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Global dynamics of partly diffusive Hindmarsh-Rose equations in neurodynamics

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Abstract. Global dynamics of the partly diffusive Hindmarsh-Rose equations as a new mathematical model in neurodynamics is presented and studied in this paper. The existence of global attractor for the solution semiflow is proved through uniform estimates showing the higher-order dissipative property and the ultimate compactness by the new approach of Kolmogorov-Riesz theorem.

Contents

1. Introduction

In this work we present and investigate the global dynamics of the partly diffusive Hindmarsh-Rose equations as a new model in neurodynamics:

$$\frac{\partial u}{\partial t} = \lambda \Delta u + \varphi(u) + v - w + J, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \psi(u) - v, \tag{1.2}$$

$$\frac{\partial w}{\partial t} = q(u - c) - rw, \tag{1.3}$$

for $t > 0,\ x \in \Omega \subset \mathbb{R}^n\ (n \leq 3)$, where Ω is a bounded domain with locally Lipschitz continuous boundary. The nonlinear terms

$$\varphi(u) = au^2 - bu^3, \quad \text{and} \quad \psi(u) = \alpha - \beta u^2. \tag{1.4}$$

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We impose the Neumann boundary conditions for the u-component,

$$\frac{\partial u}{\partial \nu}(t, x) = 0, \quad t > 0, \; x \in \partial\Omega, \tag{1.5}$$

where $\partial/\partial\nu$ stands for the outward normal derivative, and the initial conditions are denoted by

$$u(0, x) = u_0(x), \; v(0, x) = v_0(x), \; w(0, x) = w_0(x), \quad x \in \Omega. \tag{1.6}$$

In this system, the variable $u(t, x)$ refers to the membrane electric potential of a neuron cell, the variable $v(t, x)$ represents the transport rate of the ions of sodium and potassium through the fast channels and is called the spiking variable, while the variables $w(t, x)$ represents the transport rate across the cell membrane through slow channels of calcium and other ions correlated to the bursting phenomena and is called the bursting variable. All the parameters $\lambda, a, b, \alpha, \beta, q, r$ and the inject current J are positive constants, except the reference membrane potential $c \in \mathbb{R}$.

The original Hindmarsh-Rose model of three ordinary differential equations was developed in [**11**] (1984) and motivated by the discovery of neuron cells in the pond snail *Lymnaea* which generated a burst after being depolarized by a short current pulse. This model characterizes the phenomena of synaptic bursting, especially chaotic bursting, in a three-dimensional (u, v, w) space. It is mathematically different from the four-dimensional highly nonlinear Hodgkin-Huxley equations [**12**] (1952) and from the two-dimensional FitzHugh-Nagumo equations [**9**] (1961-1962) in neuron dynamics. The 2D FitzHugh-Nagumo model admits exquisite phase plane analysis showing sustained periodic spiking with refractory period, but it excludes chaotic solutions so that no chaotic bursting can be generated.

The mathematical model of partly diffusive Hindmarsh-Rose equations in this study reflects the structural feature of biological neuron cells, especially the short-branch dendrites receiving incoming signals and the long-branch axon propagating outreaching signals as well as that neurons are immersed in the aqueous biochemical solutions.

Neuronal signals are electrical pulses called spikes or the action potential. Neuron bursting of alternating phases of rapid firing spikes and then quiescence constitutes a mechanism to modulate and pace-setting for brain functionalities. Bursting patterns occur in many bio-systems such as pituitary melanotropic gland, thalamic neurons, respiratory pacemaker neurons, and insulin-secreting pancreatic β-cells, cf. [**1, 2, 4**].

The mathematical analysis of several ODE models on bursting behavior has been studied by many authors [**7, 8, 14, 18, 23, 25, 26, 28**] mainly using bifurcations together with numerical simulations. Synchronization of neural networks is another interesting topic in neurodynamics, cf. [**5, 6, 8, 19, 20, 22, 29**].

The chaotic bursting exhibited in the simulations for the Hindmarsh-Rose model of ordinary differential equations shows more rapid synchronization and more effective regularization of coupled neurons with lower threshold [**23, 25, 28**].

Recently the first two authors proved the exponential synchronization results for the neurons coupled with the gap junctions [**15**] and for the boundary coupled neuron networks [**16**] based on the partly diffusive Hindmarsh-Rose equations. Besides, we have shown in [**17**] that for the fully diffusive Hindmarsh-Rose equations there exists a global attractor. In neuron dynamics, since the ionic currents may or may not diffuse quickly while the action potential traverses along the neuron axon quickly, the partly diffusive model (1.1)-(1.3) is more interesting than the corresponding pure ODE model and the fully diffusive model.

To formulate the partly diffusive Hindmarsh-Rose equations, we define the spaces

$$H = [L^2(\Omega)]^3 = L^2(\Omega, \mathbb{R}^3) \quad \text{and} \quad E = H^1(\Omega) \times L^2(\Omega, \mathbb{R}^2).$$

The norm and inner-product of H or $L^2(\Omega)$ will simply be denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ respectively. $| \cdot |$ is to denote a vector norm or a set measure in a Euclidean space.

The initial-boundary value problem of the Hindmasrh-Rose equations (1.1)–(1.6) is formulated to an initial value problem of the evolutionary equation:

$$\frac{\partial g}{\partial t} = Ag + f(g), \quad t > 0, \quad g(0) = g_0 \in H. \tag{1.7}$$

Here $g(t)$ is the column vector of $(u(t, \cdot), v(t, \cdot), w(t, \cdot))$ and g_0 is the column vector of (u_0, v_0, w_0). The nonpositive self-adjoint operator in (1.7) is given by

$$A = \begin{pmatrix} \lambda\Delta & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -rI \end{pmatrix} : D(A) \to H, \tag{1.8}$$

where $D(A) = \{g \in H^2(\Omega) \times L^2(\Omega, \mathbb{R}^2) : \partial u/\partial \nu = 0\}$ with the graph norm, and

$$f(u, v, w) = \begin{pmatrix} \varphi(u) + v - w + J \\ \psi(u), \\ q(u - c) \end{pmatrix} : H^1(\Omega) \times L^2(\Omega, \mathbb{R}^2) \to H. \tag{1.9}$$

The operator $A : D(A) \to H$ generates a C_0-semigroup $\{e^{At}\}_{t \geq 0}$ on the Hilbert space H due to the Lumer-Philips theorem [**21**]. By the Sobolev embedding that the injections $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $H^1(\Omega) \hookrightarrow L^6(\Omega)$ are continuous for space dimension $n \leq 3$ and by the Hölder inequality, there is a constant $C_0 > 0$ such that there is a constant $C_0 > 0$ such that

$$\|\varphi(u)\| \leq C_0(1 + \|u\|^3_{L^6}), \quad \|\psi(u)\| \leq C_0(1 + \|u\|^2_{L^4}) \quad \text{for } u \in H^1(\Omega).$$

Therefore, the nonlinear mapping

$$f(u, v, w) = \begin{pmatrix} \varphi(u) + v - w + J \\ \psi(u) - v, \\ q(u - c) - rw \end{pmatrix} : E \to H \tag{1.10}$$

is locally Lipschitz continuous.

Consider the weak solution of this initial value problem (1.7) [**3**, Section XV.3], which is defined below.

Definition 1.1. A three-dimensional vector function $g(t, x), (t, x) \in [0, \tau] \times \Omega$, is called a *weak solution* to the initial value problem of the evolutionary equation (1.7), if the following conditions are satisfied:

(i) $\frac{d}{dt}(g, \zeta) = (Ag, \zeta) + (f(g), \zeta)$ is satisfied for a.e. $t \in [0, \tau]$ and any $\zeta \in E$;

(ii) $g(t, \cdot) \in C([0, \tau]; H) \cap L^2([0, \tau]; E)$ and $g(0) = g_0$.

Here $(\cdot, \cdot)$ is the dual product of the dual space E^* versus E.

The following proposition can be proved by the Galerkin approximation method.

Proposition 1.2. *For any given initial state $g_0 \in H$, there exists a unique weak solution $g(t, g_0)$, $t \in [0, \tau]$, for some $\tau > 0$ may depending on g_0, of the initial value problem* (1.7) *associated with the partly diffusive Hindmarsh-Rose equations* (1.1)-(1.6)*. The weak solution $g(t, g_0)$ continuously depends on the initial data and satisfies*

$$g \in C([0, \tau]; H) \cap C^1((0, \tau); H) \cap L^2([0, \tau]; E). \tag{1.11}$$

If the initial data $g_0 \in E$, then the weak solution becomes a strong solution on its existence time interval $[0, \tau]$, which has the regularity

$$g \in C([0, \tau]; E) \cap C^1((0, \tau); E) \cap L^2([0, \tau]; D(A)). \tag{1.12}$$

The goal of this paper is to prove the existence of a global attractor of the solution semiflow generated by the evolutionary equation (1.7). Here we list a few concepts in the theory of infinite dimensional dynamical systems [**3, 21, 24, 27**].

Definition 1.3. Let $\{S(t)\}_{t \geq 0}$ be a semiflow on a Banach space $\mathscr{X}$. A bounded set B^* of $\mathscr{X}$ is called an absorbing set for this semiflow, if for any given bounded subset $B \subset \mathscr{X}$ there is a finite time $T_B \geq 0$ such that $S(t)B \subset B^*$ for all $t \geq T_B$.

Definition 1.4. A semiflow $\{S(t)\}_{t \geq 0}$ on a Banach space $\mathscr{X}$ is called asymptotically compact, if for any bounded sequence $\{w_n\}$ in $\mathscr{X}$ and any monotone increasing sequences $0 < t_n \to \infty$, there exist subsequences $\{w_{n_k}\} \subset \{w_n\}$ and $\{t_{n_k}\} \subset \{t_n\}$ such that $\lim_{k \to \infty} S(t_{n_k}) w_{n_k}$ exists in $\mathscr{X}$.

Definition 1.5. A set $\mathscr{A}$ in a Banach space $\mathscr{X}$ is called a global attractor for a semiflow $\{S(t)\}_{t \geq 0}$ on $\mathscr{X}$, if the following two conditions are satisfied:

(i) $\mathscr{A}$ is a nonempty, compact, and invariant set in the space $\mathscr{X}$.

(ii) $\mathscr{A}$ attracts any given bounded set $B \subset \mathscr{X}$ in the sense

$$\operatorname{dist}_{\mathscr{X}}(S(t)B, \mathscr{A}) = \sup_{x \in B} \inf_{y \in \mathscr{A}} \|S(t)x - y\|_{\mathscr{X}} \to 0, \ \text{ as } \ t \to \infty.$$

The following is the main existing result on the existence of a global attractor.

Proposition 1.6. *Let $\{S(t)\}_{t \geq 0}$ be a semiflow on a Banach space $\mathscr{X}$. If the following two conditions are satisfied*:

(i) *there exists a bounded absorbing set $B^* \subset \mathscr{X}$ for $\{S(t)\}_{t \geq 0}$, and*

(ii) *the semiflow $\{S(t)\}_{t \geq 0}$ is asymptotically compact on $\mathscr{X}$,*

then there exists a unique global attractor $\mathscr{A}$ in $\mathscr{X}$ for the semiflow $\{S(t)\}_{t \geq 0}$ and

$$\mathscr{A} = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} (S(t)B^*)}. \tag{1.13}$$

The Young's inequality in the general form for any nonnegative x, y is

$$xy \leq \varepsilon x^p + C(\varepsilon, p) y^q, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad C(\varepsilon, p) = \varepsilon^{-q/p}. \tag{1.14}$$

where $p, q > 1$ and a constant $\varepsilon > 0$ can be arbitrary.

The new model of neuron dynamics studied in this paper is a system of partial-ordinary equations, for which the usual approach to prove the asymptotic compactness of a solution semiflow by the uniform Gronwall inequality that works for diffusive reaction-diffusion equations becomes ineffective. The feature of this work is a new approach to prove the ultimately compact characteristics for this hybrid system through Kolmogorov-Riesz compactness theorem [**10**] with the leverage of higher-order absorbing estimates.

2. Uniform Estimates and Absorbing Properties

In this section we shall first prove the global existence of weak solutions of the problem (1.7) in time and then we study the high-order absorbing dynamics of this system by uniform estimates of the solution semiflow.

THEOREM 2.1. *For any given initial state $g_0 = (u_0, v_0, w_0) \in H$, there exists a unique global weak solution in time, $g(t) = (u(t), v(t), w(t))$, $t \in [0, \infty)$, of the initial value problem* (1.7) *for the partly diffusive Hindmarsh-Rose equations* (1.1)-(1.3). *The weak solution turns out to be a strong solution on the interval* $(0, \infty)$. *Moreover, There exists an absorbing set in the space H for this solution semiflow.*

PROOF. Taking the L^2 inner-products $\langle (1.2), v(t) \rangle$ and $\langle (1.3), w(t) \rangle$ and by the Young's inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v\|^2 = \int_\Omega (\psi(u)v - v^2)\, dx = \int_\Omega (\alpha v - \beta u^2 v - v^2)\, dx \\ &\le \int_\Omega \left(\alpha v + \frac{1}{2}(\beta^2 u^4 + v^2) - v^2\right) dx = \int_\Omega \left(\alpha v + \frac{1}{2}\beta^2 u^4 - \frac{1}{2}v^2\right) dx \\ &\le \int_\Omega \left(2\alpha^2 + \frac{1}{8}v^2 + \frac{1}{2}\beta^2 u^4 - \frac{1}{2}v^2\right) dx = \int_\Omega \left(2\alpha^2 + \frac{1}{2}\beta^2 u^4 - \frac{3}{8}v^2\right) dx \end{aligned} \tag{2.1}$$

and

(2.2)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w\|^2 = \int_\Omega (q(u-c)w - rw^2)\, dx \\ &\le \int_\Omega \left(\frac{q^2}{2r}(u-c)^2 + \frac{1}{2}rw^2 - rw^2\right) dx \le \int_\Omega \left(\frac{q^2}{r}(u^2 + c^2) - \frac{1}{2}rw^2\right) dx \\ &\le \int_\Omega \left(\frac{u^4}{2} + \frac{q^4}{2r^2} + \frac{q^2c^2}{r} - \frac{1}{2}rw^2\right) dx \le \int_\Omega \left(u^4 - \frac{1}{2}rw^2\right) dx + \left(\frac{q^4}{r^2} + \frac{q^2c^2}{r}\right)|\Omega|. \end{aligned}$$

Take the L^2 inner-product $\langle (1.1), C_1 u(t) \rangle$ with the constant $C_1 = \frac{1}{b}(\beta^2 + 4)$ to get

$$\frac{C_1}{2}\frac{d}{dt}\|u\|^2 + C_1\lambda\|\nabla u\|^2 = \int_\Omega C_1(au^3 - bu^4 + uv - uw + Ju)\, dx, \tag{2.3}$$

where

$$\int_\Omega (-C_1 b u^4)\, dx + \int_\Omega (\beta^2 u^4)\, dx = \int_\Omega (-4u^4)\, dx,$$

and by Young's inequality we have

$$\int_\Omega C_1 a u^3\, dx \le \frac{3}{4}\int_\Omega u^4\, dx + \frac{1}{4}\int_\Omega (C_1 a)^4\, dx \le \int_\Omega u^4\, dx + (C_1 a)^4|\Omega|,$$

$$\int_\Omega C_1(uv - uw + Ju)\, dx \le \int_\Omega \left[2(C_1 u)^2 + \frac{1}{8}v^2 + \frac{(C_1 u)^2}{r} + \frac{1}{4}rw^2 + C_1 u^2 + C_1 J^2\right] dx$$

in which, for the three u^2 terms on the right-hand side,

$$\int_\Omega \left(2(C_1 u)^2 + \frac{(C_1 u)^2}{r} + C_1 u^2\right) dx \le \int_\Omega u^4\, dx + \left[C_1^2\left(2 + \frac{1}{r}\right) + C_1\right]^2 |\Omega|.$$

Substitute the above term estimates into (2.3). Then sum up the three inequalities (2.1)-(2.3) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(C_1\|u\|^2 + \|v\|^2 + \|w\|^2) + C_1\lambda\|\nabla u\|^2 \\
&\le \int_\Omega C_1(au^3 - bu^4 + uv - uw + Ju)\,dx \\
&\quad + \int_\Omega \left(2\alpha^2 + \frac{1}{2}\beta^2 u^4 - \frac{3}{8}v^2\right) dx + \int_\Omega \left(\frac{q^2}{r}(u^2 + c^2) - \frac{1}{2}rw^2\right) dx \\
&\le \int_\Omega (3-4)u^4\,dx + \int_\Omega \left(\frac{1}{8} - \frac{3}{8}\right) v^2\,dx + \int_\Omega \left(\frac{1}{4} - \frac{1}{2}\right) rw^2\,dx \\
&\quad + \left((C_1 a)^4 + C_1 J^2 + \left[C_1^2\left(2 + \frac{1}{r}\right) + C_1\right]^2 + 2\alpha^2 + \frac{q^2c^2}{r} + \frac{q^4}{r^2}\right)|\Omega| \\
&= -\int_\Omega \left(u^4(t,x) + \frac{1}{4}v^2(t,x) + \frac{1}{4}rw^2(t,x)\right) dx + C_2|\Omega|
\end{aligned}
\tag{2.4}
$$

where C_2 is the constant given by

$$
C_2 = (C_1 a)^4 + C_1 J^2 + \left[C_1^2\left(2 + \frac{1}{r}\right) + C_1\right]^2 + 2\alpha^2 + \frac{q^2c^2}{r} + \frac{q^4}{r^2}.
$$

Thus all the solutions satisfy the differential inequality

$$
\begin{aligned}
&\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + 2C_1\lambda\|\nabla u\|^2 \\
&+ \int_\Omega \left(2u^4(t,x) + \frac{1}{2}v^2(t,x) + \frac{1}{2}rw^2(t,x)\right) dx \le 2C_2|\Omega|,
\end{aligned}
\tag{2.5}
$$

for $t \in I_{max} = [0, T_{max})$, which is the maximal time interval of solution existence. Cauchy inequality implies that

$$
2u^4 \ge \frac{1}{2}\left(C_1 u^2 - \frac{C_1^2}{16}\right).
$$

From (2.5) it follows that

$$
\begin{aligned}
&\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + 2C_1\lambda\|\nabla u\|^2 \\
&+ \int_\Omega \frac{1}{2}\left(C_1u^2(t,x) + v^2(t,x) + rw^2(t,x)\right) dx \le \left(2C_2 + \frac{C_1^2}{32}\right)|\Omega|.
\end{aligned}
$$

Set $r_1 = \frac{1}{2}\min\{1, r\}$. Then we get

$$
\begin{aligned}
&\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + C_1\lambda\|\nabla u(t)\|^2 \\
&+ r_1(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) \le \left(2C_2 + \frac{C_1^2}{32}\right)|\Omega|, \quad t \in [0, T_{max}).
\end{aligned}
\tag{2.6}
$$

Apply the Gronwall inequality to the following inequality reduced from (2.6),

$$
\frac{d}{dt}(C_1\|u\|^2 + \|v\|^2 + \|w\|^2) + r_1(C_1\|u\|^2 + \|v\|^2 + \|w\|^2) \le \left(2C_2 + \frac{C_1^2}{32}\right)|\Omega|.
$$

Then we obtain the uniform estimate of all the weak solutions:

$$
C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 \le e^{-r_1 t}(C_1\|u_0\|^2 + \|v_0\|^2 + \|w_0\|^2) + M|\Omega| \tag{2.7}
$$

for any $t \in [0, T_{max})$, where

$$M = \frac{1}{r_1}\left(2C_2 + \frac{C_1^2}{32}\right).$$

The uniform bound (2.7) in time for any given solution shows that the weak solutions will never blow up at any finite time. Therefore all the weak solutions of the initial value problem (1.7) exist for $t \in [0, \infty)$.

There exists an absorbing set in the space H for the Hindmarsh-Rose semiflow $\{S(t)\}_{t \geq 0}$, which is the bounded ball

$$B^* = \{g \in H : \|g\|^2 \leq K\} \tag{2.8}$$

where

$$K = \frac{M|\Omega|}{\min\{C_1, 1\}} + 1. \tag{2.9}$$

Indeed from the uniform estimate (2.7) we see that

$$\limsup_{t \to \infty} \left(\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2\right) < K \tag{2.10}$$

for all weak solutions of (1.7) with any initial state $g_0 \in H$. And for any given bounded set $B = \{g \in H : \|g\|^2 \leq \rho\}$ in H, there is a finite time

$$T_B = \frac{1}{r_1} \log^+(\rho \max\{C_1, 1\}) \tag{2.11}$$

such that $\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 < K$ for $t \geq T_B$ and any $g_0 \in B$. Therefore the ball B^* is an absorbing set for this solution semiflow in the space H. $\square$

Note that the solution semiflow $\{S(t)\}_{t \geq 0}$ on the space H is defined by the weak solutions $g(t, g_0)$ of (1.7),

$$S(t) : g_0 \longmapsto g(t, g_0) = (u(t, \cdot), v(t, \cdot), w(t, \cdot)), \quad g_0 = (u_0, v_0, w_0) \in H, \quad t \geq 0.$$

This solution semiflow will be called the *partly diffusive Hindmarsh-Rose semflow.*

We further pursue the uniform estimates to study the absorbing property of the partly diffusive Hindmarsh-Rose semiflow in higher-order integrable space $L^4(\Omega, \mathbb{R}^3)$. It will play a key role to establish the ultimate and asymptotic compactness of this semiflow generated by the partly diffusive Hindmarsh-Rose equations (1.1)-(1.3) toward the proof of global attractor existence.

Theorem 2.2. *There exists a constant $Q > 0$ independent of any initial data such that the u-component of the weak solution $g(t, g_0) = (u(t), v(t), w(t))$ of (1.7) satisfies the absorbing property*

$$\limsup_{t \to \infty} \|u(t)\|^4_{L^4(\Omega)} < Q \tag{2.12}$$

for any initial state $g_0 \in H$.

Proof. By Theorem 2.1 and Proposition 1.2, for any given weak solution $g(t, g_0)$, $g_0 \in H$, there exists a time $t_0 \in (0, 1)$ such that, due to the Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$ for space dimension $n \leq 3$,

$$g(t_0, g_0) \in E = H^1(\Omega) \times L^2(\Omega, \mathbb{R}^2) \subset L^6(\Omega) \times L^2(\Omega, \mathbb{R}^2) \subset L^4(\Omega) \times L^2(\Omega, \mathbb{R}^2).$$

and $g(t, g_0)$ becomes a strong solution on $[t_0, \infty)$ so that

$$g(\cdot, g_0) \in C([t_0, \infty), H^1(\Omega) \times L^2(\Omega, \mathbb{R}^2)) \subset C([t_0, \infty), L^4(\Omega) \times L^2(\Omega, \mathbb{R}^2)).$$

Take the L^2 inner-product $\langle(1.1), u^3(t,\cdot)\rangle$ and use Young's inequality appropriately to obtain

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\|u(t)\|^4_{L^4} + 3\lambda\|u\nabla u\|^2_{L^2} = \frac{1}{4}\frac{d}{dt}\|u(t)\|^4_{L^4} + \frac{3}{4}\lambda\|\nabla(u^2)\|^2_{L^2} \\
&= \int_\Omega \left[au^5 - bu^6 + u^3(v - w - J)\right] dx \\
&\le \int_\Omega \left[\left(\frac{4^5 a^6}{b^5} + \frac{1}{4}bu^6\right) - bu^6 + \left(\frac{1}{4}bu^6 + \frac{4^5}{b^5}(v - w + J)^2\right)\right] dx \\
&\le C_3(a^6 + J^2)|\Omega| - \frac{1}{2}\int_\Omega bu^6\, dx + C_3\int_\Omega (v^2(t,x) + w^2(t,x))\, dx \\
&\le C_3(a^6 + J^2)|\Omega| + C_3(\|v(t)\|^2 + \|w(t)\|^2) - \frac{1}{2}\int_\Omega bu^6\, dx \\
&\le C_3(a^6 + J^2)|\Omega| + C_3 K - \frac{1}{2}\int_\Omega bu^6\, dx, \quad t \ge T_B + 1, \quad g_0 \in B,
\end{aligned} \tag{2.13}
$$

where $C_3(b) = 3(4/b)^5$ is a constants depending on the parameter b, and $T_B > 0$ is shown in (2.11) for any given bounded set B in the space H. Again by Young's inequality we have

$$
\frac{1}{2}\int_\Omega u^4\, dx \le \frac{1}{2}\left(\frac{2}{3}\int_\Omega bu^6\, fx) + \frac{1}{3}\int_\Omega \frac{1}{b^2}\, dx\right) \le \frac{1}{2}\int_\Omega bu^6\, dx + \frac{1}{2b^2}|\Omega|.
$$

Then (2.13) with the above inequality implies that

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\|u(t)\|^4_{L^4} + \frac{1}{2}\|u(t)\|^4_{L^4} \le \frac{1}{2}\frac{d}{dt}\|u(t)\|^4_{L^4} + \frac{1}{2}\int_\Omega bu^6\, dx + \frac{1}{2b^2}|\Omega| \\
&\quad \le C_3(a^6 + J^2)|\Omega| + C_3 K + \frac{1}{2b^2}|\Omega|, \quad t > T_B + 1, \ g_0 \in B,
\end{aligned} \tag{2.14}
$$

so that

$$
\frac{d}{dt}\|u(t)\|^4_{L^4} + 2\|u(t)\|^4_{L^4} < 2Q, \quad t > T_B + 1, \ g_0 \in B, \tag{2.15}
$$

for any bounded set $B \subset H$, where the constant

$$
Q = 2C_3(a^6 + J^2)|\Omega| + 2C_3 K + \frac{1}{b^2}|\Omega| + 1.
$$

Finally, we can apply the Gronwall inequality to (2.15) to conclude that

$$
\|u(t)\|^4_{L^4} < e^{-2(t-1)}\|u(1)\|^4_{L^4} + Q, \quad t > T_B + 1. \tag{2.16}
$$

It shows that (2.12) is valid when time $t \to \infty$ in (2.16). □

3. Kolmogorov-Riesz Compactness and Global Attractor

In this section, we show that the partly diffusive Hindmarsh-Rose semiflow $\{S(t)\}_{t\ge 0}$ is asymptotically compact and then reach the main result on the existence of a global attractor for this dynamical system. Our new approach for this partly diffusive system is based on the Kolmogorov-Riesz compactness Theorem shown in [**10**, Theorem 5] and stated below.

Lemma 3.1. *Let $1 \le p < \infty$ and Ω be a bounded domain with locally Lipschitz boundary in $\mathbb{R}^n$. A subset $\mathcal{F}$ in the integrable space $L^p(\Omega)$ is precompact if and only if the following two conditions are satisfied*:

1) *$\mathcal{F}$ is a bounded set in $L^p(\Omega)$.*

2) *For every $\varepsilon > 0$, there is $\eta > 0$ such that, for all $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$ with $|y| < \eta$,*

$$\int_\Omega |f(x+y) - f(x)|^p \, dx < \varepsilon^p.$$

It is a convention that $f(x) = 0$ for $x \in \mathbb{R}^n \backslash \Omega$.

Since Theorem 2.1 shows that there exists an absorbing set B^* in the space H for this partly diffusive Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$, according to Proposition 1.6, in order to prove the existence of global attractor it suffices to show that this semiflow is asymptotically compact in the space H. We now prove it by showing the ultimate compactness for all the three components of the solutions.

Lemma 3.2. *For the partly diffusive Hindmarsh-Rose semiflow, its u-component has the ultimate compactness property that, for any given bounded set $B \subset H$,*

$$\bigcup_{t \geq T_B + 1} \left(\bigcup_{g_0 \in B} u(t, \cdot) \right) \text{ is a precompact set in } L^2(\Omega) \text{ and in } L^3(\Omega), \tag{3.1}$$

where T_B is shown in (2.11).

Proof. The Laplacian operator $\lambda\Delta$ with the Neumann boundary condition (1.5) generates a parabolic semigroup $e^{\lambda\Delta t}, t \geq 0$. The u-component of the solutions to the equation (1.1) can be expressed by

$$u(t) = e^{\lambda\Delta t} u_0 + \int_0^t e^{\lambda\Delta(t-s)} (\varphi(u) + v - w + J) \, ds, \quad t \geq 0. \tag{3.2}$$

For $1 \leq p < q$, the $L^p(\Omega) \to L^q(\Omega)$ regularity of a parabolic semigroup [**21**, Theorem 38.10] indicates that, for space dimension $n \leq 3$,

$$\|e^{\lambda\Delta t} u_0\|_{L^q} \leq c(p, q)\, t^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \|u_0\|_{L^p}, \quad t > 0. \tag{3.3}$$

Here let $p = 2$ and $q = 4$ in (3.3). We have

$$\|e^{\lambda\Delta t} u_0\|_{L^4} \leq \widetilde{C}\, t^{-\frac{3}{8}} \|u_0\|_{L^2}, \quad t > 0.$$

where $\widetilde{C}$ is a positive constant depending only on Ω. Then we see

$$\|u(t)\|_{L^4} = \frac{\widetilde{C}}{t^{3/8}} \|u_0\| + \int_0^t \|e^{\lambda\Delta(t-s)} (\varphi(u(s)) + v(s) - w(s) + J)\|_{L^4} \, ds, \tag{3.4}$$

where $\varphi(u) = au^2 - bu^3$. From (2.6) we have

$$\begin{aligned} &\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + \lambda C_1 \|\nabla u(t)\|^2 \\ &+ r_1(C_1\|u(t)\|^2 + \|v(t)\|^2 + r\|w(t)\|^2) \leq (2C_2 + C_1^2)|\Omega|, \;\; t \geq 0. \end{aligned}$$

Note that $r_1 = \frac{1}{2}\min\{1, r\}$. Then it holds that

$$\begin{aligned} &\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + \min\{\lambda, r_1\} C_1 (\|\nabla u(t)\|^2 + \|u(t)\|^2) \\ &\qquad + r_1(\|v(t)\|^2 + \|w(t)\|^2) \leq (2C_2 + C_1^2)|\Omega|, \quad t \geq 0. \end{aligned} \tag{3.5}$$

Integrate (3.5) over the time interval $[0, t]$. We get

$$
\begin{aligned}
&\min\{\lambda, r_1\} \min\{C_1, 1\} \int_0^t (\|u(s)\|^2_{H^1(\Omega)} + \|v(s)\|^2 + \|w(s)\|^2)\, ds \\
&\le \int_0^t \left(\min\{\lambda, r_1\} C_1 (\|\nabla u(t)\|^2 + \|u(t)\|^2) + r_1(\|v(s)\|^2 + \|w(s)\|^2)\right) ds \\
&\le \max\{C_1, 1\} \|(u_0, v_0, w_0)\|^2 + t(2C_2 + C_1^2)|\Omega| \\
&= \max\{C_1, 1\} \|g_0\|^2 + t(2C_2 + C_1^2)|\Omega|, \quad t \ge 0.
\end{aligned} \tag{3.6}
$$

Since $H^1(\Omega)$ is continuously embedded in $L^4(\Omega)$ also in $L^6(\Omega)$, it follows from (3.4) and (3.6) that

(3.7)

$$
\begin{aligned}
&\|u(t)\|_{L^4} \le \frac{\widetilde{C}}{t^{3/8}} \|u_0\| + \int_0^t \|e^{d_1 \Delta (t-s)}\|_{\mathcal{L}(L^2, L^4)} \|au^2 - bu^3 + v - w + J\|\, ds \\
&\le \frac{\widetilde{C}\|u_0\|}{t^{3/8}} + \int_0^t \frac{\widetilde{C}}{(t-s)^{3/8}} \left[\|au^2(s) - bu^3(s) + v(s) - w(s)\| + J|\Omega|^{1/2}\right] ds \\
&\le \frac{\widetilde{C}\|u_0\|}{t^{3/8}} + \int_0^t \left[\frac{\widetilde{C}^2}{(t-s)^{3/4}} + \|au^2(s) - bu^3(s) + v(s) - w(s)\|^2\right] ds + \frac{8}{5} \widetilde{C} J t^{\frac{5}{8}} |\Omega|^{1/2} \\
&\le \widetilde{C} \left[\frac{\|u_0\|}{t^{3/8}} + 4\widetilde{C} t^{1/4} + 2J t^{\frac{5}{8}} |\Omega|^{1/2}\right] + 2\int_0^t (\|au^2(s) - bu^3(s)\|^2 + \|v(s) - w(s)\|^2)\, ds \\
&\le \widetilde{C} \left[\frac{\|u_0\|}{t^{3/8}} + 4\widetilde{C} t^{1/4} + 2J t^{\frac{5}{8}} |\Omega|^{1/2}\right] + \int_0^t (C_4 \|u(s)\|^2_{H^1(\Omega)} + 4\|v(s)\|^2 + 4\|w(s)\|^2)\, ds \\
&\le \widetilde{C} \left[\frac{\|u_0\|}{t^{3/8}} + 4\widetilde{C} t^{1/4} + 2J t^{\frac{5}{8}} |\Omega|^{1/2}\right] + [C_4 + 4] \int_0^t (\|u(s)\|^2_{H^1} + \|v(s)\|^2 + \|w(s)\|^2) ds \\
&\le \widetilde{C} \left[\frac{\|u_0\|}{t^{3/8}} + 4\widetilde{C} t^{1/4} + 2J t^{\frac{5}{8}} |\Omega|^{1/2}\right] + \frac{[C_4 + 4] \left(\max\{C_1, 1\}\|g_0\|^2 + t(2C_2 + C_1^2)|\Omega|\right)}{\min\{\lambda, r_1\} \min\{C_1, 1\}}
\end{aligned}
$$

for $t > 0$, where Cauchy inequality is used in the fourth and fifth inequalities, $C_4(a, b) > 0$ is constant depending on the structure parameters of the Sobolev embedding $H^1(\Omega)$ in $L^4(\Omega)$ and $L^6(\Omega)$. Take $t = 1$ in (3.7) and we can confirm that for any given bounded set $B \subset H$ and any initial state $g_0 \in B$, the u-component of the solution $g(t, g_0)$ at time $t = 1$ is uniformly bounded in the space $L^4(\Omega)$ and

(3.8)

$$
\|u(1)\|_{L^4} \le \widetilde{C} \left[\|\!|B|\!\|\, + 4\widetilde{C} + 2J|\Omega|^{1/2}\right] + \frac{[C_4 + 4] \left[\max\{C_1, 1\}\|g_0\|^2 + (2C_2 + C_1^2)|\Omega|\right]}{\min\{\lambda, r_1\} \min\{C_1, 1\}}
$$

where $\|\!|B|\!\| = \sup_{g_0 \in B} \|g_0\|$ is the bound of a set $B \subset H$.

The uniform boundedness (3.8) allows us to use the trajectory estimate (2.16) in Theorem 2.2 and it results in

(3.9)

$$
\begin{aligned}
&\|u(t)\|^4_{L^4} \le e^{-2(t-1)} \|u(1)\|^4_{L^4} + Q \\
&\le \left[\widetilde{C} \left[\|\!|B|\!\|\, + 4\widetilde{C} + 2J|\Omega|^{1/2}\right] + \frac{[C_4 + 4] \left[\max\{C_1, 1\}\|g_0\|^2 + (2C_2 + C_1^2)|\Omega|\right]}{\min\{\lambda, r_1\} \min\{C_1, 1\}}\right]^4 + Q
\end{aligned}
$$

for $t \geq T_B + 1$ and for any bounded set $B \subset H$ and any $g_0 \in B$. The inequality (3.9) shows that, for any given bounded set $B \subset H$,

$$\bigcup_{t \geq T_B+1} \left(\bigcup_{g_0 \in B} u(t, \cdot) \right) \quad \text{is a bounded set in } L^4(\Omega).$$

Therefore, by the compact embeddings $L^4(\Omega) \hookrightarrow L^2(\Omega)$ and $L^4(\Omega) \hookrightarrow L^3\Omega)$, we can conclude that

$$\bigcup_{t \geq T_B+1} \left(\bigcup_{g_0 \in B} u(t, \cdot) \right) \text{ is a precompact set in } L^2(\Omega) \text{ and in } L^3(\Omega). \tag{3.10}$$

The proof is completed. □

Next we treat the other two v-component and w-component of the solution $g(t, g_0)$ to (1.7). These two components satisfy the ordinary differential equations (1.2) and (1.3) respectively.

Lemma 3.3. *For the partly diffusive Hindmarsh-Rose semiflow, its v-component and w-component also satisfy the ultimate compactness property that, for any given bounded set $B \subset H$, there exist finite times $T_B^1 > 0$ and $T_B^2 > 0$ such that*

$$\bigcup_{t \geq T_B^1} \left(\bigcup_{g_0 \in B} v(t, \cdot) \right) \textit{ is a precompact set in } L^2(\Omega), \tag{3.11}$$

$$\bigcup_{t \geq T_B^2} \left(\bigcup_{g_0 \in B} w(t, \cdot) \right) \textit{ is a precompact set in } L^2(\Omega). \tag{3.12}$$

Proof. By the variation-of-constant formula for the solutions of ordinary differential equations and the form in (1.8), we have

$$\begin{aligned} v(t,x) &= e^{-t} v_0(x) + \int_0^t e^{-(t-s)} (\alpha - \beta u^2) \, ds \\ &\leq \alpha + e^{-t} v_0(x) - \beta \int_0^t e^{-(t-s)} u^2(s,x) \, ds, \\ w(t,x) &= e^{-r\,t} w_0(x) + \int_0^t e^{-r(t-s)} q(u-c) \, ds \\ &\leq \frac{q|c|}{r} + e^{-rt} w_0(x) + q \int_0^t e^{-r(t-s)} u(s,x) \, ds. \end{aligned} \tag{3.13}$$

By Lemma 3.1 (Kolmogorov-Riesz compactness Theorem) and (3.10), for any $\varepsilon > 0$, there exists $\eta > 0$ such that, for any given bounded set $B \subset H$ and all $g_0 \in B$, and for $y \in \mathbb{R}^3$ with $|y| < \eta$, it holds that

$$\int_\Omega |u(t, x+y) - u(t,x)|^3 \, dx < \varepsilon^3, \quad \text{for all } t \geq T_B + 1. \tag{3.14}$$

Consider the v-equation in (3.13) on the time interval $t \in [T_B + 1, \infty)$. Using the Hölder inequality we can infer that, for any $t \geq T_B + 1$ and any $g_0 \in B$,

(3.15)

$$\begin{aligned}
&\int_\Omega |v(t, x+y) - v(t,x)|^2 dx = e^{-(t-T_B-1)} \int_\Omega |v(T_B+1, x+y) - v(T_B+1, x)|^2 \, dx \\
&\quad + \beta \int_{T_B+1}^t e^{-(t-s)} \int_\Omega |u^2(s, x+y) - u^2(s,x)|^2 \, dx \, ds \leq 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 \\
&\quad + \beta \int_{T_B+1}^t e^{-(t-s)} \int_\Omega |u(s,x+y) - u(s,x)|^2 |u(s,x+y) + u(s,x)|^2 \, dx \, ds \\
&\leq 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 \\
&\quad + \beta \int_{T_B+1}^t e^{-(t-s)} \|(u(s, \cdot + y) - u(s, \cdot))^2\|_{L^{3/2}} \|(u(s, \cdot+y) + u(s,\cdot))^2\|_{L^3} \, ds \\
&\leq 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 \\
&\quad + 24\beta \int_{T_B+1}^t e^{-(t-s)} \|u(s, \cdot+y) - u(s,\cdot)\|_{L^3}^2 \left(\|u(s,\cdot+y)\|_{L^6}^2 + \|u(s,\cdot)\|_{L^6}^2 \right) ds \\
&\leq 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 \\
&\quad + 48\beta \int_{T_B+1}^t e^{-(t-s)} \|u(s,\cdot+y) - u(t,\cdot)\|_{L^3}^2 \|u(s,\cdot)\|_{L^6}^2 \, ds \\
&\leq 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 \\
&\quad + 48\beta \int_{T_B+1}^t e^{-(t-s)} \|u(s,\cdot+y) - u(t,\cdot)\|_{L^3}^2 (\|u(s,\cdot)\|_{L^6}^6 + 1) \, ds,
\end{aligned}$$

in which we used

$$\begin{gathered}
\|(u(s,\cdot+y) - u(s,\cdot))^2\|_{L^{3/2}} = \|u(s,\cdot+y) - u(s,\cdot)\|_{L^3}^2, \\
\|(u(s,\cdot+y) + u(s,\cdot))^2\|_{L^3} \leq \|2u^2(s,\cdot+y) + 2u^2(s,\cdot)\|_{L^3} \leq 24 \|u(s,\cdot)\|_{L^6}^2,
\end{gathered}$$

and we also used Young's inequality in the last step of (3.15),

$$\|u(s,\cdot)\|_{L^6}^2 = \left(\int_\Omega u^6(s,x)\, dx \right)^{1/3} \leq \frac{1}{3} \|u(s,\cdot)\|_{L^6}^6 + \frac{2}{3} \leq \|u(s,\cdot)\|_{L^6}^6 + 1.$$

On the other hand, integrating the second differential inequality of (2.14) together with (3.9) yields the following important bound that, for $t \geq T_B + 1$,

(3.16)

$$\begin{aligned}
&\int_{T_B+1}^t e^{-(t-s)} \int_\Omega u^6(s,x)\, dx\, ds \leq \frac{1}{b} \left(\|u(T_B+1)\|_{L^4}^4 + Q \right) \leq \frac{2Q}{b} + \\
&+ \frac{1}{b} \left[\widetilde{C} \left[|||B\,||| + 4\widetilde{C} + 2J|\Omega|^{1/2} \right] + \frac{[C_4 + 4] \left[\max\{C_1, 1\} \|g_0\|^2 + (2C_2 + C_1^2)|\Omega| \right]}{\min\{\lambda, r_1\} \min\{C_1, 1\}} \right]^4 .
\end{aligned}$$

Put (3.14) and (3.16) into the last step of (3.15). Then for any $\varepsilon > 0$, there exists $\eta > 0$ such that, for any given bounded set $B \subset H$ and all $g_0 \in B$, and for $y \in \mathbb{R}^3$

with $|y| < \eta$, we have

(3.17)
$$\begin{aligned}&\int_\Omega |v(t,x+y)-v(t,x)|^2 dx \\ &\le 2\,e^{-(t-T_B-1)}\|v(T_B+1)\|^2 + 48\,\beta \int_{T_B+1}^t e^{-(t-s)}\,\varepsilon^2 \left(\int_\Omega u(s,s)^6\,dx + 1\right) ds \\ &= 2\,e^{-(t-T_B-1)}\|v(T_B+1)\|^2 + 48\,\beta\,\varepsilon^2\left(\frac{2Q}{b} + R + 1\right), \quad t \ge T_B + 1,\ g_0 \in B,\end{aligned}$$

where the constant R is given by

(3.18)
$$R = \frac{1}{b}\left[\widetilde{C}\left[|||B\,|||\, + 4\widetilde{C} + 2J|\Omega|^{1/2}\right] + \frac{[C_4+4]\left[\max\{C_1,1\}\|g_0\|^2 + (2C_2+C_1^2)|\Omega|\right]}{\min\{\lambda, r_1\}\min\{C_1,1\}}\right]^4.$$

Moreover, there exists a time

$$T_B^1 = T_B + 1 + \log^+\left(\frac{2K}{\varepsilon^2}\right)$$

such that

$$2\,e^{-(t-T_B-1)}\|v(T_B+1)\|^2 < \varepsilon^2, \quad \text{for } t \ge T_B^1, \tag{3.19}$$

where K is given in (2.9) and (2.10). It follows from (3.17) and (3.19) that

(3.20)
$$\int_\Omega |v(t,x+y)-v(t,x)|^2 dx < \left[1 + 48\,\beta\left(\frac{2Q}{b} + R + 1\right)\right]\varepsilon^2, \quad t \ge T_B^1, \ \ g_0 \in B.$$

Since $\varepsilon > 0$ is arbitrary, according to Lemma 3.1, the estimate (3.20) ensures that

$$\bigcup_{t\,\ge\, T_B^1}\left(\bigcup_{g_0\in B} v(t,\cdot)\right) \text{ is a precompact set in } L^2(\Omega). \tag{3.21}$$

Similarly, by Lemma 3.1 and (3.10), for any $\varepsilon > 0$, there exists $\eta > 0$ such that, for any given bounded set $B \subset H$ and all $g_0 \in B$, and for $y \in \mathbb{R}^3$ with $|y| < \eta$, it holds that

$$\int_\Omega |u(t,x+y)-u(t,x)|^2\,dx < \varepsilon^2, \quad \text{for all } t \ge T_B + 1.$$

Then we can show that, for the w-component of any solution $g(t,g_0)$ of (1.7) with $g_0 \in B$, the following estimate is valid,

$$\begin{aligned}&\int_\Omega |w(t,x+y)-w(t,x)|^2 dx \le 2\,e^{-r(t-T_B-1)}\|w(T_B+1)\|^2 \\ &+ q\int_{T_B+1}^t e^{-r(t-s)}\int_\Omega |u(s,x+y)-u(s,x)|^2 dx\,ds < \left(1+\frac{q}{r}\right)\varepsilon^2, \quad t \ge T_B^2,\end{aligned} \tag{3.22}$$

where the time

$$T_B^2 = T_B + 1 + \frac{1}{r}\log^+\left(\frac{2K}{\varepsilon^2}\right).$$

According to Lemma 3.1, the estimate (3.22) shows that

$$\bigcup_{t \geq T_B^2} \left(\bigcup_{g_0 \in B} w(t, \cdot) \right) \text{ is precompact in } L^2(\Omega). \tag{3.23}$$

Thus (3.11) and (3.12) are proved by (3.21) and (3.23). The proof is completed. □

Now we reach the proof of the main result stated in the following theorem.

Theorem 3.4. *There exists a global attractor $\mathscr{A}$ in the space $H = L^2(\Omega, \mathbb{R}^3)$ for the partly diffusive Hindmarsh-Rose semiflow generated by the system of equations (1.1)-(1.3).*

Proof. It is proved in Theorem 2.1 that there exists an absorbing set B^* in the space H for this partly diffusive Hindmarsh-Rose semiflow. Lemma (3.2) and Lemma 3.3 demonstrate

$$\bigcup_{t \geq \max\{T_B+1, T_B^1, T_B^2\}} \left(\bigcup_{g_0 \in B} g(t, \cdot) \right) \text{ is a precompact set in } H. \tag{3.24}$$

Therefore, this semiflow generated by the system (1.1)-(1.3) is asymptotically compact on the space H, according to Definition 1.4. By Proposition 1.6, there exists a global attractor $\mathscr{A}$ in the space $H = L^2(\Omega, \mathbb{R}^3)$ for the partly diffusive Hindmarsh-Rose equations. □

As a remark, the featuring approach by means of the Kolmogorov-Riesz compactness Theorem in this work can be exploited to effectively prove the ultimate and asymptotic compactness for the solution semiflows associated with dissipative partly diffusive reaction-diffusion systems.

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