at one loop as embedded into a torus with A-cycle (red) and B-cycle

[6]Notation and limits depicted in figure (78) will be introduced and explained in the course of this section.

(blue), where the ratio of the respective lengths, the modular parameter, is denoted by τ.

(79)



Suitable differentials on the torus are generated by the expansion (13) of the Eisenstein–Kronecker series $F(z,\eta,\tau)$ in η. This defines – in distinction to the genus-zero scenario – an infinite number of differentials $g^{(n)}(z,\tau)dz$. The index n labelling the functions $g^{(n)}$ is called its *weight*. While $g^{(0)} = 1$ is trivial, the function $g^{(1)}$ has poles at $z \in \mathbb{Z}\tau + \mathbb{Z}$ and can be expanded in $q = \exp(2\pi i\tau)$ as follows [BMMS15]:

$$g^{(1)}(z,\tau) = \pi\cot(\pi z) + 4\pi\sum_{m=1}^{\infty}\sin(2\pi m z)\sum_{n=1}^{\infty} q^{mn}\,. \tag{80}$$

All $g^{(n)}$ with $n \geq 2$ are holomorphic in the fundamental elliptic domain $z = s + \tau t,\ s,t \in [0,1)$. The Eisenstein–Kronecker series $F(z,\eta,\tau)$ is one-periodic, but only quasi-periodic in z. Therefore, the integration kernels $g^{(n)}$ are also one-periodic

$$g^{(n)}(z+1,\tau) = g^{(n)}(z,\tau)\,, \tag{81}$$

but not τ-periodic in z, thus, they can not be elliptic functions. Furthermore, they also inherit the following symmetry property from $F(z,\eta,\tau)$:

$$g^{(n)}(-z,\tau) = (-1)^n g^{(n)}(z,\tau)\,. \tag{82}$$

Despite not being elliptic, the functions $g^{(n)}$ can be considered to be genus-one generalizations of the integration kernels defining the multiple polylogarithms (22), which also lead to meromorphic, but multi-valued functions.

The integrals over the kernels $g^{(n)}$ lead to elliptic polylogarithms [Lev97, BL11]: due to their periodicity in eq. (81) they are single-valued functions on the annulus, but can be thought of as multi-valued functions on the torus. This is equivalent to the behavior of the ordinary logarithm at genus zero: on each Riemann sheet the logarithm is single-valued, while it is a multi-valued function in the complex plane.

Due to the poles of the integration kernel $g^{(1)}$, iterated integrals of $g^{(1)}$ need to be regularized. Moreover, its poles will – in certain limits – act as the link between the string propagators at Riemann surfaces of genus zero and genus one. Corresponding to the differentials introduced in eq. (13), one can define a class of iterated integrals $\tilde{\Gamma}$ called elliptic multiple polylogarithms:

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1, n_2, \ldots, n_k \\ a_1, a_2, \ldots, a_k \end{smallmatrix}; z, \tau\right) = \int_0^z dz'\, g^{(n_1)}(z' - a_1, \tau)\, \tilde{\Gamma}\left(\begin{smallmatrix} n_2, \ldots, n_k \\ a_2, \ldots, a_k \end{smallmatrix}; z', \tau\right), \tag{83}$$

which due to their nature as iterated integrals obey shuffle relations

$$\begin{aligned} &\tilde{\Gamma}(A_1, A_2, \ldots, A_j; z, \tau)\, \tilde{\Gamma}(B_1, B_2, \ldots, B_k; z, \tau) \\ &\quad = \tilde{\Gamma}\left((A_1, A_2, \ldots, A_j) \sqcup\!\sqcup (B_1, B_2, \ldots, B_k); z, \tau\right) \end{aligned} \tag{84}$$

in terms of combined letters $A_i = \begin{smallmatrix} n_i \\ a_i \end{smallmatrix}$.

The integral over $g^{(1)}$ will be of particular interest below: $\tilde{\Gamma}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z, \tau\right)$ requires regularization because of an endpoint divergence at the lower integration boundary due to the pole at $z = 0$. The standard regularization procedure – which we are going to use here – is called *tangential basepoint regularization* and is discussed in detail for example in refs. [Del89, Bro14]. In short, we subtract the endpoint divergence by defining[7]

$$\begin{aligned} \tilde{\Gamma}_{\mathrm{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z, \tau\right) &= \lim_{\epsilon \to 0} \int_\epsilon^z dz\, g^{(1)}(z, \tau) + \log(1 - e^{2\pi i \epsilon}) \\ &= \log(1 - e^{2\pi i z}) - \pi i z + 4\pi \sum_{k,l>0} \frac{1}{2\pi k} \left(1 - \cos(2\pi k z)\right) q^{kl}. \end{aligned} \tag{85}$$

Considering $z \in (0, 1)$, the following properties can be read off from the above q-expansion

$$\begin{aligned} \tilde{\Gamma}_{\mathrm{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z \pm 1, \tau\right) &= \tilde{\Gamma}_{\mathrm{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z, \tau\right) \mp \pi i \\ \tilde{\Gamma}_{\mathrm{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; -z, \tau\right) &= \tilde{\Gamma}_{\mathrm{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z, \tau\right) + \pi i, \end{aligned} \tag{86}$$

where we place the branch cut of the logarithm such that $\log(-1) = \pi i$. This implies in particular invariance under $z \to 1 - z$ for $0 < z < 1$:

$$\tilde{\Gamma}_{\mathrm{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z, \tau\right) = \tilde{\Gamma}_{\mathrm{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; 1 - z, \tau\right). \tag{87}$$

[7] The limit $\epsilon \to 0$ is taken within the unit interval. Unless stated otherwise, the same holds for any limits in this article.

In addition, we find the following asymptotic behavior for $z \to 0$

$$\tilde{\Gamma}_{\mathrm{reg}}\left(\begin{smallmatrix}1\\0\end{smallmatrix}; z, \tau\right) \sim \log(-2\pi i z) \tag{88}$$

and $z \to 1$

$$\tilde{\Gamma}_{\mathrm{reg}}\left(\begin{smallmatrix}1\\0\end{smallmatrix}; z, \tau\right) \sim \log(-2\pi i(1-z))\,. \tag{89}$$

The above regularization procedure is an algebra homomorphism, i.e. compatible with the shuffle product. From now on, we will use the regularized iterated integrals exclusively and omit the subscript when noting $\tilde{\Gamma}$. Furthermore, we are going to keep the dependence on τ implicit for all integration kernels $g^{(n)}$ and all iterated elliptic integrals $\tilde{\Gamma}$.

In the same way as products of terms of the form $1/x_{ij}$ can be related by partial fractioning (43), there is a genus-one analogue for the Kronecker series: the Fay identity. In terms of the functions $g^{(n)}(z)$ it can be phrased as

$$\begin{aligned} &g^{(n_1)}(t-x)g^{(n_2)}(t) \\ &\quad= -(-1)^{n_1} g^{(n_1+n_2)}(x) + \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} g^{(n_2-j)}(x) g^{(n_1+j)}(t-x) \\ &\quad+ \sum_{j=0}^{n_1} \binom{n_2-1+j}{j} (-1)^{n_1+j} g^{(n_1-j)}(x) g^{(n_2+j)}(t) \end{aligned} \tag{90}$$

and derived from a similar property obeyed by the generating function $F(z, \eta, \tau)$.

For compactness, we will use a notation similar to definition (22) in terms of words from an alphabet for the elliptic multiple polylogarithms $\tilde{\Gamma}$ defined in eq. (83) with $a_1 = a_2 = \cdots = a_k = 0$. Concretely, since there are infinitely many integration kernels $g^{(n)}$, the alphabet is infinite as well and denoted by $\{x^{(0)}, x^{(1)}, \dots\}$. For a word $w = x^{(n_1)} \dots x^{(n_k)} \in \{x^{(0)}, x^{(1)}, \dots\}^{\times}$, we denote the corresponding elliptic multiple polylogarithm by

$$\tilde{\Gamma}_w(z) = \tilde{\Gamma}(x^{(n_1)} \cdots x^{(n_k)}; z) = \tilde{\Gamma}\left(\begin{smallmatrix} n_1, & \dots, & n_k \\ 0, & \dots, & 0 \end{smallmatrix}; z\right)\,. \tag{91}$$

For $w \neq (x^{(1)})^n$, one finds

$$\lim_{z \to 0} \tilde{\Gamma}_w(z) = 0\,, \tag{92}$$

while the regularization (85) implies logarithmic divergences for words $w = (x^{(1)})^n$ in the limit $z \to 0$:

$$\tilde{\Gamma}_{(x^{(1)})^n}(z) \sim \frac{1}{n!} \log(-2\pi i z)^n \,. \tag{93}$$

Due to the one-periodicity of $g^{(1)}$, this divergence also appears at the upper integration boundary for words $w = (x^{(1)})^n$ as $z \to 1$. The corresponding regularization procedure is particularly important for elliptic multiple zeta values to be discussed in the next paragraph.

Considering the limit $z \to 1$ leads to the genus-one analogues of MZVs defined in eq. (23). These so-called elliptic multiple zeta values (eMZVs) [Enr16, Mat16, BMMS15] are defined in terms of regularized iterated integrals $\tilde{\Gamma}_w$ with $w = x^{(n_1)} \dots x^{(n_k)} \in X \setminus x^{(1)}X$, i.e. $n_1 \neq 1$, at $z = 1$:

$$\omega(n_k, \dots, n_1; \tau) = \omega(w^t; \tau) = \lim_{z \to 1} \tilde{\Gamma}_w(z, \tau) = \lim_{z \to 1} \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ 0 & \dots & 0 \end{smallmatrix} ; z, \tau \right), \tag{94}$$

where w^t denotes the reversal of the word w. In order to extend this definition to all words $w \in X$, the singularity of $\tilde{\Gamma}_{x^{(1)}w}(z, \tau)$ at $z = 1$ has to be regularized. This can be done similarly as for the multiple polylogarithms in eq. (24), and is elaborated on in detail in appendix B. The main result is the following definition of the regularized eMZVs $\omega_{\text{reg}}(w^t; \tau)$: for any word $w \in X \setminus x^{(1)}X$ or $w = (x^{(1)})^n$ they are defined by

$$w \mapsto \omega_{\text{reg}}(w^t; \tau) = \begin{cases} \omega(w^t; \tau) & \text{if } w \in X \setminus (x^{(1)}X)\,, \\ 0 & \text{if } w = (x^{(1)})^n,\ n \in \mathbb{N}\,. \end{cases} \tag{95}$$

Again, the remaining cases $w \in x^{(1)}X$ and $w \neq (x^{(1)})^n$ can be related to the above situations using the shuffle algebra. As for the elliptic multiple polylogarithms, from now on unless stated otherwise, all elliptic multiple zeta values are assumed to be regularized and simply denoted by $\omega(w^t)$ omitting the subscript and the τ-dependence in $\omega_{\text{reg}}(w^t; \tau)$.

In the same way as the shuffle algebra is preserved when regularizing iterated integrals $\tilde{\Gamma}$ in eq. (85), this is true for the corresponding MZVs: (regularized) eMZVs inherit the shuffle algebra, the properties implied by the Fay identity and some further properties from the elliptic multiple polylogarithms such as the reflection identity

$$\omega(n_k, \dots, n_1) = (-1)^{n_1 + \dots + n_k} \omega(n_1, \dots, n_k) \tag{96}$$

due to property (82) of the integration kernels. Furthermore, even elliptic zeta values are related to the (genus-zero) zeta values according to

$$\omega(2m;\tau) = -2\zeta_{2m}\,. \tag{97}$$

Numerous other relations between eMZVs can be retrieved from [BMS].

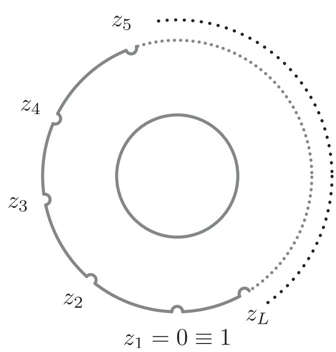
3.2. Genus-one Selberg integrals

In order to repeat the construction described for genus zero in subsection 2.2, we will need to find a genus-one generalization of the Selberg seed function defined in eq. (27) which can be used to construct genus-one Selberg integrals. The genus-one Selberg seed should depend on the positions of insertion points inserted on the boundaries of an open-string worldsheet at one loop. Such worldsheets are quotients of a genus-one Riemann surface, where the corresponding involution is induced by complex conjugation. For simplicity, we restrict our discussion to oriented worldsheets where all insertion points are located on one boundary. This scenario corresponds to planar open-string interactions at one loop with the relevant geometry being the annulus with one punctured boundary. A generalization to the non-planar case, where points are allowed on both boundaries is not expected to pose any structural obstacles. Upon embedding the annulus into a torus, the relevant boundary is identified with the A-cycle and parametrised by the unit interval. In contrast to the genus-zero labelling (26), the positions of the insertion points are going to be denoted by and ordered according to

$$0 = z_1 < z_L < z_{L-1} < \cdots < z_2 < 1 = z_1 \bmod \mathbb{Z}\,, \tag{98}$$

where we have used the symmetries of the torus to fix $z_1 = 0$.

(99)



Therefore – in analogy to the genus-zero scenario – we expect to find iterated integrals with integrands defined on the configuration space of k-punctured A-cycles $[0,1]\backslash\{z_1,\dots,z_k\}$ of tori with purely imaginary modular parameter τ and k fixed punctures:

$$\begin{aligned}&\mathcal{F}^{\tau}_{L,k} = \\ &\quad \{(z_{k+1},z_{k+2},\dots,z_L)\in([0,1]\setminus\{z_1,\dots,z_k\})^{L-k}|\forall i\neq j: z_i\neq z_j\}.\end{aligned} \tag{100}$$

Remembering the basic properties of the genus-zero Selberg seed defined in eq. (27), its generalization to genus one is straightforward. Defining

$$\tilde{\Gamma}_{ij}=\tilde{\Gamma}(\tfrac{1}{0};z_{ij},\tau)=\tilde{\Gamma}_{x^{(1)}}(z_{ij},\tau)\,, \tag{101}$$

where

$$z_{ij}=z_i-z_j\,, \tag{102}$$

one can simply replace $\log x_{ji}=G_{e_0}(x_{ji})$ in the genus-zero Selberg seed by $\tilde{\Gamma}_{ji}=\tilde{\Gamma}_{x^{(1)}}(z_{ji},\tau)$.

At this point, it is very natural to define a suitable genus-one analogue of the Selberg integrals (28):

Definition 1. Let $L\geq 2$, $0=z_1<z_L<\dots<z_2<1$ and τ the modular parameter of the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Let the empty genus-one Selberg integral (or genus-one Selberg seed) be

$$\mathrm{S}^{\tau}=\mathrm{S}^{\tau}\Big[\,\Big](z_1,\dots,z_L)=\prod_{0=z_1\leq z_i<z_j\leq z_2}\exp\Big(s_{ij}\,\tilde{\Gamma}_{ji}\Big)\,. \tag{103}$$

Genus-one Selberg integrals are then defined recursively by

$$\begin{aligned}&\mathrm{S}^{\tau}\left[\begin{smallmatrix}n_{k+1}, & \dots, & n_L\\ i_{k+1}, & \dots, & i_L\end{smallmatrix}\right](z_1,\dots,z_k)\\ &\qquad=\int_0^{z_k}dz_{k+1}\,g^{(n_{k+1})}_{k+1,i_{k+1}}\,\mathrm{S}^{\tau}\left[\begin{smallmatrix}n_{k+2}, & \dots, & n_L\\ i_{k+2}, & \dots, & i_L\end{smallmatrix}\right](z_1,\dots,z_{k+1})\end{aligned} \tag{104}$$

where $1\leq i_p<p$ for $k+1\leq p\leq L$, $n_{k+1},\dots,n_L$ are non-negative integers and

$$g^{(n)}_{ij}=g^{(n)}_{i,j}=g^{(n)}(z_i-z_j,\tau)\,. \tag{105}$$

For all genus-one Selberg integrals as well as for the genus-one Selberg seed, we will indicate the dependence on τ by the upper index and by using partial derivatives.

The sum $n_{k+1}+\cdots+n_L$ is called the *weight* of the Selberg integral. This notation, where instead of the actual shifts a_i from eq. (83) the index of a position variable z_i is used, will allow for rather compact equations when manipulating genus-one Selberg integrals. Moreover, as for the genus-zero Selberg integrals, the shift $z_{i_{k+1}}$ in the integration kernel $g^{(n_{k+1})}_{k+1,i_{k+1}}$ can only be a variable which has not yet been integrated out, which leads to the genus-one analogue of the admissibility condition in eq. (29):

$$1 \leq i_p < p \qquad \forall p \in \{k+1, \ldots, L\}\,, \tag{106}$$

whereas the corresponding integrals at genus one are again called *admissible*.

Similar to the situation for genus-zero Selberg integrals, convergence is determined by the values of the complex parameters s_{ij}. Since the pole structure of genus-one Selberg integrals matches the corresponding structure at genus zero, the conditions discussed and referred to apply for genus-one Selberg integrals as well. We will assume the parameters s_{ij} to be fixed accordingly throughout the remainder of this article.

The expression for the Selberg seed (103) is already very close to the one-loop Koba–Nielsen factor KN^τ appearing in the one-loop string amplitudes below. In particular, G_{e_0} and $\tilde{\Gamma}_{x^{(1)}}$ are the regularized integrals as defined in eqs. (24) and (85), respectively. A key observation for our construction is the relation between these two functions which is stated in eq. (93): the polylogarithm $G_{e_0}(-2\pi i z)$ describes the asymptotic behaviour of the elliptic polylogarithm $\tilde{\Gamma}_{x^{(1)}}(z,\tau)$ as $z \to 0$.

In order to be equipped for the next subsections, let us collect a couple of identities for genus-one Selberg integrals. Derivatives of the function $\tilde{\Gamma}_{ij}$ can be redirected to another index via

$$\frac{\partial}{\partial z_i}\,\tilde{\Gamma}_{ij} = g^{(1)}(z_i - z_j) = -\frac{\partial}{\partial z_j}\,\tilde{\Gamma}_{ij}\ . \tag{107}$$

In the above language, the Fay identity (90) takes the form

$$\begin{aligned} g^{(m)}_{kj}\, g^{(n)}_{ki} &= (-1)^{m+1} g^{(m+n)}_{ji} + \sum_{r=0}^{n} \binom{m+r-1}{m-1} g^{(n-r)}_{ji}\, g^{(m+r)}_{kj} \\ &\quad + \sum_{r=0}^{m} (-1)^{m-r} \binom{n+r-1}{n-1} g^{(m-r)}_{ji}\, g^{(n+r)}_{ki}\,. \end{aligned} \tag{108}$$

The left-hand side of eq. (108) is admissible, when w.l.o.g. $i < j < k$: if this condition is met, the right-hand side is a $\mathbb{Z}$-linear combination of admissible products.

The Fay identity is the reason why all integration kernels $g_{ij}^{(n)}$ are included in the definition of the genus-one Selberg integrals (104) rather than only $g_{ij}^{(1)}$: application of the Fay identity introduces weights $n \neq 1$, such that a closed system with respect to integration by parts and the Fay identity requires all integration kernels $g_{ij}^{(n)}$.

When discussing a recursive solution for genus-one Selberg integrals below, various derivatives will have to be taken with respect to insertion points z_i, which is thoroughly discussed in appendix C. Here we would like to collect some key properties used in the calculations below. Taking the regularization prescription in eq. (85) into account, we find

$$\mathrm{S}^{\tau}\,|_{z_i=z_j} = 0 \qquad \text{for } i \neq j\,, \tag{109}$$

which is the property analogous to eq. (37). Taking a derivative of the one-loop Selberg seed with respect to a particular variable yields

$$\frac{\partial}{\partial z_i}\,\mathrm{S}^{\tau} = \sum_{k\neq i} s_{ik}\, g_{ik}^{(1)}\,\mathrm{S}^{\tau}\;. \tag{110}$$

Considering the class of type-(k, L) genus-one Selberg integrals for a fixed L and a given number of integrations $L - k$, it is natural to ask for a basis. There are two operations relating different genus-one Selberg integrals: one can integrate by parts and one can apply Fay identities. The question of a basis for this type of integrals is a very old one and amounts to determining a basis of the corresponding twisted de Rham cohomology, similar to the fibration basis in genus zero mentioned in the discussion above definition (47) of the bases for genus-zero Selberg integrals. One possible representation for a basis of the twisted de Rham cohomology on genus one was suggested in [MS20b].

Since a reduction to a basis is convenient, but not necessary in our construction, we do not try to rigorously provide a genus-one analogue of the fibration basis. Instead, we note certain observations for a class of genus-one Selberg integrals with fixed L and a fixed number of integrations $L-k$:

- for an index $n_p = 0$, the corresponding integration kernel $g_{p,i_p}^{(0)} = 1$ is a constant, thus, we can always choose $i_p = 1$ in this case.

- as for the genus-zero Selberg integrals, for an index $n_p = 1$, integration by parts yields a linear equation for the integrals due to the partial derivative of the Selberg seed (110). Hence, for each index $n_p = 1$, we expect to be able to reduce the class of integrals from $1 \leq i_p < p$ to $1 \leq i_p \neq i'_p < p$ for any $1 \leq i'_p < p$ by such an integration by parts identity and applications of the Fay identity (to recover admissible integrals). However, no further such simplifications are expected for the indices $n_p > 1$.

In subsection 3.4 below, we are going to consider a differential equation for a vector of genus-one Selberg integrals of length $L-2$, which are the relevant genus-one Selberg integrals containing the one-loop and tree-level configuration-space integrals:

$$
\begin{aligned}
\mathrm{S}^{\tau}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](z_1=0, z_2) &= \int_0^{z_2} dz_3\, g^{(n_3)}_{3,i_3}\, \mathrm{S}^{\tau}\left[\begin{smallmatrix} n_4, & \dots, & n_L \\ i_4, & \dots, & i_L \end{smallmatrix}\right](z_1=0, z_2, z_3) \\
&= \int_{\mathcal{C}(z_2)} \prod_{i=3}^{L} dz_i\ \mathrm{S}^{\tau} \prod_{k=3}^{L} g^{(n_k)}_{k,i_k}\,,
\end{aligned}
\tag{111}
$$

where $1 \leq i_k < k$ and the integration region is given by (cf. eq. (31)):

$$
\mathcal{C}(z_i) = \{0 = z_1 < z_L < z_{L-1} < \dots < z_i\}\,,
\tag{112}
$$

such that the integral over this domain reads

$$
\int_{\mathcal{C}(z_2)} \prod_{i=3}^{L} dz_i = \int_0^{z_2} dz_3 \int_0^{z_3} dz_4 \cdots \int_0^{z_{L-1}} dz_L\,.
\tag{113}
$$

The integrals defined in eq. (111) are the genus-one generalization of the Selberg integrals (30) relevant for the tree-level amplitude recursion. As for this genus-zero class, the differential equation satisfied by the vector of these genus-one Selberg integrals leads to an associator equation relating one-loop to tree-level configuration-space integrals.

Using the considerations about a fibration basis above, we will at least reduce the class of iterated integrals defined in eq. (111) to a spanning set

$$
\begin{aligned}
\mathcal{B}^{\tau}_{i'_3,i'_4,\dots,i'_L} = \Big\{ \mathrm{S}^{\tau}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](0, z_2) | n_k \geq 0 \text{ and } 1 \leq i_k < k \\
\text{such that } i_k \neq i'_k \text{ if } n_k = 1 \text{ and } i_k = 1 \text{ if } n_k = 0 \Big\}
\end{aligned}
\tag{114}
$$

similar to the genus-zero basis (47). We also allow $i'_k = 0$ if we only intend to reduce the kernels with $n_k = 0$ and include all the kernels with $n_k = 1$, which certainly does not yield a basis, but a spanning set reduced by the redundant labeling of $g^{(0)}_{k,i_k} = 1$. In other words, the labels i'_k in $\mathcal{B}^{\tau}_{i'_3,i'_4,\dots,i'_L}$ denote that the integrals defined by the set $\mathcal{B}^{\tau}_{i'_3,i'_4,\dots,i'_L}$ are the genus-one Selberg integrals from eq. (111), where for $3 \leq k \leq L$ any kernel of the form $g^{(1)}_{k,i'_k}$ is rewritten in terms of the kernels $g^{(1)}_{k,i_k}$ with $1 \leq i_k < k$ and $i_k \neq i'_k$ using integration by parts and the Fay identity. Similarly, any kernel $g^{(0)}_{k,i_k} = 1$ is simply denoted by $g^{(0)}_{k,1} = 1$.

3.3. Generating function for iterated integrals $\tilde{\Gamma}$ and the KZB associator

Before writing down a differential equation of KZB type for a vector of genus-one Selberg integrals in subsection 3.4 below, which is the genus-one generalization of the KZ equation (46), let us consider its formal solution[8] in terms of the so-called (elliptic) KZB associator, originally described[9] in ref. [Enr14]. In fact, although usually represented in a language using a derivation algebra, we would like to point out that the equation as well as its formal solution is very naturally expressed in terms of the canonical iterated integrals $\tilde{\Gamma}$ on the annulus.

By following exactly the same line of arguments as in appendix A, let us start from a generating function[10]

$$\mathrm{L}^{\tau}(z) = \sum_{w \in X} w \, \tilde{\Gamma}_w(z, \tau) \tag{115}$$

[8]As for the KZ equation, we are rather interested in relating a certain regularized boundary value to another regularized boundary value using an associator equation, than completely solving the equation. A rigorous discussion on solutions of the elliptic KZB equation can e.g. be found in ref. [FV95].

[9]KZB equations are the higher-genus generalization of the KZ equation [Ber88b, Ber88a]. In this article, we exclusively consider the elliptic KZB equation and the A-cycle component of the elliptic KZB associator. Therefore, we simply refer to these genus-one objects as KZB equation and KZB associator, respectively, while the genus-zero analogues are called KZ equation and Drinfeld associator.

[10]For this subsection, we explicitly denote the τ-dependence of the functions in order to keep track of the analytic behavior of certain limits. For example, in the asymptotic behavior shown in eqs. (117) and (118), the right-hand side is τ-independent.

of elliptic multiple polylogarithms $\tilde{\Gamma}_w(z,\tau)$, which can be shown to satisfy the differential equation

$$\frac{\partial}{\partial z}\,\mathrm{L}^\tau(z) = \sum_{n\geq 0} g^{(n)}(z,\tau)x^{(n)}\,\mathrm{L}^\tau(z)\,. \tag{116}$$

This differential equation is known as the Knizhnik–Zamolodchikov–Bernard equation (or KZB equation, for short) [Ber88b, Ber88a]. As for the genus-zero case, the asymptotic behavior around $z=0$ is determined by the asymptotics of the iterated integrals in eqs. (92) and (93) which amounts to

$$\mathrm{L}^\tau(z) \sim \exp\left(x^{(1)}\,\tilde{\Gamma}(\tfrac{1}{0};z,\tau)\right) \sim (-2\pi i z)^{x^{(1)}}\,. \tag{117}$$

Due to the one-periodicity (81) of the integration kernels $g^{(n)}$, the KZB equation is invariant under $z\mapsto z-1$ and, hence, there is another solution of the differential eq. (116), $\mathrm{L}^\tau_1(z)$, with the following asymptotics near $z=1$

$$\mathrm{L}^\tau_1(z) \sim \exp\left(x^{(1)}\,\tilde{\Gamma}(\tfrac{1}{0};z,\tau)\right) \sim (-2\pi i(1-z))^{x^{(1)}}\,. \tag{118}$$

As for the genus-zero case, the associator

$$\Phi^\tau = \left(\mathrm{L}^\tau_1(z)\right)^{-1}\mathrm{L}^\tau(z) \tag{119}$$

is independent of z, which can be verified straightforwardly by taking the derivative of both sides of $L^\tau_1\,\Phi^\tau = L^\tau$ and using the differential eq. (116). Thus, the elliptic associator Φ^τ can be expressed in the limit $z\to 1$, which yields the generating series of regularized eMZVs

$$\begin{aligned}\Phi^\tau &= \lim_{z\to 1}\exp\left(-x^{(1)}\,\tilde{\Gamma}(\tfrac{1}{0};z,\tau)\right)\mathrm{L}^\tau(z)\\ &= \sum_{w\in X} w\,\omega(w^t;\tau)\,.\end{aligned} \tag{120}$$

The last equation follows from definition (115) and the cancellation of the divergent integrals due to the exponential prefactor in eq. (120). This is exactly the same mechanism which lead to the expression of the Drinfeld associator in terms of the regularized multiple zeta values in eq. (227) and effectively implements the appropriate regularization. Considering letters up to $x^{(2)}$ only, the first couple of terms of the KZB associator read

$$\begin{aligned}\Phi^\tau = 1 &+ x^{(0)}\omega(0;\tau) + x^{(1)}\omega(1;\tau) + x^{(2)}\omega(2;\tau) +\\ &+ x^{(0)}x^{(0)}\omega(0,0;\tau) + x^{(0)}x^{(1)}\omega(1,0;\tau) + x^{(0)}x^{(2)}\omega(2,0;\tau)\end{aligned}$$

$$
\begin{aligned}
&+ x^{(1)}x^{(0)}\omega(0,1;\tau) + x^{(1)}x^{(1)}\omega(1,1;\tau) + x^{(1)}x^{(2)}\omega(2,1;\tau) \\
&+ x^{(2)}x^{(0)}\omega(0,2;\tau) + x^{(2)}x^{(1)}\omega(1,2;\tau) + x^{(2)}x^{(2)}\omega(2,2;\tau) + \cdots \\
= {} & 1 + x^{(0)} - 2\zeta_2 x^{(2)} \\
&+ \frac{1}{2}x^{(0)}x^{(0)} - (x^{(0)}x^{(1)} - x^{(1)}x^{(0)})\omega(0,1;\tau) - \zeta_2(x^{(0)}x^{(2)} + x^{(2)}x^{(0)}) \\
&+ \big(x^{(1)}x^{(2)} - x^{(2)}x^{(1)}\big)\big(\omega(0,3;\tau) - 2\zeta_2\omega(0,1;\tau)\big) + 5\zeta_4 x^{(2)}x^{(2)} + \cdots
\end{aligned}
$$

The elliptic associator Φ^τ provides an associator equation similar to eq. (226) at genus zero: it connects the regularized boundary values of an arbitrary solution $\mathrm{F}^\tau(z)$ of the KZB equation

$$
\frac{\partial}{\partial z}\,\mathrm{F}^\tau(z) = \sum_{n\geq 0} g^{(n)}(z,\tau)x^{(n)}\,\mathrm{F}^\tau(z)\,, \tag{121}
$$

which are regularized in order to compensate the asymptotic behavior shown in eqs. (117) and (118)

$$
C_0^\tau = \lim_{z\to 0}(-2\pi i z)^{-x^{(1)}}\,\mathrm{F}^\tau(z),\;\; C_1^\tau = \lim_{z\to 1}(-2\pi i(1-z))^{-x^{(1)}}\,\mathrm{F}^\tau(z)\,. \tag{122}
$$

The calculation is similar to the genus-zero case (cf. eq. (224)) and the result is the genus-one associator equation

$$
\begin{aligned}
\Phi^\tau C_0^\tau &= \lim_{z\to 0}(\mathrm{L}_1^\tau(z))^{-1}\,\mathrm{L}^\tau(z)(-2\pi i z)^{-x^{(1)}}\,\mathrm{F}^\tau(z) \\
&= \lim_{z\to 1}(\mathrm{L}_1^\tau(z))^{-1}\,\mathrm{F}^\tau(z) \\
&= C_1^\tau\,.
\end{aligned} \tag{123}
$$

3.4. KZB equation for an auxiliary point

The one-loop version of the recursive construction of open-string amplitudes will again facilitate a differential equation for an extra marked point: the point z_2, which is the variable parametrizing the integration domain of the integrals in eq. (111). The relevant case for the calculation of the $(L-1)$-point genus-one configuration-space integrals below is $k{=}2$. The configuration-space integrals to be calculated are contained in boundary values corresponding to limits of the variable z_2:

- in the limit $z_2 \to 1 = z_1 \bmod \mathbb{Z}$, the integration domain closes and amounts to one complete boundary of the annulus: it leads to $(L-1)$-point genus-one configuration-space integrals with integrands defined on $\mathcal{F}^\tau_{L-1,1}$.

- when taking $z_2 \to 0 = z_1$, genus-one Selberg integrals degenerate to tree-level string corrections, since the integration domain gets confined to a genus-zero domain and the resulting integrands are defined on $\mathcal{F}_{L+1,3}$.

The two associated boundary values can be related by the genus-one associator equation (123) providing the genus-one analogue of the amplitude recursion of ref. [BSST14]. After establishing the KZB equation in this subsection, the boundary values will be discussed in subsection 3.5 below.

Consider a vector of Selberg integrals with fixed upper labels, but lower labels stretching over all admissible values:

$$\mathbf{S}^{\tau}_{(n_{k+1},\dots,n_L)} = \begin{pmatrix} \mathrm{S}^{\tau}\begin{bmatrix} n_{k+1}, & \dots, & n_L \\ 1, & \dots, & 1 \end{bmatrix}(z_1,\dots,z_k) \\ \vdots \\ \mathrm{S}^{\tau}\begin{bmatrix} n_{k+1}, & \dots, & n_L \\ k, & \dots, & L-1 \end{bmatrix}(z_1,\dots,z_k) \end{pmatrix}. \tag{124}$$

For the three-point example to be evaluated below, we have to consider integrals with $k = 2, L = 4$, such that we are going to work with vectors like

$$\mathbf{S}^{\tau}_{(2,1)} = \begin{pmatrix} \mathrm{S}^{\tau}\begin{bmatrix} 2, & 1 \\ 1, & 1 \end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\tau}\begin{bmatrix} 2, & 1 \\ 1, & 2 \end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\tau}\begin{bmatrix} 2, & 1 \\ 1, & 3 \end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\tau}\begin{bmatrix} 2, & 1 \\ 2, & 1 \end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\tau}\begin{bmatrix} 2, & 1 \\ 2, & 2 \end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\tau}\begin{bmatrix} 2, & 1 \\ 2, & 3 \end{bmatrix}(z_1,z_2) \end{pmatrix}. \tag{125}$$

The entries are going to be ordered canonically. As agreed on in the discussion of the spanning set $\mathcal{B}^{\tau}_{i'_3,i'_4,\dots,i'_L}$ defined in eq. (114), whenever there is an $n_k = 0$, we write $i_k = 1$ and we generally do not incorporate integration by parts identities to reduce the number of independent integrals, that is, we usually work with the set of integrals $\mathcal{B}^{\tau}_{0,0,\dots,0}$. Accordingly, if none of the labels $n_3,\dots,n_L$ is zero, the vector $\mathbf{S}^{\tau}_{(n_3,\dots,n_L)}$ has $(L-1)!$ components.

In establishing the KZB equation for a vector of Selberg integrals, we are going to take derivatives of $\mathbf{S}^{\tau}_{(n_3,\dots,n_L)}(z_1 = 0, z_2)$ with respect to the auxiliary point z_2. As will be pointed out below, taking a derivative of a Selberg integral of weight w will lead to a combination of genus-one Selberg integrals of weights between zero and $w+1$. Accordingly, we collect all Selberg vectors of weight w into a larger vector $\mathbf{S}^\tau_w(z_2)$:

$$\mathbf{S}^\tau_w(z_2) = \Big(\mathbf{S}^\tau_{(n_3,n_4,\ldots,n_L)}(z_1=0,z_2)\Big)_{\sum_{k=3}^L n_k = w} \tag{126}$$

and combine all those vectors $\mathbf{S}^\tau_w(z_2)$ into an infinitely large vector in order of increasing w:

$$\mathbf{S}^\tau(z_2) = \begin{pmatrix} \mathbf{S}^\tau_0 \\ \mathbf{S}^\tau_1 \\ \mathbf{S}^\tau_2 \\ \vdots \end{pmatrix}. \tag{127}$$

The vector $\mathbf{S}^\tau(z_2)$ is the genus-one analogue of the genus-zero Selberg vector $\mathbf{S}(x_3)$ defined in eq. (45), which satisfies the KZ eq. (46).

Theorem 2 (Elliptic KZB-system)**.** *Let $\mathbf{S}^\tau(z_2)$ be the vector of genus-one Selberg integrals of type $(2,L)$ with auxiliary point z_2. The derivative with respect to the auxiliary point z_2 can be written in the form*

$$\frac{\partial}{\partial z_2}\mathbf{S}^\tau(z_2) = \sum_{n\geq 0} g^{(n)}_{21} x^{(n)}\,\mathbf{S}^\tau(z_2)\,, \tag{128}$$

which is a system of elliptic KZB-type. The non-vanishing entries of the matrices $x^{(n)}$ are $\mathbb{Z}$-linear combinations of the parameters s_{ij}.

Proof. The proof is constructive, that is, the derivative of any entry of $\mathbf{S}^\tau(z_2)$ is explicitly brought in a form to fit eq. (128) by a combinatorial algorithm. The algorithm consists of two parts: in the first part the expression is rewritten in a way such that the derivative acts on the Selberg seed exclusively, which can be evaluated straighforwardly. This is achieved using integration by parts. In the second part, Fay identities are used iteratively in order to rewrite the result as linear combination of admissible Selberg integrals. The coefficients are shown to consist of polynomials of degree one in the parameters s_{ij} and a factor $g^{(n)}_{21}$.

Due to the length of describing the combinatorial algorithm, we refrain from providing the proof in the main text, rather we would like to ask the reader to consult appendix C instead. □

Example. In order to illustrate the mechanism, let us consider the z_2-derivative of the Selberg vector $\mathbf{S}^\tau_{(0,1)}(z_1=0,z_2)$:

$$\begin{aligned}
&\frac{\partial}{\partial z_2}\mathbf{S}^\tau_{(0,1)} \\
&= g^{(0)}_{21}\begin{pmatrix} -s_{24} & -s_{24} & 0 & 0\,0\,0 & -s_{23} & 0 & 0 & 0 & 0 \\ s_{14} & s_{14}+s_{34} & s_{34} & 0\,0\,0 & 0 & -s_{23}-s_{34} & s_{34} & 0 & s_{34} \\ 0 & -s_{24} & -s_{24} & 0\,0\,0 & 0 & s_{24} & -s_{23}-s_{24} & 0 & -s_{24}\end{pmatrix}\begin{pmatrix}\mathbf{S}^\tau_{(0,2)}\\ \mathbf{S}^\tau_{(1,1)}\\ \mathbf{S}^\tau_{(2,0)}\end{pmatrix} \\
&\quad + g^{(1)}_{21}\begin{pmatrix} s_{12}+s_{24} & -s_{24} & 0 \\ -s_{14} & s_{12}+s_{14} & 0 \\ 0 & 0 & s_{12}\end{pmatrix}\mathbf{S}^\tau_{(0,1)} + g^{(2)}_{21}\begin{pmatrix}-s_{24}\\ s_{14}\\ 0\end{pmatrix}\mathbf{S}^\tau_{(0,0)}\,.
\end{aligned} \tag{129}$$

An immediate observation is in place: considering the weight of the derivative to be one, taking the weight n of each function $g^{(n)}_{21}$ into account and adding the weight of the genus-one Selberg integrals, the total weight is conserved in each term of the above equation.

Starting from the above equation, one can collect all occurring vectors $\mathbf{S}^\tau_{(n_3,n_4)}$ of weight two into the weight-two-vector

$$\mathbf{S}^\tau_2 = \begin{pmatrix}\mathbf{S}^\tau_{(0,2)}\\ \mathbf{S}^\tau_{(1,1)}\\ \mathbf{S}^\tau_{(2,0)}\end{pmatrix}, \tag{130}$$

where the three subvectors are given by

$$\begin{aligned}
\mathbf{S}^\tau_{(0,2)} &= \begin{pmatrix} \mathrm{S}^\tau\begin{bmatrix}0,\,2\\1,\,1\end{bmatrix}(z_1,z_2)\\ \mathrm{S}^\tau\begin{bmatrix}0,\,2\\1,\,2\end{bmatrix}(z_1,z_2)\\ \mathrm{S}^\tau\begin{bmatrix}0,\,2\\1,\,3\end{bmatrix}(z_1,z_2)\end{pmatrix}\\
\mathbf{S}^\tau_{(2,0)} &= \begin{pmatrix} \mathrm{S}^\tau\begin{bmatrix}2,\,0\\1,\,1\end{bmatrix}(z_1,z_2)\\ \mathrm{S}^\tau\begin{bmatrix}2,\,0\\2,\,1\end{bmatrix}(z_1,z_2)\end{pmatrix}\\
\mathbf{S}^\tau_{(1,1)} &= \begin{pmatrix} \mathrm{S}^\tau\begin{bmatrix}1,\,1\\1,\,1\end{bmatrix}(z_1,z_2)\\ \mathrm{S}^\tau\begin{bmatrix}1,\,1\\1,\,2\end{bmatrix}(z_1,z_2)\\ \mathrm{S}^\tau\begin{bmatrix}1,\,1\\1,\,3\end{bmatrix}(z_1,z_2)\\ \mathrm{S}^\tau\begin{bmatrix}1,\,1\\2,\,1\end{bmatrix}(z_1,z_2)\\ \mathrm{S}^\tau\begin{bmatrix}1,\,1\\2,\,2\end{bmatrix}(z_1,z_2)\\ \mathrm{S}^\tau\begin{bmatrix}1,\,1\\2,\,3\end{bmatrix}(z_1,z_2)\end{pmatrix}.
\end{aligned} \tag{131}$$

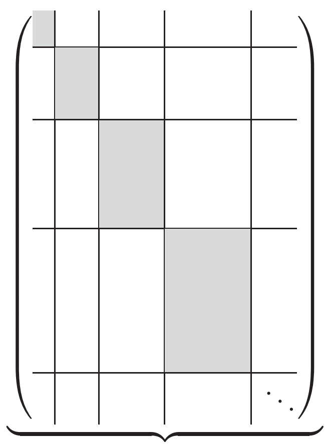
So the 11-component vector $\mathbf{S}_2^\tau$ captures the combinatorics from distributing weight two on the two slots (n_3, n_4) as well as the combinatorics of the labels i_k for each of those pairs. Neatly, the particular ordering does not play a role in the formalism to be described, however, we will follow the sorting convention in eq. (130).

Block structure of coefficient matrices $\boldsymbol{x^{(n)}}$: Let us further investigate the structure of the differential system (128) and in particular the matrices $x^{(n)}$. As visible in example (129), taking a derivative of a Selberg integral will increase the weight by one. Using the algorithm described in appendix C, one can thus write the z_2-derivative on $\mathbf{S}_w^\tau$ as

$$\frac{\partial}{\partial z_2}\mathbf{S}_w^\tau(z_2) = \sum_{n=0}^{w+1} g_{21}^{(n)} x_w^{(n)}\,\mathbf{S}_{w+1-n}^\tau(z_2)\,, \tag{132}$$

where the factor $x^{(n)}$ does not contribute to the weight, but $g_{21}^{(n)}$ does. From counting the weights, one can thus deduce that the matrices $x^{(n)}$ in eq. (128) ought to be block-(off-)diagonal, where the size of the blocks corresponds the lengths of the Selberg vectors of weight w. Schematically, one finds

$$\frac{\partial}{\partial z_2}\mathbf{S}^\tau = g_{21}^{(0)} x^{(0)}\,\mathbf{S}^\tau + g_{21}^{(1)} \tag{133}$$



$$x^{(1)} \qquad \underbrace{\begin{pmatrix} \mathbf{S}_0^\tau(z_2) \\ \hline \mathbf{S}_1^\tau(z_2) \\ \hline \mathbf{S}_2^\tau(z_2) \\ \hline \mathbf{S}_3^\tau(z_2) \\ \hline \vdots \end{pmatrix}}_{\mathbf{S}^\tau}$$

$$+\, g_{21}^{(2)} x^{(2)}\,\mathbf{S}^\tau + \cdots\,,$$

where only the blue blocks are non-vanishing. Given the blocks in the above equation, the other matrices will have the following structure:

$$
x^{(0)} = \begin{pmatrix} & x_0^{(0)} & & & \\ & & x_1^{(0)} & & \\ & & & x_2^{(0)} & \\ & & & & x_3^{(0)} \\ & & & & \ddots \end{pmatrix}, \qquad x^{(1)} = \begin{pmatrix} x_0^{(1)} & & & & \\ & x_1^{(1)} & & & \\ & & x_2^{(1)} & & \\ & & & x_3^{(1)} & \\ & & & & \ddots \end{pmatrix},
$$

(134)

$$
x^{(2)} = \begin{pmatrix} & & & & \\ x_1^{(2)} & & & & \\ & x_2^{(2)} & & & \\ & & x_3^{(2)} & & \\ & & & x_3^{(2)} & \ddots \end{pmatrix}, \qquad \ldots,
$$

where the blocks of the individual matrices are labeled by $x_w^{(n)}$.

Truncation of the KZB system. In practice, the infinitely long vector $\mathbf{S}^\tau(z_2)$ and the matrices $x^{(n)}$ of infinite dimension need to be truncated at a certain maximal total weight $w_{\max}$

$$
\mathbf{S}^\tau_{\le w_{\max}}(z_2) = \begin{pmatrix} \mathbf{S}^\tau_0 \\ \mathbf{S}^\tau_1 \\ \vdots \\ \mathbf{S}^\tau_{w_{\max}} \end{pmatrix}. \tag{135}
$$

Taking the z_2-derivative on the finite-length vector $\mathbf{S}^\tau_{\le w_{\max}}(z_2)$ leads to the differential equation

$$
\frac{\partial}{\partial z_2} \mathbf{S}^\tau_{\le w_{\max}}(z_2) = \sum_{n=0}^{w_{\max}+1} g_{21}^{(n)} x^{(n)}_{\le w_{\max}} \mathbf{S}^\tau_{\le w_{\max}}(z_2) + r_{w_{\max}} \mathbf{S}^\tau_{w_{\max}+1}(z_2)\,, \tag{136}
$$

where the remainder $r_{w_{\max}}$ prevents eq. (136) to be a complete KZB equation. However, as will be discussed below, this remainder may be disregarded when calculating one-loop configuration-space integrals up to a particular order in α'.

The matrices $x^{(n)}_{\leq w_{\max}}$ for $0 \leq n \leq w_{\max}+1$ correspond to the upper-left $(w_{\max}+1) \times (w_{\max}+1)$ block matrices of the matrices $x^{(n)}$. Explicitly:

(137)

$$
x^{(0)}_{\leq w_{\max}} = \left(\begin{array}{c|c|c|c|c}
 & x^{(0)}_0 & & & \\ \hline
 & & x^{(0)}_1 & & \\ \hline
 & & & \ddots & \\ \hline
 & & & & x^{(0)}_{w_{\max}-1} \\ \hline
 & & & &
\end{array}\right), \quad
x^{(1)}_{\leq w_{\max}} = \left(\begin{array}{c|c|c|c|c}
x^{(1)}_0 & & & & \\ \hline
 & x^{(1)}_1 & & & \\ \hline
 & & x^{(1)}_2 & & \\ \hline
 & & & \ddots & \\ \hline
 & & & & x^{(1)}_{w_{\max}}
\end{array}\right),
$$

(138)

$$
x^{(2)}_{\leq w_{\max}} = \left(\begin{array}{c|c|c|c|c}
 & & & & \\ \hline
x^{(2)}_1 & & & & \\ \hline
 & x^{(2)}_2 & & & \\ \hline
 & & \ddots & & \\ \hline
 & & & x^{(2)}_{w_{\max}} &
\end{array}\right), \dots, x^{(w_{\max}+1)}_{\leq w_{\max}} = \left(\begin{array}{c|c|c|c|c}
 & & & & \\ \hline
 & & & & \\ \hline
 & & & & \\ \hline
 & & & & \\ \hline
x^{(w_{\max}+1)}_{w_{\max}} & & & &
\end{array}\right).
$$

Moreover, the remainder $r_{w_{\max}}$ is the $(w_{\max}+1)\times 1$ block submatrix of the first $w_{\max}+1$ blocks of the $(w_{\max}+2)$-column of the matrix $x^{(0)}_{\leq w_{\max}+1}$:

(139)

$$
x^{(0)}_{\leq w_{\max}+1} = \left(\begin{array}{c|c|c|c|c|c}
 & x^{(0)}_0 & & & & \\ \hline
 & & x^{(0)}_1 & & & \\ \hline
 & & & \ddots & & \\ \hline
 & & & & x^{(0)}_{w_{\max}-1} & \\ \hline
 & & & & & x^{(0)}_{w_{\max}} \\ \hline
 & & & & &
\end{array}\right), \quad r_{w_{\max}} = \left(\begin{array}{c}
 \\ \hline
 \\ \hline
\vdots \\ \hline
 \\ \hline
x^{(0)}_{w_{\max}}
\end{array}\right).
$$

The value of $w_{\max}$ necessary for the calculation of the configuration-space integrals up to a particular order in α' is going to be determined in subsection 3.6.

3.5. Boundary values for the KZB equation

Having derived a (modified) KZB equation for the genus-one Selberg integrals, we would like to apply the genus-one associator equation (123) in order to evaluate genus-one configuration-space integrals from genus-zero configuration-space integrals. This amounts to proving the following proposition.

Proposition 3. *The regularized boundary values*

(140)
$$\mathbf{C}_1^\tau = \lim_{z_2\to 1}(-2\pi i(1-z_2))^{-x^{(1)}}\,\mathbf{S}^\tau(z_2) \ \textit{and}\ \mathbf{C}_0^\tau \ = \lim_{z_2\to 0}(-2\pi i z_2)^{-x^{(1)}}\,\mathbf{S}^\tau(z_2)$$

are related by the A-cycle component $\Phi(x^{(0)},x^{(1)},x^{(2)},\ldots)$ *of the KZB associator via*

$$\mathbf{C}_1^\tau = \Phi(x^{(0)},x^{(1)},x^{(2)},\ldots)\,\mathbf{C}_0^\tau\ . \tag{141}$$

The regularized boundary value $\mathbf{C}_1^\tau$ *contains* $(L-1)$*-point configuration-space integrals at genus one whereas* $\mathbf{C}_0^\tau$ *contains* $(L+1)$*-point configuration-space integrals at genus zero.*

Proof. While the statement in eq. (141) follows straightforwardly from the discussion in subsection 3.3 and the form of the KZB equation in theorem 2, the boundary values $\mathbf{C}_0^\tau$ and $\mathbf{C}_1^\tau$ will be explicitly constructed and shown to contain the respective configuration-space integrals below. Following definition eq. (122), we will have to evaluate

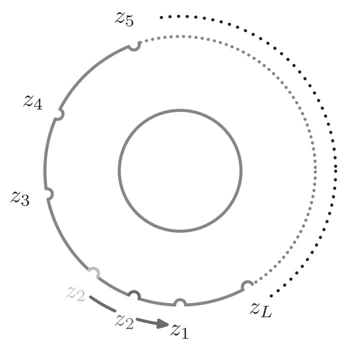
$$\begin{aligned}
\mathbf{C}_0^\tau &= \lim_{z_2\to 0}(-2\pi i z_2)^{-x^{(1)}}\,\mathbf{S}^\tau(z_2)\\
&= \lim_{z_2\to 0}\begin{pmatrix}(-2\pi i z_2)^{-x_0^{(1)}}\,\mathbf{S}_0^\tau(z_2)\\ (-2\pi i z_2)^{-x_1^{(1)}}\,\mathbf{S}_1^\tau(z_2)\\ \vdots\end{pmatrix} = \begin{pmatrix}\mathbf{C}_{0,0}^\tau\\ \mathbf{C}_{0,1}^\tau\\ \vdots\end{pmatrix},\\
\mathbf{C}_1^\tau &= \lim_{z_2\to 1}(-2\pi i(1-z_2))^{-x^{(1)}}\,\mathbf{S}^\tau(z_2)\\
&= \lim_{z_2\to 1}\begin{pmatrix}(-2\pi i(1-z_2))^{-x_0^{(1)}}\,\mathbf{S}_0^\tau(z_2)\\ (-2\pi i(1-z_2))^{-x_1^{(1)}}\,\mathbf{S}_1^\tau(z_2)\\ \vdots\end{pmatrix} = \begin{pmatrix}\mathbf{C}_{1,0}^\tau\\ \mathbf{C}_{1,1}^\tau\\ \vdots\end{pmatrix},
\end{aligned} \tag{142}$$

where $\mathbf{C}_{0,w}^\tau$ and $\mathbf{C}_{1,w}^\tau$ denote the regularized limits of the subvectors $\mathbf{S}_w^\tau(z_2)$ of weight w and the second equality in the above equations follows from the block-diagonal form of the matrix $x^{(1)}$. Switching to finite matrix size, we define

$$
\begin{aligned}
\mathbf{C}^{\tau}_{0,\leq w_{\max}} &= \lim_{z_2\to 0}(-2\pi i z_2)^{-x^{(1)}_{\leq w_{\max}}}\,\mathbf{S}^{\tau}_{\leq w_{\max}}(z_2)\,,\\
\mathbf{C}^{\tau}_{1,\leq w_{\max}} &= \lim_{z_2\to 1}(-2\pi i(1-z_2))^{-x^{(1)}_{\leq w_{\max}}}\,\mathbf{S}^{\tau}_{\leq w_{\max}}(z_2)\,.
\end{aligned}
\tag{143}
$$

Boundary value $\mathbf{C}^{\tau}_1$: Considering the limit $z_2 \to 1$, we first determine the behavior of the component integrals $\mathrm{S}^{\tau}\left[\begin{smallmatrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\right](z_2)$ and include the regularization factor $-2\pi i(1-z_2)^{-x^{(1)}}$ afterwards.

(144)



According to eq. (89) and using (87), the genus-one Selberg seed degenerates as

$$
\begin{aligned}
\lim_{z_2\to 1}(-2\pi i(1-z_2))^{-s_{12}}\,\mathrm{S}^{\tau} &= \prod_{0=z_1<z_i<z_j<z_2}\exp\left(s_{ij}\,\tilde{\Gamma}_{ji}\right)\prod_{j>2}\exp\left((s_{1j}+s_{2j})\,\tilde{\Gamma}_{j1}\right)\\
&= \mathrm{S}^{\tau}\Big|^{\tilde{s}_{ij}=s_{ij}+\delta_{i1}s_{2j}}_{(L-1)\text{-point}}\,.
\end{aligned}
\tag{145}
$$

The term on the right-hand side of the above equation is the genus-one Selberg seed for $L-1$ insertion points on the boundary of the annulus. Since the insertion points z_2 and z_1 merge in the limit, effective Mandelstam variables $\tilde{s}_{ij} = s_{ij} + \delta_{i1}s_{2j}$ for $i,j \in \{1,\ldots,L\}\setminus\{2\}$, $i<j$ have to be assigned to the insertion points, such that $\tilde{s}_{1j}$ associated to z_1 includes the contribution from z_2:

$$
\tilde{s}_{1j} = s_{1j} + s_{2j}\,. \tag{146}
$$

The behavior is the same as in the genus-zero case in eq. (60): the momentum of the external state which corresponds to one of the fixed insertion points receives two contributions, one coming from the state at $z_1 = 0$ and the other from a state at the same position of the boundary of the annulus $z_2 = z_1 \mod \mathbb{Z}$ due to the merged auxiliary insertion point z_2.

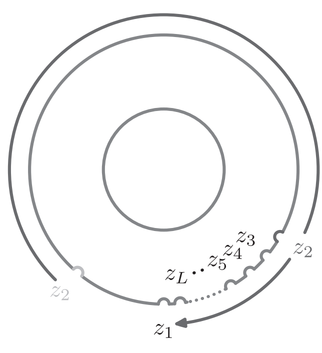
Accordingly, the genus-one Selberg integral defined in eq. (111) as a function on the configuration space of the A-cycle with two positions fixed, degenerates at lowest order in $(1-z_2)$ to an integral with integrand defined on $\mathcal{F}^{\tau}_{L-1,1}$

$$\begin{aligned}
&\lim_{z_2\to 1}(-2\pi i(1-z_2))^{-s_{12}}\, \mathrm{S}^{\tau}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](z_1=0,z_2)\\
&= \int_{\mathcal{C}(z_2\to 1)} \prod_{i=3}^{L} dz_i\ \mathrm{S}^{\tau}\Big|^{\tilde{s}_{1j}=s_{1j}+s_{2j}}_{(L-1)\text{-point}} \prod_{k=3}^{L} g^{(n_k)}_{k,i_k}\Big|_{z_2\equiv z_1=0}\\
&= \mathrm{S}^{\tau}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](0,z_2=1)\Big|^{\tilde{s}_{1j}=s_{1j}+s_{2j}}_{z_2\equiv z_1=0}\,.
\end{aligned} \tag{147}$$

Similar to eq. (59), the relevant eigenvalues of the matrices $x^{(1)}_w$ appearing in the regularization factor $-2\pi i(1-z_2)^{-x^{(1)}_w}$ in $\mathbf{C}^{\tau}_1$ from eq. (142) are s_{12}, such that the regularization factor contributes the factor $(-2\pi i(1-z_2))^{-s_{12}}$ from eq. (147) and the entries of $\mathbf{C}^{\tau}_1$ are given by the degenerate genus-one Selberg integrals in eq. (147). They are the $(L-1)$-point genus-one configuration-space integrals [BMMS15]. This concludes the proof of the statement about the boundary value $\mathbf{C}^{\tau}_1$ in proposition 3. In ref. [BKS20] this and the analogous statement for the boundary value $\mathbf{C}^{\tau}_0$ below are proven using generating series of the open-string configuration-space integrals.

Boundary value $\mathbf{C}^{\tau}_0$: The boundary value $\mathbf{C}^{\tau}_0$ is obtained by confining the region of integration to an infinitesimal interval as $z_2 \to 0 = z_1$. As for the genus-zero calculation in eq. (64), the main tool to investigate this degeneration and the corresponding behavior of genus-one Selberg integrals is a change of variables $z_i = z_2 x_i$, where x_i are points in the unit interval on the real line whereas z_i are located on the boundary of an annulus.

(148)



According to the discussion below definition 1 in subsection 3.2 and as a consequence of eq. (88), the genus-one Selberg seed S^τ degenerates at lowest order in z_2 for $z_2 \to 0$ up to a proportionality factor to the genus-zero Selberg seed S for the L points $0 = x_1 < x_L < x_{L-1} < \cdots < x_2 = 1$ on the unit interval, which is (cf. eq. (71)) precisely the (L+1)-point genus-zero Koba–Nielsen factor defined in eq. (70):

$$\begin{aligned}\lim_{z_2\to 0}(-2\pi i z_2)^{-s_{12\dots L}}\,\mathrm{S}^\tau &= \prod_{0=x_1<x_i<x_j<x_2=1} x_{ji}^{s_{ij}}\\ &= \mathrm{S}\,|_{L\text{-point}}\\ &= \mathrm{KN}\,|_{(L+1)\text{-point}}\,. \end{aligned} \tag{149}$$

The discussion of the eigenvalues of $x_w^{(1)}$ is similar to the genus-zero case and thoroughly addressed in ref. [BKS20]. It turns out that the maximal and therefore dominant eigenvalue of $x_w^{(1)}$ is $s_{12\dots L}$, such that the regularization $(-2\pi i z_2)^{-x_w^{(1)}}$ contributes the factor $(-2\pi i z_2)^{-s_{12\dots L}}$ in eq. (149). Thus, the entries of $\mathbf{C}_0^\tau$ are given by

$$\begin{aligned}&\lim_{z_2\to 0}(-2\pi i z_2)^{-s_{12\dots L}}\,\mathrm{S}^\tau\left[\begin{smallmatrix} n_3, & \dots, & n_L\\ i_3, & \dots, & i_L\end{smallmatrix}\right](z_1=0,z_2)\\ &= \lim_{z_2\to 0} z_2^{L-2}\int_{\mathcal{C}(x_2=1)}\prod_{i=3}^{L}dx_i\;\mathrm{S}\,|_{L\text{-point}}\prod_{k=3}^{L}g^{(n_k)}(z_2 x_{k,i_k},\tau)\\ &= \begin{cases}\int_{\mathcal{C}(x_2=1)}\prod_{i=3}^{L}dx_i\;\mathrm{S}\,|_{L\text{-point}}\prod_{k=3}^{L}\frac{1}{x_{ki_k}} & \text{if } n_1=n_2=\cdots=n_k=1\,,\\ 0 & \text{otherwise}\,.\end{cases}\end{aligned} \tag{150}$$

The only non-vanishing entries are the ones for which all integration kernels have weight one, i.e. $n_k = 1$, since only their pole can compensate for the z_2^{L-2} factor from the measure.

A similar behavior was observed for the genus-zero boundary value which led to eq. (66). Moreover, these simple poles ensure that the only non-vanishing integrals are exactly the degenerate genus-zero Selberg integrals $\mathrm{S}[i_3, i_4, \dots, i_L](0, 1, x_2 = 1)$ found in the genus-zero regularized boundary values $\mathbf{C}_0$ and $\mathbf{C}_1$ in eqs. (66) and (60), respectively. However, here we recover integrals with integrands defined on $\mathcal{F}_{L+1,3}$ with the L+1 insertion points $0 = x_1 < x_L < x_{L+1} < \cdots < x_2 = 1 < x_{L+1} = \infty$ (cf. eq. (26)). As discussed in subsection 2.5, these integrals are related to the (L+1)-point genus-zero configuration-space integrals by a basis transformation. This concludes the proof of proposition 3. $\square$

Mandelstam variables: In contrast to both the genus-zero discussion and the limiting situation $\mathbf{C}_1^\tau$, in the boundary value $\mathbf{C}_0^\tau$ the Mandelstam variables s_{2j} in eq. (149) associated to the auxiliary insertion point z_2 are not redundant: the auxiliary genus-one momentum $k_2^{\text{1-loop}}$ associated to z_2 encodes the genus-zero momentum k_2^{tree} associated to the tree-level insertion point x_2

(151) $$k_2^{\text{1-loop}} = k_2^{\text{tree}} \,.$$

In order to keep track of how this momentum contributes to the one-loop momenta, two distinct processes have to be considered: first, the topological change by the identification of x_1 with x_{L+1} giving the genus-one insertion point z_1 depicted in figure (78) and second, the merging of $z_2 \to 1 = z_1 \bmod \mathbb{Z}$ shown in figure (144). In the first case, the momenta k_1^{tree} and k_{L+1}^{tree} associated to x_1 and x_{L+1}, respectively, yield the joint contribution to the one-loop momentum associated to z_1

(152) $$k_1^{\text{1-loop}} = k_1^{\text{tree}} + k_{L+1}^{\text{tree}} \,.$$

The second limit is the merging of z_2 to z_1, which adds the momentum $k_2^{\text{1-loop}}$ associated to z_2 to the momentum $k_1^{\text{1-loop}}$ and we expect to find the effective momentum

(153) $$\tilde{k}_1^{\text{1-loop}} = k_1^{\text{1-loop}} + k_2^{\text{1-loop}} = k_1^{\text{tree}} + k_{L+1}^{\text{tree}} + k_2^{\text{tree}}$$

for the insertion point $z_1 = z_2 \bmod \mathbb{Z}$ of the $(L-1)$-point one-loop interaction in the regularized boundary value $\mathbf{C}_1^\tau$, where we denote the one-loop momenta $k_i^{\text{1-loop}}$ in the limit $z_2 \to z_1 = 1 \bmod \mathbb{Z}$ by a tilde as depicted in figure (158). However, from our calculations of $\mathbf{C}_1^\tau$ in eq. (145) we see that the Mandelstam variables associated to $\tilde{k}_1^{\text{1-loop}}$ are

(154) $$\tilde{s}_{1j} = s_{1j} + s_{2j} \,.$$

Therefore, the actual one-loop momentum associated to $z_1 = z_2 \bmod \mathbb{Z}$ turns out to be

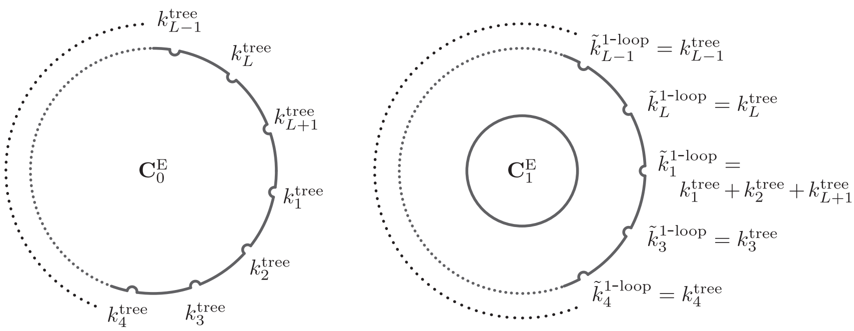
(155) $$\tilde{k}_1^{\text{1-loop}} = k_1^{\text{tree}} + k_2^{\text{tree}} \,.$$

This is in agreement with simultaneous momentum conservation in the tree-level and one-loop interaction if and only if

(156) $$k_{L+1}^{\text{tree}} = 0 \,,$$

which can be interpreted as follows: reversing the first procedure discussed above (see figure (78)), that is, going in the direction from genus one to genus zero, the momentum $k_1^{\text{1-loop}}$ associated to z_1 is expected to split in a certain way and to contribute to the two tree-level momenta k_1^{tree} and k_{L+1}^{tree} accordingly. From eq. (156) follows that these two contributions are very unequal: while the momentum associated to x_1 obtains the full contribution $k_1^{\text{tree}} = k_1^{\text{1-loop}}$, the momentum associated to x_{L+1} goes away empty-handed $k_{L+1}^{\text{tree}} = 0$. Note that the momenta associated the remaining tree-level insertion points x_i for $i = 3, 4, \ldots, L$ are exactly the one-loop momenta associated to the punctures z_i for any $0 < z_2 \leq 1 = z_1 \mod \mathbb{Z}$:

$$\tilde{k}_i^{\text{1-loop}} = k_i^{\text{1-loop}} = k_i^{\text{tree}} \quad \text{for } i = 3, 4, \ldots, L\,. \tag{157}$$



(158)

Summary of subsection: The regularized boundary value $\mathbf{C}_0^\tau$ is found to only have finitely many non-vanishing entries which are degenerate genus-zero Selberg integrals and hence linear combinations of $(N+2)$-point tree-level configuration-space integrals, where $N = L - 1$. In turn, as will be discussed in detail in the next subsection, the entries of $\mathbf{C}_1^\tau$ given by eq. (147) contain the N-point one-loop configuration-space integrals.

Therefore, the genus-one Selberg vector $\mathbf{S}^\tau(z_2)$ indeed interpolates between the genus-zero and genus-one configuration-space integrals and the corresponding associator equation

$$\mathbf{C}_1^\tau = \Phi^\tau \ \mathbf{C}_0^\tau \tag{159}$$

provides a link between genus-zero and genus-one integrals. Combining this associator equation with the genus-zero recursion for the genus-zero configuration-space integrals from ref. [BSST14, Kad20] based on the genus-zero associator equation (10) yields a recursion in genus and the number of external states to calculate the genus-one configuration-space integrals.

The consideration about the contributions of the insertion points defining the genus-one Selberg integrals to the Mandelstam variables in the configuration-space integrals appearing in the boundary values $\mathbf{C}_0^\tau$ and $\mathbf{C}_1^\tau$ leads to a geometric interpretation of the associator eq. (159): the N-point one-loop worldsheet is obtained from the $(N{+}2)$-point tree-level worldsheet by an effective gluing of the two legs corresponding to the insertion points $x_1 = 0$ and $x_{L+1} = \infty$ on the Riemann sphere. By momentum conservation the Mandelstam variables associated to the insertion point z_1 in the one-loop configuration-space integrals of $\mathbf{C}_1^\tau$ are given by the sum $\tilde{s}_{1j} = s_{1j} + s_{2j}$.

3.6. Open-string amplitudes at genus one

Let us finally employ the associator eq. (159) to calculate the α'-expansion of N-point one-loop configuration-space integrals [GSB82, DG09, BMMS15] in open string theory up to any desired order in α' from $(N{+}2)$-point tree-level configuration-space integrals.

While relating various entries of the regularized boundary values to known representations of configuaration-space integrals at genus zero and genus one, we will simultaneously single out the relevant parts of the matrix equation (159).

The main goal is the calculation of the N-point one-loop configuration-space integral up to a desired maximal order in α' denoted by $o_{\max}^{\text{1-loop}}$. As observed in the previous subsection, integrals with integrands defined on $\mathcal{F}_{N,1}^\tau$ for the N-point one-loop configuration-space integrals arise in the $z_2 \to 1$ limit of genus-one Selberg integrals with $L = N{+}1$ marked points. Simultaneously, $(N{+}2)$-point tree-level configuration-space integrals are encoded in the $z_2 \to 0$ limit of the same genus-one Selberg integrals.

As pointed out at the end of subsection 3.4 above, for practical calculations the infinite genus-one Selberg vector has to be truncated to $\mathbf{S}_{\leq w_{\max}}^\tau(z_2)$. Given the target values N and $o_{\max}^{\text{1-loop}}$ for the calculation, let us determine $w_{\max}$ as well as various other parameters for the calculation.

Each of the objects on the right-hand side in eq. (159) has an expansion in the parameter α': since $x^{(n)} \propto \alpha'$ (cf. eq. (2), the expansion in word length of the elliptic KZB associator is exactly its α'-expansion. The α'-expansion of the tree-level integrals in $\mathbf{C}_0^E$ can be obtained from the recursions in refs. [MS17, BSST14]. Therefore, the maximal target α'-order $o_{\max}^{\text{1-loop}}$ of the one-loop configuration-space integrals on the right-hand side is reached, when the KZB-associator is expanded up to α'-order

$$l_{\max} = o_{\max}^{\text{1-loop}} - o_{\min}^{\text{tree}} \tag{160}$$

where $o_{\min}^{\text{tree}}$ denotes the leading (e.g. minimal) order in the α'-expansion of tree-level integrals in $\mathbf{C}_0^\tau$. This order turns out to be given by [MSS13]

$$o_{\min}^{\text{tree}} = 2 - L = 3 - N\,. \tag{161}$$

In order to determine $w_{\max}$, we need to think about the positions of the relevant information within the vectors $\mathbf{C}_0^\tau$ and $\mathbf{C}_1^\tau$: on the one hand, according to eq. (150) the non-vanishing subvector of $\mathbf{C}_0^\tau$ which includes the tree-level configuration-space integrals is contained in the weight

$$w_0 = L - 2 \tag{162}$$

subvector $\mathbf{C}_{0,w_0}^\tau$ of $\mathbf{C}_0^\tau$. On the other hand, the one-loop configuration-space integrals are contained in the weight

$$w_1 = L - 5 - d \tag{163}$$

subvector $\mathbf{C}_{1,w_1}^\tau$. The quantity d denotes the number of additional factors of $g^{(n)}$ appearing in higher-point one-loop configuration-space integrals and is given by $d = 0$ for $L \leq 8$ and $d \geq 0$ otherwise [BMMS15]. For all calculations in this article, $d = 0$ holds. The relevant part of the elliptic KZB associator is the submatrix Φ_{w_1,w_0}^τ, which satisfies the equation

$$\mathbf{C}_{1,w_1}^\tau = \Phi_{w_1,w_0}^\tau\,\mathbf{C}_{0,w_0}^\tau\,. \tag{164}$$

Since for all amplitude situations we find $w_1 < w_0$, the submatrix Φ_{w_1,w_0}^τ is located above the diagonal of Φ^τ.

Here comes the block-(off-)diagonal form of the matrices $x^{(n)}$ depicted in (134) into play, which ensures that for a certain word length l, only finitely many words $w = x^{(n_1)} \dots x^{(n_l)}$ contribute non-trivially to Φ_{w_1,w_0}^τ. The non-trivial contribution to Φ_{w_1,w_0}^τ at each word length l, which is the order l in the α'-expansion of the associator since $x^{(n)} \propto \alpha'$, is a finite sum $\sum_w w\,\omega(w^t)$ of products $w = x_{\leq w_{\max}(l)}^{(n_1)} x_{\leq w_{\max}(l)}^{(n_2)} \dots x_{\leq w_{\max}(l)}^{(n_l)}$, where

$$w_{\max}(l) = \max(l + w_1 - w_0, w_0) \tag{165}$$

and $(n_1, n_2, \dots, n_l)$ is a length-l, ordered partition of $w_{\max}(l)$, i.e. $n_1 + n_2 + \dots + n_l = w_{\max}(l)$, which satisfies for each $r \in \{1, 2, \dots, l\}$ the additional conditions

$$0 \leq i - \sum_{s=1}^{r-1}(n_s - 1) \leq w_{\max}(l)\,, \qquad 0 \leq j + n_l - 1 \leq w_{\max}(l)\,. \tag{166}$$

Therefore, the α'-expansion of $\Phi^\tau_{w_1,w_0}$ up to some maximal order $l_{\max}$ in α' or maximal word length, respectively, can be calculated by finite-dimensional submatrices of $x^{(n)}$, which are the matrices $x^{(n)}_{\le w_{\max}}$ for the maximal weight $w_{\max}(l_{\max})$:

$$\Phi^\tau(x^{(n)})_{w_1,w_0} = \Phi^\tau(x^{(n)}_{\le w_{\max}(l_{\max})})_{w_1,w_0} + \mathcal{O}\Big((\alpha')^{l_{\max}+1}\Big)\,. \tag{167}$$

In other words, the associator submatrix $\Phi^\tau(x^{(n)})_{w_1,w_0}$ can be deduced from a truncated associator, which is determined by evaluating the matrix products of truncated representations of letters, taking only words up to length $l_{\max}$ and weight $w_{\max}$ into account. The truncated matrix representations $x^{(n)}_{\le w_{\max}}$ of the letters can be obtained from the modified KZB eq. (136). Since word length $l_{\max}$ and maximal weight $w_{\max}(l_{\max})$ are finite quantities, all sums consist of a finite number of terms and all matrices are of finite size. The process yields the finite-dimensional, truncated associator equation

$$\mathbf{C}^\tau_{1,\le w_{\max}(l_{\max})} + \mathcal{O}\Big((\alpha')^{o^{\text{1-loop}}_{\max}+1}\Big) = \Phi^\tau_{l_{\max}}(x^{(n)}_{\le w_{\max}(l_{\max})})\; \mathbf{C}^\tau_{0,\le w_{\max}(l_{\max})}\,, \tag{168}$$

where $\Phi^\tau_{l_{\max}}$ is the truncation of Φ^τ at the maximal word length $l_{\max}$. The finite subvectors

$$\begin{aligned} \mathbf{C}^\tau_{0,\le w_{\max}(l_{\max})} &= \lim_{z_2\to 0} (-2\pi i z_2)^{-x^{(1)}_{\le w_{\max}(l_{\max})}}\, \mathbf{S}^\tau_{\le w_{\max}(l_{\max})}(z_2)\,, \\ \mathbf{C}^\tau_{1,\le w_{\max}(l_{\max})} &= \lim_{z_2\to 1} (-2\pi i (1-z_2))^{-x^{(1)}_{\le w_{\max}(l_{\max})}}\, \mathbf{S}^\tau_{\le w_{\max}(l_{\max})}(z_2) \end{aligned} \tag{169}$$

of $\mathbf{C}^\tau_0$ and $\mathbf{C}^\tau_1$, respectively, contain the $(L+1)$-point tree-level string corrections at weight $w_0 = L-2 \le w_{\max}(l_{\max})$ and the $(L-1)$-point one-loop corrections at $w_1 = L-5-d$. Denoting by $\Phi^\tau_{l_{\max}}(x^{(n)}_{\le w_{\max}(l_{\max})})_{w_1,w_0}$ the weight-(w_1,w_0) submatrix of the truncated KZB associator $\Phi^\tau_{l_{\max}}\big(x^{(n)}_{\le w_{\max}(l_{\max})}\big)$, the relevant truncated vector equation which relates the string corrections to each other is

$$\mathbf{C}^\tau_{1,w_1} + \mathcal{O}\Big((\alpha')^{o^{\text{1-loop}}_{\max}+1}\Big) = \Phi^\tau_{l_{\max}}(x^{(n)}_{\le w_{\max}(l_{\max})})_{w_1,w_0}\; \mathbf{C}^\tau_{0,w_0}\,, \tag{170}$$

where $\Phi^\tau_{l_{\max}}(x^{(n)}_{\le w_{\max}(l_{\max})})_{w_1,w_0}$ is the weight-(w_1,w_0) submatrix of the truncated elliptic KZB associator $\Phi^\tau_{l_{\max}}(x^{(n)}_{\le w_{\max}(l_{\max})})$.

3.7. Relating boundary values at genus zero and one

In this section, we briefly discuss how the regularized boundary value C_0^τ of a function satisfying a KZB equation is related to a corresponding genus-zero limit C_0 of a solution of a KZ equation. This provides an explanation of why in the recursion described in the previous section, genus-zero configuration-space integrals are obtained from genus-one Selberg integrals.

Before we focus on genus-one quantities, we determine the origin of the regularization used for the regularized genus-zero boundary value

$$C_0 = \lim_{x \to 0} x^{-e_0} F(x) \tag{171}$$

of a solution $F(x)$ of the KZ equation

$$\frac{d}{dx} F(x) = \left(\frac{e_0}{x} + \frac{e_1}{x-1} \right) F(x) \,. \tag{172}$$

For $0 < x \ll 1$, the KZ equation can be written as

$$\frac{d}{dx} F(x) = \left(\frac{e_0}{x} - e_1 + \mathcal{O}(x) \right) F(x) \tag{173}$$

up to linear order in x. Using this differential equation and the fact that $[e_0, e^{xe_1}] = \mathcal{O}(x)$, the function $F(x)$ can be approximated by

$$F(x) = e^{-xe_1} x^{e_0} f_0 + \mathcal{O}(x) \tag{174}$$

for some constant f_0 in a neighborhood of zero. The regularization in C_0 ensures that this constant is exactly the regularized boundary value

$$C_0 = f_0 \,. \tag{175}$$

The genus-one calculation can be carried out analogously, which naturally leads to a close relation to the constant f_0. For a function $F^\tau(z)$ satisfying the KZB equation

$$\frac{\partial}{\partial z} F^\tau(z) = \sum_{n \geq 0} g^{(n)}(z, \tau) x^{(n)} F^\tau(z) \,, \tag{176}$$

letting $0 < z \ll 1$ leads to a similar situation as above: from the q-expansion of the integration kernels $g^{(n)}(z,\tau)$, we find that [BMMS15]

(177)
$$g^{(n)}(z,\tau) = \begin{cases} 1 & \text{if } n = 0\,, \\ \frac{1}{z} + \mathcal{O}(z) & \text{if } n = 1\,, \\ -2\zeta_{2m} - 2\frac{(-2\pi i)^{2m}}{(2m-1)!} \sum\limits_{k,l>0} l^{2m-1} q^{kl} + \mathcal{O}(z^2) & \text{if } n = 2m > 0\,, \\ \mathcal{O}(z) & \text{if } n = 2m+1 > 1\,. \end{cases}$$

Therefore, we can assemble the even generators $x^{(2m)}$ and the corresponding order-zero prefactors into

$$x^{(e)}(\tau) = x^{(0)} - 2\sum_{m>0}\left(\zeta_{2m} + \frac{(-2\pi i)^{2m}}{(2m-1)!}\sum_{k,l>0} l^{2m-1}q^{kl}\right)x^{(2m)} \tag{178}$$

in order to write the KZB eq. (176) as

$$\frac{d}{dz}F^\tau(z) = \left(\frac{x^{(1)}}{z} + x^{(e)}(\tau) + \mathcal{O}(z)\right)F^\tau(z)\,. \tag{179}$$

This is a differential equation of the form (173) of the KZ equation in the same regime. In other words, for small z, the operator

$$\nabla^{\mathrm{KZB}}\big(x^{(n)}\big) = \sum_{n\geq 0} g^{(n)}(z)x^{(n)} \tag{180}$$

on the right-hand side in the KZB equation (176) degenerates to the operator

$$\nabla^{\mathrm{KZ}}(e_0,e_1) = \frac{e_0}{z} + \frac{e_1}{z-1} \tag{181}$$

in the KZ equation (172) with $e_0 = x^{(1)}$ and $e_1 = x^{(e)}$:

$$\nabla^{\mathrm{KZB}}\big(x^{(n)}\big) = \nabla^{\mathrm{KZ}}\big(x^{(1)},x^{(e)}\big) + \mathcal{O}(z)\,. \tag{182}$$

Thus, as before for $F(x)$, the function $F^\tau(z)$ can be approximated by

$$F^\tau(z) = e^{zx^{(e)}(\tau)}z^{x^{(1)}}f_0^\tau + \mathcal{O}(z)\,, \tag{183}$$

where f_0^τ is some constant. Note that a similar degeneration to the genus-zero framework occurs for the generating series $L^\tau(z)$ of elliptic multiple polylogarithms defined in eq. (115): according to eq. (117), for $e_0 = x^{(1)}$ the series has at lowest order the same behavior as the generating series $L(z)$ of the multiple polylogarithms

$$L^\tau(z) = (-2\pi i z)^{x^{(1)}}(1+\mathcal{O}(z)) = (-2\pi i)^{x^{(1)}} L(z)|_{e_0=x^{(1)}}(1+\mathcal{O}(z))\,. \tag{184}$$

We can conclude that the regularized boundary value

$$C_0^\tau = \lim_{z\to 0} z^{-x^{(1)}} F^\tau(z) = f_0^\tau \tag{185}$$

is indeed independent of τ and, upon comparing eq. (174) with eq. (183), it is proportional (up to a constant matrix) to the corresponding genus-zero boundary value $C_0 = f_0$ for a function $F(x)$ satisfying a KZ equation with $e_0 = x^{(1)}$

$$C_0^\tau = \lim_{z\to 0} z^{-x^{(1)}} F^\tau(z) = f_0^\tau \propto f_0 = C_0\,. \tag{186}$$

Note that in the case of matrix Lie algebras with $e_0 \neq x^{(1)}$, but they have the same maximal eigenvalue, then the above argument modifies slightly but still applies analogously such that the elements of C_0^τ turn out to be some linear combinations of the elements of C_0, which is exactly the situation observed in the recursion described in the previous section.

4. Examples

4.1. Example: two points

As a first example, let us calculate the two-point one-loop string correction up to order $o_{\text{max}}^{\text{1-loop}} = 2$ in α'. While all essential steps are noted in this subsection, several lengthy details are outsourced to appendix D. The two-loop correction is non-trivial only, if the Mandelstam variables s_{ij} are treated as independent parameters of the integrals, which do not satisfy any constraints like momentum conservation. The two-point configuration-space integral reads [MS20b]

$$S_{\text{2-point}}^{\text{1-loop}}(\tilde{s}_{13}) = \int_0^1 dz_3 \exp\left(\tilde{s}_{13}\,\tilde{\Gamma}_{31}\right) = \sum_{n\geq 0} \tilde{s}_{13}^n\, \omega(\underbrace{1,\ldots,1}_{n},0)\,, \tag{187}$$

where $\tilde{s}_{13}$ is the Mandelstam variable associated to the loop momentum. Since the integral requires two vertex insertion points, the appropriate genus-one Selberg integral with an extra insertion point z_2 is of length $L = 3$ and the insertion points are ordered as

$$0 = z_1 < z_3 < z_2 < 1 = z_1 \bmod \mathbb{Z} \tag{188}$$

on the boundary of the annulus. Indeed, in the limit $z_2 \to 1$, the punctures z_2 and z_1 merge, leaving the two punctures relevant for the one-loop string corrections. Thus, we consider the iterated integrals

$$\begin{aligned} \mathrm{S}^\tau \left[\begin{smallmatrix} n_3 \\ i_3 \end{smallmatrix}\right](0, z_2) &= \int_0^{z_2} dz_3 \, \mathrm{S}^\tau \, g_{3i_3}^{(n_3)} \,, \quad 1 \leq i_3 < 3 \,, \\ \mathrm{S}^\tau &= \exp\left(s_{13}\,\tilde{\Gamma}_{31} + s_{12}\,\tilde{\Gamma}_{21} + s_{23}\,\tilde{\Gamma}_{23}\right) . \end{aligned} \tag{189}$$

According to eq. (163), the two-point one-loop correction can be found in the weight $w_1 = 0$ entry $\mathbf{C}^\tau_{1,w_1}$, while the tree-level correction resides at weight $w_0 = 1$ (cf. eq. (162)). The α'-expansion of the four-point tree-level correction turns out to start at order $o^{\text{tree}}_{\text{min}} = -1$, (cf. eq. (197)). Therefore, consulting eqs. (160) and (165), it is sufficient to consider the truncated Selberg vector at maximal weight $w_{\max} = 2$ to calculate the one-loop string corrections up to second order in α', i.e. we only need to consider the vector

$$\mathbf{S}^\tau_{\leq 2}(z_2) = \begin{pmatrix} \mathrm{S}^\tau \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](0, z_2) \\ \mathrm{S}^\tau \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](0, z_2) \\ \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right](0, z_2) \\ \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right](0, z_2) \end{pmatrix} \tag{190}$$

where we use the reduced set of integrals $\mathcal{B}^\tau_2$ obtained from the relations

$$\begin{aligned} \mathrm{S}^\tau \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](0, z_2) &= \mathrm{S}^\tau \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right](0, z_2) \,, \\ s_{13}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](0, z_2) &= -s_{23}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right](0, z_2) \end{aligned} \tag{191}$$

to exclude the integrals $\mathrm{S}^\tau \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right](0, z_2)$ and $\mathrm{S}^\tau \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right](0, z_2)$ from our analysis.

Before we can explicitly check that the regularized boundary values indeed reproduce the tree-level and one-loop string corrections and apply the associator eq. (170), we have to determine the matrices $x^{(0)}_{\leq 2}$, $x^{(1)}_{\leq 2}$ and $x^{(2)}_{\leq 2}$ appearing in the modified KZB equation satisfied by $\mathbf{S}^\tau_{\leq 2}(z_2)$. Following the general algorithm in appendix C and performing the corresponding calculations shown in appendix D, the partial differential equation can indeed be written in the form (136):

$$\frac{\partial}{\partial z_2}\,\mathbf{S}^{\tau}_{\leq 2}(z_2) = \left(g^{(0)}_{21}\,x^{(0)}_{\leq 2} + g^{(1)}_{21}\,x^{(1)}_{\leq 2} + g^{(2)}_{21}\,x^{(2)}_{\leq 2}\right)\mathbf{S}^{\tau}_{\leq 2}(z_2) + r_2\,\mathbf{S}^{\tau}_{3}(z_2)\,, \tag{192}$$

where $\mathbf{S}^{\tau}_{3}(z_2) = \left(\mathrm{S}^{\tau}\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right](0,z_2), \mathrm{S}^{\tau}\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right](0,z_2)\right)^{T}$ and the matrices are given by

$$x^{(0)}_{\leq 2} = \begin{pmatrix} 0 & s_{13} & 0 & 0 \\ 0 & 0 & -s_{23} & -s_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x^{(1)}_{\leq 2} = \begin{pmatrix} s_{12} & 0 & 0 & 0 \\ 0 & s_{123} & 0 & 0 \\ 0 & 0 & s_{12}+s_{23} & -s_{23} \\ 0 & 0 & -s_{13} & s_{12}+s_{13} \end{pmatrix} \tag{193}$$

and

$$x^{(2)}_{\leq 2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -s_{23} & 0 & 0 & 0 \\ 0 & s_{13} & 0 & 0 \\ 0 & s_{13} & 0 & 0 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -2s_{23} & -s_{23} \\ -s_{13} & 2s_{13} \end{pmatrix}. \tag{194}$$

Now, we can evaluate the relevant entries of the regularized boundary values $\mathbf{C}^{\tau}_{0,w_0=1}$ and $\mathbf{C}^{\tau}_{1,w_1=0}$ explicitly: the latter involves the weight $w_1 = 0$ eigenvalue $x^{(1)}_0 = s_{12}$ of $x^{(1)}_{\leq 2}$ in the regularization factor $(-2\pi i(1-z_2))^{-x^{(1)}_{\leq 2}}$, which leads to the boundary value

$$\mathbf{C}^{\tau}_{1,0} = \lim_{z_2\to 1}(-2\pi i(1-z_2))^{-s_{12}}\,\mathrm{S}^{\tau}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2) = S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13})\,, \tag{195}$$

given by the one-loop string correction $S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13})$ with effective Mandelstam variable

$$\tilde{s}_{13} = s_{13} + s_{23}\,, \tag{196}$$

which is in agreement with our general considerations in eq. (145). On the other hand, the relevant eigenvalue of $x^{(1)}_{\leq 2}$ for the boundary value $\mathbf{C}^{\tau}_{0,1}$ is $x^{(1)}_1 = s_{123}$, such that

$$\mathbf{C}^{\tau}_{0,1} = \lim_{z_2\to 0}(-2\pi i z_2)^{-s_{123}}\,\mathrm{S}^{\tau}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2) = \frac{1}{s_{13}}\frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} \tag{197}$$

yields indeed the well-known Veneziano amplitude for the four-point amplitude of open strings at tree-level. Since each Mandelstam variable comes with a factor of α', we find the leading order to be $o^{\text{tree}}_{\text{min}} = -1$.

Since according to eq. (160), the maximal order in α' or, equivalently, the maximal word length in the KZB associator is $l_{\max} = 3$, the truncated associator eq. (168) reads

$$\begin{pmatrix} S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13}) \\ * \\ * \\ * \end{pmatrix} + \mathcal{O}\Big((\alpha')^3\Big) = \Phi^\tau_3(x^{(n)}_{\leq 2}) \begin{pmatrix} 0 \\ \frac{1}{s_{13}} \frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} \\ 0 \\ 0 \end{pmatrix}. \tag{198}$$

From the matrices given in eqs. (193) and (194) and the truncation Φ^τ_3 of the associator Φ^τ given by the generating series of eMZVs in eq. (120), we find that the only words contributing to the relevant $(w_1, w_0) = (0,1)$-submatrix $\Phi^\tau_3(x^{(n)}_{\leq 2})_{0,1}$ are at

- word length 1: $x^{(0)}_{\leq 2}$
- word length 2: the commutator

$$[x^{(1)}_{\leq 2}, x^{(0)}_{\leq 2}] = \begin{pmatrix} 0 & -s_{13}(s_{13}+s_{23}) & 0 & 0 \\ 0 & 0 & -2s_{13}s_{23} & -2s_{23}^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- word length 3: the nested commutator

$$[x^{(1)}_{\leq 2}, [x^{(1)}_{\leq 2}, x^{(0)}_{\leq 2}]] = \begin{pmatrix} 0 & s_{13}(s_{13}+s_{23})^2 & 0 & 0 \\ 0 & 0 & -2s_{13}s_{23}(s_{13}+s_{23}) & 2s_{23}^2(s_{13}+s_{23}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  and the products

$$x^{(0)}_{\leq 2}x^{(0)}_{\leq 2}x^{(2)}_{\leq 2} = \begin{pmatrix} 0 & -2s_{13}^2 s_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x^{(0)}_{\leq 2}x^{(2)}_{\leq 2}x^{(0)}_{\leq 2} = \begin{pmatrix} 0 & -s_{13}^2 s_{23} & 0 & 0 \\ 0 & 0 & 2s_{13}s_{23}^2 & 2s_{13}s_{23}^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The above list of contributions can be easily obtained from our general analysis in eqs. (165) and (166).

Evaluating all matrix products, the relevant (w_1, w_0)-submatrix of the truncated KZB associator is explicitly given by the entry

$$\begin{aligned}\Phi_3^\tau\left(x_{\leq 2}^{(n)}\right)_{0,1} &= s_{13}\big(\omega(0) + (s_{13}+s_{23})\omega(1,0) + (s_{13}+s_{23})^2\omega(1,1,0) \\ &\quad - s_{13}s_{23}(\omega(0,2,0) + 2\omega(2,0,0))\big)\,. \end{aligned} \tag{199}$$

The α'-expansion of the Veneziano amplitude can be obtained from the identity

$$\begin{aligned}\frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} &= \exp\left(\sum_{n\geq 2}(-1)^n\frac{\zeta_n}{n}(s_{13}^n + s_{23}^n - (s_{13}+s_{23})^n)\right) \\ &= 1 - \zeta_2 s_{13}s_{23} + \mathcal{O}\Big((\alpha')^3\Big)\,. \end{aligned} \tag{200}$$

Using these two α'-expansions, the right-hand side of the relevant part of the truncated associator eq. (198) is given by

$$\begin{aligned} &S_{\text{2-point}}^{\text{1-loop}}(\tilde{s}_{13}) + \mathcal{O}\Big((\alpha')^3\Big) \\ &\qquad = \Phi_3^\tau\left(x_{\leq 2}^{(n)}\right)_{0,1}\frac{1}{s_{13}}\frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} \\ &\qquad = 1 + (s_{13}+s_{23})\omega(1,0) + (s_{13}+s_{23})^2\omega(1,1,0) + \mathcal{O}\Big((\alpha')^3\Big)\,, \end{aligned} \tag{201}$$

where we have used the identity $\omega(0,2,0) = -\zeta_2 - 2\omega(2,0,0)$ for the regularized eMZVs [BMS]. This reproduces indeed the two-point one-loop string correction $S_{\text{2-point}}^{\text{1-loop}}(\tilde{s}_{13})$ given in eq. (187) with the effective Mandelstam variable $\tilde{s}_{13} = s_{13} + s_{23}$ up to second order in α'. Simultaneously, this result approves the validity of the (relevant part) of the truncated associator eq. (201).

We have performed the calculation up to order $o_{\max}^{\text{1-loop}} = 4$ in α'. In order to compare our result with the literature, in particular with ref. [MS20b], we translate our result into iterated integrals of Eisenstein series[11] γ_0 and use the one-loop open Green's function

$$\mathcal{G}_{ij} = \tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; |z_{ij}|, \tau) + \omega(0,1) \tag{202}$$

[11]The conversion from the ω-form of eMZVs to their representation in terms of iterated integrals of Eisenstein series γ_0 is thoroughly explained in ref. [BMS16].

in the definition (103) of the Selberg seed S^τ and in the one-loop string corrections $S^{\text{1-loop}}_{N\text{-point}}(\tilde{s}_{ij})$ rather than just $\tilde{\Gamma}_{ij}$. The additional term $\omega(0,1)$ vanishes in the sum $\sum_{i<j} s_{ij}\left(\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix};|z_{ij}|,\tau)+\omega(0,1)\right)$ if momentum conservation is imposed and is, thus, physically irrelevant. Using these two adjustments, we find that the relevant part of the right-hand side of the associator eq. (198) up to order $(\alpha')^4$ is given by

$$
\begin{aligned}
S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13})\Big|_{\mathcal{G}_{ij}} = 1 &+ \tilde{s}_{13}^2\left(\frac{1}{4}\zeta_2 - 3\gamma_0(4,0)\right)\\
&+ \tilde{s}_{13}^3\left(10\gamma_0(6,0,0) - 24\zeta_2\gamma_0(4,0,0) - \frac{1}{4}\zeta_3\right)\\
&+ \tilde{s}_{13}^4\Big(9\gamma_0(4,0,4,0) - 18\gamma(4,4,0,0) - 126\gamma_0(8,0,0,0)\\
&\qquad - \frac{3}{4}\zeta_2\gamma_0(4,0) - 144\zeta_4\gamma_0(4,0,0,0)\\
&\qquad + 240\zeta_2\gamma_0(6,0,0,0) + \frac{19}{64}\zeta_4\Big) + \mathcal{O}\Big((\alpha')^5\Big)\,.
\end{aligned}
\tag{203}
$$

Note that eqs. (201) and (203) show nicely on a simple example, how using the associator eq. (159) relating the $(L{+}1)$-point tree-level to $(L{-}1)$-point one-loop string corrections may geometrically be interpreted in terms of a gluing mechanism of worldsheets as discussed at the end of subsection 3.5: starting with the four-point Veneziano amplitude, gluing together the external legs of the string worldsheet which correspond to the two external states labelled by the positions $x_1=0$ and $x_4=\infty$ on the Riemann sphere yields a two-point genus-one worldsheet with punctures $z_1=z_2 \bmod \mathbb{Z}$ and z_3. The effective momentum propagating between $z_1=z_2 \bmod \mathbb{Z}$ and z_3 yields the Mandelstam variable $\tilde{s}_{13}=s_{13}+s_{23}$ of the two-point one-loop interaction.

4.2. Example: three points

The calculation for three points proceeds in analogy to the two-point example without structural difficulties and complications. Naturally, the dimensionality of the relevant matrices and vectors is larger, such that we do not write them down explicitly but rather provide the results of the computation.

The recursive algorithm requires one extra point on top of the three insertion points present in three-point one-loop string correction integrals. Correspondingly, we are going to consider the class of genus-one Selberg integrals with $L=4$. The relevant integral is of the form

$$S^{\text{1-loop}}_{\text{3-point}}(\tilde{s}_{ij}) = \int_0^1 dz_3 \int_0^{z_3} dz_4 \exp\Big(\tilde{s}_{13}\,\tilde{\Gamma}_{31} + \tilde{s}_{14}\,\tilde{\Gamma}_{41} + \tilde{s}_{34}\,\tilde{\Gamma}_{34}\Big)\,. \tag{204}$$

The above integral resides in the weight $w_1=0$ subvector of $\mathbf{C}^\tau_1$. We are going to perform the calculation up to order $o^{\text{1-loop}}_{\text{max}}=3$ in α'. Since the corresponding five-point tree-level integrals start at order $o^{\text{tree}}_{\text{min}}=-2$ and appear at weight $w_0=2$ in $\mathbf{C}^\tau_0$, the required maximal weight for the truncation of the genus-one Selberg vector is $w_{\text{max}}=3$ according to eq. (165). The relevant finite-dimensional matrices $x^{(n)}_{\leq 3}$ for $n=0,1,2,3$ are obtained from the algorithm in appendix C, which leads to the modified KZB equation

$$\frac{\partial}{\partial z_2}\,\mathbf{S}^\tau_{\leq 3}(z_2) = \sum_{n=0}^{4} g^{(n)}_{21}\, x^{(n)}_{\leq 3}\,\mathbf{S}^\tau_{\leq 3}(z_2) + r_3\,\mathbf{S}^\tau_4(z_2)\,. \tag{205}$$

Regularized boundary values can be calculated from the $x^{(1)}_{w_0=2}$ and $x^{(1)}_{w_1=0}$ submatrices of $x^{(1)}_{\leq 3}$, which results in the expected subvectors

$$\mathbf{C}^\tau_{0,2} = \lim_{z_2\to 0}(-2\pi i z_2)^{-x^{(1)}_2}\,\mathbf{S}^\tau_2(z_2) = \begin{pmatrix} 0\\ 0\\ 0\\ \mathrm{S}[1,1](0,1,x_2=1)\\ \vdots\\ \mathrm{S}[2,3](0,1,x_2=1)\\ 0\\ 0 \end{pmatrix} \tag{206}$$

containing the five-point, genus-zero Selberg integrals for $z_2\to 0$ at weight $w_0=2$ and the three-point one-loop string correction for $z_2\to 1$ at weight $w_1=0$:

$$\mathbf{C}^\tau_{1,0} = \lim_{z_2\to 1}(-2\pi i(1-z_2))^{-x^{(1)}_0}\,\mathbf{S}^\tau_0(z_2) = \Big(S^{\text{1-loop}}_{\text{3-point}}(\tilde{s}_{ij})\Big) \tag{207}$$

with the effective Mandelstam variables

$$\tilde{s}_{1j} = s_{1j} + s_{2j}\,, \qquad \tilde{s}_{ij} = s_{ij} \tag{208}$$

for $i,j \in \{3,4\}$. The truncation of the KZB associator at $l_{\max} = 5$ (cf. eq. (160)), is required in order to use the finite associator eq. (170)

$$\mathbf{C}^{\tau}_{1,0} + \mathcal{O}\Big((\alpha')^4\Big) = \Phi^{\tau}_{5}\Big(x^{(n)}_{\leq 3}\Big)_{0,2} \, \mathbf{C}^{\tau}_{0,2} \,. \tag{209}$$

The words contributing to the weight-$(0,2)$ submatrix $\Phi^{\tau}_{5}(x^{(n)}_{\leq 3})_{0,2}$ of this truncation are determined with the mechanism described in subsection 3.6. The resulting α'-expansion of the right-hand side of eq. (209) up to order $o^{\text{1-loop}}_{\max} = 3$ reads in terms of iterated integrals of Eisenstein series and the redefinition $\tilde{\Gamma}_{ij} \mapsto \tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; |z_{ij}|, \tau) + \omega(0,1) = \mathcal{G}_{ij}$ in the Selberg seed as follows:

$$\begin{aligned}
&S^{\text{1-loop}}_{\text{3-point}}(\tilde{s}_{ij})\Big|_{\mathcal{G}_{ij}} \\
&= \frac{1}{2} + \frac{1}{8}\left(\tilde{s}^2_{13} + \tilde{s}^2_{14} + \tilde{s}^2_{34}\right)\left(\zeta_2 - 12\gamma_0(4,0)\right) \\
&\quad + \frac{1}{8}\Big(- \tilde{s}_{13}\tilde{s}_{34}\tilde{s}_{14}\big(- 240\gamma_0(6,0,0) + 144\zeta_2\gamma_0(4,0,0) + \zeta_3\big) \\
&\qquad\qquad - \left(\tilde{s}^3_{13} + \tilde{s}^3_{14} + \tilde{s}^3_{34}\right)\left(-40\gamma_0(6,0,0) + 96\zeta_2\gamma_0(4,0,0) + \zeta_3\right)\Big) \\
&\quad + \mathcal{O}\Big((\alpha')^4\Big) \,,
\end{aligned} \tag{210}$$

which agrees with the known α'-expansion of the three-point string correction.

4.3. Example: four points

If momentum conservation is imposed at the one-loop level, the first non-trivial example is the four-point one-loop string correction. It is given by the integral [GSB82]

$$S^{\text{1-loop}}_{\text{4-point}}(\tilde{s}_{ij}) = \int_0^1 dz_3 \int_0^{z_3} dz_4 \int_0^{z_4} dz_5 \prod_{0 \leq z_i < z_j \leq z_3} \exp\left(\tilde{s}_{ij}\,\tilde{\Gamma}_{ji}\right) \,, \tag{211}$$

where $i,j \in \{1,3,4,5\}$. The calculation of the α'-expansion is exactly the same as for the previous integrals: the one-loop integral is found in the weight $w_1 = 0$ subvector of $\mathbf{C}^{\tau}_1$ and the six-point tree-level integrals at the weight $w_0 = 3$ with $o^{\text{tree}}_{\min} = -3$. Hence, in order to obtain the expansion up to order $o^{\text{1-loop}}_{\max} = 2$, the KZB associator can be truncated at the maximal word length $l_{\max} = 5$ and eq. (165) requires the maximal weight $w_{\max} = w_0 = 3$.

The matrices $x^{(n)}_{\leq 3}$ for $n = 0, 1, 2, 3$ are obtained by forming the modified KZB eq. (136)

$$\frac{\partial}{\partial z_2}\mathbf{S}^{\tau}_{\leq 3}(z_2) = \sum_{n=0}^{4} g^{(n)}_{21} x^{(n)}_{\leq 3}\,\mathbf{S}^{\tau}_{\leq 3}(z_2) + r_3\,\mathbf{S}^{\tau}_{4}(z_2)\,. \tag{212}$$

As before, the subvectors of the regularized boundary values which contain the six-point, three-level Selberg integrals for $z_2 \to 0$ at weight $w_0 = 3$ and the four-point one-loop string correction for $z_2 \to 1$ at weight $w_1 = 0$ can be calculated using the appropriate submatrices of $x^{(1)}_{\leq 3}$ and read

$$\mathbf{C}^{\tau}_{0,3} = \lim_{z_2 \to 0} (-2\pi i z_2)^{-x^{(1)}_3}\,\mathbf{S}^{\tau}_{3}(z_2) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{S}[1,1,1](0,1,x_2=1) \\ \vdots \\ \mathrm{S}[2,3,4](0,1,x_2=1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{213}$$

and

$$\mathbf{C}^{\tau}_{1,0} = \lim_{z_2 \to 1} (-2\pi i(1-z_2))^{-x^{(1)}_0}\,\mathbf{S}^{\tau}_{0}(z_2) = \left(S^{\text{1-loop}}_{\text{4-point}}(\tilde{s}_{ij})\right)\,, \tag{214}$$

respectively, with the effective Mandelstam variables

$$\tilde{s}_{1j} = s_{1j} + s_{2j}\,, \qquad \tilde{s}_{ij} = s_{ij} \tag{215}$$

for $i, j \in \{3, 4, 5\}$. The truncated elliptic KZB associator at the maximal length $l_{\max} = 5$, with the contributing words calculated as usually, leads to the finite associator eq. (170)

$$\mathbf{C}^{\tau}_{1,0} + \mathcal{O}\Big((\alpha')^3\Big) = \Phi^{\tau}_{5}\Big(x^{(n)}_{\leq 3}\Big)_{0,3}\,\mathbf{C}^{\tau}_{0,3}\,. \tag{216}$$

Expressed in terms of iterated integrals of Eisenstein series and using the redefinition $\tilde{\Gamma}_{ij} \mapsto \tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; |z_{ij}|, \tau) + \omega(0,1) = \mathcal{G}_{ij}$

$$S^{\text{1-loop}}_{\text{4-point}}(\tilde{s}_{ij})\Big|_{\mathcal{G}_{ij}} = \frac{1}{6} - \frac{\zeta(3)}{4\pi^2}\,(\tilde{s}_{1,2} - 2\tilde{s}_{1,3} + \tilde{s}_{1,4} + \tilde{s}_{2,3} - 2\tilde{s}_{2,4} + \tilde{s}_{3,4})$$

$$
\begin{aligned}
&- 6\gamma_0(4,0,0)\,(\tilde{s}_{1,2} - 2\tilde{s}_{1,3} + \tilde{s}_{1,4} + \tilde{s}_{2,3} - 2\tilde{s}_{2,4} + \tilde{s}_{3,4}) \\
&+ \mathcal{O}\Big((\alpha')^2\Big)\,,
\end{aligned} \tag{217}
$$

which agrees up to order $(\alpha')^2$ with the α'-expansion of the four-point configuration-space integral.

5. Summary and outlook

In this article, we have generalized the recursive formalism for the evaluation of genus-zero Selberg integrals by Aomoto and Terasoma to genus one. After establishing and discussing the genus-one formalism, we have put it to work to evaluate several one-loop open-string scattering amplitudes.

The original construction at genus zero is based on relating two boundary values of a Knizhnik–Zamolodchikov equation by the Drinfeld associator. The boundary values arise as two different limits of Selberg integrals and can be shown to contain integrals constituting the N-point and $(N-1)$-point open-string tree-level amplitudes respectively. Accordingly, the method allows to determine all tree-level string corrections at arbitrary order in α' recursively using a suitable representation of the Drinfeld associator.

Our genus-one formalism is based on canonical generalizations of the above construction: at the heart there is now the elliptic Knizhnik–Zamolodchikov–Bernard equation, whose boundary values are related by the genus-one analogue of the Drinfeld associator, the elliptic KZB associator. The boundary values arise as limits of genus-one Selberg integrals and can be shown to contain the one-loop N-point and the tree-level $(N+2)$-point open-string configuration-space integrals. Thus all one-loop open-string corrections can be calculated using the elliptic associator equation (159) to any desired order in α'. Our results obtained match the known expressions at multiplicity two, three and four.

The original recursion at genus zero as well as our recursion at genus one have clear geometrical interpretations in terms of degenerations of the worldsheets: the extra marked point serves as variable in the KZ and KZB equations and thereby simultaneously parametrizes the degeneration of the worldsheets in the limits, which in turn define the boundary values. The class of iterated integrals leading to the Selberg integrals as well as the respective integration domains are very naturally defined in terms of the de Rham cohomology of the configuration spaces in question: at genus zero, the twisted forms appearing in the Selberg integrals give rise to a basis of the twisted de Rham cohomology of the configuration space of punctured

Riemann spheres with fixed points on the real line. Similarly, the forms in the genus-one Selberg integrals form a closed system with respect to integration by parts, the Fay identity and taking derivatives.

The following points deserve further investigation:

- Very likely, recursions with an extra marked point can not only be constructed for corrections to open-string amplitudes as done in this article. Rather, it seems the formalism is extendable to a wide range of string- and quantum field theories. An application or translation to the calculation of scattering amplitudes in $\mathcal{N} = 4$ super-Yang–Mills theory in the multi-Regge limit might be a first testing ground: several recursive structures as well as numerous formal similarities are already visible in refs. [DDDD$^+$16, DDDD$^+$20]. Another environment for amplitude recurrences, similar to our current construction, is discussed and applied in refs. [PS16b, PS16a]. It would be very interesting to understand the relation between the two approaches.
- Considering the step from genus zero to genus one, all generalizations have been completely canonical. We do not see any structural obstructions for establishing a similar recursion for higher genera. Given the algebraic complexity of the genus-one construction already, combinatorics will not only cause large matrix sizes, but also originate from considering three geometric parameters in the period matrix at genus two.
- Our construction makes use of several genus-zero tools developed in the context of [Miz19], the most prominent example being the matching of dimensions of the respective matrices, which correspond to a basis of Selberg vectors w.r.t. partial fraction and integration by parts: the respective dimensions are exactly as predicted by twisted de Rham theory.
- A substantial part in establishing our genus-one recursion was devoted to finding a useful and feasible way to single out a basis for Selberg vectors. For higher orders in α' as well as for higher multiplicity, a formulation of genus-one Selberg integrals in terms of weighted graphs and Fay identities using weighted adjacency matrices analogous to the genus-zero description in [Kad20] might be the correct computational framework.
- Most importantly, a formalism for calculating one-loop open-string scattering amplitudes from a differential equation has been put forward in refs. [MS20a, MS20b]. The constructions are formally rather similar: both rely on an elliptic KZB equation. While we are using an extra insertion point as differentiation variable, Mafra and Schlotterer employ

the modular parameter τ for this purpose. Our formulation employs iterated integrals for the insertion points and the ω-representations of eMZVs, while in refs. [MS20a, MS20b] iterated τ-integrals, Eisenstein series and the γ_0-representation of eMZVs is employed. There is little doubt that the formalisms can be shown to finally be equivalent: at an algebraic level, several steps have been undertaken in ref. [BKS20].

- Our genus-one recursion is tailored to the calculation of *planar* open-string corrections, where vertex insertions are allowed on only one of the boundaries of the annulus. An extension to non-planar open-string amplitudes is expected to be straightforward: in particular one ought to use doubly-periodic integration kernels instead of the functions $g^{(n)}$. A construction for non-planar one-loop string corrections already exists in refs. [MS20a, MS20b].

Appendix A. Generating function for polylogarithms and the Drinfeld associator

Let us introduce the general strategy to relate two regularized boundary values of a KZ equation such as (46) by considering a representation of some Lie algebra generators e_0 and e_1, as well as a function $\mathrm{F}(x)$ with $x \in (0,1)$ and values in the vector space the representations e_0 and e_1 act upon and which satisfies the KZ equation

$$\frac{d}{dx}\,\mathrm{F}(x) = \left(\frac{e_0}{x} + \frac{e_1}{x-1}\right)\mathrm{F}(x)\,. \tag{218}$$

Given this situation, one is often interested in calculating the limit of $\mathrm{F}(x)$ for $x \to 1$ while knowing the boundary value as $x \to 0$. As will be reviewed in this section, there is an operator, the Drinfeld associator $\Phi(e_0, e_1)$ [Dri89, Dri91], which parallel transports the (regularized) boundary value of $\mathrm{F}(x)$ at $x \to 0$ to its (regularized) value at $x \to 1$. It turns out that the Drinfeld associator is the generating series of the regularized MZVs, which was originally shown in ref. [LM96] and which is reviewed in this paragraph following the lines of ref. [Bro13].

In order to construct the Drinfeld associator, we first investigate the following generating function of multiple polylogarithms

$$\mathrm{L}(x) = \sum_{w \in \{e_0,e_1\}^\times} w\, G_w(x)\,. \tag{219}$$

The iterative definition (20) and the corresponding regularization prescription of the multiple polylogarithms implies that the series $\mathrm{L}(x)$ satisfies the KZ equation

$$\frac{d}{dx}\,\mathrm{L}(x) = \left(\frac{e_0}{x} + \frac{e_1}{x-1}\right)\mathrm{L}(x)\,, \tag{220}$$

with the asymptotic behavior as $x \to 0$

$$\mathrm{L}(x) \sim x^{e_0}\,. \tag{221}$$

By the symmetry $x \mapsto 1-x$ of the KZ equation, there is another solution L_1 of (220) with the asymptotic behavior

$$\mathrm{L}_1(x) \sim (1-x)^{e_1} \tag{222}$$

as $x \to 1$. Now, let $\mathrm{F}(x)$ be an arbitrary solution of the KZ equation (220). For this solution, regularized boundary values are defined via

$$C_0 = \lim_{x\to 0} x^{-e_0}\,\mathrm{F}(x)\,, \quad C_1 = \lim_{x\to 1}(1-x)^{-e_1}\,\mathrm{F}(x)\,. \tag{223}$$

For two functions $\mathrm{F}_0(x)$ and $\mathrm{F}_1(x)$ satisfying the KZ equation (218) the product $(\mathrm{F}_1)^{-1}\,\mathrm{F}_0$ is independent of x, and by the asymptotics (221), (222) of $\mathrm{L}(x)$ and $\mathrm{L}_1(x)$, respectively, the calculation

$$(\mathrm{L}_1(x))^{-1}\,\mathrm{L}(x)C_0 = \lim_{x\to 0}(\mathrm{L}_1(x))^{-1}\,\mathrm{F}(x) = \lim_{x\to 1}(\mathrm{L}_1(x))^{-1}\,\mathrm{F}(x) = C_1 \tag{224}$$

shows that the product

$$\Phi(e_0,e_1) = (\mathrm{L}_1(x))^{-1}\,\mathrm{L}(x) \tag{225}$$

maps the regularized boundary value C_0 to the regularized boundary value C_1

$$C_1 = \Phi(e_0,e_1)\,C_0\,. \tag{226}$$

The operator $\Phi(e_0,e_1)$ is the Drinfeld associator which is defined in terms of the generating series of multiple polylogarithms $\mathrm{L}(x)$ and the corresponding solution $\mathrm{L}_1(x)$. In order to write it as a generating series of MZVs, its definition (225) can be evaluated in the limit $x \to 1$, since $\Phi(e_0,e_1)$ is independent of x: it is a product of a function satisfying the KZ equation and an inverse of such a function. This leads to the relation of the Drinfeld associator to the MZVs discovered in ref. [LM96],

$$
\begin{aligned}
\Phi(e_0, e_1) &= \lim_{x\to 1}(1-x)^{-e_1}\,\mathrm{L}(x) \\
&= \sum_{w\in\{e_0,e_1\}^\times} w\zeta_w \\
&= 1 - \zeta_2[e_0,e_1] - \zeta_3[e_0+e_1,[e_0,e_1]] \\
&\quad + \zeta_4([e_1,[e_1,[e_1,e_0]]] + \tfrac{1}{4}[e_1,[e_0,[e_1,e_0]]] \\
&\quad - [e_0,[e_0,[e_0,e_1]]] + \tfrac{5}{4}[e_0,e_1]^2) + \ldots \quad ,
\end{aligned} \tag{227}
$$

i.e. the Drinfeld associator is a generating series for the (regularized) MZVs ζ_w defined in and below eq. (23). The limit $x \to 1$ is chosen to correspond to taking the tangential base point in negative direction at 1, such that the contributions from $(1-x)^{-e_1}$ lead to the discussed regularization $\zeta_{e_1} = 0$ of the divergent terms in $\mathrm{L}(x)$ by cancelling the positive integer powers of $\log(1-x)$ in the divergent multiple polylogarithms $G_w(x)$.

Appendix B. Regularization of elliptic multiple zeta values

In this section, we give a brief description how eMZVs may be regularized analogously to the regularization of the (genus-zero) MZVs.

The reversal of the ordering in the definition (94) and the regularization of the iterated integrals $\tilde{\Gamma}$ implies that only the eMZVs

$$
\omega(n_k,\ldots,n_1;\tau) = \omega(w^t;\tau) = \lim_{z\to 1}\tilde{\Gamma}_w(z,\tau) = \lim_{z\to 1}\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ 0 & \cdots & 0\end{smallmatrix}; z,\tau\right), \tag{228}
$$

labelled by the word $w = x^{(n_1)}\ldots x^{(n_k)} \in X$ with $n_1 = 1$ inherit the end point divergence at the upper integration boundary due to the $1/(z-1)$ asymptotics of $g^{(1)}(z,\tau)$ in the limit $z \to 1$. For example the definition (85) and the asymptotic behavior (89) imply that if we would allow for $n_1 = 1$ in the definition of the eMZVs, then

$$
\omega(1;\tau) = \lim_{z\to 1}\tilde{\Gamma}\left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}; z,\tau\right) = \lim_{z\to 1}\log(-2\pi i(1-z)) \tag{229}
$$

$$
\omega(\underbrace{1,\ldots,1}_{n};\tau) = \frac{1}{n!}\omega(1;\tau)^n \tag{230}
$$

are divergent and the q-expansion of $g^{(1)}$ implies

$$\begin{aligned}
\omega(0,1;\tau) &= \lim_{z\to 1} \tilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}; z,\tau\right) \qquad (231)\\
&= \lim_{z\to 1} \int_0^z dz'\, g^{(1)}(z',\tau) z' \\
&= \lim_{z\to 1} \log(-2\pi i(1-z)) - \frac{i\pi}{2} - 2\sum_{k,l>0} \frac{q^{kl}}{k}\,, \qquad (232)
\end{aligned}$$

such that

$$\begin{aligned}
\omega(1,0;\tau) &= \lim_{z\to 1} \tilde{\Gamma}\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}; z,\tau\right) \\
&= \lim_{z\to 1} \Big(\tilde{\Gamma}\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}; z,\tau\right)\tilde{\Gamma}\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}; z,\tau\right) - \tilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}; z,\tau\right)\Big) \\
&= \omega(1;\tau) - \omega(0,1;\tau) \\
&= \frac{i\pi}{2} + 2\sum_{k,l>0} \frac{q^{kl}}{k} \qquad (233)
\end{aligned}$$

is free of any logarithmic divergence. Using the shuffle algebra, any (divergent) elliptic multiple zeta value can be expanded in powers of $\omega(1;\tau)$, such that the regularized eMZVs ω_{reg} can be defined as being the convergent coefficient (of 1) in this expansion. For example from above, we find at depth one

$$\omega_{\mathrm{reg}}(1;\tau) = 0\,, \qquad (234)$$

at depth two

$$\omega(0,1;\tau) = -\omega(1,0;\tau) + \omega(0)\omega(1;\tau) \qquad (235)$$

such that

$$\omega_{\mathrm{reg}}(0,1;\tau) = -\omega(1,0;\tau) = -\omega_{\mathrm{reg}}(1,0;\tau) \qquad (236)$$

and further examples of divergent eMZVs are at depth three and weight one

$$\begin{aligned}
\omega(0,0,1;\tau) &= -\omega(0,1,0;\tau) - \omega(1,0,0;\tau) + \omega(0,0;\tau)\omega(1;\tau) \\
&= -\omega(1,0,0;\tau) + \omega(0,0;\tau)\omega(1;\tau) \qquad (237)
\end{aligned}$$

and at weight 2

$$\omega(1,0,1;\tau) = -2\omega(1,1,0;\tau) + \omega(1,0;\tau)\omega(1;\tau)\,, \tag{238}$$

as well as

$$\begin{aligned}\omega(0,1,1;\tau) &= -\omega(1,1,0;\tau) - \omega(1,0,1;\tau) + \omega(0;\tau)\omega(1,1;\tau)\\ &= \omega(1,1,0;\tau) - \omega(1,0;\tau)\omega(1;\tau) + \omega(0;\tau)\omega(1,1;\tau)\,,\end{aligned} \tag{239}$$

such that

$$\begin{aligned}\omega_{\mathrm{reg}}(0,0,1;\tau) &= -\omega_{\mathrm{reg}}(1,0,0;\tau)\,,\\ \omega_{\mathrm{reg}}(1,0,1;\tau) &= -2\omega_{\mathrm{reg}}(1,1,0;\tau)\,,\\ \omega_{\mathrm{reg}}(0,1,1;\tau) &= \omega_{\mathrm{reg}}(1,1,0;\tau)\,.\end{aligned} \tag{240}$$

As for the regularized elliptic multiple polylogarithms, we generally omit the subscript in ω_{reg} and always refer to the regularized versions when we write an elliptic multiple zeta value ω.

Appendix C. Proof of theorem 2

Proof. It is to be shown, that the z_2-derivative of a genus-one (k=2)-Selberg integral (cf. eq. (111)) is expressible as linear combination of admissible Selberg integrals with coefficients composed from $g_{21}^{(n)}$ and $\mathbb{Z}$-linear combinations of Mandelstam variables in order to recover the (matrix) KZB equation (128) for the Selberg vector $\mathbf{S}^\tau(z_2)$.

In order to prove the statement, we provide a constructive algorithm involving two steps: the first one is based on integration by parts such that any partial derivative in the integrand of $\frac{\partial}{\partial z_2}\mathrm{S}^\tau\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L\end{smallmatrix}\right](0,z_2)$ only acts on the Selberg seed $\mathrm{S}^\tau = \prod_{0\le z_i<z_j\le z_2}\exp\left(s_{ij}\,\tilde{\Gamma}_{ji}\right)$. The second step is an iterative application of the Fay identity to recover admissible products $\prod_{k=3}^L g_{k,i_k}^{(n_k)}$ in the integrand, such that the integral can be written as a linear combination of genus-one Selberg integrals.

Step 1: In order to conveniently describe the evaluation of

$$\frac{\partial}{\partial z_2}\mathrm{S}^\tau\left[\begin{smallmatrix} n_3, & \dots, & n_N \\ i_3, & \dots, & i_N\end{smallmatrix}\right](0,z_2)$$

let us start with a couple of definitions, which are reminiscent to the graphical notation in ref. [Kad20]. A product of the form

$$\prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i}\,, \qquad \text{where} \quad k_{i+1} > k_i\,, \tag{241}$$

is called a *g-chain* from k_1 to k_r with *weights* $(n_{k_2}, n_{k_3}, \dots, n_{k_r})$. Furthermore, a *g-chain with a branch* at k_j is a product of the form

$$\left(\prod_{i=1}^{j-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i}\right) g^{(n_{l_1})}_{l_1,k_j} \prod_{i=1}^{s-1} g^{(n_{l_{i+1}})}_{l_{i+1},l_i}\, g^{(n_{m_1})}_{m_1,k_j} \prod_{i=1}^{t-1} g^{(n_{m_{i+1}})}_{m_{i+1},m_i}\,, \tag{242}$$

with the *g-subchains* from k_1 to k_j, from k_j to l_s and from k_j to m_t. If there exists a g-chain in the product $\prod_{k=3}^{L} g^{(n_k)}_{k,i_k}$ from k_1 to k_s, k_s is said to be *g-chain connected* to k_1. In order to formulate the first step in the algorithm, we define for $1 \leq k \leq L$ the set of all the integers which are g-chain connected to k

$$U_k^{\vec{n},\vec{i}} = \{k \leq k' \leq L | k' \text{ is } g\text{-chain connected to } k \text{ in } \prod_{k=3}^{L} g^{(n_k)}_{k,i_k}\}\,, \tag{243}$$

which, as indicated by the superscripts $\vec{n} = (n_3, \dots, n_L)$ and $\vec{i} = (i_3, \dots, i_L)$, depends on the product $\prod_{k=3}^{L} g^{(n_k)}_{k,i_k}$ and is the genus-one analogue of the set defined in eq. (40). Similarly, we define the set of all the integers to which k is g-connected

$$D_k^{\vec{n},\vec{i}} = \{3 \leq k' \leq k | k \text{ is } g\text{-chain connected to } k' \text{ in } \prod_{k=3}^{L} g^{(n_k)}_{k,i_k}\}\,. \tag{244}$$

Thus, the set $U_k^{\vec{n},\vec{i}}$ goes up the g-chain with possible branches beginning at k and the set $D_k^{\vec{n},\vec{i}}$ goes down the g-chain beginning at k.

Using the above notions, the derivative of $\mathrm{S}^\tau\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](0, z_2)$ with respect to z_2 can be expressed as

$$\frac{\partial}{\partial z_2} \mathrm{S}^\tau\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](0, z_2) = \int_{\mathcal{C}(z_2)} \prod_{i=3}^{L} dz_i \left(\sum_{l \in U_2^{\vec{n},\vec{i}}} \frac{\partial}{\partial z_l} \mathrm{S}^\tau \right) \prod_{l=3}^{L} g^{(n_k)}_{k,i_k}$$

$$= \int_{\mathcal{C}(z_2)} \prod_{i=3}^{L} dz_i \ \mathrm{S}^\tau \left(\sum_{l \in U_2^{\vec{n},\vec{i}}} \sum_{j \in U_1^{\vec{n},\vec{i}}} s_{lj} g_{lj}^{(1)} \right) \prod_{k=3}^{L} g_{k,i_k}^{(n_k)} , \tag{245}$$

where in the first line the derivative has been rewritten to act on the Selberg seed only and in the second line those derivatives have been performed explicitly using (110).

The validity of the manipulations can be seen as follows: the admissibility condition $1 \le i_k < k$ implies that the product of differential forms in the integrand of the Selberg integral is a product of g-chains starting at 1 and g-chains starting at 2

$$\prod_{k=3}^{L} g_{k,i_k}^{(n_k)} = \prod_{k \in U_1^{\vec{n},\vec{i}}, k \ge 3} g_{k,i_k}^{(n_k)} \prod_{k \in U_2^{\vec{n},\vec{i}}, k \ge 3} g_{k,i_k}^{(n_k)} . \tag{246}$$

The z_2-derivative of the integrand of $\mathrm{S}^\tau \left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix} \right] (0, z_2)$ acts on S^τ and the g-chains starting at 2 only:

$$\begin{aligned} &\frac{\partial}{\partial z_2} \left(\mathrm{S}^\tau \prod_{k=3}^{L} g_{k,i_k}^{(n_k)} \right) \\ &= \left(\frac{\partial}{\partial z_2} \mathrm{S}^\tau \right) \prod_{k=3}^{L} g_{k,i_k}^{(n_k)} + \mathrm{S}^\tau \prod_{k \in U_1^{\vec{n},\vec{i}}, k \ge 3} g_{k,i_k}^{(n_k)} \left(\frac{\partial}{\partial z_2} \prod_{k \in U_2^{\vec{n},\vec{i}}, k \ge 3} g_{k,i_k}^{(n_k)} \right) . \end{aligned} \tag{247}$$

Moreover, the first product in the last term of the above equation can be split into a product of all the (disjoint) g-chains (possibly with branches) starting at 2 and ending at some $k \in U_2^{\vec{n},\vec{i}}$ (or several such terminal values in case of branches). If we consider one such g-chain without a branch $g_{k,k_r}^{n_k} \prod_{i=1}^{r-1} g_{k_{i+1},k_i}^{(n_{k_{i+1}})} g_{k_1,2}^{n_{k_1}}$ for $k > k_{i+1} > k_i > 2$, the partial derivative with respect to z_2 acts as follows

$$\begin{aligned} &\mathrm{S}^\tau \left(\frac{\partial}{\partial z_2} g_{k,k_r}^{n_k} \prod_{i=1}^{r-1} g_{k_{i+1},k_i}^{(n_{k_{i+1}})} g_{k_1,2}^{n_{k_1}} \right) \\ &= \mathrm{S}^\tau \, g_{k,k_r}^{n_k} \prod_{i=1}^{r-1} g_{k_{i+1},k_i}^{(n_{k_{i+1}})} \frac{\partial}{\partial z_2} g_{k_1,2}^{n_{k_1}} \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{S}^\tau\, g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} \Big(-\frac{\partial}{\partial z_{k_1}} g^{n_{k_1}}_{k_1,2} \Big) \\
&= \Big(\frac{\partial}{\partial z_{k_1}} \mathrm{S}^\tau \Big)\, g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} g^{n_{k_1}}_{k_1,2} + \mathrm{S}^\tau\, g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} \Big(\frac{\partial}{\partial z_{k_1}} g^{n_{k_2}}_{k_2,k_1} \Big) g^{n_k}_{k_1,2}
\end{aligned}
$$

(248)

$$
= \Big(\frac{\partial}{\partial z_{k_1}} \mathrm{S}^\tau \Big)\, g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} g^{n_{k_1}}_{k_1,2} + \mathrm{S}^\tau\, g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} \Big(-\frac{\partial}{\partial z_{k_2}} g^{n_{k_2}}_{k_2,k_1} \Big) g^{n_k}_{k_1,2}
$$

where we have used integration by parts for the second-to-last equation and omitted the boundary terms, since they vanish in the iterated integral $\mathrm{S}^\tau \left[\begin{smallmatrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L \end{smallmatrix} \right](0, z_2)$. The above manipulation can iteratively be repeated until any partial derivative only acts on the factor S^τ, such that due to the product rule of the derivative we obtain

(249)

$$
\mathrm{S}^\tau \left(\frac{\partial}{\partial z_2} g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} g^{n_{k_1}}_{k_1,2} \right) = \left(\left(\sum_{i=1}^{r} \frac{\partial}{\partial z_{k_i}} + \frac{\partial}{\partial z_k} \right) \mathrm{S}^\tau \right) \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} g^{n_{k_1}}_{k_1,2} \,.
$$

The product rule ensures that the same holds for the g-chains with branches as well. Therefore, we can continue with the calculation (247) and use the above procedure such that all the partial derivatives only act on the Selberg seed. The calculation is the following

$$
\begin{aligned}
&\frac{\partial}{\partial z_2} \left(\mathrm{S}^\tau \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} \right) \\
&= \Big(\frac{\partial}{\partial z_2} \mathrm{S}^\tau \Big) \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} + \mathrm{S}^\tau \prod_{k \in U_1^{\vec{n},\vec{i}}, k \geq 3} g^{(n_k)}_{k,i_k} \left(\frac{\partial}{\partial z_2} \prod_{k \in U_2^{\vec{n},\vec{i}}, k \geq 3} g^{(n_k)}_{k,i_k} \right) \\
&= \Big(\frac{\partial}{\partial z_2} \mathrm{S}^\tau \Big) \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} + \left(\sum_{l \in U_2^{\vec{n},\vec{i}}, l \geq 3} \frac{\partial}{\partial z_l} \mathrm{S}^\tau \right) \prod_{k \in U_1^{\vec{n},\vec{i}}, k \geq 3} g^{(n_k)}_{k,i_k} \prod_{k \in U_2^{\vec{n},\vec{i}}, k \geq 3} g^{(n_k)}_{k,i_k} \\
&= \left(\sum_{l \in U_2^{\vec{n},\vec{i}}} \frac{\partial}{\partial z_l} \mathrm{S}^\tau \right) \prod_{k=3}^{L} g^{(n_k)}_{k,i_k}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{S}^\tau \left(\sum_{l \in U_2^{\vec{n},\vec{\imath}}} \sum_{j=1, j\neq l}^{L} s_{lj} g_{lj}^{(1)} \right) \prod_{k=3}^{L} g_{k,i_k}^{(n_k)} \\
&= \mathrm{S}^\tau \left(\sum_{l \in U_2^{\vec{n},\vec{\imath}}} \left(\sum_{j \in U_2^{\vec{n},\vec{\imath}} \setminus \{l\}} s_{lj} g_{lj}^{(1)} + \sum_{j \in U_1^{\vec{n},\vec{\imath}}} s_{lj} g_{lj}^{(1)} \right) \right) \prod_{k=3}^{L} g_{k,i_k}^{(n_k)} \\
&= \mathrm{S}^\tau \left(\sum_{l \in U_2^{\vec{n},\vec{\imath}}} \sum_{j \in U_1^{\vec{n},\vec{\imath}}} s_{lj} g_{lj}^{(1)} \right) \prod_{k=3}^{L} g_{k,i_k}^{(n_k)} ,
\end{aligned}
\tag{250}
$$

where we have used the antisymmetry $g_{lj}^{(1)} = -g_{jl}^{(1)}$ for the last equality. This completes the proof of eq. (245).

Step 2: The integrals in eq. (245) do not yet have the desired form, i.e. a factor of $g_{21}^{(n)}$ times a product of the form $g_{k,i_k}^{(n_k)}$ with $1 \leq i_k < k$ for all $k \in \{3, \ldots, L\}$. This form can be obtained in a second step using the Fay identity (108). Due to the decomposition in eq. (246), any term in eq. (245) can be split into a product of a g-chain from 1 to j labeled by $D_j^{\vec{n},\vec{\imath}} = \{j_1 < j_2 < \cdots < j_s < j\}$ and a g-chain from 2 to l labeled by $D_l^{\vec{n},\vec{\imath}} = \{l_1 < l_2 < \cdots < l_r < l\}$ and the remaining factors:

$$
\begin{aligned}
&s_{lj} g_{lj}^{(1)} \prod_{k=3}^{L} g_{k,i_k}^{(n_k)} \\
&\quad = s_{kj} g_{kj}^{(1)} g_{j,j_s}^{(n_j)} \prod_{i=1}^{s-1} g_{j_{i+1} j_i}^{(n_{j_{i+1}})} g_{j_1,1}^{(n_{j_1})} g_{l,l_r}^{(n_l)} \prod_{i=1}^{r-1} g_{l_{i+1} l_i}^{(n_{l_{i+1}})} g_{l_1,2}^{(n_{l_1})} \prod_{k=3, k \notin D_l^{\vec{n},\vec{\imath}} \cup D_j^{\vec{n},\vec{\imath}}}^{L} g_{k,i_k}^{(n_k)} .
\end{aligned}
\tag{251}
$$

The factor $g_{lj}^{(1)}$ connects the two g-chains starting at 1 and 2, such that applying the Fay identity iteratively, the product

$$
g_{lj}^{(1)} g_{j,j_s}^{(n_j)} \prod_{i=1}^{s-1} g_{j_{i+1} j_i}^{(n_{j_{i+1}})} g_{j_1,1}^{(n_{j_1})} g_{l,l_r}^{(n_l)} \prod_{i=1}^{r-1} g_{l_{i+1} l_i}^{(n_{l_{i+1}})} g_{l_1,2}^{(n_{l_1})}
\tag{252}
$$

can be written as a factor $g_{21}^{(n)}$ times a linear combination of admissible factors. The complete procedure is the following:

- First, assume (without loss of generality, rename the labels otherwise) that $l < j$, such that the subscript j in $g_{lj}^{n_l}$ can be lowered to j_s using the Fay identity as follows:

$$g_{lj}^{n_l} g_{j,j_s}^{n_j} = (-1)^{n_l} g_{jl}^{n_l} g_{j,j_s}^{n_j} = (-1)^{n_l} g_{l,j_1} \begin{pmatrix} g_{j,j_s} \\ g_{jl} \end{pmatrix}_{n_l,n_j} , \tag{253}$$

  where the product on the right-hand side is defined to be the sum obtained by the Fay identity (108). It is a $\mathbb{Z}$-linear combination of $g_{l,j_s}^{(n_l+n_j-i)} g_{j,j_s}^{(i)}$ and $g_{l,j_s}^{(n_l+n_j-i)} g_{jl}^{(i)}$ for $0 \leq i \leq n_k + n_j$ with integer coefficients. Importantly, it is a linear combination of admissible factors and the index j in $g_{lj}^{(n_l)}$ has been lowered to j_s.
- If $l < j_s$, we repeat this step with the products $g_{l,j_s}^{(n_l+n_j-i)} g_{j_s,j_{s-1}}^{(n_{j_s})}$. Similarly for lower indices j_t, unless we arrive at $g_{j_1,1}^{(n_{j_1})}$, where another application of the Fay identity leaves us with a linear combination of $g_{l,1}^{(n)}$ and admissible factors times the product $g_{l,l_r}^{(n_l)} \prod_{i=1}^{r-1} g_{l_{i+1},l_i}^{(n_{l_{i+1}})} g_{l_1,2}^{(n_{l_1})}$. The same procedure can be applied to $g_{l,1}^{(n)} g_{l,l_r}^{(n_l)} \prod_{i=1}^{r-1} g_{l_{i+1},l_i}^{(n_{l_{i+1}})} g_{l_1,2}^{(n_{l_1})}$ such that we are left with a linear combination of admissible factors times a factor $g_{21}^{(n)}$ and some integer coefficients. However, if we arrive at some j_t such that $l > j_t$, we have to apply the Fay identity earlier to the product $g_{l,l_r}^{(n_l)} \prod_{i=1}^{r-1} g_{l_{i+1},l_i}^{(n_{l_{i+1}})} g_{l_1,2}^{(n_{l_1})}$ in order to recover admissible factors.
- Thus, if we arrive at some j_t with $l > j_t$, we apply the above procedure to the product $g_{l,l_r}^{(n_l)} \prod_{i=1}^{r-1} g_{l_{i+1},l_i}^{(n_{l_{i+1}})} g_{l_1,2}^{(n_{l_1})}$ beginning with the factor

$$g_{l,j_t}^{(n)} g_{l,l_r}^{(n_l)} = g_{j_t,l_r} \begin{pmatrix} g_{l,l_r} \\ g_{l,j_t} \end{pmatrix}_{n,n_l} . \tag{254}$$

  As above, this process can be applied to lower l_i unless we arrive either at $g_{l_1,2}^{(n_{l_1})}$ or at $l_i < j_t$. In the latter case, we again proceed with the application of the Fay identity with respect to the j_t index as in the previous step. In the former case, we arrive at a linear combination of $g_{j_t,2}^{(m)}$ and we are left with applying the procedure to the j_t index unless we hit j_1.
- The above procedure terminates once we could rewrite the product in eq. (252) as a linear combination of $g_{21}^{(n)}$ times solely admissible factors and some integer coefficients.

After applying step 1 and step 2 to any z_2-derivative of an admissible genus-one Selberg integral, one will obtain the desired form, that is, one component of the matrix equation (128).

Writing the weights of the genus-one Selberg vector as a sequence of the form $\vec{w} = (w_3, \ldots, w_L) \in \mathbb{N}^{L-2}$, such that the total weight is given by $w = |\vec{w}| = w_3 + \ldots w_L$, and the admissible labelings $\vec{i} = (i_3, \ldots, i_L) \in \mathbb{N}^{L-2}$ with $1 \leq i_k < k$, this algorithm converts the derivative of the genus-one Selberg integral $\mathrm{S}^\tau\left[\begin{smallmatrix} w_3, & \ldots, & w_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\right](0, z_2) = \mathrm{S}^\tau\left[\begin{smallmatrix} \vec{w} \\ \vec{i} \end{smallmatrix}\right](0, z_2)$ given in eq. (245) to a form similar to the KZB equation

(255)
$$\frac{\partial}{\partial z_2} \mathrm{S}^\tau\left[\begin{smallmatrix} \vec{w} \\ \vec{i} \end{smallmatrix}\right](0, z_2) = \sum_{n=0}^{w+1} g_{21}^{(n)} \sum_{\vec{m} \in \mathbb{N}^{L-2} : m = w+1-n} \sum_{\vec{j} \text{ adm}} x_{\vec{m},\vec{j}}^{\vec{w},\vec{i}} \mathrm{S}^\tau\left[\begin{smallmatrix} \vec{m} \\ \vec{j} \end{smallmatrix}\right](0, z_2) ,$$

where $m = |\vec{m}|$ and the sum over $\vec{j} \in \mathbb{N}^{L-2}$ runs over the admissible labelings, i.e. the vectors $\vec{j}$ such that $1 \leq (\vec{j})_i = j_i < 2 + i$. Each coefficient $x_{\vec{m},\vec{j}}^{\vec{w},\vec{i}} \in \mathbb{Z}[s_{ij}]$ either vanishes or is a $\mathbb{Z}$-linear combination of the Mandelstam variables, determined by the above algorithm. Note that all the terms $g_{21}^{(n)} \mathrm{S}^\tau\left[\begin{smallmatrix} \vec{m} \\ \vec{j} \end{smallmatrix}\right](0, z_2)$ are of total weight $w+1 = n+m$, since $m = w+1-n$. This is a consequence of the above algorithm: the partial derivatives in the last line of eq. (245) only act on the Selberg seed S^τ, which effectively multiplies S^τ with some $g_{lj}^{(1)}$. Hence, the integrand $\mathrm{S}^\tau \prod_{k=3}^{L} g_{k,i_k}^{(n_k)}$ is multiplied with $g_{lj}^{(1)}$ which increases the total weight by one. The application of the Fay identity in the second step of the algorithm preserves this weight, which leads to the differential eq. (255). This completes the proof. □

Example for Step 1: As an example for step 1, let us consider $L = 6$ and the following product $p(z)$ with a branch at $k = 3$

(256)
$$p(z) = \mathrm{S}^\tau\, g_{62}^{(n_6)} g_{53}^{(n_5)} g_{43}^{(n_4)} g_{32}^{(n_3)} .$$

Upon discarding boundary terms, the partial derivative of $p(z)$ with respect to z_2 is

$$\begin{aligned} \frac{\partial}{\partial z_2} p(z) &= \frac{\partial}{\partial z_2}\left(\mathrm{S}^\tau\, g_{62}^{(n_6)} g_{53}^{(n_5)} g_{43}^{(n_4)} g_{32}^{(n_3)}\right) \\ &= \left(\frac{\partial}{\partial z_2}\mathrm{S}^\tau\right) g_{62}^{(n_6)} g_{53}^{(n_5)} g_{43}^{(n_4)} g_{32}^{(n_3)} + \mathrm{S}^\tau\left(\frac{\partial}{\partial z_2} g_{62}^{(n_6)}\right) g_{53}^{(n_5)} g_{43}^{(n_4)} g_{32}^{(n_3)} \end{aligned}$$

$$
\begin{aligned}
&+\mathrm{S}^\tau\, g_{62}^{(n_6)} g_{53}^{(n_5)} g_{43}^{(n_4)} \left(\frac{\partial}{\partial z_2} g_{32}^{(n_3)}\right)\\
&= \left(\frac{\partial}{\partial z_2}\mathrm{S}^\tau\right) g_{62}^{(n_6)} g_{53}^{(n_5)} g_{43}^{(n_4)} g_{32}^{(n_3)} + \mathrm{S}^\tau \left(-\frac{\partial}{\partial z_6} g_{62}^{(n_6)}\right) g_{53}^{(n_5)} g_{43}^{(n_4)} g_{32}^{(n_3)}\\
&+\mathrm{S}^\tau\, g_{62}^{(n_6)} g_{53}^{(n_5)} g_{43}^{(n_4)} \left(-\frac{\partial}{\partial z_3} g_{32}^{(n_3)}\right)\\
&= \left(\left(\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_6}+\frac{\partial}{\partial z_3}\right)\mathrm{S}^\tau\right) g_{62}^{(n_6)} g_{53}^{(n_5)} g_{43}^{(n_4)} g_{32}^{(n_3)}\\
&+\mathrm{S}^\tau\, g_{62}^{(n_6)} \left(\frac{\partial}{\partial z_3} g_{53}^{(n_5)}\right) g_{43}^{(n_4)} g_{32}^{(n_3)} + \mathrm{S}^\tau\, g_{62}^{(n_6)} g_{53}^{(n_5)} \left(\frac{\partial}{\partial z_3} g_{43}^{(n_4)}\right) g_{32}^{(n_3)}\\
&= \left(\left(\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_6}+\frac{\partial}{\partial z_3}\right)\mathrm{S}^\tau\right) g_{62}^{(n_6)} g_{53}^{(n_5)} g_{43}^{(n_4)} g_{32}^{(n_3)}\\
&+\mathrm{S}^\tau\, g_{62}^{(n_6)} \left(-\frac{\partial}{\partial z_5} g_{53}^{(n_5)}\right) g_{43}^{(n_4)} g_{32}^{(n_3)} + \mathrm{S}^\tau\, g_{62}^{(n_6)} g_{53}^{(n_5)} \left(-\frac{\partial}{\partial z_4} g_{43}^{(n_4)}\right) g_{32}^{(n_3)}\\
&= \left(\left(\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_6}+\frac{\partial}{\partial z_3}+\frac{\partial}{\partial z_5}+\frac{\partial}{\partial z_4}\right)\mathrm{S}^\tau\right) g_{62}^{(n_6)} g_{53}^{(n_5)} g_{43}^{(n_4)} g_{32}^{(n_3)}
\end{aligned}
$$

$$
= \mathrm{S}^\tau \left(\sum_{k=2}^{L} s_{k1} g_{k1}^{(1)}\right) g_{62}^{(n_6)} g_{53}^{(n_5)} g_{43}^{(n_4)} g_{32}^{(n_3)}\,, \tag{257}
$$

which is the result expected from eq. (245) since $U_2^{\vec{n},(2,3,3,2)} = \{2,3,4,5,6\}$.

Appendix D. Explicit calculation of the one-loop two-point configuration-space integrals

In this appendix, detailed calculations for the two-point example in subsection 4.1 are provided. The configuration-space contribution to the two-point amplitude is described by the $L = 3$ Selberg integrals in eq. (189). The two-point one-loop amplitude with Mandelstam variable $s = s_{13} + s_{23}$ is reproduced for $n = 0$, $i_3 = 1$ as the first entry of the boundary value

$$
\mathbf{C}_1^\tau = \lim_{z_2\to 1} (-2\pi i(1-z_2)) \begin{pmatrix} \mathrm{S}^\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix}(0,z_2) \\ \mathrm{S}^\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}(0,z_2) \\ \mathrm{S}^\tau \begin{bmatrix} 2 \\ 1 \end{bmatrix}(0,z_2) \\ \mathrm{S}^\tau \begin{bmatrix} 2 \\ 2 \end{bmatrix}(0,z_2) \\ \vdots \end{pmatrix}. \tag{258}
$$

In order to evaluate the first entry of $\mathbf{C}_1^\tau$ we can use the block-diagonal form of $x^{(1)}$ with the first block being $x_0^{(1)} = s_{12}$ as shown below. Thus, the relevant entry of the regularization factor for $z_2 \to 1$ is $(-2\pi i(1-z_2))^{-x_1^{(1)}} \sim e^{-s_{12}\tilde{\Gamma}_{21}}$ and the integral is given by

$$\begin{aligned}
&\lim_{z_2\to 1}(-2\pi i(1-z_2))^{-s_{12}}\,\mathrm{S}^\tau\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2)\\
&= \lim_{z_2\to 1} e^{-s_{12}\tilde{\Gamma}_{21}}\int_0^{z_2} dz_3 \exp\left(s_{13}\,\tilde{\Gamma}_{31} + s_{12}\,\tilde{\Gamma}_{21} + s_{23}\,\tilde{\Gamma}_{23}\right)\\
&= \int_0^1 dz_3 \exp\left((s_{13}+s_{23})\,\tilde{\Gamma}_{31}\right)\\
&= \sum_{n\geq 0}\frac{(s_{13}+s_{23})^n}{n!}\int_0^1 dz_3\,\tilde{\Gamma}_{31}^n\\
&= \sum_{n\geq 0}\frac{(s_{13}+s_{23})^n}{n!}\int_0^1 dz_3\, n!\,\tilde{\Gamma}(\underbrace{\begin{smallmatrix}1 & \cdots & 1\\ 0 & \cdots & 0\end{smallmatrix}}_{n};z_3,\tau)\\
&= \sum_{n\geq 0}(s_{13}+s_{23})^n\,\omega(\underbrace{1,\ldots,1}_{n},0)\,.
\end{aligned} \tag{259}$$

The regularization of the above boundary value corresponds to the first eigenvalue s_{12} of $x^{(1)}$, which can be determined by bringing the derivative of $\mathrm{S}^\tau\left[\begin{smallmatrix}n_3\\i_3\end{smallmatrix}\right](0,z_2)$ in KZB form

$$\begin{aligned}
\frac{\partial}{\partial z_2}\mathrm{S}^\tau\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2) &= \int_0^{z_2} dz_3 s_{21} g_{21}^{(1)} S + \int_0^{z_2} dz_3 s_{23} g_{23}^{(1)} S\\
&= s_{21} g_{21}^{(1)}\,\mathrm{S}^\tau\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2) + \int_0^{z_2} dz_3 s_{31} g_{31}^{(1)} S\\
&= s_{21} g_{21}^{(1)}\,\mathrm{S}^\tau\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2) + s_{31} g_{31}^{(0)}\,\mathrm{S}^\tau\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2)\,,
\end{aligned} \tag{260}$$

such that the first columns of the matrices $x^{(0)}$ and $x^{(1)}$ are given by

$$x^{(0)} = \begin{pmatrix} 0 & s_{31} & 0 & 0 & \ldots\\ \vdots & & & & \end{pmatrix}\,, \qquad x^{(1)} = \begin{pmatrix} s_{21} & 0 & 0 & 0 & \ldots\\ \vdots & & & & \end{pmatrix}\,. \tag{261}$$

Note that we have used the integration by parts identity

$$s_{23}\,\mathrm{S}^\tau\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right](0,z_2) + s_{13}\,\mathrm{S}^\tau\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2) = 0\,. \tag{262}$$

The boundary value for $z_2 \to 0$ is more subtle. In this limit, the one-loop propagator degenerates to the tree level propagator and, in particular, loses its τ-dependence at the lowest order in z_2

$$\tilde{\Gamma}_{\mathrm{reg}}(\begin{smallmatrix}1\\0\end{smallmatrix}; z_2, \tau) = \log(-2\pi i z_2) + \mathcal{O}\left(z_2^2\right), \qquad g^{(1)}(z_2, \tau) = \frac{1}{z_2} + \mathcal{O}(z_2) \tag{263}$$

such that, using the change of variables $z_i = z_2 w_i$, the lowest order in z_2 for $n_3 = 1$, $i_3 = 1$ is given by

$$\begin{aligned}
&\mathrm{S}^\tau\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0, z_2)\\
&= \int_0^{z_2} dz_3 \exp\left(s_{13}\,\tilde{\Gamma}_{31} + s_{12}\,\tilde{\Gamma}_{21} + s_{23}\,\tilde{\Gamma}_{23}\right) g_{31}^{(1)}\\
&= \int_0^1 dw_3\, z_2(-2\pi i z_2 w_3)^{s_{13}}(-2\pi i z_2)^{s_{12}}(-2\pi i z_2(1-w_3))^{s_{23}} \frac{1}{z_2 w_3}(1 + \mathcal{O}(z_2))\\
&= (-2\pi i z_2)^{s_{123}} \int_0^1 dw_3\, w_3^{s_{13}}(1-w_3)^{s_{23}} \frac{1}{w_3}(1 + \mathcal{O}(z_2))\\
&= (-2\pi i z_2)^{s_{123}} \left(\frac{1}{s_{13}} \frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})}\right)(1 + \mathcal{O}(z_2))\,.
\end{aligned} \tag{264}$$

Therefore, at the lowest order in z_2, the integral $\mathrm{S}^\tau\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0, z_2)$ degenerates to the four-point tree-level amplitude with Mandelstam variables s_{13} and s_{23}. Now, let us check that the regularization by the factor $(-2\pi i z_2)^{-x^{(1)}}$ projects out that lowest-order coefficient of z_2. In order to obtain the appropriate eigenvalue of $x^{(1)}$, the differential equation satisfied by $\mathrm{S}^\tau\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0, z_2)$ has to be brought in KZB form and the coefficient of $\mathrm{S}^\tau\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0, z_2)$ itself has to be determined

$$\begin{aligned}
&\frac{\partial}{\partial z_2}\mathrm{S}^\tau\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0, z_2)\\
&\quad= \int_0^{z_2} dz_3 \exp\left(s_{13}\,\tilde{\Gamma}_{31} + s_{12}\,\tilde{\Gamma}_{21} + s_{23}\,\tilde{\Gamma}_{23}\right) g_{31}^{(1)}\left(s_{12} g_{21}^{(1)} + s_{23} g_{23}^{(1)}\right)\\
&\quad= s_{12} g_{21}^{(1)}\,\mathrm{S}^\tau\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0, z_2) - s_{23}\int_0^{z_2} dz_3\ \mathrm{S}^\tau\, g_{31}^{(1)} g_{32}^{(1)}\,.
\end{aligned} \tag{265}$$

In order to bring the second integral into the appropriate form, the Fay identity

$$g_{31}^{(1)} g_{32}^{(1)} = g_{21}^{(2)} + g_{31}^{(2)} + g_{32}^{(2)} + g_{21}^{(1)} g_{32}^{(1)} - g_{21}^{(1)} g_{31}^{(1)} \tag{266}$$

has to be used, followed by an application of eq. (262)

$$\begin{aligned}
&\frac{\partial}{\partial z_2} \mathrm{S}^\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}(0, z_2) \\
&\quad = -s_{23} g_{21}^{(2)} \mathrm{S}^\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix}(0, z_2) - s_{23} g_{21}^{(0)} \mathrm{S}^\tau \begin{bmatrix} 2 \\ 1 \end{bmatrix}(0, z_2) - s_{23} g_{21}^{(0)} \mathrm{S}^\tau \begin{bmatrix} 2 \\ 2 \end{bmatrix}(0, z_2) \\
&\qquad + s_{12} g_{21}^{(1)} \mathrm{S}^\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}(0, z_2) - s_{23} g_{21}^{(1)} \mathrm{S}^\tau \begin{bmatrix} 1 \\ 2 \end{bmatrix}(0, z_2) + s_{23} g_{21}^{(1)} \mathrm{S}^\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}(0, z_2) \\
&\quad = -s_{23} g_{21}^{(2)} \mathrm{S}^\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix}(0, z_2) - s_{23} g_{21}^{(0)} \mathrm{S}^\tau \begin{bmatrix} 2 \\ 1 \end{bmatrix}(0, z_2) - s_{23} g_{21}^{(0)} \mathrm{S}^\tau \begin{bmatrix} 2 \\ 2 \end{bmatrix}(0, z_2) \\
&\qquad + (s_{12} + s_{13} + s_{23})\, g_{21}^{(1)} \mathrm{S}^\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}(0, z_2)\,.
\end{aligned} \tag{267}$$

Therefore, we find that the appropriate eigenvalue of $x^{(1)}$ is indeed $s_{123} = s_{12}+s_{13}+s_{23}$, such that according to eq. (264) the second, i.e. the weight-one, entry of $\mathbf{C}_0^\tau$ is given by the four-point tree-level amplitude

$$\mathbf{C}_0^\tau = \lim_{z_2 \to 0} e^{-x^{(1)} \tilde{\Gamma}_{21}} \begin{pmatrix} \mathrm{S}^\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix}(0, z_2) \\ \mathrm{S}^\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}(0, z_2) \\ \mathrm{S}^\tau \begin{bmatrix} 2 \\ 1 \end{bmatrix}(0, z_2) \\ \mathrm{S}^\tau \begin{bmatrix} 2 \\ 2 \end{bmatrix}(0, z_2) \\ \vdots \end{pmatrix} = \begin{pmatrix} * \\ \frac{1}{s_{13}} \frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} \\ * \\ * \\ \vdots \end{pmatrix}. \tag{268}$$

As discussed in subsection 3.5, since the eigenvalue of $x^{(1)}$ can not be bigger than s_{123} and we can only compensate the Jacobian z_2 in eq. (264) from the change of variables $z_3 = z_2 w_3$ by the singular asymptotic behavior of $g^{(1)}(z_3, \tau) \to \frac{1}{z_2 w_3}$ for $z_2 \to 0$, if there would be another integration kernel $g^{(n_3)}(z_{3i_3}, \tau)$ with $n_3 \neq 1$ which is regular close to the origin, there would not be such a compensation. Thus, all other entries of the boundary value $\mathbf{C}_0^\tau$ which do not correspond to a singular integration kernel $g^{(1)}(z_{3i_3}, \tau)$ vanish and we obtain

$$\mathbf{C}_0^\tau = \begin{pmatrix} 0 \\ \frac{1}{s_{13}}\frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} \\ 0 \\ 0 \\ \vdots \end{pmatrix} . \tag{269}$$

In order to check the consistency of the first entry of the vector equation

$$\mathbf{C}_1^\tau = \Phi^\tau \ \mathbf{C}_0^\tau \tag{270}$$

up to order $(\alpha')^2$, we also need to calculate the derivative of $\mathbf{S}_2^\tau(z_2)$, which includes the following two derivatives: the first one is

$$\begin{aligned} \frac{\partial}{\partial z_2} \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right](0, z_2) &= \int_0^{z_2} dz_3 \ \mathrm{S}^\tau \ g_{31}^{(2)} \left(s_{21} g_{21}^{(1)} + s_{23} g_{23}^{(1)} \right) \\ &= s_{12} g_{21}^{(1)} \ \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right](0, z_2) - s_{23} \int_0^{z_2} dz_3 \ \mathrm{S}^\tau \ g_{31}^{(2)} g_{32}^{(1)} , \end{aligned} \tag{271}$$

where we can apply again the Fay identity

$$\begin{aligned} g_{32}^{(1)} g_{31}^{(2)} &= -(-1)^2 g_{12}^{(3)} + \sum_{r=0}^{2} \binom{r}{0} g_{21}^{(2-r)} g_{k2}^{(1+r)} + \sum_{r=0}^{1} \binom{r+1}{1} g_{12}^{(1-r)} g_{k1}^{(2+r)} \\ &= g_{21}^{(3)} + g_{21}^{(2)} g_{32}^{(1)} + g_{21}^{(1)} g_{32}^{(2)} + g_{21}^{(0)} g_{32}^{(3)} - g_{21}^{(1)} g_{31}^{(2)} + 2 g_{12}^{(0)} g_{31}^{(3)} . \end{aligned} \tag{272}$$

Therefore, we find

$$\begin{aligned} &\frac{\partial}{\partial z_2} \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] \\ &\quad = s_{12} g_{21}^{(1)} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] - s_{23} \big(g_{21}^{(3)} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] + g_{21}^{(2)} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] + g_{21}^{(1)} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] \\ &\qquad + g_{21}^{(0)} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right] - g_{21}^{(1)} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] + 2 g_{12}^{(0)} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right] \big) \\ &\quad = g_{21}^{(0)} \left(-2 s_{23} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right] - s_{23} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right] \right) + g_{21}^{(1)} \left((s_{12} + s_{23}) \,\mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] - s_{23} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] \right) \\ &\qquad + g_{21}^{(2)} \left(-s_{32} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] \right) + g_{21}^{(3)} \left(-s_{32} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \right) \\ &\quad = g_{21}^{(0)} \left(-2 s_{23} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right] - s_{23} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right] \right) + g_{21}^{(1)} \left((s_{12} + s_{23}) \,\mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] - s_{23} \,\mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] \right) \end{aligned}$$

(273)
$$+\, g_{21}^{(2)} \left(s_{13}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \right) + g_{21}^{(3)} \left(-s_{32}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \right)$$

and similarly

$$\begin{aligned}
&\frac{\partial}{\partial z_2} \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right](0, z_2) \\
&\quad = \int_0^{z_2} dz_3\ \mathrm{S}^\tau\, g_{32}^{(2)} \left(s_{21} g_{21}^{(1)} + s_{23} g_{23}^{(1)} \right) + \int_0^{z_2} dz_3\ \mathrm{S}^\tau\, \frac{\partial}{\partial z_2} g_{32}^{(2)} \\
&\quad = s_{21} g_{21}^{(1)}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right] - s_{23} \int_0^{z_2} dz_3\ \mathrm{S}^\tau\, g_{32}^{(2)} g_{32}^{(1)} - \int_0^{z_2} dz_3\ \mathrm{S}^\tau\, \frac{\partial}{\partial z_3} g_{32}^{(2)} \\
&\quad = s_{21} g_{21}^{(1)}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right] - s_{23} \int_0^{z_2} dz_3\ \mathrm{S}^\tau\, g_{32}^{(2)} g_{32}^{(1)} \\
&\qquad\quad + \int_0^{z_2} dz_3\ \mathrm{S}^\tau (s_{31} g_{31}^{(1)} + s_{32} g_{32}^{(1)}) g_{32}^{(2)} \\
&\quad = s_{21} g_{21}^{(1)}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right] + s_{13} \int_0^{z_2} dz_3\ \mathrm{S}^\tau\, g_{32}^{(2)} g_{31}^{(1)} ,
\end{aligned} \tag{274}$$

where we can again use

$$\begin{aligned}
g_{31}^{(1)} g_{32}^{(2)} &= g_{21}^{(3)} + g_{12}^{(2)} g_{31}^{(1)} + g_{12}^{(1)} g_{31}^{(2)} + g_{12}^{(0)} g_{31}^{(3)} - g_{12}^{(1)} g_{32}^{(2)} + 2 g_{21}^{(0)} g_{32}^{(3)} \\
&= -g_{21}^{(3)} + g_{21}^{(2)} g_{31}^{(1)} - g_{21}^{(1)} g_{31}^{(2)} + g_{21}^{(0)} g_{31}^{(3)} + g_{21}^{(1)} g_{32}^{(2)} + 2 g_{21}^{(0)} g_{32}^{(3)} ,
\end{aligned} \tag{275}$$

such that

$$\begin{aligned}
&\frac{\partial}{\partial z_2} \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right](0, z_2) \\
&= g_{21}^{(0)} \left(s_{13}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right] + 2 s_{13}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \right] \right) + g_{21}^{(1)} \left(-s_{13}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right] + (s_{12} + s_{13})\, \mathrm{S}^\tau \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right] \right) \\
&\quad + g_{21}^{(2)} \left(s_{13}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \right) + g_{21}^{(3)} \left(-s_{13}\, \mathrm{S}^\tau \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \right) .
\end{aligned} \tag{276}$$

The relevant 4×4-submatrices $x_{\leq 2}^{(n)}$ of $x^{(n)}$ for $n \in \{0, 1, 2\}$ appearing in the differential eq. (192) of $\mathbf{S}_{\leq 2}^\tau(z_2)$, i.e.

$$\frac{\partial}{\partial z_2} \mathbf{S}_{\leq 2}^\tau(z_2) = \left(g_{21}^{(0)} x_{\leq 2}^{(0)} + g_{21}^{(1)} x_{\leq 2}^{(1)} + g_{21}^{(2)} x_{\leq 2}^{(2)} \right) \mathbf{S}_{\leq 2}^\tau(z_2) + r_2\, \mathbf{S}_3^\tau(z_2) , \tag{277}$$

can now be read off from the differential equations (260), (267), (273) and (276), which gives the matrices in eqs. (193) and (194).

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