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Well-posedness for the initial-boundary-value problem for the Benney-Luke equation in a quarter plane

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Abstract. We study the local and global well posedness for the initial-boundary-value problem associated with the Benney-Luke equation on the half line on suitable Sobolev type spaces, imposing some compatibility conditions on the initial-boundary-data. The solution mapping associated to the appropriate initial-boundary-data is Lipschitz between appropriate Banach spaces.

Contents

1. Introduction

In this paper we consider the initial-boundary-value problem associated with the Benney-Luke equation on the half line

$$
\tag{1.1}
\begin{cases}
u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + pu_t u_x^{p-1} u_{xx} + 2u_x^p u_{xt} & = 0, \\
u_x(0,t) = h_1(t), \quad u_t(0,t) & = h_2(t) \\
u_x(x,0) = f_1(x), \quad u_t(x,0) & = f_2(x),
\end{cases}
$$

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where functions f_i and h_i belong to suitable Sobolev type spaces. For $p = 1$, this equation is a formally valid approximation for describing small-amplitude, long water waves in water of finite depth. This equation is the one dimensional version of the model derived by J. Quintero and R. Pego in [**4**] as an isotropic model for three-dimensional water waves (see also [**1**]), where the parameters $a,\ b > 0$ are such that $a - b = \sigma - \frac{1}{3}$, with σ being assoaciated with the surface tension) (the Bond number). We will assume $a > b > 0$ throughout this paper, which corresponds to small or zero surface tension ($\sigma > \frac{1}{3}$). In contrast to one-way equations such as the KdV, or BBM equations, we point out that the model (1.1) is an approximation formally valid for describing *two-way* water wave propagation in the case $p = 1$.
The local well-posedness for the initial value problem for the Benney-Luke equation (1.1) was obatined by J. Quintero in [**2**] (see also [**3**]) with initial data (u_0, u_1) such that $u_0 \in \dot{H}^{s+1} = \{f \in \mathcal{D}'(\mathbb{R}) : f' \in H^s(\mathbb{R})\}$ and $u_1 \in H^s(\mathbb{R})$ for $s \geq s(p)$, where $\mathcal{D}'(\mathbb{R})$ denotes the space of distributions on $\mathbb{R}$. For $p = 1$, it can be seen that $s(p) = 1$. In particular, if u is the local solution on $[0, T]$ we have that

$$u_x \in C([0,T], H^s(\mathbb{R})), \quad u_t \in C([0,T], H^s(\mathbb{R})) \cap C^1([0,T], H^{s-1}(\mathbb{R})).$$

The result follows by standard arguments using semigroup theory and the existence a smoothing effect on the nonlinear part. The global well-posedness for the initial value problem for the Benney-Luke equation (1.1) was established using the fact that the Hamiltonian structure associated with the Benney-Luke is conserved in time for mild solutions (see [**2**]).

It is important to mention that the study of the initial-boundary-value problems (**IBVP**) for dispersive water wave models has recently brought the attention to some researcher due to the need for looking those models in a finite domains or in the half line, and also due to its importance in the theory of controllability of those models (see [**5**], [**6**], [**7**], [**8**], [**9**], [**10**], [**11**], [**12**], [**13**]). For instance, the **IBVP** for the KdV equation

$$\begin{cases} \partial_t u - \partial_x^3 u + u\partial_x u &= 0, \quad x \in \mathbb{R},\ t \geq 0, \quad k \in \mathbb{N} \\ u(x,0) &= \varphi(x) \end{cases} \tag{1.2}$$

was addressed for different mathematicians. Using the boundary forcing methods for initial data $(\varphi, h) \in H^s(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+)$, J. Colliander and C. Kenig in ([**6**]) with $s \geq 0$, and J. Holmer in [**7**] for $s \geq -\frac{3}{4}$ established a local well-posedness result for the (**IBVP**) (1.2). J. Bona, S. Sun, and B. Zhang in [**5**] using a Laplace transform technique studied the local well-posedness $(\varphi, h) \in H^s(\mathbb{R}^+) \times H_{loc}^{\frac{s+1}{3}}(\mathbb{R}^+)$ with $s \geq \frac{3}{4}$.
As it is known, besides the KdV equation, there are different models used to describe the dynamics of an irrotational incompressible fluid in a bounded domain or in the half plane, as the "good Boussinesq equation" and the Benney-Luke model. The local well-posedness for (**IBVP**) for the "good Boussinesq equation"

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + \left(u^2\right)_{xx} &= 0, \quad x > 0, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) &= h(x), \end{cases} \tag{1.3}$$

was established by R. Xue in ([**9**]) using the contraction principle and a Laplace transform technique, as the one used by J. Bona, S. Sun, and B. Zhang in [**5**] in the case of the (**IBVP**) for the KdV equation. More exactly, the local well-posedness was obtained for initial data $(f,h) \in H^s(\mathbb{R}^+) \times H^{s-1}(\mathbb{R}^+)$ and boundary condition $(h_1,h_2) \in H^{\frac{s}{2}+\frac{1}{4}}(\mathbb{R}^+) \times H^{\frac{s}{2}-\frac{3}{4}+\epsilon}(\mathbb{R}^+)$, under some compatibility conditions, for $s > \frac{1}{2}$ and $\epsilon > 0$ small. Moreover, the global well-posedness was established in the case of zero boundary data and initial conditions $(f,h) \in H_0^s(\mathbb{R}^+) \times H_0^{s-1}(\mathbb{R}^+)$ for $s \geq 1$ with $\|f\|_{H_0^1(\mathbb{R}^+)} + \|h\|_{L^2(\mathbb{R}^+)}$ small.

The aim of this work is to establish a well-posedness result for (**IBVP**) associated with the Benney-Luke equation on the half line, following the approach used by R. Xue for the "good Boussinesq equation" (1.3) and J. Bona, S. Sun, and B. Zhang in [**5**] in the case of the KdV equation. In other words, we will use the contraction mapping principle and a Laplace transform technique to study the (**IBVP**) for the Benney-Luke equation (1.1).

2. Notation and Preliminaries

Before we go further, we state the basic notation and some important results used in the development of the paper. Let $H^s(\mathbb{R})$ be the Sobolev space defined as

$$H^s(\mathbb{R}) = \{f \in \mathcal{D}'(\mathbb{R}) : (1+|\zeta|)^s \hat{f}(\zeta) \in L^2(\mathbb{R})\}$$

where $\mathcal{D}'(\mathbb{R})$ denotes the space of distributions on $\mathbb{R}$ and $\hat{f}$ denotes the Fourier transform with respect to the spatial variable x. We also define the Sobolev type space

$$\mathcal{H}^{\alpha,\beta}(\mathbb{R}) = \{f \in \mathcal{D}'(\mathbb{R}) : |\zeta|^\alpha (1+|\zeta|)^{\beta-\alpha} \hat{f}(\zeta) \in L^2(\mathbb{R})\}$$

The space $\dot{H}^s(\mathbb{R})$ be the space defined as

$$\dot{H}^s(\mathbb{R}) = \{f \in \mathcal{D}'(\mathbb{R}) : |\zeta|^s \hat{f}(\zeta) \in L^2(\mathbb{R})\}$$

Now, for $s \geq 0$, we define the spaces $H^s(\mathbb{R}^+)$ and $\dot{H}^s(\mathbb{R}^+)$ by

$$H^s(\mathbb{R}^+) = \{f = F|_{\mathbb{R}^+} : F \in H^s(\mathbb{R})\}, \quad \|f\|_{H^s(\mathbb{R}^+)} = \inf\{\|F\|_{H^s(\mathbb{R})} : f = F|_{\mathbb{R}^+}\}$$
$$\dot{H}^s(\mathbb{R}^+) = \{f = F|_{\mathbb{R}^+} : F \in \dot{H}^s(\mathbb{R})\}, \quad \|f\|_{\dot{H}^s(\mathbb{R}^+)} = \inf\{\|F\|_{\dot{H}^s(\mathbb{R})} : f = F|_{\mathbb{R}^+}\}$$

We note that for $f \in H^s(\mathbb{R})$, then we have that $f|_{\mathbb{R}^+} \in H^s(\mathbb{R}^+)$ and $\|f|_{\mathbb{R}^+}\|_{H^s(\mathbb{R}^+)} \leq \|f|_{\mathbb{R}}\|_{H^s(\mathbb{R})}$. For $s < 0$, $H^s(\mathbb{R}^+)$ denotes the space of bounded linear transformations g defined on $C_0^\infty(\mathbb{R}^+)$ with

$$\|g\|_{H^s(\mathbb{R}^+)} = \sup\{|g(f)| : f \in C_0^\infty(\mathbb{R}^+) \text{ and } \|f\|_{H^{-s}(\mathbb{R}^+)} = 1\}.$$

In a similar fashion, $\dot{H}^s(\mathbb{R}^+)$ denotes the space of bounded linear transformations g defined on $C_0^\infty(\mathbb{R}^+)$ with

$$\|g\|_{\dot{H}^s(\mathbb{R}^+)} = \sup\{|g(f)| : f \in C_0^\infty(\mathbb{R}^+) \text{ and } \|f\|_{\dot{H}^{-s}(\mathbb{R}^+)} = 1\}.$$

For $s \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, we define the spaces

$$\begin{aligned} H_0^s(\mathbb{R}^+) &= \{f \in H_0^s(\mathbb{R}) : supp(f) \subset [0,\infty)\} \\ \dot{H}_0^s(\mathbb{R}^+) &= \{f \in \dot{H}_0^s(\mathbb{R}) : \mathrm{supp}(f) \subset [0,\infty)\} \\ \mathcal{H}_0^{\alpha,\beta}(\mathbb{R}^+) &= \{f \in \mathcal{H}^{\alpha,\beta}(\mathbb{R}) : supp(f) \subset [0,\infty)\} \end{aligned}$$

We summarize the result obtained by R. Xue regarding these spaces (see Lemma 2.1, Lemma 2.2 and Lemma 2.3 in [**9**]).

Lemma 2.1. *(1) For $s < 0$, we have that $\dot{H}^s(\mathbb{R}^+) = \dot{H}_0^s(\mathbb{R}^+)$.*

(2) For $s \leq \frac{1}{2}$, we have that $H^s(\mathbb{R}^+) = H_0^s(\mathbb{R}^+)$.

(3) For $k + \frac{1}{2} < s \leq k + \frac{3}{2}$ for some integer k, we have that

$$H_0^s(\mathbb{R}^+) = \{f \in H^s(\mathbb{R}) : Tr(\partial_x^j f) = 0,\ j : 0, 1, ...k\}.$$

where $Tr(\partial_x^j f) = \partial_x^j F$ for $F \in H^s(\mathbb{R})$ and $f = F|_{\mathbb{R}^+}$.

(4) For $\alpha < 0$ and $\beta \leq \frac{1}{2}$ with $\alpha \leq \beta$, we have that $\mathcal{H}^{\alpha,\beta}(\mathbb{R}^+) = \mathcal{H}_0^{\alpha,\beta}(\mathbb{R}^+)$.

(5) For $\alpha < 0$ and $k + \frac{1}{2} < \beta \leq k + \frac{3}{2}$ for some integer k, we have that

$$\mathcal{H}_0^{\alpha,\beta}(\mathbb{R}^+) = \{f \in \mathcal{H}^{\alpha,\beta}(\mathbb{R}^+) : Tr(\partial_x^j f) = 0,\ j : 0, 1, ...k\}.$$

where we set $Tr(\partial_x^j f) = \partial_x^j F$ for $F \in H^\beta(\mathbb{R})$ and $f = F|_{\mathbb{R}^+}$.

For $l, k \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, and $A = \mathbb{R}$, $\mathbb{R}^+$, we set $Y^l(A)$, $Y_0^l(A)$, $Y^{l,k}(A)$, $Y_0^{l,k}(A)$ and $Y^{\alpha,\beta}$ as

$$\begin{aligned} Y^l(A) &= H^l(A) \times H^l(A), & Y_0^l(A) &= H_0^l(A) \times H_0^l(A) \\ Y^{l,k}(A) &= H^l(A) \times H^k(A), & Y_0^{l,k}(A) &= H_0^l(A) \times H_0^k(A) \\ \mathcal{Y}^{\alpha,\beta}(\mathbb{R}^+) &= \mathcal{H}^{\alpha,\beta}(\mathbb{R}^+) \times \mathcal{H}^{\alpha,\beta}(\mathbb{R}^+). \end{aligned}$$

Hereafter, χ denotes the characteristic function on the set $\mathbb{R}^+$ satisfying $\chi(x) = 1$ for $x > 0$ and $\chi(x) = 0$ for $x \leq 0$.

3. Linear estimates for the IBVP

We begin this section rewriting the Benney-Luke equation (1.1) as a first order equation. To do this, we consider the following variables $q = u_x$ and $r = u_t$. In this case, we see formally that $q_t = r_x$ and that the first equation in (1.1) can be expressed as

$$r_t - q_x + aq_{xxx} - br_{xxt} + prq^{p-1}q_x + 2q^p r_x = 0,$$

which is equivalent to the equation,

$$(I - b\partial_x^2)r_t - (I - a\partial_x^2)q_x + prq^{p-1}q_x + 2q^p r_x = 0.$$

Now, if we set the linear operators $A = I - a\partial_x^2$ and $B = I - b\partial_x^2$, then we have that r satisfies the equation

$$r_t = B^{-1}Aq_x - B^{-1}(prq^{p-1}q_x + 2q^p r_x).$$

So, the initial-boundary-value problem (1.1) can be written as the first order initial-boundary-value problem

$$\begin{cases} q_t &= r_x, \quad x>0, \quad t>0, \\ r_t &= B^{-1}Aq_x - B^{-1}(prq^{p-1}q_x + 2q^p r_x) \\ q(0,t) &= h_1(t), \quad r(0,t) = h_2(t) \\ q(x,0) &= f_1(x), \quad r(x,0) = f_2(x), \end{cases}$$

or equivalent to the **IBVP**

$$\begin{cases} \partial_t X(x,t) &= MX(x,t) + G(q,r)x,t), \quad x>0, \quad t>0, \\ X(0,t) &= \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}, \quad X(x,0) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \end{cases} \tag{3.1}$$

where X, M and G are given by

$$X = \begin{pmatrix} q \\ r \end{pmatrix} M = \begin{pmatrix} 0 & \partial_x \\ (B^{-1}A)\partial_x & 0 \end{pmatrix}, \quad G(q,r) = \begin{pmatrix} 0 \\ -B^{-1}(prq^{p-1}q_x + 2q^p r_x) \end{pmatrix}$$

REMARK 3.1. Before we go further, we want to point out that in the variables $q = u_x$ and $r = u_t$, we have that the quantity

$$\mathcal{M}(q)(t) = \int_{\mathbb{R}} q(t,x)\,dx$$

is conserved in time for classical solutions and even for mild solutions, if $r(0,t) = 0$ as long as the solution exists. So, if we consider the Cauchy problem associated with the system in the variable (q,r) with the initial data $q_0 \in H^s(\mathbb{R}^+)$ with mean zero property

$$\int_0^\infty q_0(x)\,dx = 0.$$

So, as long as the solution exists for t, Then we have that

$$\int_0^\infty q(x,t)\,dx = 0,$$

meaning that $q(\cdot,t)$ has the mean zero property as long as the solution exists for t. In this case, the function defined by $u(x,t) = \partial_x^{-1}q(x,t) \in \mathcal{V}^{s+1}$ is such that $q(x,t) = u_x(x,t)$ and $r(x,t) = u_t(x,t)$ where

$$\mathcal{V}^{s+1} = \{f \in \mathcal{S} : f_x \in H^s(\mathbb{R}^+)\}, \qquad \partial_x^{-1}(f)(x) = \int_0^x f(y)\,dy.$$

So, we focus in the local and global well posedness for the Cauchy problem associated with system in the variable (q,r), and establish global well posedness for the Cauchy problem associated with the Benney-Luke model in the case of homogeneous boundary conditions ($h_1 = h_2 = 0$).

3.1. Linear Homogeneous case ($G \equiv 0$). We will study first the linear homogeneous case ($G \equiv 0$). In other words, we will consider the system

$$\begin{cases} \partial_t X(x,t) &= MX(x,t), \quad x>0, \quad t>0, \\ X(0,t) &= \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}, \quad X(x,0) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}. \end{cases} \tag{3.2}$$

The analysis of this (**IBVP**) will be divided in three subproblems. First, we will look for the solution $W_b(h_1,h_2)$ of the (**IBVP**) on the half line

$$\begin{cases} \partial_t X(x,t) &= MX(x,t), \quad x>0, \quad t>0, \\ X(0,t) &= \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}, \quad X(x,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases}$$

Second, we look for the solution $W_R(f_1,f_2)$ of the initial value problem (**IVP**) on the real line

$$\begin{cases} \partial_t X(x,t) &= MX(x,t), \quad x\in\mathbb{R}, \quad t>0, \\ X(x,0) &= \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \end{cases}$$

Third, we seek for the solution $W_C(f_1,f_2)$ of the (**IBVP**) on the half line

$$\begin{cases} \partial_t X(x,t) &= MX(x,t), \quad x>0, \quad t>0, \\ X(0,t) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad X(x,0) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \end{cases}$$

We note that the function

$$W(f_1,f_2,h_1,h_2) = W_b(f_1,f_2) + W_R(f_1,f_2) + W_C(f_1,f_2)$$

is solution of the (**IBVP**) (3.2) on the half line. We will obtain the linear estimates for initial data and boundary conditions on the space $Y_0^l(\mathbb{R}^+)$, and use appropriate modifications to obtain the estimates for initial data and boundary conditions on the space $Y^l(\mathbb{R}^+)$, as done by Xue in [**9**].

3.2. The homogeneous IBVP : Case $f_1\equiv f_2\equiv 0$. We will consider the **IBVP** lineal homogeneous for $f_1\equiv f_2\equiv 0$, $x>0$ y $t>0$. In other words, we will study the problem,

$$\begin{cases} \partial_t X(x,t) &= MX(x,t), \quad x>0, \quad t>0, \\ X(0,t) &= \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}, \quad X(x,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{cases} \tag{3.3}$$

LEMMA 3.2. *For $h_1,h_2\in C_0^\infty(\mathbb{R}^+)$, the solutions $W_b(h_1,h_2)$ of the* (**IBVP**) *for (3.3) has the explicit formula*

$$X(x,t) = W_b(h_1,h_2)(x,t) = \begin{pmatrix} U_1(x,t)+U_2(x,t)+\overline{U_1(x,t)}+\overline{U_2(x,t)} \\ V_1(x,t)+V_2(x,t)+\overline{V_1(x,t)}+\overline{V_2(x,t)} \end{pmatrix}, \tag{3.4}$$

where U_i and V_i are given for $s(\mu)=\sqrt{\frac{a\mu^2-1}{1-b\mu^2}}$ as

$$U_1(x,t) = \frac{-a}{2\pi}\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} \rho_1(x,t,\mu)\,d\mu, \qquad U_2(x,t) = \frac{-\sqrt{a}}{2\pi}\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} \rho_2(x,t,\mu)\,d\mu,$$

$$V_1(x,t) = \frac{-a}{2\pi i}\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} s(\mu)\rho_1(x,t,\mu)\,d\mu, \qquad V_2(x,t) = \frac{-a}{2\pi}\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} \mu\rho_2(x,t,\mu)\,d\mu.$$

where,

$$\rho_1(x,t,\mu) = \frac{e^{i\mu s(\mu)t}e^{-\mu x}s(\mu)\rho_3(\mu)}{(a\mu^2-1)}\left(\int_0^\infty \left(\mu h_1(\xi) - \frac{h_2(\xi)}{\sqrt{a}}\right)e^{-i\mu s(\mu)\xi}\,d\xi\right)$$

$$\rho_2(x,t,\mu) = \frac{e^{i\mu s(\mu)t}e^{\frac{is(\mu)x}{\sqrt{a}}}s(\mu)\rho_3(\mu)}{(a\mu^2-1)}\left(\int_0^\infty (is(\mu)h_1(\xi)+h_2(\xi))\,e^{-i\mu s(\mu)\xi}\,d\xi\right)$$

$$\rho_3(\mu) = \left(\mu - \frac{is(\mu)}{\sqrt{a}}\right).$$

Proof. Using the Laplace transform with respect the the t variable, we see that the linear homogeneous problem (3.3) becomes

(3.5)
$$\begin{cases} \lambda\widetilde{q}(x,\lambda) &= \widetilde{r}_x(x,\lambda), \quad \lambda\widetilde{r}(x,\lambda) = B^{-1}A\widetilde{q}_x(x,\lambda),\ \Re\lambda>0,\ x>0,\ j=0,1,\\ \widetilde{q}(0,\lambda) &= \widetilde{h}_1(\lambda),\ \ \widetilde{r}(0,\lambda)=\widetilde{h}_2(\lambda),\ \ \partial_x^j\widetilde{q}(+\infty,\lambda)=\partial_x^j\widetilde{r}(+\infty,\lambda)=0,. \end{cases}$$

where λ is the dual variable dual for t, $\widetilde{q}(x,\lambda),\widetilde{r}(x,\lambda),\widetilde{h}_1(\lambda)$, and $\widetilde{h}_2(\lambda)$ are the Laplace transform of $q(x,t),r(x,t),h_1(t)$ y $h_2(t)$ with respect to the t variable, respectively. From system (3.5), we have that,

$$\begin{aligned} \lambda\widetilde{r}_x &= B^{-1}A\widetilde{q}_{xx} \\ \lambda^2\widetilde{q} &= B^{-1}A\widetilde{q}_{xx} \quad\Leftrightarrow\quad \lambda^2B\widetilde{q} = A\widetilde{q}_{xx}. \end{aligned}$$

From these equations, we conclude that q satisfies the fourth order differential equation

$$a\widetilde{q}_{xxxx} - (1+b\lambda^2)\widetilde{q}_{xx} + \lambda^2\widetilde{q} = 0,$$

whose general solution is given by

$$\widetilde{q}(x,\lambda) = c_1e^{\gamma_{1A}x} + c_2e^{\gamma_{2A}x} + c_3e^{\gamma_{3A}x} + c_4e^{\gamma_{4A}x} = c_1e^{\gamma_{1A}x} + c_2e^{\gamma_{2A}x},$$

where γ_{1A}, γ_{2A}, γ_{3A}, and γ_{4A} are the four roots of the characteristic equation

$$a\gamma^4 - (1+b\lambda^2)\gamma^2 + \lambda^2 = 0,\quad \lambda\in A = \{\omega : \Re(\omega) > \lambda_+\}, \tag{3.6}$$

with $\lambda_\pm = \frac{\sqrt{2a-b\pm2\sqrt{a(a-b)}}}{b} > 0$, ordered so that $\Re(\gamma_{1A}) < 0$, $\Re(\gamma_{2A}) < 0$, $\Re(\gamma_{3A}) > 0$, and $\Re(\gamma_{4A}) > 0$. We see for $j = 1,2$ that the roots are given by

$$\gamma_{jA} = -\sqrt{\frac{(1+b\lambda^2)+(-1)^{j+1}\sqrt{(1+b\lambda^2)^2-4a\lambda^2}}{2a}},\quad \gamma_{3A} = -\gamma_{2A},\quad \gamma_{4A} = -\gamma_{1A}.$$

It is clear for $j = 1,2,3,4$ that $\Re(\gamma_{jA})$ is analytic for $\Re(\lambda) > \lambda_+$ and continuous for $\Re(\lambda) \geq \lambda_+$, except for $\lambda = \lambda_+$.
In the same fashion, we see that r can be expressed as

$$\widetilde{r}(x,\lambda) = d_1e^{\gamma_{1A}x} + d_2e^{\gamma_{2A}x}.$$

Now, from (3.5) we have that $\lambda\widetilde{q}(x,\lambda) = \widetilde{r}_x(x,\lambda)$. In other words,

$$c_1\lambda e^{\gamma_{1A}x} + c_2\lambda e^{\gamma_{2A}x} = d_1\gamma_{1A}e^{\gamma_{1A}x} + d_2\gamma_{2A}e^{\gamma_{2A}x},$$

So, we easily see that

$$c_1 = \frac{\gamma_{1A}}{\lambda}d_1 \quad \text{y} \quad c_2 = \frac{\gamma_{2A}}{\lambda}d_2. \tag{3.7}$$

So, replacing c_1 and c_2 in the initial conditions for (3.5) we get that d_1 and d_2 satisfy the system

$$\begin{pmatrix} \gamma_{1A} & \gamma_{2A} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} \lambda\widetilde{h}_1 \\ \widetilde{h}_2 \end{pmatrix},$$

whose solution is given by

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \frac{1}{\gamma_1 - \gamma_2} \begin{pmatrix} 1 & -\gamma_{2A} \\ -1 & \gamma_{1A} \end{pmatrix} \begin{pmatrix} \lambda\widetilde{h}_1 \\ \widetilde{h}_2 \end{pmatrix}.$$

So, we see that d_1 and d_2 are given by

$$\begin{aligned} d_1 &= \frac{1}{\gamma_{1A} - \gamma_{2A}}(\lambda\widetilde{h}_1 - \gamma_{2A}\widetilde{h}_2) \\ d_2 &= \frac{1}{\gamma_{1A} - \gamma_{2A}}(\gamma_{1A}\widetilde{h}_2 - \lambda\widetilde{h}_1). \end{aligned} \tag{3.8}$$

To compute c_i, we replace (3.8) in (3.7), to get

$$\begin{aligned} c_1 &= \frac{\gamma_{1A}}{\lambda(\gamma_{1A} - \gamma_{2A})}(\lambda\widetilde{h}_1 - \gamma_{2A}\widetilde{h}_2) \\ c_2 &= \frac{\gamma_{2A}}{\lambda(\gamma_{1A} - \gamma_{2A})}(\gamma_{1A}\widetilde{h}_2 - \lambda\widetilde{h}_1). \end{aligned}$$

In other words, we compute $\widetilde{q}$ and $\widetilde{r}$ explicitly as

$$\widetilde{q}(x,\lambda) = \frac{1}{\lambda(\gamma_{1A} - \gamma_{2A})}\big[\gamma_{1A}(\lambda\widetilde{h}_1 - \gamma_{2A}\widetilde{h}_2)e^{\gamma_{1A}x} - \gamma_{2A}(\lambda\widetilde{h}_1 - \gamma_{1A}\widetilde{h}_2)e^{\gamma_{2A}x}\big], \tag{3.9}$$

$$\widetilde{r}(x,\lambda) = \frac{1}{\gamma_{1A} - \gamma_{2A}}\big[(\lambda\widetilde{h}_1 - \gamma_{2A}\widetilde{h}_2)e^{\gamma_{1A}x} - (\lambda\widetilde{h}_1 - \gamma_{1A}\widetilde{h}_2)e^{\gamma_{2A}x}\big]. \tag{3.10}$$

So, for any p with $\Re(p) > \lambda_+$, and defining $\Gamma_A(\lambda) = \frac{(\gamma_{1A}+\gamma_{2A})}{2\pi i(\gamma_{1A}^2 - \gamma_{2A}^2)}e^{\lambda t}$ we are able to use the representations of q and r for $x > 0$ and $t > 0$,

$$q(x,t) = \int_{p-i\infty}^{p+i\infty} \frac{\Gamma_A(\lambda)}{\lambda}\big[\gamma_{1A}(\lambda\widetilde{h}_1 - \gamma_{2A}\widetilde{h}_2)e^{\gamma_{1A}x} - \gamma_{2A}(\lambda\widetilde{h}_1 - \gamma_{1A}\widetilde{h}_2)e^{\gamma_{2A}x}\big]d\lambda, \tag{3.11}$$

$$r(x,t) = \int_{p-i\infty}^{p+i\infty} \Gamma_A(\lambda)\big[(\lambda\widetilde{h}_1 - \gamma_{2A}\widetilde{h}_2)e^{\gamma_{1A}x} - (\lambda\widetilde{h}_1 - \gamma_{1A}\widetilde{h}_2)e^{\gamma_{2A}x}\big]d\lambda. \tag{3.12}$$

Now, we observe that

$$|\gamma_{1A}^2 - \gamma_{2A}^2| = \left|\frac{\sqrt{(1+b\lambda^2)^2 - 4a\lambda^2}}{a}\right| = b\frac{\sqrt{|\lambda - \lambda_+||\lambda + \lambda_+||\lambda^2 - \lambda_-^2|}}{a},$$

then $|\gamma_{1A}^2 - \gamma_{2A}^2| = O(|\lambda - \lambda_+|^{\frac{1}{2}})$ as $\lambda \to \lambda_+$ with $\Re(\lambda) > \lambda_+$. Moreover, for given positive constants $C_1 < C_2$, with $C_1 < \Re(\lambda) < C_2$, we observe that

$$\gamma_{jA} = \frac{-\sqrt[4]{(1+b\lambda^2)^2 - 4a\lambda^2}}{\sqrt{2a}}\sqrt{\frac{(1+b\lambda^2)}{\sqrt{(1+b\lambda^2)^2 - 4a\lambda^2}} + (-1)^{j+1}}, \quad j = 1, 2,$$

then we also have as $|\lambda| \to \infty$ that

$$\gamma_{1A} \to \frac{-\sqrt[4]{(1+b\lambda^2)^2 - 4a\lambda^2}}{\sqrt{2a}}, \qquad \gamma_{2A} \to \frac{-i\sqrt[4]{(1+b\lambda^2)^2 - 4a\lambda^2}}{\sqrt{2a}}.$$

Using this estimates, we see taking $p \to \lambda_+$ into (3.11) and (3.12) that

(3.13)

$$q(x,t) = \int_{\lambda_+ - i\infty}^{\lambda_+ + i\infty} \frac{\Gamma_A(\lambda)}{\lambda} \big[\gamma_{1A}(\lambda\widetilde{h}_1 - \gamma_{2A}\widetilde{h}_2)e^{\gamma_{1A}x} - \gamma_{2A}(\lambda\widetilde{h}_1 - \gamma_{1A}\widetilde{h}_2)e^{\gamma_{2A}x}\big] d\lambda,$$

(3.14)

$$r(x,t) = \int_{\lambda_+ - i\infty}^{\lambda_+ + i\infty} \Gamma_A(\lambda)\big[(\lambda\widetilde{h}_1 - \gamma_{2A}\widetilde{h}_2)e^{\gamma_{1A}x} - (\lambda\widetilde{h}_1 - \gamma_{1A}\widetilde{h}_2)e^{\gamma_{2A}x}\big] d\lambda.$$

Now, let γ_{1B}, γ_{2B}, γ_{3B}, and γ_{4B} be the four roots of the characteristic equation

$$a\gamma^4 - (1+b\lambda^2)\gamma^2 + \lambda^2 = 0, \quad \lambda \in B = \{\omega : \lambda_- < \Re(\omega) < \lambda_+\} \tag{3.15}$$

ordered so that $\Re(\gamma_{1B}) < 0$, $\Re(\gamma_{2B}) < 0$, $\Re(\gamma_{3B}) > 0$, $\Re(\gamma_{4B}) > 0$. As above, for $i = 1,2,3,4$ that $\Re(\gamma_{iB})$ is analytic for $\lambda_- < \Re(\omega) < \lambda_+$ and continuous for $\lambda_- \le \Re(\omega) \le \lambda_+$, except for $\omega = \lambda_\pm$. Using the uniqueness and the continuity of the root of the characteristic equation $a\gamma^4 - (1+b\lambda^2)\gamma^2 + \lambda^2 = 0$ on the half lines $\Gamma_+ = \{\omega : \Re(\omega) = \lambda_+,\ \Im(\omega) > 0\}$ and $\Gamma_- = \{\omega : \Re(\omega) = \lambda_-,\ \Im(\omega) < 0\}$, we are allowed to assume that

$$\begin{aligned}
&\gamma_{1A} = \gamma_{1B}, \quad \gamma_{2A} = \gamma_{2B}, \quad \lambda \in \Gamma_+,\\
&\gamma_{1A} = \gamma_{1B}, \quad \gamma_{2A} = \gamma_{2B}, \quad \lambda \in \Gamma_-, \quad \text{or} \quad \gamma_{2A} = \gamma_{1B}, \quad \gamma_{1A} = \gamma_{2B}, \quad \lambda \in \Gamma_-.
\end{aligned}$$

From the symmetry of formulas (3.13), (3.14), and for $\Gamma_B(\lambda) = \frac{(\gamma_{1B}+\gamma_{2B})}{2\pi i(\gamma_{1B}^2 - \gamma_{2B}^2)} e^{\lambda t}$, we conclude that

(3.16)

$$q(x,t) = \int_{\lambda_+ - i\infty}^{\lambda_+ + i\infty} \frac{\Gamma_B(\lambda)}{\lambda} \big[\gamma_{1B}(\lambda\widetilde{h}_1 - \gamma_{2B}\widetilde{h}_2)e^{\gamma_{1B}x} - \gamma_{2B}(\lambda\widetilde{h}_1 - \gamma_{1B}\widetilde{h}_2)e^{\gamma_{2B}x}\big] d\lambda,$$

(3.17)

$$r(x,t) = \int_{\lambda_+ - i\infty}^{\lambda_+ + i\infty} \Gamma_B(\lambda)\big[(\lambda\widetilde{h}_1 - \gamma_{2B}\widetilde{h}_2)e^{\gamma_{1B}x} - (\lambda\widetilde{h}_1 - \gamma_{1B}\widetilde{h}_2)e^{\gamma_{2B}x}\big] d\lambda.$$

It is straightforward to see that

$$\begin{aligned}
|\gamma_{1B}^2 - \gamma_{2B}^2| &= O(|\lambda - \lambda_+|^{\frac{1}{2}}), \quad \lambda \to \lambda_+, \quad \lambda_- < \Re(\lambda) < \lambda_+.\\
|\gamma_{1B} - \gamma_{2B}| &= O(\sqrt{(1+b\lambda^2)^2 - 4a\lambda^2}), \quad \lambda \to \lambda_-, \quad \lambda_- < \Re(\lambda) < \lambda_+.\\
\gamma_{1B} &\to -\sqrt[4]{(1+b\lambda^2)^2 - 4a\lambda^2}, \quad |\lambda| \to \infty, \quad \lambda \in B.\\
\gamma_{2B} &\to -i\sqrt[4]{(1+b\lambda^2)^2 - 4a\lambda^2}, \quad |\lambda| \to \infty, \quad \lambda \in B.
\end{aligned}$$

Now, from the Cauchy's Theorem with respect to the region B, we are able to write

(3.18)
$$q(x,t)=\int_{\lambda_- - i\infty}^{\lambda_- + i\infty}\frac{\Gamma_B(\lambda)}{\lambda}\big[\gamma_{1B}(\lambda\widetilde{h}_1-\gamma_{2B}\widetilde{h}_2)e^{\gamma_{1B}x}-\gamma_{2B}(\lambda\widetilde{h}_1-\gamma_{1B}\widetilde{h}_2)e^{\gamma_{2B}x}\big]d\lambda,$$

(3.19)
$$r(x,t)=\int_{\lambda_- - i\infty}^{\lambda_- + i\infty}\Gamma_B(\lambda)\big[(\lambda\widetilde{h}_1-\gamma_{2B}\widetilde{h}_2)e^{\gamma_{1B}x}-(\lambda\widetilde{h}_1-\gamma_{1B}\widetilde{h}_2)e^{\gamma_{2B}x}\big]d\lambda.$$

Now, in a similar fashion, performing similar estimates and using the Cauchy's Theorem with respect to the region $C=\{\omega : 0<\Re(\omega)<\lambda_-\}$, we see that

(3.20)
$$q(x,t)=\int_{0-i\infty}^{0+i\infty}\frac{\Gamma_C(\lambda)}{\lambda}\big[\gamma_{1C}(\lambda\widetilde{h}_1-\gamma_{2C}\widetilde{h}_2)e^{\gamma_{1C}x}-\gamma_{2C}(\lambda\widetilde{h}_1-\gamma_{1C}\widetilde{h}_2)e^{\gamma_{2C}x}\big]d\lambda,$$

(3.21)
$$r(x,t)=\int_{0-i\infty}^{0+i\infty}\Gamma_C(\lambda)\big[(\lambda\widetilde{h}_1-\gamma_{2C}\widetilde{h}_2)e^{\gamma_{1C}x}-(\lambda\widetilde{h}_1-\gamma_{1C}\widetilde{h}_2)e^{\gamma_{2C}x}\big]d\lambda.$$

Now, if we set U_1 and U_2 by

$$U_1(x,t)=\frac{1}{2\pi i}\int_{0+i0}^{0+i\infty}\frac{e^{\lambda t}}{\lambda(\gamma_{1C}^2-\gamma_{2C}^2)}\big[\gamma_{1C}(\gamma_{1C}+\gamma_{2C})(\lambda\widetilde{h}_1-\gamma_{2C}\widetilde{h}_2)e^{\gamma_{1C}x}\big]d\lambda$$

$$U_2(x,t)=-\frac{1}{2\pi i}\int_{0+i0}^{0+i\infty}\frac{e^{\lambda t}}{\lambda(\gamma_{1C}^2-\gamma_{2C}^2)}\big[\gamma_{2C}(\gamma_{1C}+\gamma_{2C})(\lambda\widetilde{h}_1-\gamma_{1C}\widetilde{h}_2)e^{\gamma_{2C}x}\big]d\lambda.$$

Then we have for $x,t>0$ we have that

$$q(x,t)=U_1(x,t)+U_2(x,t)+\overline{U_1(x,t)}+\overline{U_2(x,t)}. \tag{3.22}$$

As done for q, we can compute an explicit formula for r. In this case, we define functions V_1 and V_2 as

$$V_1(x,t)=\frac{1}{2\pi i}\int_{0+i0}^{0+i\infty}\frac{e^{\lambda t}}{\gamma_{1C}^2-\gamma_{2C}^2}\big[(\gamma_{1C}+\gamma_{2C})(\lambda\widetilde{h}_1-\gamma_{2C}\widetilde{h}_2)e^{\gamma_{1C}x}\big]d\lambda$$

$$V_2(x,t)=-\frac{1}{2\pi i}\int_{0-i0}^{0+i\infty}\frac{e^{\lambda t}}{\gamma_{1C}^2-\gamma_{2C}^2}\big[(\gamma_{1C}+\gamma_{2C})(\lambda\widetilde{h}_1-\gamma_{1C}\widetilde{h}_2)e^{\gamma_{2C}x}\big]d\lambda.$$

We see in this case for $x,t>0$ that

$$r(x,t)=V_1(x,t)+V_2(x,t)+\overline{V_1(x,t)}+\overline{V_2(x,t)}, \tag{3.23}$$

meaning that

$$W_b(h_1,h_2)(x,t) = \begin{pmatrix} q(x,t) \\ r(x,t) \end{pmatrix} = \begin{pmatrix} U_1(x,t)+U_2(x,t)+\overline{U_1(x,t)}+\overline{U_2(x,t)} \\ V_1(x,t)+V_2(x,t)+\overline{V_1(x,t)}+\overline{V_2(x,t)} \end{pmatrix}.$$

Finally, we need to recall that γ is a root of the characteristic polynomial

$$a\gamma^4 - (1+b\lambda^2)\gamma^2 + \lambda^2 = 0,$$

which is equivalent to have λ expressed as

$$\lambda^2 = \gamma^2\left(\frac{a\gamma^2-1}{b\gamma^2-1}\right).$$

A simple computation shows that if we define

$$\lambda = i\mu\sqrt{\frac{a\mu^2-1}{1-b\mu^2}}$$

with $\frac{1}{\sqrt{a}} \le \mu \le \frac{1}{\sqrt{b}}$ $(a>b)$, then the roots are given explicitly by

$$\gamma_1(\mu) = -\mu = -\gamma_3(\mu), \qquad \gamma_2(\mu) = \frac{i}{\sqrt{a}}\sqrt{\frac{a\mu^2-1}{1-b\mu^2}} = -\gamma_4(\mu)$$

It is easy to verify that the image $s(\mu) = \sqrt{\frac{a\mu^2-1}{1-b\mu^2}}$ with $\frac{1}{\sqrt{a}} \le \mu \le \frac{1}{\sqrt{b}}$ is $[0,\infty)$. In other word, $\lambda = i\mu\sqrt{\frac{a\mu^2-1}{1-b\mu^2}} = i\mu s(\mu)$ runs along the imaginary axes from zero to $+\infty$. In this case, we have that

$$d\lambda = \frac{i(ab\mu^4 - 2a\mu^2 + 1)\,d\mu}{(1-b\mu^2)^{3/2}(a\mu^2-1)^{1/2}}.$$

On the other hand,

$$\gamma_2^2 - \gamma_1^2 = \frac{(ab\mu^4 - 2a\mu^2 + 1)}{a(1-b\mu^2)}$$

So, we conclude that

$$\frac{d\lambda}{\gamma_2^2-\gamma_1^2} = \frac{ia\,d\mu}{(1-b\mu^2)^{1/2}(a\mu^2-1)^{1/2}}$$

and also that

$$\frac{d\lambda}{\lambda(\gamma_2^2-\gamma_1^2)} = \frac{a\,d\mu}{\mu(a\mu^2-1)}.$$

Moreover, we also have that

$$\gamma_1(\gamma_1+\gamma_2) = -\mu\left(\mu - \frac{is(\mu)}{\sqrt{a}}\right), \quad \gamma_2(\gamma_1+\gamma_2) = -\frac{is(\mu)}{\sqrt{a}}\left(\mu - \frac{is(\mu)}{\sqrt{a}}\right).$$

Now, from the definition of U_i, we have that

$$U_1(x,t) = \frac{-a}{2\pi}\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} \rho_1(x,t,\mu)\,d\mu, \quad U_2(x,t) = \frac{-\sqrt{a}}{2\pi}\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} \rho_2(x,t,\mu)\,d\mu,$$

with

$$\rho_1(x,t,\mu) = \frac{e^{i\mu s(\mu)t}e^{-\mu x}s(\mu)\rho_3(\mu)}{(a\mu^2-1)}\left(\int_0^\infty \left(\mu h_1(\xi) - \frac{h_2(\xi)}{\sqrt{a}}\right)e^{-i\mu s(\mu)\xi}\,d\xi\right)$$
$$\rho_2(x,t,\mu) = \frac{e^{i\mu s(\mu)t}e^{\frac{is(\mu)x}{\sqrt{a}}}s(\mu)\rho_3(\mu)}{(a\mu^2-1)}\left(\int_0^\infty (is(\mu)h_1(\xi)+h_2(\xi))\,e^{-i\mu s(\mu)\xi}\,d\xi\right)$$
$$\rho_3(\mu) = \left(\mu - \frac{is(\mu)}{\sqrt{a}}\right).$$

The expression for $V_i \quad (i=1,2)$ are obtained in a similar fashion. □

Lemma 3.3. *Let $s \geq 0$ and $W_b(h_1,h_2) = (q,r)^t$. If $h_1 \in H_0^{s-\frac{3}{2}}(\mathbb{R}^+)$ and $h_2 \in H_0^{s-\frac{5}{2}}(\mathbb{R}^+)$, then we have the following estimate*

$$\sup_{t\geq 0}\|W_b(h_1,h_2)(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \leq C\left(\|h_1\|_{H_0^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|h_2\|_{H_0^{s-\frac{5}{2}}(\mathbb{R}^+)}\right).$$

Proof. From previous result, we need to estimate U_i and V_i for $i=1,2$. In order to estimate U_1, we consider the operator T_1 define on $L^2\left(\frac{1}{\sqrt{a}},\frac{1}{\sqrt{b}}\right)$ by

$$T_1(g)(x,t) = \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} e^{i\mu s(\mu)t}e^{\gamma_1(\mu)x}g(\mu)d\mu.$$

If we set g_1 and g_2 by

$$g_1(\mu) = \left(\frac{\mu s(\mu)(\sqrt{a}\mu - is(\mu))}{\sqrt{a}(a\mu^2-1)}\right)\left(\int_0^\infty h_1(\xi)e^{-i\mu s(\mu)\xi}\,d\xi\right)$$
$$g_2(\mu) = \left(\frac{s(\mu)(\sqrt{a}\mu - is(\mu))}{a(a\mu^2-1)}\right)\left(\int_0^\infty h_2(\xi)e^{-i\mu s(\mu)\xi}\,d\xi\right),$$

then we see that

$$U_1 = \frac{a}{2\pi}(T_1(g_1) - T_1(g_2)), \tag{3.24}$$

meaning that to estimate U_1 requires to estimate $T_1(g_1)$ y $T_1(g_2)$. We first establish the estimate for $T_1(g)$ with $g \in L^2\left(\frac{1}{\sqrt{a}},\frac{1}{\sqrt{b}}\right)\bigcap L^1\left(\frac{1}{\sqrt{a}},\frac{1}{\sqrt{b}}\right)$. Now, for $s \geq 0$ we choose $n = [s]+1$. So, for $k = 0,1,2,...,n$ we have that

$$\partial_x^k T_1(g)(x,t) = \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} [\gamma_1(\mu)]^k e^{i\mu s(\mu)t}e^{\gamma_1(\mu)x}g(\mu)d\mu.$$

From the integral Minkowski inequality, we have that

$$
\begin{aligned}
\|\partial_x^k T_1(g)(\cdot,t)\|_{L_x^2(\mathbb{R}^+)} &\leq \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} |\gamma_1(\mu)|^k |e^{i\mu s(\mu)t}| \|e^{\gamma_1(\mu)x}\|_{L_x^2(\mathbb{R}^+)} |g(\mu)| d\mu \\
&\leq \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} |\gamma_1(\mu)|^k \Big(\int_0^{+\infty} |e^{\gamma_1(\mu)x}|^2 dx \Big)^{1/2} |g(\mu)| d\mu \\
&\leq \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} |\gamma_1(\mu)|^k \left(\frac{1}{2|Re(\gamma_1(\mu))|} \right)^{1/2} |g(\mu)| d\mu \\
&\leq \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} |\gamma_1(\mu)|^k \left(\frac{1}{2|\gamma_1(\mu)|} \right)^{1/2} |g(\mu)| d\mu \\
&\leq C \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} |\gamma_1(\mu)|^{k-\frac{1}{2}} |g(\mu)| d\mu \\
&\leq C \big\| \gamma_1^{k-\frac{1}{2}} g \big\|_{L^1(1/\sqrt{a},1/\sqrt{b})}.
\end{aligned} \tag{3.25}
$$

Using estimate (3.25), we have that

$$
\|T_1(g)(\cdot,t)\|_{L^2(\mathbb{R}^+)} \leq C \big\| |\gamma_1|^{-\frac{1}{2}} g \big\|_{L^1(1/\sqrt{a},1/\sqrt{b})},
$$

which correspond to the particular cases for $k = 0$ in (3.25). Moreover,

$$
\begin{aligned}
\|T_1(g)(\cdot,t)\|_{H^n(\mathbb{R}^+)} &= \left(\sum_{k=1}^n \big\|\partial_x^k T_1(g)(\cdot,t)\big\|_{L^2(\mathbb{R}^+)}^2 \right)^{1/2} \\
&\leq \left[\sum_{k=1}^n C \big\| \gamma_1^{k-\frac{1}{2}} g \big\|_{L^1(1/\sqrt{a},1/\sqrt{b})}^2 \right]^{1/2} \\
&\leq \left[nC \big\| \gamma_1^{n-\frac{1}{2}} g \big\|_{L^1(1/\sqrt{a},1/\sqrt{b})}^2 \right]^{1/2} \\
&\leq C \big\| \gamma_1^{n-\frac{1}{2}} g \big\|_{L^1(1/\sqrt{a},1/\sqrt{b})}
\end{aligned}
$$

From these estimates and the Calderon-Lions interpolation theorem, we conclude that,

$$
\|T_1(g)(\cdot,t)\|_{H^s(\mathbb{R}^+)} \leq C \big\| |\gamma_1|^{s-\frac{1}{2}} g \big\|_{L^1(1/\sqrt{a},1/\sqrt{b})}. \tag{3.26}
$$

Applying (3.26) to $g = g_1$ y $g = g_2$, we see respectively that

$$
\begin{aligned}
\|T_1(g_1)(\cdot,t)\|_{H^s(\mathbb{R}^+)} &\leq C \big\| |\gamma_1|^{s-\frac{1}{2}} g_1 \big\|_{L^1(1/\sqrt{a},1/\sqrt{b})} \\
&\leq C \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} \mu^{s-\frac{1}{2}} |g_1(\mu)| d\mu \\
&\leq C \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} \mu^{s+\frac{1}{2}} \sigma(\mu) \left| \int_0^\infty h_1(\xi) e^{-i\mu s(\mu)\xi}\, d\xi \right| d\mu
\end{aligned} \tag{3.27}
$$

and

$$\begin{aligned}\|T_1(g_2)(\cdot,t)\|_{H^s(\mathbb{R}^+)} &\leq C\left\||\gamma_1|^{s-\frac{1}{2}}g_2\right\|_{L^1(1/\sqrt{a},1/\sqrt{b})}\\ &\leq C\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}}\mu^{s-\frac{1}{2}}|g_2(\mu)|d\mu\\ &\leq C\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}}\mu^{s-\frac{1}{2}}\sigma(\mu)\left|\int_0^\infty h_2(\xi)e^{-i\mu s(\mu)\xi}\,d\xi\right|d\mu,\end{aligned}\tag{3.28}$$

with $\sigma(\mu)=\frac{s(\mu)\sqrt{a\mu^2+s^2(\mu)}}{a\mu^2-1}$. Now, we consider the following change of variables $\eta=\mu s(\mu)$. Then we see that

$$\eta^2=\frac{\mu^2(a\mu^2-1)}{1-b\mu^2},\tag{3.29}$$

Moreover, we have that

$$2\eta d\eta=\frac{-2\mu[a\mu^2(b\mu^2-1)-(a\mu^2-1)]}{(1-b\mu^2)^2}d\mu,$$

meaning that

$$\begin{aligned}d\eta &= \frac{[a\mu^2(1-b\mu^2)+(a\mu^2-1)]}{(a\mu^2-1)^{1/2}(1-b\mu^2)^{3/2}}d\mu\\ &= \frac{(a\mu^2+s^2(\mu))}{(a\mu^2-1)^{1/2}(1-b\mu^2)^{1/2}}d\mu.\end{aligned}$$

Using previous estimates, estimates (3.27) and (3.28), we see that

$$\begin{aligned}\|T_1(g_1)(\cdot,t)\|_{H^s(\mathbb{R}^+)} &\leq C\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}}\mu^{s+\frac{1}{2}}\sigma(\mu)\left(\frac{(a\mu^2-1)^{1/2}(1-b\mu^2)^{1/2}}{(a\mu^2+s^2(\mu))}\right)\\ &\quad\left|\int_0^\infty h_1(\xi)e^{-i\mu s(\mu)\xi}\,d\xi\right|\left(\frac{(a\mu^2+s^2(\mu))}{(a\mu^2-1)^{1/2}(1-b\mu^2)^{1/2}}\right)d\mu\\ &\leq C\int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}}\mu^{s+\frac{1}{2}}\left(\frac{1}{\sqrt{a\mu^2+s^2(\mu)}}\right)\\ &\quad\left|\int_0^\infty h_1(\xi)e^{-i\mu s(\mu)\xi}\,d\xi\right|\left(\frac{(a\mu^2+s^2(\mu))}{(a\mu^2-1)^{1/2}(1-b\mu^2)^{1/2}}\right)d\mu\end{aligned}$$

and from similar computations,

$$\begin{aligned}\|T_1(g_2)(\cdot,t)\|_{H^s(\mathbb{R}^+)} &\leq C\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}}\mu^{s-\frac{1}{2}}\left(\frac{1}{\sqrt{a\mu^2+s^2(\mu)}}\right)\\ &\quad\left|\int_0^\infty h_2(\xi)e^{-i\mu s(\mu)\xi}\,d\xi\right|\left(\frac{(a\mu^2+s^2(\mu))}{(a\mu^2-1)^{1/2}(b\mu^2-1)^{1/2}}\right)d\mu\end{aligned}$$

Recall from (3.29) that $a\mu^4+(b\eta^2-1)\mu^2-\eta^2=0$ and that $\frac{1}{\sqrt{a}}\leq\mu\leq\frac{1}{\sqrt{b}}$. So, if we assume that $\eta\neq 0$, then we conclude that

$$\sqrt{a\mu^2+s^2(\mu)}\approx C\sqrt{1+\eta^2}\approx C(1+\eta),\quad \mu^{s\pm\frac{1}{2}}\left(\frac{1}{\sqrt{a\mu^2+s^2(\mu)}}\right)\approx(1+\eta)^{-1}.$$

From this, we have for $i=1,2$ that

$$\|T_1(g_i)(\cdot,t)\|_{H^s(\mathbb{R}^+)}\leq C\int_0^{+\infty}|1+\eta|^{-1}\left|\int_0^\infty h_i(\xi)e^{-i\eta\xi}\,d\xi\right|d\eta\tag{3.30}$$

So, using Hölder inequality, we get that

$$\begin{aligned} \|T_1(g_i)(\cdot,t)\|_{H^s(\mathbb{R}^+)} &\leq C\big\||1+\eta|^{-1}\big\|_{L^2_\eta(\mathbb{R}^+)} \left\|\int_0^\infty h_i(\xi)e^{-i\eta\xi}\,d\xi\right\|_{L^2_\eta(\mathbb{R}^+)} \\ &\leq C\|h_i\|_{L^2(\mathbb{R}^+)} \end{aligned} \tag{3.31}$$

Putting this together, we conclude that

$$\begin{aligned} \|U_1(\cdot,t)\|_{H^s(\mathbb{R}^+)} &\leq \|T_1(g_1)(\cdot,t)\|_{H^s(\mathbb{R}^+)} + \|T_1(g_2)(\cdot,t)\|_{H^s(\mathbb{R}^+)} \\ &\leq C\left(\|h_1\|_{L^2(\mathbb{R}^+)} + \|h_2\|_{L^2(\mathbb{R}^+)}\right) \end{aligned} \tag{3.32}$$

Now, in order to estimate $U_2(x,t)$, we proceed as above by defining the operator T_2,

$$T_2(g)(x,t) = \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} e^{i\mu s(\mu)t}e^{\gamma_2(\mu)x}g(\mu)d\mu.$$

We set in this case functions g_3 and g_4 as

$$\begin{aligned} g_3(\mu) &= \left(\frac{s^2(\mu)(\sqrt{a}\mu - is(\mu))}{\sqrt{a}(a\mu^2-1)}\right)\left(\int_0^\infty h_1(\xi)e^{-i\mu s(\mu)\xi}\,d\xi\right) \\ g_4(\mu) &= \left(\frac{s(\mu)(\sqrt{a}\mu - is(\mu))}{\sqrt{a}(a\mu^2-1)}\right)\left(\int_0^\infty h_2(\xi)e^{-i\mu s(\mu)\xi}\,d\xi\right), \end{aligned}$$

and so, we have that

$$U_2(x,t) = \frac{\sqrt{a}}{2\pi}(iT_2(g_3) + T_2(g_4)). \tag{3.33}$$

Now, we estimate $T_2(g)$ for $g \in L^2\left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}\right)$. Let $s \geq 0$ be given and choose $n = [s]+1$. Then for $k = 0, 1, 2, ..., n$ we have that

$$\partial_x^k T_2(g)(x,t) = \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} [\gamma_2(\mu)]^k e^{i\mu s(\mu)t}e^{\gamma_2(\mu)x}g(\mu)d\mu.$$

As done by J. Bona, M Sun, and B. Zheng in [**5**] (Lemma 3.2), we define

$$\xi(\mu) = \frac{1}{\sqrt{a}}s(\mu), \qquad G(\mu) = [\gamma_2(\mu)]^k e^{i\mu s(\mu)t}g(\mu).$$

From these notations, we have that

$$\partial_x^k T_2(g)(x,t) = \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} e^{i\xi(\mu)x}G(\mu)d\mu.$$

The first observation is that $\xi'(\mu) \neq 0$ and $|\xi'(\mu)| > \dfrac{\sqrt{ab}}{a-b}$ in the interval $\left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}\right)$. In fact,

$$\xi'(\mu) = \frac{(a-b)\mu}{\sqrt{a}(a\mu^2-1)^{\frac{1}{2}}(1-b\mu^2)^{\frac{3}{2}}} \neq 0.$$

Now, for $\mu \in \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}\right)$, then we have that

$$a\mu^2 - 1 \leq \frac{a-b}{b}, \quad 1 - b\mu^2 \leq \frac{a-b}{a}.$$

So, we conclude that

$$|\xi'(\mu)| \geq \frac{\sqrt{ab}}{a-b}$$

Then we are able to perform this change of variables $\omega = \xi(\mu)$ to get

$$\begin{aligned}\partial_x^k T_2(g)(x,t) &= \int_{\xi\left(\frac{1}{\sqrt{a}}\right)}^{\xi\left(\frac{1}{\sqrt{b}}\right)} e^{i\omega x} G(\xi^{-1}(\omega)) \frac{1}{\xi'\left(\xi^{-1}(\omega)\right)} d\omega \\ &= \int_{-\infty}^{+\infty} e^{i\omega x} G(\xi^{-1}(\omega)) \frac{1}{\xi'\left(\xi^{-1}(\omega)\right)} \chi_{\left(\xi\left(\frac{1}{\sqrt{a}}\right), \xi\left(\frac{1}{\sqrt{b}}\right)\right)} d\omega,\end{aligned}$$

where χ_A denote de characteristic function on A. So, from Parseval's formula we have that

$$\begin{aligned}\|\partial_x^k T_2(g)(\cdot,t)\|^2_{L^2(\mathbb{R})} &= \|\widehat{\partial_x^k T_2(g)(\cdot,t)}\|^2_{L^2(\mathbb{R})} \\ &= \int_{-\infty}^{+\infty} \left[G(\xi^{-1}(\omega)) \frac{1}{\xi'\left(\xi^{-1}(\omega)\right)} \chi_{\left(\xi\left(\frac{1}{\sqrt{a}}\right), \xi\left(\frac{1}{\sqrt{b}}\right)\right)} \right]^2 d\omega \\ &= \int_{\xi\left(\frac{1}{\sqrt{a}}\right)}^{\xi\left(\frac{1}{\sqrt{b}}\right)} \left[G(\xi^{-1}(\omega)) \frac{1}{\xi'\left(\xi^{-1}(\omega)\right)} \right]^2 d\omega \\ &= \int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}} |G(\mu)|^2 \frac{1}{|\xi'(\mu)|^2} d\mu.\end{aligned}$$

From this, we conclude that

$$\|\partial_x^k T_2(g)(\cdot,t)\|_{L^2(\mathbb{R}^+)} \leq C \left\| \gamma_2^k \frac{g}{\xi'} \right\|_{L^2(1/\sqrt{a}, 1/\sqrt{b})}. \tag{3.34}$$

On the other hand, we have that

$$\begin{aligned}\|T_2(g)(\cdot,t)\|_{H^n(\mathbb{R}^+)} &= \left(\sum_{k=1}^n \left\| \partial_x^k T_2(g)(\cdot,t) \right\|^2_{L^2(\mathbb{R}^+)} \right)^{1/2} \\ &\leq \left(\sum_{k=1}^n C \left\| |\gamma_2|^k \tfrac{g}{\xi'} \right\|^2_{L^2(1/\sqrt{a}, 1/\sqrt{b})} \right)^{1/2} \\ &\leq \left(nC \left\| |\gamma_2|^n \tfrac{g}{\xi'} \right\|^2_{L^2(1/\sqrt{a}, 1/\sqrt{b})} \right)^{1/2} \\ &\leq C \left\| |\gamma_2|^n \tfrac{g}{\xi'} \right\|_{L^2(1/\sqrt{a}, 1/\sqrt{b})}\end{aligned}$$

As above, from the Calderon-Lions interpolation theorem, we have that

$$\|T_2(g)(\cdot,t)\|_{H^s(\mathbb{R}^+)} \leq C \left\| |\gamma_2|^s \frac{g}{\xi'} \right\|_{L^2(1/\sqrt{a}, 1/\sqrt{b})}. \tag{3.35}$$

Now, we note for some constant $C = C(a,b)$ that

$$\begin{aligned}\frac{|g_3(\mu)|^2}{|\xi'(u)|^2}\left(\frac{d\mu}{d\eta}\right) &= \frac{(a\mu^2-1)^{\frac{3}{2}}(1-b\mu^2)^{\frac{3}{2}}}{(a-b)^2\mu^2}\left|\int_0^\infty h_1(\xi)e^{-i\mu s(\mu)\xi}d\xi\right|^2\\ &\le \frac{C}{s^3(\mu)}\left|\int_0^\infty h_1(\xi)e^{-i\mu s(\mu)\xi}d\xi\right|^2\\ \frac{|g_4(\mu)|^2}{|\xi'(u)|^2}\left(\frac{d\mu}{d\eta}\right) &= \frac{(a\mu^2-1)^{\frac{1}{2}}(1-b\mu^2)^{\frac{5}{2}}}{(a-b)^2\mu^2}\left|\int_0^\infty h_2(\xi)e^{-i\mu s(\mu)\xi}d\xi\right|^2\\ &\le \frac{C}{s^5(\mu)}\left|\int_0^\infty h_2(\xi)e^{-i\mu s(\mu)\xi}d\xi\right|^2.\end{aligned}$$

So, applying (3.35) to $g = g_3$, we have that

(3.36)

$$\begin{aligned}\|T_2(g_3)(\cdot,t)\|_{H^s(\mathbb{R}^+)} \le\ & C\left\||\gamma_2|^s\tfrac{g_3}{\xi'}\right\|_{L^2(1/\sqrt{a},1/\sqrt{b})}\\ \le\ & C\left[\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}}|\gamma_2(\mu)|^{2s}\frac{|g_3(\mu)|^2}{|\xi'(u)|^2}d\mu\right]^{1/2}\\ \le\ & C\left[\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}}|s(\mu)|^{2s-3}\left|\int_0^\infty h_1(\xi)e^{-i\mu s(\mu)\xi}d\xi\right|^2\frac{d\eta}{d\mu}d\mu\right]^{1/2}\end{aligned}$$

Moreover, applying (3.35) to $g = g_4$, we have that

(3.37)

$$\begin{aligned}\|T_2(g_4)(\cdot,t)\|_{H^s(\mathbb{R}^+)} \le\ & C\left\||\gamma_2|^s\tfrac{g_4}{\xi'}\right\|_{L^2(1/\sqrt{a},1/\sqrt{b})}\\ \le\ & C\left[\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}}|\gamma_2(\mu)|^{2s}\frac{|g_4(\mu)|^2}{|\xi'(u)|^2}d\mu\right]^{1/2}\\ \le\ & C\left[\int_{\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{b}}}|s(\mu)|^{2s-5}\left|\int_0^\infty h_2(\xi)e^{-i\mu s(\mu)\xi}\,d\xi\right|^2\frac{d\eta}{d\mu}d\mu\right]^{1/2}.\end{aligned}$$

We now note that $\mu \in \left[\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}\right]$, which implies that $\eta = \mu s(\mu) \approx s(\mu)$. So, using the change of variables $\eta = \mu s(\mu)$ in (3.36) and (3.37), and previous estimates, we conclude that

$$\begin{aligned}\|T_2(g_3)(\cdot,t)\|_{H^s(\mathbb{R}^+)} \le\ & C\left[\int_0^\infty \eta^{2s-3}\left(\int_0^\infty h_1(\xi)e^{-i\eta\xi}\,d\xi\right)^2d\eta\right]\\ \le\ & C\left\|(1+\eta)^{2s-3}\int_0^\infty h_1(\xi)e^{-i\eta\xi}\,d\xi\right\|_{L^2_\eta(0,+\infty)}\\ \le\ & C\|h_1\|_{H_0^{s-\frac{3}{2}}(\mathbb{R}^+)}.\end{aligned}$$

In a similar fashion, we have that

$$\|T_2(g_4)(\cdot,t)\|_{H^s(\mathbb{R}^+)} \le C\|h_2\|_{H_0^{s-\frac{5}{2}}(\mathbb{R}^+)}.$$

Putting together those estimates, we conclude that

$$\begin{aligned}\|U_2(\cdot,t)\|_{H^s(\mathbb{R}^+)} &\le \|T_2(g_3)(\cdot,t)\|_{H^s(\mathbb{R}^+)} + \|T_2(g_4)(\cdot,t)\|_{H^s(\mathbb{R}^+)}\\ &\le C\left(\|h_1\|_{H_0^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|h_2\|_{H_0^{s-\frac{5}{2}}(\mathbb{R}^+)}\right).\end{aligned} \tag{3.38}$$

So, from estimates (3.22), (3.32), and (3.38), we finally get

$$\|q(\cdot,t)\|_{H^s(\mathbb{R}^+)} \leq C\left(\|h_1\|_{H_0^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|h_2\|_{H_0^{s-\frac{5}{2}}(\mathbb{R}^+)}\right).$$

By following the same type of arguments (see formulas (3.9) and (3.10)), we can see that

$$\|r(\cdot,t)\|_{H^s(\mathbb{R}^+)} \leq C\left(\|h_1\|_{H_0^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|h_2\|_{H_0^{s-\frac{5}{2}}(\mathbb{R}^+)}\right).$$

□

3.3. The Homogeneous Initial Value Problem on the line. Now, we consider the initial-value problem (**IVP**) in the line

$$\left\{\begin{array}{ll} \partial_t X(x,t) &= MX(x,t), \quad x\in\mathbb{R}, \quad t>0,\\ X(x,0) &= \begin{pmatrix} f_1(x)\\ f_2(x)\end{pmatrix}, \end{array}\right. \tag{3.39}$$

LEMMA 3.4. *Let $s\in\mathbb{R}$ and $f_1, f_2\in H^s(\mathbb{R})$, then the solution $W_R(f_1,f_2)(x,t)$ of the (**IVP**) (3.39) is given by the explicit formula*

$$W_R(f_1,f_2)(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{\widehat{M}t}\begin{pmatrix}\hat{f}_1(\xi)\\ \hat{f}_2(\xi)\end{pmatrix} e^{ix\xi}\, d\xi. \tag{3.40}$$

where

$$e^{\widehat{M}t} = \begin{pmatrix} (e^{i\sqrt{\Lambda(\xi)}\xi t} + e^{-i\sqrt{\Lambda(\xi)}\xi t}) & \dfrac{(e^{i\sqrt{\Lambda(\xi)}\xi t} - e^{-i\sqrt{\Lambda(\xi)}\xi t})}{\sqrt{\Lambda(\xi)}}\\ \sqrt{\Lambda(\xi)}(e^{i\sqrt{\Lambda(\xi)}\xi t} - e^{-i\sqrt{\Lambda(\xi)}\xi t}) & (e^{i\sqrt{\Lambda(\xi)}\xi t} + e^{-i\sqrt{\Lambda(\xi)}\xi t})\end{pmatrix}$$

PROOF. Let write $W_R(f_1,f_2) = (q,r)^t$. If we take Fourier transform in (3.39) with respect to the x variable, we conclude that

$$\left\{\begin{array}{ll} \partial_t \widehat{X}(\xi,t) &= \widehat{M}\widehat{X}(\xi,t), \quad \xi\in\mathbb{R}, \quad t>0,\\ \widehat{X}(\xi,0) &= \begin{pmatrix} \widehat{f_1}(\xi)\\ \widehat{f_2}(\xi)\end{pmatrix}, \end{array}\right. \tag{3.41}$$

where ξ denotes the dual variable of x, and $\widehat{q}$, $\widehat{r}$, $\widehat{f_1}$ y $\widehat{f_2}$ are the Fourier transform of q, r, f_1 y f_2 with respect to x, respectively, and

$$\widehat{M} = \begin{pmatrix} 0 & i\xi\\ i\xi\Lambda(\xi) & 0\end{pmatrix},$$

where $\Lambda(\xi) = \frac{1+a\xi^2}{1+b\xi^2}$. So, we have that the general solutions for (3.5) has the form

$$\widehat{X}(\xi,t) = e^{\widehat{M}t}\begin{pmatrix}\widehat{f_1}(\xi)\\ \widehat{f_2}(\xi)\end{pmatrix} \tag{3.42}$$

Now, a direct computation shows that

$$e^{\widehat{M}t} = \begin{pmatrix} \cos(\sqrt{\Lambda(\xi)}\xi t) & \dfrac{i\sin(\sqrt{\Lambda(\xi)}\xi t)}{\sqrt{\Lambda(\xi)}}\\ i\sqrt{\Lambda(\xi)}\sin(\sqrt{\Lambda(\xi)}\xi t) & \cos(\sqrt{\Lambda(\xi)}\xi t)\end{pmatrix}.$$

Using this formula in (3.42), we obtain that

$$\begin{aligned}\widehat{X}(\xi,t) &= \begin{pmatrix} \cos(\sqrt{\Lambda(\xi)}\xi t) & \dfrac{i\sin(\sqrt{\Lambda(\xi)}\xi t)}{\sqrt{\Lambda(\xi)}} \\ i\sqrt{\Lambda(\xi)}\sin(\sqrt{\Lambda(\xi)}\xi t) & \cos(\sqrt{\Lambda(\xi)}\xi t)\end{pmatrix}\begin{pmatrix}\widehat{f}_1(\xi)\\ \widehat{f}_2(\xi)\end{pmatrix} \\ &= \begin{pmatrix} \cos(\sqrt{\Lambda(\xi)}\xi t)\widehat{f}_1(\xi) + \dfrac{i\sin(\sqrt{\Lambda(\xi)}\xi t)\widehat{f}_2(\xi)}{\sqrt{\Lambda(\xi)}} \\ i\sqrt{\Lambda(\xi)}\sin(\sqrt{\Lambda(\xi)}\xi t)\widehat{f}_1(\xi) + \cos(\sqrt{\Lambda(\xi)}\xi t)\widehat{f}_2(\xi)\end{pmatrix}.\end{aligned}$$

So, using inverse Fourier transform, we see that $W_R(f_1, f_2)$ has the explicit formula

$$W_R(f_1,f_2)(x,t) = \int_{-\infty}^{\infty} e^{\widehat{M}t}\begin{pmatrix}\hat{f}_1(\xi)\\ \hat{f}_2(\xi)\end{pmatrix} e^{ix\xi}d\xi. \tag{3.43}$$

Now, using that

$$\cos(w) = \frac{1}{2}\left(e^{iw}+e^{-iw}\right), \quad \sin(w) = \frac{1}{2i}\left(e^{iw}-e^{-iw}\right)$$

we obtain the desired conclusion. □

Before going forward, we note that the components of the solution of the (**IVP**) are

$$\begin{aligned} q(x,t) =& \frac{1}{2}\int_{-\infty}^{\infty}\Bigg[(e^{i\sqrt{\Lambda(\xi)}\xi t}+e^{-i\sqrt{\Lambda(\xi)}\xi t})\widehat{f}_1(\xi) + \frac{1}{\sqrt{\Lambda(\xi)}}(e^{i\sqrt{\Lambda(\xi)}\xi t} \\ & -e^{-i\sqrt{\Lambda(\xi)}\xi t})\widehat{f}_2(\xi)\Bigg] e^{ix\xi}d\xi \\ r(x,t) =& \frac{1}{2}\int_{-\infty}^{\infty}\Big[\sqrt{\Lambda(\xi)}(e^{i\sqrt{\Lambda(\xi)}\xi t}-e^{-i\sqrt{\Lambda(\xi)}\xi t})\widehat{f}_1(\xi) + (e^{i\sqrt{\Lambda(\xi)}\xi t} \\ & +e^{-i\sqrt{\Lambda(\xi)}\xi t})\widehat{f}_2(\xi)\Big] e^{ix\xi}d\xi. \end{aligned}$$

LEMMA 3.5. *Let $s \in (-\infty, +\infty)$ and $f_1, f_2 \in H^s(\mathbb{R})$, then we have that*

$$\sup_{t>0}\|W_R(f_1,f_2)(\cdot,t)\|_{Y^s(\mathbb{R})} \le C\left(\|f_1\|_{H^s(\mathbb{R})}+\|f_2\|_{H^s(\mathbb{R})}\right).$$

PROOF. First we note that $\Lambda(\xi)$ is bounded by

$$\frac{\min(1,a)}{\max(1,b)} \le \frac{1+a\xi^2}{1+b\xi^2} \le \frac{\max(1,a)}{\min(1,b)}.$$

So, from previous formulas and the bound for $\Lambda(\xi)$,

$$\begin{aligned}\|q(\cdot,t)\|_{H^s(\mathbb{R})} &\le \frac{1}{2}\Big\|(1+|\xi|)^s\Big|(e^{i\sqrt{\Lambda(\xi)}\xi t}+e^{-i\sqrt{\Lambda(\xi)}\xi t})\widehat{f}_1(\xi)\\ &+\frac{1}{\sqrt{\Lambda(\xi)}}(e^{i\sqrt{\Lambda(\xi)}\xi t}-e^{-i\sqrt{\Lambda(\xi)}\xi t})\widehat{f}_2(\xi)\Big|\Big\|_{L^2_\xi(\mathbb{R})}\\ &\le \frac{1}{2}\Big\|(1+|\xi|)^s\left(\left|e^{i\sqrt{\Lambda(\xi)}\xi t}\right|+\left|e^{-i\sqrt{\Lambda(\xi)}\xi t}\right|\right)\widehat{f}_1(\xi)\Big\|_{L^2_\xi(\mathbb{R})}\\ &+\frac{1}{2}\left\|\frac{(1+|\xi|)^s}{\sqrt{\Lambda(\xi)}}\left(\left|e^{i\sqrt{\Lambda(\xi)}\xi t}\right|+\left|e^{-i\sqrt{\Lambda(\xi)}\xi t}\right|\right)\widehat{f}_2(\xi)\right\|_{L^2_\xi(\mathbb{R})}\\ &\le \left\|(1+|\xi|)^s\widehat{f}_1(\xi)\right\|_{L^2_\xi(\mathbb{R})}+C\left\|(1+|\xi|)^s\widehat{f}_2(\xi)\right\|_{L^2_\xi(\mathbb{R})}\\ &\le C\left(\|f_1\|_{H^s(\mathbb{R})}+\|f_2\|_{H^s(\mathbb{R})}\right).\end{aligned}$$

In a similar fashion, we see that

$$\|r(\cdot,t)\|_{H^s(\mathbb{R})}\le C\left(\|f_1\|_{H^s(\mathbb{R})}+\|f_2\|_{H^s(\mathbb{R})}\right).$$

□

REMARK 3.6. Now, we are interested in estimating for $l\in\mathbb{R}$,

$$\sup_{x>0}\|W_R(f_1,f_2)(x,\cdot)\|_{H^l(\mathbb{R})}.$$

To do this, we see that $W_R(f_1,f_2)$ can be rewritten in terms of the auxiliary function $\phi(\xi)=\xi\sqrt{\Lambda(\xi)}$. In fact, note that

$$\begin{aligned}q(x,t)=&\frac{1}{2}\int_{-\infty}^{\infty}\left[\left(\widehat{f}_1(\xi)+\frac{1}{\sqrt{\Lambda(\xi)}}\widehat{f}_2(\xi)\right)e^{i\phi(\xi)t}+\left(\widehat{f}_1(\xi)-\frac{1}{\sqrt{\Lambda(\xi)}}\widehat{f}_2(\xi)\right)\right.\\ &\left.e^{-i\phi(\xi)t}\right]e^{ix\xi}d\xi\\ r(x,t)=&\frac{1}{2}\int_{-\infty}^{\infty}\left[\left(\sqrt{\Lambda(\xi)}\widehat{f}_1(\xi)+\widehat{f}_2(\xi)\right)e^{i\phi(\xi)t}-\left(\sqrt{\Lambda(\xi)}\widehat{f}_1(\xi)-\widehat{f}_2(\xi)\right)\right.\\ &\left.e^{-i\phi(\xi)t}\right]e^{ix\xi}d\xi\end{aligned}$$

Now, for $f,f_1,f_2\in\mathcal{S}(\mathbb{R})$, we consider the operators

$$\begin{aligned}V(f)(x,t)&=\int_{-\infty}^{\infty}\widehat{f}(\xi)e^{ix\xi}e^{i\phi(\xi)t}d\xi\\ V_1(f_1,f_2)(x,t)&=\frac{1}{2}\int_{-\infty}^{\infty}\left[\widehat{A}(\xi)\,e^{i\phi(\xi)t}+\widehat{B}(\xi)\,e^{-i\phi(\xi)t}\right]e^{ix\xi}d\xi\\ V_2(f_1,f_2)(x,t)&=\frac{1}{2}\int_{-\infty}^{\infty}\left[\widehat{C}(\xi)\,e^{i\phi(\xi)t}+\widehat{D}(\xi)\,e^{-i\phi(\xi)t}\right]e^{ix\xi}d\xi\end{aligned}$$

where functions A, B, C and D are given by

$$\begin{aligned}&\widehat{A}(\xi)=\widehat{f}_1(\xi)+\frac{1}{\sqrt{\Lambda(\xi)}}\widehat{f}_2(\xi),\quad \widehat{B}(\xi)=\widehat{f}_1(\xi)-\frac{1}{\sqrt{\Lambda(\xi)}}\widehat{f}_2(\xi)\\ &\widehat{C}(\xi)=\sqrt{\Lambda(\xi)}\widehat{f}_1(\xi)+\widehat{f}_2(\xi),\quad \widehat{D}(\xi)=-\sqrt{\Lambda(\xi)}\widehat{f}_1(\xi)+\widehat{f}_2(\xi)\end{aligned}$$

If we set $\xi = \xi(\eta)$ as the root of the equation $\phi(\xi) = \eta$, then using this change of variables, we see that

$$V(f)(x,t) = \int_{-\infty}^{\infty} e^{i\eta t}\widehat{f}(\xi(\eta))e^{ix\xi(\eta)}\frac{d\xi(\eta)}{d\eta}d\eta.$$

In particular, the Fourier transform for $V(f)(x,\cdot)$ with respect to the variable $t \in \mathbb{R}$ with dual variable η is given by

$$\mathcal{F}^t(V(f))(x,\eta) = \widehat{f}(\xi(\eta))e^{ix\xi(\eta)}\frac{d\xi(\eta)}{d\eta}.$$

Now assume that $\alpha, \beta \in \mathbb{R}$, then using the change of variable $\eta = \phi(\xi)$ with $\xi \in \mathbb{R}$ in $V(f)(x,t)$, we have that

$$\begin{aligned}
\|V(f)(x,\cdot)\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})} &= \left\| \eta^{\alpha}(1+|\eta|)^{\beta-\alpha}\widehat{f}(\xi(\eta))e^{ix\xi(\eta)}\frac{d\xi(\eta)}{d\eta}\right\|_{L^2_\eta(\mathbb{R})} \\
&= \left(\int_{-\infty}^{\infty} |\eta|^{2\alpha}(1+|\eta|)^{2\beta-2\alpha}\left|\widehat{f}(\xi(\eta))\right|^2\left|\frac{d\xi(\eta)}{d\eta}\right|^2 d\eta\right)^{1/2} \\
&= \left(\int_{-\infty}^{\infty} |\phi(\xi)|^{2\alpha}(1+|\phi(\xi)|)^{2\beta-2\alpha}\left|\widehat{f}(\xi)\right|^2\left|\frac{d\xi}{d\eta}\right|^2 \frac{d\eta}{d\xi}d\xi\right)^{1/2} \\
&= \left(\int_{-\infty}^{\infty} |\phi(\xi)|^{2\alpha}(1+|\phi(\xi)|)^{2\beta-2\alpha}\left|\widehat{f}(\xi)\right|^2\left|\frac{d\xi}{d\eta}\right| d\xi\right)^{1/2}.
\end{aligned}$$

On the other hand, we now that

$$\eta^2 = \xi^2\left(\frac{1+a\xi^2}{1+b\xi^2}\right), \quad \xi \in \mathbb{R},$$

So, we have also that

$$\frac{d\eta}{d\xi} = \frac{1}{\sqrt{\Lambda(\xi)}}\left(\frac{(1+2a\xi^2+ab\xi^4)}{(1+b\xi^2)^2}\right) \quad \Leftrightarrow \quad \frac{d\xi}{d\eta} = \sqrt{\Lambda(\xi)}\left(\frac{(1+b\xi^2)^2}{1+2a\xi^2+ab\xi^4}\right).$$

Using the bound for $\Lambda(\xi)$, we conclude that

$$\left|\frac{d\xi}{d\eta}\right| \le \sqrt{\Lambda(\xi)}\left(\frac{(1+b\xi^2)^2}{1+2a\xi^2+a^2\xi^4}\right) \le C(a,b).$$

Plugging this inequality in previous estimation, we conclude that

$$\begin{aligned}
\|V(f)(x,\cdot)\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})} &\le C\left(\int_{-\infty}^{\infty} |\xi|^{2\alpha}(1+|\xi|)^{2\beta-2\alpha}\left|\widehat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\le C\left\| |\xi|^{\alpha}(1+|\xi|)^{\beta-\alpha}\widehat{f}(\xi)\right\|_{L^2_\xi(\mathbb{R})} \\
&\le C\left\|f\right\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})}.
\end{aligned}$$

In particular, for $\alpha = 0$, we obtain that

$$\|V(f)(x,\cdot)\|_{H^s(\mathbb{R})} \le C\left\|f\right\|_{H^s(\mathbb{R})}.$$

Moreover, due to the form of functions A, B, C, and D, we conclude for $i = 1, 2$ that

$$\|V_i(f_1, f_2)(x,\cdot)\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})} \leq C\left(\|f_1\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})} + \|f_2\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})}\right)$$
$$\|V_i(f_1, f_2)(x,\cdot)\|_{H^s(\mathbb{R})} \leq C\left(\|f_1\|_{H^s(\mathbb{R})} + \|f_2\|_{H^s(\mathbb{R})}\right)$$

using that $\Lambda(\xi)$ is bounded away from zero. Now, if $F \in \{A, B, C, D\}$, then we see that

$$|\widehat{F}| \leq C(a,b)\left(|\widehat{f}_1| + |\widehat{f}_2|\right).$$

In other words, we have the following estimates,

LEMMA 3.7. *Let $s \in (-\infty, +\infty)$, $\alpha, \beta \in \mathbb{R}$, $k \in \mathbb{N}$, and $f_1, f_2 \in S(\mathbb{R})$. Then we have that*

$$\begin{aligned} \sup_{x>0} \|W_R(f_1, f_2)(x,\cdot)\|_{Y^{\alpha,\beta}(\mathbb{R})} &\leq C\|f_1\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})} + C\|f_2\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{R})}, \\ \sup_{x>0} \|W_R(f_1, f_2)(x,\cdot)\|_{Y^s(\mathbb{R})} &\leq C\|f_1\|_{H^s(\mathbb{R})} + C\|f_2\|_{H^s(\mathbb{R})} \\ \sup_{x>0} \left\|M^k W_R(f_1, f_2)(x,\cdot)\right\|_{Y^s(\mathbb{R})} &\leq C\|f_1\|_{H^{s+k}(\mathbb{R})} + C\|f_2\|_{H^{s+k}(\mathbb{R})}. \end{aligned}$$

PROOF. For the first two estimates, we note that $W_R(f_1, f_2)(x,t) = (q, r)^t(x,t)$ and that

$$q(x,t) = \frac{1}{2}V_1(f_1, f_2)(x,t), \quad r(x,t) = \frac{1}{2}V_2(f_1, f_2)(x,t).$$

Now, to get the third estimate, we need to observe that

$$\partial_x(e^{i\xi x}) = i\xi e^{i\xi x}, \quad B^{-1}A\partial_x(e^{i\xi x}) = i\xi\Lambda(\xi)e^{i\xi x}.$$

Now, from the representation (3.43), we easily conclude that

(3.44)

$$\begin{aligned} &MW_R(f_1, f_2)(x,t) \\ &= \int_{-\infty}^{\infty} \begin{pmatrix} i\sqrt{\Lambda(\xi)}\sin(\sqrt{\Lambda(\xi)}\xi t)\widehat{\partial_x f}_1(\xi) + \cos(\sqrt{\Lambda(\xi)}\xi t)\widehat{\partial_x f}_2(\xi) \\ \cos(\sqrt{\Lambda(\xi)}\xi t)\widehat{B^{-1}A\partial_x} f_1(\xi) + \dfrac{i\sin(\sqrt{\Lambda(\xi)}\xi t)}{\sqrt{\Lambda(\xi)}}\widehat{B^{-1}A\partial_x} f_2(\xi) \end{pmatrix} e^{ix\xi}d\xi \\ &= W_R(M(f_1, f_2)^t)(x,t) \end{aligned}$$

So, from the second estimate and using that ∂_x and $B^{-1}A\partial_x$ are order one operators, we conclude that

$$\sup_{x>0} \|MW_R(f_1, f_2)(x,\cdot)\|_{Y^s(\mathbb{R})} \leq C\|f_1\|_{H^{s+1}(\mathbb{R})} + C\|f_2\|_{H^{s+1}(\mathbb{R})}.$$

The same argument shows the estimate for $k \in \mathbb{N}$. □

3.4. The Homogeneous (IBVP): case $h_1 \equiv h_2 \equiv 0$. Here we consider the homogeneous linear (**IBVP**) in the first quarter

$$\begin{cases} \partial_t X(x,t) &= MX(x,t), \quad x > 0, \quad t > 0, \\ X(0,t) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad X(x,0) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}. \end{cases}$$

We use the explicit solutions W_b y W_R with initial data in the space $H_0^l(\mathbb{R}^+)$. In fact, given functions $f_1, f_2 \in C_0^\infty(\mathbb{R}^+)$, we consider extensions $\widetilde{f_i}$ for f_i from $\mathbb{R}^+$ to $\mathbb{R}$ such that $\widetilde{f_i}(x) = 0$ for $x \notin \mathbb{R}^+$. Now, we define functions $\widetilde{h}_i$ as

$$\begin{pmatrix} \tilde{h}_1(t) \\ \tilde{h}_2(t) \end{pmatrix} = W_R(\tilde{f}_1, \tilde{f}_2)(0,t).$$

We note that the solution W_C of the (**IBVP**) with boundary data $h_i \equiv 0$ is given by

$$W_C(f_1, f_2)(x,t) = W_R(\tilde{f}_1, \tilde{f}_2)(x,t) - W_b(\chi\tilde{h}_1, \chi\tilde{h}_2)(x,t).$$

From the estimates for W_b and W_R we are able to obtain estimates using the representation for W_C.

LEMMA 3.8. *For $s > \frac{1}{2}$ and $f_1, f_2 \in H_0^s(\mathbb{R}^+)$ we have the following estimate,*

$$\sup_{t>0} \|W_C(f_1, f_2)(\cdot, t)\|_{Y^s(\mathbb{R}^+)} \leq C\left(\|f_1\|_{H_0^s(\mathbb{R}^+)} + \|f_2\|_{H_0^s(\mathbb{R}^+)}\right).$$

PROOF. By definition, we have that $\tilde{f}_1, \tilde{f}_2 \in C_0^\infty(\mathbb{R}) \subset H^\beta(\mathbb{R})$ for any $\beta > 0$, and so,

$$\sup_{t>0} \|W_R(\tilde{f}_1, \tilde{f}_2)(\cdot, t)\|_{Y^\beta(\mathbb{R})} \leq C\left(\|f_1\|_{H_0^\beta(\mathbb{R}^+)} + \|f_2\|_{H_0^\beta(\mathbb{R}^+)}\right)$$

Moreover, since $u(x,t) = W_R(\tilde{f}_1, \tilde{f}_2)(x,t)$ is a classical solution for the initial-value problem in the line (3.5) with $X(x,0) = (\tilde{f}_1(x), \tilde{f}_2(x))^t$. We see that

$$(\tilde{h}_1(0), \tilde{h}_2(0)) = W_R(\tilde{f}_1, \tilde{f}_2)(0,0) = (\tilde{f}_1(0), \tilde{f}_2(0)) = (0,0).$$

On the other hand, from Lemma (2.1) and Lemma (3.7) we know that
$\chi(t)(\tilde{h}_1(t), \tilde{h}_2(t)) \in Y_0^\beta(\mathbb{R}^+)$, and that for $\beta > \frac{1}{2}$

$$\|\chi(t)(\tilde{h}_1(t), \tilde{h}_2(t))\|_{Y_0^\beta(\mathbb{R}^+)} \leq C\left(\|f_1\|_{H_0^\beta(\mathbb{R}^+)} + \|f_2\|_{H_0^\beta(\mathbb{R}^+)}\right).$$

As a consequence of the previous Lemma and Lemma (3.3), we conclude that

$$\sup_{t>0} \|W_b(\chi(t)\tilde{h}_1(t), \chi(t)\tilde{h}_2(t))(\cdot, t)\|_{Y^\beta(\mathbb{R}^+)} \leq C\left(\|f_1\|_{H_0^\beta(\mathbb{R}^+)} + \|f_2\|_{H_0^\beta(\mathbb{R}^+)}\right),$$

which implies that for any $f_i \in C_0^\infty(\mathbb{R}^+)$ with $i = 1, 2$ we have that

$$\sup_{t>0} \|W_C(f_1, f_2)(\cdot, t)\|_{Y^\beta(\mathbb{R}^+)} \leq C\left(\|f_1\|_{H_0^\beta(\mathbb{R}^+)} + \|f_2\|_{H_0^\beta(\mathbb{R}^+)}\right),$$

Using the density of $C_0^\infty(\mathbb{R})$ en $H_0^\beta(\mathbb{R}^+)$, we conclude for $f_i \in H_0^\beta(\mathbb{R}^+)$ with $i = 1, 2$, that

$$\sup_{t>0} \|W_C(f_1, f_2)(\cdot, t)\|_{Y^\beta(\mathbb{R})} \leq C\left(\|f_1\|_{H_0^\beta(\mathbb{R}^+)} + \|f_2\|_{H_0^\beta(\mathbb{R}^+)}\right).$$

□

3.5. The linear Homogeneous (IBVP): The general case. Now, we return to the general linear homogeneous (**IBVP**),

$$\left\{\begin{array}{ll} \partial_t X(x,t) & = MX(x,t), \quad x>0, \quad t>0, \\ X(0,t) & = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}, \quad X(x,0) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}. \end{array}\right. \tag{3.45}$$

From the discussion above, the solution of the (**IBVP**) necessarily depends on the explicit solutions $W_b(h_1,h_2)$, $W_R(f_1,f_2)$, y $W_C(f_1,f_2)$, after appropriate modifications to interchange the space $H_0^l(\mathbb{R}^+)$ for the space $H^l(\mathbb{R}^+)$. It is important to point out that given functions $f_i, h_i \in H^l(\mathbb{R}^+)$, it is necessary to define in a convenient way functions $\tilde{f}, \tilde{h}_i \in H_0^l(\mathbb{R}^+)$ depending on $l \in \mathbb{R}$, and to establish some compatibility conditions between functions f_i and h_i for $i=1,2$ in $x=0$ and $t=0$, as we will see in the coming result.

s-Compatibility conditions

In the coming result, we assume for $\frac{1}{2} < s \le \frac{9}{2}$ that $f_1, f_2 \in H^s(\mathbb{R}^+)$ and $h_1 \in H^{s-\frac{3}{2}}(\mathbb{R}^+)$, $h_2 \in H^{s-\frac{5}{2}}(\mathbb{R}^+)$.

(sC1) For $\frac{1}{2} < s \le \frac{3}{2}$, $f_i(0) = h_i(0)$ for $i=1,2$.

(sC2) For $\frac{3}{2} < s \le \frac{5}{2}$, $f_i(0) = h_i(0)$ for $i=1,2$ and

$$h_2(0) = 2h_1(0) + f_1'(0), \quad f_2'(0) = -h_1(0),$$

(sC3) For $\frac{5}{2} < s \le \frac{7}{2}$, $f_i(0) = h_i(0)$ $(i=1,2)$, and

$$\begin{aligned} &f_2(0) = f_1(0) + h_1'(0), h_2(0) = 2f_1(0) + 3f_1'(0) + f_1''(0) - h_1'(0), \\ &f_2'(0) = h_1'(0), f_2''(0) = -f_1(0) - f_1'(0) - h_1'(0). \end{aligned}$$

(sC4) For $\frac{7}{2} < s \le \frac{9}{2}$, $f_i(0) = h_i(0)$ $(i=1,2)$, and

$$\begin{aligned} &h_2(0) = f_1(0) - 2f_1''(0) - f_1'''(0) - h_1'(0), f_2'(0) = h_1'(0), \\ &f_2''(0) = -f_1(0) - f_1'(0) - h_1'(0), h_2'(0) = 2h_1'(0) + 2f_1''(0) + f_1'''(0), \\ &f_2'''(0) = 2f_1'''(0) + 5f_1''(0) + 6f_1'(0) + f_1(0) + h_1'(0), h_1''(0) = -2h_1'(0) - f_1(0). \end{aligned}$$

Theorem 3.9. *Let $\frac{1}{2} < s \le \frac{9}{2}$. If $f_1, f_2 \in H^s(\mathbb{R}^+)$, $h_1 \in H^{s-\frac{3}{2}}(\mathbb{R}^+)$, and $h_2 \in H^{s-\frac{5}{2}}(\mathbb{R}^+)$ satisfy one of the compatibility conditions* **(sC1)**-**(sC4)**, *then the solution $W(f_1,f_2,h_1,h_2)$ of the general (***IBVP***) (3.45) satisfies the estimate*

$$\begin{aligned} &\sup_{t>0} \|W(f_1,f_2,h_1,h_1)(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \\ &\qquad \le C\left(\|f_1\|_{H^s(\mathbb{R}^+)} + \|f_2\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|h_2\|_{H^{s-\frac{5}{2}}(\mathbb{R}^+)}\right). \end{aligned}$$

Proof. First assume that $\frac{1}{2} < s \le \frac{3}{2}$. In this case, we write the solution of the general (**IBVP**) $W(f_1,f_2,h_1,h_2)$ as

$$W(f_1,f_2,h_1,h_2)(x,t) = V(x,t) - K(x,t), \quad K(x,t) = -(f_1(0), f_2(0))^t e^{-x-t}.$$

A direct computation shows that V satisfies the (**IBVP**)

$$\left\{\begin{array}{ll} \partial_t V(x,t) & = MV(x,t) + K_1(x,t), \quad x>0, \quad t>0, \\ V(0,t) & = \begin{pmatrix} \tilde{h}_1(t) \\ \tilde{h}_2(t) \end{pmatrix}, \quad V(x,0) = \begin{pmatrix} \tilde{f}_1(x) \\ \tilde{f}_2(x) \end{pmatrix} \end{array}\right. \tag{3.46}$$

where K_1 satisfies that

$$K_1(x,t) = \partial_t K - MK = \begin{pmatrix} (f_1(0)+f_2(0))e^{-x} \\ f_2(0)e^{-x} + f_1(0)B^{-1}A(e^{-x}) \end{pmatrix} e^{-t},$$

and the boundary and initial data for the linear homogeneous (**IBVP**) (3.46) are given by

$$\tilde{H}(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} - \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} e^{-t}, \tilde{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} - \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} e^{-x}.$$

Now, we take $\psi \in H^{10}(\mathbb{R})$ to be an extension of e^{-x} from $\mathbb{R}^+$ to $\mathbb{R}$, and consider the function $K_2(\cdot,t) \in (L^1_t H^s_0(\mathbb{R}^+))^2$ defined by

$$K_2(x,t) = (\tilde{K}_1(x) - \tilde{K}_1(0)e^{-x})e^{-t}, \tilde{K}_1(x) = \begin{pmatrix} (f_1(0)+f_2(0))\psi(x) \\ f_2(0)\psi(x) + f_1(0)B^{-1}A(\psi(x)) \end{pmatrix}.$$

Now, from Lemma (2.1), we have that $(\tilde{h}_1, \tilde{h}_2)^t \in H_0^{s-\frac{3}{2}}(\mathbb{R}^+) \times H_0^{s-\frac{5}{2}}(\mathbb{R}^+)$ and we also have that $\tilde{F} \in H^s_0(\mathbb{R}^+)$. Then using Lemma (3.3) and Lemma (3.8) for V_2 (replacing K_1 for K_2), we have that

$$\begin{aligned}
&\sup_{t>0} \|V_2(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \le \sup_{t>0} \left(\|W_b(\tilde{h}_1, \tilde{h}_2)(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \right) \\
&+ \sup_{t>0} \left(\|W_C(\tilde{f}_1, \tilde{f}_2)(\cdot,t)\|_{Y^s(\mathbb{R}^+)} + \int_0^t \|W_C(K_2(\cdot,\tau))(\cdot,t-\tau)(\cdot,t)\|_{Y^s(\mathbb{R}^+)}\, d\tau \right) \\
&\le C \left(\|\tilde{h}_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|\tilde{h}_2\|_{H^{s-\frac{5}{2}}(\mathbb{R}^+)} + \|\tilde{f}_1\|_{H^s(\mathbb{R}^+)} + \|\tilde{f}_2\|_{H^s(\mathbb{R}^+)} \right. \\
&\left. + \int_0^\infty \left(\|K_2(\cdot,\tau))\|_{Y^s(\mathbb{R}^+)} \right) d\tau \right) \\
&\le C_1 \left(\|\tilde{h}_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|\tilde{h}_2\|_{H^{s-\frac{5}{2}}(\mathbb{R}^+)} + \|\tilde{f}_1\|_{H^s(\mathbb{R}^+)} + \|\tilde{f}_2\|_{H^s(\mathbb{R}^+)} \right. \\
&\left. + (|f_1(0)| + |f_2(0)|) \int_0^\infty e^{-\tau}\, d\tau \right).
\end{aligned}$$

Now, it is straightforward to see that $Z_2 = V - V_2$ satisfies the (**IBVP**)

$$\text{(3.47)} \qquad \begin{cases} \partial_t V(x,t) &= MV(x,t) + \tilde{K}_1(0)e^{-x-t}, \quad x>0, \quad t>0, \\ V(0,t) &= 0, \quad V(x,0) = 0 \end{cases}$$

whose solution is given by

$$Z_2(x,t) = \tilde{K}_1(0) \int_0^t W_R(\psi)(x,t-\tau)e^{-t+\tau}\, d\tau.$$

Then we have that,

$$\begin{aligned}
\|Z_2(\cdot,t)\|_{Y^s} &\le \|\tilde{K}_1(0)\| \int_0^t \|W_R(\psi)(\cdot,t-\tau)\|_{Y^s} e^{-t+\tau}\, d\tau \\
&\le C(|f_1(0)| + |f_2(0)|) \int_0^\infty e^{-\tau}\, d\tau.
\end{aligned}$$

From this fact, we conclude that $V = V_2 + Z_2$ in $Y^s(\mathbb{R}^+)$, and also that

$$\begin{aligned}\sup_{t>0} \|V(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \leq & C_2 \left(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|h_2\|_{H^{s-\frac{5}{2}}(\mathbb{R}^+)} \right. \\ & \left. + \|f_1\|_{H^s(\mathbb{R}^+)} + \|f_2\|_{H^s(\mathbb{R}^+)} \right).\end{aligned}$$

where we are using that $|f_i(0)| \leq C\|f_i\|_{H^s(\mathbb{R}^+)}$ for $s > \frac{1}{2}$. From these estimates, we get that

$$\begin{aligned}&\sup_{t>0} \|W(f_1, f_1, h_1, h_2)(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \leq \sup_{t>0} \left((|f_1(0)| + |f_2(0)|) \| \|e^{-t-x}\|_{H^s(\mathbb{R}^+)} \right. \\ &\quad \left. + \|V(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \right) \\ &\quad \leq C \left(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty} + \|\tilde{h}_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|\tilde{h}_2\|_{H^{s-\frac{5}{2}}(\mathbb{R}^+)} \right. \\ &\quad \left. + \|\tilde{f}_1\|_{H^s(\mathbb{R}^+)} + \|\tilde{f}_2\|_{H^s(\mathbb{R}^+)} + \|f_1\|_{H^s(\mathbb{R}^+)} + \|f_2\|_{H^s(\mathbb{R}^+)} \right).\end{aligned}$$

On the other hand, we have that

$$\|e^{-x-\tau}\|_{H^s(\mathbb{R}^+)} \leq Ce^{-\tau}.$$

Now, using that $H^s(\mathbb{R}^+) \subset C_b(\mathbb{R}^+)$ for $s > \frac{1}{2}$, that $h_i(0) = f_i(0)$, and that

$$\begin{aligned}\|\tilde{f}_i\|_{H^s(\mathbb{R}^+)} &\leq \|f_i\|_{H^s(\mathbb{R}^+)} + |f_i(0)| \leq \|f_i\|_{H^s(\mathbb{R}^+)} + \|f_i\|_{L^\infty} \leq C\|f_i\|_{H^s(\mathbb{R}^+)}, \\ \|\tilde{h}_i\|_{H^{s-l_i}(\mathbb{R}^+)} &\leq \|h_i\|_{H^{s-l_i}(\mathbb{R}^+)} + |h_i(0)| \\ &\leq \|h_i\|_{H^{s-l_i}(\mathbb{R}^+)} + \|f_i\|_{L^\infty} \leq C(\|h_i\|_{H^{s-l_i}(\mathbb{R}^+)} + \|f_i\|_{H^s(\mathbb{R}^+)}),\end{aligned}$$

where $l_1 = \frac{3}{2}$ and $l_2 = \frac{5}{2}$. From those computations, we obtain that

$$\begin{aligned}&\sup_{t>0} \|W(f_1, f_2, h_1, h_1)(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \\ &\qquad \leq C \left(\|f_1\|_{H^s(\mathbb{R}^+)} + \|f_2\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|h_2\|_{H^{s-\frac{5}{2}}(\mathbb{R}^+)} \right).\end{aligned}$$

Let assume now that $\frac{3}{2} < s \leq \frac{5}{2}$. In this case, we see that $W(f_1, f_2, h_1, h_2)$ can be written as

$$\begin{aligned}W(f_1, f_2, h_1, h_2)(x,t) =& W_b(\tilde{h}_1, \tilde{h}_2)(x,t) + W_C(\tilde{f}_1, \tilde{f}_2)(x,t) \\ & + W_E(x,t) + \begin{pmatrix} \varphi_1(x)e^{-t} \\ \varphi_2(x)e^{-t} \end{pmatrix}\end{aligned}$$

where $\varphi_1 \in H^{10}(\mathbb{R})$ is an extension from $\mathbb{R}^+$ to $\mathbb{R}$ of the function $(h_1(0) + x(f_1'(0) + h_1(0)))e^{-x}$, $\varphi_2 \in H^{10}(\mathbb{R})$ is an extension from $\mathbb{R}^+$ to $\mathbb{R}$ of the function $(2h_1(0) + f_1'(0) + x(f_1'(0) + h_1(0)))e^{-x}$, and $\tilde{f}_i$, $\tilde{h}_i$ are chosen appropriately and W_E is solution of the nonhomogeneous (**IVP**) on the line

$$\left\{ \begin{aligned} \partial_t X(x,t) &= MX(x,t) + L(x,t), \quad -\infty < x < \infty, \quad t > 0, \\ X(x,0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \right. \tag{3.48}$$

with $\varphi_1 = -\varphi_2'$ and

$$L(x,t) = \begin{pmatrix} 0 \\ \psi(x)e^{-t} \end{pmatrix}.$$

The first observation is $\psi = B^{-1}A\varphi_2'' + \varphi_2 \in H^5(\mathbb{R})$, and that the solution W_E of the (**IVP**) (3.48) has the form

$$W_E(x,t) = \int_0^t W_R(0, \psi(\cdot)e^{-\tau})(x, t-\tau)\, d\tau.$$

On the other hand, we have for $x \in \mathbb{R}^+$ that

$$\begin{aligned}
\widetilde{F}(x) &= \begin{pmatrix} f_1(x) - (h_1(0) + x(f_1'(0) + h_1(0)))e^{-x} \\ f_2(x) - (2h_1(0) + f_1'(0) + x(f_1'(0) + h_1(0)))e^{-x} \end{pmatrix} \\
\widetilde{F}'(x) &= \begin{pmatrix} f_1'(x) - (f_1'(0) - x(f_1'(0) + h_1(0)))e^{-x} \\ f_2(x) - (-h_1(0) - x(f_1'(0) + h_1(0)))e^{-x} \end{pmatrix} \\
\widetilde{H}(t) &= H(t) - W_E(0,t) - \begin{pmatrix} h_1(0)e^{-t} \\ (2h_1(0) + f_1'(0))e^{-t} \end{pmatrix}.
\end{aligned}$$

So, from the compatibility conditions, we have that $\tilde{F} \in Y_0^s(\mathbb{R}^+)$, and also that $\widetilde{H} \in H_0^{s-\frac{3}{2}}(\mathbb{R}^+) \times H_0^{s-\frac{5}{2}}(\mathbb{R}^+)$, since $W_E(0,0) = (0,0)^t$. It follows that the function $W_{b,C} = W_b(\tilde{h}_1, \tilde{h}_2) + W_C(\tilde{f}_1, \tilde{f}_2)$ satisfies the (**IBVP**) (3.46) with $K_1 = 0$.

Now, we want to bound the solution W_E. Note that by Lemma (3.5), we have that

$$\begin{aligned}
\sup_{t>0} \|W_E(\cdot,t)\|_{Y^s(\mathbb{R})} &\leq \int_0^t \|W_R(0,\psi)(\cdot, t-\tau)\|_{Y^s(\mathbb{R})} e^{-\tau}\, d\tau \\
&\leq \int_0^\infty \|\psi\|_{H^s(\mathbb{R})} e^{-\tau}\, d\tau \\
&\leq C\left(|h_1(0)| + |f_1(0)|\right) \\
&\leq 2C\|f_1\|_{H^s(\mathbb{R}^+)},
\end{aligned} \tag{3.49}$$

where we are using that $f_1(0) = h_1(0)$, and for $x > 0$ that
$\psi = B^{-1}A\varphi_2'' + \varphi_2 \in H^5(\mathbb{R})$. Now, we note that

$$\begin{aligned}
\sup_{t>0} \|W_E(0,t)\|_{L_t^2(\mathbb{R})} &\leq \int_0^t \sup_{x\in\mathbb{R}} \|W_R(0,\psi(\cdot))(x, t-\tau)\|_{L_t^2(\mathbb{R})} e^{-\tau}\, d\tau \\
&\leq C \int_0^\infty \sup_{x\in\mathbb{R}} \|\psi\|_{L_x^2(\mathbb{R})} e^{-\tau}\, d\tau \\
&\leq C(|h_1(0)| + |f_1'(0)|) \\
&\leq C\left(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|f_1\|_{H^s(\mathbb{R}^+)}\right)
\end{aligned}$$

On the other hand, we also have that if W_R is a solution of the (**IBVP**) (3.39), then we have that $MW_R(f_1, f_2) = W_R(M(f_1, f_2)^t)$ (see (3.44)), and so

$$\begin{aligned}
MW_E(x,t) &= \int_0^t W_R\left(M\begin{pmatrix} 0 \\ \psi e^{-\tau} \end{pmatrix}\right)(x, t-\tau)\, d\tau \\
&= \int_0^t W_R(\partial_x \psi e^{-\tau}, 0)(x, t-\tau)\, d\tau.
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}\|MW_E(0,t)\|_{L^2_t(\mathbb{R}^+)} &\leq \int_0^t \sup_{x\in\mathbb{R}} \|W_R(M(\partial_x\psi e^\tau,0))(x,t-\tau)\|_{L^2_t(\mathbb{R}^+)}\,d\tau \\ &\leq C\int_0^\infty \|\psi\|_{H^2(\mathbb{R})}e^{-\tau}\,d\tau \\ &\leq C\left(|h_1(0)|+|f_1(0)|\right) \\ &\leq C\left(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)}+\|f_1\|_{H^s(\mathbb{R}^+)}\right). \end{aligned} \tag{3.50}$$

Now, we observe that

$$\partial_t^2 W_E = M^2 W_E + M\begin{pmatrix}0\\ \psi(x)e^{-t}\end{pmatrix} - \begin{pmatrix}0\\ \psi(x)e^{-t}\end{pmatrix}$$

From these facts, Lemma (3.5), and Lemma (3.7), we see that

$$\begin{aligned}\|\partial_t^2 W_E(0,\cdot)\|_{L^2_t(\mathbb{R}^+)} &\leq \sup_{x\geq 0}\left(\|M^2W_E(x,\cdot)\|_{L^2_t(\mathbb{R})}\right) + C\|\psi\|_{H^3(\mathbb{R})} \\ &\leq C\left(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)}+\|f_1\|_{H^s(\mathbb{R}^+)}\right) + C(|h_1(0)+|f_1(0)|) \\ &\leq C_1\left(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)}+\|f_1\|_{H^s(\mathbb{R}^+)}\right)\end{aligned}$$

Using an interpolation argument, we conclude that

$$\|W_E(0,\cdot)\|_{Y^s(\mathbb{R}^+)} \leq C_1\left(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)}+\|f_1\|_{H^s(\mathbb{R}^+)}\right)$$

By the discussion above, we know that

$$W(f_1,f_2,h_1,h_2)(x,t) = W_{b,C}(x,t) + W_E(x,t) + \begin{pmatrix}\varphi_1(x)e^{-t}\\ \varphi_2(x)e^{-t}\end{pmatrix},$$

where $W_{b,C} = W_b(\tilde{h}_1,\tilde{h}_2) + W_C(\tilde{f}_1,\tilde{f}_2)$. First, we note that

$$\sup_{t>0}\|\varphi_i e^{-t}\|_{H^s(\mathbb{R}^+)} \leq C(|h_1(0)|+|f_1'(0)|) \leq C(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)}+\|f_1\|_{H^s(\mathbb{R}^+)})$$

On the other hand, we have that

$$\begin{aligned}\sup_{t>0}\|W_{b,C}(\cdot,t)\|_{Y^s(\mathbb{R}^+)} &\leq C\left(\|\tilde{h}_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)}+\|\tilde{h}_2\|_{H^{s-\frac{5}{2}}(\mathbb{R}^+)}\right) \\ &\leq C\left(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)}+\|h_2\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)}\right),\end{aligned}$$

where we are using the definition of $\widetilde{H}$ and the compatibility conditions (**sC2**). So, from those computations and the estimates for W_E given by (3.50), we obtain that

$$\begin{aligned}&\sup_{t>0}\|W(f_1,f_2,h_1,h_1)(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \\ &\qquad\leq C\left(\|f_1\|_{H^s(\mathbb{R}^+)}+\|f_2\|_{H^s(\mathbb{R}^+)}+\|h_1\|_{H^{s-\frac{5}{2}}(\mathbb{R}^+)}+\|h_2\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)}\right).\end{aligned}$$

Assume now that $\frac{5}{2} < s \le \frac{7}{2}$. In this case we use the following decomposition of the solution $W(f_1, f_2, h_1, h_2)$ given by

$$W(f_1, f_2, h_1, h_2)(x,t) = W_1(x,t) + W_b(\tilde{h}_1, \tilde{h}_2)(x,t) + W_C(\tilde{f}_1, \tilde{f}_2)(x,t) + \begin{pmatrix} \varphi_1(x,t) \\ \varphi_2(x,t) \end{pmatrix} \tag{3.51}$$

where for $i = 1, 2$,

$$\varphi_i(x,t) = (\varphi_{i,1}(x) + t\varphi_{i,2}(x))e^{-t-x} = (\tilde{\varphi}_{i,1}(x) + t\tilde{\varphi}_{i,2}(x))e^{-t}$$

with $\tilde{\varphi}_{11}$ and $\tilde{\varphi}_{12}$ are extensions from $\mathbb{R}^+$ to $\mathbb{R}$ of the functions

$$\left(f_1(0) + (f_1(0) + f_1'(0))x + \frac{1}{2}(f_1(0) + 2f_1'(0) + f_1''(0))x^2 \right) e^{-x}$$

and $(h_1'(0) + f_1(0))e^{-x}$, respectively. Functions $\tilde{f}_i$, $\tilde{h}_i$ are chosen appropriately and W_1 is solution of the nonhomogeneous (**IVP**) (3.48) on the line with

$$L(x,t) = \begin{pmatrix} 0 \\ (\psi_1(x) + t\psi_2(x))e^{-t} \end{pmatrix},$$

where the functions ψ_1, ψ_2 are defined as

$$\psi_1 = B^{-1}A\partial_x\tilde{\varphi}_{1,1} + \tilde{\varphi}_{2,1} - \tilde{\varphi}_{2,2}, \quad \psi_2 = B^{-1}A\partial_x\tilde{\varphi}_{1,2} + \tilde{\varphi}_{2,2}.$$

It is straightforward to see that $\varphi_{2,i}$ can be taken for $x > 0$ as

$$\begin{aligned} \tilde{\varphi}_{2,1}(x) &= (2f_1(0) + 3f_1'(0) + f_1''(0) - h_1(0) + (2f_1(0) + 3f_1'(0) + f_1''(0))x \\ &\quad + \frac{1}{2}(f_1(0) + 2f_1'(0) + f_1''(0))x^2)e^{-x} \\ \tilde{\varphi}_{2,2}(x,t) &= -(f_1(0) + h_1'(0))e^{-x}. \end{aligned}$$

The first observation is that W_1 has the form

$$W_1(x,t) = \int_0^t W_R(0, (\psi_1(\cdot) + \tau\psi_2(\cdot))e^{-\tau})(x, t-\tau)\, d\tau.$$

On the other hand, for $x \in \mathbb{R}^+$

$$\begin{aligned} \widetilde{F}^{(k)}(x) &= \begin{pmatrix} f_1^{(k)}(x) - \tilde{\varphi}_{1,1}^{(k)}(x) \\ f_2^{(k)}(x) - \tilde{\varphi}_{1,2}^{(k)}(x) \end{pmatrix}, k = 0, 1, 2, \\ \widetilde{H}^{(k)}(t) &= H^{(k)}(t) - \partial_t^{(k)} W_1(0,t) - \partial_t^{(k)} \begin{pmatrix} \varphi_1(0,t)e^{-t} \\ \varphi_2(0,t)e^{-t} \end{pmatrix}, k = 0, 1. \end{aligned}$$

From the compatibility conditions, we easily see that $\tilde{F} \in Y_0^s(\mathbb{R}^+)$, and that $\widetilde{H} \in Y_0^{s-\frac{3}{2}, s-\frac{5}{2}}(\mathbb{R}^+)$. So, we have that $W_{b,C} = W_b(\tilde{h}_1, \tilde{h}_2) + W_C(\tilde{f}_1, \tilde{f}_2)$ satisfies the (**IBVP**) (3.46) with $K_1 = 0$.

We first bound the solution W_1. By Lemma (3.5), we have that

$$\begin{aligned}
\|W_1(\cdot,t)\|_{Y^s(\mathbb{R})} &\leq \int_0^t \|W_R(0,\psi)\|_{Y^s(\mathbb{R})}e^{-\tau}\,d\tau \\
&\leq \int_0^\infty \|\psi\|_{H^s(\mathbb{R})}e^{-\tau}\,d\tau \\
&\leq \int_0^\infty (\|\psi_1\|_{H^s(\mathbb{R})} + \tau\|\psi_2\|_{H^s(\mathbb{R})})e^{-\tau}\,d\tau \\
&\leq C\left(|h_1(0)| + |f_1(0)| + |f_1'(0)| + |f_1''(0)|\right) \\
&\leq C\left(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|f_1\|_{H^s(\mathbb{R}^+)}\right)
\end{aligned}$$

where we are using that $f_2'(0) = h_1'(0)$, $s > \frac{5}{2}$, $s - \frac{3}{2} > \frac{1}{2}$, and

$$\begin{aligned}
||\psi_i||_{H^s} &\leq ||\tilde\varphi_{1,1}||_{H^{s+1}} + ||\tilde\varphi_{1,1}||_{H^{s+1}} + ||\tilde\varphi_{2,1}||_{H^s} + ||\tilde\varphi_{2,2}|_{H^s} \\
&\leq C\left(|h_1(0)| + |f_1(0)| + |f_1'(0)| + |f_1''(0)|\right) \\
&\leq C\left(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|f_1\|_{H^s(\mathbb{R}^+)}\right).
\end{aligned}$$

On the other hand, we note as above that

$$\|\varphi_i(\cdot,t)\|_{Y^s(\mathbb{R})} \leq C\left(\|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|f_1\|_{H^s(\mathbb{R}^+)}\right).$$

Now, using that $\tilde F \in Y_0^s(\mathbb{R}^+)$, and $\tilde H \in Y_0^{s-\frac{3}{2},s-\frac{5}{2}}(\mathbb{R}^+)$ from the compatibility condition (**sC3**), that $W(f_1,f_2,h_1,h_2)$ has the form (3.51), and similar computations as in previous cases, we see that

$$\begin{aligned}
&\sup_{t>0}\|W(f_1,f_2,h_1,h_1)(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \\
&\qquad\leq C\left(\|f_1\|_{H^s(\mathbb{R}^+)} + \|f_2\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|h_2\|_{H^{s-\frac{5}{2}}(\mathbb{R}^+)}\right).
\end{aligned}$$

Now, we consider $\frac{7}{2} < s \leq \frac{9}{2}$. In this case, we decompose $W(f_1,f_2,h_1,h_2)$ by

$$\begin{aligned}
(3.52)\quad &W(f_1,f_2,h_1,h_2)(x,t) \\
&\qquad = W_2(x,t) + W_b(\tilde h_1,\tilde h_2)(x,t) + W_C(\tilde f_1,\tilde f_2)(x,t) + \begin{pmatrix}\varphi_1(x,t)\\ \varphi_2(x,t)\end{pmatrix}
\end{aligned}$$

where for $i = 1,2$,

$$\varphi_i(x,t) = (\varphi_{i,1}(x) + t\varphi_{i,2}(x))e^{-t-x} = (\tilde\varphi_{i,1}(x) + t\tilde\varphi_{i,2}(x))e^{-t}$$

with $\tilde\varphi_{11}$ and $\tilde\varphi_{12}$ are extensions from $\mathbb{R}^+$ to $\mathbb{R}$ of the functions

$$\begin{aligned}
&\Big[f_1(0) + (f_1(0) + f_1'(0))x + \frac{1}{2}(f_1(0) + 2f_1'(0) + f_1''(0))x^2 \\
&\qquad\qquad + \frac{1}{6}(f_1(0) + 3f_1'(0) + 3f''(0) + f_1'''(0))x^3\Big]\,e^{-x}
\end{aligned}$$

and $(h_1'(0)+f_1(0))e^{-x}$, respectively. It is straightforward to see that $\varphi_{2,i}$ can be taken for $x>0$ as

$$\tilde{\varphi}_{2,1}(x) = (f_1(0) - 2f_1''(0) - f_1'''(0) - h_1'(0) + (f_1(0) - 2f_1''(0) - f_1'''(0))x \\ -\frac{1}{2}(f_1'(0)+2f_1''(0)+f_1'''(0))x^2 + \frac{1}{6}(f_1'''(0)+3f_1''(0)+3f_1'(0)-f_1(0))x^3)e^{-x},$$

$\tilde{\varphi}_{2,2}(x,t) = (f_1(0)+h_1'(0))e^{-x}$.
Let W_2 be defined by

$$W_2(x,t) = \int_0^t W_R(0,(\psi_1(\cdot)+\tau\psi_2(\cdot))e^{-\tau})(x,t-\tau)\,d\tau.$$

then, W_2 is the solution of the nonhomogeneous (**IVP**) (3.48) on the line with

$$L(x,t) = \begin{pmatrix} 0 \\ (\psi_1(x)+t\psi_2(x))e^{-t} \end{pmatrix},$$

where the functions ψ_1, ψ_2 are defined as

$$\psi_1 = B^{-1}A\partial_x\tilde{\varphi}_{1,1} + \tilde{\varphi}_{2,1} - \tilde{\varphi}_{2,2}, \quad \psi_1(0)=0, \quad \psi_2 = B^{-1}A\partial_x\tilde{\varphi}_{1,2} + \tilde{\varphi}_{2,2},$$

with $\psi_1(0)=0$. From the compatibility conditions (**sC4**), we have directly that $\tilde{F} \in Y_0^s(\mathbb{R}^+)$ and that $\widetilde{H} \in Y_0^{s-\frac{3}{2},s-\frac{5}{2}}(\mathbb{R}^+)$. So, using similar arguments and computations as in previous cases, we see that

$$\sup_{t>0} \|W(f_1,f_2,h_1,h_1)(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \\ \leq C\left(\|f_1\|_{H^s(\mathbb{R}^+)} + \|f_2\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{s-\frac{3}{2}}(\mathbb{R}^+)} + \|h_2\|_{H^{s-\frac{5}{2}}(\mathbb{R}^+)}\right).$$

□

Finally, using the result for the linear non homogeneous problem, we have the following result for the nonhomogeneous problem

THEOREM 3.10. *Let* $\frac{1}{2} < s \leq \frac{9}{2}$ *and let* $F \in L^1([0,T]:Y^s(\mathbb{R}^+))$ *for* $T>0$. *Then the Initial-boundary-value problem*

$$(3.53)\qquad \begin{cases} \partial_t X(x,t) &= MX(x,t)+F(x,t), \quad x>0, \quad 0\leq t\leq T, \\ X(x,0) &= \begin{pmatrix}0\\0\end{pmatrix}, \quad X(0,t)=\begin{pmatrix}0\\0\end{pmatrix}. \end{cases}$$

possesses a unique solution $W_I \in C([0,T]:Y^s(\mathbb{R}^+))$ *such that*

$$\sup_{0\leq t\leq T}\|W_I(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \leq C\int_0^T \|F(\cdot,\tau)\|_{Y^s(\mathbb{R}^+)}\,d\tau.$$

PROOF. Before we go further, let's consider the additional problem by taking $F(x,t)=g(t)e^{-x}$ with $F\in L^1([0,T]:Y^s(\mathbb{R}^+))$ for $T>0$. Now, let $\psi\in H^{10}(\mathbb{R})$ be an extension for e^{-x} with $x>0$, and choose $g_j \in (C_0^\infty(0,T))^2$ such that

$$\int_0^T |g_j(t)-g(t)|\,dt \to 0, \quad \text{as } j\to\infty.$$

We define W_j by

$$W_j(x,t)=\int_0^t W_R(g_j(\tau)\psi)(x,t-\tau)\,d\tau.$$

We see directly that W_j is a classical solution of the pure-initial-value problem

$$\left\{\begin{array}{ll}\partial_t W(x,t) &= MW(x,t)+g_j(t)\psi(x), \quad x\in\mathbb{R},\quad t>0,\\ W(x,0) &= (0,0)^t\end{array}\right. \tag{3.54}$$

As in Lemma (3.7), we have that

$$\sup_{x\in\mathbb{R}}\|W_j(x,\cdot)\|_{Y^{s-\frac{3}{2}}(\mathbb{R})}\leq C\int_0^T \|g_j(\tau)\psi(\cdot)\|_{Y^{s-\frac{3}{2}}(\mathbb{R})}\,d\tau\leq \int_0^T \|g_j(\tau))\|\,d\tau.$$

Moreover, since $W_j(0,0)=\partial_t W_j(0,0)=0$, we also have that

$$\|W_j(0,\cdot)\|_{Y_0^{s-\frac{3}{2}}(\mathbb{R})}\leq C\int_0^T \|g_j(\tau)\psi(\cdot)\|_{Y^{s-\frac{3}{2}}(\mathbb{R})}\,d\tau\leq \int_0^T \|g_j(\tau))\|\,d\tau \tag{3.55}$$

Now, we define $Z_j(x,t)=W_j(x,t)-W_b(W_j(0,\cdot))(x,t)$. Then we have that the function Z_j satisfies the initial-boundary-value problem (3.53) with nonhomogeneous term $G(x,t)=g_j(t)\psi(x)$, and also that

$$\sup_{0\leq t\leq T}\|Z_j(\cdot,t)\|_{Y^s(\mathbb{R}^+)}\leq C\int_0^T \|g_j(\tau)\|\,d\tau.$$

Moreover, we also have from Lemma (3.7) and (3.55) that

$$\begin{aligned}\sup_{0\leq t\leq T}\|W_j(\cdot,t)\|_{Y^s(\mathbb{R}^+)} &\leq \sup_{0\leq t\leq T}\|Z_j(\cdot,t)\|_{Y^s(\mathbb{R}^+)}+\sup_{0\leq t\leq T}\|W_b(W_j(0,\cdot))\|_{Y^s(\mathbb{R}^+)}\\ &\leq C\int_0^T \|g_j(\tau)\|\,d\tau+\sup_{0\leq t\leq T}\|W_j(0,\cdot)\|_{Y^{s-\frac{3}{2}}(\mathbb{R}^+)}\\ &\leq C_1\int_0^T \|g_j(\tau)\|\,d\tau\end{aligned}$$

Finally, we see that $(W_j)_j$ is a Cauchy sequence in $(C_0^\infty([0,T]:Y^s(\mathbb{R}^+)))^2$ with limit W. In fact,

$$\|W_j(\cdot,t)-W_k(\cdot,t)\|_{Y^s(\mathbb{R}^+)}\leq \int_0^t \|g_j(\tau)-g_k(\tau)\|\,d\tau.$$

So, from this fact, previous computation and taking limit as $j\to\infty$, we conclude that $W(x,t)=\lim_{j\to\infty}W_j(x,t)$ is the solution of the (**IBVP**) (3.54) replacing g_j for g, and also that

$$\sup_{0\leq t\leq T}\|W(\cdot,t)\|_{Y^s(\mathbb{R}^+)}\leq C\int_0^T \|g(\tau)\|\,d\tau.$$

Let assume that $\frac{1}{2}<s\leq\frac{3}{2}$, and define $\tilde{F}(x,t)=F(x,t)-F(0,t)e^{-x}$. We consider U and V the solutions of the initial-boundary-value problem (3.53) with non homogeneous part $G(x,t)=\tilde{F}(x,t)$ and $G(x,t)=F(0,t)e^{-x}=g(t)e^{-x}$, respectively. From the Duhamel formula, we know that the solution U has the form

$$U(x,t)=\int_0^t W(\tilde{F}(\cdot,\tau),0,0)(x,t-\tau)\,d\tau,$$

where W stands for the solution of the (**IBVP**) (3.45). Since $U(x,0)=0$ and $U(0,t)=0$, satisfies the hypotheses of Theorem (3.9), we have directly that

$$\begin{aligned}\sup_{0\le t\le T}\|U(\cdot,t)\|_{Y^s(\mathbb{R}^+)} &\le C\sup_{0\le t\le T}\int_0^T \|W(\tilde{F}(\cdot,\tau)\|_{Y^s(\mathbb{R}^+)}\,d\tau\\ &\le C\int_0^T \left(\|F(\cdot,\tau)\|_{Y^s(\mathbb{R}^+)}+\|F(0,\tau)\|\right)d\tau\\ &\le C_1\int_0^T \|F(\cdot,\tau)\|_{Y^s(\mathbb{R}^+)}\,d\tau\end{aligned}$$

where we are using that $H^s(\mathbb{R}^+)\subset C_b(\mathbb{R}^+)$ for $s>\frac{1}{2}$. On the other hand, applying previous arguments for $g(t)=F(0,t)$, we also have that

$$\sup_{0\le t\le T}\|V(\cdot,t)\|_{Y^s(\mathbb{R}^+)}\le C\int_0^T\|F(0,\tau)\|\,d\tau\le C_1\int_0^T\|F(\cdot,\tau)\|_{Y^s(\mathbb{R}^+)}\,d\tau.$$

Since we know that $W_I=U+V$, we conclude directly the desired estimate.

In the remaining case we use the same approach by considering $\tilde{F}$ appropriately. In the case $\frac{3}{2}<s\le\frac{5}{2}$, we use

$$\tilde{F}(x,t)=F(x,t)-\left(F(0,t)+(\partial_xF(0,t)+F(0,t))x\right)e^{-x},$$

for $\frac{5}{2}<s\le\frac{7}{2}$, we take

$$\begin{aligned}\widetilde{F}(x,t)=F(x,t)-\Big(F(0,t)+(\partial_xF(0,t)+F(0,t))x+&\\ \frac{1}{2}(\partial_x^2F(0,t)+2\partial_xF(0,t)+F(0,t))x^2\Big)&\,e^{-x},\end{aligned}$$

and finally, for $\frac{7}{2}<s\le\frac{9}{2}$, we take

$$\begin{aligned}\widetilde{F}(x,t)=F(x,t)-e^{-x}\Big(F(0,t)+(\partial_xF(0,t)+F(0,t))x+\frac{1}{2}(\partial_x^2F(0,t)+\\ 2\partial_xF(0,t)+F(0,t))x^2+\frac{1}{6}(\partial_x^3F(0,t)+3\partial_x^2F(0,t)+3\partial_xF(0,t)+F(0,t))x^3\Big)\end{aligned}$$

□

4. Nonlinear case

The main goal in this section is to establish a local existence result for the initial value with boundary conditions (**IBVP**)

$$(4.1)\qquad \begin{cases}\partial_tX(x,t) &= MX(x,t)+G(X)(x,t),\quad x>0,\quad t>0,\\ X(x,0) &= F(x),\quad X(0,t)=H(t)\end{cases},$$

and also to prove global well posedness with homogeneous boundary. We set the space $\mathcal{Y}^s$ as

$$\mathcal{Y}^s(\mathbb{R}^+)=Y^s(\mathbb{R}^+)\times Y^{s-\frac{3}{2},s-\frac{5}{2}}(\mathbb{R}^+).$$

Now, for a given $T>0$ and $\frac{1}{2}<s\le\frac{9}{2}$, we define the space $Z_T^s=C([0,T]:Y^s)$ with the norm,

$$\|U\|_{Z_T^s}=\sup_{t\in[0,T]}\|U(\cdot,t)\|_{Y^s}.$$

Before going forward, we need to recall that the nonlinear part is given by

$$G(q,r) = G_1(q,r) + G_2(q,r)$$

where G_1 and G_2 are defined by,

$$G_1(q,r) = \begin{pmatrix} 0 \\ -B^{-1}(r(q^p)_x) \end{pmatrix}, \quad G_2(q,r) = \begin{pmatrix} 0 \\ -B^{-1}(2q^p r_x) \end{pmatrix}.$$

On the other hand, using the s**-compatible conditions for** $s > \frac{1}{2}$ applied to the couple of functions $(F,H) \in \mathcal{Y}^s(\mathbb{R}^+)$, we have the following existence result,

THEOREM 4.1. *Let* $\frac{1}{2} < s \le \frac{9}{2}$ *and* $(F,H) \in \mathcal{Y}^s$ *satisfying one of the s-compatible conditions (***sC1***)-(***sC4***). Then there exists*

$$T_0 = T_0\left(\|F\|_{Y^s(\mathbb{R}^+)}, \|H\|_{Y^{s-\frac{3}{2},s-\frac{5}{2}}(\mathbb{R}^+)}\right) > 0$$

*such that the initial value with boundary conditions (***IBVP***) (4.1) has a unique solution* $X = \mathcal{K}(F,H) \in Z^s_{T_0}$*. Moreover, for* $0 < T_1 < T_0$*, there exists a neighborhood* $\mathcal{U}_\epsilon$ *of* $(F,H) \in \mathcal{Y}^s$ *such that the solution application* $\mathcal{K} : U_\epsilon \to Z^s_T$ *is Lipschitz.*

PROOF. Let $\mathcal{X}^R_T = \{U \in Z^s_T : \|U\|_{Z^s_T} \le R\}$ be the ball of radius $R > 0$ on Z^s_T, where T and R are positive constants that will be determined later on. We consider the application Φ defined as

$$\Phi(U) = W(F,H) + W_I(G(U)).$$

We will see for $(F,H) \in \mathcal{Y}^s$ satisfying one of the s-compatible conditions that Φ is a contraction from $\mathcal{X}^R_T$ into $\mathcal{X}^R_T$ for appropriated $T > 0$ and $R > 0$. We set

$$R_0 = \|F\|_{Y^s(\mathbb{R}^+)} + \|H\|_{Y^{s-\frac{3}{2},s-\frac{5}{2}}(\mathbb{R}^+)}.$$

Now for given $U, V \in X^R_T$, we have from Theorem (3.9) and Theorem (3.10) that

$$\begin{aligned} \|\Phi(U)\|_{Z^s_T} &\le \|W(F,H)\|_{Z^s_T} + \|W_I(G(U))\|_{Z^s_T} \\ &\le CR_0 + C\int_0^T \|G(U)(\cdot,\tau)\|_{Y^s(\mathbb{R}^+)}\, d\tau \\ &\le CR_0 + CT \sup_{0\le\tau\le T} \|G(U)(\cdot,\tau)\|_{Y^s(\mathbb{R}^+)} \end{aligned} \tag{4.2}$$

Now, to get the nonlinear estimates, we need use the following *Sobolev Multiplication Law*: Let $d \ge 1$ and let s_1, s_2, t be such that either

$$s_1 + s_2 \ge 0, \quad t \le s_1 + s_2, \quad t < s_1 + s_2 - \frac{d}{2},$$

or $s_1 + s_2 > 0$, $t < s_1 + s_2$, $t \le s_1 + s_2 - \frac{d}{2}$,then we have that

$$\|\psi\varphi\|_{H^t(\mathbb{R}^d)} \le \|\psi\|_{H^{s_1}(\mathbb{R}^d)}\|\varphi\|_{H^{s_2}(\mathbb{R}^d)}. \tag{4.3}$$

So, let $V_i = (q_i, r_i) \in Y^s(\mathbb{R}^+)$ $(i = 1,2)$. Using that B^{-1} has order (-2), and $t = s-2$, $s_1 = s$ and $s_2 = s-1$ en (4.3), we have that

(4.4)

$$\begin{aligned}
\|G_1(V_1) - G_1(V_2)\|_{Y^s(\mathbb{R}^+)} &= \| - B^{-1}\left(r_1(q_1^p)_x - r_2(q_2^p)_x\right)\|_{H^s(\mathbb{R}^+)} \\
&\le C(b)\left(\|(r_1-r_2)(q_1^p)_x\|_{H^{s-2}(\mathbb{R}^+)} + \|r_2(q_1^p - q_2^p)_x\|_{H^{s-2}(\mathbb{R}^+)}\right) \\
&\le C(b)\left(\|r_1-r_2\|_{H^s(\mathbb{R}^+)}\|(q_1^p)_x\|_{H^{s-1}(\mathbb{R}^+)} + \|r_2\|_{H^s(\mathbb{R}^+)}\|(q_1^p - q_2^p)_x\|_{H^{s-1}(\mathbb{R}^+)}\right) \\
&\le C(b)\Big(\|r_1-r_2\|_{H^s(\mathbb{R}^+)}\|q_1\|^p_{H^s(\mathbb{R}^+)} + \|r_2\|_{H^s(\mathbb{R}^+)}\|q_1^p - q_2^p\|_{H^s(\mathbb{R}^+)}\Big) \\
&\le 2C(b)\Big(2\|r_1-r_2\|_{H^s(\mathbb{R}^+)}\|q_1\|^p_{H^s(\mathbb{R}^+)} \\
&\qquad + p\|r_2\|_{H^s(\mathbb{R}^+)}\|q_1-q_2\|_{H^s(\mathbb{R}^+)}\Big(\|q_1\|^{p-1}_{H^s(\mathbb{R}^+)} + \|q_2\|^{p-1}_{H^s(\mathbb{R}^+)}\Big)\Big) \\
&\le C_1(p,b)\|V_1 - V_2\|_{Y^s(\mathbb{R}^+)}\Big(\|V_1\|^p_{Y^s(\mathbb{R}^+)} + \|V_2\|^p_{Y^s(\mathbb{R}^+)}\Big).
\end{aligned}$$

A similar estimate shows that

(4.5)

$$\|G_2(V_1) - G_2(V_2)\|_{Y^s(\mathbb{R}^+)} \le C_1(p,b)\|V_1 - V_2\|_{Y^s(\mathbb{R}^+)}\Big(\|V_1\|^p_{Y^s(\mathbb{R}^+)} + \|V_2\|^p_{Y^s(\mathbb{R}^+)}\Big).$$

Moreover, for $V_1 = V$ and $V_2 = 0$, we have that

$$\|G(V)\|_{Y^s(\mathbb{R}^+)} \le C_1(p,b)\Big(\|V\|^{p+1}_{Y^s(\mathbb{R}^+)}\Big).$$

From these facts, we have that

$$\|\Phi(U)\|_{Z_T^s} \le CR_0 + C_1(p,b)T\|U\|^{p+1}_{Z_T^s}. \tag{4.6}$$

and also that

$$\begin{aligned}
\|\Phi(U) - \Phi(V)\|_{Z_T^s} &\le \|W_I(G(U) - G(V))\|_{Z_T^s} \\
&\le C\int_0^T \|G(U)(\cdot,\tau) - G(V)(\cdot,\tau)\|_{Y^2(\mathbb{R}^+)}\,d\tau \\
&\le C_1(p,b)\int_0^T \|U(\tau) - V(\tau)\|_{Y^s(\mathbb{R}^+)}\Big(\|U(\tau)\|^p_{Y^s(\mathbb{R}^+)} + \|V(\tau)\|^p_{Y^s(\mathbb{R}^+)}\Big)\,d\tau \\
&\le C_1(p,b)T\sup_{0\le\tau\le T}\|U(\tau) - V(\tau)\|_{Y^s(\mathbb{R}^+)}\Big(\|U(\tau)\|^p_{Y^s(\mathbb{R}^+)} + \|V(\tau)\|^p_{Y^s(\mathbb{R}^+)}\Big) \\
&\le C_1(p,b)T\|U - V\|_{Z_T^s}\Big(\|U\|^p_{Z_T^s} + \|V\|^p_{Z_T^s}\Big) \\
&\le 2C_1(p,b)R^pT\|U - V\|_{Z_T^s}.
\end{aligned} \tag{4.7}$$

We set $R = 2CR_0$ and choose $T = T_0 > 0$ such that

$$2C_1(p,b)R^pT_0 = \frac{1}{N}, \tag{4.8}$$

with $N > 2^p + 1$ and $N \in \mathbb{N}$. From (4.8), and estimates (4.6) and (4.7), we have that Φ is a contraction from $X_{T_0}^R$ into $X_{T_0}^R$, and so, the Contraction mapping Theorem guaranties the existence and uniqueness of a local solution to the (**IBVP**) in the space $X_{T_0}^R$. It is not hard to extend this result for the large class $C([0;T_0] : Y^s(\mathbb{R}^+))$. Now, suppose that for any $0 < T_1 < T_0$, the mapping $\mathcal{K}$ from $\mathcal{U}_\epsilon$ to $C([0;T_1] : Y^s(\mathbb{R}^+))$ is Lipschitz, where $\mathcal{U}_\epsilon$ is a neighborhood of $(F,H) \in \mathcal{Y}^s$ and

ϵ will be determined. Let $(\tilde{F}_i, \tilde{H}_i) \in \mathcal{U}_\epsilon$ for $i = 1, 2$. Let U, U_1, and U_2 be the corresponding solutions with initial-boundary data $\mathcal{G} = (F, H)$, $\mathcal{G}_1 = (\tilde{F}_1, \tilde{H}_1)$ and $\mathcal{G}_2 = (\tilde{F}_2, \tilde{H}_2)$, respectively. Then we have that

$$U(x,t) - U_1(x,t) = W(F - F_1, H - H_1)(x,t) + W_I(G(U) - G(U_1))(x,t).$$

So, following the same type of estimates as in (4.4) and (4.5), we see for $T \le T_1 < T_0$ that

$$\begin{aligned} \|U - U_1\|_{Z^s_T} &\le C\|\mathcal{G} - \mathcal{G}_1\|_{\mathcal{Y}^s(\mathbb{R}^+)} + 2C_1(p,b)R^pT_1\|U - U_1\|_{Z^s_T} \\ &\le \epsilon C + \frac{1}{N}\|U - V\|_{Z^s_T}. \end{aligned} \tag{4.9}$$

Now, we set the function

$$l(x) = -(N-1)x + NC\epsilon.$$

We note that $l(0) = NC\epsilon > 0$ and $l\left(\frac{NC\epsilon}{N-1}\right) = 0$. Moreover, from inequality (4.9) at T_1, we have that $l(y) \ge 0$ for $y = \|U - U_1\|_{Z^s_{T_1}}$. From these fact, we have that $\|U - U_1\|_{Z^s_{T_1}} \le \frac{NC\epsilon}{N-1}$. So, we conclude that

$$\|U_1\|_{Z^s_{T_1}} \le \|U - U_1\|_{Z^s_{T_1}} + \|U\|_{Z^s_{T_1}} \le R + \frac{NC\epsilon}{N-1}. \tag{4.10}$$

Similarly,

$$\|U_2\|_{Z^s_{T_1}} \le R + \frac{NC\epsilon}{N-1}. \tag{4.11}$$

Note that

$$U_1(x,t) - U_2(x,t) = W(F_1 - F_2, H_1 - H_2)(x,t) + W_I(G_1(U) - G(U_2))(x,t).$$

Choose $\epsilon > 0$ so small that $\frac{NC\epsilon}{N-1} < R$. From a similar argument that in (4.7) we have that

$$\begin{aligned} \|U_1 - U_2\|_{Z^s_{T_1}} &\le C\|\mathcal{G}_1 - \mathcal{G}_2\|_{\mathcal{Y}^s(\mathbb{R}^+)} + C_1(p,b)T_1\|U_1 - U_2\|_{Z^s_{T_1}}\left(\|U_1\|^p_{Z^s_{T_1}} + \|U_2\|^p_{Z^s_{T_1}}\right) \\ &\le C\|\mathcal{G}_1 - \mathcal{G}_2\|_{\mathcal{Y}^s(\mathbb{R}^+)} + 2C_1(p,b)T_1\left(R + \frac{NC\epsilon}{N-1}\right)^p\|U_1 - U_2\|_{Z^s_{T_1}} \\ &\le C\|\mathcal{G}_1 - \mathcal{G}_2\|_{\mathcal{Y}^s(\mathbb{R}^+)} + 2C_1(p,b)R^pT_12^p\|U_1 - U_2\|_{Z^s_{T_1}} \\ &\le C\|\mathcal{G}_1 - \mathcal{G}_2\|_{\mathcal{Y}^s(\mathbb{R}^+)} + \frac{2^p}{N}\|U_1 - U_2\|_{Z^s_{T_1}}, \end{aligned}$$

which implies

$$\|U_1 - U_2\|_{Z_{T_1^s}} \le \left(\frac{N}{N-2^p}\right)\|\mathcal{G} - \mathcal{G}_1\|_{\mathcal{Y}^s(\mathbb{R}^+)}.$$

meaning that the solution mapping $\mathcal{K}$ from $\mathcal{U}_\epsilon$ to $Z^s_{T_1}$ is Lipschitz. ☐

Local wellposedness for $s > \frac{9}{2}$.

Before we go further, we need to introduce some compatibility conditions needed in order to relate the initial data with the boundary condition. For a given $(F,H)\in\mathcal{Y}^s$ and $k:0,1,2,...$, we set

$$F_0(x)=F(x) \tag{4.12}$$

$$H_0(x)=H(x) \tag{4.13}$$

$$H_k(t)=\partial_t^k H(t) \tag{4.14}$$

$$F_k(x)=MF_{k-1}(x)-G_{1,k-1}(x)-2G_{2,k-1}(x) \tag{4.15}$$

where $G_{1,k-1}$ and $G_{2,k-1}$ are given by

$$G_{1,k-1}(x)=\left(\begin{array}{c}0\\ B^{-1}\left(\sum_{m=0}^{k-1}\frac{(k-1)!}{m!(k-1-m)}\partial_x\left(\sum_{j=0}^{m}\frac{w^{(j)}(f_{1,0})}{j!}\kappa(m,j)\right)f_{2,k-m-1}\right)\end{array}\right)$$

$$G_{2,k-1}(x)=\left(\begin{array}{c}0\\ B^{-1}\left(\sum_{m=0}^{k-1}\frac{(k-1)!}{m!(k-1-m)}\left(\sum_{j=0}^{m}\frac{w^{(j)}(f_{1,0})}{j!}\kappa(m,j)\right)\partial_x f_{2,k-m-1}\right)\end{array}\right)$$

with $w(y)=y^p$, $w^{(j)}$ being the j-derivative of w, $f_{l,\alpha}$ being the l component of F_α, and

$$\kappa(m,j)=\sum_{A(m,j)}\frac{m!}{\alpha_1!....\alpha_i!}f_{1,\alpha_1}...f_{1,\alpha_i},$$

where $A(m,j)=\{(\alpha_1,...\alpha_j):\alpha_1+..+\alpha_j=m,\alpha_i\geq 1,\ 1\leq i\leq j\}$.

s-Compatibility conditions for $s>\frac{9}{2}$.
For $s>\frac{1}{2}$, we define $\frac{1}{2}<\tilde{s}\leq\frac{9}{2}$ by $s=4m+\tilde{s}$, where m is a nonnegative integer given by $m=\left[\frac{2s-1}{8}\right]$. We say that $(F,H)\in\mathcal{Y}^s(\mathbb{R}^+)$ is s-compatible if,

$$F_k(0)=H_k(0),$$

holds for $k:1,2,.....,4m-1$ and one of the following conditions holds for $k=4m$:

(sC1k) For $\frac{1}{2}<\tilde{s}\leq\frac{9}{2}$, $f_{i,k}(0)=h_{i,k}(0)$ for $i=1,2$.
(sC2k) For $\frac{3}{2}<\tilde{s}\leq\frac{5}{2}$, $f_{i,k}(0)=h_{i,k}(0)$ for $i=1,2$ and

$$h_{2,k}(0)=2h_{1,k}(0)+f'_{1,k}(0),\quad f'_{2,k}(0)=-h_{1,k}(0),$$

(sC3k) For $\frac{5}{2}<\tilde{s}\leq\frac{7}{2}$, $f_{i,k}(0)=h_{i,k}(0)$ for $i=1,2$, and

$$\begin{aligned}f_{2,k}(0)&=f_{1,k}(0)+h'_{1,k}(0)\\ f'_{2,k}(0)&=h'_{1,k}(0)\\ f''_{2,k}(0)&=-f_{1,k}(0)-f'_{1,k}(0)-h'_{1,k}(0)\\ h_{2,k}(0)&=2f_{1,k}(0)+3f'_{1,k}(0)+f''_{1,k}(0)-h'_{1,k}(0)\end{aligned}$$

(sC4k) For $\frac{7}{2} < \tilde{s} \le \frac{9}{2}$, $f_{i,k}(0) = h_{i,k}(0)$ $(i = 1, 2)$, and

$$
\begin{aligned}
f_{2,k}(0) &= f_{1,k}(0) - 2f''_{1,k}(0) - f'''_{1,k}(0) - h'_{1,k}(0),\\
f'_{2,k}(0) &= h'_{1,k}(0)\\
f''_{2,k}(0) &= -f_{1,k}(0) - f'_{1,k}(0) - h'_{1,k}(0)\\
f'''_{2,k}(0) &= 2f'''_{1,k}(0) + 5f''_{1,k}(0) + 6f'_{1,k}(0) + f_{1,k}(0) + h'_{1,k}(0)\\
h'_{2,k}(0) &= 2h'_{1,k}(0) + 2f''_{1,k}(0) + f'''_{1,k}(0)\\
h''_{1,k}(0) &= -2h'_{1,k}(0) - f_{1,k}(0).
\end{aligned}
$$

For $T > 0$ and such a value s, let Z^s_T be the collection of all functions

$$U \in C^k([0, T]; Y^s(\mathbb{R}^+))$$

for $k = 0, 1, \cdots, 4m - 1$ and $\partial_t^{4m} U \in C([0, T]; Y^{\tilde{s}}(\mathbb{R}^+))$ with

$$\|U\|_{Z^s_T} = \sup_{t\in[0,T]} \|\partial_t^{4m} U\|_{Y^{\tilde{s}}(\mathbb{R}^+)} + \sum_{k=0}^{4m-1} \sup_{t\in[0,T]} \|\partial_t^{k} U\|_{Y^{\tilde{s}}(\mathbb{R}^+)}$$

Theorem 4.2. *Let $s > \frac{9}{2}$ and $(F_0, H_0) \in \mathcal{Y}^s$ satisfying one of the s-compatible conditions (**sC1k**)-(**sC4k**). Then there exists $T_0 = T_0\left(\|(F_0, H_0)\|_{\mathcal{Y}^s(\mathbb{R}^+)}\right) > 0$ such that the initial value with boundary conditions (**IBVP**)*

$$
\begin{cases} \partial_t X(x, t) &= MX(x, t) + G(X)(x, t), \quad x > 0, \quad t > 0,\\ X(x, 0) &= F_0(x), \quad X(0, t) = H_0(t) \end{cases} \tag{4.16}
$$

has a unique solution $X = \mathcal{K}(F_0, H_0) \in Z^s_{T_0}$. Moreover, for $0 < T_1 < T_0$, there exists a neighborhood $\mathcal{U}_\epsilon = \{(F, H) \in \mathcal{Y}^s : \|(F, H) - (F_0, H_0)\|_{\mathcal{Y}^s} < \epsilon\}$ at (F_0, H_0) such that the solution application $\mathcal{K} : \mathcal{U}_\epsilon \to Z^s_{T_1}$ is Lipschitz.

Proof. Let $(F, H) \in \mathcal{Y}^s$ satisfying one of the s-compatible conditions (**sC1k**)-(**sC4k**), let T and R be positive constants to be determined after, and define $\mathcal{X}^R_T$ be the ball of radius $R > 0$ on Z^s_T. In other words,

$$\mathcal{X}^R_T = \{U \in Z^s_T : \|U\|_{Z^s_T} \le R\}.$$

We consider the application Φ defined as

$$\Phi(U) = W(F, H) + W_I(G(U)).$$

Now, for $k = 0, 1, 2, ..., 4m - 1$ and $U \in Z^s_T$ we define $V^{(k)} = \partial_t^k \Phi(U)$. So, we see directly that $V^{(k)}$ satisfies the (**IBVP**)

$$
\begin{cases} \partial_t V^{(k)}(x, t) &= MV^{(k)}(x, t) + G_k(U)(x, t), \quad x > 0, \quad t > 0,\\ V^{(k)}(0, x) &= F_k(x), \quad V^{(k)}(t, 0) = H_k(t) \end{cases} \tag{4.17}
$$

where the nonlinear term is given by

$$G_k(U) = \begin{pmatrix} 0 \\ -B^{-1}\left(\sum_{j=0}^{k} \frac{k!}{j!(k-j)} \left(\partial_t^j((q^p)_x)\partial_t^{k-j} r) + 2\partial_t^j(q^p)\partial_t^{k-j}(r_x)\right)\right) \end{pmatrix}$$

and the boundary-initial value conditions are given (4.15)-(4.14). From the discussion above, we have that $V^{(k)}$ can be expressed as

$$V^{(k)}(x, t) = W(F_k, H_k)(x, t) + W_I(G_k(U))(x, t).$$

Now, we know that the initial-boundary (F_k, H_k) satisfies the s-compatibility condition for $k = 0, 1, ..., 4m-1$ and the $\tilde{s}$-compatibility condition for $k = 4m$. So, Theorem (3.9) and Theorem (3.10), we have for $k = 0, 1, ..., 4m-1$ that,

$$\begin{aligned}\|V^{(k)}(\cdot,t)\|_{Y^s(\mathbb{R}^+)} &\leq C\|(F_k, H_k)\|_{\mathcal{Y}^s(\mathbb{R}^+)} + C\int_{\mathbb{R}} \|G_k(U)(\cdot,\tau)\|_{Y^s(\mathbb{R}^+)}\, d\tau \\ &\leq C\|(F_k, H_k)\|_{\mathcal{Y}^s(\mathbb{R}^+)} + CT \sup_{0\leq t\leq T} \|G_k(U)(\cdot,t)\|_{Y^s(\mathbb{R}^+)}\end{aligned}$$

First we note for $0 \leq l \leq k$ that,

$$\begin{aligned}\left\|\partial_t^l(q^p)\right\|_{H^{s-2}(\mathbb{R}^+)} &= \left\|\sum_{j=0}^{l} C(p,j)q^{p-j} \sum_{A(j,l)} \partial_t^{\alpha_1} q \cdots \partial_t^{\alpha_j} q\right\|_{H^{s-2}(\mathbb{R}^+)} \\ &\leq C\sum_{j=0}^{l} \|q\|^{p-j}_{H^{s-2}(\mathbb{R}^+)} \sum_{A(j,l)} \|\partial_t^{\alpha_1} q\|_{H^{s-2}(\mathbb{R}^+)} \cdots \left\|\partial_t^{\alpha_j} q\right\|_{H^{s-2}(\mathbb{R}^+)} \\ &\leq C\sum_{j=0}^{l} \|q\|^{p-j}_{H^{s-2}(\mathbb{R}^+)} \left(\sum_{j=1}^{l} \|\partial_t^j q\|_{H^{s-2}(\mathbb{R}^+)}\right)^j \\ &\leq C\left(\sum_{j=0}^{l} \|\partial_t^j q\|_{H^{s-2}(\mathbb{R}^+)}\right)^p,\end{aligned}$$

where $A(j,l) = \{(\alpha_1, \cdots, \alpha_j) : \alpha_1 + \cdots + \alpha_j = l,\ \alpha_i \geq 1,\ 1 \leq i \leq j\}$. So, from this we also have that

$$\begin{aligned}\left\|-B^{-1}\left(\partial_t^j(r)\partial_t^{k-j}((q^p)_x)\right)\right\|_{H^s(\mathbb{R}^+)} &\leq C\left\|\partial_t^j(r)\partial_t^{k-j}((q^p)_x)\right\|_{H^{s-2}(\mathbb{R}^+)} \\ &\leq C\left\|\partial_t^j(r)\right\|_{H^{s-2}(\mathbb{R}^+)} \left\|\partial_t^{k-j}(q^p)\right\|_{H^{s-1}(\mathbb{R}^+)} \\ &\leq C\left(\sum_{j=0}^{2m-2} \|\partial_t^j q\|_{H^{s-2}(\mathbb{R}^+)}\right)^p \left(\sum_{j=0}^{2m-2} \|\partial_t^j q\|_{H^{s-2}(\mathbb{R}^+)}\right) \\ &\leq C\|U\|^{p+1}_{Y^s(\mathbb{R}^+)}.\end{aligned}$$

Following the same arguments, we have that

$$\left\|-B^{-1}\left(\partial_t^j(q^p)\partial_t^{k-j}(r_x)\right)\right\|_{H^s(\mathbb{R}^+)} \leq C\|U\|^{p+1}_{Y^s(\mathbb{R}^+)},$$

Using this and previous estimates, we have for $k = 0, 1, \cdots, 4m-1$ that

$$\|G_k(U)(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \leq C\|U\|^{p+1}_{Y^s(\mathbb{R}^+)}.$$

So, we conclude for $k = 0, 1, \cdots, 4m-1$ that

$$\sup_{0\leq t\leq T} \|V^{(k)}(\cdot,t)\|_{Y^s(\mathbb{R}^+)} \leq C\|(F_k, H_k)\|_{\mathcal{Y}^s(\mathbb{R}^+)} + CT\|U\|^{p+1}_{Z_T^s}. \tag{4.18}$$

Now, for $k = 4m$, we have that

$$\|V^{(k)}(\cdot,t)\|_{Y^{\tilde{s}}(\mathbb{R}^+)} \leq C\|(F_k, H_k)\|_{Y^{\tilde{s}}(\mathbb{R}^+)} + CT \sup_{0\leq t\leq T} \|G_k(U)(\cdot,t)\|_{Y^{\tilde{s}}(\mathbb{R}^+)}$$

Following the same type of computations in previous case, and splitting the sum by adding from $j = 0$ to $j = 4m - 1$, we conclude that

$$\sup_{0\le t\le T} \|V^{(k)}(\cdot,t)\|_{Y^{\tilde{s}}(\mathbb{R}^+)} \le C\|(F_k,H_k)\|_{\mathcal{Y}^s(\mathbb{R}^+)} + CT\|U\|^{p+1}_{Z^s_T}. \tag{4.19}$$

From estimates (4.18) and (4.19), we conclude that

$$\|\Phi(U)\|_{Z^s_T} \le C\|(F_k,H_k)\|_{\mathcal{Y}^s(\mathbb{R}^+)} + CT\|U\|^{p+1}_{Z^s_T}.$$

Now, with a little more work, it if possible to establish for $U, V \in X^R_T$ that

$$\|\Phi(U) - \Phi(V)\|_{Z^s_T} \le CT(\|U\|^p_{Z^s_T} + \|V\|^p_{Z^s_T})\|U - V\|^{p+1}_{Z^s_T}.$$

If we choose $4NCRT_0 = 1$, for $N \in \mathbb{N}$ large enough and $R^p = 2CR_0^p$ with R_0 is taking as $R_0 = \|(F_k,H_k)\|_{\mathcal{Y}^s(\mathbb{R}^+)}$, then the fixed point principle implies that Φ has a unique fixed point in the set X^R_T. The last part of the proof follows using similar arguments used as those used ithe last part of Theorem (4.1). □

5. Global existence

In this section we consider for $s \ge 1$ the global existence of the initial value problem with homogeneous boundary condition (**IBVP**)

$$\begin{cases} \partial_t X(x,t) &= MX(x,t) + G(X)(x,t), \quad x > 0, \quad t > 0, \\ X(x,0) &= F(x), \quad X(0,t) = 0. \end{cases} \tag{5.1}$$

where the initial data $F = (f_1, f_2), \in Y^s_0(\mathbb{R}^+)$ are such that

$$\int_0^\infty f_1(x)\,dx = 0.$$

From the discussion in previous sections, we have that the (**IBVP**) (5.1) is locally well-posed and the time existence T^* depends on the quantity $\|F\|_{Y^s_0(\mathbb{R}^+)}$, but it is clear that the larger is $\|F\|_{Y^s_0(\mathbb{R}^+)}$, the smaller will be T^*. We will see that the (**IBVP**) (5.1) has a global solution by establishing the existence of the conserved quantities with respect to $t > 0$

$$\mathcal{M}(q)(x,t) = \int_0^\infty q(x,t)\,dx, \tag{5.2}$$

$$\mathcal{E}(q,r)(x,t) = \int_0^\infty \left(q^2 + aq_x^2 + r^2 + br_x^2\right)\,dx \tag{5.3}$$

In fact, assume that (q,r) is a smooth solution of the system (3.1) such that $(q,r)(\cdot,t) \in Y^s(\mathbb{R}^+)$. By integrating the first equation, we get formally that,

$$\partial_t\left[\int_0^\infty q(x,t)\,dx\right] = \int_0^\infty q_t(x,t)\,dx = \int_0^\infty r_x(x,t)\,dx = -r(0,t) = -h_2(t) = 0,$$

as long as the solution exists. Now, multiplying the second equation by r and integrating, we have that

$$\int_0^\infty (I - b\partial_x^2)r_t r\,dx = \int_0^\infty \left((I - a\partial_x^2)q_x + prq^{p-1}q_x + 2q^p r_x\right) r\,dx.$$

First we note that as long as the solution exists

$$\begin{aligned}\int_0^\infty \left(prq^{p-1}q_x + 2q^p r_x\right) r\,dx &= \int_0^\infty \left(r^2(q^p)_x + q^p(r^2)_x\right)\,dx \\ &= \int_0^\infty \left(r^2 q^p\right)_x\,dx = -r^2(0,t)q^p(0,t) = 0.\end{aligned}$$

On the other hand, as long as the solution exists we have that

$$\begin{aligned}\int_0^\infty \left((I-a\partial_x^2)q_x\right) r\,dx &= \int_0^\infty \left(q_x r - a\partial_x^3 q r\right)\,dx \\ &= (q-a\partial_x^2 q)r\Big|_0^\infty - \int_0^\infty \left(qr_x - a\partial_x^2 q r_x\right)\,dx \\ &= -\int_0^\infty \left(qq_t - a\partial_x^2 q q_t\right)\,dx \\ &= -\frac{1}{2}\partial_t\left(\int_0^\infty q^2\,dx\right) + a(q_x q_t)\Big|_0^\infty - a\int_0^\infty (q_x q_{xt})\,dx \\ &= -\frac{1}{2}\partial_t\left(\int_0^\infty (q^2+aq_x^2)\,dx\right) + a(q_x q_t)\Big|_0^\infty \\ &= -\frac{1}{2}\partial_t\left(\int_0^\infty (q^2+aq_x^2)\,dx\right)\end{aligned}$$

since $q_t(0,t) = h_1(t) = 0$ for $t \geq 0$. Finally, we see that as long as the solution exists

$$\begin{aligned}\int_0^\infty \left((I-b\partial_x^2)r_t\right) r\,dx &= \frac{1}{2}\partial_t\left(\int_0^\infty r^2\,dx\right) - b\left(r_{xt}r\Big|_0^\infty - \int_0^\infty r_{xt}r_x\,dx\right) \\ &= \frac{1}{2}\partial_t\left(\int_0^\infty (r^2+br_x^2)\,dx\right).\end{aligned}$$

In other words, if (q,r) is a smooth solution of the homogeneous (**IBVP**) (5.1), we have that the quantities (5.2) and (5.3) are conserved in time, meaning that the existence of the conserved quantities with respect to $t > 0$

$$\begin{aligned}\mathcal{E}(q,r)(x,t) = \mathcal{E}(q,r)(0,t) &= \int_0^\infty \left(f_1^2 + a(\partial_x f_1)^2 + f_2^2 + b(\partial_x f_2)^2\right)dx \\ &\leq C\|F\|^2_{Y_0^1(\mathbb{R}^+)}.\end{aligned} \tag{5.4}$$

THEOREM 5.1. *For any given $F \in Y_0^1(\mathbb{R}^+)$, the initial-boundary-value problem with homogeneous conditions (5.1) has a unique global solution $U \in C([0,\infty) : Y_0^1(\mathbb{R}^+))$ satisfying that*

$$\sup_{t\geq 0}\|U\|_{Y_0^1(\mathbb{R}^+)} \leq C\|F\|_{Y_0^1(\mathbb{R}^+)}.$$

PROOF. Let $(f_{1,j}, f_{2,j}) \in C_0^\infty(\mathbb{R}^+)$ such that

$$\|f_i - f_{i,j}\|_{H_0^1(\mathbb{R}^+)} \to 0, \quad j\to\infty, \quad i = 1,2.$$

We set $U_j = (q_j, r_j)$ to be the solution of the (**IBVP**) (5.1) with initial conditions $F_j = (f_{1,j}, f_{2,j})$. So, we know that

$$U_j(x,t) = W(F_j,0)(x,t) + W_I(G(U_j))(x,t). \tag{5.5}$$

Moreover, we also have that $U_j \in C^2([0,T] : Y^4(\mathbb{R}^+))$ and using that the solution map is Lipschitz we get that

$$\sup_{0\le t\le T} \|U_j(\cdot,t) - U_k(\cdot,t)\|_{Y^1(\mathbb{R}^+)} \to 0, \quad j,k \to +\infty. \tag{5.6}$$

On the other hand for $1 \le k \le 4$, we have that $\partial_x^k U_j \in C([0,T] : Y^{4-k}(\mathbb{R}^+))$. Due to the fact that $Y^1(\mathbb{R}^+) \subset C(\mathbb{R}^+)$, we also have that
$\partial_x^k U_j \in C^2([0,T] : C(\mathbb{R}^+))$ for $0 \le k \le 3$. In particular, we conclude that

$$\lim_{x\to\infty} \partial_x^k U_j(x,t) = 0, \quad t \in [0,T], \quad 0 \le k \le 3.$$

Let $U \in C([0,T] : Y^1(\mathbb{R}^+))$ be the limit of the sequence $(U_j)_j$, as $j \to +\infty$ (see (5.6)). So from (5.5), we have that

$$U(x,t) = W(F,0)(x,t) + W_I(G(U))(x,t), \quad \lim_{j\to\infty} U_j(0,t) = U(0,t) = 0$$

meaning that U is a solution of the (**IBVP**) (5.1) such that
$U \in C([0,T] : Y_0^1(\mathbb{R}^+))$. On the other hand, from the discussion above (see (5.4)), we know that energy

$$\mathcal{E}(U_j)(t) = \mathcal{E}(F_j), \quad 0 \le t \le T.$$

Moreover, taking limit as $j \to +\infty$ in the energy, we conclude that

$$\mathcal{E}(U)(t) = \mathcal{E}(F), \quad 0 \le t \le T.$$

Now, a simple computation shows that there is a positive constant C_1 such that

$$C_1^{-1}\|F\|_{Y^1(\mathbb{R}^+)} \le \sqrt{\mathcal{E}(F)} \le C_1\|F\|_{Y^1(\mathbb{R}^+)}.$$

This fact guarantees that any local solution can be extended in time. In other words, we have that U is a solution of the (**IBVP**) (5.1) such that $U \in C([0,+\infty) : Y_0^1(\mathbb{R}^+))$.
□

Finally, we have a global existence result for the Benney-Luke equation which follows directly from previous result, the existence the quantity $\mathcal{M}$ given by (5.2) which is conserved in time, and the remark (3.1) that allows us to establish the equivalence between the initial-boundary-value problem (**IBVP**) with homogeneous boundary condition (5.1), and the the initial-boundary-value problem with homogeneous boundary condition for the Benney-Luke model

$$\begin{cases} u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + pu_t u_x^{p-1} u_{xx} + 2u_x^p u_{xt} &= 0, \ x > 0, \ t > 0, \\ u_x(0,t) = 0, \quad u_t(0,t) &= 0 \\ u_x(x,0) = f_1(x), \quad u_t(x,0) &= f_2(x), \end{cases}$$

Corollary 5.2. For any given $F = (f_1, f_2)^t \in Y_0^1(\mathbb{R}^+)$, the initial-boundary-value problem with homogeneous boundary conditions for the Benney-Luke equation (1.1) has a unique global solution $u \in C([0,\infty) : \mathcal{V}^2(\mathbb{R}^+))$ satisfying that

$$\sup_{t\ge 0} \|u(\cdot,t)\|_{\mathcal{V}^2(\mathbb{R}^+)} \le C\|F\|_{H^1(\mathbb{R}^+)}.$$

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