On Spectrum of the Linearized 3D Euler Equation

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ABSTRACT. We investigate essential spectrum of the Euler equation linearized about an arbitrary smooth steady flow in dimension 3. It is proved that for every Lyapunov-Oseledets exponent μ of the associated bicharacteristic-amplitude system, the circle of radius $e^{\mu t}$ has a common point with the spectrum. If, in addition, μ is attained on an aperiodic point, then the spectrum contains the entire circle.

CONTENTS

1. Introduction

Studying stability of equilibrium solutions to a nonlinear PDE, it is crucial to understand asymptotic behavior of the corresponding linearized equation. For the system of Euler equations governing the motion of an ideal incompressible fluid,

$$(1.1) \qquad u_t = -(u, \nabla)u - \nabla p,$$

$$(1.2) \qquad \operatorname{div} u = 0$$

this has been the subject of a classical theory originated in works of Rayleigh, Kelvin, Love and others [**3, 6, 13, 17**]. In its early stage a prime emphasis was given to the search of unstable eigenvalues (i.e. eigenvalues with positive real part)

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for particular examples of equilibria such as parallel shear flow. We refer to recent works [1, 9, 10, 15] for modern developments in this direction. The results we will present concern examination of so-called essential part of the spectrum, which may as well become a source of (linear) instability.

So, we consider a steady solution to (1.1), (1.2), denoted $u(x)$, on the torus $\mathbb{T}^n$. Our choice of boundary conditions is motivated by purely technical reasons. Other types such as the free space or bounded domain should not present major difficulties. We also assume sufficient smoothness for $u(x)$ where necessary, and this will be our only restriction on u.

Neglecting non-linear terms, the evolution equation for small perturbations of u is written as follows

$$\text{(1.3)} \qquad\qquad v_t = -(u, \nabla)v - (v, \nabla)u - \nabla p$$

$$\text{(1.4)} \qquad\qquad \operatorname{div} v = 0.$$

The nature of the spectrum of the Euler operator defined from (1.3) by

$$Lv = -(u, \nabla)v - (v, \nabla)u - \nabla p$$

and the corresponding evolution semigroup $G_t = e^{tL}$, $t \geq 0$, is sensitive to the choice of norm in which instability is measured. For our discussion, we take the physically reasonable energy norm. So, the semigroup acts on the space of divergence-free square-integrable vector fields on the torus, written $L^2_{\mathrm{div}}(T^n)$, $n = 2, 3$.

For a general bounded operator G on a Banach space, its spectrum $\sigma(G)$ splits into union of discrete and essential parts. Following Browder [2] we call $\lambda \in \sigma(G)$ a point of discrete spectrum, $\sigma_{\mathrm{disc}}(G)$, if (the third condition given in [2] is redundant for bounded operators)

(1d) λ is an isolated point in $\sigma(G)$,

(2d) the corresponding spectral subspace, determined by the image of the Riesz projection

$$P = \frac{1}{2\pi i} \int_{|z-\lambda|<\delta} \frac{dz}{z - G},$$

is finite dimensional.

The *essential spectrum*, denoted by $\sigma_{\mathrm{ess}}(G)$, is defined as $\sigma(G) \backslash \sigma_{\mathrm{disc}}(G)$. Similar classification can be given for any unbounded closed operator.

A complete description for the essential spectrum of the linear Euler equation in dimension two has been recently given in [19]. To state the result, let $\varphi_t : \mathbb{T}^n \to \mathbb{T}^n$ be the integral measure-preserving flow of the steady field $u(x)$. The maximal Lyapunov exponent of the Jacobi matrix $\partial\varphi_t(x) = (\frac{\partial\varphi_t^i(x)}{\partial x_j})_{i,j=1}^n$ is defined by the formula

$$\Lambda = \lim_{t\to\infty} \frac{1}{t} \log \max_{x\in\mathbb{T}^n} \|\partial\varphi_t(x)\|.$$

Then, for $n = 2$, the following identities hold:

$$\sigma_{\mathrm{ess}}(G_t) = \{e^{-t\Lambda} \leq |z| \leq e^{t\Lambda}\},$$
$$\sigma_{\mathrm{ess}}(L) = \{|\mathrm{Re}\, z| \leq \Lambda\}.$$

Consequently, we obtain the spectral mapping theorem:

$$\sigma(G_t)\backslash\{0\} = e^{t\sigma(L)}.$$

In dimension three, however, description of essential spectrum becomes increasingly difficult. The problem arises from the presence of additional vortex stretching term. In the vorticity formulation of (1.3) it adds a generically bounded perturbation to the advective derivative, whereas in two dimensions it adds only compact one. To bypass this difficulty a new method was deployed in [11, 21] based on WKB-type expansion of localized highly oscillating perturbations. More specifically, suppose initially

$$v(x, t = 0) = b_0(x, \xi_0)e^{i\xi_0 \cdot x/\delta},$$

where δ is small, the amplitude b_0 is localized around x_0 and orthogonal to the wave vector ξ_0 to insure incompressibility. The corresponding solution can be expanded into a formal series in powers of δ,

$$v(x, t) = b(x, \xi_0, t)e^{iS(x,\xi_0,t)/\delta} + ...$$

Substituting this into (1.3) and disregarding the lower powers of δ we obtain the following system of ODE written in Lagrangian coordinates

$$(1.5) \qquad \begin{cases} \dot{x} & = u(x), & x(0) = x_0 \in \mathbb{T}^3 \\ \dot{\xi} & = -\partial u^\top \xi, & \xi(0) = \xi_0 \in (\mathbb{R}^3)^* \backslash \{0\} \\ \dot{b} & = -\partial u \cdot b + 2(\partial u \cdot b, \frac{\xi}{|\xi|})\frac{\xi}{|\xi|}, & b(0) = b_0 \in \mathbb{R}^3, \ b_0 \perp \xi_0 \end{cases},$$

where $\xi = \nabla S$. It is referred to as the bicharacteristic-amplitude system associated with $u(x)$ (see [8] for details). Its dynamical content is the following.

The first two equations define a flow on $\mathbb{K} = \mathbb{T}^3 \times \mathbb{PR}^2$ given by

$$(x, \xi) \rightarrow \left(\varphi_t(x), \frac{\partial \varphi_t^{-\top}(x)\xi}{|\partial \varphi_t^{-\top}(x)\xi|} \right),$$

where $'-\top'$ means 'inverse transpose', and $\mathbb{PR}^2$ denotes the two dimensional projective space. The corresponding mapping

$$(x_0, \xi_0, b_0) \rightarrow b(t, x_0, \xi_0, b_0)$$

generated by solutions of the third equation in (1.5) defines a linear cocycle over this flow (see [4, 14]):

$$B_t(x_0, \xi_0)b_0 = b(t, x_0, \xi_0, b_0).$$

Since the b-equation is homogenous in ξ and smooth in both variables, so is the cocycle. Furthermore, since $\mathbb{K}$ is compact, the classical Oseledets' Multiplicative Ergodic Theorem [4, 16] applies to yield a set of points μ, called Lyapunov-Oseledets exponents, for each of which there is a $(x_0, \xi_0) \in \mathbb{K}$ and $b_0 \in \mathbb{R}^3$ such that

$$(1.6) \qquad \mu = \lim_{t \to \infty} \frac{1}{t} \log |B_t(x_0, \xi_0)b_0|.$$

It can be shown that the maximal exponent exists and is equal to

$$\mu_{\max} = \lim_{t \to \infty} \frac{1}{t} \log \max_{(x,\xi) \in \mathbb{K}} \|B_t(x, \xi)\|$$

(see [4]). This dynamical quantity serves as a replacement for the upper Lyapunov exponent of the basic flow φ_t in three dimensions. In fact, the following formula for the essential spectral radius was proved in [20]

$$r_{\text{ess}}(G_t) = e^{t\mu_{\max}},$$

(see also [**18**]). This formula was effectively used in a number of examples to detect instabilities caused by essential spectrum, [**9, 11**]. For us it gives a piece of information about the structure of the spectrum, namely $\sigma_{\mathrm{ess}}(G_t) \cap \{|z| = e^{t\mu_{\max}}\} \neq \emptyset$. Our first result is a direct generalization of this fact.

THEOREM 1.1. *Suppose $u(x)$ is a smooth equilibrium solution of the Euler equation* (1.1) *on $\mathbb{T}^3$. Let μ be any Lyapunov-Oseledets exponent of the associated system* (1.5)*. Then for all $t > 0$,*

(1.7) $$\sigma_{\mathrm{ess}}(G_t) \cap \{|z| = e^{t\mu}\} \neq \emptyset.$$

We prove this theorem in two steps. First, we establish a general sufficient condition for (1.7) based on existence of a weakly null sequence of unit vectors $\{x_n\}_{n=1}^{\infty}$ such that for each $t > 0$, $\|G_t x_n\|$ tends to $f(t)$, which as a function of t has Lyapunov index μ. Then, we adopt a construction from [**18, 20**] and produce such a sequence for the Euler semigroup.

By analogy with the two dimensional case, one may conjecture that

(1.8) $$\sigma_{\mathrm{ess}}(G_t) = \{e^{-t\mu_{\max}} \leq |z| \leq e^{t\mu_{\max}}\}.$$

This amounts to proving two facts: $\sigma_{\mathrm{ess}}(G_t)$ has no circular gaps, and $\sigma_{\mathrm{ess}}(G_t)$ is rotationally invariant. It was shown in [**18, 20**] (see Section 2) that, modulo a compact perturbation, the Euler semigroup is representable in a form very similar to that of the evolution Mather semigroup associated with the b-cocycle $B_t(x, \xi)$. Due to de la Llave's theorem [**5**], a Mather semigroup has no circular spectral gaps when defined on a space of divergence-free vector fields. Even though it remains open whether such a property is inherited by the Euler semigroup, this at least suggests the validity of the first fact.

As to the second, unfortunately, all known mechanisms producing rotational invariance of the spectrum require presence of long periodic or non-periodic orbits in the underlying flow. In two dimensions, this is automatically provided by existence of a hyperbolic point whenever $\Lambda > 0$. Fully three dimensional equilibria, on the other hand, may exhibit a much more complicated topology in which even periodic flows can produce positive Lyapunov exponents.

This examination shows that more truthful, although wide open to date, statement would be the annular hull version of (1.8),

$$\bigcup_{0 \leq \theta \leq 2\pi} e^{i\theta} \sigma_{\mathrm{ess}}(G_t) = \{e^{-t\mu_{\max}} \leq |z| \leq e^{t\mu_{\max}}\}.$$

The second part of our paper addresses a special case, in which more refined description of the spectrum can be given.

A point $x_0 \in \mathbb{T}^3$ is called *aperiodic* if its every neighborhood is crossed by an orbit of $\{\varphi_t\}$ with arbitrary large (possibly infinite) period. A Lyapunov-Oseledets exponent is called *aperiodic* if the point x_0 in formula (1.6) can be chosen aperiodic. As discussed previously, availability of long orbits is suggestive to rotational invariance. Indeed, the following result is true.

THEOREM 1.2. *Suppose $u(x)$ is a smooth equilibrium solution of the Euler equation* (1.1) *on $\mathbb{T}^3$. Let μ be an aperiodic Lyapunov-Oseledets exponent of* (1.5)*. Then for all $t > 0$,*

$$\{|z| = e^{t\mu}\} \subset \sigma_{\mathrm{ess}}(G_t).$$

We leave the essential spectrum of the generator for future study. Let us remark that the highly oscillating functions used in the method present the main obstacle in proving similar results for L. It is possible to suppress the oscillations in the situation when given Lyapunov exponent is attained for ξ_0 and x_0 satisfying $\xi_0 \perp u(x_0)$. This special case is treated in article [**14**], where, in particular, the authors prove that the exponential growth type of the semigroup and the spectral bound of the generator are equal.

2. Hyperbolic Semigroups

We start with a few general remarks on C_0-semigroups. Our immediate goal is to find a simple sufficient condition for non-hyperbolicity, which will eventually lead us to the proof of Theorem 1.1.

Let us recall that a C_0-semigroup $\{G_t\}_{t\geq 0}$ of bounded operators on a Banach space X is called *hyperbolic* provided X can be split into a direct sum of two subspaces X_s and X_u invariant with respect to the semigroup, and such that for some $\varepsilon > 0$ and $M > 0$ one has

(1h) $\|G_t x\| \leq M e^{-\varepsilon t}\|x\|$, for all $x \in X_s$ and $t > 0$;
(2h) $\|G_t x\| \geq M^{-1} e^{\varepsilon t}\|x\|$, for all $x \in X_u$ and $t > 0$.

The semigroup $\{G_t\}_{t\geq 0}$is hyperbolic if and only if $\sigma(G_t) \cap \mathbb{T} = \emptyset$ for all – or, equivalently, for some $t > 0$ (see [**7**]). If this is the case, the operators

$$P_s = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{dz}{z - G_t}$$
$$P_u = I - P_s$$

define projection maps onto the corresponding subspaces X_s, X_u, and

$$\sigma(G_t|_{X_s}) = \sigma(G_t) \cap \{|z| \leq e^{-\varepsilon t}\}$$
$$\sigma(G_t|_{X_u}) = \sigma(G_t) \cap \{|z| \geq e^{\varepsilon t}\}$$

A simple rescaling argument shows that for every $\lambda \in \mathbb{R}$, the new semigroup operator $e^{\lambda t}G_t$ has no spectrum inside the annulus $\{e^{(\lambda-\varepsilon)t} \leq |z| \leq e^{(\lambda+\varepsilon)t}\}$.

Analogously, we say that the semigroup $\{G_t\}_{t\geq 0}$ is *essentially hyperbolic* if $\sigma_{\text{ess}}(G_t)\cap\mathbb{T} = \emptyset$ for all $t > 0$. Equivalently, there are invariant subspaces X_s, X_c, X_u which split X into a direct sum, such that conditions (1h), (2h) are satisfied for X_s, X_u, and X_c is finite dimensional.

In this terminology, Theorem 1.1 simply states that $\{e^{-\mu t}G_t\}_{t\geq 0}$ is not essentially hyperbolic for every Lyapunov-Oseledets exponent μ of the bicharacteristic system (1.5). So, to prove the theorem we first establish a simple sufficient condition for non-essential hyperbolicity of a general C_0-semigroup. Then based on construction from [**18, 20**] we verify this condition for the Euler semigroup.

PROPOSITION 2.1. *Let $\{G_t\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Suppose there is a sequence of unit vectors $\{x_n\}_{n=1}^{\infty} \subset X$ such that*

(1p) $\lim_{n\to\infty} \|G_t x_n\| = f(t)$ *for all $t > 0$,*
(2p) $\lim_{t\to\infty} \frac{1}{t} \log f(t) = \mu$.

Then $\{e^{-\mu t}G_t\}_{t\geq 0}$ is not hyperbolic. If, in addition, $x_n \to 0$ weakly in X, then $\{e^{-\mu t}G_t\}_{t\geq 0}$ is not essentially hyperbolic, i.e.

$$\sigma_{\text{ess}}(G_t) \cap e^{\mu t}\mathbb{T} \neq \emptyset.$$

PROOF. After rescaling we can assume that $\mu = 0$. Also, assume on the contrary that $\{G_t\}_{t\geq 0}$ is hyperbolic. Then there exist invariant subspaces X_s and X_u which satisfy conditions (1h), (2h) with some $\varepsilon > 0$. Let $x_n = x_n^s + x_n^u$ where $x_n^s \in X_s$ and $x_n^u \in X_u$. We show that $\|x_n^u\| \to 0$.

Indeed, fix an arbitrary $\delta > 0$. According to (2p) there is a $t > 0$ such that $0 < f(t) < e^{\varepsilon t/2}$. Furthermore, by (1p), for sufficiently large $n \in \mathbb{N}$ we find $\|G_t x_n\| \leq 2f(t)$. This implies, via (1h) and (2h), that for some positive c,

$$2e^{\varepsilon t/2} \quad > \quad \|G_t x_n\| \quad \geq \quad c\|G_t x_n^s\| \; + \; c\|G_t x_n^u\| \quad \geq \quad cM^{-1}e^{\varepsilon t}\|x_n^u\|.$$

Thus, $\|x_n^u\| \leq 2c^{-1}Me^{-\varepsilon t/2}$. Choosing t so that $2c^{-1}Me^{-\varepsilon t/2} < \delta$ we obtain the desired result.

Since $\|x_n^u\| \to 0$ as $n \to \infty$, condition (1p) can now be replaced by $\lim_{n\to\infty}\|G_t\, x_n^s\| = f(t)$ for each $t > 0$. Then in view of (1h), $f(t) \leq ce^{-\varepsilon t}$ for all $t > 0$, which contradicts (2p).

Let us assume that $x_n \to 0$ weakly, and $\sigma(G_t) \cap \mathbb{T} \subset \sigma_{\mathrm{disc}}(G_t)$. Then the spectral subspace X_c corresponding to the part of the spectrum lying on $\mathbb{T}$ is finite dimensional. The other two subspaces X_s and X_u satisfy (1h) and (2h) as before. Also, $X = X_s \oplus X_c \oplus X_u$. We decompose each x_n into the sum $x_n^u + x_n^c + x_n^s$. If $\dim X_c = N$, we let

$$x_n^c = \sum_{j=1}^{N} y_j^*(x_n)y_j$$

for some basis $\{y_j\}_{j=1}^N$ in X_c and $\{y_j\}_{j=1}^N \subset X^*$. Since $x_n \to 0$ weakly, $y_j^*(x_n) \to 0$ for every j, and hence, $\|x_n^c\| \to 0$. This implies that the restricted semigroup $\{G_t|_{X_s \oplus X_u}\}_{t\geq 0}$ satisfies the same hypotheses (1p),(2h), but is hyperbolic now. By the preceding argument it is not possible.

This contradiction finishes the proof. $\square$

Let us turn to the proof of Theorem 1.1.

In papers [18, 20] a weakly null sequence of divergence free vector fields $\{x_n\} \subset L^2_{\mathrm{div}}(T^3)$ satisfying the conditions of Proposition 2.1 was constructed. We reproduce the main steps of the construction here for convenience of the reader.

Let us fix a Lyapunov-Oseledets exponent μ of the b-cocycle determined by (1.5). Then there exists a point $(x_0, \xi_0) \in \mathbb{K}$ and a vector $b_0 \in \mathbb{R}^3$ such that

$$\mu = \lim_{t\to\infty} \frac{1}{t}\log|B_t(x_0, \xi_0)b_0|.$$

It is shown in [18] that every representative of the Euler semigroup, G_t, is equal, modulo compact perturbation, to the following pseudo-differential operator

$$P_t w(x) = \sum_{q\in\mathbb{Z}^3\setminus\{0\}} B_t(\varphi_{-t}(x), q)\hat{w}(q)e^{iq\cdot\varphi_{-t}(x)}.$$

Let us take a scalar function $h \in L^2(\mathbb{T}^3)$, $\|h\| = 1$ with Fourier transform supported in the ball of radius R, and its weight concentrated around x_0. Fix a $\delta > 0$ and choose $q_\delta \in \mathbb{Z}^3$ so that $|\xi_0 - \frac{q_\delta}{|q_\delta|}| < \delta$, and $q_\delta/\delta \in \mathbb{Z}^3$. We define

$$w_\delta = \delta\nabla \times \left(\frac{iq_\delta}{|q_\delta|^2} \times b_0 \cdot h(y)e^{i\frac{q_\delta}{\delta}\cdot y}\right)$$
$$= b_0 h(x)e^{i\frac{q_\delta}{\delta}\cdot x} + O(\delta).$$

So, $\|w_\delta\| = 1 + O(\delta)$. Furthermore,

$$\hat{w}_\delta(q) = -\delta \cdot q \times \left(\frac{q_\delta}{|q_\delta|^2} \times b_0 \right) \hat{h}(q - q_\delta/\delta).$$

Using the homogeneity and smoothness of B_t in the second variable, we obtain

$$P_t w_\delta(x) = - \sum_{q \in \mathbb{Z}^3 \setminus \{0\}} B_t(x, q) \, \delta q \times \left(\frac{q_\delta}{|q_\delta|^2} \times b_0 \right) \hat{h}(q - q_\delta/\delta) \, e^{iq \cdot x}$$

$$= -e^{i\frac{q_\delta}{\delta} \cdot x} \sum_{|q| \le R} B_t(x, \delta q + q_\delta)(\delta q + q_\delta) \times \left(\frac{q_\delta}{|q_\delta|^2} \times b_0 \right) \hat{h}(q) \, e^{iq \cdot x}$$

$$= -e^{i\frac{q_\delta}{\delta} \cdot x} \sum_{|q| \le R} B_t(x, q_\delta) q_\delta \times \left(\frac{q_\delta}{|q_\delta|^2} \times b_0 \right) \hat{h}(q) \, e^{iq \cdot x} + O(\delta)$$

$$= e^{i\frac{q_\delta}{\delta} \cdot x} \sum_{|q| \le R} B_t(x, q_\delta) b_0 \hat{h}(q) \, e^{iq \cdot x} + O(\delta)$$

$$= e^{i\frac{q_\delta}{\delta} \cdot x} B_t(x, q_\delta) b_0 h(x) + O(\delta).$$

We can let $R \to \infty$ as $\delta \to 0$ slow enough to preserve the limits $O(\delta) \to 0$ and $w_\delta \to 0$ weakly, and so that $h = h_\delta$ eventually concentrates at x_0 as $\delta \to 0$. Thus,

$$\|P_t w_\delta\| = \|B_t(\varphi_{-t}(\cdot), q_\delta) b_0 h_\delta(\varphi_{-t}(\cdot))\| + O(\delta) \to |B_t(x_0, \xi_0) b_0|.$$

On the other hand,

$$\|G_t w_\delta - P_t w_\delta\| \to 0, \text{ as } \delta \to 0,$$

since the difference is compact. So, to conclude the proof we set $x_n = w_{1/n}$ and $f(t) = |B_t(x_0, \xi_0) b_0|$.

3. Aperiodic Case

In this section we outline a scheme of proving Theorem 1.2.

As we mentioned earlier, the presence of high oscillations makes it impossible to single out one particular point $z \in \mathbb{C}$ in the actual spectrum of L. Instead, we let z vary on the vertical line $\text{Re} z = \mu$, for some aperiodic Lyapunov-Oseledets exponent μ, and we show that

$$\|(L - \mu - i/\varepsilon)^{-1}\| \to \infty,$$

as $\varepsilon \to 0$. Choosing $\varepsilon = (2\pi k + \zeta)^{-1}$ for $k = 1, 2, \ldots$ and ζ arbitrary, one gets under the hypothesis of Gearhart's theorem [12], which implies that $e^{(\mu + i\zeta)t} \in \sigma(G_t)$.

Our basic idea is to construct a sequence of vector fields $\omega_{\varepsilon, N} \in L^2_{\text{div}}(\mathbb{T}^3)$, $\varepsilon, N > 0$, localized about the orbit of point x_0 given by (1.6), so that

$$\limsup_{\varepsilon \to 0} \|(L - \mu - i/\varepsilon)\omega_{\varepsilon, N}\| / \|\omega_{\varepsilon, N}\| \le \alpha(N),$$

where $\alpha(N)$ is some function not bounded away from zero as $N \to \infty$. In our construction, parameter N will be naturally restricted to the largest period of an orbit passing near x_0. So, in order to let N assume large values, we have to have long orbits in any neighborhood of x_0. This will be achieved by the aperiodicity condition.

So, let us fix an aperiodic Lyapunov-Oseledets exponent μ with x_0, ξ_0, and b_0 as in (1.6). For technical reasons, if $\xi_0 \perp u(x_0)$, then we fix another ξ_1 close to

ξ_0 and not perpendicular to $u(x_0)$. This assumption is not restrictive and will be eliminated later.

Since μ is aperiodic we can choose x_0 to be aperiodic. In this case, for any given neighborhood U of x_0 and $N > 0$ we find an x_1 with period greater than $2N$. Clearly, if U is sufficiently small, then $\xi_1 \not\perp u(x_1)$.

Denote by Σ a planar tile centered at x_1 and perpendicular to ξ_1. Let us take Σ of size $\sqrt[4]{\varepsilon} \times \sqrt[4]{\varepsilon}$ for ε so small that every point in the flow-box $\mathcal{F}_{\varepsilon,N} = \{\varphi_t(\Sigma)\}_{0 \le t \le N}$ is uniquely determined by the corresponding $\sigma \in \Sigma$ and $t \in [0, N]$.

Our function $\omega_{\varepsilon,N}$ is defined inside this flow-box $\mathcal{F}_{\varepsilon,N}$ by the following expression

$$\omega_{\varepsilon,N} = \frac{\varepsilon}{i} \, \nabla \times \left(\frac{\nabla S \times f}{|\nabla S|^2} \, e^{iS/\varepsilon} \right),$$

where S is a scalar-valued frequency function, and f is a vector amplitude field supported in the flow-box. Notice that $\omega_{\varepsilon,N}$ being the rotor of a vector field is automatically divergence free.

Let us specify S and f.

Since ∇S plays the role of a wave-vector, we want it to evolve along the orbit according to the ξ-equation in (1.5) with initial condition being a multiple of ξ_1. Setting

$$S(x = \varphi_t(\sigma)) = t,$$

for all $x \in \mathcal{F}_{\varepsilon,N}$, we see that $S|_\Sigma \equiv 0$ and hence, $\nabla S|_\Sigma$ is indeed proportional to ξ_1. Moreover, $S(\varphi_t(x)) - S(x) = t$ for all $x \in \mathcal{F}_{\varepsilon,N}$. Taking the gradient at $x = \sigma$ we obtain

$$\partial \varphi_t^\top(\sigma) \nabla S(\varphi_t(\sigma)) = \nabla S(\sigma)$$

$$\nabla S(\varphi_t(\sigma)) = \partial \varphi_t^{-\top}(\sigma) \nabla S(\sigma).$$

The last identity expresses the fact that $\nabla S(\varphi_t(\sigma))$ is a solution to the ξ-equation in t. Also, directly from the definition we infer

$$(3.1) \qquad\qquad (u, \nabla) S = \frac{dS(\varphi_t(\sigma))}{dt} \equiv 1.$$

For functions $\beta \in C_0^\infty(\Sigma)$ and $\gamma \in C_0^\infty(0, N)$ to be specified later, we define the amplitude field by

$$(3.2) \qquad\qquad f(x = \varphi_t(\sigma)) = \beta(\sigma) \gamma(t) e^{-\mu t} b(t) \int_0^t \frac{\gamma(s)}{e^{-\mu s} |b(s)|} ds$$

where $\sigma \in \Sigma$, $0 \le t \le N$, and $b(t) = b(t, \sigma, \xi_1, b_0) = B_t(\sigma, \xi_1) b_0$ is the solution to the b-equation (1.5) with initial data σ, ξ_1, b_0. Recall that $\sigma \sim x_0$, $\xi_1 \sim \xi_0$ and b_0 is taken from (1.6).

It will be beneficial to represent $\omega_{\varepsilon,N}$ as a sum of oscillating and negligible terms. Namely, by direct computation,

$$\omega_{\varepsilon,N} = -f \cdot e^{iS/\varepsilon} + \frac{\varepsilon}{i} \, e^{iS/\varepsilon} \nabla \times \left(\frac{\nabla S \times f}{|\nabla S|^2} \right).$$

Denoting $g = f \cdot e^{iS/\varepsilon}$ and $\omega_0 = \nabla \times \left(\frac{\nabla S \times f}{|\nabla S|^2} \right)$ we rewrite

$$(3.3) \qquad\qquad \omega_{\varepsilon,N} = -g + \frac{\varepsilon}{i} \, e^{iS/\varepsilon} \omega_0.$$

Let $z = \mu + i/\varepsilon$ be our spectral parameter. Then, the formula for g reads explicitly

$$(3.4) \qquad g(x = \varphi_t(\sigma)) = \beta(\sigma)\gamma(t)e^{-zt}b(t) \int_0^t \frac{\gamma(s)}{e^{-\mu s}|b(s)|} ds.$$

In the next section we investigate the action of $(L - z)$ on $\omega_{\varepsilon,N}$. In particular, we adjust the functions β and γ properly to make the energy norm $\|(L - z)\omega_{\varepsilon,N}\|$ small relative to $\|\omega_{\varepsilon,N}\|$.

4. Action of L

Using formulas (3.1), (3.3), and (3.4), after tedious but straightforward computations, we obtain at $x = \varphi_t(\sigma)$

$$(-(u, \nabla) - \partial u)\omega_{\varepsilon,N} = \beta(\sigma)\gamma'(t)e^{-zt}b(t) \int_0^t \frac{\gamma(s)}{e^{-\mu s}|b(s)|} ds - zg$$

$$(4.1) \qquad + \beta(\sigma)\gamma(t)e^{-zt}b'(t) \int_0^t \frac{\gamma(s)}{e^{-\mu s}|b(s)|} ds + \beta(\sigma)e^{-zt}b(t)\frac{\gamma^2(t)}{e^{-\mu t}|b(t)|}$$

$$+ \partial u \cdot g - e^{iS/\varepsilon}\omega_0 + \frac{\varepsilon}{i} e^{iS/\varepsilon}((u, \nabla) + \partial u)\omega_0.$$

Since the Leray projection is orthogonal, we have

$$\|(L - z)\omega_{\varepsilon,N}\| \leq \|(-(u, \nabla) - \partial u)\omega_{\varepsilon,N} - z\omega_{\varepsilon,N} - \nabla p\|,$$

for any scalar function p. Let us choose

$$p = \frac{\varepsilon}{i} e^{iS/\varepsilon}2 \left(\partial u \cdot f, \frac{\nabla S}{|\nabla S|^2} \right) = \frac{\varepsilon}{i} e^{iS/\varepsilon}q.$$

Then

$$\nabla p = \nabla S e^{iS/\varepsilon}q + \frac{\varepsilon}{i} e^{iS/\varepsilon}\nabla q.$$

By construction, $b(t)$ solves the b-equation, i.e.

$$b' = \partial u \cdot b + 2 \left(\partial u \cdot b, \frac{\nabla S}{|\nabla S|} \right) \frac{\nabla S}{|\nabla S|}.$$

Combining the last two identities with (4.1), we evaluate at $x = \varphi_t(\sigma)$

$$(-(u, \nabla) - \partial u)\omega_{\varepsilon,N} - z\omega_{\varepsilon,N} - \nabla p = \beta(\sigma)\gamma'(t)e^{-zt}b(t) \int_0^t \frac{\gamma(s)}{e^{-\mu s}|b(s)|} ds$$

$$+ \beta(\sigma)e^{-zt}b(t)\frac{\gamma^2(t)}{e^{-\mu t}|b(t)|} - \frac{\varepsilon}{i} e^{iS/\varepsilon}((u, \nabla)\omega_0 - \nabla q + \partial u \cdot \omega_0).$$

Suppressing variables we get the following inequality,

$$\|(L - z)\omega_{\varepsilon,N}\| \leq \|\beta\gamma'e^{-\mu t}b \int_0^t \| + \|\beta\gamma^2\| + \varepsilon\|(u, \nabla)\omega_0 - \nabla q + \partial u \cdot \omega_0\|$$

$$= I_1 + I_2 + \varepsilon I_3.$$

Our next goal is to make the right hand side infinitely small in comparison with the norm of $\omega_{\varepsilon,N}$ as $\varepsilon \to 0$ and $N \to \infty$. We achieve it by a proper choice of functions β and γ, which are still at our disposal. Their definition would be more transparent if to work with σ and t rather than x variable. To this end, let us introduce suitable transformation of coordinates and change variables in the integrals I_1, I_2.

We define a map from $\mathcal{S} = [0, N] \times \Sigma$ onto $\mathcal{F}_{\varepsilon, N}$ by

$$H(t, \sigma) = \varphi_t(\sigma).$$

A direct computation shows that

$$\partial H(t, \sigma) = \left[u \circ \varphi_t(\sigma) \; \vdots \; \partial_{\sigma_1} \varphi_t(\sigma) \; \vdots \; \partial_{\sigma_2} \varphi_t(\sigma) \right],$$

where $\sigma = (\sigma_1, \sigma_2)$. Let e_k be the unit vector in the σ_k-th direction. Then

$$\partial H(t, \sigma) = \partial \varphi_t(\sigma) \left[u(\sigma) \; \vdots \; e_1 \; \vdots \; e_2 \right].$$

Consequently, the quantity

$$|\det \partial H(t, \sigma)| = |(u(\sigma), e_1 \times e_2)|,$$

denoted $d(\sigma)$ in the sequel, does not depend on t. Besides, $d(0) \neq 0$ due to our assumption that $\xi_1 \not\perp u(x_1)$.

So, making the change of variables in I_1 and I_2 we get

$$I_1^2 = \int_{\mathcal{S}} d(\sigma) \beta^2(\sigma) \left| e^{-\mu t} b(t) \gamma'(t) \int_0^t \frac{\gamma(s)}{e^{-\mu s} |b(s)|} ds \right|^2 dt d\sigma$$

$$I_2^2 = \int_{\mathcal{S}} d(\sigma) \beta^2(\sigma) \gamma^4(t) \, dt d\sigma.$$

Let us choose $\beta \in L^2(\Sigma, d\sigma)$ so that $\|\beta\|_{L^2(\Sigma, d\sigma)} = 1$ and

$$\|\partial^{(\alpha)} \beta\|_{L^2(\Sigma, d\sigma)} \leq 1/\sqrt{\varepsilon},$$

for $|\alpha| \leq 2$. The latter is possible due to our assumption on the size of Σ. Since all the functions involved in I_3 contain at most the second derivative of β, we immediately obtain $\varepsilon I_3 \to 0$ as $\varepsilon \to 0$. Also,

$$I_1^2 \to d(0) \int_0^N \left| e^{-\mu t} b(t) \gamma'(t) \int_0^t \frac{\gamma(s)}{e^{-\mu s} |b(s)|} ds \right|^2 dt \quad \overset{\text{def}}{=} d(0) J_1(N)$$

$$I_2^2 \to d(0) \int_0^N \gamma^4(t) dt \qquad\qquad\qquad\qquad \overset{\text{def}}{=} d(0) J_2(N).$$

As to the functions $\omega_{\varepsilon, N}$, using representation (3.3) and performing similar estimates, we obtain

$$\|\omega_{\varepsilon, N}\|^2 \to d(0) \int_0^N \left| e^{-\mu t} b(t) \gamma(t) \int_0^t \frac{\gamma(s)}{e^{-\mu s} |b(s)|} ds \right|^2 dt \overset{\text{def}}{=} d(0) J(N).$$

We now define γ as follows,

$$\gamma(t) = \begin{cases} 0, & t \leq 0, \\ t/\sqrt{N}, & 0 < t \leq 1, \\ 1/\sqrt{N}, & 1 < t \leq N-1, \\ (N-t)/\sqrt{N}, & N-1 < t \leq N, \end{cases}$$

smoothed out at $0, 1, N-1, N$.

This implies that $J_2(N) \sim 1/N$, where $A \sim B$ means there is a constant c such that $c^{-1} A < B < cA$.

To estimate J_1 and J first notice that $b(t) \sim b(\tau)$ whenever $|t - \tau| < 1$. So, denoting $h(t) = h(t, x_1, \xi_1, b_0) = e^{-\mu t} |b(t, x_1, \xi_1, b_0)|$ we have

$$J_1(N) \sim \frac{1}{N^2} \left[\int_0^N \frac{h(N)}{h(t)} dt \right]^2$$

$$J(N) \sim \frac{1}{N^2} \int_0^N \left[\int_0^t \frac{h(t)}{h(s)} ds \right]^2 dt.$$

Combining the obtained estimates we see that

$$\limsup_{\varepsilon \to 0} \|(L - z)\omega_{\varepsilon,N}\|^2 / \|\omega_{\varepsilon,N}\|^2 \lesssim \frac{d(0)J_1(N) + d(0)J_2(N)}{d(0)J(N)}$$

$$\sim \frac{\frac{1}{N} \left[\int_0^N \frac{h(N)}{h(t)} dt \right]^2 + 1}{\frac{1}{N} \int_0^N \left[\int_0^t \frac{h(t)}{h(s)} ds \right]^2 dt} = \frac{S_1(N) + 1}{S(N)} = \alpha(N).$$

Hence,

(4.2) $$\liminf_{\varepsilon \to 0} \|(L - z)^{-1}\| \gtrsim \alpha^{-1/2}(N).$$

By the construction, $\alpha(N)$ depends only on x_1, ξ_1, b_0 but not on the angle between ξ_1 and $u(x_1)$. So we can let $\xi_1 \to \xi_0$ in (4.2) preserving the inequality. Furthermore, due to our aperiodicity assumption, the point x_1 can be chosen as close to x_0 as we wish. Hence, we can let $x_1 \to x_0$ too. As a result, inequality (4.2) remains valid for all $N > 0$ and for x_0, ξ_0, b_0 originally given by (1.6). In particular, the function h in the formula for α has Lyapunov index 0, i.e

$$\lim_{t \to \infty} \frac{1}{t} \log h(t) = 0.$$

So, it remains to prove that $\alpha(N)$ is not bounded away from zero as $N \to \infty$.

5. Functions with Zero Lyapunov Index

Let us rewrite $\alpha(N)$ as the sum

$$\alpha(N) = \frac{S_1(N)}{S(N)} + \frac{1}{S(N)}.$$

Clearly, it suffices to show that $S_1(N)/S(N)$ is not bounded away from zero and $S(N) \to \infty$ as $N \to \infty$.

The proof of the first assertion is based on the Grönwall Lemma, while the second one requires a more delicate probabilistic argument.

LEMMA 5.1. *Let $h : \mathbb{R} \to \mathbb{R}^+$ be any function of zero Lyapunov index and let*

$$S_1(N) = \frac{1}{N} \int_0^N \left[\int_0^t \frac{h(t)}{h(s)} ds \right]^2 dt,$$

$$S(N) = \frac{1}{N} \left[\int_0^N \frac{h(N)}{h(t)} dt \right]^2.$$

Then $S_1(N)/S(N)$ is not bounded away from zero as $N \to \infty$.

PROOF. Assume, on the contrary, that there is a constant $c > 0$ such that $S_1(N) \geq cS(N)$. Denoting

$$f(t) = \left[\int_0^t \frac{h(t)}{h(s)} ds \right]^2$$

we can rewrite the inequality above as

$$f(N) \geq c \int_0^N f(t) dt = c_1 + c \int_1^N f(t) dt.$$

So, by the Grönwall Lemma

$$f(N) \gtrsim e^{cN},$$

which, possibly with different c, implies

$$\frac{1}{N} \int_0^N \frac{h(N)}{h(s)} ds \gtrsim \frac{1}{N} e^{cN}.$$

Thus, there is an $s \in [0, N]$ such that

$$h(N) \gtrsim h(s) \frac{1}{N} e^{cN}.$$

Since h has Lyapunov index zero, we can write

$$h(s) \gtrsim e^{-cs/2}.$$

Hence,

$$h(N) \gtrsim \frac{1}{N} e^{cN/2},$$

which leads to a contradiction. □

For the second assertion, it is more convenient to rewrite $S(N)$ in an equivalent discrete form. Namely,

$$S(N) \sim \frac{1}{N} \sum_{n=1}^N \left[\sum_{k=1}^n \frac{h(n)}{h(k)} \right]^2.$$

This readily follows from the fact that $h(t) \sim h(\tau)$, whenever $|t - \tau| < 1$.

LEMMA 5.2. *Let $h : \mathbb{N} \to \mathbb{R}^+$ be any function of zero Lyapunov index. Then*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \left[\sum_{k=1}^n \frac{h(n)}{h(k)} \right]^2 = \infty$$

.

PROOF. Let us fix an arbitrary $M \in \mathbb{N}$ and introduce the following sets

$$P_n = \{ k \leq n : \frac{h(n)}{h(k)} > 1/2 \}$$

$$Q_N = \{ n \leq N : |P_n| > M \}$$

It suffices to prove that

(5.1) $\liminf_{N \to \infty} |Q_N|/N \geq 1/8.$

Indeed, then for sufficiently large N we have

$$\frac{1}{N}\sum_{n=1}^{N}\left[\sum_{k=1}^{n}\frac{h(n)}{h(k)}\right]^{2} \geq \frac{1}{N}\sum_{n\in Q_N}\left[\sum_{k\in P_n}\frac{h(n)}{h(k)}\right]^{2}$$

$$\geq \frac{1}{N}\sum_{n\in Q_N}(|P_n|/2)^{2}$$

$$\geq \frac{1}{N}|Q_N|M^{2}/4 \geq \frac{1}{N}\frac{N}{40}M^{2} = M^{2}/40.$$

In order to prove (5.1) we argue as follows.

Suppose (5.1) fails. Then we can find an arbitrarily large N such that $|Q_N| < N/8$ or, taking the complement in $[0, N]$, $|Q_N^c| \geq 7N/8$. Let as fix one such N.

Obviously, there exists an $n_1 \in Q_N^c$ such that $n_1 \geq 7N/8$. Hence, $|P_{n_1}| \leq M$.

We continue by selecting the first integer n_2 to the left of n_1 that belongs to $P_{n_1}^c \cap Q_N^c$. Then $|P_{n_2}| \leq M$, $(n_2, n_1) \cap P_{n_1}^c \cap Q_N^c = \emptyset$ and

$$h(n_1) \leq \frac{1}{2}h(n_2).$$

Similarly, we find $n_3 < n_2$ the first in $P_{n_2}^c \cap Q_N^c$. Then $|P_{n_3}| \leq M$ and

$$h(n_2) \leq \frac{1}{2}h(n_3).$$

On the k-th step, we take $n_k < n_{k-1}$ to be the first in $P_{n_{k-1}}^c \cap Q_N^c$ so that $|P_{n_k}| \leq M$ and

$$h(n_{k-1}) \leq \frac{1}{2}h(n_k).$$

The process will continue at least until

(5.2) $n_k \leq N/8 + M,$

for otherwise $|P_{n_k}^c| > n_k - M > N/8$ and $|Q_N^c| \geq 7N/8$, so $P_{n_k}^c \cap Q_N^c \neq \emptyset$. Let k be the first integer satisfying (5.2).

Then,

(5.3) $h(n_1) \leq \frac{1}{2^{k-1}}h(n_k).$

On the other hand, by our construction,

$$(n_{i+1}, n_i) \cap P_{n_i}^c \cap Q_N^c = \emptyset$$
$$(n_{i+1}, n_i) = (P_{n_i} \cup Q_N) \cap (n_{i+1}, n_i).$$

So,

$$n_i - n_{i+1} \leq M + |Q_N \cap (n_{i+1}, n_i)|.$$

Consequently,

$$7N/8 - N/8 - M \leq n_1 - n_k = \sum_{i=1}^{k-1} n_i - n_{i+1}$$

$$\leq (k-1)M + \sum_{i=1}^{k-1}|Q_N \cap (n_{i+1}, n_i)|$$

$$\leq (k-1)M + |Q_N| < (k-1)M + N/8.$$

From this we get the necessary estimate on k :

$$k \geq \frac{5N}{8M}.$$

Using (5.3) we obtain

$$\frac{1}{n_1} \log \left[\frac{h(n_1)}{h(n_k)} \right] \leq \frac{k-1}{n_1} \log(1/2) \leq \left(\frac{5N}{8M} - 1 \right) \frac{8}{7N} \log(1/2)$$
$$= \left(\frac{5}{7M} - \frac{8}{7N} \right) \log(1/2)$$

Passing to the limit as $N \to \infty$ shows that the Lyapunov index of h is negative, which contradicts our hypothesis. $\qquad\square$

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