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On global attractor of 3D Klein-Gordon equation with several concentrated nonlinearities

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Abstract. The global attraction is proved for solutions to 3D Klein-Gordon equation coupled to several nonlinear point oscillators. Our main result is a convergence of each finite energy solution to the set of all solitary waves as $t \to \pm\infty$. This attraction is caused by the nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent dispersion radiation.

We justify this mechanism by the following strategy based on *inflation of spectrum by the nonlinearity*. We show that any *omega-limit trajectory* has the time-spectrum in the spectral gap $[-m, m]$ and satisfies the original equation. Then the application of the Titchmarsh convolution theorem reduces the time-spectrum to a single harmonic $\omega \in [-m, m]$.

Contents

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1. Introduction

The paper concerns a nonlinear interaction of the Klein-Gordon field with point oscillators. In 3D case a rigorous definition of point interactions was introduced by Berezin and Faddeev [6]. We refer to [5] for numerous literature concerning the models with a point interactions.

We consider the system governed by the following equations

$$\begin{cases} \ddot\psi(x,t) = (\Delta - m^2)\psi(x,t) + \sum\limits_{1\le j\le n} \zeta_j(t)\delta(x-x_j) \\ \\ \lim\limits_{x\to x_j} (\psi(x,t) - \zeta_j(t) g(x-x_j)) = F_j(\zeta_j(t)) \end{cases} \tag{1.1}$$

where $m>0$, $t\in\mathbb{R}$, $x_j\in\mathbb{R}^3$, $j=1,...,n$, $n\ge 1$, and $g(x)$ is the Green's function of the operator $-\Delta+m^2$ in $\mathbb{R}^3$, i.e.

$$g(x) = \frac{e^{-m|x|}}{4\pi|x|}. \tag{1.2}$$

The nonlinearities $F_j(z)$ admit real-valued potentials:

$$F_j(z) = \partial_{\overline{z}} U_j(z), \quad U_j \in C^2(\mathbb{C}), \quad j=1,...,n. \tag{1.3}$$

We assume that the potentials $U_j(z)$ are U(1)-invariant, where U(1) stands for the unitary group $e^{i\theta}$ with $\theta\in\mathbb{R}$, i.e.,

$$U_j(z) = u_j(|z|^2), \quad z\in\mathbb{C}. \tag{1.4}$$

Conditions (1.3) and (1.4) imply that

$$F_j(z) = b_j(|z|^2)z, \quad z\in\mathbb{C}, \tag{1.5}$$

where $b_j(\cdot) = u_j'(\cdot)\in C^1(\mathbb{R})$ are real-valued. Therefore

$$F_j(e^{i\theta}z) = e^{i\theta}F_j(z), \quad \theta\in\mathbb{R}, \quad z\in\mathbb{C}. \tag{1.6}$$

This symmetry implies that $e^{i\theta}\psi(x,t)$ is a solution to (1.1) if $\psi(x,t)$ is. The system (1.1) admits soliton solutions $\psi_\omega(x)e^{-i\omega t}$ with some $\omega\in(-m,m)$ and $\psi_\omega\in L^2(\mathbb{R}^3)$. We denote by $\mathbf{S}$ the set of all amplitudes $\psi_\omega(x)$. Our main goal is the global attraction

$$\psi(\cdot,t)\to\mathbf{S}, \quad t\to\pm\infty, \tag{1.7}$$

for all solutions from the Hilbert space $\mathcal{D}_F$ (see Definition 2.1), where the asymptotics hold in local L^2-seminorms.

Similar global attraction was established for the first time in [**9**]–[**14**] for 1D wave and 1D Klein-Gordon equations coupled to a nonlinear oscillator, and in [**15, 16**] for $U(1)$-invariant nD Klein-Gordon and Dirac equations with mean field interaction.

In the context of the Schrödinger and wave equations point interaction of type (1.1) was introduced in [**1, 2, 5, 23, 24**], where the well-posedness of the Cauchy problem and blow up solutions were studied. The well-posedness for the system (1.1) has been proved in [**20**]

The asymptotic stability of solitary waves has been obtained in [**6, 17, 18**] for 1D Schrödinger equation coupled to nonlinear oscillator, in [**19**] for 1D discrete Klein-Gordon equation coupled to nonlinear oscillator, and in [**3, 4**] for 3D Schrödinger equation with concentrated nonlinearity.

Global attraction to stationary states for 3D wave and 3D Klein-Gordon equations with one concentrated nonlinearity has been proved for the first time in [**21, 22**]. The case of several concentrated nonlinearities was not considered previously.

Let us comment on our approach. First, we split the solution into a sum of dispersive and singular components: $\psi(x,t) = \psi_f(x,t) + \psi_s(x,t)$. The dispersive component $\psi_f(x,t)$ is a solution to the free Klein-Gordon equation, and singular component $\psi_s(x,t)$ is a solution to the Klein-Gordon equations with delta-like sources. The dynamics of the sources is governed by first-order nonlinear integro-differential equations.

The dispersive component vanishes asymptotically for large times in local seminorms and one remains with the contribution of the singular part only: we should show that the singular component converges in the chosen topology to a solitary wave which is a standing wave with a single frequency.

For this purpose we first prove the "omega-compactness", i.e. that each sequence $\psi_s(x, t+s_l)$ with $s_l \to \infty$ contains a converging subsequence $\psi_s(x, t+s_{l'}) \to \beta(x,t)$. Now to prove the attraction (1.7) it suffices to show that each "omega-limiting trajectory" β lies on the set $\mathbf{S}$. Equivalently, it suffices to reduce the spectrum of β to a single point ω. The spectrum is defined as the support of the vector-distribution $\tilde{\beta}(\omega)$ which is the Fourier transform in time of $\beta(t) = \beta(\cdot, t)$. The analysis of the Fourier transform in time is the key point of our approach.

The first step in this reduction is the proof of absolute continuity of the spectral density of ψ_s outside the spectral gap $[-m, m]$. The absolute continuity is a nonlinear version of Kato's theorem on the absence of embedded eigenvalues and provides the dispersion decay for the high energy component. It allows to reduce the spectrum of β to the spectral gap. To reduce the spectrum further to a single point of this gap we show that β is a solution to original system (1.1). This system implies the *spectral inclusion*: the spectrum of the nonlinear term is contained in the spectrum of β. Here we use the theory of quasimeasures developed in [**13**].

Finally, we apply the Titchmarsh convolution theorem (see [**8**, Theorem 4.3.3]) to conclude that each omega-limit trajectory is a singleton, i.e. its spectrum has a single frequency. The Titchmarsh theorem controls the inflation of spectrum by the nonlinearity. Physically, these arguments justify the following binary mechanism of the energy radiation, which is responsible for the attraction to solitary waves: (i) nonlinear energy transfer from lower to higher harmonics, and (ii) subsequent dispersion decay caused by the energy radiation to infinity.

The general scheme of the proof bring to mind the approach of [**7, 13, 14**]. Nevertheless 3D Klein-Gordon equation with point interactions requires new ideas due to a more singular character. As a consequence, the formulation of the problem and the techniques used are not a straightforward generalization of the one-dimensional result [**13**] and the result [**7, 14**] for 3D equations with mean field interaction. Moreover, the present paper is not a straightforward extension of [**22**] which concerns the Klein-Gordon equation with one concentrated nonlinearity. The case of several nonlinearities required novel arguments.

Our paper is organized as follows. In Section 2 we formulate the main theorem, and in Section 3 we separate the dispersive component and study its decay properties. In Section 4 we construct spectral representation for the remaining singular component, and prove absolute continuity of its spectrum outside the spectral gap.

In Section 5 we establish compactness for the singular component. In Section 6 we reduce the spectrum of omega-limit trajectories to the spectral gap $[-m, m]$ and establish the key spectral inclusion. In Section 7 we reduce the spectrum of omega-limit trajectories to one point. In Section 8 we prove the main theorem.

2. Main results

Model. Denote $g_j(x) = g(x-x_j)$. We fix some nonlinear functions $F_j : \mathbb{C} \to \mathbb{C}$ and define the domain

$$D_F = \{\psi \in L^2(\mathbb{R}^3) : \psi(x) = \psi_{reg}(x) + \sum_{1\le j\le n} \zeta_j g_j(x),\ \psi_{reg} \in H^2(\mathbb{R}^3),\ \zeta_j \in \mathbb{C},$$
$$\lim_{x\to x_j} (\psi(x) - \zeta_j g_j(x)) = F_j(\zeta_j)\}.$$

Note that D_F generally is not a linear space. Let H_F be a nonlinear operator on the domain D_F defined by

$$H_F\psi = (\Delta - m^2)\psi_{reg}, \quad \psi \in D_F. \tag{2.1}$$

The system (1.1) for $\psi(t) \in C(\mathbb{R}, D_F)$ reads

$$\ddot\psi(x,t) = H_F\psi(x,t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \tag{2.2}$$

where the derivatives are understood in the sense of distributions. Let us introduce the phase space for equation (2.2). Denote the space

$$\dot D = \{\pi \in L^2(\mathbb{R}^3) : \pi(x) = \pi_{reg}(x) + \sum_{1\le j\le n} \nu_j g_j(x),\ \pi_{reg} \in H^1(\mathbb{R}^3),\ \nu_j \in \mathbb{C}\}.$$

Obviously, $D_F \subset \dot D$.

Definition 2.1. (1) $\mathcal{D}_F$ is the space of the states $\Psi = (\psi, \pi) \in D_F \oplus \dot D$ equipped with the finite norm

$$\|\Psi\|^2_{\mathcal{D}_F} := \|\psi_{reg}\|^2_{H^2(\mathbb{R}^3)} + \|\pi_{reg}\|^2_{H^1(\mathbb{R}^3)} + \sum_{1\le j\le n} |\zeta_j|^2 + \sum_{1\le j\le n} |\nu_j|^2.$$

(2) $\mathcal{X}$ is the Hilbert space of the states $\Psi = (\psi, \pi) \in H^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3)$ equipped with the finite norm

$$\|\Psi\|^2_{\mathcal{X}} := \|\psi\|^2_{H^2(\mathbb{R}^3)} + \|\pi\|^2_{H^1(\mathbb{R}^3)}.$$

Definition 2.2. $H^s_{loc} = H^s_{loc}(\mathbb{R}^3)$, $s = 0, 1, 2, ...$, denotes the Fréchet space with finite seminorms

$$\|\psi\|_{H^s_R} := \|\psi\|_{H^s(B_R)}, \quad R > 0,$$

where B_R is the ball of radius R.

Denote $L^2_{loc} = H^0_{loc}$, $\mathcal{L}^2_{loc} = L^2_{loc} \oplus L^2_{loc}$ and $\mathcal{X}_{loc} = H^2_{loc} \oplus H^1_{loc}$. We set for $\Psi = (\psi, \pi)$, and

$$\|\Psi\|^2_{\mathcal{L}^2_R} = \|\psi\|^2_{L^2_R} + \|\pi\|^2_{L^2_R}, \qquad \|\Psi\|^2_{\mathcal{X}_R} = \|\psi\|^2_{H^2_R} + \|\pi\|^2_{H^1_R}, \quad R > 0.$$

Remark 2.3. The space $\mathcal{L}^2_{loc}$ is metrisable. The metrics can be defined by

$$\mathrm{dist}_{\mathcal{L}^2_{loc}}(\Psi_1, \Psi_2) = \sum_{R=1}^{\infty} 2^{-R} \frac{\|\Psi_1 - \Psi_2\|_{\mathcal{L}^2_R}}{1 + \|\Psi_1 - \Psi_2\|_{\mathcal{L}^2_R}}. \tag{2.3}$$

Global well-posedness. Let $G = \{g_{jk}\}$ be a matrix with the entries

$$g_{jk} := \begin{cases} \frac{e^{-m|x_j - x_k|}}{4\pi|x_j - x_k|}, \text{ if } j \neq k \\ \\ 0, \text{ if } j = k \end{cases} \tag{2.4}$$

and let $\mathcal{G}(\zeta) = (G\zeta, \zeta) = \sum\limits_{1\le k,j\le n} g_{jk}\zeta_j\overline{\zeta}_k$, where $\zeta = (\zeta_1, ..., \zeta_n)$. We assume that

$$\sum_{1\le j\le n} U_j(\zeta_j) - \mathcal{G}(\zeta) \ge b|\zeta|^2 - a, \quad \text{for } \zeta \in \mathbb{C}^n, \quad \text{where } b > 0 \text{ and } a \in \mathbb{R}. \tag{2.5}$$

Denote $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^3)}$. The next theorem is proved in [**20**].

Theorem 2.4. *Let conditions (1.3), (1.4) and (2.5) hold. Then*

(1) *For every initial data $\Psi(0) = \Psi_0 = (\psi_0, \pi_0) \in \mathcal{D}$ the Cauchy problem for (2.2) has a unique solution $\psi(t)$ such that*

$$\Psi(t) = (\psi(t), \dot\psi(t)) \in C(\mathbb{R}, \mathcal{D}_F).$$

(2) *The energy is conserved:*

$$\mathcal{H}(\Psi(t)) := \frac{1}{2}\Big(\|\dot\psi(t)\|^2 + \|\nabla\psi_{reg}(t)\|^2 + m^2\|\psi_{reg}(t)\|^2\Big) + \sum_{1\le j\le n} U_j(\zeta_j) = \text{Const}, \ t \in \mathbb{R}.$$

(3) *The following a priori bound holds*

$$|\zeta(t)| \le C(\Psi_0), \quad t \in \mathbb{R}. \tag{2.6}$$

Solitary waves and the main theorem.

Definition 2.5. (i) The solitary waves of equation (2.2) are solutions of the form

$$\psi(x,t) = e^{-i\omega t}\psi_\omega(x), \quad \omega \in \mathbb{R}, \quad \psi_\omega \in L^2(\mathbb{R}^3). \tag{2.7}$$

(ii) The solitary manifold is the set $\mathbf{S} = \{\Psi_\omega = (\psi_\omega, -i\omega\psi_\omega) : \omega \in \mathbb{R}\}$, where ψ_ω are the amplitudes of solitary waves.

The identity (1.6) implies that the set $\mathbf{S}$ is invariant under multiplication by $e^{i\theta}$, $\theta \in \mathbb{R}$. Let us note that $F(0) = 0$ by (1.5). Hence, for any $\omega \in \mathbb{R}$ there is a zero solitary wave with $\psi_\omega(x) \equiv 0$.

Lemma 2.6. *Assume that $F(\zeta)$ satisfies (1.5). Then nonzero solitary waves may exist only for $\omega \in (-m, m)$. The amplitudes of solitary waves are given by*

$$\psi_\omega(x) = \sum_{1\le j\le n} q_j \frac{e^{-\varkappa(\omega)|x-x_j|}}{4\pi|x - x_j|} \in L^2(\mathbb{R}^3), \tag{2.8}$$

where $\varkappa(\omega) = \sqrt{m^2 - \omega^2}$, and $q_j = q_j(\omega)$ are solutions to

$$\sum_{k\neq j} q_k \frac{e^{-\varkappa(\omega)|x_k - x_j|}}{4\pi|x_k - x_j|} + q_j \frac{m - \sqrt{m^2 - \omega^2}}{4\pi} = b(|q_j|^2)q_j. \tag{2.9}$$

Proof. Substituting (2.7) into the first equation of (1.1), we get

$$e^{-i\omega t}(-\Delta + m^2 - \omega^2)\psi_\omega(x) = \sum_{1\le j\le n} \zeta_j(t)\delta(x - x_j).$$

Taking into account the linear independence of functions $\delta(x-x_j)$ for different x_j, we arrive at the equation

$$(-\Delta+m^2-\omega^2)\psi_\omega(x)=\sum_{1\le j\le n}q_j\delta(x-x_j),$$

where q_j do not depend of t. The solution of the last equation is given by (2.8). Now the solitary wave $\psi(x,t)=e^{-i\omega t}\sum\limits_{1\le j\le n}q_j\frac{e^{-\varkappa(\omega)|x-x_j|}}{4\pi|x-x_j|}$ can be represented as

$$\psi(x,t)=\psi_{reg}(x,t)+\sum_{1\le j\le n}\zeta_j(t)g_j(x),$$

where

$$\psi_{reg}(x,t)=e^{-i\omega t}\sum_{1\le j\le n}q_j\frac{e^{-\varkappa(\omega)|x-x_j|}-e^{-m|x-x_j|}}{4\pi|x-x_j|}.$$

Evidently, $\psi_{reg}(\cdot,t)\in H^2(\mathbb{R}^3)$ for $t\in\mathbb{R}$. Finally, the second equation of (1.1) together with (1.5) give (2.9). □

For $\omega\in[-m,m]$, we denote

$$\alpha_{jk}(\omega)=\begin{cases}\frac{e^{-\varkappa(\omega)|x_j-x_k|}}{|x_j-x_k|}, & j\ne k,\\ 0, & j=k.\end{cases} \tag{2.10}$$

Definition 2.7. For $1\le n'\le n/2$, we define

$$Z_{n'}=\{\omega\in[-m,m]:\exists I\subset\{1,...,n\},J\subset\{1,...,n\}\setminus I,|I|=|J|=n': \det\{\alpha_{jk}(\omega)\}_{j\in I,k\in J}=0\} \tag{2.11}$$

Denote

$$Z_*=\bigcup_{1\le n'\le n/2}Z_{n'}.$$

We assume that the matrix (2.10) satisfies the following condition.

Condition A $\qquad Z_*\cap[-m,m]=\emptyset.$

Evidently, this condition holds in the case $1\le n\le 3$. We show in Section 9 that it holds for almost all $(x_1,x_2,...,x_n)\in\mathbb{R}^{3n}$, if $m>0$ is sufficiently small.

At last, we assume that the nonlinearity is polynomial. This condition is crucial in our argument since it will allow us to apply the Titchmarsh convolution theorem. Now all our conditions on F can be summarized as follows.

Condition B $\quad F_j(z)=\partial_{\overline{z}}U_j(z),\ U_j(z)=\sum\limits_{k=0}^{N_j}u_{k,j}|z|^{2k},\ u_{k,j}\in\mathbb{R},\ u_{N_j,j}>0,\ N_j\ge 2.$

Our main result is the following theorem.

Theorem 2.8 (Main Theorem). *Let conditions* **A** *and* **B** *hold. Then any solution* $\Psi(t)\in C(\mathbb{R},\mathcal{D}_F)$ *to* (2.2) *converges to solitary manifold* $\mathbf{S}$ *in the space* $\mathcal{L}^2_{loc}$*:*

$$\lim_{t\to\pm\infty}\mathrm{dist}_{\mathcal{L}^2_{loc}}(\Psi(t),\mathbf{S})=0, \tag{2.12}$$

where $\mathrm{dist}_{\mathcal{L}^2_{loc}}(\cdot,\cdot)$ *is defined in (2.3).*

It suffices to prove Theorem 2.8 for $t\to+\infty$.

3. Dispersive component

In [**20**] we proved that the solution $\psi(x,t)$ to (2.2) with initial data

$$\psi_0=\psi_{0,reg}+\sum_{1\le j\le n}\zeta_{0j}g_j\in D_F,\quad \pi_0=\pi_{0,reg}+\sum_{1\le j\le n}\nu_{0j}g_j\in \dot D$$

is given by

$$\psi(x,t) \;=\; \psi_f(x,t)+\sum_{1\le j\le n}\Big[\frac{\theta(t-|x-x_j|)}{4\pi|x-x_j|}\zeta_j(t-|x-x_j|)$$

$$-\frac{m}{4\pi}\int_0^t\frac{\theta(t-s-|x-x_j|)J_1(m\sqrt{(t-s)^2-|x-x_j|^2})}{\sqrt{(t-s)^2-|x-x_j|^2}}\zeta_j(s)ds\Big]. \tag{3.1}$$

Here J_1 is the Bessel function, θ is the Heaviside function,
$\psi_f(x,t)\in C([0,\infty),L^2(\mathbb{R}^3))$ is a unique solution to the Cauchy problem for the free Klein-Gordon equation

$$\ddot\psi_f(x,t)=(\Delta-m^2)\psi_f(x,t),\quad \psi_f(x,0)=\psi_0(x),\quad \dot\psi_f(x,0)=\pi_0(x), \tag{3.2}$$

and $\zeta(t)=(\zeta_j(t),...,\zeta_n(t))\in C^1([0,\infty))$ is a unique solution to the Cauchy problem for the following first-order system of nonlinear ntegro-differential equations with delay

$$\lambda_j(t)=\frac{1}{4\pi}(\dot\zeta_j(t)-m\zeta_j(t))+F_j(\zeta(t))$$

$$+\sum_{1\le k\le n}\frac{m}{4\pi}\int_0^t\frac{\theta(t-s-|x_j-x_k|)J_1(m\sqrt{(t-s)^2-|x_j-x_k|^2})}{\sqrt{(t-s)^2-|x_j-x_k|^2}}\zeta_k(s)ds$$

$$-\sum_{k\ne j}\frac{\theta(t-|x_j-x_k|)\zeta_k(t-|x_j-x_k|)}{4\pi|x_j-x_k|},\quad \zeta_j(0)=\zeta_{0j},\quad j=1,...,n, \tag{3.3}$$

where $\lambda_j(t):=\lim\limits_{x\to x_j}\psi_f(x,t)\in C([0,\infty))$. Note that the limit is well defined, $\lambda(t)$ is continuous for $t>0$, and it admits a limit as $t\to+0$ (see [**20**]). The integral in (3.3) is bounded for all $t\ge 0$ due to well known properties of the Bessel function J_1: $J_1(r)\sim r^{-1/2}$ for $r\to\infty$, and $J_1(r)\sim r$ as $r\to 0$ (see for example [**25**]).
Now we study the decay properties of the dispersive component $\psi_f(x,t)$ for $t\to\infty$.

Proposition 3.1. *$\psi_f(x,t)$ decays in $\mathcal{X}_{loc}$ seminorms. That is, $\forall R>0$*

$$\|(\psi_f(t),\dot\psi_f(t)\|_{\mathcal{X}_R}\to 0,\qquad t\to\infty. \tag{3.4}$$

Proof. We split $\psi_f(x,t)$ as

$$\psi_f(x,t)=\psi_{f,reg}(x,t)+\sum_{1\le j\le n}\psi_{f,j}(x,t),\quad t\ge 0,$$

where $\psi_{f,reg}$ and $\psi_{f,j}$ are defined as solutions to the following Cauchy problems:

$$\ddot\psi_{f,reg}(x,t)=(\Delta-m^2)\psi_{f,reg}(x,t),\quad (\psi_{f,reg},\dot\psi_{f,reg})|_{t=0}=(\psi_{0,reg},\pi_{0,reg}). \tag{3.5}$$

$$\ddot\psi_{f,j}(x,t)=(\Delta-m^2)\psi_{f,j}(x,t),\qquad (\psi_{f,j},\dot\psi_{f,j})|_{t=0}=(\zeta_{0j}g_j,\nu_{0j}g_j). \tag{3.6}$$

Since $(\psi_{0,reg},\pi_{0,reg})\in\mathcal{X}$, we have

$$(\psi_{f,reg},\dot\psi_{f,reg})\in C_b([0,\infty),\mathcal{X}). \tag{3.7}$$

The well known decay in local seminorms for the free Klein-Gordon equation (see [**13**, Lemma 3.1]) implies

$$\|(\psi_{f,reg}(\cdot,t),\dot\psi_{f,reg}(\cdot,t))\|_{\mathcal{X}_R}\to 0,\quad t\to\infty \tag{3.8}$$

for $\forall R>0$. It remains to consider $\psi_{f,j}$.

LEMMA 3.2. *(cf. [**22**, Lemma 3.3 and Corollary 3.4])*

$$\psi_{f,j}(x,t)\in C_b([0,\infty),L^2(\mathbb{R}^3)),\quad j=1,...,n,$$

and $\forall R>0$ *the following decay holds*

$$\|(\psi_{f,j}(t),\dot\psi_{f,j}(t)\|_{\mathcal{X}_R}\to 0,\quad t\to\infty,\quad j=1,...,n. \tag{3.9}$$

The lemma has been proved in [**22**] for the case $x_j=0$. Nevertheless, the proof remains true also for arbitrary $x_j\in\mathbb{R}^3$. Finally, (3.8) and (3.9) imply (3.4). □

COROLLARY 3.3. *From (3.4) immediately follows that*

$$\lambda_j(t)=\psi_f(x_j,t)\to 0,\quad t\to\infty. \tag{3.10}$$

4. Singular component

4.1. Complex Fourier-Laplace transform. For $t\ge 0$, we rewrite (3.1) as

$$\psi(x,t)=\psi_f(x,t)+\psi_s(x,t),\quad \psi_s(x,t):=\sum_{1\le j\le n}\psi_{s,j}(x,t), \tag{4.1}$$

where

$$\begin{aligned}\psi_{s,j}(x,t):&=\frac{\theta(t-|x-x_j|)}{4\pi|x-x_j|}\zeta_j(t-|x-x_j|)\\ &-\frac{m}{4\pi}\int_0^t\frac{\theta(s-|x-x_j|)J_1(m\sqrt{s^2-|x-x_j|^2})}{\sqrt{s^2-|x-x_j|^2}}\zeta_j(t-s)ds,\quad j=1,...,n.\end{aligned} \tag{4.2}$$

It is easy to verify that $\psi_{s,j}(x,t)\in C([0,\infty),L^2(\mathbb{R}^3))$ and it is the solution to the Cauchy problem

$$\ddot\psi_{s,j}(x,t)=(\Delta-m^2)\psi_{s,j}(x,t)+\zeta_j(t)\delta(x),\quad \psi_{s,j}(x,0)=0,\quad \dot\psi_{s,j}(x,0)=0. \tag{4.3}$$

According to Theorem 2.4, $\psi(t)\in C_b([0,\infty),L^2(\mathbb{R}^3))$. Hence (3.1), (3.7) and Lemma 3.2 give that

$$\psi_s(t)\in C_b([0,\infty),L^2(\mathbb{R}^3)). \tag{4.4}$$

Let us analyze the Fourier-Laplace transform of $\psi_s(x,t)$:

$$\tilde\psi_s(x,\omega)=\mathcal{F}_{t\to\omega}[\theta(t)\psi_s(x,t)]:=\int_0^\infty e^{i\omega t}\psi_s(x,t)\,dt,\quad \omega\in\mathbb{C}^+,\quad x\in\mathbb{R}^3, \tag{4.5}$$

where $\mathbb{C}^+:=\{z\in\mathbb{C}:\ \operatorname{Im}z>0\}$. Note that $\tilde\psi_s(\cdot,\omega)$ is an L^2-valued analytic function of $\omega\in\mathbb{C}^+$ due to (4.4). Equations (4.3) implies that

$$-\omega^2\tilde\psi_s(x,\omega)=(\Delta-m^2)\tilde\psi_s(x,\omega)+\sum_{1\le j\le n}\tilde\zeta_j(\omega)\delta(x-x_j),\quad \omega\in\mathbb{C}^+,\quad x\in\mathbb{R}^3, \tag{4.6}$$

where $\tilde\zeta_j(\omega)$ is the Fourier-Laplace transform of $\zeta_j(t)$:

$$\tilde\zeta_j(\omega)=\mathcal{F}_{t\to\omega}[\theta(t)\zeta_j(t)]=\int_0^\infty e^{i\omega t}\zeta_j(t)\,dt,\quad j=1,...,n. \tag{4.7}$$

Applying the Fourier transform $\mathcal{F}_{x\to\xi}$ to (4.6), we get

$$\hat{\tilde{\psi}}_s(\xi,\omega)=\sum_{1\le j\le n}\frac{e^{i\xi x_j}\tilde{\zeta}_j(\omega)}{\xi^2+m^2-\omega^2},\qquad \xi\in\mathbb{R}^3,\qquad \omega\in\mathbb{C}^+. \tag{4.8}$$

Denote

$$k(\omega)=\sqrt{\omega^2-m^2},\qquad \operatorname{Im}k(\omega)>0,\qquad \omega\in\mathbb{C}^+. \tag{4.9}$$

Then $k(\omega)$ is the analytic function on $\mathbb{C}^+$, and $\tilde{\psi}_s(x,\omega)$ equals

$$\tilde{\psi}_s(x,\omega)=\sum_{1\le j\le n}\tilde{\zeta}_j(\omega)V_j(x,\omega),\quad V_j(x,\omega)=\frac{e^{ik(\omega)|x-x_j|}}{4\pi|x-x_j|},\qquad \omega\in\mathbb{C}^+. \tag{4.10}$$

We then have, formally, for any $\varepsilon>0$:

$$\begin{aligned}\psi_s(x,t)&=\mathcal{F}^{-1}_{\omega\to t}\big[\sum_{1\le j\le n}\tilde{\zeta}_j(\omega)V_j(x,\omega)\big]=\frac{1}{2\pi}\sum_{1\le j\le n}\int_{\operatorname{Im}\omega=\varepsilon}e^{-i\omega t}\tilde{\zeta}_j(\omega)V_j(x,\omega)\,d\omega\\&=\frac{1}{2\pi}\sum_{1\le j\le n}\int_{\mathbb{R}}e^{-i\omega t}\tilde{\zeta}_j(\omega+i0)V_j(x,\omega+i0)\,d\omega.\end{aligned}$$

4.2. Traces on real line. By (4.4) the Fourier transform $\tilde{\psi}_s(\cdot,\omega)$ is a tempered L^2-valued distribution of $\omega\in\mathbb{R}$. It is the boundary value of the analytic function (4.5) in the following sense:

$$\tilde{\psi}_s(\cdot,\omega)=\lim_{\varepsilon\to0+}\tilde{\psi}_s(\cdot,\omega+i\varepsilon),\qquad \omega\in\mathbb{R}, \tag{4.11}$$

where the convergence holds in $\mathcal{S}'(\mathbb{R},L^2(\mathbb{R}^3))$. Indeed,

$$\tilde{\psi}_s(\cdot,\omega+i\varepsilon)=\mathcal{F}_{t\to\omega}[\theta(t)\psi_s(\cdot,t)e^{-\varepsilon t}],$$

where $\theta(t)\psi_s(\cdot,t)e^{-\varepsilon t}\underset{\varepsilon\to0+}{\longrightarrow}\theta(t)\psi_s(\cdot,t)$ in $\mathcal{S}'(\mathbb{R},L^2(\mathbb{R}^3))$. Therefore, (4.11) holds by the continuity of the Fourier transform $\mathcal{F}_{t\to\omega}$ in $\mathcal{S}'(\mathbb{R})$. Similarly,

$$\tilde{\zeta}_j(\omega)=\lim_{\varepsilon\to0+}\tilde{\zeta}_j(\omega+i\varepsilon),\quad \omega\in\mathbb{R},\quad j=1,...,n \tag{4.12}$$

in the sence of distributions, since the functions $\theta(t)\zeta_j(t)$ are bounded. The convergence holds in the space of tempered distributions $\mathcal{S}'(\mathbb{R})$.

Now we can justify the representation (4.10) for $\omega\in\mathbb{R}\setminus\{-m;m\}$, where the multiplication in (4.10) is understood in the sense of distribution (see [**13**]). Namely,

Lemma 4.1. *$V_j(x,\omega)$, $j=1,...,n$ are smooth functions of $\omega\in\mathbb{R}\setminus\{-m;m\}$ for any fixed $x\in\mathbb{R}^3\setminus\{x_j\}$, and the identity*

$$\tilde{\psi}_s(x,\omega)=\sum_{1\le j\le n}\tilde{\zeta}_j(\omega)V_j(x,\omega),\quad \omega\in\mathbb{R}\setminus\{-m;m\} \tag{4.13}$$

holds in the sense of distributions.

Proof. This lemma follows from (4.11) and (4.12) by the smoothness of $V_j(x,\omega)$ for $\omega\ne\pm m$. □

4.3. Absolutely continuous spectrum. Note that $\mathbb{R}\setminus(-m,m)$ coincides with the continuous spectrum of the free Klein-Gordon equation.

PROPOSITION 4.2. *(cf.* [**7**, Proposition 3.3]*) For any finite open interval I with $I\cap[-m,m]=\emptyset$*

$$\int\limits_{S^2\times I}\Big|\sum_{1\le j\le n}e^{ik(\omega)\theta\cdot x_j}\tilde\zeta_j(\omega)\Big|^2 d\theta d\omega\le C(I)<\infty, \tag{4.14}$$

where S^2 is the sphere of radius one in $\mathbb{R}^3$.

PROOF. It suffices to consider the case $I\subset(m,\infty)$. The Parseval identity applied to

$$\tilde\psi_s(x,\omega+i\epsilon)=\int_0^\infty\psi_s(x,t)e^{i\omega t-\epsilon t}\,dt,\quad \epsilon>0,$$

gives

$$\int_{\mathbb{R}}\|\tilde\psi_s(\cdot,\omega+i\epsilon)\|^2_{L^2(\mathbb{R}^3)}\,d\omega=2\pi\int_0^\infty\|\psi_s(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\,e^{-2\epsilon t}\,dt.$$

Since $\sup_{t\ge0}\|\psi_s(\cdot,t)\|_{L^2(\mathbb{R})}<\infty$ by (4.4), we may bound the right-hand side by C_1/ϵ, with some $C_1>0$. Taking into account (4.10), we arrive at the key inequality

$$\int_{\mathbb{R}}\|\sum_{1\le j\le n}\tilde\zeta_j(\omega+i\epsilon)V_j(\cdot,\omega+i\epsilon)\|^2_{L^2(\mathbb{R}^3)}\,d\omega\le\frac{C_1}{\epsilon}. \tag{4.15}$$

Note that

$$\hat V_j(\xi,\omega+i\epsilon)=e^{i\xi\cdot x_j}G(|\xi|,\omega+i\epsilon),\quad G(|\xi|,\omega+i\epsilon)=\frac{1}{|\xi|^2+m^2-(\omega+i\epsilon)^2}.$$

Hence, we can rewrite (4.15) as

$$\begin{aligned}&\int\limits_{\mathbb{R}}\epsilon\|\sum_{1\le j\le n}\tilde\zeta_j(\omega+i\epsilon)V_j(\cdot,\omega+i\epsilon)\|^2_{L^2(\mathbb{R}^3)}\,d\omega\\ &=\int\limits_{\mathbb{R}^3\times\mathbb{R}}\epsilon|G(|\xi|,\omega+i\epsilon)\sum_{1\le j\le n}e^{i\xi\cdot x_j}\tilde\zeta_j(\omega+i\epsilon)|^2\,d\omega\frac{d\xi}{(2\pi)^3}\le C_1.\end{aligned} \tag{4.16}$$

Denote

$$W_\epsilon=\{(\xi,\omega)\in\mathbb{R}^n\times I:|\omega-\sqrt{|\xi|^2+m^2}|\le\epsilon\}.$$

LEMMA 4.3. *There exists a constant $C_2=C_2(I)$ such that*

$$\int\limits_{W_\epsilon}\epsilon|G(|\xi|,\omega+i\epsilon)\sum_{1\le j\le n}e^{ik(\omega)\theta_\xi\cdot x_j}\tilde\zeta_j(\omega+i\epsilon)|^2\,d\omega\frac{d\xi}{(2\pi)^3}\le C_2,\ \theta_\xi=\xi/|\xi|. \tag{4.17}$$

PROOF. Using the triangle inequality in the form $|\|a\|-\|b\||\le\|a-b\|$, we get:

$$\begin{aligned}&\Big|\Big(\int\limits_{W_\epsilon}\epsilon|G(|\xi|,\omega+i\epsilon)\sum_{1\le j\le n}e^{i\xi\cdot x_j}\tilde\zeta_j(\omega+i\epsilon)|^2\,d\omega d\xi\Big)^{1/2}\\ &-\Big(\int\limits_{W_\epsilon}\epsilon|G(|\xi|,\omega+i\epsilon)\sum_{1\le j\le n}e^{ik(\omega)\theta_\xi\cdot x_j}\tilde\zeta_j(\omega+i\epsilon)|^2\,d\omega d\xi\Big)^{1/2}\Big|\\ &\le\Big(\int\limits_{W_\epsilon}\epsilon|G(|\xi|\omega+i\epsilon)|^2\sum_{1\le j\le n}|\tilde\zeta_j(\omega+i\epsilon)|^2|e^{i(k(\omega)\theta_\xi-\xi)\cdot x_j}-1|^2\,d\omega d\xi\Big)^{1/2}.\end{aligned} \tag{4.18}$$

According to (2.6) and (4.7), $|\tilde{\zeta}_j(\omega+i\epsilon)|\le C\epsilon^{-1}$. Further, for $(\xi,\omega)\in W_\epsilon$ one has

$$|e^{i(k(\omega)\theta_\xi-\xi)\cdot x_j}-1)|\le |x_j||k(\omega)\theta_\xi-\xi|\le |x_j||k(\omega)-|\xi||\le C\epsilon$$

with some constant $C>0$ independent on $\epsilon\in(0,1)$. Moreover,

$$\begin{aligned}|G(|\xi|,\omega+i\epsilon)|^2 &= \frac{1}{||\xi|^2+m^2-(\omega+i\epsilon)^2|^2}\le\frac{1}{(\mathrm{Im}\,(|\xi|^2+m^2-(\omega+i\epsilon)^2)^2}\\ &= \frac{1}{4\epsilon^2\omega^2}\le\frac{1}{4\epsilon^2m^2}. \qquad (4.19)\end{aligned}$$

Hence,

$$\begin{aligned}&\int\limits_{W_\epsilon}\epsilon|\hat V(|\xi|,\omega+i\epsilon)|^2\sum_{1\le j\le n}|\tilde{\zeta}_j(\omega+i\epsilon)|^2|e^{i(k(\omega)\theta_\xi-\xi)\cdot x_j}-1|^2\,d\omega d\xi\\ &\le C\int\limits_{W_\epsilon}\frac{d\omega d\xi}{\epsilon}\le C(I)\end{aligned}$$

since the thickness of W_ϵ in the ω-direction equiv 2ϵ, and W_ϵ is bounded by $C(I)$ in the ξ-direction. □

LEMMA 4.4. *There exists $\epsilon_I\in(0,1)$ such that for any $0<\epsilon\le\epsilon_I$*

$$\int\limits_{\Delta_\epsilon(\omega)}\epsilon|G(r,\omega+i\epsilon)|^2r^2dr\ge\pi/2,\quad \omega\in I, \qquad (4.20)$$

where $\Delta_\epsilon(\omega)=\{r\in\mathbb{R}_+:|\omega-\sqrt{r^2+m^2}|\le\epsilon\}$.

PROOF. Due to (4.19) one has

$$\int\limits_{\Delta_\epsilon(\omega)}\epsilon|G(r,\omega+i\epsilon)|^2r^2dr\ge\frac{1}{4m^2\epsilon}\int\limits_{\Delta_\epsilon(\omega)}r^2dr, \qquad (4.21)$$

Denote $a=\inf I>m$, $\epsilon_I=\min\{m,(a-m)/2\}$. For $r\in\Delta_\epsilon(\omega)$, and $\epsilon\le\epsilon_I$, we have

$$\sqrt{\omega^2-m^2+\epsilon^2-2\omega\epsilon}\le r\le\sqrt{\omega^2-m^2+\epsilon^2+2\omega\epsilon} \qquad (4.22)$$

Hence,

$$\begin{aligned}|\Delta_\epsilon(\omega)| &= \sqrt{\omega^2-m^2+\epsilon^2+2\omega\epsilon}-\sqrt{\omega^2-m^2+\epsilon^2-2\omega\epsilon}\\ &\ge \frac{4\omega\epsilon}{2\sqrt{\omega^2-m^2+\epsilon^2+2\omega\epsilon}}\ge\frac{4\omega\epsilon}{2(\omega+\epsilon)}>\epsilon.\end{aligned}$$

Taking into account (4.21), we obtain

$$\int\limits_{\Delta_\epsilon(\omega)}\epsilon|G(r,\omega+i\epsilon)|^2r^2dr\ge\frac{1}{4m^2}\min_{\Delta_\epsilon(\omega)}(r^2)\ge\frac{1}{8} \qquad (4.23)$$

since by (4.22)

$$\min_{\Delta_\epsilon(\omega)}(r^2)=(\omega-\epsilon)^2-m^2\ge(a-\frac{a-m}{2})^2-m^2=\frac{a^2+m^2}{4}+m(a-m)\ge m^2/2.$$

□

In the spherical coordinates, (4.17) reads

$$\int\limits_{W_\epsilon} \epsilon|G(|\xi|,\omega+i\epsilon)\sum_{1\le j\le n} e^{ik(\omega)\theta_\xi\cdot x_j}\tilde\zeta_j(\omega+i\epsilon)|^2\,d\omega\frac{d\xi}{(2\pi)^3}$$

$$=\frac{1}{(2\pi)^3}\int\limits_{S^2\times I}\Big|\sum_{1\le j\le n} e^{ik(\omega)\theta\cdot x_j}\tilde\zeta_j(\omega+i\epsilon)\Big|^2\Big(\int\limits_{\Delta_\epsilon(\omega)}\epsilon|G(r,\omega+i\epsilon)|^2r^2dr\Big)d\theta d\omega\le C_2.$$

Applying Lemma 4.4, we obtain

$$\int\limits_{S^2\times I}\Big|\sum_{1\le j\le n} e^{ik(\omega)\theta\cdot x_j}\tilde\zeta_j(\omega+i\epsilon)\Big|^2 d\theta d\omega\le 8(2\pi)^3C_2,\quad 0<\epsilon\le\epsilon_I.$$

We conclude that the set of functions

$$g_{I,\epsilon}(\theta,\omega)=\sum_{1\le j\le n} e^{ik(\omega)\theta\cdot x_j}\tilde\zeta_j(\omega+i\epsilon),\quad 0<\epsilon\le\epsilon_I,$$

defined for $\theta\in S^2$, $\omega\in I$, is bounded in the Hilbert space $L^2(S^2\times I)$, and, by the Banach Theorem, is weakly compact. The convergence of the distributions (4.12) implies the weak convergence $g_{I,\epsilon}\underset{\epsilon\to 0+}{\rightharpoondown} g_I$ in the Hilbert space $L^2(S^2\times I)$. The limit function $g_I\in L^2(S^2\times I)$ coincides with the distribution $\sum\limits_{1\le j\le n} e^{ik(\omega)\theta\cdot x_j}\tilde\zeta_j(\omega)$ on $S^2\times I$. This proves the bound (4.14). □

Proposition 4.5. *The distributions $\tilde\zeta_j(\omega)$, $1\le j\le n$, are locally L^2 for $\omega\in\mathbb{R}\setminus[-m,m]$.*

Proof. Similar result is proved in [**7**] (see Proposition 3.8) in another context. We reproduce some arguments of [**7**] for the convenience of readers. The proof is based on Proposition 4.2 and the following two Lemmas from [**7**].

Lemma 4.6. *(cf.* [**7**, Lemma 3.9]*) Let $\varkappa\ne 0$. Assume that the vectors $x_j\in\mathbb{R}^3$, $1\le j\le N$, are pairwise different. Then there exist vectors $\theta_k\in S^2$, $1\le k\le N$, such that*

$$\det_{1\le k,j\le N} e^{-i\varkappa\theta_k\cdot x_j}\ne 0.$$

Lemma 4.7. *(cf.* [**7**, Lemma 3.10]*) For any $\omega_0\in\mathbb{R}\setminus[-m,m]$, there is an open neighborhood $I\subset\mathbb{R}\setminus[-m,m]$ of ω_0 and a family of diffeomorphisms $\Theta_k: B^2\to\Omega_k\in S^2$, where B^2 is a unit open ball in $\mathbb{R}^2$ and Ω_k are open neighborhoods of S^2, such that*

$$\det_{1\le k,j\le N} e^{-ik(\omega)\Theta_k(\tau)\cdot x_j}\ne 0 \text{ for all } \omega\in\overline{I},\ \tau\in B^2.$$

Now let $\omega_0\in\mathbb{R}\setminus[-m,m]$, and let I be an open neighborhood of ω_0 from Lemma 4.7. Pick a function $\xi\in C_0^\infty(B^2)$ such that

$$\int_{B^2}\xi(\tau)d\tau=1. \tag{4.24}$$

Let $R_{kj}(\omega,\tau)$, $\omega\in\overline{I}$, $\tau\in B^2$ be the matrix inverse to $A_{kj}(\omega,\tau)=e^{-ik(\omega)\Theta_k(\tau)\cdot x_j}$. Denote

$$R_k(\omega,\theta)=\int\limits_{B^2}\sum_{j=1}^N R_{kj}(\omega,\tau)\delta_{\Theta_j(\tau)}(\theta)\xi(\tau)d\tau,\quad \omega\in\overline{I},\quad \theta\in S^2, \tag{4.25}$$

where $\delta_{\theta_0}(\theta)$ is a delta-function on S^2 supported at $\theta_0 \in S^2$. Then for each $1 \le j \le N$, the operator

$$\mathcal{R}_k : u(\omega,\theta) \to \mathcal{R}_k u(\omega) := \int\limits_{\Omega_k} R_k(\omega,\theta)u(\omega,\theta)d\Omega_\theta \tag{4.26}$$

acts continuously from $L^2(I\times S^2)$ to $L^2(I)$, Indeed, for a given value $\omega \in I$, let $T_k(\theta)$ be the inverse function to $\Theta_k(\tau)$ defined on the neighborhood $\{\Theta_k(\tau) : \ \tau \in B^2\} \subset S^2$. Then function $R_k(\omega,\theta)$ is smooth, since

$$\delta_{\Theta_k(\tau)}(\theta)\xi(\tau) = \frac{\delta(\tau - T_k(\theta))}{\left|\det \frac{\partial\Theta_k(\tau)}{\partial\tau}\right|}\xi(\tau).$$

Further, (4.24)- (4.26) imply

$$\mathcal{R}_k\Big(\sum_{l=1}^N e^{-ik(\omega)\theta\cdot x_l}\tilde{\zeta}_l(\omega)\Big) = \sum_{l=1}^N \int\limits_{S^2} R_k(\omega,\theta)e^{-ik(\omega)\theta\cdot x_l}\tilde{\zeta}_l(\omega)d\Omega_\theta$$

$$= \sum_{l=1}^N \int\limits_{S^2}\int\limits_{B^2}\sum_{j=1}^N R_{k.j}(\omega,\tau)\delta(\theta-\Theta_j(\tau))\xi(\tau)e^{-ik(\omega)\theta\cdot x_l}\tilde{\zeta}_l(\omega)d\tau d\Omega_\theta$$

$$= \sum_{l=1}^N \int\limits_{B^2}\sum_{j=1}^N R_{k.j}(\omega,\tau)\xi(\tau)e^{-ik(\omega)\Theta_j(\tau)\cdot x_l}\tilde{\zeta}_l(\omega)d\tau$$

$$= \sum_{l=1}^N \int\limits_{B^2} \delta_{kl}\xi(\tau)\tilde{\zeta}_l(\omega)d\tau = \tilde{\zeta}_k(\omega).$$

By Proposition 4.2

$$\sum_{l=1}^N e^{-ik(\omega)\theta\cdot x_l}\tilde{\zeta}_l(\omega) \in L^2(I\times S^2).$$

Moreover, $\mathcal{R}_k$ is continuous from $L^2(I\times S^2)$ to $L^2(I)$. Hence, $\zeta_k(\omega) \in L^2(I)$. □

5. Compactness

We are going to prove compactness of the set of translations of $\{\psi_s(x,t+s) : \ s \ge 0\}$. We start from the following lemma

LEMMA 5.1. *For any sequence $s_l \to \infty$ there exists an infinite subsequence (which we also denote by s_l) such that*

$$\zeta_j(t+s_l) \to \eta_j(t), \quad l\to\infty, \quad t\in\mathbb{R}, \quad j=1,...,n, \tag{5.1}$$

for some $\eta_j \in C_b(\mathbb{R})$. The convergence is uniform on $[-T,T]$ for any $T>0$. Moreover, $\eta_j(t)$ are solutions to the system

$$F_j(\eta(t)) + \frac{1}{4\pi}(\dot\eta_j(t) - m\eta_j(t)) - \sum_{k\ne j}\frac{\eta_k(t-|x_j-x_k|)}{4\pi|x_j-x_k|}$$

$$+\sum_{1\le k\le n}\frac{m}{4\pi}\int\limits_{|x_j-x_k|}^{\infty}\frac{J_1(m\sqrt{(s^2-|x_j-x_k|^2})}{\sqrt{(s^2-|x_j-x_k|^2}}\eta_k(t-s)ds = 0, \quad j=1,...,n, \tag{5.2}$$

where $\eta(t) = (\eta_1(t),...,\eta_n(t))$.

PROOF. Theorem 2.4-iii), Corollary 3.3 and equation (3.3) imply that $\zeta \in C_b^1(\mathbb{R})$. Then (5.1) follows from the Arzelá-Ascoli theorem. Further, for any $t \in \mathbb{R}$, we get

$$\int_{|x_j-x_k|}^{t+s_l} \frac{J_1(m\sqrt{(s^2-|x_j-x_k|^2)}}{\sqrt{(s^2-|x_j-x_k|^2}}\zeta_j(t+s_l-s)ds$$
$$\to \int_{|x_j-x_k|}^{\infty} \frac{J_1(m\sqrt{(s^2-|x_j-x_k|^2)}}{\sqrt{(s^2-|x_j-x_k|^2}}\eta_j(t-s)ds, \quad l\to\infty$$

by the Lebesgue dominated convergence theorem. Then system (3.3) together with (3.10) imply (5.2). □

Lemma 5.1 implies

LEMMA 5.2. *The following convergences hold as $l \to \infty$:*

$$\psi_{s,j}(\cdot,t+s_l) \to \beta_j(\cdot,t) := \frac{\eta_j(t-|x-x_j|)}{4\pi|x-x_j|} - \frac{m}{4\pi}\int_0^{\infty} \frac{\theta(s-|x-x_j|)J_1(m\sqrt{s^2-|x-x_j|^2})}{\sqrt{s^2-|x-x_j|^2}}\eta_j(t-s)ds, \tag{5.3}$$

$$\dot\psi_{s,j}(\cdot,t+s_l) \to \dot\beta_j(\cdot,t) = \frac{\dot\eta_j(t-|x-x_j|)}{4\pi|x-x_j|} - \frac{m}{4\pi}\int_0^{\infty} \frac{\theta(s-|x-x_j|)J_1(m\sqrt{s^2-|x-x_j|^2})}{\sqrt{s^2-|x-x_j|^2}}\dot\eta_j(t-s)ds \tag{5.4}$$

in the topology of $C_b([-T,T],L^2_{loc})$ for any $T>0$.

PROOF. The convergence (5.3) follows immediately from (4.2), (5.1) and the Lebesgue dominated convergence theorem. Let us prove (5.4). Equations (3.3) and (5.2) imply that

$$\dot\zeta_j(t+s_l) \to \dot\eta_j(t), \quad l\to\infty, \tag{5.5}$$

uniformly on $[-T,T]$ for any $T>0$. Further, differentiating (4.2) for $t>|x-x_j|$, we obtain

$$\dot\psi_{s,j}(x,t) = \frac{\dot\zeta_j(t-|x-x_j|)}{4\pi|x-x_j|} - \frac{m}{4\pi}\frac{J_1(m\sqrt{t^2-|x-x_j|^2})}{\sqrt{t^2-|x-x_j|^2}}\zeta_j(0) - \frac{m}{4\pi}\int_0^t \frac{\theta(s-|x-x_j|)J_1(m\sqrt{s^2-|x-x_j|^2})}{\sqrt{s^2-|x-x_j|^2}}\dot\zeta_j(t-s)ds,$$

which imply (5.4) by (5.5). □

Now Lemma 5.2 implies that

$$
\begin{aligned}
\psi_s(\cdot,t+s_l) &= \sum_{j=1}^n \psi_{s,j}(\cdot,t+s_l) \to \beta(\cdot,t) := \sum_{j=1}^n \beta_j(\cdot,t),\\
\dot\psi_s(\cdot,t+s_l) &= \sum_{j=1}^n \dot\psi_{s,j}(\cdot,t+s_l) \to \dot\beta(\cdot,t) = \sum_{j=1}^n \dot\beta_j(\cdot,t).
\end{aligned} \tag{5.6}
$$

Remark 5.3. (4.4) implies that $\beta(\cdot,t) \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^3))$.

6. Nonlinear spectral analysis

Proposition 3.1 demonstrates that the long-time asymptotics of the solution $\psi(x,t)$ in L^2_{loc} depends only on the singular component $\psi_s(x,t)$. We call *an omega-limit trajectory* of $\psi_s(x,t)$ any function $\beta(x,t)$ that can appear as a limit in (5.6). The convergences (5.6), and system (1.1) together with (3.1), (3.4) and (3.10) imply that any $\beta(x,t)$ is a solution to (1.1) with $\eta_j(t)$ instead $\zeta_j(t)$:

$$
\left\{
\begin{array}{l}
\ddot\beta(x,t) = (\Delta - m^2)\beta(x,t) + \sum\limits_{1\le j\le n} \eta_j(t)\delta(x-x_j)\\[2ex]
\lim\limits_{x\to x_j} (\beta(x,t) - \eta_j(t)g(x-x_j)) = F_j(\eta(t))
\end{array}
\right| \quad t \in \mathbb{R}.
$$

Below we prove the following proposition, which implies our main Theorem 2.8.

Proposition 6.1. *Every omega-limit trajectory is a solitary wave, that is,*

$$
\beta(x,t) = \psi_{\omega_*}(x)e^{-i\omega_* t}, \quad x\in\mathbb{R}^3, \quad t\in\mathbb{R}, \tag{6.1}
$$

with some $\omega_* \in \mathbb{R}$.

6.1. Reduction of spectrum. First, we reduce the spectrum of $\eta_j(t)$ which is the support of the Fourier transform $\tilde\eta_j(\omega)$.

Lemma 6.2. $\operatorname{supp}\tilde\eta_j \subset [-m,m]$ *for each* $1\le j\le n$.

Proof. Due to (5.1) and the continuity of the Fourier transform in $\mathcal{S}'(\mathbb{R})$, we have

$$
\alpha(\omega)\tilde\zeta_j(\omega)e^{-i\omega s_l} \xrightarrow{\mathcal{S}'} \alpha(\omega)\tilde\eta_j(\omega), \quad l\to\infty, \quad 1\le j\le n.
$$

for any $\alpha \in C_0^\infty(\mathbb{R})$. Assume that $\operatorname{supp}\alpha \cap [-m,m] = \emptyset$. Since $\tilde\zeta_j(\omega)$ is locally L^2 for $\omega\in\mathbb{R}\setminus[-m,m]$ by Proposition 4.5, the product $\alpha(\omega)\tilde\zeta_j(\omega)$ is in $L^1(\mathbb{R})$. Then $\eta_j(\omega) = 0$ for $\omega\notin[-m,m]$ by the Riemann-Lebesgue Theorem. □

Using (4.13) and taking into account that $V_j(x,\omega)$ is smooth for $\omega\neq\pm m$ and $x\neq x_j$, we obtain the following relation, which holds in the sense of distributions:

$$
\tilde\beta(x,\omega) = \sum_{1\le j\le n} \tilde\eta_j(\omega)V_j(x,\omega), \qquad \omega\in\mathbb{R}\setminus\{\pm m\}. \tag{6.2}
$$

Since $V_j(x,\omega)\neq 0$ for $\omega\in\mathbb{R}\setminus\{\pm m\}$, it follows from Lemma 6.2 that

$$
\operatorname{supp}\tilde\beta(x,\cdot) \subset [-m,m]. \tag{6.3}
$$

6.2. Spectral inclusion. We will derive (6.1) from the following identities

$$\eta_j(t) = C_j e^{-i\omega_* t}, \quad t \in R, \quad \omega_* \in [-m, m], \quad j = 1, ..., n, \tag{6.4}$$

which will be proved below. We start with an investigation of $\operatorname{supp}\tilde{\eta}_j$. Denote

$$\omega^- = \min_{1\le j\le n} \inf \operatorname{supp}\tilde{\eta}_j, \quad \omega^+ = \max_{1\le j\le n} \sup \operatorname{supp}\tilde{\eta}_j.$$

LEMMA 6.3. *The following spectral inclusions hold:*

$$\operatorname{supp}\widetilde{F_j(\eta_j)} \subset [\omega^-, \omega^+], \quad 1 \le j \le n. \tag{6.5}$$

PROOF. Applying the Fourier transform to (5.2), we get by the theory of quasimeasures (see [**13**]) that

$$\begin{aligned} \widetilde{F_j(\eta_j)}(\omega) &= \frac{1}{4\pi}(i\omega + m - m\tilde{K}(0,\omega))\tilde{\eta}_j(\omega) \\ &- \frac{1}{4\pi}\sum_{k\ne j}\big(m\tilde{K}(x_j - x_k, \omega) - \frac{e^{i\omega|x_j - x_k|}}{|x_j - x_k|}\big)\tilde{\eta}_k(\omega), \quad |\omega| \le m, \end{aligned}$$

where $\tilde{\eta}_j(\omega)$ are quasimeasures, and we denote for $|\omega| \le m$

$$\tilde{K}(x,\omega) = \begin{cases} \frac{e^{i\omega|x|} - e^{-\sqrt{m^2-\omega^2}|x|}}{m|x|}, & x \ne 0 \\ \frac{\sqrt{m^2-\omega^2}+i\omega}{m}, & x = 0 \end{cases} \tag{6.6}$$

which is the Fourier transforms of the functions

$$K(x,t) = \theta(t - |x|)\frac{J_1(m\sqrt{t^2 - |x|^2})}{\sqrt{t^2 - |x|^2}} \in L^1(\mathbb{R}) \tag{6.7}$$

(see [**22**, Appendix A]). Therefore, for $|\omega| \le m$, we obtain

$$\widetilde{F_j(\eta_j)}(\omega) = \frac{1}{4\pi}(m - \sqrt{m^2 - \omega^2})\tilde{\eta}_j(\omega) + \frac{1}{4\pi}\sum_{k\ne j}\frac{e^{-\sqrt{m^2-\omega^2}|x_j - x_k|}}{|x_j - x_k|}\tilde{\eta}_k(\omega), \tag{6.8}$$

which implies (6.5). □

7. Titchmarsh convolution theorem

LEMMA 7.1. *If* $\sup\operatorname{supp}\tilde{\eta}_j = \omega^+$, *then* $\operatorname{supp}\tilde{\eta}_j = \{\omega^+\}$. *If* $\inf\operatorname{supp}\tilde{\eta}_j = \omega^-$, *then* $\operatorname{supp}\tilde{\eta}_j = \{\omega^-\}$.

PROOF. Condition **B** implies that the function $F_j(\eta(t))$ admits the representation

$$F_j(\eta_j(t)) = a_j(\eta_j(t))\eta_j(t), \tag{7.1}$$

where

$$a_j(z) = \sum_{n=1}^{N_j - 1} 2nu_{n,j}|z|^{2n}. \tag{7.2}$$

The functions $\eta_j(t)$ and $a_j(\eta_j(t))$ are bounded continuous functions in $\mathbb{R}$ by Lemma 5.1. Hence, $\eta_j(t)$ and $a_j(\eta_j(t))$ are tempered distributions. According to (6.3) $\operatorname{supp}\tilde{\eta}_j \subset [-m, m]$, $\operatorname{supp}\tilde{\overline{\eta}}_j \subset [-m, m]$, and then $\widetilde{a_j(\eta_j)}$ also has a bounded support.

Hence, the Titchmarsh convolution theorem (see [**8**, Theorem 4.3.3]) implies that for any compactly supported distributions $f, g \in \mathcal{S}'(\mathbb{R})$, we have

$$\inf \operatorname{supp}(f*g) = \inf \operatorname{supp} f + \inf \operatorname{supp} g, \quad \sup \operatorname{supp}(f*g) = \sup \operatorname{supp} f + \sup \operatorname{supp} g.$$

Therefore, (6.5), (7.1) and (7.2) imply

$$\begin{aligned} \sup \operatorname{supp} \widetilde{F_j(\eta_j)} &= \sup \operatorname{supp} \tilde\eta_j + (N_j - 1)(\sup \operatorname{supp} \tilde\eta_j + \sup \operatorname{supp} \tilde{\bar\eta}_j) \\ &= \sup \operatorname{supp} \tilde\eta_j + (N_j - 1)(\sup \operatorname{supp} \tilde\eta_j - \inf \operatorname{supp} \tilde\eta_j) \le \omega^+ \end{aligned} \tag{7.3}$$

$$\begin{aligned} \inf \operatorname{supp} \widetilde{F_j(\eta_j)} &= \inf \operatorname{supp} \tilde\eta_j + (N_j - 1)(\inf \operatorname{supp} \tilde\eta_j + \inf \operatorname{supp} \tilde{\bar\eta}_j) \\ &= \inf \operatorname{supp} \tilde\eta_j - (N_j - 1)(\sup \operatorname{supp} \tilde\eta_j - \inf \operatorname{supp} \tilde\eta_j) \ge \omega^- \end{aligned} \tag{7.4}$$

Suppose now, that $\sup \operatorname{supp} \tilde\eta_j = \omega^+$. Then $\inf \operatorname{supp} \tilde\eta_j = \sup \operatorname{supp} \tilde\eta_j = \omega^+$ by (7.3). Similarly, if $\inf \operatorname{supp} \tilde\eta_j = \omega^-$, then $\sup \operatorname{supp} \tilde\eta_j = \inf \operatorname{supp} \tilde\eta_j = \omega^-$ by (7.4). □

COROLLARY 7.2. *If* $\sup \operatorname{supp} \tilde\eta_j = \omega^+$, *then* $\operatorname{supp} \widetilde{F_j(\eta_j)} = \{\omega^+\}$. *If* $\inf \operatorname{supp} \tilde\eta_j = \omega^-$, *then* $\operatorname{supp} \widetilde{F_j(\eta_j)} = \{\omega^-\}$.

Now we suppose that

$$\omega^- < \omega^+, \tag{7.5}$$

and show that this condition leads to a contradiction. condition (7.5) and Lemma 7.1 imply that there exist disjoint open neighborhoods $\mathcal{O}^+$ and $\mathcal{O}^-$ of ω^+ and ω^-, respectively, so that

$$\mathcal{O}^\pm \cap \operatorname{supp} \tilde\eta_j \subset \{\omega^\pm\}, \quad 1 \le j \le n.$$

Let $\chi^\pm \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp} \chi^\pm \subset \mathcal{O}^\pm$.

LEMMA 7.3. *There exist* $A_j^\pm \in \mathbb{C}$, $1 \le j \le n$, *such that*

$$\chi^\pm(\omega)\tilde\eta_j(\omega) = A_j^\pm \delta(\omega - \omega^\pm), \tag{7.6}$$

PROOF. The inclusions $\operatorname{supp} \chi^\pm \tilde\eta_j \subset \{\omega^\pm\}$ implies that $\chi^\pm \eta_j$ is a linear combination of $\delta(\omega - \omega^\pm)$ and its derivatives. But the derivatives are forbidden due to the boundedness of $\eta_j(t)$. □

Condition (7.5) implies that at least one of A_j^- and at least one of A_j^+ are nonzero. Moreover, if $A_j^- \neq 0$ for some j, then $A_j^+ = 0$ and vice versa. Let there exist k nonzero elements in the set of all A_j^-, and l nonzero elements in the set of all A_j^+, where $k + l \le n$. Also assume that $k \le l$. The case $l \le k$ can be considered similarly. Without loss of generality, we can renumber the functions η_j (and then the coefficients $A_j^\pm$) to have

$$A_j^- \neq 0 \quad \text{for } j = 1, 2, ..., k, \quad \text{and } A_j^+ \neq 0 \quad \text{for } k+1 \le j \le k+l.$$

Now Lemma 7.1 and (7.6) imply that

$$\tilde\eta_j(\omega) = \begin{cases} A_j^- \delta(\omega - \omega^-), & 1 \le j \le k \\ A_j^+ \delta(\omega - \omega^+), & k+1 \le j \le k+l \end{cases} \tag{7.7}$$

Therefore,

$$\chi^-(\omega)\tilde\eta_j(\omega) = 0, \quad k+1 \le j \le k+l. \tag{7.8}$$

According to Corollary 7.2, $\operatorname{supp}\widetilde{F_j(\eta_j)} = \{\omega^+\}$, for $k+1 \le j \le k+l$, and hence

$$\chi^-(\omega)\widetilde{F_j(\eta_j)}(\omega) = 0, \quad k+1 \le j \le k+l. \tag{7.9}$$

Multiplying equations (6.8) with $k+1 \le j \le k+l$ by $\chi^-(\omega)$ and using (7.7)– (7.9), we obtain

$$\begin{cases} 0 = \alpha_{k+1,1}(\omega^-)A_1^- + \alpha_{k+1,2}(\omega^-)A_2^- + ... + \alpha_{k+1,k}(\omega^-)A_k^- \\ 0 = \alpha_{k+2,1}(\omega^-)A_1^- + \alpha_{k+2,2}(\omega^-)A_2^- + ... + \alpha_{k+2,k}(\omega^-)A_k^- \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 0 = \alpha_{k+l,1}(\omega^-)A_1^- + \alpha_{k+l,2}(\omega^-)A_2^- + ... + \alpha_{k+l,k}(\omega^-)A_k^- \end{cases}$$

where α_{jk} are defined in (2.10). Due to condition **A**

$$\det\{\alpha_{k+i,j}\}_{i,j=1,...,k} \neq 0.$$

Hence $A_1^- = A_1^- = ... = A_k^- = 0$, which contradicts (7.5). It follows that $\omega^- = \omega^+ = \omega_*$, and $\operatorname{supp}\tilde\eta_j \subset \{\omega_*\}$ for $j = 1,...,n$. Therefore, $\tilde\eta_j$ is a finite linear combination of $\delta(\omega-\omega_*)$ and its derivatives. But the derivatives could not be present because of the boundedness of $\eta_j(t)$. Thus $\tilde\eta_j \sim \delta(\omega-\omega_*)$, which implies (6.4).

8. Proofs of main results

8.1. Proof of Proposition 6.1. Substituting (6.4) into the RHS of (5.6), we obtain by (5.3) and (6.6) -(6.7) that

$$\begin{aligned} \beta(x,t) &= \sum_{j=1}^n C_j\Big(\frac{e^{-i\omega_*(t-|x-x_j|)}}{4\pi|x-x_j|} \\ &- \frac{m}{4\pi}\int_0^\infty \frac{\theta(s-|x-x_j|)J_1(m\sqrt{s^2-|x-x_j|^2})}{\sqrt{s^2-|x-x_j|^2}}e^{-i\omega_*(t-s)}ds\Big) \\ &= \sum_{j=1}^n C_j\Big(\frac{e^{i\omega_*|x-x_j|)}}{|x-x_j|} - m\tilde K(x-x_j,\omega_*)\Big)\frac{e^{-i\omega_* t}}{4\pi} \\ &= \sum_{j=1}^n \frac{C_j e^{-\sqrt{m^2-\omega_*^2}|x-x_j|}}{4\pi|x-x_j|}e^{-i\omega_* t}. \end{aligned}$$

Hence, (6.1) holds and $\beta(x,t)$ is a solitary wave. □

REMARK 8.1. If (6.4) holds with some $|\omega_*| < m$, then (6.1) follows immediately from (6.2).

8.2. Proof of Theorem 2.8. Due to Proposition 3.1 it suffices to prove that

$$\lim_{t\to\infty} \operatorname{dist}_{\mathcal{L}^2_{loc}}(\psi_s(t), \mathbf{S}) = 0,$$

where $\psi_s(t) = (\psi_s(t), \dot\psi_s(t))$. Assume by contradiction that there exists a sequence $s_j \to \infty$ such that

$$\operatorname{dist}_{\mathcal{L}^2_{loc}}(\psi_s(s_j), \mathbf{S}) \ge \delta, \quad \forall j \tag{8.1}$$

for some $\delta > 0$. According to Lemmas 5.1 and 5.2, and formula (6.1) there exist a subsequence s_{j_k} of the sequence s_j and an amplitude ψ_{ω_+} such that the following convergences hold

$$\psi_s(t+s_{j_k}) \to (\psi_{\omega_*}e^{-i\omega_* t}, -i\omega_*\psi_{\omega_*}e^{-i\omega_* t}), \quad k\to\infty, \quad t\in\mathbb{R}.$$

This implies that $\psi_s(s_{j_k}) \to (\psi_{\omega_*}, -i\omega\psi_{\omega_*})$, which contradicts (8.1). This completes the proof of Theorem 2.8. □

9. Example

Here we construct an example of matrix for which condition **A** is satisfied. Suppose that $(x_1,...,x_n)\in\mathbb{R}^{3n}$, such that for any $I\subset\{1,2,...,n\}$, and any $J\subset\{1,2,...,n\}\setminus I$, $|I|=|J|$, one has

$$\det\{\frac{1}{|x_j-x_k|}\}_{j\in I,k\in J}\neq 0. \tag{9.1}$$

Such sets exist since these minors vanishes on proper analytic subsets of $\mathbb{R}^{3n}$, which follows by induction in $|I|$. Denote $\overline{x} = \max\limits_{1\le j<k\le n}|x_j-x_k|$. Suppose that

$$m\overline{x}\le\delta$$

with some small positive $\delta<1$. Then

$$\begin{aligned}\alpha_{jk}(\omega) &= \frac{e^{-\sqrt{m^2-\omega^2}|x_j-x_k|}}{|x_j-x_k|} = \frac{1}{|x_j-x_k|}(1-\sqrt{m^2-\omega^2}|x_j-x_k|+...)\\ &= \frac{1}{|x_j-x_k|}(1+\Delta_{jk}(\omega)),\end{aligned}$$

where

$$\max_{1\le j<k\le n}|\Delta_{jk}(\omega)|\le\delta.$$

Hence, condition **A** holds for sufficiently small δ by (9.1).

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