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Global Dynamics of Nonlinear Wave Equations With Cubic Non-Monotone Damping

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ABSTRACT. For nonlinear wave equations with cubic non-monotone damping, it is proved that the weak solutions exist globally and generate a semiflow. The dissipativity of the semiflow is shown by the asymptotic bootstrap method. Then it is shown that the global weak attractor exists.

CONTENTS

1. Introduction

In recent decades, many challenging issues have been emerging from the studies of dynamics governed by various nonlinear hyperbolic evolutionary equations with various kinds of weak dissipation such as localized damping, boundary damping, or non-monotone damping. Represented typically by nonlinear wave equations and nonlinear Petrovsky equations, the latter describing elastic vibrations of beams, plates or shells, a general form of nonlinear hyperbolic evolutionary equations with damping is given by

$$u_{tt} + Au + f(u) + g\left(u_t\right) = h(t,x), \quad t > 0, x \in \Omega,$$

where $\Omega \subset \Re^n$, $n \leq 3$, the operator A stands for $-a\Delta$ or $a\Delta^2$ with an appropriate homogeneous boundary condition, $f(s)$ and $g(s)$ are in general nonlinear scalar

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functions, and their corresponding Nemytsky mappings are denoted by the same notation, and $h(t,x)$ is an external input. The long-time behavior of solutions to the initial value problems of this second-order evolutionary equation draws investigation interests of many authors in both theoretical and application contexts, cf. [1–8], [10], and [11].

Usually the assumptions on the damping function $g \in C^1(\Re)$ are

$$g(0) = 0 \text{ and } g'(s) \geq g_0 > 0 \text{ for } s \in \Re.$$

This type of damping is called monotone damping, with which the dissipativity of solutions can be handled without substantial difficulty, provided that the nonlinear term $f(u)$ satisfies certain pro-dissipative conditions.

Some research efforts in recent years have been devoted to the cases with different kinds of weak damping. Although without any official definition, a weak damping is in the sense that the damping effect is substantially weakened either due to a localized damping, namely, the support of the damping function g is only a small portion of Ω, cf. [5] and [10], or due to a non-monotone structure of the damping function g, which is the topic of this work. Besides, a widely open problem is concerning the global dynamics of nonlinear wave equations and Petrovsky equations with a purely boundary damping or finitely many interior pointwise dampings, other than some known results of stability that all solutions converge to the zero equilibrium under certain strong conditions.

2. Local Solutions

In this paper, we shall consider the following nonlinear wave equation with a cubic non-monotone damping and the associated initial-boundary value problem:

$$\begin{aligned} u_{tt} - a\Delta u + f(u) + g(u_t) &= h(t,x), \quad t > 0, x \in \Omega, \\ u(t,x)|_\Gamma &= 0, \quad t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) &= u_1(x), \quad x \in \Omega. \end{aligned} \tag{2.1}$$

We assume that $\Omega \subset \Re^n$, $n = 1$ or 2, is a bounded domain with locally Lipschitz continuous boundary denoted by Γ. Assume that the scalar function $g(s)$ is given by

$$g(s) = -\alpha s + \beta s^3, \tag{2.2}$$

and a, α, and β are positive constants. Assume that $f(s)$ is a scalar function in $C^1(\Re)$, $f(0) = 0$, and the corresponding Nemytsky operator f (with the same notation) defined by

$$f(\varphi)(x) = f(\varphi(x)), \quad x \in \Omega,$$

is a gradient operator from $V = H_0^1(\Omega)$ into $H = L^2(\Omega)$ with its antiderivative $F : V \to \Re$, such that the following conditions are satisfied:

$$d_1 \|u\|_{2p}^{2p} - d_0 \leq \langle f(u), u \rangle, \text{ for } u \in V, \tag{2.3a}$$

$$-C_0 \leq F(u) \leq D_1 \|u\|_{2p}^{2p} + D_0, \text{ for } u \in V, \tag{2.3b}$$

where $p \geq 1$ is a fixed integer, the constants d_0, d_1, C_0, D_0 and D_1 are all positive.

The norm and inner product of the Hilbert space $H = L^2(\Omega)$ are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. In view of the homogeneous boundary condition, we shall use $\|\nabla \varphi\|$ as the equivalent norm of the Hilbert space V, for $\varphi \in V$. For any

$q \neq 2$, the norm of $L^q(\Omega)$ is denoted by $\| \cdot \|_q$ or $\| \cdot \|_{L^q(\Omega)}$. We shall use $|\varphi|$ for the absolute value of $\varphi(t, x)$.

Note that the conditions in (1.3a) and (1.3b) are satisfied by polynomials $f(s)$ of odd degree with a positive leading coefficient. Also note that these two conditions are satisfied by the Sine-Gordon equations and the Klein-Gordon equations. Here $h(t, x)$ is a given function to be specified later, which can be time-invariant.

The equation (2.1) in one spatial dimension ($n = 1$) with $f(s) = 0$ and $g(s)$ given in (2.2) has been exploited as a mathematical model of galloping vibrations of power lines in distant electric transmission, cf. [12] and [13]. The corresponding results on the existence of global solutions and dissipative dynamics properties have been proved in [14]. We noticed from [12] and [13] that the primary concern from the engineering viewpoint is whether for any initial status with finite energy a solution exists globally for $t > 0$ without blow-up, and whether every global solution remains to be bounded in the energy space E, here $E = V \times H$.

Let us define three product Hilbert spaces

$$E_1 = \big(H^2(\Omega) \cap V\big) \times V, \quad E = V \times H, \quad E_{-1} = H \times H^{-1}(\Omega).$$

The E-norm of $\operatorname{col}(\varphi, \psi)$ is equivalently defined to be

$$\| \operatorname{col}(\varphi, \psi) \| = \left(a\|\nabla\varphi\|^2 + \|\psi\|^2\right)^{1/2}.$$

As usual, define a linear operator $A : D(A) \to H$ by $A\varphi = -a\Delta\varphi$, with $D(A) = H^2(\Omega) \cap V$. Then A is a coercively positive, self-adjoint operator in H with compact resolvent. The initial-boundary value problem (2.1) is formulated into an initial value problem of the first-order nonlinear evolutionary equation:

$$\frac{dw}{dt} = \Lambda w + R(t, w), \quad t > 0, \qquad w(0) = w_0, \tag{2.4}$$

where $w(t) = \operatorname{col}(u(t, \cdot), v(t, \cdot))$, with $v = u_t$, $w_0 = \operatorname{col}(u_0, u_1) \in E$, and $\Lambda : D(\Lambda) \to E$ is the closed linear operator

$$\Lambda = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \text{ with } D(\Lambda) = E_1, \tag{2.5}$$

where R is the nonlinear operator

$$R(t, w) = \begin{bmatrix} 0 \\ -f(u) - g(v) + h(t, \cdot) \end{bmatrix}. \tag{2.6}$$

It is well known that the operator Λ generates a C_0-semigroup $e^{\Lambda t}$, $t \geq 0$, on E and this semigroup is extended to a C_0 unitary group on E.

Note that even if $h(t, x) = 0$, then R becomes autonomous, due to the cubic term in g one cannot claim that R maps E into E. On the other hand, both the semigroup $e^{\Lambda t}$ and its bounded perturbation incorporating the linear damping term αu_t are not analytic semigroups on E. Thus one is unable to establish the local existence and uniqueness of a solution to this initial value problem (2.4) simply by the standard mild solution theory.

We take the Bubnov-Galerkin approach, cf. [9] and [10], to address this issue of local solutions first.

Theorem 2.1. *Assume that for some* $\tau > 0$,

$$h \in L^\infty(0, \tau; V) \text{ and } h_t \in L^2(0, \tau; H).$$

Then for any $w_0 \in E_1$, there exists a unique solution (called a strong solution)

$$w \in L^\infty(0,\tau;E_1) \cap C([0,\tau);E) \tag{2.7}$$

of the IVP (2.4). Moreover, the first component u of w satisfies Eq. (2.1) in H for almost every $t \in (0,\tau)$, and one has

$$u_{tt} \in L^\infty(0,\tau;H). \tag{2.8}$$

PROOF. First let us show the uniqueness. If both u and z are solutions of (2.1) with the same initial data, then $\varphi = u - z$ satisfies the equation

$$\varphi_{tt} - a\Delta\varphi + f(u) - f(z) + g(u_t) - g(z_t) = 0.$$

Taking the inner product in H of the above equation with φ_t, one has

$$\begin{aligned}
0 &= \frac{1}{2}\frac{d}{dt}\left(\|\varphi_t\|^2 + a\|\nabla\varphi\|^2\right) + \int_\Omega (f(u) - f(z))(u_t - z_t)\,dx \\
&\quad + \int_\Omega (g(u_t) - g(z_t))(u_t - z_t)\,dx \\
&\geq \frac{1}{2}\frac{d}{dt}\left(\|\varphi_t\|^2 + a\|\nabla\varphi\|^2\right) + \int_\Omega f'(\xi)(u-z)(u_t - z_t)\,dx - \alpha\|\varphi_t\|^2,
\end{aligned}$$

where the scalar ξ is between $u(t,x)$ and $z(t,x)$, because

$$\beta\left(u_t^3 - z_t^3\right)(u_t - z_t) \geq \frac{3}{4}\beta u_t^2 (u_t - z_t)^2 \geq 0.$$

Hence,

$$\frac{d}{dt}\left(\|\varphi_t\|^2 + a\|\nabla\varphi\|^2\right) \leq 2\alpha\|\varphi_t\|^2 + 2K_f\|\varphi\|\,\|\varphi_t\|,$$

where

$$K_f = \sup\{|f'(\xi)| : |\xi| \leq B_{u,z}\}$$

and

$$B_{u,z} = \|u\|_{L^\infty(0,\tau;H^2(\Omega))} + \|z\|_{L^\infty(0,\tau;H^2(\Omega))}.$$

Here note that $H^2(\Omega)$ is imbedded in $C_B(\Omega)$, for $n \leq 2$. By the Poincaré inequality, there is a constant $c > 0$ such that $\|\varphi\|^2 \leq c\|\nabla\varphi\|^2$. It follows that

$$\frac{d}{dt}\left(\|\varphi_t\|^2 + a\|\nabla\varphi\|^2\right) \leq \max\left\{2\alpha + K_f, K_f\frac{c}{a}\right\}\left(\|\varphi_t\|^2 + a\|\nabla\varphi\|^2\right), \tag{2.9}$$

which implies that, by the Gronwall inequality, $\|\text{col}\,(\varphi,\varphi_t)\|_E = 0$, $t \in [0,\tau)$. Thus the uniqueness is proved.

Next we show the existence of a strong solution. Let $\{e_i\}$ be the complete set of orthonormal eigenvectors of the operator A associated with eigenvalues $\{\lambda_i\}$, where $\{\lambda_i\}$ is increasing, each repeated to its multiplicity, and $\lambda_i \to \infty$ as $i \to \infty$. For integer $m \geq 1$, let $H_m = \text{Span}\,\{e_1,\ldots,e_m\}$. Choose two sequences $\{u_{0m}\}$ and $\{u_{1m}\}$ in H_m such that

$$u_{0m} \to u_0 \text{ in } D(A) \text{ (equipped with the graph norm)},$$
$$\text{and } u_{1m} \to u_1 \text{ in } V, \text{ as } m \to \infty.$$

Let $u_m(t)$ be the solution of the following IVP in H_m:

(2.10)

$$\langle \partial_{tt} u_m(t), e_i \rangle + a \langle \nabla u_m(t), \nabla e_i \rangle + \langle f(u_m(t)), e_i \rangle + \langle g(\partial_t u_m), e_i \rangle = \langle h(t,\cdot), e_i \rangle, \quad i = 1, \ldots, m,$$

$$u_m(0) = u_{0m} \text{ and } \partial_t u_m(0) = u_{1m}.$$

In the nonlinear ODEs of the initial value problem (2.10), the nonlinear terms satisfy the local Lipschitz condition in u_m and $\partial_t u_m$. So there is a unique solution $u_m(t)$ of the IVP (2.10) on $[0, \tau_m)$, $0 < \tau_m \le \tau$. Replacing e_i by $\partial_t u_m(t)$ in (2.10), for $t \in [0, \tau_m)$, we get

$$\frac{1}{2} \frac{d}{dt} \left[\|\partial_t u_m\|^2 + a \|\nabla u_m\|^2 + 2F(u_m) \right] + \beta \int_\Omega |\partial_t u_m(t)|^4 \, dx = \langle h(t,\cdot), \partial_t u_m(t) \rangle + \alpha \|\partial_t u_m\|^2.$$

Using the first inequality in (1.3b), we can add some terms to the right hand side of the above equality to get

$$\frac{1}{2} \frac{d}{dt} \left[\|\partial_t u_m\|^2 + a \|\nabla u_m\|^2 + 2F(u_m) \right] + \beta \int_\Omega |\partial_t u_m(t)|^4 \, dx$$
$$\le \frac{1}{2} \left(\|h(t,\cdot)\|^2 + \|\partial_t u_m(t)\|^2 \right) + \alpha \|\partial_t u_m\|^2 + a \|\nabla u_m\|^2 + 2 \left(F(u_m) + C_0 \right).$$

Integrating the above inequality over $[0, t] \subset [0, \tau_m)$ and by the Gronwall inequality, we find that for any $t \in [0, \tau_m)$,

$$\|\partial_t u_m\|^2 + a \|\nabla u_m\|^2 \le C(\tau), \tag{2.11a}$$

$$\int_0^t \int_\Omega |\partial_t u_m(t)|^4 \, dx ds \le C(\tau), \tag{2.11b}$$

where $C(\tau)$ is a positive constant only depending on the data $\{u_0, u_1, h\}$ as well as τ, but it is uniform in m. This implies that $\tau_m = \tau$ and the solution u_m will not blow up on the time interval $[0, \tau)$, for all $m = 1, 2, \cdots$. Again, (2.11) holds for all $m \ge 1$ and all $t \in [0, \tau)$.

Now we conduct another *a priori* estimate by replacing e_i with $-\Delta(\partial_t u_m)$ in (2.10), it follows that

$$\begin{aligned} &\langle \nabla(\partial_{tt} u_m), \nabla(\partial_t u_m) \rangle + a \langle \Delta u_m, \Delta(\partial_t u_m) \rangle + \langle \nabla f(u_m), \nabla(\partial_t u_m) \rangle \\ &+ \langle \nabla g(\partial_t u_m), \nabla(\partial_t u_m) \rangle = \langle \nabla h(t,\cdot), \nabla(\partial_t u_m) \rangle, t \in [0, \tau). \end{aligned} \tag{2.12}$$

In (2.12) we see that

$$\langle \nabla g(\partial_t u_m), \nabla(\partial_t u_m) \rangle = 3\beta \int_\Omega |\partial_t u_m|^2 |\nabla(\partial_t u_m)|^2 \, dx - \alpha \|\nabla(\partial_t u_m)\|^2,$$

and

$$\langle \nabla f(u_m), \nabla(\partial_t u_m) \rangle = \int_\Omega f'(u_m) |\nabla(\partial_t u_m)|^2 \, dx.$$

Note that due to (2.11a) we have a constant $K_0 = K_0(u_0, u_1, h, \tau)$ such that

$$\|u_m(t)\|_{L^\infty(\Omega)} \le K_0$$

so that there exists a constant $K_1 = K_1(f)$, which is independent of $t \in [0, \tau)$ and of $m \geq 1$, with the property $|f'(u_m(t))| \leq K_1$. Hence we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla(\partial_t u_m)\|^2 + a\|\Delta u_m\|^2\right) + 3\beta\int_\Omega |\partial_t u_m|^2 |\nabla(\partial_t u_m)|^2\, dx \tag{2.13}$$
$$\leq (\alpha + K_1)\|\nabla(\partial_t u_m)\|^2 + \langle \nabla h(t,\cdot), \nabla(\partial_t u_m)\rangle, t \in [0,\tau).$$

Now applying the Cauchy-Schwarz inequality to the last term in (2.13) and noting that $h \in L^\infty(0,\tau;V)$ (which implies $\nabla h \in L^\infty(0,\tau;H)$), we can use the Gronwall inequality again to get

$$\|\nabla(\partial_t u_m)\|^2 + a\|\Delta u_m\|^2 \leq K(\tau), \tag{2.14}$$

for all $m \geq 1$ and all $t \in [0,\tau)$, where $K(\tau)$ is a positive constant only depending on $\{u_0, u_1, h, f\}$.

Combining (2.11) and (2.14), we can confirm that

$$\{u_m\}_{m=1}^\infty \subset \text{a bounded set } B_0 \subset L^\infty\left(0,\tau; H^2(\Omega)\right), \tag{2.15a}$$
$$\{\partial_t u_m\}_{m=1}^\infty \subset \text{a bounded set } B_1 \subset L^\infty\left(0,\tau; H_0^1(\Omega)\right), \tag{2.15b}$$

Here in (2.15a) we invoked a result that for a domain Ω with locally Lipschitz continuous boundary Γ, the following inequality holds,

$$\|\varphi\|_{H^2(\Omega)} \leq \text{const}\,\|\Delta\varphi\|,$$

for any $\varphi \in H_0^1(\Omega)$ such that $\Delta\varphi \in L^2(\Omega)$, cf. [9].

Next we can use a bootstrap argument to differentiate the ODEs of (2.10) in t to obtain

$$\langle \partial_{ttt} u_m(t), e_i\rangle + a\langle \nabla(\partial_t u_m(t)), \nabla e_i\rangle + \langle f'(u_m(t))\,\partial_t u_m(t), e_i\rangle \tag{2.16}$$
$$+3\beta\langle |\partial_t u_m(t)|^2\, \partial_{tt} u_m(t), e_i\rangle - \alpha\langle \partial_{tt} u_m(t), e_i\rangle = \langle \partial_t h(t,\cdot), e_i\rangle,\ i = 1,\dots,m.$$

Replacing e_i by $\partial_{tt} u_m(t)$ in (2.16), we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\partial_{tt} u_m\|^2 + a\|\nabla(\partial_t u_m)\|^2\right) + \frac{3}{4}\beta\int_\Omega \left|\frac{\partial}{\partial t}(\partial_t u_m(t))^2\right|^2 dx \tag{2.17}$$
$$\leq K_1\|\partial_t u_m(t)\|\,\|\partial_{tt} u_m(t)\| + \alpha\|\partial_{tt} u_m(t)\|^2 + \|\partial_t h(t,\cdot)\|\,\|\partial_{tt} u_m(t)\|^2.$$

By the assumption on h_t, and by the fact that $u_{0m} \to u_0$ in $D(A)$ and $u_{1m} \to u_1$ in V as $m \to \infty$ imply $\|\partial_{tt} u_m(0)\|$ and $\|\nabla(\partial_t u_m)(0)\|$ are uniformly bounded for $m \geq 1$, we can use the Cauchy-Schwarz inequality and then the Gronwall inequality to deduce from (2.17) that

$$\{\partial_{tt} u_m\}_{m=1}^\infty \subset \text{a bounded set } B_2 \subset L^\infty\left(0,\tau; L^2(\Omega)\right), \tag{2.18a}$$
$$\left\{\frac{\partial}{\partial t}|\partial_t u_m(t)|^2\right\}_{m=1}^\infty \subset \text{a bounded set } B_3 \subset L^2\left(0,\tau; L^2(\Omega)\right). \tag{2.18b}$$

From the estimates (2.11), (2.14) and the results (2.15) and (2.18), it follows that there exists a subsequence of $\{u_m\}$, which is relabled as $\{u_m\}$, such that as $m \to \infty$, one has

(2.19a) $$u_m \to u \text{ weak}^* \text{ in } L^\infty\left(0,\tau; H^2(\Omega)\cap H_0^1(\Omega)\right),$$

(2.19b) $$\partial_t u_m \to \partial_t u \text{ weak}^* \text{ in } L^\infty\left(0,\tau; H_0^1(\Omega)\right) \text{ and weakly in } L^2\left(0,\tau; H_0^1(\Omega)\right),$$

(2.19c) $$\partial_{tt} u_m \to \partial_{tt} u \text{ weak}^* \text{ in } L^\infty\left(0,\tau; L^2(\Omega)\right),$$

and

(2.20) $$|\partial_t u_m|^2 \to \psi \text{ weakly in } H^1\left(0,\tau; L^2(\Omega)\right),$$

where the limit function $u = u(t,x)$ and the vector function $w = \text{col}\,(u, \partial_t u)$ satisfy (2.7) and (2.8). Note that the continuity, $w \in C([0,\tau); E)$, follows directly from (2.19) and Lemma 1.2 of Chapter 1 in [9].

In (2.20), ψ is a function in $H^1\left(0,\tau; L^2(\Omega)\right)$. We shall prove that there exists a subsequence of $\{u_m\}$, (always relabeled as the same and we shall not repeat), such that

(1.21a) $$\partial_t u_m \to \partial_t u \text{ strongly in } L^2\left(0,\tau; L^2(\Omega)\right).$$

Thus, $\psi = |\partial_t u|^2$. Indeed, (1.21a) means

(1.21b) $$\int_0^\tau \|\partial_t u_m - \partial_t u\|^2\, dt \to 0, \text{ as } m \to \infty.$$

This can be shown by the dominated convergence theorem, since we can verify the following two conditions.

First, (2.19b) implies that for any $\eta(x) \in H_0^1(\Omega)$,

(2.22) $$\int_0^\tau \langle \partial_t u_m - \partial_t u, \eta\rangle_{H_0^1(\Omega)} dt \to 0, \text{ as } m \to \infty.$$

Since $H_0^1(\Omega)$ is separable, there is a countable dense set $\{\eta_k\}$ in it, and (2.22) holds for $\eta = \eta_k$. Via the convergence in measure, it implies that there is a null set $N \subset [0,\tau)$ and a subsubsequence of $\{u_m\}$ such that for any $t \in [0,\tau)\setminus N$,

(2.23) $$\langle \partial_t u_m - \partial_t u, \eta\rangle_{H_0^1(\Omega)} \to 0,\ \forall \eta \in H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ (for $n \le 2$), the weak convergence in (2.23) implies that there is a subsequence of $\{u_m\}$ with

$$\partial_t u_m - \partial_t u \to 0 \text{ strongly in } L^2(\Omega), \text{ as } m \to \infty.$$

Thus, in (1.21b), $\|\partial_t u_m - \partial_t u\|^2 \to 0$, for almost every $t \in [0,\tau)$.

Second, from (2.18b) and (2.20) we have

$$|\partial_t u_m(t)|^2 = |\partial_t u_m(0)|^2 + \int_0^t \frac{\partial}{\partial s}|\partial_s u_m(s)|^2\, ds,\ t \in [0,\tau),$$

so that

$$\|\partial_t u_m(t)\|^2 \le \|\partial_t u_m(0)\|^2 + \tau \left\| \frac{\partial}{\partial t}|\partial_t u_m(t)|^2 \right\|^2_{L^2(0,\tau;L^2(\omega))} \le \text{const}\,(u_1,\tau),$$

where the constant is uniform for all m. It follows that in (1.21b) $\|\partial_t u_m - \partial_t u\|^2 \le$ const. Then by the dominated convergence theorem (1.21b) and (1.21a) hold.

Finally, let $\{u_m\}$ be the chosen subsequence which satisfies (2.18)–(1.21), put them in (2.10), and take the limit as $m \to \infty$. Then we find that the limit function u satisfies the equation

$$\langle \partial_{tt} u(t) - a\Delta u(t) + f(u(t)) + g(\partial_t u(t)), e_i \rangle = \langle h(t,\cdot), e_i \rangle,\ i = 1, 2, \cdot,$$
$$u(0) = u_0, \quad \partial_t u(0) = u_1.$$

These equations hold in the space $L^\infty[0,\tau)$. Since $\{e_i\}$ is an orthonormal basis for $H^2(\Omega)$, it means that the vector function $w(t) = \text{col}\,(u, \partial_t u)$ is a strong solution of the initial value problem (2.4). The proof of Theorem 2.1 is completed. □

The next result concerns the weak solution of the IVP (2.4) under the different assumptions on the data (w_0, h).

THEOREM 2.2. *Assume that for some $\tau > 0$,*

$$h \in L^\infty(0,\tau; H). \tag{2.24}$$

Then for any $w_0 \in E$, there exists a unique solution (*called a weak solution*)

$$w \in L^\infty(0,\tau; E) \cap C([0,\tau); E_{-1}) \tag{2.25}$$

of the IVP (2.4). *The first component u of w satisfies Eq.* (2.1) *in $H^{-1}(\Omega)$ for almost every $t \in (0,\tau)$, and it holds that*

$$u_{tt} \in L^1\left(0,\tau; H^{-1}(\Omega)\right). \tag{2.26}$$

PROOF. The proof of this theorem is also based on the Bubnov-Galerkin method and is quite parallel to the proof of Theorem 2.1. We shall show some key steps only. Let $u_m(t)$ be the solution of the IVP (2.10) with the initial data (u_{0m}, u_{1m}) satisfying

$$(u_{0m}, u_{1m}) \to (u_0, u_1) \text{ in } E = V \times H, \text{ as } m \to \infty. \tag{2.27}$$

The same estimates in (2.11) remain valid and, together with (2.27), implies that

$$\{u_m\}_{m=1}^\infty \subset \text{a bounded set } S_0 \subset L^\infty(0,\tau; V), \tag{2.28a}$$

$$\{\partial_t u_m\}_{m=1}^\infty \subset \text{a bounded set } S_1 \subset L^\infty(0,\tau; H) \cap L^4\left(0,\tau; L^4(\Omega)\right). \tag{2.28b}$$

Hence, one can extract a subsequence (always relabeled as the same) $\{u_m\}$ such that as $m \to \infty$, one has

$$u_m \to u \text{ weak}^* \text{ in } L^\infty(0,\tau; V), \tag{2.29a}$$

$$\partial_t u_m \to \partial_t u \text{ weak}^* \text{ in } L^\infty(0,\tau; H), \tag{2.29b}$$

$$\partial_t u_m \to \partial_t u \text{ weakly in } L^4\left(0,\tau; L^4(\Omega)\right). \tag{2.29c}$$

Thus the limit function u and its time derivative $\partial_t u$ satisfies

$$w = \text{col}\,(u, \partial_t u) \in L^\infty(0,\tau; E).$$

Consider the $f(u_m(t))$ term in (2.10). Since $f(u_m(t)) = f(u_m(t)) - f(0) = f'(\xi_{m,t})\, u_m(t)$, and by (2.28a), we have

$$|\xi_{m,t}| \le |u_m(t,x)| \text{ (for a.e. } x) \le \|u_m(t)\|_{L^\infty(\Omega)} \le \|u_m(t)\|_V \text{ (for a.e. } t) \le C_\tau,$$

where C_τ is a constant independent of m. Thus there is a uniform constant K_τ such that $|f'(\xi_{m,t})| \le K_\tau$, a.e. Consequently,

$$\{f(u_m)\}_{m=1}^\infty \subset \text{a bounded set } S_2 \subset L^\infty(0,\tau; H).$$

However, due to the coefficient $f'(\xi_{m,t})$, one cannot claim $f(u_m)$ is uniformly bounded in $L^\infty(0,\tau;V)$. However, there exists a further subsequence of $\{u_m\}$, such that

$$f(u_m) \to \varphi \text{ weak}^* \text{ in } L^\infty(0,\tau;H), \tag{2.30}$$

so that

$$f(u_m) - f(u) \to \varphi - f(u) \text{ weak}^* \text{ in } L^\infty(0,\tau;H).$$

On the other hand, from (2.29a) and the similar boundedness of $\{\zeta_{m,t}\}$ we can get

$$f(u_m) - f(u) = f'(\zeta_{m,t})(u_m - u) \to 0 \text{ weak}^* \text{ in } L^\infty(0,\tau;H).$$

The last two convergence relations yield the result $\varphi = f(u)$.

Next consider the $g(\partial_t u_m)$ term in (2.10). In view of (2.2), (2.29b) and (2.29c), we have

$$\begin{aligned} &-\alpha\partial_t u_m \to -\alpha\partial_t u \text{ weak}^* \text{ in } L^\infty(0,\tau;H), \\ &\beta(\partial_t u_m)^3 \to \beta(\partial_t u)^3 \text{ weakly in } L^{4/3}\left(0,\tau;L^{4/3}(\Omega)\right), \end{aligned} \tag{2.31}$$

which, in turn, implies that

$$\beta(\partial_t u_m)^3 \to \beta(\partial_t u)^3 \text{ strongly in } L^1\left(0,\tau;H^{-1}(\Omega)\right),$$

because of the compact imbedding property and through possibly a further extraction of the subsequence. Thus, we get

$$\langle g(\partial_t u_m), e_i\rangle \to (g(\partial_t u), e_i)_d \text{ in } L^1(0,\tau), \text{ as } m \to \infty, \tag{2.32}$$

where $(\cdot,\cdot)_d$ stands for the dual product of an element in $H^{-1}(\Omega)$ and an element in $V = H_0^1(\Omega)$, cf. [10, Section 3.6.2].

Since the Laplacian operator $\Delta \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$, from (2.29) we can get

$$a\langle\nabla u_m, \nabla e_i\rangle \to -a(\Delta u, e_i)_d \text{ in } L^\infty(0,\tau), \text{ as } m \to \infty. \tag{2.33}$$

From (2.30), (2.32) and (2.33) it follows that

$$u_{tt} \in L^1\left(0,\tau;H^{-1}(\Omega)\right), \tag{2.34}$$

and the equalities

$$(u_{tt} - a\Delta u + f(u) + g(u_t), e_i)_d = \langle h(t,\cdot), e_i\rangle = (h(t,\cdot), e_i)_d, \quad i = 1,2,\cdots,$$

holds in $L^1(0,\tau)$. Hence we conclude that $w = \text{col}\,(u,u_t)$ satisfies Eq. (2.4) in E_{-1} almost everywhere. Again by Lemma 1.2 of [9, Chap. 1], the facts (2.29) and (2.34) imply that $w \in C([0,\tau);E_{-1})$. This result, together with (2.29) and (2.34), ensures that

$$\begin{aligned} &u_m(0) = u_{0m} \to u(0) \text{ weakly in } V, \text{ so that strongly in } H, \\ &\partial_t u_m(0) = u_{1m} \to u_t(0) \text{ weakly in } H, \text{ so that strongly in } V' = H^{-1}(\Omega). \end{aligned}$$

Thus, by (2.27) and the above facts, we have $u(0) = u_0$ and $u_t(0) = u_1$. The initial conditions of (2.4) are also satisfied. □

3. Global Existence and Boundness of Solutions

In this section, we study the global existence of strong and weak solutions and the boundedness of each individual solution. We shall use the method of a quasi-energy functional to conduct a priori estimates in the following lemma. Let us assume

$$h \in L^\infty(0,\infty;V) \text{ and } h_t \in L^2(0,\tau;H).$$

Denote the norm of $h(t,\cdot)$ in $L^\infty(0,\infty;H)$ by $\|h\|_\infty$.

Define a functional $\Psi(u,v)$ (which can be called a quasi-energy functional) by

$$\Psi(u,v) = a\|\nabla u\|^2 + \|v\|^2 + \varepsilon\langle u,v\rangle - \frac{1}{2}\varepsilon\alpha\|u\|^2 + 2F(u) \tag{3.1}$$

where $\varepsilon > 0$ is an undetermined constant, and $F(u)$ is the antiderivative of f mentioned in the set-up of Section 1.

Lemma 3.1. *For any $w_0 \in E_1$, the strong solution $w(t) = \mathrm{col}\,(u,u_t)$ of the initial value problem* (2.4) *satisfies the differential inequality,*

$$\begin{aligned}\frac{d}{dt}\Psi\,(u,u_t) + \frac{1}{4}\,\varepsilon\min\{1,(2d_1/D_1)\}\Psi\,(u,u_t) \\ \le C\beta\,\|u_t\|^3_{L^4(\Omega)}\,(\varepsilon\|\nabla u\| - 1) + K\left(1 + \|h\|^2_\infty\right), \\ \text{for } t \in I_{\max}, \text{ and } 0 < \varepsilon < \varepsilon_0,\end{aligned} \tag{3.2}$$

where $C > 0$, $K > 0$ are constants, $I_{\max}$ is the maximal interval of existence of this solution, and ε_0 is the constant given by

$$\varepsilon_0 = \min\left\{1, \frac{a\lambda_1}{1+\alpha}\right\}, \tag{3.3}$$

where λ_1 is the smallest eigenvalue of the operator $-\Delta$ with the given homogeneous Dirichlet boundary condition, and d_1, D_1 are the constants in (2.3a) *and* (2.3b).

Proof. Taking the inner-product in H of Eq. (2.1) with $2u_t + \varepsilon_u$, we find

$$\begin{aligned}&\langle u_{tt} - a\Delta u + f(u) - \alpha u_t + \beta\,(u_t)^3\,, 2u_t + \varepsilon u\rangle \\ &= \frac{d}{dt}\,\|u_t\|^2 + a\,\frac{d}{dt}\|\nabla u\|^2 + 2\frac{d}{dt}\,F(u) + 2\beta\int_\Omega |u_t|^4\,dx - 2\alpha\,\|u_t\|^2 + \varepsilon\frac{d}{dt}\langle u_t,u\rangle \\ &-\varepsilon\,\|u_t\|^2 + \varepsilon a\|\nabla u\|^2 + \varepsilon\langle f(u),u\rangle + \varepsilon\beta\int_\Omega (u_t)^3\,u\,dx - \frac{1}{2}\varepsilon\alpha\frac{d}{dt}\|u\|^2 = \langle h, 2u_t + \varepsilon\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d}{dt}\Psi\,(u,u_t) = \frac{d}{dt}\left[\|u_t\|^2 + a\|\nabla u\|^2 + \varepsilon\langle u_t,u\rangle - \frac{1}{2}\,\varepsilon\alpha\|u\|^2 + 2F(u)\right] \\ + \left\{2\beta\int_\Omega |u_t|^4\,dx - (2\alpha+\varepsilon)\,\|u_t\|^2 + \varepsilon a\|\nabla u\|^2 + \varepsilon\beta\int_\Omega (u_t)^3\,u\,dx + \varepsilon\langle f(u),u\rangle\right\} \\ = \langle h, 2u_t + \varepsilon\rangle, \text{ for } I_{\max}.\end{aligned} \tag{3.4}$$

By the assumptions in (2.3) and the fact $0 < \varepsilon < \varepsilon_0$, with ε_0 given in (3.3), we have

$$\begin{aligned}
\Psi(u,u_t) &= \|u_t\|^2 + a\|\nabla u\|^2 + \varepsilon\langle u_t, u\rangle - \frac{1}{2}\varepsilon\alpha\|u\|^2 + 2F(u) \\
&\geq \|u_t\|^2 + a\|\nabla u\|^2 - \frac{1}{2}\varepsilon\left(\|u_t\|^2 + \|u\|^2\right) - \frac{1}{2}\varepsilon\alpha\|u\|^2 - 2C_0 \\
&\geq a\|\nabla u\|^2 - \frac{1}{2}\varepsilon(1+\alpha)\|u\|^2 + \left(1 - \frac{1}{2}\varepsilon\right)\|u_t\|^2 - 2C_0 \\
&\geq \frac{1}{2}\left(\|u_t\|^2 + a\|\nabla u\|^2\right) + \frac{1}{2}\left(a\lambda_1 - \varepsilon(1+\alpha)\right)\|u\|^2 + \frac{1}{2}(1-\varepsilon)\|u_t\|^2 - 2C_0 \\
&\geq \frac{1}{2}\left(\|u_t\|^2 + a\|\nabla u\|^2\right) - 2C_0
\end{aligned} \tag{3.5}$$

and, on the other hand,

$$\begin{aligned}
\Psi(u,u_t) &\leq \|u_t\|^2 + a\|\nabla u\|^2 + \frac{1}{2}\varepsilon\left(\|u_t\|^2 + \|u\|^2\right) + 2D_1\|u\|_{2p}^{2p} + 2D_0 \\
&\leq 2\left(\|u_t\|^2 + a\|\nabla u\|^2 + D_1\|u\|_{2p}^{2p} + D_0\right).
\end{aligned} \tag{3.6}$$

Now we treat the terms in the brace of (3.4). By the Hölder inequality, we get

$$\left|\varepsilon\beta\int_\Omega (u_t)^3\, u\, dt\right| \leq \varepsilon\beta\,\|u_t\|_{L^4(\Omega)}^3\,\|u\|_{L^4(\Omega)} \leq \varepsilon\beta C\,\|u_t\|_{L^4(\Omega)}^3\,\|\nabla u\|,$$

where $C > 0$ is a uniform constant due to the imbedding $H_0^1(\Omega)$ in $L^4(\Omega)$, for $n \leq 2$. We have

$$|\langle h, 2u_t + \varepsilon\rangle| \leq \alpha\,\|u_t\|^2 + \frac{1}{4}\varepsilon\alpha\lambda_1\|u\|^2 + \left(\alpha^{-1} + \frac{\varepsilon}{a\lambda_1}\right)\|h\|_\infty^2.$$

Substituting the above two inequalities into (3.4), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Psi(u,u_t) + \Big\{2\beta\,\|u_t\|_{L^4(\Omega)}^4 - (3\alpha+\varepsilon)\,\|u_t\|^2 \\
&\qquad + \varepsilon a\|\nabla u\|^2 - \frac{1}{4}\varepsilon a\lambda_1\|u\|^2 + \varepsilon d_1\|u\|_{L^{2p}(\Omega)}^{2p}\Big\} \\
&\leq \varepsilon\beta C\,\|u_t\|_{L^4(\Omega)}^3\,\|\nabla u\| + \left(\alpha^{-1} + \frac{\varepsilon}{a\lambda_1}\right)\|h\|_\infty^2 + d_0,
\end{aligned}$$

where we have used the inequality $\frac{1}{4}\varepsilon a\lambda_1\|u\|^2 \leq \frac{1}{2}\varepsilon a\|\nabla u\|^2$. Then it follows that

$$\begin{aligned}
&\frac{d}{dt}\Psi(u,u_t) + \left\{2\beta\,\|u_t\|_{L^4(\Omega)}^4 - (3\alpha+\varepsilon)\,\|u_t\|^2 + \frac{1}{2}\varepsilon a\|\nabla u\|^2 + \varepsilon d_1\|u\|_{L^{2p}(\Omega)}^{2p}\right\} \\
&\leq \varepsilon\beta C\,\|u_t\|_{L^4(\Omega)}^3\,\|\nabla u\| + \left(\alpha^{-1} + \frac{\varepsilon}{a\lambda_1}\right)\|h\|_\infty^2 + d_0.
\end{aligned} \tag{3.7}$$

By using the Young inequality, we find

$$C\,\|u_t\|_{L^4(\Omega)}^3 = C\left(\int_\Omega |u_t|^4\, dx\right)^{3/4} \leq \|u_t\|_{L^4(\Omega)}^4 + C^*,$$

with $C^* = \left(3^3/4^4\right) C^3$. Applying this inequality to the first term in the brace of (3.7), we get

$$\begin{aligned}
&\frac{d}{dt}\Psi(u,u_t) + \left(\beta C \|u_t\|^3_{L^4(\Omega)} - \beta C^*\right) + \Big\{\beta \|u_t\|^4_{L^4(\Omega)} - 3(\alpha+\varepsilon)\|u_t\|^2 \\
&\qquad + 2\varepsilon \|u_t\|^2 + \frac{1}{2}\varepsilon a \|\nabla u\|^2 + \varepsilon d_1 \|u\|^{2p}_{L^{2p}(\Omega)}\Big\} \\
&\le \varepsilon\beta C \|u_t\|^3_{L^4(\Omega)} \|\nabla u\| + \left(\alpha^{-1} + \frac{\varepsilon}{a\lambda_1}\right)\|h\|^2_\infty + d_0.
\end{aligned} \tag{3.8}$$

Moreover, since $\|u_t\|_{L^4(\Omega)} \ge |\Omega|^{-1/4}\|u_t\|$, we have

$$\begin{aligned}
&\beta \|u_t\|^4_{L^4(\Omega)} - 3(\alpha+\varepsilon)\|u_t\|^2 \ge \beta|\Omega|^{-1}\|u_t\|^4 - 3(\alpha+\varepsilon)\|u_t\|^2 \\
&= \beta|\Omega|^{-1}\left[\|u_t\|^2 - \frac{3(\alpha+\varepsilon)}{2\beta|\Omega|^{-1}}\right]^2 - \frac{9(\alpha+\varepsilon)^2|\Omega|}{4\beta} \ge -\frac{9(\alpha+\varepsilon)^2|\Omega|}{4\beta}.
\end{aligned} \tag{3.9}$$

Besides, by (3.6), we have

$$\begin{aligned}
&2\varepsilon\|u_t\|^2 + \frac{1}{2}\varepsilon a\|\nabla u\|^2 + \varepsilon d_1\|u\|^{2p}_{L^{2p}(\Omega)} \\
&\ge \frac{1}{2}\varepsilon\left[4\|u_t\|^2 + a\|\nabla u\|^2 + (2d_1/D_1)\left(D_1\|u\|^{2p}_{L^{2p}(\Omega)} + D_0\right)\right] - \frac{\varepsilon d_1 D_0}{D_1} \\
&\ge \frac{1}{4}\varepsilon\min\{1\,(2d_1/D_1)\}\Psi(u,u_t) - \frac{\varepsilon d_1 D_0}{D_1}.
\end{aligned} \tag{3.10}$$

Now substitute (3.9) and (3.10) into (3.8) to obtain

$$\begin{aligned}
&\frac{d}{dt}\Psi(u,u_t) + \frac{1}{4}\varepsilon\min\{1,(2d_1/D_1)\}\Psi(u,u_t) \le \beta C\|u_t\|^3_{L^4(\Omega)}(\varepsilon\|\nabla u\| - 1) \\
&\quad + \left(\alpha^{-1} + \frac{\varepsilon}{a\lambda_1}\right)\|h\|^2_\infty + \beta C^* + \frac{9(\alpha+\varepsilon)^2|\Omega|}{4\beta} + \frac{\varepsilon d_1 D_0}{D_1} + d_0.
\end{aligned} \tag{3.11}$$

Let K be the uniform constant given by

$$K = \max\left\{\beta C^* + \frac{9(\alpha+\varepsilon_0)^2|\Omega|}{4\beta} + \frac{\varepsilon_0 d_1 D_0}{D_1} + d_0, \alpha^{-1} + \frac{\varepsilon_0}{a\lambda_1}\right\}.$$

Then the last differential inequality (3.11) can be written as

$$\begin{aligned}
&\frac{d}{dt}\Psi(u,u_t) + \frac{1}{4}\varepsilon\min\{1,(2d_1/D_1)\}\Psi(u,u_t) \\
&\le C\beta\|u_t\|^3_{L^4(\Omega)}(\varepsilon\|\nabla u\| - 1) + K\left(1 + \|h\|^2_\infty\right),
\end{aligned}$$

for $t \in I_{\max}$ and for any $0 < \varepsilon < \varepsilon_0$. It is exactly (3.2). Thus, the proof of this lemma is completed. □

The next result confirms the global existence and boundedness of every strong solution in the energy space E and on the time interval $[0,\infty)$. Let

$$\varepsilon^* = \frac{1}{4}\varepsilon\min\{1,(2d_1/D_1)\}, \tag{3.12}$$

which is the coefficient of the second term on the right-hand side of the differential inequality (3.2).

THEOREM 3.2. *Assume that* $h \in L^\infty(0,\infty;V)$ *and* $h_t \in L^2(0,\tau;H)$. *Then for any* $w_0 \in E_1$, *there exists a unique strong solution* $w(t) = w(t, w_0)$ *of the initial value problem* (2.4) *for* $t \in [0,\infty)$. *It satisfies*

$$\limsup_{t\to\infty} \|w(t)\|_E^2 \le \frac{2}{\varepsilon^*} K_1 \left(1 + \|h\|_\infty^2\right), \tag{3.13}$$

provided that $0 < \varepsilon < \varepsilon_0$, *with* ε_0 *given in* (3.3), *and that the initial data* $w_0 = \mathrm{col}\,(u_0, u_1)$ *satisfies*

$$\begin{aligned} &a\|\nabla u_0\|^2 + \|u_1\|^2 + D_1 \|u_0\|_{2p}^{2p} + D_0 + C_0 \\ &\qquad \le \frac{a}{4\varepsilon^2} - \frac{1}{2\varepsilon^*} K_1 \left(1 + \|h\|_\infty^2\right), \end{aligned} \tag{3.14}$$

where $K_1 = K + C_0$ *and* K *is the constant in* (3.2), C_0 *is the constant in* (2.3).

PROOF. Define another functional $\Phi(u,v)$ by

$$\Phi(u,v) = \Psi(u,v) + 2C_0. \tag{3.15}$$

Then from (3.5) and (3.6), we have

$$\frac{1}{2}\left(a\|\nabla u\|^2 + \|u_t\|^2\right) \le \Phi(u,u_t) \le 2\left(\|u_t\|^2 + a\|\nabla u\|^2 + D_1\|u\|_{2p}^{2p} + D_0 + C_0\right). \tag{3.16}$$

If the initial data $w_0 = \mathrm{col}\,(u_0,u_1)$ satisfy (3.14), then (3.16) shows

$$\Phi(u_0,u_1) \le \frac{a}{2\varepsilon^2} - \frac{1}{\varepsilon^*} K_1 \left(1 + \|h\|_\infty^2\right). \tag{3.17}$$

We are going to prove that the unique strong solution $w(t) = \mathrm{col}\,(u,u_t)$ starting from such an initial point $w_0 = \mathrm{col}\,(u_0,u_1)$ satisfies the inequality:

$$\Phi(u(t),u_t(t)) \le \frac{a}{2\varepsilon^2}, \text{ for any } t \in I_{\max}. \tag{3.18}$$

Suppose (3.18) is not true. then there is a time t_0, such that

$$t_0 = \inf\left\{t \in I_{\max} : \Psi(u(t),u_t(t)) > \frac{a}{2\varepsilon^2}\right\},$$

and we can claim $0 < t_0 < \infty$ because of (3.17) and the continuity of the strong solution in t with respect to the E-norm, as shown in Theorem 2.1. Thus we get

$$\Phi(u(t),u_t(t)) \le \frac{a}{2\varepsilon^2}, \text{ for } 0 \le t < t_0, \tag{3.19a}$$

and

$$\Phi(u(t),u_t(t)) = \frac{a}{2\varepsilon^2}. \tag{3.19b}$$

From (3.16) and (3.19a), it follows that

$$\varepsilon\|\nabla u\| \le \varepsilon\sqrt{2a^{-1}\Phi(u,u_t)} \le 1, \text{ for } 0 \le t \le t_0.$$

Then by (3.2) in Lemma 3.1, we find

$$\frac{d}{dt}\Phi(u,u_t) + \varepsilon^*\Phi(u,u_t) \le K_1\left(1 + \|h\|_\infty^2\right), \text{ for } 0 \le t \le t_0,$$

because $C\beta\|u_t\|_{L^4(\Omega)}^3 (\varepsilon\|\nabla u\| - 1) \le 0$, for any strong solution with the initial status satisfying (3.17). Integrating the above differential inequality, we obtain

$$\Phi(u(t),u_t(t)) \le \exp(-\varepsilon^* t)\,\Phi(u_0,u_1) + \frac{1}{\varepsilon^*}K_1\left(1 + \|h\|_\infty^2\right), \text{ for } 0 \le t \le t_0. \tag{3.20}$$

In particular, (3.20) and (3.17) yield that, at the point $t = t_0$,

(3.21)

$$\Phi\left(u\left(t_0\right), u_t\left(t_0\right)\right) \leq \exp\left(-\varepsilon^* t_0\right)\left[\frac{a}{2\varepsilon^2} - \frac{1}{\varepsilon^*} K_1\left(1 + \|h\|_\infty^2\right)\right] + \frac{1}{\varepsilon^*} K_1\left(1 + \|h\|_\infty^2\right) < \frac{a}{2\varepsilon^2}.$$

The contradiction of (3.19b) versus (3.21) shows that such a time t_0 does not exist. Hence the inequality (3.18) holds which, together with (3.16), in turn implies that

$$\|w(t)\|_E^2 \leq 2\Phi\left(u(t), u_t(t)\right) \leq \frac{a}{\varepsilon^2}, \text{ for any } t \in I_{\max}. \tag{3.22}$$

Therefore, for any initial data w_0 that satisfies (3.14), we have $I_{\max} = [0, \infty)$. It means that the strong solution $w(t)$ of (2.4) exists in E globally for $t \in [0, \infty)$. Consequently, the inequality (3.20) holds for all $t \in [0, \infty)$. Letting $t \to \infty$, from (3.20) and the first inequality of (3.22) we see that (3.13) is valid. □

THEOREM 3.3. *Assume that $h \in H$ is time-invariant. Then for any $w_0 \in E$, the unique weak solution $w(t) = w\left(t, w_0\right)$ of the initial value problem* (2.4) *exists globally for $t \in [0, \infty)$, and the solution $w(t)$ is bounded in E. Moreover, $\{S(t) : E \to E\}_{t \geq 0}$ defined by*

$$S(t) w_0 = w\left(t, w_0\right), \quad t \geq 0, \tag{3.23}$$

is a semiflow (*or called a solution semigroup*) *on the energy space E.*

PROOF. For any given initial point $w_0 \in E_1$ and under the assumption $h \in V$, one can always find a sufficiently small $\varepsilon = \varepsilon\left(w_0\right) > 0$ such that the condition (3.14) is satisfied, where $\varepsilon^* = \varepsilon^*(\varepsilon)$ is accordingly defined by (3.12). Then the result (3.13) in Theorem 3.2 applies to the strong solution $w(t)$ starting from this initial point w_0. Thus the global existence and E-boundedness are proved for each strong solution of the IVP (2.4).

Then this result can be extended to the weak solutions under the assumption $h \in H$. Indeed, for any $w_0 \in E$, the weak solution $w(t) = w\left(t, w_0\right)$ satisfies $w \in L^\infty\left(I_{\max}; E\right)$. Thus similar to the argument toward the inequality (2.9), we can show that for any two initial data $w_{10}, w_{20} \in E$ and two different inputs $h_1, h_2 \in H$, the corresponding weak solutions $w_1(t) = w\left(t, w_{10}, h_1\right)$ and $w_2(t) = w\left(t, w_{20}, h_2\right)$ satisfy the following inequality,

$$\left\|w_1(t) - w_2(t)\right\|_E^2 \leq \left(\left\|w_{10} - w_{20}\right\|_E^2 + t\left\|h_1 - h_2\right\|^2\right) e^{K(f,\alpha) t}, \tag{3.24}$$

for $t \in I_{\max}\left(w_1\right) \cap I_{\max}\left(w_2\right)$, where $K(f, \alpha)$ is a constant depends on f and α. Note that E_1 is dense in E and that V is dense in H. For any $w_0 \in E$, there is an $\varepsilon\left(w_0\right) > 0$ such that the condition (3.14) is satisfied, then through approximation by a sequence of strong solutions $\{w\left(t, w_{0k}, h_k\right)\}$, with $w_{0k} \to w_0$ in E and $h_k \to h$ in H, by using (3.24) we can confirm that the weak solution $w(t) = w\left(t, w_0, h\right)$ also satisfies (3.22), so that the weak solution $w(t)$ exists globally in E for $t \in [0, \infty)$. Consequently, (3.13) holds and each weak solution is bounded in E.

The inequality (3.24) also yields the continuous dependence of a weak solution $w(t)$ on the initial data w_0 with respect to the E-norm, for each given $t \geq 0$. Hence the operators $S(t)$, $t \geq 0$, in (3.23) are well-defined, and $\{S(t), t \geq 0\}$ is a semiflow on E. □

As a remark, we assume that $h \in H$ is time-invariant just in order to make Eq. (2.4) autonomous so that $S(t)$, $t \geq 0$, becomes a semiflow. Otherwise, if

$h \in L^\infty(0,\infty;H)$, the weak solution for any initial data w_0 still exists globally and is bounded, except that the equation (2.4) becomes nonautonomous.

4. Absorbing Sets

Here we shall further explore the dissipativity of the semiflow generated by the weak solutions of the nonlinear evolutionary equation (2.4) formulated from the original nonlinear wave equation (2.1). The specific objective of this section is to prove the existence of absorbing sets. For the definition of absorbing sets and relevant dissipativity, we refer to [7], [10] and [11].

Let us first identify the difficulty and the feature in an attempt to show the absorbing property for the solutions of this nonlinear evolutionary equation. In view of Theorems 3.2 and 3.3, the asymptotic bound in (3.13) for each solution is not uniform for all solutions starting from anywhere in E. In fact, it imposes a restriction that in order to make that asymptotic bound (3.13) valid, the initial data $w_0 = \text{col}\,(u_0, u_1)$ must satisfy the condition (3.14).

We see that both the bound of w_0 (the right side of (3.14)) and the asymptotic bound of the solution $w(t)$ (the right side of (3.13)) involve $\varepsilon \in (0, \varepsilon_0)$ in a reciprocal manner. Thus, in general, when the norm $\|w_0\|_E$ gets larger, correspondingly $\varepsilon > 0$ gets smaller so that the asymptotic bound will increase to infinity. This raises a substantial difficulty to show that there is a uniformly bounded set B in E such that the trajectory of every weak solution will eventually enter into that bounded set, which is called an absorbing set, if it exists. Actually this difficulty stems from the non-monotone, cubic damping term in the original wave equation (2.1).

The work of this section features an effort to overcome this difficulty. The key to this effort is an observation that the upper bound in condition (3.14) and the asymptotic bound in (3.13) are of different orders: the former has the order ε^{-2}, while the latter has the order ε^{-1}. This observation hints that some kind of asymptotic compression exists, which enables us to show the existence of absorbing sets by the **asymptotical bootstrap** method.

Theorem 4.1. *Assume that $h \in L^\infty(0,\infty;H)$. Then there exists a fixed bounded set B in the space E, such that for any $w_0 \in E$, there is a finite time $T = T(w_0) > 0$ and the corresponding weak solution $w(t) = w(t; w_0)$ satisfies*

$$\{w(t) : t > T(w_0)\} \subset B. \tag{4.1}$$

Therefore, if $h \in H$, the solution semiflow $\{S(t)\}_{t\geq 0}$ in the space E is dissipative. Then the set B is an absorbing set.

Proof. For any given initial point $w_0 = \text{col}\,(u_0, u_1) \in E$, let

$$r_0 = \Phi(u_0, u_1),$$

where $\Phi(u,v)$ is the functional defined in (3.15) with (3.1). For such a given w_0, there is an $\varepsilon > 0$ with which the inequality (3.17) is satisfied, where ε^* is the constant given in (3.12) depending on ε. Indeed, (3.17) can be equivalently written as

$$2r_0\varepsilon^2 + 8\left[\min\{1, (2d_1/D_1)\}\right]^{-1} K_1\left(1 + \|h\|_\infty^2\right)\varepsilon - a \leq 0. \tag{3.2a}$$

Let

$$M = M(h) = 4\left[\min\{1, (2d_1/D_1)\}\right]^{-1} K_1\left(1 + \|h\|_\infty^2\right).$$

Then (3.2a) is written as

(3.2b) $$2r_0\varepsilon^2 + 2M(h)\varepsilon - a \le 0.$$

Note that the quadratic polynomial $Q(\varepsilon)$ on the left-hand side of (3.2b) has the discriminant

$$\Delta(h) = 4M(h)^2 + 8ar_0 > 0,$$

and $Q(\varepsilon) = 0$ has two real roots:

$$\varepsilon_1, \varepsilon_2 = \frac{1}{2r_0}\left(-M(h) \pm \sqrt{M(h)^2 + 2ar_0}\right).$$

Therefore, if we choose

(4.3) $$\varepsilon = \varepsilon(r_0) = \min\left\{\varepsilon_0, \frac{1}{2r_0}\left(-M(h) + \sqrt{M(h)^2 + 2ar_0}\right)\right\},$$

where ε_0 is the constant in (3.3), then the inequality (3.2a) (written as (3.2b)) is satisfied.

In view of the argument from (3.17), which is (3.2) here, to (3.18) in the proof of Theorem 3.2, the choice of ε in (4.3) implies that

$$\Phi(u(t), u_t(t)) \le \frac{a}{2\varepsilon(r_0)^2}, \text{ for } t \ge 0,$$

which yields with (3.16) that

(4.4) $$\varepsilon(r_0)\|\nabla u(t)\| \le 1, \text{ for } t \ge 0.$$

Substituting (4.4) into the different inequality (3.2) and observing (3.15), we obtain

(4.5) $$\Phi(u(t), u_t(t)) \le \exp(-\varepsilon^*(r_0)t)r_0 + \frac{1}{\varepsilon^*(r_0)}K_1\left(1 + \|h\|_\infty^2\right), \text{ for } t \ge 0,$$

via integration, where

$$\varepsilon^*(r_0) = \frac{1}{4}\varepsilon(r_0)\min\{1, (2d_1/D_1)\}.$$

Hence, the weak solution $w(t) = w(t; w_0)$ satisfies

(4.6)

$$\limsup_{t\to\infty}\Phi(u(t), u_t(t)) \le \frac{1}{\varepsilon^*(r_0)}K_1\left(1 + \|h\|_\infty^2\right) = \frac{M}{\varepsilon(r_0)}$$

$$= M\max\left\{\frac{1}{\varepsilon_0}, \frac{2r_0}{-M + \sqrt{M^2 + 2ar_0}}\right\} = M\max\left\{\frac{1}{\varepsilon_0}, \frac{1}{a}\left(\sqrt{M^2 + 2ar_0} + M\right)\right\}$$

$$< M\max\left\{\frac{2}{\varepsilon_0}, \frac{1}{a}\left(2M + \sqrt{2ar_0}\right)\right\},$$

where $M = M(h)$ is shown between (3.2a) and (3.2b).

Thus there exists a time $t_1 = t_1(r_0) > 0$, such that

(4.7) $$\Phi(u, u_t) \le M\max\left\{\frac{2}{\varepsilon_0}, \frac{1}{a}\left(2M + \sqrt{2ar_0}\right)\right\}, \text{ for } t \ge t_1.$$

We can regard $w(t_1) = \text{col}\,(u(t_1), u_t(t_1))$ as a new starting point of this solution trajectory in E, and let

$$r_1 = \Phi(u(t_1), u_t(t_1)).$$

Then iterate the above argument for this solution on the time interval $[t_1, \infty)$: we can assert that there exists a time $t_2 = t_2(r_0) > t_1$, such that

$$\Phi(u, u_t) \le M \max\left\{\frac{2}{\varepsilon_0}, \frac{1}{a}\left(2M + \sqrt{2ar_1}\right)\right\}, \text{ for } t \ge t_2.$$

Keep this process going. Recurrently we obtain a positive sequence $\{r_n\}$ and an increasing time sequence $\{t_n\}$, such that

$$r_{n+1} \le M \max\left\{\frac{2}{\varepsilon_0}, \frac{1}{a}\left(2M + \sqrt{2ar_n}\right)\right\}, \quad n = 0, 1, 2, \ldots, \tag{4.8}$$

and

$$\Phi(u, u_t) \le M \max\left\{\frac{2}{\varepsilon_0}, \frac{1}{a}\left(2M + \sqrt{2ar_n}\right)\right\}, \text{ for } t \ge t_{n+1}, \quad n = 0, 1, 2, \ldots. \tag{4.9}$$

Next we prove that the sequence $\{r_n\}$ has a finite positive upper bound. Without loss of generality, let us assume that there is a positive integer $n_0 = n_0(r_0) > 0$ such that

$$\frac{2}{\epsilon_0} \le \frac{1}{a}\left(2M + \sqrt{2ar_n}\right), \text{ for } n \ge n_0, \tag{4.10}$$

since otherwise the subsequent proof can be made even easier. By (4.10) we have

$$r_{n+1} \le \frac{M}{a}\left(2M + \sqrt{2ar_n}\right), \text{ for } n \ge n_0, \tag{3.11a}$$

Let b_0 and b_1 be two constants: $b_0 = 2M^2/a$ and $b_1 = M\sqrt{2/a}$. Then (3.11a) is written as

$$r_{n+1} \le b_0 + b_1\sqrt{r_n}, \text{ for } n \ge n_0, \tag{3.11b}$$

By Lemma 4.2 to be shown after this theorem, we know that there is a uniform constant R, $0 < R < \infty$, and there is an integer $n_1 \ge n_0$, such that

$$r_n \le R, \text{ for } n \ge n_1. \tag{4.12}$$

Therefore, the above bootstrap argument shows that

$$\|w(t)\|_E^2 = a\|\nabla u(t)\|^2 + \|u_t(t)\|^2 \le 2\Phi(u, u_t) \le 2\max\left\{\frac{2M}{\varepsilon_0}, b_0 + b_1\sqrt{R}\right\}, \quad \text{for } t > t_{n_1+1} \tag{4.13}$$

due to (3.16), (4.9) and (4.12). Thus we have proved that the closed ball

$$B = B(R_0) = \{\varphi \in E : \|\varphi\|_E \le R_0\} \tag{4.14}$$

with the radius

$$R_0 = \left[2\max\left\{2M/\varepsilon_0, b_0 + b_1\sqrt{R}\right\}\right]^{1/2} \tag{4.15}$$

is an absorbing set for the semiflow $\{S(t)\}_{t\ge 0}$ generated by the weak solutions, and (4.1) is proved with $B = B(R_0)$ given in (4.14) and (4.15) and with

$$T(w_0) = t_{n_1+1}, \text{ where } n_1 \text{ depends on } r_0 = \Phi(u_0, u_1) \text{ and } h.$$

The proof of this theorem is completed. □

Now we prove a technical lemma, which was utilized in the proof of the main result Theorem 4.1, when we reached (4.12).

LEMMA 4.2. *Let $\{r_n\}$ be a positive sequence which satisfies*

$$r_{n+1} \le b_0 + b_1\sqrt{r_n}, \text{ for } n \ge n_0,$$

where n_0 is a given positive integer, $b_0 = 2M^2/a$ and $b_1 = M\sqrt{2/a}$ are constants. Then there exists a uniform constant $R = R(h)$, $0 < R < \infty$, and an integer $n_1 = n_1(w_0, h) \ge n_0$, such that

$$r_n \le R, \text{ for } n \ge n_1.$$

PROOF. Define $\{z_n\}$ to be the following sequence,

$$\begin{aligned} z_n &= r_n, \text{ for } 1 \le n \le n_0, \\ z_{n+1} &= b_0 + b_1\sqrt{z_n}, \text{ for } n \ge n_0. \end{aligned} \tag{4.16}$$

By induction, it is easy to see that

$$r_n \le z_n, \text{ for } n \ge n_0. \tag{4.17}$$

Note that $z_n \ge b_0$, for any $n \ge n_0 + 1$. It is deduced that

$$\begin{aligned} |z_{n+3} - z_{n+2}| &= b_1 \left|\sqrt{z_{n+2}} - \sqrt{z_{n+1}}\right| \\ &= \frac{b_1 |z_{n+2} - z_{n+1}|}{\left(b_0 + b_1\sqrt{z_{n+1}}\right)^{1/2} + \left(b_0 + b_1\sqrt{z_n}\right)^{1/2}} \\ &\le \frac{b_1 |z_{n+2} - z_{n+1}|}{2\left(b_0 + b_1\sqrt{b_0}\right)^{1/2}}, \text{ for } n > n_0. \end{aligned} \tag{4.18}$$

Moreover, we have

$$\frac{b_1}{\left(b_0 + b_1\sqrt{b_0}\right)^{1/2}} = \frac{b_1/\sqrt{b_0}}{\left(1 + b_1/\sqrt{b_0}\right)^{1/2}} = \frac{1}{\sqrt{2}},$$

because

$$\frac{b_1}{\sqrt{b_0}} = \frac{M\sqrt{2/a}}{\sqrt{2M^2/a}} = 1.$$

Then from (4.18) it follows that

$$|z_{n+3} - z_{n+2}| \le \frac{1}{2\sqrt{2}} |z_{n+2} - z_{n+1}|, \text{ for } n > n_0.$$

This implies $\{z_n : n \ge n_0\}$ is a Cauchy sequence. Hence it is a convergent sequence, whose limit can be calculated from (4.16), i.e.,

$$\lim_{n\to\infty} z_n = \frac{1}{4}\left(b_1 + \sqrt{|b_1|^2 + 4b_0}\right)^2. \tag{4.19}$$

Finally, let

$$R = \frac{1}{4}\left(b_1 + \sqrt{|b_1|^2 + 4b_0}\right)^2 + 1. \tag{4.20}$$

Then there exists an integer $n_1 \ge n_0$, which depends on R, such that

$$r_n \le z_n \le R, \text{ for } n \ge n_1. \tag{4.21}$$

Note that: (i) The constant R in (4.20) depends on $M(h)$ only. The constant R is independent of the initial data w_0 in E. (ii) The integer n_1 depends on both n_0

and R. Then, in view of (4.10), n_1 depends on $r_0 = \Phi(u_0, u_1)$ and h. Eventually, we find that n_1 depends on w_0 and h. Thus, we finished the proof of this lemma as well as the proof of the main result, Theorem 4.1. □

Since there exists an absorbing set $B = B(R_0) \subset E$ for the solution semiflow $S(t)$ on the Hilbert space E, which attracts every point $w_0 \in E$, the semiflow $S(t)$ on E is point dissipative.

5. Global Weak Attractor

In this section, we shall prove a generic theorem on the existence of a global weak attractor, which is then applied to the semiflow $S(t)$ generated by the weak solutions in E.

DEFINITION. Let Y be a Banach space. A global weak attractor $\mathbf{G}$ for a semiflow $\sigma(t)$, $t \geq 0$, defined on Y is such a subset in Y which satisfies the following three conditions:

(i) $\mathbf{G}$ is a nonempty, weakly compact, and invariant set;
(ii) $\mathbf{G}$ attracts a bounded neighborhood U ($\subset Y$) in the weak topology; and
(iii) $\mathbf{G}$ attracts every point $y_0 \in Y$ in the weak topology.

THEOREM 5.1. *Let Y be a reflexive, separable Banach space. If a semiflow $\sigma(t)$, $t \geq 0$, is point dissipative, then there exists a global weak attractor $\mathbf{G}$, which is given by*

$$\mathbf{G} = \bigcap_{t\geq 0} Cl_Y^w \bigcup_{\tau \geq t} \sigma(\tau)B, \tag{5.1}$$

where B is a closed absorbing (ball) set, and Cl_Y^w stands for the closure with respect to the weak topology of Y.

PROOF. The proof of this theorem will go through the following three lemmas. □

LEMMA 5.2. *In a reflexive, separable Banach space Y, the weak closure $B^* = Cl_Y^w(B)$ of a closed, bounded ball B is metrizable. Let this weak metric be d_Y, then the metric space (B^*, d_Y) is complete and the set B is d_Y-precompact.*

PROOF. First we show that a bounded ball $B = \{x \in Y : \|x\| \leq M\}$ is metrizable. Since Y is reflexive and separable if and only if its dual space Y' is reflexive and separable, there is a countable and dense set of Y'. Let $\{\varphi_n\}$ be a countable set of Y', which is dense in the closed unit ball of Y'. Define d_Y to be

$$d_Y(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\varphi_n(x - y)|, \quad x, y \in Y. \tag{5.2}$$

It is clear that this d_Y is a metric on Y. We now show that the topology associated with this metric coincides with the (relative) weak topology of the ball B.

Let $x_0 \in B$ and let U be a neighborhood of x_0 in the weak topology $\sigma(Y, Y')$. Without loss of generality, we can assume that U is defined by

$$U = \{x \in B : |\xi_i(x - x_0)| < \varepsilon_i,\ i = 1, 2, \ldots, k\}, \tag{5.3}$$

where $\varepsilon_i > 0$, $\xi_i \in Y'$, $\|\xi_i\| = 1$, and k is some positive integer. For any $i \in \{1, 2, \ldots, k\}$, one can find an integer n_i such that

$$\|\xi_i - \varphi_{n_i}\|_{Y'} < \frac{\varepsilon_i}{4M}.$$

Choose and fix an $r > 0$, which satisfies

$$2^{n_i+1} r < \varepsilon_i, \text{ for all } i = 1, 2, \ldots, k,$$

then we can verify that

$$W = \{x \in B : d_Y(x, x_0) < r\} \subset U.$$

Conversely, it can be shown that for any neighborhood $W = \{x \in B : d_Y(x, x_0) < \rho\}$, there is a neighborhood U of x_0 in the weak topology of B, such that $U \subset W$. The detail is omitted. Therefore, the topology induced by the metric d_Y is exactly the same as the weak topology of the closed, bounded ball B.

Finally, by the continuous extension, this metric d_Y can be extended to the weak closure set of B, which has been denoted by $B^* = Cl_Y^w(B)$ in the description of this lemma. Thus (B^*, d_Y) becomes a complete metric space. Moreover, since B is bounded in the strong sense, so it must be weakly precompact, which is equivalent to saying that B is d_Y-precompact. □

LEMMA 5.3. *Let $\sigma(t)$, $t \geq 0$, be a semiflow on a complete metric space (X, d). If $B \subset X$ is a precompact, absorbing set for $\sigma(t)$ with respect to this metric, then the following ω-limit set*

$$\omega(B) = \bigcap_{t \geq 0} Cl_d \bigcup_{\tau \geq t} \sigma(\tau) B, \tag{5.4}$$

is nonempty, compact, and invariant.

PROOF. Although the proof is essentially provided in [7] and [10], here we include the details for completeness. First, by the well-known characterization lemma, i.e., Lemma 21.4 in [10], it is easy to check that $\omega(B)$ is invariant with respect to $\sigma(t)$, $t \geq 0$. Let

$$N(\sigma(t)B) = Cl_d \bigcup_{\tau \geq t} \sigma(\tau) B,$$

which is the hull of $\sigma(t)B$. On the one hand, $\{N(\sigma(t)B) : t \geq 0\}$ are nonempty, closed, bounded, and monotone decreasing sets. On the other hand, by the properties of the Kuratowski measure of noncompactness κ listed in Lemma 22.2 in [10], we have

$$\kappa(N(\sigma(t)B) = \kappa(\sigma(t)B) \to \kappa(B) = 0, \text{ as } t \to \infty.$$

Then the item (5) of Lemma 22.2 in [10] directly implies that $\omega(B) = \bigcap_{t \geq 0} N(\sigma(t)B)$ is a nonempty, compact set. □

LEMMA 5.4. *Let $\sigma(t)$, $t \geq 0$, be a semiflow in a complete metric space (X, d). If $B \subset X$ is a precompact, absorbing set of $\sigma(t)$, then the ω-limit set $\omega(B)$ given by* (5.4) *attracts B.*

PROOF. Assume, on the contrary, that $\omega(B)$ given by (5.4) does not attract B. Then for some $\varepsilon_0 > 0$, there exist sequences $\{x_n\} \subset B$ and $\{t_n\}$, $t_n \to \infty$, such that

$$\delta(\sigma(t_n) x_n, \omega(B)) \geq \varepsilon_0, \text{ for all } n \geq 1. \tag{5.5}$$

Here the Hausdorff distance $\delta(\cdot,\cdot)$ is defined by

$$\delta\left(B_1, B_2\right) = \sup_{x\in B_1}\left\{\inf_{y\in B_2} d(x,y)\right\}.$$

Since $B \subset X$ is an absorbing set, B attracts any given bounded set, such as itself. Therefore, for sufficiently large n, say, $n \geq n_0$, we shall have $\sigma\left(t_n\right) x_n \in B$.

On the other hand, since B is precompact, there exists a subsequence of $\{\sigma\left(t_n\right) x_n : n \geq n_0\}$, which is relabeled as the same, such that

$$\lim_{n\to\infty} \sigma\left(t_n\right) x_n = u^* \text{ exists.}$$

By the characterization of ω-limit sets, cf. [10, Lemma 21.4], we have $u^* \in \omega(B)$. But (5.5) implies that $\delta\left(u^*, \omega(B)\right) \geq \varepsilon_0 > 0$, which is a contradiction. □

The completion of the proof of Theorem 5.1:

In the complete metric space (B^*, d_Y), where $B^* = Cl_Y^w(B)$, the absorbing set B for the semiflow $\sigma(t)$, $t \geq 0$, in Theorem 5.1 is precompact. Thus, we can apply Lemmas 5.3 and 5.4 to $\mathbf{G} = \omega(B)$ to assert that this $\mathbf{G}$ is the global weak attractor for the semiflow $\sigma(t)$. In fact, we can check that the three conditions in the aforementioned definition are satisfied: The condition (i) is satisfied due to Lemma 5.3. The condition (ii) is satisfied because of Lemma 5.4 and $\omega(B) \subset B$. The condition (iii) is satisfied, because combining the facts that B is an absorbing ball and $\omega(B)$ attracts B, we have

$$\lim_{t\to\infty} \delta\left(S(t)y_0, \omega(B)\right) = 0, \text{ for any } y_0 \in Y.$$

The proof is finished. □

As a direct application of this generic result, we can conclude with the following result on the solution semiflow generated by the concerned nonlinear wave equation with the cubic non-monotone damping.

Theorem 5.5. *Assume that $h \in H$ is time-invariant. Then there exists a global weak attractor $\mathbf{G}$ for the semiflow $S(t)$, $t \geq 0$, generated by the weak solutions of the IVP* (2.4) *in the energy space E, which is given by*

$$\mathbf{G} = \bigcap_{t\geq 0} Cl_E^w \bigcup_{\tau\geq t} S(\tau)B, \tag{5.6}$$

where B is the absorbing set shown in Theorem 4.1.

Proof. Just applying Theorem 5.1 to the solution semiflow $S(t)$, $t \geq 0$, on the space E, we reach the conclusion. □

Remark. Under the assumptions of Theorem 5.1, it has been shown that if in addition the semiflow $\sigma(t)$, $t \geq 0$, is asymptotically compact, then the global weak attractor $\mathbf{G}$ given in (5.6) turns out to be a global attractor in the strong sense. The concept of asymptotical compactness is seen in [7] and [10]. However, at this time we are unable to show that this solution semiflow $S(t)$, $t \geq 0$, is asymptotically compact, and we do not know whether it is true for the nonlinear wave equations with such a nonlinear non-monotone damping.

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