

Persistent Homoclinic Orbits for Nonlinear Schrödinger Equation Under Singular Perturbations

Y. Charles Li

Communicated by Charles Li, received January 25, 2004.

ABSTRACT. Existence of homoclinic orbits in the cubic nonlinear Schrödinger equation under singular perturbations is proved. Emphasis is placed upon the regularity of the semigroup $e^{\epsilon t \partial_x^2}$ at $\epsilon = 0$. This article is a substantial generalization of [4], and motivated by the effort of Dr. Zeng [10] [9]. The mistake of Zeng [9] is corrected with a normal form transform approach. Both one and two unstable modes cases are investigated.

CONTENTS

1. Introduction	87
2. Local Theory	89
3. Global Theory	109
References	123

1. Introduction

The recent development of chaos in partial differential equations [5] [4] [3] has brought a lot of hope to this area. Basically, for perturbed soliton equations, the above works offer a general program for proving the existence of chaos. These works focused upon perturbed nonlinear Schrödinger equations. [5] dealt with integrable theory. In particular, [5] dealt with those critical ingredients of integrable theory, which are needed for studying perturbed systems. Specifically, [5] provides explicit representations of figure-eight structures through Darboux transformations, general explicit representations of Melnikov vectors, and phase space foliations. [4] dealt with perturbed nonlinear Schrödinger equations, to prove the existence of homoclinic orbits. Two main tools are used: Melnikov analysis and Fenichel fibers. The

1991 *Mathematics Subject Classification*. Primary 35Q55, 35Q30; Secondary 37L10, 37L50.

Key words and phrases. Homoclinic orbits, Darboux transformations, Melnikov analysis, invariant manifolds, fibers.

This work is partially supported by a Guggenheim Fellowship.

locating of homoclinic orbits requires two measurements: One is given by Melnikov functions, and the other is a scale measurement (called second measurement). It turns out that such second measurement is typical in higher dimensional systems, that is, to locate homoclinic orbits in high dimensions, the second measurement is often necessary. The tool needed in the second measurement is certain normal form transform to get proper scale estimate of certain stable manifold. The located homoclinic orbits are often of the type of Silnikov. In [3], Smale horseshoes are constructed in the neighborhood of the Silnikov homoclinic orbits. Two main tools are developed: Locating the fixed points of the Poincaré map and Conley-Moser conditions. Another important tool used is smooth linearization. Such smooth linearization does not pose any limitation by virtue of the fact that the Smale horseshoes once constructed are structurally stable.

The current article is a further development of [4]. In [4], a singular perturbation $\epsilon \partial_x^2$ is replaced by its Galerkin truncation which is a bounded Fourier multiplier. In [10] [9], Zeng used exactly the same ideas of [4] and followed exactly step by step of [4] to overcome the difficulty introduced by the singular perturbation $\epsilon \partial_x^2$. A mistake was made in the second measurement argument in [10]. That is, instead of using the normal form transform to get the proper scale estimate, Zeng tried to use Liapunov functions. One of the crucial estimates on the decay of Liapunov functions is not right. Zeng claimed that he could fix the argument in [9]. The current author is not convinced both geometrically and analytically that such Liapunov function argument can ever work. One of the goals of the current article is to use normal form transform to fix the above argument. An elegant normal form transform is constructed to establish the proper scale estimate.

The singularly perturbed nonlinear Schrödinger equation is given as,

$$(1.1) \quad iq_t = q_{xx} + 2[|q|^2 - \omega^2]q + i\epsilon[q_{xx} - \alpha q + \beta] ,$$

where $q = q(t, x)$ is a complex-valued function of the two real variables t and x , t represents time, and x represents space. $q(t, x)$ is subject to periodic boundary condition of period 2π , and even constraint, i.e.

$$q(t, x + 2\pi) = q(t, x) , \quad q(t, -x) = q(t, x) .$$

ω is a positive constant, $\alpha > 0$ and $\beta > 0$ are constants, and $\epsilon > 0$ is the perturbation parameter.

The main difficulty introduced by the singular perturbation $\epsilon \partial_x^2$ is that it breaks the spectral gap condition of the unperturbed system. Therefore, standard invariant manifold results will not apply. Nevertheless, it turns out that certain invariant manifold results do hold. The regularity of such invariant manifolds at $\epsilon = 0$ is controlled by the regularity of $e^{\epsilon \partial_x^2}$ at $\epsilon = 0$. Difficulties and interesting results created by the singular perturbation term $\epsilon \partial_x^2 q$ are all clearly pointed out and/or commented in Remarks. Integrable theory for two unstable modes is developed.

The entire theory of locating a homoclinic orbit is divided into two parts. Part 1 deals with local invariant manifold theory. Part 2 deals with global theory which includes integrable theory, Melnikov analysis, etc..

The notation $|| \cdot ||$ will denote absolute value, and the notation $|| \cdot ||_s$ will denote the Sobolev H^s (i.e. $W^{s,2}$) norm of periodic function with period 2π .

The article is organized as follows: Section 2 deals with local theory which contains six subsections. Section 3 deals with global theory which contains five subsections.

2. Local Theory

Local theory is referred to a theory in a neighborhood of certain circle of fixed points, which includes local unstable fiber theorem, local center-stable manifold theorem, and size estimate of local stable manifold for certain saddle. These are some of the tools needed in locating a homoclinic orbit.

2.1. Dynamics in a 2D Invariant Subspace.

The 2D subspace Π ,

$$(2.1) \quad \Pi = \{q \mid \partial_x q = 0\},$$

is an invariant subspace under the PNLS flow (1.1). The governing equation in Π is

$$(2.2) \quad i\dot{q} = 2[|q|^2 - \omega^2]q + i\epsilon[-\alpha q + \beta],$$

where $\cdot = \frac{d}{dt}$. Dynamics of this equation is shown in Figure 1. Interesting dynamics is created through resonance in the neighborhood of the circle S_ω :

$$(2.3) \quad S_\omega = \{q \in \Pi \mid |q| = \omega\}.$$

When $\epsilon = 0$, S_ω consists of fixed points. To explore the dynamics in this neighborhood better, one can make a series of changes of coordinates. Let $q = \sqrt{I}e^{i\theta}$, then (2.2) can be rewritten as

$$(2.4) \quad \dot{I} = \epsilon(-2\alpha I + 2\beta\sqrt{I}\cos\theta),$$

$$(2.5) \quad \dot{\theta} = -2(I - \omega^2) - \epsilon\beta\frac{\sin\theta}{\sqrt{I}}.$$

There are three fixed points:

- (1) The focus O_ϵ in the neighborhood of the origin,

$$(2.6) \quad \begin{cases} I = \epsilon^2 \frac{\beta^2}{4\omega^4} + \dots, \\ \cos\theta = \frac{\alpha\sqrt{I}}{\beta}, \end{cases} \quad \theta \in (0, \frac{\pi}{2}).$$

Its eigenvalues are

$$(2.7) \quad \mu_{1,2} = \pm i\sqrt{4(\omega^2 - I)^2 - 4\epsilon\sqrt{I}\beta\sin\theta - \epsilon\alpha},$$

where I and θ are given in (2.6).

- (2) The focus P_ϵ in the neighborhood of S_ω (2.3),

$$(2.8) \quad \begin{cases} I = \omega^2 + \epsilon\frac{1}{2\omega}\sqrt{\beta^2 - \alpha^2\omega^2} + \dots, \\ \cos\theta = \frac{\alpha\sqrt{I}}{\beta}, \end{cases} \quad \theta \in (-\frac{\pi}{2}, 0).$$

Its eigenvalues are

$$(2.9) \quad \mu_{1,2} = \pm i\sqrt{\epsilon}\sqrt{-4\sqrt{I}\beta\sin\theta + \epsilon\left(\frac{\beta\sin\theta}{\sqrt{I}}\right)^2} - \epsilon\alpha,$$

where I and θ are given in (2.8).

- (3) The saddle Q_ϵ in the neighborhood of S_ω (2.3),

$$(2.10) \quad \begin{cases} I = \omega^2 - \epsilon\frac{1}{2\omega}\sqrt{\beta^2 - \alpha^2\omega^2} + \dots, \\ \cos\theta = \frac{\alpha\sqrt{I}}{\beta}, \end{cases} \quad \theta \in (0, \frac{\pi}{2}).$$

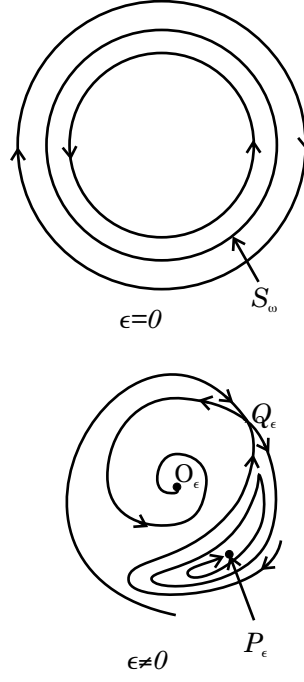


FIGURE 1. Dynamics on the invariant plane II.

Its eigenvalues are

$$(2.11) \quad \mu_{1,2} = \pm \sqrt{\epsilon} \sqrt{4\sqrt{I}\beta \sin \theta - \epsilon \left(\frac{\beta \sin \theta}{\sqrt{I}} \right)^2} - \epsilon \alpha,$$

where I and θ are given in (2.10).

Now focus our attention to order $\sqrt{\epsilon}$ neighborhood of S_ω (2.3) and let

$$J = I - \omega^2, \quad J = \sqrt{\epsilon}j, \quad \tau = \sqrt{\epsilon}t,$$

we have

$$(2.12) \quad j' = 2 \left[-\alpha(\omega^2 + \sqrt{\epsilon}j) + \beta \sqrt{\omega^2 + \sqrt{\epsilon}j} \cos \theta \right],$$

$$(2.13) \quad \theta' = -2j - \sqrt{\epsilon} \beta \frac{\sin \theta}{\sqrt{\omega^2 + \sqrt{\epsilon}j}},$$

where $' = \frac{d}{d\tau}$. To leading order, we get

$$(2.14) \quad j' = 2[-\alpha\omega^2 + \beta\omega \cos \theta],$$

$$(2.15) \quad \theta' = -2j.$$

There are two fixed points which are the counterparts of P_ϵ and Q_ϵ (2.8) and (2.10):

(1) The center P_* ,

$$(2.16) \quad j = 0, \quad \cos \theta = \frac{\alpha\omega}{\beta}, \quad \theta \in \left(-\frac{\pi}{2}, 0 \right).$$

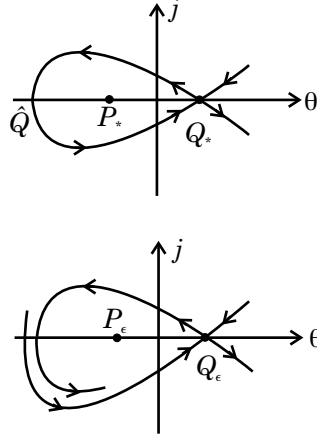


FIGURE 2. The fish-like dynamics in the neighborhood of the resonant circle S_ω .

Its eigenvalues are

$$(2.17) \quad \mu_{1,2} = \pm i 2\sqrt{\omega}(\beta^2 - \alpha^2\omega^2)^{\frac{1}{4}}.$$

(2) The saddle Q_* ,

$$(2.18) \quad j = 0, \quad \cos \theta = \frac{\alpha\omega}{\beta}, \quad \theta \in \left(0, \frac{\pi}{2}\right).$$

Its eigenvalues are

$$(2.19) \quad \mu_{1,2} = \pm 2\sqrt{\omega}(\beta^2 - \alpha^2\omega^2)^{\frac{1}{4}}.$$

In fact, (2.14) and (2.15) form a Hamiltonian system with the Hamiltonian

$$(2.20) \quad \mathcal{H} = j^2 + 2\omega(-\alpha\omega\theta + \beta\sin\theta).$$

Connecting to Q_* is a fish-like singular level set of \mathcal{H} , which intersects the axis $j = 0$ at Q_* and $\hat{Q} = (0, \hat{\theta})$,

$$(2.21) \quad \alpha\omega(\hat{\theta} - \theta_*) = \beta(\sin \hat{\theta} - \sin \theta_*), \quad \hat{\theta} \in \left(-\frac{3\pi}{2}, 0\right),$$

where θ_* is given in (2.18). See Figure 2 for an illustration of the dynamics of (2.12)-(2.15). For later use, we define a piece of each of the stable and unstable manifolds of Q_* ,

$$j = \phi_*^u(\theta), \quad j = \phi_*^s(\theta), \quad \theta \in [\hat{\theta} + \hat{\delta}, \theta_* + 2\pi],$$

for some small $\hat{\delta} > 0$, and

$$(2.22) \quad \begin{aligned} \phi_*^u(\theta) &= -\frac{\theta - \theta_*}{|\theta - \theta_*|} \sqrt{2\omega[\alpha\omega(\theta - \theta_*) - \beta(\sin \theta - \sin \theta_*)]}, \\ \phi_*^s(\theta) &= -\phi_*^u(\theta). \end{aligned}$$

$\phi_*^u(\theta)$ and $\phi_*^s(\theta)$ perturb smoothly in θ and $\sqrt{\epsilon}$ into $\phi_{\sqrt{\epsilon}}^u$ and $\phi_{\sqrt{\epsilon}}^s$ for (2.12) and (2.13).

The homoclinic orbit to be located will take off from Q_ϵ along its unstable curve, flies away from and returns to Π , lands near the stable curve of Q_ϵ and approaches Q_ϵ spirally.

2.2. Change of Coordinates. As mentioned above, interesting dynamics happens in the neighborhood of the circle S_ω (2.3). It is natural and convenient to center our coordinates around S_ω . First, write q as

$$(2.23) \quad q(t, x) = [\rho(t) + f(t, x)]e^{i\theta(t)},$$

where ρ and θ are polar coordinates on Π (2.1), and f has zero spatial mean. We use the notation $\langle \cdot \rangle$ to denote spatial mean,

$$(2.24) \quad \langle q \rangle = \frac{1}{2\pi} \int_0^{2\pi} q dx.$$

Since the L^2 -norm is an action variable when $\epsilon = 0$, it is more convenient to replace ρ by:

$$(2.25) \quad I = \langle |q|^2 \rangle = \rho^2 + \langle |f|^2 \rangle.$$

Since S_ω corresponds to $I = \omega^2$, the final pick is

$$(2.26) \quad J = I - \omega^2.$$

In terms of the new variables (J, θ, f) , Equation (1.1) can be rewritten as

$$(2.27) \quad \dot{J} = \epsilon \left[-2\alpha(J + \omega^2) + 2\beta\sqrt{J + \omega^2} \cos \theta \right] + \epsilon \mathcal{R}_2^J,$$

$$(2.28) \quad \dot{\theta} = -2J - \epsilon\beta \frac{\sin \theta}{\sqrt{J + \omega^2}} + \mathcal{R}_2^\theta,$$

$$(2.29) \quad f_t = L_\epsilon f + V_\epsilon f - i\mathcal{N}_2 - i\mathcal{N}_3,$$

where

$$(2.30) \quad L_\epsilon f = -if_{xx} + \epsilon(-\alpha f + f_{xx}) - i2\omega^2(f + \bar{f}),$$

$$(2.31) \quad V_\epsilon f = -i2J(f + \bar{f}) + i\epsilon\beta f \frac{\sin \theta}{\sqrt{J + \omega^2}},$$

$$(2.32) \quad \mathcal{R}_2^J = -2\langle |f_x|^2 \rangle + 2\beta \cos \theta \left[\sqrt{J + \omega^2 - \langle |f|^2 \rangle} - \sqrt{J + \omega^2} \right],$$

$$(2.33) \quad \begin{aligned} \mathcal{R}_2^\theta &= -\langle (f + \bar{f})^2 \rangle - \frac{1}{\rho} \langle |f|^2 (f + \bar{f}) \rangle \\ &\quad - \epsilon\beta \sin \theta \left[\frac{1}{\sqrt{J + \omega^2 - \langle |f|^2 \rangle}} - \frac{1}{\sqrt{J + \omega^2}} \right], \end{aligned}$$

$$(2.34) \quad \mathcal{N}_2 = 2\rho[2(|f|^2 - \langle |f|^2 \rangle) + (f^2 - \langle f^2 \rangle)],$$

$$(2.35) \quad \begin{aligned} \mathcal{N}_3 &= -\langle f^2 + \bar{f}^2 + 6|f|^2 \rangle f + 2(|f|^2 f - \langle |f|^2 f \rangle) \\ &\quad - \frac{1}{\rho} \langle |f|^2 (f + \bar{f}) \rangle f - 2\langle |f|^2 \rangle \bar{f} \\ &\quad - \epsilon\beta \sin \theta \left[\frac{1}{\sqrt{J + \omega^2 - \langle |f|^2 \rangle}} - \frac{1}{\sqrt{J + \omega^2}} \right] f. \end{aligned}$$

REMARK 2.1. The singular perturbation term “ $\epsilon \partial_x^2 q$ ” can be seen at two locations, L_ϵ and \mathcal{R}_2^J (2.30, 2.32). The singular perturbation term $\langle |f_x|^2 \rangle$ in \mathcal{R}_2^J does not create any difficulty. Since H^1 is a Banach algebra [1], this term is still of quadratic order, $\langle |f_x|^2 \rangle \sim \mathcal{O}(\|f\|_1^2)$.

LEMMA 2.2. *The nonlinear terms have the orders:*

$$\begin{aligned} |\mathcal{R}_2^J| &\sim \mathcal{O}(\|f\|_s^2), \quad |\mathcal{R}_2^\theta| \sim \mathcal{O}(\|f\|_s^2), \\ \|\mathcal{N}_2\|_s &\sim \mathcal{O}(\|f\|_s^2), \quad \|\mathcal{N}_3\|_s \sim \mathcal{O}(\|f\|_s^3), \quad (s \geq 1). \end{aligned}$$

PROOF. The proof is an easy direct verification. \square

2.3. Normal Form Transformation. In locating a homoclinic orbit to Q_ϵ (2.10), we need to estimate the size of the local stable manifold of Q_ϵ . The size of the variable J is of order $\mathcal{O}(\sqrt{\epsilon})$. The size of the variable θ is of order $\mathcal{O}(1)$. To be able to track a homoclinic orbit, we need the size of the variable f to be of order $\mathcal{O}(\epsilon^\mu)$, $\mu < 1$. Such an estimate can be achieved, if the quadratic term \mathcal{N}_2 (2.34) in (2.29) can be removed through a normal form transformation. In [10], Zeng tried to avoid such normal form transform, instead to use certain Liapunov function to achieve the size estimate. Geometrically and analytically, such normal form transform is very intuitive for establishing the size estimate, while the intuition for the success of the Liapunov function argument is elusive.

In terms of Fourier transforms,

$$f = \sum_{k \neq 0} \hat{f}(k) e^{ikx}, \quad \bar{f} = \sum_{k \neq 0} \overline{\hat{f}(-k)} e^{ikx},$$

and the two terms in \mathcal{N}_2 can be written as,

$$\begin{aligned} f^2 - \langle f^2 \rangle &= \sum_{k+\ell \neq 0} \hat{f}(k) \hat{f}(\ell) e^{i(k+\ell)x}, \\ |f|^2 - \langle |f|^2 \rangle &= \sum_{k+\ell \neq 0} \hat{f}(k) \overline{\hat{f}(-\ell)} e^{i(k+\ell)x} \\ (2.36) \quad &= \frac{1}{2} \sum_{k+\ell \neq 0} [\hat{f}(k) \overline{\hat{f}(-\ell)} + \hat{f}(\ell) \overline{\hat{f}(-k)}] e^{i(k+\ell)x}. \end{aligned}$$

It turns out to be convenient to work with the symmetrized form (2.36). We will search for a normal form transformation of the general form,

$$(2.37) \quad g = f + K(f, f),$$

where

$$\begin{aligned} K(f, f) &= \sum_{k+\ell \neq 0} \left[\hat{K}_1(k, \ell) \hat{f}(k) \hat{f}(\ell) + \hat{K}_2(k, \ell) \hat{f}(k) \overline{\hat{f}(-\ell)} \right. \\ &\quad \left. + \hat{K}_2(\ell, k) \overline{\hat{f}(-k)} \hat{f}(\ell) + \hat{K}_3(k, \ell) \overline{\hat{f}(-k)} \overline{\hat{f}(-\ell)} \right] e^{i(k+\ell)x}, \end{aligned}$$

$\hat{K}_j(k, \ell)$, ($j = 1, 2, 3$) are the unknown coefficients to be determined, and $\hat{K}_j(k, \ell) = \hat{K}_j(\ell, k)$, ($j = 1, 3$).

LEMMA 2.3. *For $\omega \in (\frac{1}{2}, \frac{3}{2})/S$, S is a finite subset, there exists a normal form transformation of the form (2.37) that transforms the equation*

$$f_t = L_\epsilon f - i\tilde{\mathcal{N}}_2,$$

into an equation with a cubic nonlinearity

$$g_t = L_\epsilon g + \mathcal{O}(\|g\|_s^3), \quad (s \geq 1),$$

where L_ϵ is given in (2.30), and $\tilde{\mathcal{N}}_2$ has the expression (cf: (2.34)),

$$(2.38) \quad \tilde{\mathcal{N}}_2 = 2\omega[2(|f|^2 - \langle |f|^2 \rangle) + (f^2 - \langle f^2 \rangle)].$$

PROOF. Denote the operator $i\partial_t - iL_\epsilon$ by \mathcal{L}_ϵ . We have

$$\begin{aligned} \mathcal{L}_\epsilon g &= \mathcal{L}_\epsilon f + \mathcal{L}_\epsilon K(f, f) \\ &= \mathcal{L}_\epsilon f - iL_\epsilon K(f, f) + iK(L_\epsilon f, f) + iK(f, L_\epsilon f) \\ &\quad + iK(\partial_t f - L_\epsilon f, f) + iK(f, \partial_t f - L_\epsilon f), \end{aligned}$$

where $\mathcal{L}_\epsilon f = \tilde{\mathcal{N}}_2$ and $K(\partial_t f - L_\epsilon f, f)$ and $K(f, \partial_t f - L_\epsilon f)$ will be shown to be cubic in f . To eliminate the quadratic terms, we need to set

$$iL_\epsilon K(f, f) - iK(L_\epsilon f, f) - iK(f, L_\epsilon f) = \tilde{\mathcal{N}}_2,$$

which takes the explicit form:

$$(2.39) \quad (\sigma_1 + i\sigma)\hat{K}_1(k, \ell) + b\hat{K}_2(k, \ell) + \overline{b\hat{K}_2(\ell, k)} + \overline{b\hat{K}_3(-k, -\ell)} = -2\omega,$$

$$(2.40) \quad -b\hat{K}_1(k, \ell) + (\sigma_2 + i\sigma)\hat{K}_2(k, \ell) + \overline{b\hat{K}_2(-\ell, -k)} + b\hat{K}_3(k, \ell) = -2\omega,$$

$$(2.41) \quad -b\hat{K}_1(k, \ell) + \overline{b\hat{K}_2(-k, -\ell)} + (\sigma_3 + i\sigma)\hat{K}_2(\ell, k) + b\hat{K}_3(k, \ell) = -2\omega,$$

$$(2.42) \quad \overline{b\hat{K}_1(-k, -\ell)} - b\hat{K}_2(k, \ell) - b\hat{K}_2(\ell, k) + (\sigma_4 + i\sigma)\hat{K}_3(k, \ell) = 0,$$

where

$$\begin{aligned} b &= -2\omega^2, \quad \sigma = \epsilon(2k\ell - \alpha), \quad \sigma_1 = 2(k\ell + \omega^2), \quad \sigma_2 = 2(\ell^2 + k\ell - \omega^2), \\ \sigma_3 &= 2(k^2 + k\ell - \omega^2), \quad \sigma_4 = 2(k^2 + \ell^2 + k\ell - 3\omega^2). \end{aligned}$$

Since these coefficients are even in (k, ℓ) , we will search for even solutions, i.e.

$$\hat{K}_j(-k, -\ell) = \hat{K}_j(k, \ell), \quad j = 1, 2, 3.$$

(2.39)+(2.42), (2.40)+(2.42)⁻, and (2.41)+(2.42)⁻ lead to

$$\begin{aligned} (\sigma_1 + i\sigma)\hat{K}_1(k, \ell) + \overline{b\hat{K}_1(k, \ell)} &= -K, \\ (\sigma_2 + i\sigma)\hat{K}_2(k, \ell) - \overline{b\hat{K}_2(k, \ell)} &= -\bar{K}, \\ (\sigma_3 + i\sigma)\hat{K}_2(\ell, k) - \overline{b\hat{K}_2(\ell, k)} &= -\bar{K}, \end{aligned}$$

where

$$K = 2\omega + (\sigma_4 + i\sigma)\hat{K}_3(k, \ell) + \overline{b\hat{K}_3(k, \ell)}.$$

Therefore we can express $\hat{K}_j(k, \ell)$ in terms of K as,

$$(2.43) \quad \hat{K}_1(k, \ell) = (\sigma_1^2 + \sigma^2 - b^2)^{-1}[b\bar{K} - (\sigma_1 - i\sigma)K],$$

$$(2.44) \quad \hat{K}_2(k, \ell) = (\sigma_2^2 + \sigma^2 - b^2)^{-1}[-b\bar{K} - (\sigma_2 - i\sigma)K],$$

$$(2.45) \quad \hat{K}_2(\ell, k) = (\sigma_3^2 + \sigma^2 - b^2)^{-1}[-b\bar{K} - (\sigma_3 - i\sigma)K],$$

$$(2.46) \quad \hat{K}_3(k, \ell) = (\sigma_4^2 + \sigma^2 - b^2)^{-1}[(\sigma_4 - i\sigma)(K - 2\omega) - b(\bar{K} - 2\omega)].$$

Substituting these expressions into (2.42), we get

$$(2.47) \quad K = (|U|^2 - |V|^2)^{-1}(W\bar{U} - \bar{W}V),$$

where

$$(2.48) \quad U = \frac{b^2}{\sigma_1^2 + \sigma^2 - b^2} + \frac{b(\sigma_2 - i\sigma)}{\sigma_2^2 + \sigma^2 - b^2} + \frac{b(\sigma_3 - i\sigma)}{\sigma_3^2 + \sigma^2 - b^2} + \frac{\sigma_4^2 + \sigma^2}{\sigma_4^2 + \sigma^2 - b^2},$$

$$(2.49) \quad V = -\frac{b(\sigma_1 + i\sigma)}{\sigma_1^2 + \sigma^2 - b^2} + \frac{b^2}{\sigma_2^2 + \sigma^2 - b^2} + \frac{b^2}{\sigma_3^2 + \sigma^2 - b^2} - \frac{b(\sigma_4 + i\sigma)}{\sigma_4^2 + \sigma^2 - b^2},$$

$$(2.50) \quad W = 2\omega(\sigma_4^2 + \sigma^2 - b^2)^{-1}[\sigma_4^2 + \sigma^2 - b(\sigma_4 + i\sigma)].$$

For $\omega \in (\frac{1}{2}, \frac{3}{2})$, the denominators in (2.43)-(2.46) and (2.48)-(2.50), and σ_j ($1 \leq j \leq 4$) vanish at ω in a finite subset. For ω not in this finite subset, σ^2 is a $\mathcal{O}(\epsilon^2)$ small perturbation of $\sigma_j^2 - b^2$ ($1 \leq j \leq 4$), and σ is a $\mathcal{O}(\epsilon)$ small perturbation of σ_j ($1 \leq j \leq 4$). Setting $\sigma = 0$, we have $K = 2\omega$, which leads to the solution given in [4]. To the order $\mathcal{O}(\epsilon)$,

$$(2.51) \quad K = 2\omega \left[1 + ib\sigma \left(\frac{1}{\sigma_1^2 - b^2} + \frac{1}{\sigma_2^2 - b^2} + \frac{1}{\sigma_3^2 - b^2} \right) \cdot \left(\frac{b}{\sigma_1 - b} + \frac{b}{\sigma_2 + b} + \frac{b}{\sigma_3 + b} + \frac{\sigma_4}{\sigma_4 - b} \right)^{-1} \right].$$

We will show that the denominator in (2.51) does not vanish except for ω in a finite subset. Denote the denominator by D . As $k \rightarrow \pm\infty$, or $\ell \rightarrow \pm\infty$,

$$(2.52) \quad D \rightarrow 1.$$

We also know that

$$\begin{aligned} D &= 1 + b \left[\frac{1}{\sigma_1 - b} + \frac{1}{\sigma_2 + b} + \frac{1}{\sigma_3 + b} + \frac{1}{\sigma_4 - b} \right] \\ &= 1 + \frac{b}{2} \left[\frac{(k + \ell)^2 - 4\omega^2}{(\ell^2 + k\ell - 2\omega^2)(k^2 + k\ell - 2\omega^2)} + \frac{(k + \ell)^2}{(k\ell + 2\omega^2)(k^2 + \ell^2 + k\ell - 2\omega^2)} \right] \\ &= \frac{-2(2\omega^2)^4 + \chi_1(2\omega^2)^3 + \chi_2(2\omega^2)^2 + \chi_3(2\omega^2) + \chi_4}{(\ell^2 + k\ell - 2\omega^2)(k^2 + k\ell - 2\omega^2)(k\ell + 2\omega^2)(k^2 + \ell^2 + k\ell - 2\omega^2)}, \end{aligned}$$

where χ_j ($1 \leq j \leq 4$) are polynomials in k and ℓ . For each k and ℓ , the numerator vanishes at most at four values of $2\omega^2$. Together with the fact (2.52), we have that for $\omega \in (\frac{1}{2}, \frac{3}{2})$, D does not vanish except for ω in a finite subset.

The denominator in (2.47) has the representation:

$$|U|^2 - |V|^2 = \text{Re}\{(U + V)(\bar{U} - \bar{V})\},$$

where

$$\begin{aligned} U + V &= \frac{\sigma_4}{\sigma_4 + b} - ib\sigma \left[\frac{1}{\sigma_1^2 - b^2} + \frac{1}{\sigma_2^2 - b^2} + \frac{1}{\sigma_3^2 - b^2} + \frac{1}{\sigma_4^2 - b^2} \right] \\ &\quad + \text{higher order terms in } \epsilon, \\ U - V &= D + ib\sigma \left[\frac{1}{\sigma_1^2 - b^2} - \frac{1}{\sigma_2^2 - b^2} - \frac{1}{\sigma_3^2 - b^2} + \frac{1}{\sigma_4^2 - b^2} \right] \\ &\quad + \text{higher order terms in } \epsilon. \end{aligned}$$

Then

$$|U|^2 - |V|^2 = \frac{\sigma_4}{\sigma_4 + b} D + \mathcal{O}(\epsilon^2).$$

Therefore, for $\omega \in (\frac{1}{2}, \frac{3}{2})$, the denominator in (2.47), $|U|^2 - |V|^2$ does not vanish except for ω in a finite subset. (2.43)-(2.47) give the solution to the linear system (2.39)-(2.42) for $\omega \in (\frac{1}{2}, \frac{3}{2})/S$, where S is a finite subset.

As in [4], $K(f, f)$ is also a bounded bilinear map:

$$\|K(f, f)\|_s \leq C\|f\|_s^2, \quad (s \geq 1).$$

We can invert the equation

$$g = f + K(f, f)$$

to obtain

$$f = g + \mathcal{K}(g),$$

where \mathcal{K} is of order $\mathcal{O}(\|g\|_s^2)$, ($s \geq 1$). Thus, terms like $K(\partial_t f - L_\epsilon f, f)$ and $K(f, \partial_t f - L_\epsilon f)$ are cubic terms in g . \square

REMARK 2.4. In this remark, we would like to make a comparison between the above normal form transform with that in [4], and in particular to comment on why the above normal form transform is necessary when singular perturbation $\epsilon \partial_x^2 f$ is studied. In [4], the linear operator L_ϵ is replaced by L_0 (i.e. setting $\epsilon = 0$ in L_ϵ) in constructing normal form transform. The corresponding normal form transform is given by

$$(2.53) \quad K = 2\omega, \quad \hat{K}_3(k, \ell) = 0, \quad \hat{K}_1(k, \ell) = -\frac{\omega}{k\ell}, \quad \hat{K}_2(k, \ell) = -\frac{\omega}{\ell(k+\ell)}.$$

When such a normal form transform is applied to Equation (2.29), the singular perturbation term $\epsilon \partial_x^2 f$ will introduce the following term in the equation for g :

$$(2.54) \quad \epsilon \partial_x^2 \mathcal{K}(f, f)$$

which is actually an unbounded bilinear operator. Therefore, we have to work with L_ϵ for a normal form transform. On the other hand, in [4], the singular perturbation $\epsilon \partial_x^2$ is mollified into a bounded pseudo-differential operator (actually a bounded Fourier multiplier) $\epsilon \hat{\partial}_x^2$. The term (2.54) is replaced by

$$\epsilon \hat{\partial}_x^2 \mathcal{K}(f, f)$$

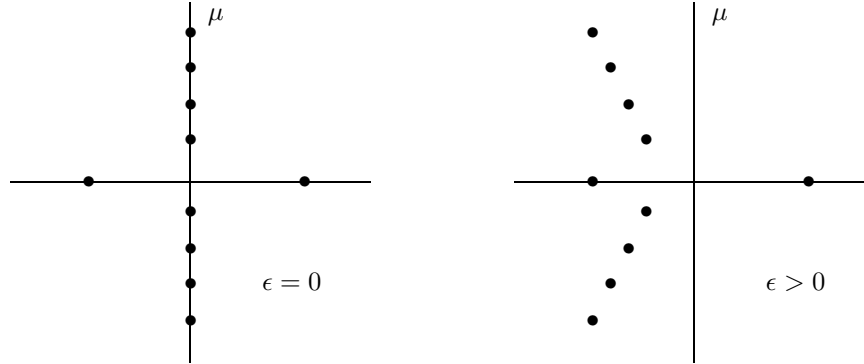
which is of order $\mathcal{O}(\epsilon \|f\|_s^2)$ sufficient for the estimate on the size of the local stable manifold of Q_ϵ . Although the normal form transform (2.43)-(2.47) has a more complicated expression than (2.53), they have the same asymptotic nature in k and ℓ . $\epsilon \partial_x^2 f$ only introduces small perturbations in the expressions of $\hat{K}_j(k, \ell)$ ($1 \leq j \leq 3$).

We apply the normal form transform given by (2.43)-(2.47) to the full equation (2.29), and the full system (2.27)-(2.29) is transformed into:

$$(2.55) \quad \dot{J} = \epsilon \left[-2\alpha(J + \omega^2) + 2\beta\sqrt{J + \omega^2} \cos \theta \right] + \epsilon \mathcal{R}_2^J,$$

$$(2.56) \quad \dot{\theta} = -2J - \epsilon\beta \frac{\sin \theta}{\sqrt{J + \omega^2}} + \mathcal{R}_2^\theta,$$

$$(2.57) \quad g_t = L_\epsilon g + V_\epsilon g + \mathcal{N},$$

FIGURE 3. The point spectra of the linear operator L_ϵ .

where L_ϵ , V_ϵ , \mathcal{R}_2^J and \mathcal{R}_2^θ are given in (2.30)-(2.33) with $f = g + \mathcal{K}(g)$, and

$$\begin{aligned}
 \mathcal{N} &= V_\epsilon \mathcal{K}(g) - i(\mathcal{N}_2 - \tilde{\mathcal{N}}_2) - i\mathcal{N}_3 + K(\partial_t f - L_\epsilon f, f) + \mathcal{K}(f, \partial_t f - L_\epsilon f) \\
 (2.58) \quad &= V_\epsilon \mathcal{K}(g) - i(\mathcal{N}_2 - \tilde{\mathcal{N}}_2) - i\mathcal{N}_3 + K(V_\epsilon f - i\mathcal{N}_2 - i\mathcal{N}_3, f) \\
 &\quad + K(f, V_\epsilon f - i\mathcal{N}_2 - i\mathcal{N}_3),
 \end{aligned}$$

where \mathcal{N}_2 , $\tilde{\mathcal{N}}_2$ and \mathcal{N}_3 are given in (2.34), (2.38) and (2.35) with $f = g + \mathcal{K}(g)$. \mathcal{N} has the estimate,

$$(2.59) \quad \|\mathcal{N}\|_s \sim \mathcal{O}(\|J\| \|g\|_s^2 + \epsilon \|g\|_s^2 + \|g\|_s^3), \quad (s \geq 1).$$

2.4. Unstable Fibers. Under regular perturbations as in [4], center-stable, center-unstable, and center manifolds, and Fenichel stable and unstable fibers persist as in the standard theory. Under the singular perturbation, what are the objects that persist? We start with the linear operator L_ϵ .

2.4.1. *The Spectrum of L_ϵ .* The spectrum of L_ϵ consists of only point spectrum. The eigenvalues of L_ϵ are:

$$(2.60) \quad \mu_k^\pm = -\epsilon(\alpha + k^2) \pm k\sqrt{4\omega^2 - k^2}, \quad (k = 1, 2, \dots).$$

When $\omega \in (\frac{1}{2}, 1)$, only μ_1^\pm are real, and μ_k^\pm are complex for $k > 1$. When $\omega \in (1, \frac{3}{2})$, only μ_1^\pm and μ_2^\pm are real, and μ_k^\pm are complex for $k > 2$.

REMARK 2.5. The main difficulty introduced by the singular perturbation $\epsilon \partial_x^2 f$ is the breaking of the spectral gap condition. Figure 3 shows the distributions of the eigenvalues when $\epsilon = 0$ and $\epsilon \neq 0$. It clearly shows the breaking of the stable spectral gap condition. As a result, center and center-unstable manifolds do not necessarily persist. On the other hand, the unstable spectral gap condition is not broken. This gives the hope for the persistence of center-stable manifold. Another case of persistence can be described as follows: Notice that the plane Π (2.1) is invariant under the PNLS flow (1.1). When $\epsilon = 0$, there is an unstable fibration with base points in a neighborhood of the circle S_ω (2.3) in Π , as an invariant sub-fibration of the unstable Fenichel fibration with base points in the center manifold. When $\epsilon > 0$, the center manifold may not persist, but Π persists, moreover, the unstable spectral gap condition is not broken, therefore, the unstable sub-fibration with base points in Π may persist. This is the topics of this subsection. Since the semiflow generated by PNLS (1.1) is not a C^1 perturbation of that generated

by the unperturbed NLS due to the singular perturbation $\epsilon \partial_x^2$, standard results on persistence can not be applied.

From now on, we will take the case of two unstable eigenvalues as our example to conduct the arguments. The case of one unstable eigenvalue is easier. The eigenfunctions corresponding to the real eigenvalues are:

$$(2.61) \quad e_k^\pm = e^{\pm i\vartheta_k} \cos kx, \quad e^{\pm i\vartheta_k} = \frac{k \mp i\sqrt{4\omega^2 - k^2}}{2\omega}, \quad k = 1, 2.$$

Notice that they are independent of ϵ . The eigenspaces corresponding to the complex conjugate pairs of eigenvalues are given by:

$$E_k = \text{span}_{\mathbb{C}}\{\cos kx\}.$$

and have real dimension 2.

2.4.2. The Set-Up of Equations. For the goal of this subsection, we need to single out the eigen-directions (2.61). Let

$$g = \sum_{\pm; k=1,2} \xi_k^\pm e_k^\pm + h,$$

where ξ_k^\pm are real variables, and

$$\langle h \rangle = \langle h \cos x \rangle = \langle h \cos 2x \rangle = 0.$$

In terms of the coordinates $(\xi_k^\pm, J, \theta, h)$, (2.55)-(2.57) can be rewritten as:

$$(2.62) \quad \dot{\xi}_k^+ = \mu_k^+ \xi_k^+ + V_k^+ \xi_k^+ + \mathcal{N}_k^+, \quad (k = 1, 2),$$

$$(2.63) \quad \dot{J} = \epsilon \left[-2\alpha(J + \omega^2) + 2\beta\sqrt{J + \omega^2} \cos \theta \right] + \epsilon \mathcal{R}_2^J,$$

$$(2.64) \quad \dot{\theta} = -2J - \epsilon\beta \frac{\sin \theta}{\sqrt{J + \omega^2}} + \mathcal{R}_2^\theta,$$

$$(2.65) \quad h_t = L_\epsilon h + V_\epsilon h + \tilde{\mathcal{N}},$$

$$(2.66) \quad \dot{\xi}_k^- = \mu_k^- \xi_k^- + V_k^- \xi_k^- + \mathcal{N}_k^-, \quad (k = 1, 2),$$

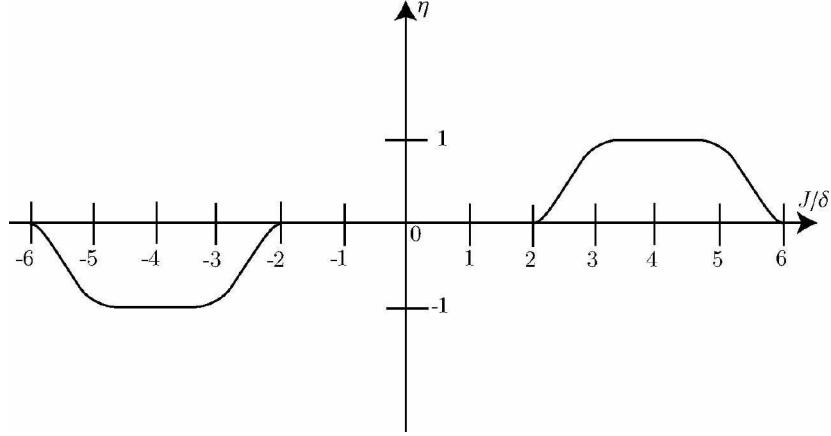
where μ_k^\pm are given in (2.60), \mathcal{N}_k^\pm and $\tilde{\mathcal{N}}$ are projections of \mathcal{N} to the corresponding directions, and

$$\begin{aligned} V_k^+ \xi_k^+ &= 2c_k J (\xi_k^+ + \xi_k^-) + \epsilon\beta \frac{\sin \theta}{\sqrt{J + \omega^2}} (c_k^+ \xi_k^+ - c_k^- \xi_k^-), \\ V_k^- \xi_k^- &= -2c_k J (\xi_k^+ + \xi_k^-) + \epsilon\beta \frac{\sin \theta}{\sqrt{J + \omega^2}} (c_k^- \xi_k^+ - c_k^+ \xi_k^-), \\ c_k &= \frac{k}{\sqrt{4\omega^2 - k^2}}, \quad c_k^+ = \frac{2\omega^2 - k^2}{k\sqrt{4\omega^2 - k^2}}, \quad c_k^- = \frac{2\omega^2}{k\sqrt{4\omega^2 - k^2}}. \end{aligned}$$

2.4.3. Statement of the Unstable Fiber Theorem. The main unstable fiber theorem can be stated as follows.

THEOREM 2.6. *There exists an annular neighborhood \mathcal{A} of the circle S_ω (2.3) in Π (2.1), for any $p \in \mathcal{A}$, there is a local unstable fiber \mathcal{F}_p^+ which is a 2D surface. \mathcal{F}_p^+ has the following properties:*

- (1) \mathcal{F}_p^+ is a C^1 smooth surface in $\|\cdot\|_n$ norm, $\forall n \geq 1$.
- (2) \mathcal{F}_p^+ is also C^1 smooth in $\epsilon, \alpha, \beta, \omega$, and p in $\|\cdot\|_n$ norm, for any $n \geq 1$, $\epsilon \in [0, \epsilon_0)$ for some $\epsilon_0 > 0$.

FIGURE 4. The bump function η .

- (3) $p \in \mathcal{F}_p^+$, \mathcal{F}_p^+ is tangent to $\text{span}\{e_1^+, e_2^+\}$ at p when $\epsilon = 0$, where e_k^+ ($k = 1, 2$) are defined in (2.61).
- (4) \mathcal{F}_p^+ has the exponential decay property: Let S^t be the evolution operator of (2.62)-(2.66), $\forall p_1 \in \mathcal{F}_p^+$,

$$\frac{\|S^t p_1 - S^t p\|_n}{\|p_1 - p\|_n} \leq C e^{\frac{\mu^+}{3}t}, \quad \forall t \leq 0,$$

where $\mu^+ = \min\{\mu_1^+, \mu_2^+\}$.

- (5) $\{\mathcal{F}_p^+\}_{p \in \mathcal{A}}$ forms an invariant family of unstable fibers,

$$S^t \mathcal{F}_p^+ \subset \mathcal{F}_{S^t p}^+, \quad \forall t \in [-T, 0],$$

and $\forall T > 0$ (T can be $+\infty$), such that $S^t p \in \mathcal{A}$, $\forall t \in [-T, 0]$.

2.4.4. Proof of the Unstable Fiber Theorem. There are two main approaches in establishing invariant manifolds and fibrations: 1. Lyapunov-Perron's method [7], 2. Hadamard's method [2]. Here we will adopt the Lyapunov-Perron's method, pay special attention to non-standard applications of the method, and focus on the difficulties generated by the singular perturbation $\epsilon \partial_x^2$.

DEFINITION 2.7. For any $\delta > 0$, we define the annular neighborhood of the circle S_ω (2.3) as

$$\mathcal{A}(\delta) = \{(J, \theta) \mid |J| < \delta\}.$$

To apply the Lyapunov-Perron's method, it is standard and necessary to modify the J equation so that $\mathcal{A}(4\delta)$ is overflowing invariant. Let $\eta \in C^\infty(R, R)$ be a "bump" function:

$$\eta = \begin{cases} 0, & \text{in } (-2, 2) \cup (-\infty, -6) \cup (6, \infty), \\ 1, & \text{in } (3, 5), \\ -1, & \text{in } (-5, -3), \end{cases}$$

as shown in Figure 4, $|\eta'| \leq 2$, $|\eta''| \leq C$. We modify the J equation (2.63) as follows:

$$(2.67) \quad \dot{J} = \epsilon b \eta(J/\delta) + \epsilon \left[-2\alpha(J + \omega^2) + 2\beta\sqrt{J + \omega^2} \cos \theta \right] + \epsilon \mathcal{R}_2^J,$$

where $b > 2(2\alpha\omega^2 + 2\beta\omega)$. Then $\mathcal{A}(4\delta)$ is overflowing invariant. There are two main points in adopting the bump function:

- (1) One needs $\mathcal{A}(4\delta)$ to be overflowing invariant so that a Lyapunov-Perron type integral equation can be set up along orbits in $\mathcal{A}(4\delta)$ for $t \in (-\infty, 0)$.
- (2) One needs the vector field inside $\mathcal{A}(2\delta)$ to be unchanged so that results for the modified system can be claimed for the original system in $\mathcal{A}(\delta)$.

REMARK 2.8. Due to the singular perturbation, the real part of μ_k^\pm approaches $-\infty$ as $k \rightarrow \infty$. Thus the h equation (2.65) can not be modified to give overflowing flow. This rules out the construction of unstable fibers with base points having general h coordinates.

For any $(J_0, \theta_0) \in \mathcal{A}(4\delta)$, let

$$(2.68) \quad J = J_*(t), \quad \theta = \theta_*(t), \quad t \in (-\infty, 0],$$

be the backward orbit of the modified system (2.67) and (2.64) with the initial point (J_0, θ_0) . If

$$(\xi_k^+(t), J_*(t) + \tilde{J}(t), \theta_*(t) + \tilde{\theta}(t), h(t), \xi_k^-(t))$$

is a solution of the modified full system, then one has

$$(2.69) \quad \dot{\xi}_k^+ = \mu_k^+ \xi_k^+ + F_k^+, \quad (k = 1, 2)$$

$$(2.70) \quad u_t = Au + F,$$

where

$$u = \begin{pmatrix} \tilde{J} \\ \tilde{\theta} \\ h \\ \xi_1^- \\ \xi_2^- \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_\epsilon & 0 & 0 \\ 0 & 0 & 0 & \mu_1^- & 0 \\ 0 & 0 & 0 & 0 & \mu_2^- \end{pmatrix}, \quad F = \begin{pmatrix} F_J \\ F_\theta \\ F_h \\ F_1^- \\ F_2^- \end{pmatrix},$$

$$F_k^+ = V_k^+ \xi_k^+ + \mathcal{N}_k^+,$$

$$F_J = \epsilon b [\eta(J/\delta) - \eta(J_*(t)/\delta)] + \epsilon \left[-2\alpha\tilde{J} + 2\beta\sqrt{J + \omega^2} \cos \theta - 2\beta\sqrt{J_*(t) + \omega^2} \cos \theta_*(t) \right] + \epsilon \mathcal{R}_2^J,$$

$$F_\theta = -\epsilon\beta \frac{\sin \theta}{\sqrt{J + \omega^2}} + \epsilon\beta \frac{\sin \theta_*(t)}{\sqrt{J_*(t) + \omega^2}} + \mathcal{R}_2^\theta,$$

$$F_h = V_\epsilon h + \tilde{\mathcal{N}},$$

$$F_k^- = V_k^- \xi_k^- + \mathcal{N}_k^-,$$

$$J = J_*(t) + \tilde{J}, \quad \theta = \theta_*(t) + \tilde{\theta}.$$

System (2.69)-(2.70) can be written in the equivalent integral equation form:

$$(2.71) \quad \xi_k^+(t) = \xi_k^+(t_0)e^{\mu_k^+(t-t_0)} + \int_{t_0}^t e^{\mu_k^+(t-\tau)} F_k^+(\tau) d\tau, \quad (k = 1, 2)$$

$$(2.72) \quad u(t) = e^{A(t-t_0)} u(t_0) + \int_{t_0}^t e^{A(t-\tau)} F(\tau) d\tau.$$

By virtue of the gap between μ_k^+ and the real parts of the eigenvalues of A , one can introduce the following space: For $\sigma \in \left(\frac{\mu^+}{100}, \frac{\mu^+}{3}\right)$, $\mu^+ = \min\{\mu_1^+, \mu_2^+\}$, and $n \geq 1$, let

$$G_{\sigma,n} = \left\{ g(t) = (\xi_k^+(t), u(t)) \mid t \in (-\infty, 0], g(t) \text{ is continuous} \right. \\ \left. \text{in } t \text{ in } H^n \text{ norm, } \|g\|_{\sigma,n} = \sup_{t \leq 0} e^{-\sigma t} \left[\sum_{k=1,2} |\xi_k^+(t)| + \|u(t)\|_n \right] < \infty \right\}.$$

$G_{\sigma,n}$ is a Banach space under the norm $\|\cdot\|_{\sigma,n}$. Let $\mathcal{B}_{\sigma,n}(r)$ denote the ball in $G_{\sigma,n}$ centered at the origin with radius r . Since A only has point spectrum, the spectral mapping theorem is valid. It is obvious that for $t \geq 0$,

$$\|e^{At}u\|_n \leq C(1+t)\|u\|_n,$$

for some constant C . Thus, if $g(t) \in \mathcal{B}_{\sigma,n}(r)$, $r < \infty$ is a solution of (2.71)-(2.72), by letting $t_0 \rightarrow -\infty$ in (2.72) and setting $t_0 = 0$ in (2.71), one has

$$(2.73) \quad \xi_k^+(t) = \xi_k^+(0)e^{\mu_k^+t} + \int_0^t e^{\mu_k^+(t-\tau)} F_k^+(\tau) d\tau, \quad (k = 1, 2)$$

$$(2.74) \quad u(t) = \int_{-\infty}^t e^{A(t-\tau)} F(\tau) d\tau.$$

For $g(t) \in \mathcal{B}_{\sigma,n}(r)$, let $\Gamma(g)$ be the map defined by the right hand side of (2.73)-(2.74). Then a solution of (2.73)-(2.74) is a fixed point of Γ . For any $n \geq 1$ and $\epsilon < \delta^2$, and δ and r are small enough, F_k^+ and F are Lipschitz in g with small Lipschitz constants. Standard arguments of the Lyapunov-Perron's method readily imply the existence of a fixed point g_* of Γ in $\mathcal{B}_{\sigma,n}(r)$. The difficulties lie in the investigation on the regularity of g_* with respect to $(\epsilon, \alpha, \beta, \omega, J_0, \theta_0, \xi_k^+(0))$. That is our focus. The most difficult one is the regularity with respect to ϵ due to the singular perturbation, which is our further focus. Formally differentiating g_* in (2.73)-(2.74) with respect to ϵ , one gets

$$(2.75) \quad \xi_{k,\epsilon}^+(t) = \int_0^t e^{\mu_k^+(t-\tau)} \left[\partial_u F_k^+ \cdot u_\epsilon + \sum_{\ell=1,2} \partial_{\xi_\ell^+} F_k^+ \cdot \xi_{\ell,\epsilon}^+ \right](\tau) d\tau \\ + \mathcal{R}_k^+(t), \quad (k = 1, 2)$$

$$(2.76) \quad u_\epsilon(t) = \int_{-\infty}^t e^{A(t-\tau)} \left[\partial_u F \cdot u_\epsilon + \sum_{\ell=1,2} \partial_{\xi_\ell^+} F \cdot \xi_{\ell,\epsilon}^+ \right](\tau) d\tau + \mathcal{R}(t),$$

where

$$(2.77) \quad \begin{aligned} \mathcal{R}_k^+(t) &= \xi_k^+(0) \mu_{k,\epsilon}^+ t e^{\mu_k^+ t} + \int_0^t \mu_{k,\epsilon}^+ (t-\tau) e^{\mu_k^+ (t-\tau)} F_k^+(\tau) d\tau \\ &\quad + \int_0^t e^{\mu_k^+ (t-\tau)} [\partial_\epsilon F_k^+ + \partial_{u_*} F_k^+ \cdot u_{*,\epsilon}] (\tau) d\tau, \end{aligned}$$

$$(2.78) \quad \begin{aligned} \mathcal{R}(t) &= \int_{-\infty}^t (t-\tau) A_\epsilon e^{A(t-\tau)} F(\tau) d\tau \\ &\quad + \int_{-\infty}^+ e^{A(t-\tau)} [\partial_\epsilon F + \partial_{u_*} F \cdot u_{*,\epsilon}] (\tau) d\tau, \end{aligned}$$

$$(2.79) \quad \mu_{k,\epsilon}^+ = -(\alpha + k^2), \quad k = 1, 2,$$

$$(2.80) \quad A_\epsilon = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha + \partial_x^2 & 0 & 0 \\ 0 & 0 & 0 & -(\alpha + 1) & 0 \\ 0 & 0 & 0 & 0 & -(\alpha + 4) \end{pmatrix},$$

$$(2.81) \quad u_* = (J_*, \theta_*, 0, 0, 0)^T,$$

where T = transpose, and (J_*, θ_*) are given in (2.68). The troublesome terms are the ones containing A_ϵ or $u_{*,\epsilon}$ in (2.77)-(2.78).

$$(2.82) \quad \|A_\epsilon F\|_n \leq C \|F\|_{n+2} \leq \tilde{c} (\|u\|_{n+2} + \sum_{k=1,2} |\xi_k^+|),$$

where \tilde{c} is small when (\cdot) on the right hand side is small.

$$\begin{aligned} \partial_{J_*} F_J \cdot J_{*,\epsilon} &= \frac{\epsilon}{\delta} b [\eta'(J/\delta) - \eta'(J_*/\delta)] J_{*,\epsilon} \\ &\quad + \epsilon [\beta \frac{\cos \theta}{\sqrt{J + \omega^2}} - \beta \frac{\cos \theta_*}{\sqrt{J_* + \omega^2}}] J_{*,\epsilon} \\ &\quad + \text{easier terms.} \\ |\partial_{J_*} F_J \cdot J_{*,\epsilon}| &\leq \frac{\epsilon}{\delta^2} b \sup_{0 \leq \hat{\gamma} \leq 1} |\eta''([\hat{\gamma} J_* + (1 - \hat{\gamma}) J]/\delta)| |\tilde{J}| |J_{*,\epsilon}| \\ &\quad + \epsilon \beta C (|\tilde{J}| + |\tilde{\theta}|) |J_{*,\epsilon}| + \text{easier terms} \\ &\leq C_1 (|\tilde{J}| + |\tilde{\theta}|) |J_{*,\epsilon}| + \text{easier terms,} \\ \sup_{t \leq 0} e^{-\sigma t} |\partial_{J_*} F_J \cdot J_{*,\epsilon}| &\leq C_1 \sup_{t \leq 0} [e^{-(\sigma + \tilde{\nu})t} (|\tilde{J}| + |\tilde{\theta}|)] \sup_{t \leq 0} [e^{\tilde{\nu}t} |J_{*,\epsilon}|] \\ &\quad + \text{easier terms,} \end{aligned}$$

where $\sup_{t \leq 0} e^{\tilde{\nu}t} |J_{*,\epsilon}|$ can be bounded when ϵ is sufficiently small for any fixed $\tilde{\nu} > 0$, through a routine estimate on Equations (2.67) and (2.64) for $(J_*(t), \theta_*(t))$. Other terms involving $u_{*,\epsilon}$ can be estimated similarly. Thus, the $\|\cdot\|_{\sigma,n}$ norm of terms involving $u_{*,\epsilon}$ has to be bounded by $\|\cdot\|_{\sigma+\tilde{\nu},n}$ norms. This leads to the standard rate condition for the regularity of invariant manifolds. That is, the regularity is controlled by the spectral gap. The $\|\cdot\|_{\sigma,n}$ norm of the term involving A_ϵ has to be bounded by $\|\cdot\|_{\sigma,n+2}$ norms. This is a new phenomenon caused by the singular perturbation. This problem is resolved by virtue of a special property of the fixed point g_* of Γ . Notice that if $\sigma_2 \geq \sigma_1$, $n_2 \geq n_1$, then $G_{\sigma_2, n_2} \subset G_{\sigma_1, n_1}$. Thus by

the uniqueness of the fixed point, if g_* is the fixed point of Γ in G_{σ_2, n_2} , g_* is also the fixed point of Γ in G_{σ_1, n_1} . Since g_* exists in $G_{\sigma, n}$ for an fixed $n \geq 1$ and $\sigma \in (\frac{\mu^+}{100}, \frac{\mu^+}{3} - 10\tilde{\nu})$ where $\tilde{\nu}$ is small enough,

$$\begin{aligned}\|\mathcal{R}_k^+\|_{\sigma, n} &\leq C_1 + C_2 \|g_*\|_{\sigma + \tilde{\nu}, n}, \\ \|\mathcal{R}\|_{\sigma, n} &\leq C_3 \|g_*\|_{\sigma, n+2} + C_4 \|g_*\|_{\sigma + \tilde{\nu}, n} + C_5,\end{aligned}$$

where C_j ($1 \leq j \leq 5$) depend upon $\|g_*(0)\|_n$ and $\|g_*(0)\|_{n+2}$. Let

$$M = 2(\|\mathcal{R}_k^+\|_{\sigma, n} + \|\mathcal{R}\|_{\sigma, n}),$$

and Γ' denote the linear map defined by the right hand sides of (2.75) and (2.76). Since the terms $\partial_u F_k^+$, $\partial_{\xi_\ell^+} F_k^+$, $\partial_u F$, and $\partial_{\xi_\ell^+} F$ all have small $\|\cdot\|_n$ norms, Γ' is a contraction map on $\mathcal{B}(M) \subset L(\mathcal{R}, G_{\sigma, n})$, where $\mathcal{B}(M)$ is the ball of radius M . Thus Γ' has a unique fixed point $g_{*, \epsilon}$. Next one needs to show that $g_{*, \epsilon}$ is indeed the partial derivative of g_* with respect to ϵ . That is, one needs to show

$$(2.83) \quad \lim_{\Delta\epsilon \rightarrow 0} \frac{\|g_*(\epsilon + \Delta\epsilon) - g_*(\epsilon) - g_{*, \epsilon} \Delta\epsilon\|_{\sigma, n}}{\Delta\epsilon} = 0.$$

This has to be accomplished directly from Equations (2.73)-(2.74), (2.75)-(2.76) satisfied by g_* and $g_{*, \epsilon}$. The most troublesome estimate is still the one involving A_ϵ . First, notice the fact that $e^{\epsilon \partial_x^2}$ is holomorphic in ϵ when $\epsilon > 0$, and not differentiable at $\epsilon = 0$. Then, notice that $g_* \in G_{\sigma, n}$ for any $n \geq 1$, thus, $e^{\epsilon \partial_x^2} g_*$ is differentiable, up to certain order m , in ϵ at $\epsilon = 0$ from the right, i.e.

$$(d^+ / d\epsilon)^m e^{\epsilon \partial_x^2} g_*|_{\epsilon=0}$$

exists in H^n . Let

$$\begin{aligned}z(t, \Delta\epsilon) &= e^{(\epsilon + \Delta\epsilon)t \partial_x^2} g_* - e^{\epsilon t \partial_x^2} g_* - (\Delta\epsilon) t \partial_x^2 e^{\epsilon t \partial_x^2} g_* \\ &= e^{\epsilon t \partial_x^2} w(\Delta\epsilon),\end{aligned}$$

where $t \geq 0$, $\Delta\epsilon > 0$, and

$$w(\Delta\epsilon) = e^{(\Delta\epsilon)t \partial_x^2} g_* - g_* - (\Delta\epsilon) t \partial_x^2 g_*.$$

Since $w(0) = 0$, by the Mean Value Theorem, one has

$$\|w(\Delta\epsilon)\|_n = \|w(\Delta\epsilon) - w(0)\|_n \leq \sup_{0 \leq \lambda \leq 1} \left\| \frac{dw}{d\Delta\epsilon}(\lambda \Delta\epsilon) \right\|_n |\Delta\epsilon|,$$

where at $\lambda = 0$, $\frac{d}{d\Delta\epsilon} = \frac{d^+}{d\Delta\epsilon}$, and

$$\frac{dw}{d\Delta\epsilon} = t[e^{(\Delta\epsilon)t \partial_x^2} \partial_x^2 g_* - \partial_x^2 g_*].$$

Since $\frac{dw}{d\Delta\epsilon}(0) = 0$, by the Mean Value Theorem again, one has

$$\left\| \frac{dw}{d\Delta\epsilon}(\lambda \Delta\epsilon) \right\|_n = \left\| \frac{dw}{d\Delta\epsilon}(\lambda \Delta\epsilon) - \frac{dw}{d\Delta\epsilon}(0) \right\|_n \leq \sup_{0 \leq \lambda_1 \leq 1} \left\| \frac{d^2 w}{d\Delta\epsilon^2}(\lambda_1 \lambda \Delta\epsilon) \right\|_n |\lambda| |\Delta\epsilon|,$$

where

$$\frac{d^2 w}{d\Delta\epsilon^2} = t^2 [e^{(\Delta\epsilon)t \partial_x^2} \partial_x^4 g_*].$$

Therefore, one has the estimate

$$(2.84) \quad \|z(t, \Delta\epsilon)\|_n \leq |\Delta\epsilon|^2 t^2 \|g_*\|_{n+4}.$$

This estimate is sufficient for handling the estimate involving A_ϵ . The estimate involving $u_{*,\epsilon}$ can be handled in a similar manner. For instance, let

$$\tilde{z}(t, \Delta\epsilon) = F(u_*(t, \epsilon + \Delta\epsilon)) - F(u_*(t, \epsilon)) - \Delta\epsilon \partial_{u_*} F \cdot u_{*,\epsilon},$$

then

$$\|\tilde{z}(t, \Delta\epsilon)\|_{\sigma,n} \leq |\Delta\epsilon|^2 \sup_{0 \leq \lambda \leq 1} \|[u_{*,\epsilon} \cdot \partial_{u_*}^2 F \cdot u_{*,\epsilon} + \partial_{u_*} F \cdot u_{*,\epsilon\epsilon}](\lambda \Delta\epsilon)\|_{\sigma,n}.$$

From the expression of F (2.70), one has

$$\begin{aligned} & \|u_{*,\epsilon} \cdot \partial_{u_*}^2 F \cdot u_{*,\epsilon} + \partial_{u_*} F \cdot u_{*,\epsilon\epsilon}\|_{\sigma,n} \\ & \leq C_1 \|g_*\|_{\sigma+2\bar{\nu},n} [(\sup_{t \leq 0} e^{\bar{\nu}t} |u_{*,\epsilon}|)^2 + \sup_{t \leq 0} e^{2\bar{\nu}t} |u_{*,\epsilon\epsilon}|], \end{aligned}$$

and the term [] on the right hand side can be easily shown to be bounded. In conclusion, let

$$h = g_*(\epsilon + \Delta\epsilon) - g_*(\epsilon) - g_{*,\epsilon} \Delta\epsilon,$$

one has the estimate

$$\|h\|_{\sigma,n} \leq \tilde{\kappa} \|h\|_{\sigma,n} + |\Delta\epsilon|^2 \tilde{C}(\|g_*\|_{\sigma,n+4}; \|g_*\|_{\sigma+2\bar{\nu},n}),$$

where $\tilde{\kappa}$ is small, thus

$$\|h\|_{\sigma,n} \leq 2|\Delta\epsilon|^2 \tilde{C}(\|g_*\|_{\sigma,n+4}; \|g_*\|_{\sigma+2\bar{\nu},n}).$$

This implies that

$$\lim_{\Delta\epsilon \rightarrow 0} \frac{\|h\|_{\sigma,n}}{|\Delta\epsilon|} = 0,$$

which is (2.83).

Let $g_*(t) = (\xi_k^+(t), u(t))$. First, let me comment on $\frac{\partial u}{\partial \xi_k^+(0)} \Big|_{\xi_k^+(0)=0, \epsilon=0} = 0$.

From (2.74), one has

$$\left\| \frac{\partial u}{\partial \xi_k^+(0)} \right\|_{\sigma,n} \leq \kappa_1 \left\| \frac{\partial g_*}{\partial \xi_k^+(0)} \right\|_{\sigma,n},$$

by letting $\xi_k^+(0) \rightarrow 0$ and $\epsilon \rightarrow 0^+$, $\kappa_1 \rightarrow 0$. Thus

$$\left. \frac{\partial u}{\partial \xi_k^+(0)} \right|_{\xi_k^+(0)=0, \epsilon=0} = 0.$$

I shall also comment on “exponential decay” property. Since $\|g_*\|_{\frac{\mu^+}{3},n} \leq r$,

$$\|g_*(t)\|_n \leq r e^{\frac{\mu^+}{3}t}, \quad \forall t \leq 0.$$

DEFINITION 2.9. Let $g_*(t) = (\xi_k^+(t), u(t))$, where

$$u(0) = \int_{-\infty}^0 e^{A(t-\tau)} F(\tau) d\tau$$

depends upon $\xi_k^+(0)$. Thus

$$u_*^0 : \xi_k^+(0) \mapsto u(0),$$

defines a $2D$ surface, which we call an unstable fiber denoted by \mathcal{F}_p^+ , where $p = (J_0, \theta_0)$ is the base point, $\xi_k^+(0) \in [-r, r] \times [-r, r]$.

Let S^t denote the evolution operator of (2.69)-(2.70), then

$$S^t \mathcal{F}_p^t \subset \mathcal{F}_{S^t p}^t, \quad \forall t \leq 0.$$

That is, $\{\mathcal{F}_p^+\}_{p \in \mathcal{A}(4\delta)}$ is an invariant family of unstable fibers. The proof of the Unstable Fiber Theorem is finished. \square

REMARK 2.10. If one replaces the base orbit $(J_*(t), \theta_*(t))$ by a general orbit for which only $\|\cdot\|_n$ norm is bounded, then the estimate (2.82) will not be possible. The $\|\cdot\|_{\sigma, n+2}$ norm of the fixed point g_* will not be bounded either. In such case, g_* may not be smooth in ϵ due to the singular perturbation.

REMARK 2.11. Smoothness of g_* in ϵ at $\epsilon = 0$ is the key point of the entire argument in this article. In the global theory in later sections, information is known at $\epsilon = 0$. This key point will link “ $\epsilon = 0$ ” information to “ $\epsilon \neq 0$ ” studies. Only continuity in ϵ at $\epsilon = 0$ is not enough for the study. The beauty of the entire theory is reflected by the fact that although $e^{\epsilon \partial_x^2}$ is not holomorphic at $\epsilon = 0$, $e^{\epsilon \partial_x^2} g_*$ can be smooth at $\epsilon = 0$ from the right, up to certain order depending upon the regularity of g_* . This is the beauty of the singular perturbation.

2.5. Center-Stable Manifold. We start with Equations (2.62)-(2.66), let

$$(2.85) \quad v = \begin{pmatrix} J \\ \theta \\ h \\ \xi_1^- \\ \xi_2^- \end{pmatrix}, \quad \tilde{v} = \begin{pmatrix} J \\ h \\ \xi_1^- \\ \xi_2^- \end{pmatrix},$$

and let $E_n(r)$ be the tubular neighborhood of S_ω (2.3):

$$(2.86) \quad E_n(r) = \{(J, \theta, h, \xi_1^-, \xi_2^-) \in H^n \mid \|\tilde{v}\|_n \leq r\}.$$

$E_n(r)$ is of codimension 2 in the entire phase space coordinatized by $(\xi_1^+, \xi_2^+, J, \theta, h, \xi_1^-, \xi_2^-)$.

2.5.1. *Statement of the Center-Stable Manifold Theorem.*

THEOREM 2.12. *There exists a C^1 smooth codimension 2 locally invariant center-stable manifold W_n^{cs} in H^n for any $n \geq 1$. W_n^{cs} can be represented as the graph of a C^1 function $\xi_*^+ : E_n(r) \rightarrow \mathbb{R}^2$, for some $r > 0$.*

- (1) *At points in the subset W_{n+4}^{cs} of W_n^{cs} , W_n^{cs} is C^1 smooth in ϵ for $\epsilon \in [0, \epsilon_0]$ and some $\epsilon_0 > 0$. That is, if $v \in E_{n+4}(r) \subset E_n(r)$, then $\xi_*^+(v)$ is C^1 smooth in ϵ , in H^n norm. Moreover, $\partial_\epsilon \xi_*^+(v)$ is uniformly bounded in $v \in E_{n+4}(r)$ and $\epsilon \in [0, \epsilon_0]$.*
- (2) *W_n^{cs} is C^1 smooth in (α, β, ω) .*
- (3) *The annular neighborhood \mathcal{A} in Theorem 2.6 is included in W_n^{cs} , i.e. $\xi_*^+(J, \theta, 0, 0, 0) = 0$. Along the circle S_ω (2.3), W_n^{cs} is tangent to $E_n(r)$ when $\epsilon = 0$, i.e. $\partial_v \xi_*^+(0, \theta, 0, 0) = 0$ when $\epsilon = 0$. W_n^{cs} is C^1 close to $E_n(r)$, i.e. $\|\partial_v \xi_*^+\| \leq Cr$.*

REMARK 2.13. C^1 regularity in ϵ is crucial in locating a homoclinic orbit. As can be seen later, one has detailed information on certain unperturbed (i.e. $\epsilon = 0$) homoclinic orbit, which will be used in tracking candidates for a perturbed homoclinic orbit. In particular, Melnikov measurement will be needed. Melnikov measurement measures zeros of $\mathcal{O}(\epsilon)$ signed distances, thus, the perturbed orbit needs to be $\mathcal{O}(\epsilon)$ close to the unperturbed orbit in order to perform Melnikov measurement.

2.5.2. *Proof of the Center-Stable Manifold Theorem.* Let $\chi \in C^\infty(R, R)$ be a “cut-off” function:

$$\chi = \begin{cases} 0, & \text{in } (-\infty, -4) \cup (4, \infty), \\ 1, & \text{in } (-2, 2). \end{cases}$$

We apply the cut-off

$$\chi_\delta = \chi(\|\tilde{v}\|_n/\delta)\chi(\xi_1^+/\delta)\chi(\xi_2^+/\delta)$$

to Equations (2.62)-(2.66), so that the equations in a tubular neighborhood of the circle S_ω (2.3) are unchanged, and linear outside a bigger tubular neighborhood. The modified equations take the form:

$$(2.87) \quad \dot{\xi}_k^+ = \mu_k^+ \xi_k^+ + \tilde{F}_k^+, \quad (k = 1, 2)$$

$$(2.88) \quad v_t = Av + \tilde{F},$$

where A is given in (2.70),

$$\begin{aligned} \tilde{F}_k^+ &= \chi_\delta[V_k^+ \xi_k^+ + \mathcal{N}_k^+], \\ \tilde{F} &= (\tilde{F}_J, \tilde{F}_\theta, \tilde{F}_h, \tilde{F}_1^-, \tilde{F}_2^-)^T, \quad T = \text{transpose}, \\ \tilde{F}_J &= \chi_\delta \in [-2\alpha(J + \omega^2) + 2\beta\sqrt{J + \omega^2} \cos \theta + \mathcal{R}_2^J], \\ \tilde{F}_\theta &= \chi_\delta \left[-\epsilon\beta \frac{\sin \theta}{\sqrt{J + \omega^2}} + \mathcal{R}_2^\theta \right], \\ \tilde{F}_h &= \chi_\delta[V_\epsilon h + \tilde{\mathcal{N}}], \quad (k = 1, 2) \\ \tilde{F}_k^- &= \chi_\delta[V_k^- \xi_k^- + \mathcal{N}_k^-], \end{aligned}$$

Equations (2.87)-(2.88) can be written in the equivalent integral equation form:

$$(2.89) \quad \xi_k^+(t) = \xi_k^+(t_0)e^{\mu_k^+(t-t_0)} + \int_{t_0}^t e^{\mu_k^+(t-\tau)} \tilde{F}_k^+(\tau) d\tau,$$

$$(2.90) \quad v(t) = e^{A(t-t_0)}v(t_0) + \int_{t_0}^t e^{A(t-\tau)} \tilde{F}(\tau) d\tau.$$

We introduce the following space: For $\sigma \in \left(\frac{\mu^+}{100}, \frac{\mu^+}{3}\right)$, $\mu^+ = \min\{\mu_1^+, \mu_2^+\}$, and $n \geq 1$, let

$$\begin{aligned} \tilde{G}_{\sigma,n} &= \left\{ g(t) = (\xi_k^+(t), v(t)) \mid t \in [0, \infty), g(t) \text{ is continuous in } t \right. \\ &\quad \left. \text{in } H^n \text{ norm, } \|g\|_{\sigma,n} = \sup_{t \geq 0} e^{-\sigma t} \left[\sum_{k=1,2} |\xi_k^+(t)| + \|v(t)\|_n \right] < \infty \right\}. \end{aligned}$$

$\tilde{G}_{\sigma,n}$ is a Banach space under the norm $\|\cdot\|_{\sigma,n}$. Let $\tilde{\mathcal{A}}_{\sigma,n}(r)$ denote the closed tubular neighborhood of S_ω (2.3):

$$\tilde{\mathcal{A}}_{\sigma,n}(r) = \left\{ g(t) = (\xi_k^+(t), v(t)) \in \tilde{G}_{\sigma,n} \mid \sup_{t \geq 0} e^{-\sigma t} \left[\sum_{k=1,2} |\xi_k^+(t)| + \|\tilde{v}(t)\|_n \right] \leq r \right\},$$

where \tilde{v} is defined in (2.85). If $g(t) \in \tilde{\mathcal{A}}_{\sigma,n}(r)$, $r < \infty$, is a solution of (2.89)-(2.90), by letting $t_0 \rightarrow +\infty$ in (2.89) and setting $t_0 = 0$ in (2.90), one has

$$(2.91) \quad \xi_k^+(t) = \int_{+\infty}^t e^{\mu_k^+(t-\tau)} \tilde{F}_k^+(\tau) d\tau, \quad (k = 1, 2)$$

$$(2.92) \quad v(t) = e^{At} v(0) + \int_0^t e^{A(t-\tau)} \tilde{F}(\tau) d\tau.$$

For any $g(t) \in \tilde{\mathcal{A}}_{\sigma,n}(r)$, let $\tilde{\Gamma}(g)$ be the map defined by the right hand side of (2.91)-(2.92). In contrast to the map Γ defined in (2.73)-(2.74), $\tilde{\Gamma}$ contains constant terms of order $\mathcal{O}(\epsilon)$, e.g. \tilde{F}_J and \tilde{F}_θ both contain such terms. Also, $\tilde{\mathcal{A}}_{\sigma,n}(r)$ is a tubular neighborhood of the circle S_ω (2.3) instead of the ball $\mathcal{B}_{\sigma,n}(r)$ for Γ . Fortunately, these facts will not create any difficulty in showing $\tilde{\Gamma}$ is a contraction on $\tilde{\mathcal{A}}_{\sigma,n}(r)$. For any $n \geq 1$ and $\epsilon < \delta^2$, and δ and r are small enough, \tilde{F}_k^+ and \tilde{F} are Lipschitz in g with small Lipschitz constants. $\tilde{\Gamma}$ has a unique fixed point \tilde{g}_* in $\tilde{\mathcal{A}}_{\sigma,n}(r)$, following from standard arguments. For the regularity of \tilde{g}_* with respect to $(\epsilon, \alpha, \beta, \omega, v(0))$, the most difficult one is of course with respect to ϵ due to the singular perturbation. Formally differentiating \tilde{g}_* in (2.91)-(2.92) with respect to ϵ , one gets

$$(2.93) \quad \xi_{k,\epsilon}^+(t) = \int_{+\infty}^t e^{\mu_k^+(t-\tau)} \left[\sum_{\ell=1,2} \partial_{\xi_\ell^+} \tilde{F}_k^+ \cdot \xi_{\ell,\epsilon}^+ + \partial_v \tilde{F}_k^+ \cdot v_\epsilon \right] (\tau) d\tau + \tilde{R}_k^+(t), \quad (k = 1, 2)$$

$$(2.94) \quad v(t) = \int_0^t e^{A(t-\tau)} \left[\sum_{\ell=1,2} \partial_{\xi_\ell^+} \tilde{F} \cdot \xi_{\ell,\epsilon}^+ + \partial_v \tilde{F} \cdot v_\epsilon \right] (\tau) d\tau + \tilde{R}(t),$$

where

$$(2.95) \quad \tilde{R}_k^+(t) = \int_{+\infty}^t \mu_{k,\epsilon}^+(t-\tau) e^{\mu_k^+(t-\tau)} \tilde{F}_k^+(\tau) d\tau + \int_{+\infty}^t e^{\mu_k^+(t-\tau)} \partial_\epsilon \tilde{F}_k^+(\tau) d\tau,$$

$$(2.96) \quad \tilde{R}(t) = t A_\epsilon e^{At} v(0) + \int_0^t (t-\tau) A_\epsilon e^{A(t-\tau)} \tilde{F}(\tau) d\tau + \int_0^t e^{A(t-\tau)} \partial_\epsilon \tilde{F}(\tau) d\tau,$$

and $\mu_{k,\epsilon}^+$ and A_ϵ are given in (2.79)-(2.81). The troublesome terms are the ones containing A_ϵ in (2.96). These terms can be handled in the same way as in the Proof of the Unstable Fiber Theorem. The crucial fact utilized is that if $v(0) \in H^{n_1}$, then \tilde{g}_* is the unique fixed point of $\tilde{\Gamma}$ in both \tilde{G}_{σ,n_1} and \tilde{G}_{σ,n_2} for any $n_2 \leq n_1$.

REMARK 2.14. In the Proof of the Unstable Fiber Theorem, the arbitrary initial data in (2.73)-(2.74) are $\xi_k^+(0)$ ($k = 1, 2$) which are scalars. Here the arbitrary initial datum in (2.91)-(2.92) is $v(0)$ which is a function of x . If $v(0) \in H^{n_2}$ but not H^{n_1} for some $n_1 > n_2$, then $\tilde{g}_* \notin \tilde{G}_{\sigma,n_1}$, in contrast to the case of (2.73)-(2.74) where $g_* \in G_{\sigma,n}$ for any fixed $n \geq 1$. The center-stable manifold W_n^{cs} stated in the Center-Stable Manifold Theorem will be defined through $v(0)$. This already illustrates why W_n^{cs} has the regularity in ϵ as stated in the theorem.

We have

$$\begin{aligned} \|\tilde{R}_k^+\|_{\sigma,n} &\leq \tilde{C}_1, \\ \|\tilde{R}\|_{\sigma,n} &\leq \tilde{C}_2 \|\tilde{g}_*\|_{\sigma,n+2} + \tilde{C}_3, \end{aligned}$$

for $\tilde{g}_* \in \tilde{\mathcal{A}}_{\sigma, n+2}(r)$, where \tilde{C}_j ($j = 1, 2, 3$) are constants depending in particular upon the cut-off in \tilde{F}_k^+ and \tilde{F} . Let $\tilde{\Gamma}'$ denote the linear map defined by the right hand sides of (2.93)-(2.94). If $v(0) \in H^{n+2}$ and $\tilde{g}_* \in \tilde{\mathcal{A}}_{\sigma, n+2}(r)$, standard argument shows that $\tilde{\Gamma}'$ is a contraction map on a closed ball in $L(R, \tilde{G}_{\sigma, n})$. Thus $\tilde{\Gamma}'$ has a unique fixed point $\tilde{g}_{*, \epsilon}$. Furthermore, if $v(0) \in H^{n+4}$ and $\tilde{g}_* \in \tilde{\mathcal{A}}_{\sigma, n+4}(r)$, one has that $\tilde{g}_{*, \epsilon}$ is indeed the derivative of \tilde{g}_* in ϵ , following the same argument as in the Proof of the Unstable Fiber Theorem. Here one may be able to replace the requirement $v(0) \in H^{n+4}$ and $\tilde{g}_* \in \tilde{\mathcal{A}}_{\sigma, n+4}(r)$ by just $v(0) \in H^{n+2}$ and $\tilde{g}_* \in \tilde{\mathcal{A}}_{\sigma, n+2}(r)$. But we are not interested in sharper results, and the current result is sufficient for our purpose.

DEFINITION 2.15. For any $v(0) \in E_n(r)$ where r is sufficiently small and $E_n(r)$ is defined in (2.86), let $\tilde{g}_*(t) = (\xi_k^+(t), v(t))$ be the fixed point of $\tilde{\Gamma}$ in $\tilde{G}_{\sigma, n}$, where one has

$$\xi_k^+(0) = \int_{-\infty}^0 e^{\mu_k^+(t-\tau)} \tilde{F}_k^+(\tau) d\tau, \quad (k = 1, 2)$$

which depend upon $v(0)$. Thus

$$\xi_*^+ : v(0) \mapsto \xi_k^+(0), \quad (k = 1, 2)$$

defines a codimension 2 surface, which we call center-stable manifold denoted by W_n^{cs} .

The regularity of the fixed point \tilde{g}_* immediately implies the regularity of W_n^{cs} . We have sketched the proof of the most difficult regularity, i.e. with respect to ϵ . Uniform boundedness of $\partial_\epsilon \xi_*^+$ in $v(0) \in E_{n+4}(r)$ and $\epsilon \in [0, \epsilon_0)$, is obvious. Other parts of the detailed proof is completely standard. We have that W_n^{cs} is a C^1 locally invariant submanifold which is C^1 in (α, β, ω) . W_n^{cs} is C^1 in ϵ at point in the subset W_{n+4}^{cs} . From Equation (2.91), Claim 3 in the Theorem immediately follows. \square

REMARK 2.16. Let S^t denote the evolution operator of the perturbed nonlinear Schrödinger equation (1.1). The proofs of the Unstable Fiber Theorem and the Center-Stable Manifold Theorem also imply the following: S^t is a C^1 map on H^n for any fixed $t > 0$, $n \geq 1$. S^t is also C^1 in (α, β, ω) . S^t is C^1 in ϵ as a map from H^{n+4} to H^n for any fixed $n \geq 1$, $\epsilon \in [0, \epsilon_0)$, $\epsilon_0 > 0$.

2.6. Stable Manifold of Q_ϵ . As mentioned earlier, the homoclinic orbit to be located will be asymptotic to the saddle Q_ϵ (2.10). Dynamics on the invariant plane Π (2.1) on which Q_ϵ lives, is governed by Equations (2.4)-(2.5) which are equivalent to Equation (2.27)-(2.28) with $f = 0$. The eigenvalues of Q_ϵ are given by (2.11) on Π and (2.60) off Π . Thus Q_ϵ has three unstable eigenvalues of two scales: One unstable eigenvalue of order $\mathcal{O}(\sqrt{\epsilon})$ with eigen-direction in Π , the other two unstable eigenvalues of order $\mathcal{O}(1)$ with eigen-directions off Π . On Π , Q_ϵ has the unstable curve $\phi_{\sqrt{\epsilon}}^u$ with approximate representation (2.22). Thus the 3D unstable manifold of Q_ϵ , $W^u(Q_\epsilon)$ has the representation

$$W^u(Q_\epsilon) = \cup_{p \in \phi_{\sqrt{\epsilon}}^u} \mathcal{F}_p^+$$

where \mathcal{F}_p^+ is the unstable fiber given in Theorem 2.6. The scales of the stable eigenvalues of Q_ϵ range from $\mathcal{O}(\epsilon)$ to $\mathcal{O}(\infty)$. The stable eigenvalue with eigen-direction in Π has order $\mathcal{O}(\sqrt{\epsilon})$. On Π , Q_ϵ has the stable curve $\phi_{\sqrt{\epsilon}}^s$ with approximate representation (2.22). From the standard stable manifold theorem, Q_ϵ has a C^1 stable

manifold $W_n^s(Q_\epsilon)$ in H^n for any $n \geq 1$. In fact, the codimension 3 stable manifold of Q_ϵ , $W_n^s(Q_\epsilon)$ intersects Π along $\phi_{\sqrt{\epsilon}}^s$. In order to locate a homoclinic orbit, we need the size of $W_n^s(Q_\epsilon)$ large enough. Along $\phi_{\sqrt{\epsilon}}^s$, the size of $W_n^s(Q_\epsilon)$ is $\mathcal{O}(1)$ sufficient for our purpose. One can view $W_n^s(Q_\epsilon)$ as a wall with base $\phi_{\sqrt{\epsilon}}^s$. As can be seen later in the Second Measurement, one needs the size of $W_n^s(Q_\epsilon)$ off Π to be of order $\mathcal{O}(\epsilon^\kappa)$, $\kappa < 1$ in order to overcome the order $\mathcal{O}(\epsilon)$ “fuzz” between certain perturbed and unperturbed ($\epsilon = 0$) orbits to locate a perturbed homoclinic orbit. Starting from the system (2.27)-(2.29), one can only get the size of $W_n^s(Q_\epsilon)$ off Π to be $\mathcal{O}(\epsilon)$ from standard stable manifold theorems. As discussed previously in the subsection on Normal Form Transformation, an estimate of order $\mathcal{O}(\epsilon^\kappa)$, $\kappa < 1$ can be achieved if the quadratic term \mathcal{N}_2 (2.34) in (2.29) can be removed through a normal form transformation. Such a normal form transformation has been found in that subsection.

THEOREM 2.17. *The size of $W_n^s(Q_\epsilon)$ off Π is of order $\mathcal{O}(\sqrt{\epsilon})$ for $\omega \in (\frac{1}{2}, \frac{3}{2})/S$, where S is a finite subset.*

PROOF. For $\omega \in (\frac{1}{2}, \frac{3}{2})/S$, where S is a finite subset, we apply the normal form transform given by (2.43)-(2.47) to Equation (2.29), then the system (2.27)-(2.29) is transformed into the system (2.55)-(2.57). By virtue of the estimate (2.59), the theorem follows from standard argument. For details, see [4]. \square

As discussed in the subsection on Center-Stable Manifold, the center-stable manifold W_n^{cs} is unique. Thus $W_n^s(Q_\epsilon)$ is a codimension 1 submanifold of W_n^{cs} .

3. Global Theory

Global Theory is referred to a theory global in phase space, which includes integrable theory, Melnikov measurement, and the so called second measurements. These are tools necessary in locating a homoclinic orbit.

The entire process of locating the homoclinic orbit can be briefly summarized as follows: The integrable theory will provide explicit representations for certain family of homoclinic orbits asymptotic to periodic orbits on the invariant plane Π . Local unstable fiber theorem will provide ways of picking orbits in the local unstable manifold of Q_ϵ , that are close to certain unperturbed homoclinic orbits. Our main strategy is to use the unperturbed homoclinic orbits to trace the candidates for a perturbed homoclinic orbit. The procedure is splinted into two steps:

Step 1. Find an orbit that is in $W^u(Q_\epsilon) \cap W_n^{cs}$.

Step 2. Find out when this orbit is also in $W^u(Q_\epsilon) \cap W_n^s(Q_\epsilon)$, where $W_n^s(Q_\epsilon)$ is a codimension 1 submanifold of W_n^{cs} .

Step 1 will be accomplished through Melnikov measurement. The Melnikov vectors will be provided by integrable theory. The Melnikov integrals will be evaluated along the unperturbed homoclinic orbits mentioned above. In contrast to the work [4], the new feature in Step 1 is that W_n^{cs} is not C^1 in ϵ everywhere rather only at its subset W_{n+4}^{cs} . This difficulty is overcome by the fact that $W^u(Q_\epsilon) \subset H^n$ for any fixed $n \geq 1$ by virtue of the unstable fiber theorem. Step 2 will be accomplished by the so called second measurement. It turns out that one can trace the perturbed orbit in $W^u(Q_\epsilon) \cap W_n^{cs}$ through an unperturbed homoclinic orbit to an order $\mathcal{O}(\epsilon |\ln \epsilon|)$ neighborhood of Π (2.1). In order to check when this orbit can be

in $W^u(Q_\epsilon) \cap W_n^s(Q_\epsilon)$, one needs the size of $W_n^s(Q_\epsilon)$ off Π to be large enough, and $\mathcal{O}(\sqrt{\epsilon})$ is sufficient.

3.1. Integrable Theory. Consider the integrable 1D cubic focusing nonlinear Schrödinger equation ($\epsilon = 0$ in (1.1)),

$$(3.1) \quad iq_t = q_{xx} + 2[|q|^2 - \omega^2]q .$$

Its Lax pair is given by the Zakharov-Shabat linear system,

$$(3.2) \quad \psi_x = U\psi ,$$

$$(3.3) \quad \psi_t = V\psi ,$$

where

$$U = i \begin{pmatrix} \lambda & q \\ \bar{q} & -\lambda \end{pmatrix} ,$$

$$V = i \begin{pmatrix} 2\lambda^2 - |q|^2 + \omega^2 & 2\lambda q - iq_x \\ 2\lambda \bar{q} + i\bar{q}_x & -2\lambda^2 + |q|^2 - \omega^2 \end{pmatrix} .$$

3.1.1. Isospectral Theory. Focusing one's attention on the spatial part (3.2) of the Lax pair (3.2,3.3), one can define the fundamental matrix solution $M(x)$, s.t. $M(0)$ is the 2×2 identity matrix. Then the Floquet discriminant Δ is defined as

$$\Delta = \text{trace } M(2\pi) .$$

$\Delta = \Delta(\lambda, q)$, as a functional in q for any $\lambda \in \mathbb{C}$, provides enough functionally independent constants of motion to make NLS (3.1) integrable in the classical Liouville sense. For each fixed q , there is a sequence of special points $\{\lambda_j^s, j \in Z\}$ of $\lambda \in \mathbb{C}$ called simple points for which $|\Delta(\lambda_j^s, q)| = 2$. There is also a sequence of critical points $\{\lambda_j^c, j \in Z\}$ of $\lambda \in \mathbb{C}$ for which $\frac{\partial}{\partial \lambda} \Delta(\lambda_j^c, q) = 0$. When some λ_j^s coincides with some λ_l^c , a double point is formed. The geometric multiplicity is the dimension of the eigenspace of (3.2) at the double point.

DEFINITION 3.1. The sequence of constants of motion F_j is defined as

$$(3.4) \quad F_j = \Delta(\lambda_j^c, q) , \quad j \in Z .$$

F_j 's provide a sequence of Melnikov functions. More importantly, the gradients of F_j 's, which will be the Melnikov vectors, have a simple representation,

$$(3.5) \quad \frac{\delta F_j}{\delta \vec{q}} = i \frac{\sqrt{\Delta^2 - 4}}{W(\psi^+, \psi^-)} \begin{pmatrix} \psi_2^+ \psi_2^- \\ -\psi_1^+ \psi_1^- \end{pmatrix} , \quad \text{at } \lambda = \lambda_j^c, \quad j \in Z ,$$

where $\vec{q} = (q, \bar{q})^T$, $\psi^\pm = (\psi_1^\pm, \psi_2^\pm)^T$ are two eigenfunctions at $\lambda = \lambda_j^c$, and $W(\psi^+, \psi^-) = \psi_1^+ \psi_2^- - \psi_2^+ \psi_1^-$ is the Wronskian. For more details on the isospectral theory of NLS, we refer the readers to [5].

3.1.2. Bäcklund-Darboux Transformation. The particular form of the Bäcklund-Darboux transformation for NLS (3.1), that is useful for our purpose, is due to David Sattinger and V. Zurkowski [8].

THEOREM 3.2. Let $q(t, x)$ be a solution of NLS (3.1), ν is a complex double point of geometric multiplicity 2. Let ϕ^\pm be two linearly independent eigenfunctions of the Lax pair (3.2,3.3) at $\lambda = \nu$. Denote by ϕ the general solution

$$\phi = \phi(t, x, \nu, c_+, c_-) = c_+ \phi^+ + c_- \phi^- ,$$

We use ϕ to define a Gauge transformation matrix

$$(3.6) \quad G = G(\lambda; \nu, \phi) = \Gamma \begin{pmatrix} \lambda - \nu & 0 \\ 0 & \lambda - \bar{\nu} \end{pmatrix} \Gamma^{-1},$$

where

$$\Gamma = \begin{pmatrix} \phi_1 & -\overline{\phi_2} \\ \phi_2 & \overline{\phi_1} \end{pmatrix}.$$

Then we define Q and Ψ by

$$Q = q + 2(\nu - \bar{\nu}) \frac{\phi_1 \overline{\phi_2}}{|\phi_1|^2 + |\phi_2|^2},$$

and

$$\Psi = G\psi,$$

where ψ solves the Lax pair (3.2,3.3) at (λ, q) . Then Ψ solves the Lax pair (3.2,3.3) at (λ, Q) , and Q also solves NLS (3.1).

3.1.3. *Figure Eight Structures.* Consider the special solution of NLS (3.1),

$$(3.7) \quad q_c = ae^{i\theta(t)}, \quad \theta(t) = -[2(a^2 - \omega^2)t + \gamma].$$

The corresponding Floquet discriminant is given by

$$\Delta(\lambda, q_c) = 2 \cos(2\pi k), \quad k = \sqrt{a^2 + \lambda^2},$$

and two eigenfunctions (Bloch functions) are

$$(3.8) \quad \psi^\pm = \begin{pmatrix} ae^{i\frac{\theta}{2}} \\ (\pm k - \lambda)e^{-i\frac{\theta}{2}} \end{pmatrix} \exp\{\pm i2\lambda kt \pm ikx\}.$$

When k is real, to have temporal growth (and decay) in ψ^\pm , one needs λ to be purely imaginary. The temporal growth (and decay) in ψ^\pm is connected to the linear instability of q_c , since quadratic products of ψ^\pm solve linearized NLS [5]. The temporal growth is also necessary for constructing homoclinic solutions through the Bäcklund-Darboux transformation. Specifically, the double points of Δ are given by

$$k = \sqrt{a^2 + \lambda^2} = j/2, \quad j \in Z/\{0\}.$$

If one requires that a lies in the interval

$$a \in (1/2, 1),$$

then there is only one pair of complex double points

$$\lambda = \pm\nu = \pm i\sigma, \quad \sigma = \sqrt{a^2 - 1/4}.$$

If one requires that a lies in the interval

$$a \in (1, 3/2),$$

then there are two pairs of complex double points

$$\lambda = \pm\nu = \pm i\sigma, \text{ and } \lambda = \pm\hat{\nu} = \pm i\hat{\sigma}, \hat{\sigma} = \sqrt{a^2 - 1}.$$

Next we will construct homoclinic orbits, starting from the special solution q_c , through the Bäcklund-Darboux transformation. Notice that building the Bäcklund-Darboux transformation at $\lambda = \nu$ v.s. at $\lambda = -\nu$ and at $\lambda = \hat{\nu}$ v.s. at $\lambda = -\hat{\nu}$ always lead to equivalent results. We will choose $\lambda = \nu$ and $\lambda = \hat{\nu}$.

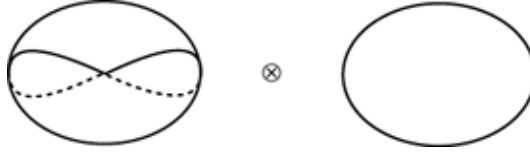


FIGURE 5. Figure eight structure of noneven data with one unstable mode.



FIGURE 6. Figure eight structure of even data with one unstable mode.

One Pair of Complex Double Points Case

Let $\phi^\pm = \psi^\pm(t, x, \nu)$ defined in (3.8), and let

$$(3.9) \quad \phi = c^+ \phi^+ + c^- \phi^- .$$

Applying the Bäcklund-Darboux transformation given in Theorem 3.2, one gets a new solution,

$$(3.10) \quad \begin{aligned} Q = & q_c \left[1 + \sin \vartheta_0 \operatorname{sech} \tau \cos y \right]^{-1} \cdot \left[\cos 2\vartheta_0 - i \sin 2\vartheta_0 \tanh \tau \right. \\ & \left. - \sin \vartheta_0 \operatorname{sech} \tau \cos y \right] , \end{aligned}$$

where

$$(3.11) \quad c^+/c^- = e^{\rho+i\vartheta} , \quad \frac{1}{2} + \nu = a e^{i\vartheta_0} , \quad \tau = 2\sigma t - \rho , \quad y = x + \vartheta - \vartheta_0 + \pi/2 .$$

As $t \rightarrow \pm\infty$,

$$(3.12) \quad Q \rightarrow q_c e^{\mp i 2\vartheta_0} .$$

Thus Q is asymptotic to q_c up to phase shifts as $t \rightarrow \pm\infty$. We say Q is a homoclinic orbit asymptotic to the periodic orbit given by q_c . For a fixed amplitude a of q_c , the phase γ of q_c and the Bäcklund parameters ρ and ϑ parametrize a 3-dimensional submanifold with a figure eight structure. For an illustration, see Figure 5. If one restricts the Bäcklund parameter ϑ by $\vartheta - \vartheta_0 + \pi/2 = 0$, or π , one gets Q to be even in x ,

$$(3.13) \quad \begin{aligned} Q = & q_c \left[1 \pm \sin \vartheta_0 \operatorname{sech} \tau \cos x \right]^{-1} \\ & \cdot \left[\cos 2\vartheta_0 - i \sin 2\vartheta_0 \tanh \tau \mp \sin \vartheta_0 \operatorname{sech} \tau \cos x \right] , \end{aligned}$$

where the upper sign corresponds to 0. Then for a fixed amplitude a of q_c , the phase γ of q_c and the Bäcklund parameter ρ parametrize a 2-dimensional submanifold with a figure eight structure. For an illustration, see Figure 6.

Two Pairs of Complex Double Points Case

Let $\hat{\phi}^\pm = \psi^\pm(t, x, \hat{\nu})$ defined in (3.8), and let

$$(3.14) \quad \hat{\phi} = \hat{c}^+ \hat{\phi}^+ + \hat{c}^- \hat{\phi}^- .$$

In this “two pairs of complex double points” case, to get the complete foliation of the figure eight structure, one needs to iterate the Bäcklund-Darboux transformation. First one needs to apply the Bäcklund-Darboux transformation at $\lambda = \nu$, then one needs to iterate the Bäcklund-Darboux transformation at $\lambda = \hat{\nu}$. Switching the order between ν and $\hat{\nu}$ leads to the same result. At $\lambda = \nu$, the Gauge transform $G = G(\lambda; \nu, \phi)$ (3.6), then one defines

$$(3.15) \quad \hat{\Phi}^\pm = G(\hat{\nu}; \nu, \phi) \hat{\phi}^\pm .$$

Let

$$(3.16) \quad \hat{\Phi} = G(\hat{\nu}; \nu, \phi) \hat{\phi} = \hat{c}^+ \hat{\Phi}^+ + \hat{c}^- \hat{\Phi}^- .$$

After an iteration on the Bäcklund-Darboux transformation, one gets the solution of NLS (3.1) with the representation,

$$\tilde{Q} = q_c + 2(\nu - \bar{\nu}) \frac{\phi_1 \bar{\phi}_2}{|\phi_1|^2 + |\phi_2|^2} + 2(\hat{\nu} - \bar{\hat{\nu}}) \frac{\hat{\Phi}_1 \bar{\hat{\Phi}}_2}{|\hat{\Phi}_1|^2 + |\hat{\Phi}_2|^2} .$$

Explicit formula for \tilde{Q} is,

$$(3.17) \quad \tilde{Q} = Q + q_c \frac{\mathcal{W}_2 \sin \hat{\vartheta}_0}{\mathcal{W}_1} ,$$

where Q is given in (3.10),

$$\begin{aligned} \mathcal{W}_1 = & \left[(\sin \hat{\vartheta}_0)^2 (1 + \sin \vartheta_0 \operatorname{sech} \tau \cos y)^2 + \frac{1}{8} (\sin 2\vartheta_0)^2 (\operatorname{sech} \tau)^2 (1 - \cos 2y) \right] \\ & \cdot (1 + \sin \hat{\vartheta}_0 \operatorname{sech} \hat{\tau} \cos \hat{y}) \\ & - \frac{1}{2} \sin 2\vartheta_0 \sin 2\hat{\vartheta}_0 \operatorname{sech} \tau \operatorname{sech} \hat{\tau} (1 + \sin \vartheta_0 \operatorname{sech} \tau \cos y) \sin y \sin \hat{y} \\ & + (\sin \vartheta_0)^2 \left[1 + 2 \sin \vartheta_0 \operatorname{sech} \tau \cos y + [(\cos y)^2 - (\cos \vartheta_0)^2] (\operatorname{sech} \tau)^2 \right] \\ & \cdot (1 + \sin \hat{\vartheta}_0 \operatorname{sech} \hat{\tau} \cos \hat{y}) \\ & - 2 \sin \hat{\vartheta}_0 \sin \vartheta_0 \left[\cos \hat{\vartheta}_0 \cos \vartheta_0 \tanh \hat{\tau} \tanh \tau + (\sin \vartheta_0 + \operatorname{sech} \tau \cos y) \right. \\ & \left. \cdot (\sin \hat{\vartheta}_0 + \operatorname{sech} \hat{\tau} \cos \hat{y}) \right] (1 + \sin \vartheta_0 \operatorname{sech} \tau \cos y) , \end{aligned}$$

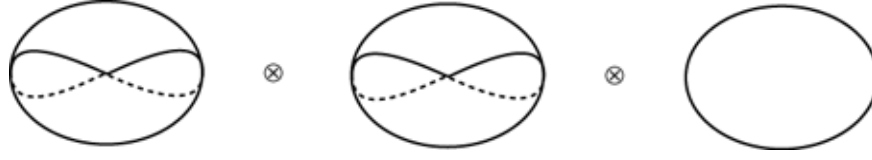


FIGURE 7. Figure eight structure of noneven data with two unstable modes.

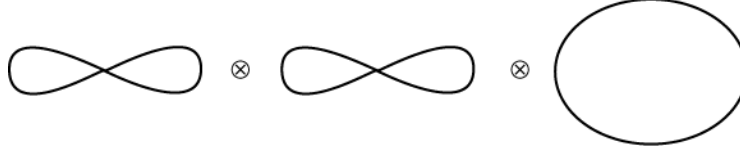


FIGURE 8. Figure eight structure of even data with two unstable modes.

$$\begin{aligned}
 \mathcal{W}_2 = & \left[-2(\sin \hat{\vartheta}_0)^2(1 + \sin \vartheta_0 \operatorname{sech} \tau \cos y)^2 + \frac{1}{4}(\sin 2\vartheta_0)^2(\operatorname{sech} \tau)^2(1 - \cos 2y) \right] \\
 & \cdot (\sin \hat{\vartheta}_0 + \operatorname{sech} \hat{\tau} \cos \hat{y} + i \cos \hat{\vartheta}_0 \tanh \hat{\tau}) \\
 & + 2(\sin \vartheta_0)^2(-\cos \vartheta_0 \tanh \tau + i \sin \vartheta_0 + i \operatorname{sech} \tau \cos y)^2 \\
 & \cdot (\sin \hat{\vartheta}_0 + \operatorname{sech} \hat{\tau} \cos \hat{y} - i \cos \hat{\vartheta}_0 \tanh \hat{\tau}) \\
 & + 2 \sin \vartheta_0 (\sin \vartheta_0 + \operatorname{sech} \tau \cos y + i \cos \vartheta_0 \tanh \tau) \\
 & \cdot \left[2 \sin \hat{\vartheta}_0 (1 + \sin \vartheta_0 \operatorname{sech} \tau \cos y)(1 + \sin \hat{\vartheta}_0 \operatorname{sech} \hat{\tau} \cos \hat{y}) \right. \\
 & \left. - \sin 2\vartheta_0 \cos \hat{\vartheta}_0 \operatorname{sech} \tau \operatorname{sech} \hat{\tau} \sin y \sin \hat{y} \right],
 \end{aligned}$$

and the notations are given by

$$1 + \hat{\nu} = ae^{i\hat{\vartheta}_0}, \quad \hat{c}^+/\hat{c}^- = e^{\hat{\rho} + i\hat{\vartheta}}, \quad \hat{\tau} = 4\hat{\sigma}t - \hat{\rho}, \quad \hat{y} = 2x + \hat{\vartheta} - \hat{\vartheta}_0 + \pi/2.$$

The asymptotic phase of Q is as follows, as $t \rightarrow \pm\infty$,

$$(3.18) \quad \tilde{Q} \rightarrow q_c e^{\mp i2(\vartheta_0 + \hat{\vartheta}_0)}.$$

Thus \tilde{Q} is asymptotic to q_c up to phase shifts as $t \rightarrow \pm\infty$. We say \tilde{Q} is a homoclinic orbit asymptotic to the periodic orbit given by q_c . For a fixed amplitude a of q_c , the phase γ of q_c and the Bäcklund parameters ρ , ϑ , $\hat{\rho}$, and $\hat{\vartheta}$ parametrize a 5-dimensional submanifold with a figure eight structure. For an illustration, see Figure 7. If one put restrictions on the Bäcklund parameters ϑ and $\hat{\vartheta}$, s.t.

$$(3.19) \quad \vartheta - \vartheta_0 + \pi/2 = \begin{cases} 0 & \begin{cases} \hat{\vartheta} - \hat{\vartheta}_0 + \pi/2 = 0, \\ \hat{\vartheta} - \hat{\vartheta}_0 + \pi/2 = \pi, \end{cases} \\ \pi & \begin{cases} \hat{\vartheta} - \hat{\vartheta}_0 + \pi/2 = 0, \\ \hat{\vartheta} - \hat{\vartheta}_0 + \pi/2 = \pi, \end{cases} \end{cases}$$

then Q is even in x . Then for a fixed amplitude a of q_c , the phase γ of q_c and the Bäcklund parameters ρ and $\hat{\rho}$ parametrize a 3-dimensional submanifold with a figure eight structure. For an illustration, see Figure 8.

3.1.4. *Melnikov Vectors.* Notice that (3.5) evaluated at (ν, Q) and $(\bar{\nu}, Q)$ are linearly dependent. Same is true for (ν, \tilde{Q}) or $(\bar{\nu}, \tilde{Q})$.

One Pair of Complex Double Points Case

In this case, the Melnikov vector is $\frac{\delta F_1}{\delta \vec{q}}$, (3.5) at $\lambda = \nu$, evaluated along the homoclinic orbit Q (3.10) or (3.13).

$$\frac{\delta F_1}{\delta \vec{q}} = i \frac{\sqrt{\Delta^2(\nu) - 4}}{W(\Phi^+, \Phi^-)} \begin{pmatrix} \Phi_2^+ \Phi_2^- \\ -\Phi_1^+ \Phi_1^- \end{pmatrix},$$

where (cf: (3.6)),

$$(3.20) \quad \begin{aligned} \Phi^\pm &= G(\nu; \nu, \phi) \phi^\pm \\ &= \pm c^\mp W(\phi^+, \phi^-) \frac{\nu - \bar{\nu}}{|\phi_1|^2 + |\phi_2|^2} \begin{pmatrix} \bar{\phi}_2 \\ -\phi_1 \end{pmatrix}. \end{aligned}$$

By L'Hospital's rule,

$$\frac{\sqrt{\Delta^2 - 4}}{W(\Phi^+, \Phi^-)} = \frac{\sqrt{\Delta(\nu)\Delta''(\nu)}}{(\nu - \bar{\nu})W(\phi^+, \phi^-)}.$$

ϕ (3.9) can be rewritten as

$$(3.21) \quad \phi_1 = 2\sqrt{c^+ c^-} a e^{i\theta/2} u_1, \quad \phi_2 = 2\sqrt{c^+ c^-} a e^{-i\theta/2} u_2,$$

where

$$\begin{aligned} u_1 &= \cosh \frac{\tau}{2} \cos z - i \sinh \frac{\tau}{2} \sin z, \\ u_2 &= -\sinh \frac{\tau}{2} \cos(z - \vartheta_0) + i \cosh \frac{\tau}{2} \sin(z - \vartheta_0), \end{aligned}$$

where

$$z = x/2 + \vartheta/2,$$

and other notations have been defined in (3.7, 3.11, 3.17). Finally, one gets the explicit representation for the Melnikov vector,

$$(3.22) \quad \frac{\delta F_1}{\delta \vec{q}} = \frac{1}{4} a^{-2} i (\nu - \bar{\nu}) \sqrt{\Delta(\nu)\Delta''(\nu)} \frac{1}{(|u_1|^2 + |u_2|^2)^2} \begin{pmatrix} \bar{q}_c \bar{u}_1^2 \\ -q_c \bar{u}_2^2 \end{pmatrix}.$$

Two Pairs of Complex Double Points Case

In this case, the Melnikov vectors are $\frac{\delta F_1}{\delta \vec{q}}$ and $\frac{\delta F_2}{\delta \vec{q}}$ (3.5) at $\lambda = \nu$ and $\lambda = \hat{\nu}$ respectively, evaluated along the homoclinic orbit \tilde{Q} (3.17) or (3.19). We know that $\hat{\Phi}$ is defined in (3.16). Then we use $\hat{\Phi}$ to define a Gauge matrix $G(\lambda; \hat{\nu}, \hat{\Phi})$. Let

$$(3.23) \quad \Phi^{(\pm, *)} = G(\nu; \hat{\nu}, \hat{\Phi}) \Phi^\pm, \quad \hat{\Phi}^{(\pm, *)} = G(\hat{\nu}; \hat{\nu}, \hat{\Phi}) \hat{\Phi}^\pm,$$

where Φ^\pm and $\hat{\Phi}^\pm$ are defined in (3.20, 3.15). Then the Melnikov vectors are

$$(3.24) \quad \frac{\delta F_1}{\delta \vec{q}} = i \frac{\sqrt{\Delta^2(\nu) - 4}}{W(\Phi^{(+, *)}, \Phi^{(-, *)})} \begin{pmatrix} \Phi_2^{(+, *)} \Phi_2^{(-, *)} \\ -\Phi_1^{(+, *)} \Phi_1^{(-, *)} \end{pmatrix},$$

$$(3.25) \quad \frac{\delta F_2}{\delta \vec{q}} = i \frac{\sqrt{\Delta^2(\hat{\nu}) - 4}}{W(\hat{\Phi}^{(+, *)}, \hat{\Phi}^{(-, *)})} \begin{pmatrix} \hat{\Phi}_2^{(+, *)} \hat{\Phi}_2^{(-, *)} \\ -\hat{\Phi}_1^{(+, *)} \hat{\Phi}_1^{(-, *)} \end{pmatrix},$$

By L'Hospital's rule,

$$\frac{\sqrt{\Delta^2(\nu) - 4}}{W(\Phi^{(+,*)}, \Phi^{(-,*)})} = \frac{\sqrt{\Delta(\nu)\Delta''(\nu)}}{(\nu - \bar{\nu})(\nu - \hat{\nu})(\nu - \bar{\hat{\nu}})W(\phi^+, \phi^-)} ,$$

$$\frac{\sqrt{\Delta^2(\hat{\nu}) - 4}}{W(\hat{\Phi}^{(+,*)}, \hat{\Phi}^{(-,*)})} = \frac{\sqrt{\Delta(\hat{\nu})\Delta''(\hat{\nu})}}{(\hat{\nu} - \bar{\hat{\nu}})(\hat{\nu} - \nu)(\hat{\nu} - \bar{\nu})W(\hat{\phi}^+, \hat{\phi}^-)} .$$

We know that ϕ (3.9) can be rewritten as (3.21). $\hat{\phi}$ (3.14) can also be rewritten as

$$(3.26) \quad \hat{\phi}_1 = 2\sqrt{\hat{c}^+\hat{c}^-}ae^{i\theta/2}v_1 , \quad \hat{\phi}_2 = 2\sqrt{\hat{c}^+\hat{c}^-}ae^{-i\theta/2}v_2 ,$$

where

$$\begin{aligned} v_1 &= \cosh \frac{\hat{\tau}}{2} \cos \hat{z} - i \sinh \frac{\hat{\tau}}{2} \sin \hat{z} , \\ v_2 &= -\sinh \frac{\hat{\tau}}{2} \cos(\hat{z} - \hat{\vartheta}_0) + i \cosh \frac{\hat{\tau}}{2} \sin(\hat{z} - \hat{\vartheta}_0) , \end{aligned}$$

where

$$\hat{z} = x + \hat{\vartheta}/2 ,$$

and other notations have been defined in (3.7, 3.11, 3.17). Using (3.21, 3.26), one can get the representation for $\hat{\Phi}$ (3.16),

$$\hat{\Phi}_1 = 2\sqrt{\hat{c}^+\hat{c}^-}ae^{i\theta/2}V_1 , \quad \hat{\Phi}_2 = 2\sqrt{\hat{c}^+\hat{c}^-}ae^{-i\theta/2}V_2 ,$$

where V_1 and V_2 are defined as

$$(3.27) \quad V_1 = \frac{1}{|u_1|^2 + |u_2|^2} \left[[(\hat{\nu} - \nu)|u_1|^2 + (\hat{\nu} - \bar{\nu})|u_2|^2]v_1 + (\bar{\nu} - \nu)u_1\bar{u}_2v_2 \right] ,$$

$$(3.28) \quad V_2 = \frac{1}{|u_1|^2 + |u_2|^2} \left[(\bar{\nu} - \nu)\bar{u}_1u_2v_1 + [(\hat{\nu} - \bar{\nu})|u_1|^2 + (\hat{\nu} - \nu)|u_2|^2]v_2 \right] .$$

Finally, one gets the explicit representations

$$(3.29) \quad \frac{\delta F_1}{\delta \vec{q}} = \frac{1}{4}a^{-2}i(\nu - \bar{\nu})(\nu - \hat{\nu})^{-1}(\nu - \bar{\hat{\nu}})^{-1}\sqrt{\Delta(\nu)\Delta''(\nu)} \begin{pmatrix} \bar{q}_c S_2^2 \\ -q_c \hat{S}_1^2 \end{pmatrix} ,$$

$$(3.30) \quad \frac{\delta F_2}{\delta \vec{q}} = \frac{1}{2}a^{-2}i(\hat{\nu} - \bar{\hat{\nu}})(\hat{\nu} - \nu)(\hat{\nu} - \bar{\nu})\sqrt{\Delta(\hat{\nu})\Delta''(\hat{\nu})} \begin{pmatrix} \bar{q}_c \hat{S}_2^2 \\ -q_c \hat{S}_1^2 \end{pmatrix} ,$$

where S_l and \hat{S}_l ($l = 1, 2$) are independent of the phase γ of q_c , and have the representations

$$(3.31) \quad \begin{aligned} S_1 &= \frac{1}{(|u_1|^2 + |u_2|^2)(|V_1|^2 + |V_2|^2)} \left[[(\nu - \hat{\nu})|V_1|^2 + (\nu - \bar{\nu})|V_2|^2] \overline{u_2} \right. \\ &\quad \left. - (\bar{\nu} - \hat{\nu})V_1 \overline{V_2 u_1} \right], \end{aligned}$$

$$(3.32) \quad \begin{aligned} S_2 &= \frac{1}{(|u_1|^2 + |u_2|^2)(|V_1|^2 + |V_2|^2)} \left[(\bar{\nu} - \hat{\nu})\overline{V_1} V_2 \overline{u_2} - [(\nu - \bar{\nu})|V_1|^2 \right. \\ &\quad \left. + (\nu - \hat{\nu})|V_2|^2] \overline{u_1} \right], \end{aligned}$$

$$(3.33) \quad \hat{S}_1 = \frac{\overline{V_2}}{|V_1|^2 + |V_2|^2},$$

$$(3.34) \quad \hat{S}_2 = \frac{\overline{V_1}}{|V_1|^2 + |V_2|^2}.$$

3.2. Melnikov Analysis. Let p be any point on $\phi_{\sqrt{\epsilon}}^u$ (2.22) which is the unstable curve of Q_ϵ in Π (2.1). Let $q_\epsilon(0)$ and $q_0(0)$ be any two points on the unstable fibers $\mathcal{F}_p^+|_\epsilon$ and $\mathcal{F}_p^+|_{\epsilon=0}$, with the same ξ_k^+ coordinates. By the Unstable Fiber Theorem, \mathcal{F}_p^+ is C^1 in ϵ for $\epsilon \in [0, \epsilon_0)$, $\epsilon_0 > 0$, thus

$$\|q_\epsilon(0) - q_0(0)\|_{n+8} \leq C\epsilon.$$

The key point here is that $\mathcal{F}_p^+ \subset H^s$ for any fixed $s \geq 1$. By Remark 2.16, the evolution operator of the perturbed NLS equation (1.1) S^t is C^1 in ϵ as a map from H^{n+4} to H^n for any fixed $n \geq 1$, $\epsilon \in [0, \epsilon_0)$, $\epsilon_0 > 0$. Also S^t is a C^1 map on H^n for any fixed $t > 0$, $n \geq 1$. Thus

$$\|q_\epsilon(T) - q_0(T)\|_{n+4} = \|S^T(q_\epsilon(0)) - S^T(q_0(0))\|_{n+4} \leq C_1\epsilon,$$

where $T > 0$ is large enough so that

$$q_0(T) \in W_{n+4}^{cs}|_{\epsilon=0}.$$

Our goal is to determine when $q_\epsilon(T) \in W_n^{cs}$ through Melnikov measurement. Let $q_\epsilon(T)$ and $q_0(T)$ have the coordinate expressions

$$(3.35) \quad q_\epsilon(T) = (\xi_k^{+, \epsilon}, v_\epsilon), \quad q_0(T) = (\xi_k^{+, 0}, v_0).$$

Let $\tilde{q}_\epsilon(T)$ be the unique point on W_{n+4}^{cs} , which has the same v -coordinate as $q_\epsilon(T)$,

$$\tilde{q}_\epsilon(T) = (\tilde{\xi}_k^{+, \epsilon}, v_\epsilon) \in W_{n+4}^{cs}.$$

By the Center-Stable Manifold Theorem, at points in the subset W_{n+4}^{cs} , W_n^{cs} is C^1 smooth in ϵ for $\epsilon \in [0, \epsilon_0)$, $\epsilon_0 > 0$, thus

$$(3.36) \quad \|q_\epsilon(T) - \tilde{q}_\epsilon(T)\|_n \leq C_2\epsilon.$$

Also our goal now is to determine when the signed distances

$$\xi_k^{+, \epsilon} - \tilde{\xi}_k^{+, \epsilon}, \quad (k = 1, 2)$$

are zero through Melnikov measurement. Equivalently, one can define the signed distances

$$\begin{aligned} d_k &= \langle \nabla F_k(q_0(T)), q_\epsilon(T) - \tilde{q}_\epsilon(T) \rangle \\ &\equiv \partial_q F_k(q_0(T))(q_\epsilon(T) - \tilde{q}_\epsilon(T)) \\ &\quad + \partial_{\bar{q}} F_k(q_0(T))(q_\epsilon(T) - \tilde{q}_\epsilon(T))^{-}, \quad k = 1, 2, \end{aligned}$$

where F_k and ∇F_k are given in the subsection on Integrable Theory, $q_0(t)$ is the homoclinic orbit also given in the same subsection. In fact, $q_\epsilon(t)$, $\tilde{q}_\epsilon(t)$, $q_0(t) \in H^n$, for any fixed $n \geq 1$. The rest of the derivation for Melnikov integrals is completely standard. For details, see [4] [6].

$$(3.37) \quad d_k = \epsilon M_k + o(\epsilon), \quad k = 1, 2,$$

where

$$\begin{aligned} M_k &= \int_{-\infty}^{+\infty} \int_0^{2\pi} [\partial_q F_k(q_0(t))(\partial_x^2 q_0(t) - \alpha q_0(t) + \beta) \\ &\quad + \partial_{\bar{q}} F_k(q_0(t))(\partial_x^2 \overline{q_0(t)} - \alpha \overline{q_0(t)} + \beta)] dx dt, \end{aligned}$$

where $q_0(t)$, $\partial_q F_k$, and $\partial_{\bar{q}} F_k$ are given in the subsection on Integrable Theory.

THEOREM 3.3. *There exists $\epsilon_0 > 0$, such that for any $\epsilon \in (0, \epsilon_0)$, there exists a domain $\mathcal{D}_\epsilon \subset \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ where $\omega \in (\frac{1}{2}, \frac{3}{2})/S$, S is a finite subset, and $\alpha\omega < \beta$. For any $(\alpha, \beta, \omega) \in \mathcal{D}_\epsilon$, there exists another orbit in $W^u(Q_\epsilon) \cap W_n^{cs}$ other than the unstable curve $\phi_{\sqrt{\epsilon}}^u$ (2.22) of Q_ϵ , for the perturbed nonlinear Schrödinger equation (1.1).*

PROOF. This theorem follows immediately from the explicit computation in the subsection on Evaluation of Melnikov Integrals and Second Distance, and the implicit function theorem. \square

3.3. The Second Measurement. The second measurement starts with the orbit obtained in Theorem 3.3, i.e. $q_\epsilon(t)$ where $q_\epsilon(T) = \tilde{q}_\epsilon(T)$. The goal is to determine when $q_\epsilon(t)$ is also in $W_n^s(Q_\epsilon)$. Recall that $W_n^s(Q_\epsilon)$ can be visualized as a codimension-one wall in W_n^{cs} with base curve in Π and with $\mathcal{O}(\sqrt{\epsilon})$ height. Thus we have to continue to follow $q_\epsilon(t)$ and $q_0(t)$ to a smaller neighborhood of Π . From the explicit expression of $q_0(t)$, we know that $q_0(t)$ approaches Π at the rate $\mathcal{O}(e^{-\mu t})$,

$$(3.38) \quad \mu = \min\{\sqrt{4\omega^2 - 1}, 4\sqrt{\omega^2 - 1}\}$$

(cf: (2.60)). Thus

$$(3.39) \quad \text{distance} \left\{ q_0\left(T + \frac{1}{\mu} |\ln \epsilon|\right), \Pi \right\} < C\epsilon.$$

LEMMA 3.4. *For all $t \in \left[T, T + \frac{1}{\mu} |\ln \epsilon|\right]$,*

$$(3.40) \quad \|q_\epsilon(t) - q_0(t)\|_n \leq \tilde{C}_1 \epsilon |\ln \epsilon|^2,$$

where $\tilde{C}_1 = \tilde{C}_1(T)$.

PROOF. We start with the system (2.91)-(2.92). Let

$$\begin{aligned} q_\epsilon(t) &= (\xi_k^{+, \epsilon}(t), J^\epsilon(t), \theta^\epsilon(t), h^\epsilon(t), \xi_k^{-, \epsilon}(t)), \\ q_0(t) &= (\xi_k^{+, 0}(t), J^0(t), \theta^0(t), h^0(t), \xi_k^{-, 0}(t)). \end{aligned}$$

Let $T_1(> T)$ be a time such that

$$(3.41) \quad \|q_\epsilon(t) - q_0(t)\|_n \leq \tilde{C}_2 \epsilon |\ln \epsilon|^2,$$

for all $t \in [T, T_1]$, where $\tilde{C}_2 = \tilde{C}_2(T)$ is independent of ϵ . From (3.36), such a T_1 exists. The proof will be completed through a continuation argument. For $t \in [T, T_1]$,

$$(3.42) \quad \begin{aligned} & \sum_{k=1,2} (|\xi_k^{+,0}(t)| + |\xi_k^{-,0}(t)|) + \|h^0(t)\|_n \leq C_3 r e^{-\frac{1}{2}\mu(t-T)}, \quad |J^0(t)| \leq C_4 \sqrt{\epsilon}, \\ & |J^\epsilon(t)| \leq |J^0(t)| + |J^\epsilon(t) - J^0(t)| \leq |J^0(t)| + \tilde{C}_2 \epsilon |\ln \epsilon|^2 \leq C_5 \sqrt{\epsilon}, \\ & \sum_{k=1,2} (|\xi_k^{+,\epsilon}(t)| + |\xi_k^{-,\epsilon}(t)|) + \|h^\epsilon(t)\|_n \leq C_3 r e^{-\frac{1}{2}\mu(t-T)} + \tilde{C}_2 \epsilon |\ln \epsilon|^2, \end{aligned}$$

where r is small. Since actually $q_\epsilon(t), q_0(t) \in H^n$ for any fixed $n \geq 1$, by Theorem 2.12,

$$(3.43) \quad |\xi_k^{+,\epsilon}(t) - \xi_k^{+,0}(t)| \leq C_6 \|v_\epsilon(t) - v_0(t)\|_n + C_7 \epsilon,$$

whenever $v_\epsilon(t), v_0(t) \in E_{n+4}(r)$, where $v_\epsilon(T) = v_\epsilon$ and $v_0(T) = v_0$ are defined in (3.35). Thus we only need to estimate $\|v_\epsilon(t) - v_0(t)\|_n$. From (2.92), we have for $t \in [T, T_1]$ that

$$(3.44) \quad v(t) = e^{A(t-T)} v(T) + \int_T^t e^{A(t-\tau)} \tilde{F}(\tau) d\tau.$$

Let $\Delta v(t) = v_\epsilon(t) - v_0(t)$. Then

$$(3.45) \quad \begin{aligned} \Delta v(t) &= [e^{A(t-T)} - e^{A|_{\epsilon=0}(t-T)}] v_0(T) + e^{A(t-T)} \Delta v(T) \\ &+ \int_T^t e^{A(t-\tau)} [\tilde{F}(\tau) - \tilde{F}(\tau)|_{\epsilon=0}] d\tau \\ &+ \int_T^t [e^{A(t-\tau)} - e^{A|_{\epsilon=0}(t-\tau)}] \tilde{F}(\tau)|_{\epsilon=0} d\tau. \end{aligned}$$

By the condition (3.42), we have for $t \in [T, T_1]$ that

$$(3.46) \quad \|\tilde{F}(t) - \tilde{F}(t)|_{\epsilon=0}\|_n \leq [C_8 \sqrt{\epsilon} + C_9 r e^{-\frac{1}{2}\mu(t-T)}] \epsilon |\ln \epsilon|^2.$$

Then

$$(3.47) \quad \|\Delta v(t)\|_n \leq C_{10} \epsilon (t - T) + C_{11} r \epsilon |\ln \epsilon|^2 + C_{12} \sqrt{\epsilon} (t - T)^2 \epsilon |\ln \epsilon|^2.$$

Thus by the continuation argument, for $t \in [T, T + \frac{1}{\mu} |\ln \epsilon|]$, there is a constant $\hat{C}_1 = \hat{C}_1(T)$,

$$(3.48) \quad \|\Delta v(t)\|_n \leq \hat{C}_1 \epsilon |\ln \epsilon|^2.$$

□

By Lemma 3.4 and estimate (3.39),

$$(3.49) \quad \text{distance} \left\{ q_\epsilon(T + \frac{1}{\mu} |\ln \epsilon|), \Pi \right\} < \tilde{C} \epsilon |\ln \epsilon|^2.$$

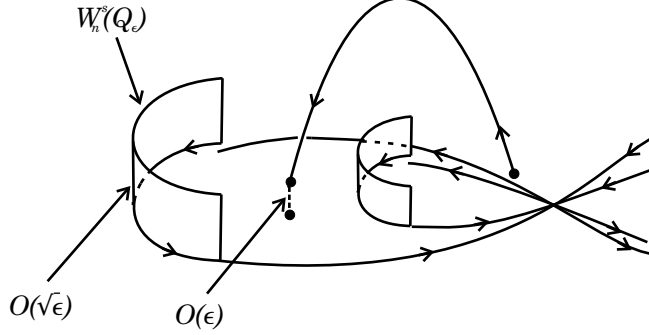


FIGURE 9. The second measurement.

Recall the fish-like singular level set given by \mathcal{H} (2.20), the width of the fish is of order $\mathcal{O}(\sqrt{\epsilon})$, and the length of the fish is of order $\mathcal{O}(1)$. Notice also that $q_0(t)$ has a phase shift

$$(3.50) \quad \theta_1^0 = \theta^0(T + \frac{1}{\mu} |\ln \epsilon|) - \theta^0(0).$$

For fixed β , changing α can induce $\mathcal{O}(1)$ change in the length of the fish, $\mathcal{O}(\sqrt{\epsilon})$ change in θ_1^0 , and $\mathcal{O}(1)$ change in $\theta^0(0)$. See Figure 9 for an illustration. The leading order signed distance from $q_\epsilon(T + \frac{1}{\mu} |\ln \epsilon|)$ to $W_n^s(Q_\epsilon)$ can be defined as

$$(3.51) \quad \begin{aligned} \tilde{d} &= \mathcal{H}(j_0, \theta^0(0)) - \mathcal{H}(j_0, \theta^0(0) + \theta_1^0) \\ &= 2\omega [\alpha\omega\theta_1^0 + \beta[\sin \theta^0(0) - \sin(\theta^0(0) + \theta_1^0)]] , \end{aligned}$$

where \mathcal{H} is given in (2.20). The common zero of M_k (3.37) and \tilde{d} and the implicit function theorem imply the existence of a homoclinic orbit asymptotic to Q_ϵ . Much detailed arguments have been given to the signed distance \tilde{d} and the second measurement in [4] and [6].

3.4. Evaluation of Melnikov Integrals and Second Distance. It turns out that to the leading order, one can evaluate M_k (3.37) at $q_0(t)$ where $a = \omega$. Our goal in this subsection is to find the common zero of M_k (3.37) and \tilde{d} (3.51).

One Pair of Complex Double Points Case

$M_1 = 0$ and $\tilde{d} = 0$ lead to

$$(3.52) \quad M_1 = M^{(1)} + \alpha M^{(2)} + \beta \cos \gamma M^{(3)} = 0 ,$$

$$(3.53) \quad \beta \cos \gamma = \frac{\alpha\omega(\Delta\gamma)}{2 \sin \frac{\Delta\gamma}{2}} ,$$

where $\Delta\gamma = -4\vartheta_0$, $M^{(j)} = M^{(j)}(\omega)$, ($j = 1, 2, 3$), and

$$M^{(1)} = \omega^2 \int_{-\infty}^{+\infty} \int_0^{2\pi} (|u_1|^2 + |u_2|^2)^{-2} [\bar{u}_1^2 \partial_x^2 P - \bar{u}_2^2 \partial_x^2 \bar{P}] dx dt ,$$

$$M^{(2)} = \omega^2 \int_{-\infty}^{+\infty} \int_0^{2\pi} (|u_1|^2 + |u_2|^2)^{-2} [\bar{u}_2^2 \bar{P} - \bar{u}_1^2 P] dx dt ,$$

$$M^{(3)} = \omega \int_{-\infty}^{+\infty} \int_0^{2\pi} (|u_1|^2 + |u_2|^2)^{-2} [\bar{u}_1^2 - \bar{u}_2^2] dx dt ,$$

and P is given by $Q = q_c P$, and Q is given in (3.13). Equations (3.52) and (3.53) define a codimension-one surface in the space of (α, β, ω) , given by

$$\alpha = \frac{1}{\kappa(\omega)} ,$$

where

$$\kappa(\omega) = -[2M^{(2)} \sin \frac{\Delta\gamma}{2} + M^{(3)} \omega(\Delta\gamma)] [2M^{(1)} \sin \frac{\Delta\gamma}{2}]^{-1} ,$$

and its graph is plotted in Figure 10.

Two Pairs of Complex Double Points Case

$M_j = 0$ ($j = 1, 2$) and $\tilde{d} = 0$ lead to

$$(3.54) \quad M_j = M_j^{(1)} + \alpha M_j^{(2)} + \beta \cos \gamma M_j^{(3)} + \beta \sin \gamma M_j^{(4)} = 0 , \quad (j = 1, 2)$$

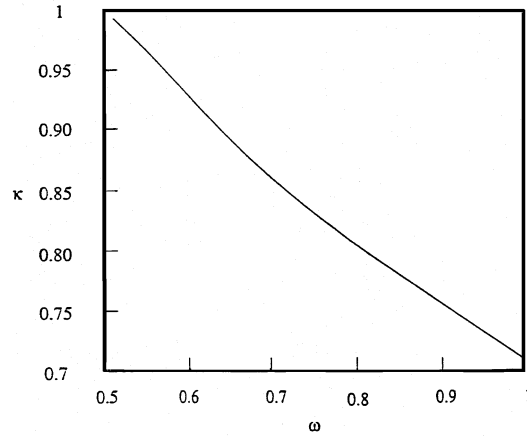
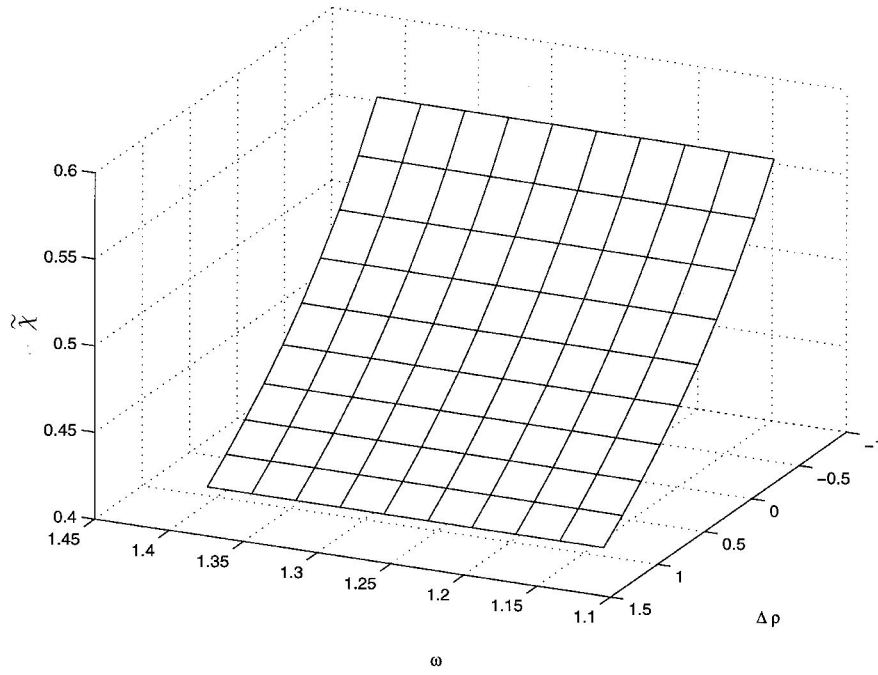
$$(3.55) \quad \beta \cos \gamma = \frac{\alpha \omega \widetilde{\Delta\gamma}}{2 \sin \frac{\widetilde{\Delta\gamma}}{2}} ,$$

where $\widetilde{\Delta\gamma} = -4(\vartheta_0 + \hat{\vartheta}_0)$, $M_j^{(l)} = M_j^{(l)}(\omega, \Delta\rho)$, ($j = 1, 2$, $l = 1, 2, 3, 4$), $\Delta\rho = 2\hat{\sigma}\sigma^{-1}\rho - \hat{\rho}$, and

$$\begin{aligned} M_1^{(1)} &= \omega^2 \int_{-\infty}^{+\infty} \int_0^{2\pi} [S_2^2 \partial_x^2 \tilde{P} - S_1^2 \partial_x^2 \bar{\tilde{P}}] dx dt , \\ M_1^{(2)} &= \omega^2 \int_{-\infty}^{+\infty} \int_0^{2\pi} [S_1^2 \bar{\tilde{P}} - S_2^2 \tilde{P}] dx dt , \\ M_1^{(3)} &= \omega \int_{-\infty}^{+\infty} \int_0^{2\pi} [S_2^2 - S_1^2] dx dt , \\ M_1^{(4)} &= i\omega \int_{-\infty}^{+\infty} \int_0^{2\pi} [S_2^2 + S_1^2] dx dt , \end{aligned}$$

and \tilde{P} is given by $\tilde{Q} = q_c \tilde{P}$, and \tilde{Q} is given in (3.17) and (3.19). $M_2^{(l)}$ can be obtained from $M_1^{(l)}$ ($l = 1, 2, 3, 4$) by replacing S_m by \hat{S}_m ($m = 1, 2$). Equations (3.54) and (3.55) define a codimension-one surface in the space of (α, β, ω) , given by

$$\begin{aligned} \alpha &= \frac{1}{\tilde{\chi}(\omega, \Delta\rho)} , \\ \beta &= \beta(\omega, \Delta\rho) = \left[(\alpha \omega \widetilde{\Delta\gamma})^2 (2 \sin \frac{\widetilde{\Delta\gamma}}{2})^{-2} \right. \\ &\quad \left. + (M_1^{(4)})^{-2} [M_1^{(1)} + \alpha(M_1^{(2)} + M_1^{(3)} \omega \widetilde{\Delta\gamma} (2 \sin \frac{\widetilde{\Delta\gamma}}{2})^{-1})]^2 \right]^{1/2} , \end{aligned}$$

FIGURE 10. The curve of $\kappa(\omega)$.FIGURE 11. The surface of $\tilde{\chi}(\omega, \Delta\rho)$.

where

$$\begin{aligned} \tilde{\chi}(\omega, \Delta\rho) = & (M_2^{(1)} M_1^{(4)} - M_1^{(1)} M_2^{(4)})^{-1} \left[(M_1^{(2)} M_2^{(4)} - M_2^{(2)} M_1^{(4)}) \right. \\ & \left. + \omega \widetilde{\Delta\gamma} (2 \sin \frac{\widetilde{\Delta\gamma}}{2})^{-1} (M_1^{(3)} M_2^{(4)} - M_2^{(3)} M_1^{(4)}) \right], \end{aligned}$$

and its graph is plotted in Figure 11.

3.5. Statement of the Main Theorem.

THEOREM 3.5 (Main Theorem). *There exists a $\epsilon_0 > 0$, such that for any $\epsilon \in (0, \epsilon_0)$, there exists a codimension 1 surface in the space of $(\alpha, \beta, \omega) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ where $\omega \in (\frac{1}{2}, \frac{3}{2})/S$, S is a finite subset, and $\alpha\omega < \beta$. For any (α, β, ω) on the codimension-one surface, the perturbed nonlinear Schrödinger equation (1.1) possesses a homoclinic orbit asymptotic to the saddle Q_ϵ (2.10). The codimension 1 surface has the approximate representation given in the subsection on Evaluation of Melnikov Integrals and Second Measurement.*

PROOF. From the explicit computation in last subsection and the implicit function theorem, d_k and \tilde{d} are zero for the parameter values specified in the theorem \square .

References

- [1] R. Adams. *Sobolev Space*. Academic Press, New York, 1975.
- [2] J. Hadamard. Sur Litteration et les Solutions Asymptotiques des Equations Differentielles. *Bull. Soc. Math. France*, 29:224–228, 1901.
- [3] Y. Li. Smale Horseshoes and Symbolic Dynamics in Perturbed Nonlinear Schrödinger Equations. *J. Nonlinear Sci.*, 9, no.4:363–415, 1999.
- [4] Y. Li, D. McLaughlin, J. Shatah, and S. Wiggins. Persistent Homoclinic Orbits for a Perturbed Nonlinear Schrödinger equation. *Comm. Pure Appl. Math.*, XLIX:1175–1255, 1996.
- [5] Y. Li and D. W. McLaughlin. Morse and Melnikov Functions for NLS Pde's. *Commun. Math. Phys.*, 162:175–214, 1994.
- [6] Y. Li and D. W. McLaughlin. Homoclinic Orbits and Chaos in Perturbed Discrete NLS System. Part I Homoclinic Orbits. *Journal of Nonlinear Sciences*, 7:211–269, 1997.
- [7] O. Perron. Die Stabilitätsfrage bei Differentialgleichungssysteme. *Math. Z.*, 32:703–728, 1930.
- [8] D. Sattinger and V. Zurburg. Gauge Theory of Bäcklund Transformations. II. *Phys. D.*, 26, no.1-3:225–250, 1987.
- [9] C. Zeng. Erratum: Homoclinic Orbits for a Perturbed Nonlinear Schrödinger Equation. *Comm. Pure Appl. Math.*, 53, no.12:1603–1605, 2000.
- [10] C. Zeng. Homoclinic Orbits for a Perturbed Nonlinear Schrödinger Equation. *Comm. Pure Appl. Math.*, 53, no.10:1222–1283, 2000.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211
E-mail address: cli@math.missouri.edu