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Investigation of the long time dynamics of a diffusive three species aquatic model

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ABSTRACT. We consider Upadhay's three species aquatic model with the inclusion of spatial spread. We show the existence of a $H^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)$ bounded absorbing set in the phase space $L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)$. We then derive uniform estimates to tackle the question of asymptotic compactness of the semi-group for the system in the Sobolev space $H^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)$. Via these we demonstrate the existence of a global attractor for the system which is compact in $H^2(\Omega)$ and attracts all bounded sets in $L^2(\Omega)$ in the $H^2(\Omega)$ topology.

CONTENTS

1. Introduction

One of the most important areas in ecology, and perhaps the biological sciences in general, is the analysis and modelling of food chains. These essentially comprise of the predator-prey relations between species in a given ecosystem [2]. The understanding of food chain models, or lack thereof has intrigued biologists for quite a while [11], [32]. These models have also attracted considerable interest from mathematicians. A possible reason being that these systems have all the ingredients needed of chaos. This includes high dimensional dynamics, non-linearities and coupling. Chaos being of immense interest to

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the dynamical systems community makes the study of food chain models a natural hunting ground for mathematicians. An interesting subclass of such models are three species coupled models. As the name suggests they are models for the food chain dynamics between three given species. They include cases where there is both a specialist predator and a generalist predator. Or the case where there are two competing preys, or perhaps two competing predators. See [4]. These have immense application in modeling tri-trophic food environments [6]. They have also been used successfully employed in understanding pattern formation in various biological contexts [44], [45], [46]. On another note, these models have also been in the limelight due to the following unanswered question. Why is the chaotic behavior of such systems although established theoretically, by dynamicists, not often observed in the wild, by ecologists [1], [47], [48]? Also see [5],[3], [47], [28] to this end. Thus there is a plethora of reasons why one might consider the study of food chain models to be both an intriguing and challenging intellectual pursuit.

The current manuscript is our second in a series of works concerned with one such three species model. Namely the well known Upadhyay model [7]. In [7], Upadhyay proposed a generalized model for an aquatic ecological food chain system consisting of TPP, Zooplankton and Molluscs. He introduced mutual interference in all the three populations, by adding an extra mortality term in the Zooplankton population. He also took into account the realistic toxin liberation process of the TPP population. His work thus generalized several other well known models in the literature see [3] and [29]. However, Upadhyay's model as it stands, is an ODE model. In [10], we incorporated spatial spread into the model, thus proposing a diffusive three species aquatic model. Our reasoning being that spatial spread of the species is quite natural and needs to be accounted for. We were thus faced with analysing an infinite dimensional dynamical system. Via PDE techniques we were able to establish the well posedness of this diffusive model. We showed the existence of a unique weak solution to the system, which is actually a strong solution via further regularity.

Before we outline our current goals, lets recall certain relevant findings from [7]. These will essentially set the tone for our current investigations. Upadhyay observed that an increase in the strength of toxic substance released by toxin producing phytoplankton population, represented by the parameter θ reduces the propensity of chaotic dynamics and changes the state of chaos to limit cycle and finally settles down to stable focus. This is seen clearly in the bifurcation diagram in figure 1, where the successive maxima of x_3 labelled z is mapped against θ.

The simulations from [7] show chaos for various ranges of the parameter space, including the mutual interference parameters and the rate of toxin release. He also finds states of extinction for certain species in certain parameter ranges, just as stable focus and limit cycles are also found. In [7] the non diffusive system is integrated numerically using sixth order Runge-Kutta method along with a predictor corrector method. It is observed that the system has a chaotic solution at the following set of parameter values $a_1 = 1.93$, $b_1 = 0.06$, $w_0 = 1$, $D_0 = 10$, $a_2 = 1$,$w_1 = 2$, $D_1 = 10$, $w_2 = 0.405$, $D_2 = 10$, $c = 0.03$, $w_3 = 1$, $D_3 = 20$, $m_1 = 1$, $m_2 = 1$, $m_3 = 2$, $\theta = 0$. See figure 1 for a attractor of the system for these values. The chaos is continually observed for small values of θ and for $1 < m_i \leq 3$.

It is well known that usually, under the action of diffusion, dynamical systems tend to smooth out. This mechanism is commonly referred to as "dissipation" [12], [47]. Thus we are lead to believe that even in the diffusive case there should be a global attractor, that supports these dynamics. This would include states of extinction and stable focus for

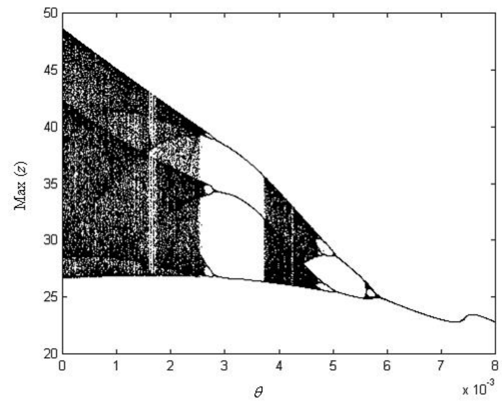

FIGURE 1. Bifurcation diagram for Holling type II response

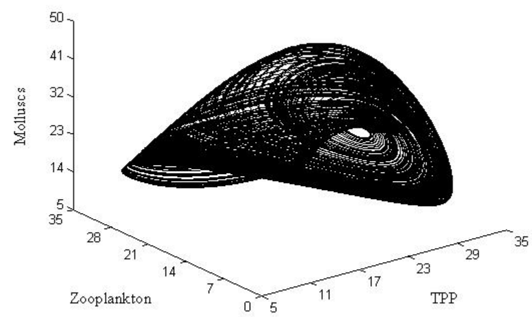

FIGURE 2. Attractor of the non-diffusive model

large values of θ, limit cycle for intermediate values, and chaos for small values and for $1 < m_i \leq 3$, as reported in [7].

Our goal in the current manuscript is to show this precisely. We take our cue from the non-diffusive case and investigate the long time dynamics of the diffusive model. The global attractor which is the object that encompasses the long-time dynamics, is by definition a compact invariant set in the phase space, to which all trajectories eventually evolve. To study this object in the PDE case often times involves making detailed estimates of various functional norms. Heuristically, the goal behind these is to show the existence of a bounded absorbing set in the phase space, and then to establish asymptotic compactness of the semi-group for the system of equations.

There is quite a bit of literature on the global and asymptotic dynamics of PDE's arising in ecological modelling, as applied to predator prey systems. Results for general cross diffusion systems were reported in [39]. More recent work presenting general theory has been done by Shim [40], [41]. Also of much interest have been systems with time delays [36], [33], [21],[22], [23]. More specific cross diffusion systems have been explored in [37], and systems with stage structure in [35],[34]. [43] has explored diffusion in tri-trophic food models. Recently results for coupled models, modelling invasive aquatic species, including the investigation of long time dynamics have been reported in [8], [9].

Zhang et al also investigated a cross diffusion PDE model with Holling type III functional response. They investigated the long time dynamics via construction of appropriate Lyapunov functions. Ko and Ryu have also investigated predator prey models with Holling type II responses, including the long time dynamics. Here there is scope of extensive prey refuge [38]. Questions of persistence and existence in two species models have been explored in [20] . Much has also been done on questions regarding stability and boundedness of solutions in long time of some of these model systems. See [25]. Pao has investigated in some detail the global dynamics of diffusive competition systems [17], [19],

Our approach in the current manuscript is as follows. We derive the the existence of a bounded absorbing sets in $H^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)$. We also derive the uniform estimates by means of which we tackle the question of asymptotic compactness of the semi-group for the model in $H^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)$. See [18] where similar techniques have been used, albeit in the context of fluids in porous media. Armed with these we will demonstrate the existence of a global attractor for the model. To this end we provide our main results, Theorems 5.3 and 5.7. We next show that this global attractor is finite dimensional and derive upper bounds on both it's Hausdorff and fractal dimensions. Thus entailing our result, Theorem 6.2. We lastly make some concluding remarks, tying our present findings to the findings in [7]. The devil however is in the details, all of which we will present subsequently.

2. The Mathematical Model

In [10], we considered Upadhyay's three species aquatic, with the inclusion of spatial spread. The diffusive model takes the following form,

$$\frac{\partial x_1}{\partial t} = \Delta x_1 + a_1 x_1 - b_1 x_1^2 - w_0\left(\frac{x_1}{x_1+D_0}\right)^{m_1}(x_2)^{m_2}, \tag{1}$$

$$\frac{\partial x_2}{\partial t} = \Delta x_2 - a_2 x_2 + w_1\left(\frac{x_1}{x_1+D_1}\right)^{m_1}(x_2)^{m_2} - w_2\left(\frac{x_2}{x_2+D_2}\right)^{m_2}(x_3)^{m_2} - \theta f_1(x_1)x_2, \tag{2}$$

$$\frac{\partial x_3}{\partial t} = \Delta x_3 + c x_3^{m_3} - w_3 f_2(x_2) x_3^{m_3}. \tag{3}$$

The problem is posed on $\Omega\subset\mathbb{R}^3$. Ω is bounded, and $\partial\Omega$ is assumed to be lipschitz. We consider Dirchlet boundary conditions

$$x_1 = 0 \text{ on } \partial\Omega,\ x_2 = 0 \text{ on } \partial\Omega,\ \ x_3 = 0 \text{ on } \partial\Omega, \tag{4}$$

We also impose suitable initial conditions

$$x_1(x,0) = x_{10},\ x_2(x,0) = x_{20},\ x_3(x,0) = x_{30}. \tag{5}$$

As is customary in most biological systems we assume x_1, x_2 and x_3 are bounded by their carrying capacities K_1, K_2 and K_3. We will assume

$$||x_1||_\infty \le K, \tag{6}$$

$$||x_2||_\infty \le K, \tag{7}$$

and

$$||x_3||_\infty \leq K. \tag{8}$$

Where

$$K = \max(K_1, K_2, K_3). \tag{9}$$

Also we assume

$$c \leq \frac{w_3}{D_3 + K} \leq w_3 f_2(x_2). \tag{10}$$

This prevents finite time blow up of (3). Essentially we model a food chain where a prey population x_1 is predated by a population x_2. The population x_2, in turn serves as favourite food for a population x_3. This interaction is represented by the above system. Here $m_i > 0$ for $i = 1,2,3$, also $a_1, a_2, b_1, w_0, w_1, w_2, w_3, c$ and D_0, D_1, D_2, D_3, D_4 are the positive constants. The parameters m_i for $i = 1,2,3$ are mutual interference parameters that model the inter species competition among predators when hunting for prey [**26**],[**27**],[**31**], [**30**]. The model hopes to effectively capture the dynamics between TPP population (prey) denoted x_1, which serves as the only food source for the specialist predator Zooplankton denoted x_2, which in turn, serves as the favourite food for the generalist predator Molluscs denoted x_3. However we enable spatial spread of all three species, TPP, Zooplankton and Molluscs. See [**7**], [**10**] for details. Before one begins the analysis aimed at investigating the long time dynamics of a model, it is customary to settle questions of well posedness for the model. To this end we recall certain results of interest from [**10**].

THEOREM 2.1. *Consider the diffusive three species aquatic model as defined via* (1)-(3)*. For initial data in $L^2(\Omega)$ there exists a unique weak solution (x_1, x_2, x_3) to the system such that*

$$(x_1, x_2, x_3) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega)) \tag{11}$$

and

$$\frac{\partial x_1}{\partial t} \in L^2(0, T; H^{-1}(\Omega)), \tag{12}$$

Also there exists a time t^*, depending on the $L^2(\Omega)$ norm of the initial data such that we have the following further regularity for the solutions,

$$(x_1, x_2, x_3) \in L^\infty(t^*, T; H_0^1(\Omega)) \cap L^2(t^*, T; H^2(\Omega)) \tag{13}$$

Furthermore (x_1, x_2, x_3) are continuous with respect to initial data.

LEMMA 2.2. *Consider two distinct nonlinear terms as they appear in* (1)*, from the diffusive three species aquatic model*

$$F(x_1, x_2) = -w_0 \left(\frac{x_1}{x_1 + D_0}\right)^{m_1} (x_2)^{m_2}, \tag{14}$$

$$F(y_1, y_2) = -w_0 \left(\frac{y_1}{y_1 + D_0}\right)^{m_1} (y_2)^{m_2} \tag{15}$$

Then for $m_1 \leq 3$ and $m_2 \leq 3$, the following estimate holds

$$|F(x_1,x_2)-F(y_1,y_2)|_2 \leq C_1|x_1-y_1|_2+C_2|x_2-y_2|_2 \tag{16}$$

THEOREM 2.3. *Consider the diffusive three species aquatic model, (1)- (3). Then for every $(x_{10},x_{20},x_{30}) \in H_0^1(\Omega)$, there is a $T>0$ such that the mild solution of the system in $H_0^1(\Omega)$, is a unique strong solution in $H_0^1(\Omega)$ with*

$$(x_1,x_2,x_3) \in C[t_0,T;H_0^1(\Omega)) \cap C_{loc}^{t_0,1-r}(0,T;H^{2-\delta}(\Omega)) \cap C(t_0,T;D(A)) \tag{17}$$

for $0 \leq r < 1$, $0 < \delta << 1$ and $t_0 > \max(t^*,t_1^*)$ [1]

3. Absorbing Sets in the Phase Space

The first step in proving the existence of a global attractor is the construction of bounded absorbing sets in the phase space in various norms. Let us say we have a semi-group $S(t): H \rightarrow H$. Recall the following definition

DEFINITION 3.1. *A bounded set $\mathscr{B}$ in a phase space H is called a bounded absorbing set if for each bounded subset U of H, there is a time $T=T(U)$, such that $S(t)U \subset \mathscr{B}$ for all $t>T$.*

We begin by making uniform estimates that will demonstrate the existence of such sets in various norms. In all the calculations made henceforth C,C_1,C_2,C_3 are generic constants that can change in their value from line to line, and sometimes within the same line if so required.

3.1. Existence of Absorbing Sets in $L^2(\Omega)$. We begin by multiplying (1) by x_1 and integrating by parts over Ω. This yields,

$$\frac{1}{2}\frac{d}{dt}|x_1|_2^2 = -|\nabla x_1|_2^2 + a_1|x_1|_2^2 - b_1|x_1|_3^3 - w_0\int_\Omega \frac{x_1^{m_1+1}}{(x_1+D_0)^{m_1}}x_2^{m_2}d\mathbf{x}, \tag{18}$$

We then use Holder's inequality, followed by Young's inequality to yield

$$\frac{1}{2}\frac{d}{dt}|x_1|_2^2 = -|\nabla x_1|_2^2 + b_1|x_1|_3^3 + C - b_1|x_1|_3^3 - w_0\int_\Omega \frac{x_1^{m_1+1}}{(x_1+D_0)^{m_1}}x_2^{m_2}d\mathbf{x}, \tag{19}$$

We now use Poincaire's inequality and the positivity of x_1 and x_2, thus the fact that

$$0 < \int_\Omega \frac{x_1^{m_1+1}}{(x_1+D_0)^{m_1}}x_2^{m_2}d\mathbf{x}, \tag{20}$$

to yield

$$\frac{d}{dt}|x_1|_2^2 + C_1|x_1|_2^2 \leq C_2, \tag{21}$$

here C_2 depends explicitly on b_1 and a_1.

Thus application of Gronwall's Lemma gives us the following estimate

[1]Here t^*,t_1^* depend only on the $L^2(\Omega)$ norm of the initial data, and have been explicitly worked out in [10]

$$|x_1|_2^2 \le e^{-C_1 t}|x_1(0)|_2^2 + \frac{C_2}{C_1}. \tag{22}$$

The above implies the existence of a time t_1 given explicitly by

$$t_1 = \frac{\ln(|x_1(0)|_2^2)}{C_1}, \tag{23}$$

such that for all $t \ge t_1$ the following estimate holds uniformly

$$|x_1|_2^2 \le 1 + \frac{C_2}{C_1}. \tag{24}$$

We now make a local in time estimate for ∇x_1. Integrating (19) in the time interval $[t_1, t_1+1]$ we obtain

$$\int_{t_1}^{t_1+1} |\nabla x_1|_2^2 dt \le |x_1(t_1)|_2^2 + \int_{t_1}^{t_1+1} C_2 dt \le C. \tag{25}$$

Thus via a mean value theorem for integrals there exists a time $t_2 \in [t_1, t_1+1]$ such that the following estimate holds

$$|\nabla x_1(t_2)|_2^2 \le C. \tag{26}$$

We now move on to showing existence of absorbing set for x_2 in $L^2(\Omega)$. We proceed by multiplying equation (2) by x_2 and integrating by parts over Ω to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|x_2|_2^2 \\ = \ & -|\nabla x_2|_2^2 - a_2|x_2|_2^2 + w_1\int_\Omega \frac{x_1^{m_1}}{(x_1+D_1)^{m_1}} x_2^{m_2+1} d\mathbf{x} \\ - \ & w_2 \int_\Omega \frac{x_2^{m_2+1}}{(x_2+D_2)^{m_2}} x_3^{m_2} d\mathbf{x} - \int_\Omega \theta f_1(x_1)(x_2)^2 d\mathbf{x}. \end{aligned} \tag{27}$$

Recall via the positivity of x_1, x_2 ,x_3 ,θ, w_2 and D_2 that

$$w_2 \int_\Omega \frac{x_2^{m_2+1}}{(x_2+D_2)^{m_2}} x_3^{m_2} d\mathbf{x} > 0, \tag{28}$$

and

$$\int_\Omega \theta (x_2)^2 f_1(x_1) d\mathbf{x} > 0. \tag{29}$$

We now use Poincaire's inequality in conjunction with the above estimates to yield

$$\frac{d}{dt}|x_2|_2^2 + C|x_2|_2^2 \le w_1 \int_\Omega \frac{x_1^{m_1}}{(x_1+D_1)^{m_1}} x_2^{m_2+1} d\mathbf{x} \le w_1 (K)^{m_2+1}|\mathbf{\Omega}|. \tag{30}$$

Gronwall's Lemma applied on the above yields

$$|x_2|_2^2 \leq e^{-Ct}|x_2(0)|_2^2 + \frac{w_1(K)^{m_2+1}|\Omega|}{C}. \tag{31}$$

The above implies the existence of a time t_3 given explicitly by

$$t_3 = \frac{\ln(|x_2(0)|_2^2)}{C}, \tag{32}$$

such that for all $t \geq t_3$ the following estimate holds uniformly

$$|x_2|_2^2 \leq 1 + \frac{w_1(K)^{m_2+1}|\Omega|}{C}. \tag{33}$$

We next derive local in time estimates for ∇x_2. From the earlier estimates without resorting to Poincaire's inequality we have

$$\frac{d}{dt}|x_2|_2^2 + C|\nabla x_2|_2^2 \leq w_1(K)^{m_2+1}|\Omega|. \tag{34}$$

Integrating the above in the time interval $[t_3, t_3+1]$ we obtain

$$\int_{t_3}^{t_3+1} |\nabla x_2|_2^2 dt \leq |x_2(t_3)|_2^2 + \int_{t_1}^{t_1+1} w_1(K)^{m_2+1}|\Omega| dt \leq C. \tag{35}$$

Thus using a mean value theorem for integrals there exists a time $t_4 \in [t_3, t_3+1]$ such that the following estimate holds

$$|\nabla x_2(t_4)|_2^2 \leq C. \tag{36}$$

We next show the existence of absorbing set for x_3 in $L^2(\Omega)$.

We multiply (3) by x_3 and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}|x_3|_2^2 = -|\nabla x_3|_2^2 + c\int_\Omega x_3^{m_3+1} d\mathbf{x} - w_3 \int_\Omega f_2(x_2) x_3^{m_3+1} d\mathbf{x}. \tag{37}$$

We now use the positivity of f_2, w_3 and x_3 , the bound on x_3, and Poincaire's inequality to obtain

$$\frac{d}{dt}|x_3|_2^2 + C|x_3|_2^2 \leq c\int_\Omega x_3^{m_3+1} d\mathbf{x} \leq c|\Omega|(K)^{m_3+1}. \tag{38}$$

Gronwall's inequality implies

$$|x_3|_2^2 \leq e^{-Ct}|x_3(0)|_2^2 + \frac{|\Omega|(K)^{m_3+1}}{C}. \tag{39}$$

The above implies the existence of a time t_5 given explicitly by

$$t_5 = \frac{\ln(|x_3(0)|_2^2)}{C} \tag{40}$$

such that for all $t \geq t_5$ the following estimate holds uniformly

$$|x_3|_2^2 \leq 1 + \frac{|\Omega|(K)^{m_3+1}}{C}. \tag{41}$$

We next make local in time estimates for ∇x_3.

From the earlier estimates without resorting to Poincaire's inequality we have

$$\frac{d}{dt}|x_3|_2^2 + C|\nabla x_3|_2^2 \leq w_1(K)^{m_3+1}|\Omega|. \tag{42}$$

Integrating the above in the time interval $[t_5, t_5+1]$ we obtain

$$\int_{t_5}^{t_5+1} |\nabla x_3|_2^2 dt \leq |x_3(t_5)|_2^2 + \int_{t_5}^{t_5+1} w_1(K)^{m_3+1}|\Omega| dt \leq C. \tag{43}$$

Thus using a mean value theorem for integrals there exists a time $t_6 \in [t_5, t_5+1]$ such that the following estimate holds

$$|\nabla x_3(t_6)|_2^2 \leq C. \tag{44}$$

Via the above estimates we can now state the following Lemma,

LEMMA 3.2. *Let (x_1, x_2, x_3) be solutions to the diffusive three species aquatic model with $(x_{10}, x_{20}, x_{30}) \in L^2(\Omega)$. There exists a time $t^* = \max(t_1, t_3, t_5)$, and a constant C independent of time and initial data, and dependent only on $K, m_i, a_i, b_1, c, w_i, D_i, \theta$, for $0 \leq i \leq 4$ such that for any $t > t^*$ the following uniform estimates hold :*

$$|x_1|_2^2 \leq C, \tag{45}$$

$$|x_2|_2^2 \leq C, \tag{46}$$

$$|x_3|_2^2 \leq C. \tag{47}$$

3.2. Existence of Absorbing Sets in $H_0^1(\Omega)$. We multiply (1) by $-\Delta x_1$ and integrate by parts over Ω to obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|\nabla x_1|_2^2 + |\Delta x_1|_2^2 \\ = \quad & -a_1 \int_\Omega x_1 \Delta x_1 d\mathbf{x} + b_1 \int_\Omega (x_1)^2 \Delta x_1 d\mathbf{x} \\ + \quad & w_0 \int_\Omega \frac{x_1^{m_1}}{(x_1+D_0)^{m_1}} x_2^{m_2} \Delta x_1 d\mathbf{x}. \end{aligned} \tag{48}$$

Integration by parts on the third term on the right hand side, followed by the application of Cauchy Schwartz and Young's inequalities on the last term on the right hand side yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla x_1|_2^2+|\Delta x_1|_2^2\\
\leq\ & -b_1\int_\Omega(2(\nabla x_1)^2x_1d\mathbf{x}+\frac{3}{4}|\Delta x_1|_2^2+4a_1^2|\Omega|K^2+4b_1^2|\Omega|K^4+w_0|\Omega|K^{2m_2}\\
\leq\ & \frac{3}{4}|\Delta x_1|_2^2+C. \qquad (49)
\end{aligned}$$

This follows via the positivity of b_1 and x_1 and thus implies

$$b_1\int_\Omega(2(\nabla x_1)^2x_1d\mathbf{x}>0. \qquad (50)$$

Thus we obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla x_1|_2^2+\frac{C}{4}|\nabla x_1|_2^2\leq C. \qquad (51)$$

Gronwall's Lemma applied to the above via integrating in the time interval $[t_2,t]$ yields

$$|\nabla x_1|_2^2\leq e^{-\frac{C}{2}(t-t_2)}|\nabla x_1(t_2)|_2^2+C. \qquad (52)$$

This implies the existence of a time t_7 defined explicitly by

$$t_7=\frac{2t_2+2\ln(|\nabla x_1(t_2)|_2^2)}{C}, \qquad (53)$$

such that for any $t>t_7$ the following estimate holds uniformly

$$|\nabla x_1|_2^2\leq C. \qquad (54)$$

We next show existence of absorbing set for x_2 in $H_0^1(\Omega)$. We multiply (2) by $-\Delta x_2$ and integrate by parts over Ω to yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla x_2|_2^2\\
\leq\ & -a_2|\nabla x_2|_2^2-|\Delta x_2|_2^2+w_1\int_\Omega(x_2)^{m_2}|\Delta x_2|d\mathbf{x}\\
+\ & w_2\int_\Omega x_3^{m_2}|\Delta x_2|d\mathbf{x}+\theta\int_\Omega x_2|\Delta x_2|d\mathbf{x}. \qquad (55)
\end{aligned}$$

Application of Cauchy Schwartz and Young's inequalities, and the bounds on x_2 and x_3 yield

$$\frac{1}{2}\frac{d}{dt}|\nabla x_2|_2^2+|\Delta x_2|_2^2\leq w_1K^{m_2}|\Omega|+w_2K^{m_2}|\Omega|+\theta K^2|\Omega|+\frac{1}{2}|\Delta x_2|_2^2. \qquad (56)$$

Poincaire's inequality now yields

$$\frac{1}{2}\frac{d}{dt}|\nabla x_2|_2^2+\frac{C}{2}|\nabla x_2|_2^2\leq(w_1+w_2+\theta)K^{m_2}|\Omega|. \qquad (57)$$

Gronwall's Lemma applied to the above via integrating in the time interval $[t_4,t]$ yields

$$|\nabla x_2|_2^2 \leq e^{-\frac{C}{2}(t-t_4)}|\nabla x_2(t_4)|_2^2 + C. \tag{58}$$

This implies the existence of a time t_8 defined explicitly by

$$t_8 = \frac{2t_4 + 2\ln(|\nabla x_2(t_4)|_2^2)}{C}, \tag{59}$$

such that for any $t > t_8$ the following estimate holds uniformly

$$|\nabla x_2|_2^2 \leq C. \tag{60}$$

We now show existence of absorbing set for x_3 in $H_0^1(\Omega)$. This is done by multiplying (3) by $-\Delta x_3$ and integrating by parts over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla x_3|_2^2 = -|\Delta x_3|_2^2 - c\int_\Omega (x_3)^{m_3}\Delta x_3 d\mathbf{x} + w_3\int_\Omega x_3^{m_2}\Delta x_3 d\mathbf{x}. \tag{61}$$

Application of Cauchy Schwartz and Young's inequalities on the last couple of terms on the right hand, and the use of the bounds on x_3 yield

$$\frac{1}{2}\frac{d}{dt}|\nabla x_3|_2^2 + C|\Delta x_3|_2^2 \leq \frac{C}{4}|\Delta x_3|_2^2 + C_1|\Omega|(K)^{2m_3} + \frac{C}{4}|\Delta x_3|_2^2 + C_2|\Omega|(K)^{2m_3}. \tag{62}$$

Now Poincaire's inequality yields

$$\frac{d}{dt}|\nabla x_3|_2^2 + \frac{C}{2}|\nabla x_3|_2^2 \leq C_3|\Omega|(K)^{2m_3}. \tag{63}$$

Gronwall's Lemma applied to the above via integrating in the time interval $[t_6, t]$ yields

$$|\nabla x_3|_2^2 \leq e^{-\frac{C}{2}(t-t_6)}|\nabla x_3(t_6)|_2^2 + C. \tag{64}$$

This implies the existence of a time t_{10} defined explicitly by

$$t_{10} = \frac{2t_6 + 2\ln(|\nabla x_3(t_6)|_2^2)}{C}, \tag{65}$$

such that for any $t > t_{10}$ the following estimate holds uniformly

$$|\nabla x_3|_2^2 \leq C. \tag{66}$$

Via the above estimates we can now state the following Lemma,

LEMMA 3.3. *Let (x_1, x_2, x_3) be solutions to the diffusive three species aquatic model with $(x_{10}, x_{20}, x_{30}) \in L^2(\Omega)$. There exists a time $t^{**} = \max(t_6, t_8, t_{10})$, and a constant C independent of time and initial data, and dependent only on $K, m_i, a_i, b_1, c, w_i, D_i, \theta$ for $0 \leq i \leq 4$, such that for any $t > t^{**}$ the following uniform estimates hold:*

$$|\nabla x_1|_2^2 \leq C, \tag{67}$$

$$|\nabla x_2|_2^2 \leq C, \tag{68}$$

$$|\nabla x_3|_2^2 \leq C. \tag{69}$$

3.3. Existence of Absorbing Sets in $H^2(\Omega)$. We begin by multiplying (1) by $\Delta^2 x_1$ and integrating by parts over Ω. This yields,

$$\begin{aligned} & \frac{d}{dt}|\Delta x_1|_2^2 \\ = & -|\nabla\Delta x_1|_2^2 - a_1\int_\Omega \nabla(\Delta x_1)\cdot\nabla x_1 d\mathbf{x} + b_1\int_\Omega 2x_1\nabla x_1\nabla(\Delta x_1)d\mathbf{x} \\ + & w_0\int_\Omega \nabla\left(\frac{x_1^{m_1}}{(x_1+D_0)^{m_1}}x_2^{m_2}\right)\nabla(\Delta x_1)d\mathbf{x} \end{aligned} \tag{70}$$

We then use Holder's inequality, followed by Young's inequality to yield

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|\Delta x_1|_2^2 \\ \leq & -|\nabla\Delta x_1|_2^2 + \frac{1}{4}|\nabla\Delta x_1|_2^2 + C|\nabla x_1|_2^2 + \frac{1}{4}|\nabla\Delta x_1|_2^2 + CKb_1|\nabla x_1|_2^2 \\ - & w_0\int_\Omega \frac{x_1^{m_1}}{(x_1+D_0)^{m_1}} m_2 x_2^{m_2-1}\nabla x_2\nabla(\Delta x_1)d\mathbf{x} \\ - & w_0\int_\Omega x_2^{m_2}\left(\frac{((x_1+D_0)^{m_1}m_1x_1^{m_1-1}\nabla x_1) - (x_1^{m_1}\nabla x_1)(m_1(x_1+D_0)^{m_1-1})}{(x_1+D_0)^{2m_1}}\right)\nabla(\Delta x_1)d\mathbf{x} \\ \leq & -\frac{1}{2}|\nabla\Delta x_1|_2^2 + C_1|\nabla x_1|_2^2 + C_2|\nabla x_2|_2^2 + \frac{1}{8}|\nabla\Delta x_1|_2^2 \\ + & \frac{1}{8}|\nabla\Delta x_1|_2^2 + C_3|\nabla x_2|_2^2 \\ \leq & -\frac{1}{4}|\nabla\Delta x_1|_2^2 + C|\nabla x_1|_2^2 + C|\nabla x_2|_2^2 \end{aligned} \tag{71}$$

We now use the compact Sobolev embedding

$$H^3(\Omega) \hookrightarrow H^2(\Omega) \tag{72}$$

to yield

$$\frac{d}{dt}|\Delta x_1|_2^2 + \frac{C}{2}|\Delta x_1|_2^2 \leq C|\nabla x_1|_2^2 + C|\nabla x_2|_2^2. \tag{73}$$

Note integrating (49) in the time interval $[t^{**}, t^{**}+1]$ yields the existence of a time t_2^{**} such that for $t > t_2^{**}$ we have

$$|\Delta x_1(t_2^{**})|_2^2 \leq |\nabla x_1(t_2^{**})|_2^2 + C \leq C. \tag{74}$$

This follows via Lemma 3.3. Now Gronwall's Lemma applied to (73) via integrating in the time interval $[t_2^{**}, t]$ yields

$$|\Delta x_1|_2^2 \leq e^{-\frac{C}{2}(t-t_2^{**})}|\Delta x_1(t_2^{**})|_2^2 + C \leq Ce^{-\frac{C}{2}(t-t_2^{**})} + C. \tag{75}$$

This implies the existence of a time t_{11} defined explicitly by

$$t_{11} = \frac{2t^{**} + 2\ln(|\Delta x_1(t_2^{**})|_2^2)}{C}, \tag{76}$$

such that for any $t > t_{11}$ the following estimate holds uniformly

$$|\Delta x_1|_2^2 \leq C. \tag{77}$$

We now show existence of absorbing sets for x_2 in $H^2(\Omega)$.

We begin by multiplying (2) by $\Delta^2 x_2$ and integrating by parts over Ω. This yields,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\Delta x_2|_2^2 \\
= \quad & -|\nabla \Delta x_2|_2^2 - a_2|\Delta x_2|_2^2 - w_1 \int_\Omega \nabla\left(\frac{x_1^{m_1}}{(x_1+D_1)^{m_1}} x_2^{m_2}\right)\nabla(\Delta x_2)d\mathbf{x} \\
+ \quad & w_2 \int_\Omega \nabla\left(\frac{x_2^{m_2}}{(x_2+D_2)^{m_2}} x_3^{m_2}\right)\nabla(\Delta x_2)d\mathbf{x} \\
+ \quad & \theta \int_\Omega \nabla\left(f_1(x_1)x_2\right)\nabla(\Delta x_2)d\mathbf{x}.
\end{aligned} \tag{78}$$

We then use Holder's inequality, followed by Young's inequality to yield

$$\begin{aligned}
&\frac{d}{dt}|\Delta x_2|_2^2 \\
\leq \quad & -|\nabla \Delta x_2|_2^2 - a_2|\Delta x_2|_2^2 + \frac{1}{8}|\nabla \Delta x_2|_2^2 + C|\nabla x_2|_2^2 \\
- \quad & w_1 \int_\Omega \frac{x_1^{m_1}}{(x_1+D_1)^{m_1}} m_2 x_2^{m_2-1}\nabla x_2 \nabla(\Delta x_2)d\mathbf{x} \\
- \quad & w_1 \int_\Omega x_2^{m_2}\left(\frac{((x_1+D_1)^{m_1} m_1 x_1^{m_1-1}\nabla x_1) - (x_1^{m_1}\nabla x_1)(m_1(x_1+D_1)^{m_1-1})}{(x_1+D_1)^{2m_1}}\right)\nabla(\Delta x_2)d\mathbf{x} \\
+ \quad & w_2 \int_\Omega \frac{x_2^{m_2}}{(x_2+D_2)^{m_2}} m_2 x_3^{m_2-1}\nabla x_3 \nabla(\Delta x_2)d\mathbf{x} \\
+ \quad & w_2 \int_\Omega x_3^{m_2}\left(\frac{((x_2+D_2)^{m_2} m_2 x_2^{m_2-1}\nabla x_2) - (x_2^{m_2}\nabla x_2)(m_2(x_2+D_2)^{m_2-1})}{(x_2+D_2)^{2m_2}}\right)\nabla(\Delta x_2)d\mathbf{x} \\
\leq \quad & -|\nabla \Delta x_2|_2^2 + \frac{1}{8}|\nabla \Delta x_2|_2^2 + \frac{1}{8}|\nabla \Delta x_2|_2^2 + \frac{1}{8}|\nabla \Delta x_2|_2^2 + \frac{1}{8}|\nabla \Delta x_2|_2^2 \\
+ \quad & C_1|\nabla x_1|_2^2 + C_2|\nabla x_2|_2^2 + C_3|\nabla x_3|_2^2
\end{aligned} \tag{79}$$

We now use the compact Sobolev embedding

$$H^3(\Omega) \hookrightarrow H^2(\Omega), \tag{80}$$

to yield

$$\frac{d}{dt}|\Delta x_2|_2^2 + \frac{C}{2}|\Delta x_2|_2^2 \leq C(|\nabla x_1|_2^2 + |\nabla x_2|_2^2 + |\nabla x_3|_2^2). \tag{81}$$

Note integrating (56) in the time interval $[t^{**}, t^{**}+1]$ yields the existence of a time t_2^{**} such that for $t > t_2^{**}$ we have

$$|\Delta x_2(t_2^{**})|_2^2 \leq C. \tag{82}$$

Now Gronwall's Lemma applied to (81) via integrating in the time interval $[t_2^{**}, t]$ yields

$$|\Delta x_2|_2^2 \leq e^{-\frac{C}{2}(t-t_2^{**})}|\Delta x_2(t_2^{**})|_2^2 + C. \tag{83}$$

This implies the existence of a time t_{12} defined explicitly by

$$t_{12} = \frac{2t^{**} + 2\ln(|\Delta x_2(t_2^{**})|_2^2)}{C}, \tag{84}$$

such that for any $t > t_{12}$ the following estimate holds uniformly

$$|\Delta x_2|_2^2 \leq C. \tag{85}$$

The estimates for Δx_3 are made similarly in essence we have the existence of a time t_{13} defined explicitly by

$$t_{13} = \frac{2t_3^{**} + 2\ln(|\Delta x_3(t_3^{**})|_2^2)}{C}, \tag{86}$$

such that for any $t > t_{13}$ the following estimate holds uniformly

$$|\Delta x_3|_2^2 \leq C. \tag{87}$$

Via the above estimates we can now state the following Lemma,

LEMMA 3.4. *Let (x_1, x_2, x_3) be solutions to the diffusive three species aquatic model with $(x_{10}, x_{20}, x_{30}) \in L^2(\Omega)$. There exists a time $t^{***} = \max(t_{11}, t_{12}, t_{13})$, and a constant C independent of time and initial data, and dependent only on $K, m_i, a_i, b_1, c, w_i, D_i, \theta$, for $0 \leq i \leq 4$ such that for any $t > t^{***}$ the following uniform estimates hold :*

$$|\Delta x_1|_2^2 \leq C, \tag{88}$$

$$|\Delta x_2|_2^2 \leq C, \tag{89}$$

$$|\Delta x_3|_2^2 \leq C. \tag{90}$$

4. Further Uniform Estimates

In this section we make estimates on certain additional norms. These will be required to show the asymptotic compactness of the semigroup for the system, which is held of till later. The requisite estimates are completed next. We begin by making a uniform estimate of $\frac{\partial x_1}{\partial t}$ in $L^2(\Omega)$. The method is to apply brute force on equation (1) to yield that for $t > t^{***}$ we have

$$\begin{aligned}
&\left|\frac{\partial x_1}{\partial t}\right|_2^2 \\
= \quad & |\Delta x_1 + a_1 x_1 - b_1 x_1^2 - w_0 \left(\frac{x_1}{x_1+D_0}\right)^{m_1} (x_2)^{m_2}|_2^2 \\
\leq \quad & C\left(|\Delta x_1|_2^2 + a_1^2|x_1|_2^2 + b_1^2|x_1|_4^4 + w_0^2\left|\left(\frac{x_1}{x_1+D_0}\right)^{m_1}(x_2)^{m_2}\right|_2^2\right) \\
\leq \quad & C|\Delta x_1|_2^2 + a_1^2|x_1|_2^2 + b_1^2|x_1|_4^4 + w_0^2\left|\left(\frac{x_1}{x_1+D_0}\right)^{m_1}(x_2)^{m_2}\right|_2^2 \\
\leq \quad & C\left(|\Delta x_1|_2^2 + a_1^2|x_1|_2^2 + b_1^2|x_1|_4^4 + w_0^2|x_2|_{2m_2}^{2m_2}\right) \\
\leq \quad & C|\Delta x_1|_2^2 + C|\Delta x_1|_2^4 + C|\Delta x_2|_2^{2m_2} \\
\leq \quad & C. \qquad (91)
\end{aligned}$$

This follows via the compact Sobolev embedding of

$$H^2(\Omega) \hookrightarrow L^{2m_2}(\Omega) \hookrightarrow L^4(\Omega) \hookrightarrow L^2(\Omega) \text{ for } m_2 \leq 3. \qquad (92)$$

The similar method works for $\frac{\partial x_2}{\partial t}$ and $\frac{\partial x_3}{\partial t}$. Via the above estimates we can now state the following Lemma,

LEMMA 4.1. *Let (x_1, x_2, x_3) be solutions to the diffusive three species aquatic model with $(x_{10}, x_{20}, x_{30}) \in L^2(\Omega)$. There exists a time $t^{***} = \max(t_{11}, t_{12}, t_{13})$, and a constant C independent of time and initial data, and dependent only on $K, m_i, a_1, b_1, c, w_i, D_1$, such that for any $t > t^{***}$ the following uniform estimates hold:*

$$\left|\frac{\partial x_1}{\partial t}\right|_2^2 \leq C, \qquad (93)$$

$$\left|\frac{\partial x_2}{\partial t}\right|_2^2 \leq C, \qquad (94)$$

$$\left|\frac{\partial x_3}{\partial t}\right|_2^2 \leq C. \qquad (95)$$

We will next make a uniform estimate of $\nabla \frac{\partial x_1}{\partial t}$ in $L^2(\Omega)$. This estimate is quite tricky due to the structure of the equations. We are thus required to make an intermediate estimate. We take the partial of (1) with respect to 't' to yield

$$\begin{aligned}
&\frac{\partial^2 x_1}{\partial t^2} \\
= \quad & \frac{\partial \Delta x_1}{\partial t} + a_1\frac{\partial x_1}{\partial t} - 2b_1 x_1 \frac{\partial x_1}{\partial t} - w_0\left(\frac{x_1}{x_1+D_0}\right)^{m_1} m_2 x_2^{m_2-1}\frac{\partial x_2}{\partial t} \\
- \quad & w_0 x_2^{m_2}\left(\frac{x_1}{x_1+D_0}\right)^{m_1-1}\left(\frac{m_1}{x_1+D_0}\frac{\partial x_1}{\partial t} - \frac{m_1 x_1}{(x_1+D_0)^2}\frac{\partial x_1}{\partial t}\right). \qquad (96)
\end{aligned}$$

We now multiply the above by $\frac{\partial x_1}{\partial t}$ and integrate by parts over Ω, and use Holder's and Young's inequalities and earlier estimates to yield

$$\frac{\partial}{\partial t}\left|\frac{\partial x_1}{\partial t}\right|_2^2+\left|\frac{\partial \nabla x_1}{\partial t}\right|_2^2 \leq C\left|\frac{\partial x_1}{\partial t}\right|_2^2+C\left|\frac{\partial x_2}{\partial t}\right|_2^2, \tag{97}$$

We integrate the above in the time interval $[t^{***}, t^{***}+1]$ to yield

$$\int_{t^{***}}^{t^{***}+1}\left|\frac{\partial \nabla x_1}{\partial t}\right|_2^2 dt \leq \int_{t^{***}}^{t^{***}+1}\left(\left|\frac{\partial x_1}{\partial t}\right|_2^2+\left|\frac{\partial x_2}{\partial t}\right|_2^2\right)dt+\left|\frac{\partial x_1(t^{***})}{\partial t}\right|_2^2 \leq C. \tag{98}$$

Thus via the mean value theorem for integrals there exists a time

$$t^{****} \in [t^{***}, t^{***}+1] \tag{99}$$

such that

$$\left|\frac{\partial \nabla x_1(t^{****})}{\partial t}\right|_2^2 \leq C. \tag{100}$$

We can now proceed to make an estimate for the $L^2(\Omega)$ norm of $\nabla\frac{\partial x_1}{\partial t}$. Now the standard energy method for $\left|\frac{\partial \nabla x_1}{\partial t}\right|_2^2$ will involve an estimate of $\left|\frac{\partial \nabla x_2}{\partial t}\right|_2^2$. This we do not have, without manipulating the equation for x_2. If we attempt to manipulate the equation for x_2 to make this estimate, it in turn will involve an estimate of $\left|\frac{\partial \nabla x_3}{\partial t}\right|_2^2$. To get around this circuitous structure, we define a variable W by the following equation.

$$W=\left|\frac{\partial \nabla x_1}{\partial t}\right|_2^2+\left|\frac{\partial \nabla x_1}{\partial t}\right|_2^2+\left|\frac{\partial \nabla x_1}{\partial t}\right|_2^2. \tag{101}$$

we will attempt to derive a differential inequality for W. Recall the equation for $\frac{\partial^2 x_1}{\partial t^2}$

$$\begin{aligned}
&\frac{\partial^2 x_1}{\partial t^2}\\
=\;& \frac{\partial \Delta x_1}{\partial t}+a_1\frac{\partial x_1}{\partial t}-2b_1x_1\frac{\partial x_1}{\partial t}-w_0\left(\frac{x_1}{x_1+D_0}\right)^{m_1} m_2x_2^{m_2-1}\frac{\partial x_2}{\partial t}\\
-\;& w_0x_2^{m_2}\left(\frac{x_1}{x_1+D_0}\right)^{m_1-1}\left(\frac{m_1}{x_1+D_0}\frac{\partial x_1}{\partial t}-\frac{m_1x_1}{(x_1+D_0)^2}\frac{\partial x_1}{\partial t}\right),
\end{aligned} \tag{102}$$

We multiply the above by $-\frac{\partial \Delta x_1}{\partial t}$ and integrate by parts over Ω, and use Holder's and Young's inequalities to yield

$$\begin{aligned}
&\frac{\partial}{\partial t}\left|\frac{\partial \nabla x_1}{\partial t}\right|_2^2+\left|\frac{\partial \Delta x_1}{\partial t}\right|_2^2\\
\leq\;& -a_1\left|\frac{\partial \Delta x_1}{\partial t}\right|_2^2+C\left|\frac{\partial \nabla x_1}{\partial t}\right|_2^2+C\left(\left|\frac{\partial x_1}{\partial t}\right|_2^2+\left|\frac{\partial x_2}{\partial t}\right|_2^2\right).
\end{aligned} \tag{103}$$

Note via the compact embedding of

$$H^2(\Omega) \hookrightarrow H^1(\Omega) \tag{104}$$

We obtain

$$\frac{\partial}{\partial t}\left|\frac{\partial \nabla x_1}{\partial t}\right|_2^2 \leq C_1\left|\frac{\partial \nabla x_1}{\partial t}\right|_2^2 + C_2\left(\left|\frac{\partial x_1}{\partial t}\right|_2^2 + \left|\frac{\partial x_2}{\partial t}\right|_2^2\right). \tag{105}$$

Similarly we can derive

$$\frac{\partial}{\partial t}\left|\frac{\partial \nabla x_2}{\partial t}\right|_2^2 \leq C_1\left|\frac{\partial \nabla x_2}{\partial t}\right|_2^2 + C_2\left(\left|\frac{\partial x_2}{\partial t}\right|_2^2 + \left|\frac{\partial x_3}{\partial t}\right|_2^2\right), \tag{106}$$

and

$$\frac{\partial}{\partial t}\left|\frac{\partial \nabla x_3}{\partial t}\right|_2^2 \leq C_1\left|\frac{\partial \nabla x_3}{\partial t}\right|_2^2 + C_2\left|\nabla x_3\right|_2^2. \tag{107}$$

Adding up the above yields the following inequality for W

$$\frac{\partial W}{\partial t} \leq C_1 W + C\left(\left|\frac{\partial x_1}{\partial t}\right|_2^2 + \left|\frac{\partial x_2}{\partial t}\right|_2^2 + \left|\frac{\partial x_3}{\partial t}\right|_2^2 + |\nabla x_3|_2^2\right). \tag{108}$$

Now via earlier estimates we have obtained for $t \geq t^{****}$,

$$\int_t^{t+1}\left|\frac{\partial x_1}{\partial t}\right|_2^2 dt \leq C\,,\ \int_t^{t+1}\left|\frac{\partial x_2}{\partial t}\right|_2^2 dt \leq C,\ \int_t^{t+1}\left|\frac{\partial x_3}{\partial t}\right|_2^2 dt \leq C \tag{109}$$

Also

$$\int_t^{t+1} |\nabla x_3|_2^2 dt \leq C, \tag{110}$$

Furthermore

$$\int_t^{t+1} |\nabla\left(\frac{\partial x_1}{\partial t}\right)|_2^2 dt \leq C. \tag{111}$$

Thus we can invoke the uniform Gronwall Lemma [13] to yield that for $t \geq t^{****} > t^{***}$ the following estimate holds uniformly

$$W(t+1) \leq C. \tag{112}$$

However this trivially implies that for $t \geq t^{****}$

$$\left|\nabla\left(\frac{\partial x_1(t+1)}{\partial t}\right)\right|_2^2 \leq C. \tag{113}$$

5. Existence of Global Attractor

In this section we will prove the existence of a global attractor for the diffusive three species aquatic model. Recall the following definition

DEFINITION 5.1. *Consider a semi group $S(t)$ acting on a phase space M, then the global attractor $\mathscr{A} \subset M$ for this semigroup is an object that satisfies*
i) $\mathscr{A}$ is compact in M.
ii) $\mathscr{A}$ is invariant, i.e, $S(t)\mathscr{A} = \mathscr{A}$, $t \geq 0$
iii) If B is bounded in M then

$$dist_M(S(t)B, \mathscr{A}) \to 0,\ t \to \infty.$$

Next various preliminaries are presented, detailing the phase spaces of interest and recalling certain standard theory.

5.1. Preliminaries. Let us define our phase spaces of interest.

$$H = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega),$$

We also define the following spaces,

$$Y = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega),$$

$$X = H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega).$$

In order to prove the existence of a global attractor we are required to show that there exists a bounded absorbing set in the phase space, followed by the asymptotic compactness property of the semigroup in question. This is defined next

DEFINITION 5.2. *The semi-group $\{S(t)\}_{t\geq 0} : H \to H$ associated with a dynamical system is said to be asymptotically compact in H if for any $\{x_{i0,n}\}_{n=1}^{\infty}$ bounded in H, and a sequence of times $\{t_n \to \infty\}$, $S(t_n)x_{i0,n}$ possesses a convergent subsequence in H.*

We now state the following Theorem,

THEOREM 5.3. *Consider the diffusive three species aquatic model,* (1)- (3)*. There exists a (H,H) global attractor $\mathscr{A}$ for the system. This is compact and invariant in H, and it attracts all bounded subsets of H in the H metric.*

PROOF. We have shown that the system is well posed in [**10**]. Thus there exists a well defined semi-group $\{S(t)\}_{t\geq 0} : H \to H$. The estimates derived in Lemma 3.2 demonstrate the existence of bounded absorbing sets in H. Also note that Lemma 3.3 gives us the existence of bounded absorbing set in Y. Thus given a sequence $\{x_{i0,n}\}_{n=1}^{\infty}$, for $1 \leq i \leq 3$ that is bounded in $L^2(\Omega)$, via Lemma 3.3 we know that for $t > t^{**}$,

$$S(t)(x_{i0,n}) \subset B \subset H_0^1. \tag{114}$$

Here B is the bounded absorbing set in H_0^1. Now for n large enough $t_n > t^{**}$, thus for such t_n we have

$$S(t_n)(x_{i0,n}) \subset B \subset H_0^1. \tag{115}$$

This implies that we have the following uniform bound via Lemma 3.3

$$|S(t_n)(x_{i0,n})|_{H_0^1} \leq C, \tag{116}$$

which implies via standard functional analysis theory, see [14], [16], the existence of a subsequence still labelled $S(t_n)(x_{i0,n})$ such that

$$S(t_n)(x_{i0,n}) \rightharpoonup x_i \text{ in } H_0^1(\Omega), \tag{117}$$

Which implies via the compact Sobolev embedding of

$$Y \hookrightarrow H, \tag{118}$$

that

$$S(t_n)(x_{i0,n}) \to x_i \text{ in } L^2(\Omega). \tag{119}$$

This yields the asymptotic compactness of the semi-group $\{S(t)\}_{t\geq 0}$ in H. This proves the Theorem □

5.2. Asymptotic Compactness Property in X**.** We will show that the attractor for the diffusive three species aquatic model attracts in a topology stronger than $L^2(\Omega)$. We first present two more definitions.

DEFINITION 5.4. *Let $\mathscr{A} \subset X$. Then $\mathscr{A}$ is said to be a (H,X) global attractor if the following conditions are satisfied*
i) $\mathscr{A}$ is compact in X.
ii) $\mathscr{A}$ is invariant, i.e, $S(t)\mathscr{A} = \mathscr{A}$, $t \geq 0$
iii) If B is bounded in H then

$$dist_X(S(t)B, \mathscr{A}) \to 0,\ t \to \infty.$$

DEFINITION 5.5. *The semi-group $\{S(t)\}_{t\geq 0} : H \to H$ associated with a dynamical system is said to be asymptotically compact in X if for any $\{x_{i0,n}\}_{n=1}^{\infty}$ bounded in H, and a sequence of times $\{t_n \to \infty\}$, $S(t_n)x_{i0,n}$ possesses a convergent subsequence in X.*

REMARK 1. *Recall that if $\mathscr{A}$ is an (H,H) attractor, then all that is required to prove that it is in fact an (H,X) attractor is to show the existence of a bounded absorbing set in X, and also demonstrate the asymptotic compactness of the semi-group in X,* [13].

We will now demonstrate the asymptotic compactness property for the semigroup in X. We will perform the analysis for x_1. x_2, x_3 will follow similarly. Our strategy is to rewrite (1) as

$$\Delta x_1 = \frac{\partial x_1}{\partial t} - a_1 x_1 + b_1 x_1^2 - F(x_1,x_2) \tag{120}$$

Here

$$F(x_1,x_2) = w_0 \left(\frac{x_1^{m_1}}{(x_1+D_1)^{m_1}} x_2^{m_2} \right) \tag{121}$$

We will demonstrate that every term on the right hand side of (120) is uniformly bounded in $L^2(\Omega)$. This will show that Δx_1 is uniformly bounded in $L^2(\Omega)$. This will imply the uniform boundedness of x_1 in $H^2(\Omega)$, via elliptic regularity, [14]. This can be done for x_2, x_3 as well. Thus the asymptotic compactness in X will follow. We state the following Lemma

LEMMA 5.6. *The semi-group $\{S(t)\}_{t\geq 0}$ associated with the dynamical system for the diffusive three species aquatic model (1)- (3), is asymptotically compact in X.*

PROOF. Let us denote $x_{1n}(t) = S(t)x_{10,n}$ and $u(t_n) = \frac{\partial x_{1n}}{\partial t}|_{t=t_n}$. We have that

$$\Delta x_{1n}(t_n) = u(t_n) - a_1 x_{1n}(t_n) + b_1 x_{1n}(t_n)^2 - F(x_{1n}(t_n), x_{2n}(t_n)). \tag{122}$$

Via (113) we have for $t \geq t^{****}+1$

$$\left|\nabla \frac{\partial x_1}{\partial t}\right|_2 \leq C. \tag{123}$$

Now for n large enough, we eventually have $t_n \geq t^{****}+1$. Thus we obtain

$$\left|\nabla \frac{\partial x_{1n}(t_n)}{\partial t}\right|_2 \leq C. \tag{124}$$

Also via Lemma 3.3 we have the estimate

$$|\nabla x_1|_2 \leq C \text{ for } t > t^{**}. \tag{125}$$

Thus for n large enough again $t_n \geq t^{****}+1 > t^{**}$, and we obtain

$$|\nabla x_{1n}|_2 \leq C. \tag{126}$$

These uniform bounds allow us to extract weakly convergent subsequences. Thus we obtain

$$x_{1n}(t_n) \rightharpoonup x_1 \ \text{ in } H_0^1(\Omega). \tag{127}$$

$$u(t_n) \rightharpoonup \frac{\partial x_1}{\partial t} \ \text{ in } H_0^1(\Omega). \tag{128}$$

Now via Lemma 2.2, [10] we have the estimate

$$|F_n(x_{1n}) - F(x_1)|_2 \leq C|x_{1n} - x_1|_2. \tag{129}$$

However via the earlier estimate it is immediate that

$$x_{1n}(t_n) \rightarrow x_1 \ \text{ in } L^2(\Omega). \tag{130}$$

Thus we obtain

$$F_n(x_{1n}) \rightarrow F(x_1) \ \text{ in } L^2(\Omega). \tag{131}$$

Thus from the convergence of the nonlinear term, classical functional analysis theory, see [13], and the compact embedding of

$$H_0^1(\Omega) \hookrightarrow L^4(\Omega) \hookrightarrow L^2(\Omega), \tag{132}$$

we obtain

$$x_{1n}(t_n) \rightarrow x_1 \ \text{ in } L^2(\Omega). \tag{133}$$

$$x_{1n}^2(t_n) \rightarrow x_1^2 \ \text{ in } L^2(\Omega). \tag{134}$$

$$\frac{\partial x_{1n}}{\partial t}(t_n) \rightarrow \frac{\partial x_1}{\partial t} \ \text{ in } L^2(\Omega), \tag{135}$$

$$F_n(x_{1n}) \rightarrow F(x_1) \ \text{ in } L^2(\Omega). \tag{136}$$

Using these convergent subsequences and equation (122), we obtain

$$\Delta x_{1n} \to \Delta x_1 \ \text{ in } L^2(\Omega). \tag{137}$$

However via elliptic regularity theory [14] this implies

$$x_{1n} \to x_1 \ \text{ in } H^2(\Omega). \tag{138}$$

Thus the Lemma is proved. □

We can now state the following result

THEOREM 5.7. *Consider the diffusive three species aquatic model model, (1)- (3). There exists a (H,X) global attractor $\mathscr{A}$ for this system. This is compact and invariant in X. Furthermore it attracts all bounded subsets of H in the X metric.*

PROOF. We have shown that the system is well posed in [10]. Thus there exists a well defined semi-group $\{S(t)\}_{t\geq 0} : H \to H$. Theorem 5.3 gives us the existence of an (H,H) global attractor. In Lemma 3.4 we have shown the existence of a bounded absorbing sets in X . Last but not least, Lemma 5.6 demonstrates the asymptotic compactness of the semi-group $\{S(t)\}_{t\geq 0} : H \to H$ in X, for the dynamical system associated with the diffusive three species aquatic model (1)- (3). These results when taken in conjunction prove the Theorem. □

6. Finite Dimensionality of the Global Attractor

In this section we show that the Hausdorff and fractal dimensions of the global attractor for the diffusive three species aquatic model is finite. We will provide upper bounds on these dimensions in terms of parameters in the model. There is a standard methodology to derive these estimates [15], [13], [16]. We consider a volume element in the phase space, and try and derive conditions that will cause it to decay, as time goes forward. If $\mathscr{A}$ is the global attractor of the semi-group $\{S(t)\}_{t\geq 0}$ in H associated with the diffusive three species aquatic model, we can define

$$q_n(t) = \sup_{u_0 \in A} \sup_{\xi_i \in H, ||\xi_i||=1, 1\leq i\leq n} \frac{1}{t}\int_0^t Tr(\Delta U(\tau) + \delta U(\tau) + F^{'}(S(\tau)u_0) \circ Q_n(\tau) d\tau, \tag{139}$$

where

$$q_n = \limsup_{t\to\infty} q_n(t). \tag{140}$$

Here 'F' is the nonlinear map in (1)- (3), and 'δ' the linear map. also Q_n is the orthogonal projection of the phase space H onto the subspace spanned by $U_1(t), U_2(t), ...U_n(t)$, with

$$U_i(t) = L(S(t)u_0)\xi_i, i = 1,2,..n. \tag{141}$$

$L(S(t)u_0)$ is the Frechet derivative of the map $S(t)$ at u_0. Also for this model, $L(S(t)u_0)\xi = U(t) = (X1(t), X2(t), X3(t))$, for any $\xi = (\eta, \zeta, \kappa)$ where $u = (x_1, x_2, x_3)$ is a solution to the diffusive three species aquatic model, $\phi_j = (\phi_j^1, \phi_j^2, \phi_j^3,)$ are an orthonormal basis for the subspace $Q_n(\tau)H$ and $(X1(t), X2(t), X3(t))$ are strong solutions to the variational equations for the diffusive three species aquatic model. These have been worked out explicitly

$$
\begin{aligned}
&\frac{\partial X1}{\partial t} \\
= \;& \Delta X1 + a_1 X1 - 2b_1 x_1 X1 - w_0 \binom{m_2}{1} \frac{x_1^{m_1} x_2^{m_2-1} X2}{(x_1+D_0)^{m_1}} \\
+ \;& w_0 \binom{m_1}{1} \left(\frac{x_1^{m_1} x_2^{m_2} X1}{(x_1+D_0)^{m_1+1}} + \frac{x_1^{m_1-1} x_2^{m_2} X1}{(x_1+D_0)^{m_1}} \right),
\end{aligned} \tag{142}
$$

$$
\begin{aligned}
&\frac{\partial X2}{\partial t} \\
= \;& \Delta X2 - a_2 X2 \\
+ \;& w_1 \binom{m_2}{1} \frac{x_1^{m_1} x_2^{m_2-1} X2}{(x_1+D_1)^{m_1}} - w_1 \binom{m_1}{1} \left(\frac{x_1^{m_1} x_2^{m_2} X1}{(x_1+D_1)^{m_1+1}} + \frac{x_1^{m_1-1} x_2^{m_2} X1}{(x_1+D_1)^{m_1}} \right), \\
- \;& w_2 \binom{m_2}{1} \frac{x_2^{m_2} x_3^{m_2-1} X3}{(x_2+D_2)^{m_2}} + w_2 \binom{m_2}{1} \left(\frac{x_2^{m_2} x_3^{m_2} X2}{(x_2+D_2)^{m_2+1}} + \frac{x_2^{m_2-1} x_3^{m_2} X2}{(x_2+D_2)^{m_2}} \right) \\
- \;& \theta \frac{x_1 X2 + x_2 X1}{x_1 + D_4} - \frac{x_1 x_2 X1}{(x_1+D_4)^2},
\end{aligned} \tag{143}
$$

$$
\begin{aligned}
&\frac{\partial X3}{\partial t} \\
= \;& \Delta X3 + c \binom{m_1}{1} x_3^{m_3-1} \, X3 + w_3 \binom{m_2}{1} \frac{X3}{(x_3+D_3)^{m_2+1}},
\end{aligned} \tag{144}
$$

$$
X1(0) = \eta, \; X2(0) = \zeta, \; X3(0) = \kappa. \tag{145}
$$

We recall the following Lemma from [13], which will be usefull to derive the requisite estimates.

LEMMA 6.1. *If there is an integer n such that $q_n < 0$ then the Hausdorff and fractal dimensions of $\mathscr{A}$, denoted $d_H(\mathscr{A})$ and $d_F(\mathscr{A})$, satisfy the following estimates*

$$
d_H(\mathscr{A}) \leq n, \tag{146}
$$

$$
d_F(\mathscr{A}) \leq 2n. \tag{147}
$$

Our aim is thus clear cut. We will derive exactly which conditions enforce that $q_n < 0$ for the diffusive model. We next begin our estimates.

$$
\begin{aligned}
& Tr(\Delta U(\tau)+\delta U(\tau)+F^{'}(S(\tau)u_0)\circ Q_n(\tau) \\
= & \sum_{j=1}^{n}\left\langle \Delta\phi_j(\tau),\phi_j(\tau)\right\rangle+\sum_{j=1}^{n}\delta\left\langle \phi_j(\tau),\phi_j(\tau)\right\rangle \\
+ & \left\langle F^{'}(S(\tau)u_0)\phi_j(\tau),\phi_j(\tau)\right\rangle \\
\leq & \sum_{j=1}^{n}3|\nabla\phi_j(\tau)|^2+\sum_{j=1}^{n}\delta|\phi_j(\tau)|^2+J_1+J_2+J_3. \qquad (148)
\end{aligned}
$$

Here

$$
\begin{aligned}
& J_1 \\
\leq & \sum_{j=1}^{n}\int_{\Omega}w_0\binom{m_2}{1}\frac{x_1(\tau)^{m_1}x_2(\tau)^{m_2-1}\phi_j^1\phi_j^2}{(x_1(\tau)+D_0)^{m_1}} \\
+ & \sum_{j=1}^{n}\int_{\Omega}w_0\binom{m_1}{1}\left(\frac{x_1(\tau)^{m_1-1}x_2(\tau)^{m_2}|\phi_j^1|^2}{(x_1(\tau)+D_0)^{m_1}}-\frac{x_1(\tau)^{m_1}x_2(\tau)^{m_2}|\phi_j^1|^2}{(x_1(\tau)+D_0)^{m_1+1}}\right) \\
\leq & 2w_0\left(\binom{m_2}{1}+\binom{m_1}{1}\right)\frac{K^{m_1+m_2+1}}{D_0^{m_1+1}}\sum_{j=1}^{n}|\phi_j(\tau)|_2^2. \qquad (149)
\end{aligned}
$$

This follows via Holder's inequality and the compact Sobolev embedding of

$$
H_0^1\hookrightarrow L^4\hookrightarrow L^2. \qquad (150)
$$

$$
\begin{aligned}
& J_2 \\
\leq & \sum_{j=1}^{n}\int_{\Omega}w_1\binom{m_2}{1}\frac{x_1(\tau)^{m_1}x_2(\tau)^{m_2-1}|\phi_j^2|^2}{(x_1(\tau)+D_1)^{m_1}} \\
+ & \sum_{j=1}^{n}\int_{\Omega}w_1\binom{m_1}{1}\left(\frac{x_1(\tau)^{m_1-1}x_2(\tau)^{m_2}\phi_j^1\phi_j^2}{(x_1(\tau)+D_1)^{m_1}}-\frac{x_1(\tau)^{m_1}x_2(\tau)^{m_2}\phi_j^1\phi_j^2}{(x_1(\tau)+D_1)^{m_1+1}}\right) \\
+ & \sum_{j=1}^{n}\int_{\Omega}w_2\binom{m_2}{1}\frac{x_2(\tau)^{m_2}x_3(\tau)^{m_2-1}\phi_j^3\phi_j^2}{(x_2(\tau)+D_2)^{m_2}} \\
+ & \sum_{j=1}^{n}\int_{\Omega}w_2\binom{m_2}{1}\left(\frac{x_2(\tau)^{m_2-1}x_2(\tau)^{m_2}|\phi_j^2|^2}{(x_2(\tau)+D_2)^{m_2}}-\frac{x_2(\tau)^{m_2}x_3(\tau)^{m_2}|\phi_j^2|^2}{(x_2(\tau)+D_2)^{m_2+1}}\right) \\
+ & \sum_{j=1}^{n}\int_{\Omega}\theta\frac{x_1(\tau)|\phi_j^2|^2+x_2(\tau)\phi_j^1\phi_j^2}{x_1(\tau)+D_4}-\frac{x_1(\tau)x_2(\tau)\phi_j^1\phi_j^2}{(x_1(\tau)+D_4)^2} \\
\leq & 2w_1\left(\binom{m_2}{1}+\binom{m_1}{1}\right)\frac{K^{m_1+m_2-1}}{D_1^{m_1+1}}\sum_{j=1}^{n}|\phi_j(\tau)|_2^2 \\
+ & 2w_1\left(\binom{m_2}{1}+\binom{m_1}{1}\right)\frac{K^{m_2+m_2-1}}{D_2^{m_2+1}}\sum_{j=1}^{n}|\phi_j(\tau)|_2^2 \\
+ & \left(2\theta\frac{K}{D_4^2}\sum_{j=1}^{n}|\phi_j(\tau)|_2^2\right). \qquad (151)
\end{aligned}
$$

This follows via Holder's inequality and the compact Sobolev embedding of

$$H_0^1 \hookrightarrow L^4 \hookrightarrow L^2, \tag{152}$$

and

$$J_3 = \sum_{j=1}^{n} \int_{\Omega} c \binom{m_1}{1} x_3(\tau)^{m_3-1} \, |\phi_j^3|^2 + w_3 \binom{m_2}{1} \frac{|\phi_j^3|^2}{(x_3(\tau)+D_3)^{m_2+1}}$$
$$\leq \left(c \binom{m_1}{1} K^{m_3-1} + \frac{w_3}{(D_3)^{m_2+1}} \binom{m_2}{1} \right) \sum_{j=1}^{n} |\phi_j(\tau)|_2^2. \tag{153}$$

Thus we obtain the estimate,

$$\begin{aligned} & Tr(\Delta U(\tau) + \delta U(\tau) + F^{'}(S(\tau)u_0) \circ Q_n(\tau) \\ \leq \; & -3 \sum_{j=1}^{n} |\nabla \phi_j(\tau)|_2^2 + (a_1 + 2b_1 K - a_2) |\phi_j(\tau)|_2^2 \\ + \; & 2w_0 \left(\binom{m_2}{1} + \binom{m_1}{1} \right) \left(\frac{K^{m_1+m_2+1}}{D_0^{m_1+1}} \right) \sum_{j=1}^{n} |\phi_j(\tau)|_2^2 \\ + \; & \left(4w_1 \left(\binom{m_2}{1} + \binom{m_1}{1} \right) \left(\frac{K^{m_1+m_2-1}}{D_1^{m_1+1}} + \frac{K^{2m_2-1}}{D_2^{m_2+1}} \right) \right. \\ + \; & \left. 2\theta \frac{K}{D_4^2} + c \binom{m_1}{1} K^{m_3-1} \right) \sum_{j=1}^{n} |\phi_j(\tau)|_2^2 \\ + \; & \left(\left(\frac{w_3}{(D_3)^{m_2+1}} \binom{m_2}{1} \right) \sum_{j=1}^{n} |\phi_j(\tau)|_2^2 \right). \end{aligned}$$

Now via the generalized Sobolev-Lieb-Thirring inequalities [13] we obtain

$$\sum_{j=1}^{n} |\nabla \phi_j(\tau)|_2^2 \geq K_1 \frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}} \tag{154}$$

Here K_1 depends only on the shape and dimension of Ω. Thus we obtain

$$Tr(\Delta U(\tau) - \delta U(\tau) + F^{'}(S(\tau)u_0) \circ Q_n(\tau)$$
$$\leq -3K_1 \frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}} + (a_1 + 2b_1 K - a_2)n + f(w_0, w_1, D_0, D_1, D_3, D_4, m_1, m_2)n, \tag{155}$$

for $\tau > 0, \; u_0 \in \mathscr{A}$.

REMARK 2. *We will abbreviate*

$$f(w_0, w_1, D_0, D_1, D_2, D_3, D_4, m_1, m_2, \theta) = f(w_i, D_i, m_i)$$

hence forth. Note the exact expression for "f" is given explicitly by

$$
\begin{aligned}
& f(w_i, D_i, m_i) \\
= \quad & 2w_0 \left(\binom{m_2}{1} + \binom{m_1}{1} \right) \frac{K^{m_1+m_2+1}}{D_0^{m_1+1}} + 4w_1 \left(\binom{m_2}{1} + \binom{m_1}{1} \right) \frac{K^{m_1+m_2-1}}{D_1^{m_1+1}} \\
+ \quad & 4w_1 \left(\binom{m_2}{1} + \binom{m_1}{1} \right) \frac{K^{2m_2-1}}{D_2^{m_2+1}} \\
+ \quad & \left(2\theta \frac{K}{D_4^2} + c \binom{m_1}{1} K^{m_3-1} + \frac{w_3}{(D_3)^{m_2+1}} \binom{m_2}{1} \right). \qquad (156)
\end{aligned}
$$

We use the derived estimates to obtain

$$
\begin{aligned}
& q_n(t) \\
= \quad & \sup_{u_0 \in A} \sup_{\xi_i \in H, ||\xi_i||=1, 1 \le i \le n} \frac{1}{t} \int_0^t Tr(\Delta U(\tau) - \delta U(\tau) + F'(S(\tau)u_0) \circ Q_n(\tau) d\tau \\
\le \quad & -3DK_1 \frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}} + (f(w_i, D_i, m_i) + (a_1 + 2b_1 K - a_2))n, \ \forall t > 0. \qquad (157)
\end{aligned}
$$

This yields

$$
\limsup_{t \to \infty} q_n(t) \le -3DK_1 \frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}} + (f(w_i, D_i, m_i)) + (a_1 + 2b_1 K - a_2))n < 0, \qquad (158)
$$

if the integer n satisfies

$$
n - 1 < \left(\frac{(f(w_i, D_i, m_i) + (a_1 + 2b_1 K - a_2))}{3K_1} \right)^{\frac{3}{2}} |\Omega| < n. \qquad (159)
$$

Via the above we can now state the following result

THEOREM 6.2. *Consider the diffusive three species aquatic model,* (1)- (3). *The global attractor $\mathscr{A}$ of the diffusive three species aquatic model is of finite dimension. Furthermore explicit upper bounds for its Hausdorff and fractal dimensions are given as follows*

$$
d_H(A) \le \left(\frac{f(w_i, D_i, m_i)) + (a_1 + 2b_1 K - a_2)}{3K_1} \right)^{\frac{3}{2}} |\Omega| + 1, \qquad (160)
$$

$$
d_F(A) \le 2 \left(\frac{f(w_i, D_i, m_i)) + (a_1 + 2b_1 K - a_2)}{3K_1} \right)^{\frac{3}{2}} |\Omega| + 2 \qquad (161)
$$

PROOF. The earlier derived estimates via (158), (159) along with Lemma 6.1 allow us to obtain the desired result □

7. Conclusion

In conclusion we have shown rigorously the existence of a global attractor for the diffusive three species aquatic model. This is compact in $H^2(\Omega)$ and attracts bounded sets in $L^2(\Omega)$ in the $H^2(\Omega)$ topology. This seems in accordance with the results of [7], where evidence of various limiting behaviour was found in the ODE case. This included the existence of stable focus, limit cycles and states of extinction. Moreover all simulations were performed for the mutual interference parameters $m_i \in [1,3]$. This is well in accordance with our current results as we have relied on the compact Sobolev embedding

$$H_0^1 \hookrightarrow L^{2m_i}. \tag{162}$$

Of course this is only true in $\mathbb{R}^3$ if $m_i \leq 3$. The possible albeit unrealistic cases for $m_i > 3$ remains a current challenge, and might not be possible to analyse altogether. At least not within the scope of Sobolev embeddings.

Furthermore the question of finite dimensionality of the global attractor is in general of interest, due to various practical and numerical concerns. It essentially tells us about the degrees of freedom in the system. Having provided upper bounds on the dimension of the global attractor, we hope to pave the way for future numerical work. However our bounds are not sharp. Furthermore from a point of view of direct numerical simulation, they are perhaps not quite pragmatic. Essentially we have shown that

$$d_H(\mathscr{A}) < d_F(\mathscr{A}) < C\, K^{m_1+m_2}, \tag{163}$$

where recall "K" is the maximum of the carrying capacity for each species. This bound is immensely large, even for modest m_i, say in the range $[1,3]$. It will be of much interest if we can sharpen this to perhaps the order of K or below. Furthermore results from [7] show that the interaction between predators is a stabilizing factor, which does not seem to be violated in the diffusive case as well, via the existence of the bounded absorbing sets in the phase space that we have shown. This is probably expected due to the smoothing property of diffusive terms, which are the essential mechanism for dissipation in dissipative dynamical systems described by parabolic PDE, [16], [12].

All in all we believe that the diffusive three species aquatic model is quite a robust model, with well defined dynamics. It can perhaps be used successfully in modelling tri-trophic food environments and more general three species food chains, especially if one is interested in the long time dynamics. It is our goal to continue to investigate the nuances of this model. These could include, but would not be restricted to, time delays or even additive stochastic terms, trying to incorporate various realistic ecological scenarios such as random attacks by predators, prey refuge and cooperation. All of these directions will be left as future endeavours.

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