Dynamics of PDE, Vol.20, No.2, 117-133, 2023

Global strong solution and exponential decay to the 3D incompressible Bénard system with density-dependent viscosity and vacuum

Min Liu, Yong Li

Communicated by Dong Li, received January 12, 2023.

Abstract. In this paper, we study the Cauchy problem of the incompressible Bénard system with density-dependent viscosity on the whole three-dimensional space. We first construct a key priori exponential estimates by the energy method, and then we prove that there is a unique global strong solution for the 3D Cauchy problem under the assumption that initial energy is suitably small. In particular, it is not required to be smallness condition for the initial density which contains vacuum and even has compact support. Finally, we obtain the exponential decay rates for the gradients of velocity, temperature field and pressure.

Contents

1. Introduction 118
2. Preliminaries 121
3. Convergence rate of the solution. 122
References 132

1. Introduction

We consider the 3D nonhomogeneous incompressible Bénard system with density-dependent viscosity as follows:

$$
\begin{cases}
\rho_t + \operatorname{div}(\rho u) = 0,\\
(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p - \operatorname{div}(\mu(\rho)\nabla u) = \rho\theta e_3,\\
(\rho\theta)_t + \operatorname{div}(\rho u\theta) - \kappa \triangle \theta = \rho u \cdot e_3,\\
\operatorname{div} u = 0,
\end{cases} \tag{1.1}
$$

2020 *Mathematics Subject Classification.* Primary 37, 76, 92, 70; Secondary 34, 35, 82, 80.
Key words and phrases. Bénard system, global well-posedness, density-dependent, exponential decay rates, vacuum.

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the spatial coordinate, $t \geq 0$ is time, $\rho = \rho(x,t)$, $u = (u^1, u^2, u^3)(x,t)$ are the density and velocity, respectively; $\theta = \theta(x,t)$ stands for the temperature of the fluid, and $p = p(x,t)$ denotes the pressure; $\mu(\rho) > 0$ is the viscosity coefficient which depends on the density; the constant $\kappa > 0$ represents the heat conductivity coefficient. $e^3 = (0,0,1)^t$ is the write vertical vector. The viscosity coefficient $\mu(\rho)$ satisfies the following hypothesis:

$$\mu(\rho) \in C^1[0,\infty) \quad and \quad 0 < \underline{\mu} \leq \mu(\rho) \leq \bar{\mu} < \infty, \quad \forall \rho \in [0,\infty). \tag{1.2}$$

The initial data and far field conditions are given by

$$\begin{cases} \rho(x,0) = \rho_0(x), \rho u(x,0) = \rho_0 u_0(x), \rho\theta(x,0) = \rho\theta_0(x), & x \in \mathbb{R}^3, \\ (\rho, u, \theta)(x,t) \to 0, \quad as \quad |x| \to \infty. \end{cases} \tag{1.3}$$

The equations (1.1) is called Bénard system, which describes the Rayleigh-Bénard convective motion in a heated fluid [**1, 2**]. In fact, Bénard [**3**] first proposed the Bénard convection problem in experimental observations. Later on, many scholars modified and innovated it, which can be refree [**4, 5, 6, 7, 8**]. When ρ is a constant, the system (1.1) is homogeneous viscid fluid and has been considered extensively, if we ignore the thermal effects and remove the vertical disturbed velocity component $u \cdot e_3$, the Bénard system becomes the Boussinesq equations. If the initial data (u_0, θ_0) has finite energy, bounded vorticity and the initial temperature $\theta_0 \in L^2 \cap B^{-1}_{\infty,1}$, Danchin and Paicu [**9**] proved that the inviscid Bénard system has a global unique solution for Yudovich's type data. The authors also showed the global existence in the case of infinite energy velocity field which admits the vortex-patches-like structures and achieve another further improvement to the less regular data as a extension of the important work of Yudovich [**10**]. Wu-Xue [**11**] obtained the global unique solution for Yudovich's type data for the 2D inviscid Bénard system with fractional diffusivity. Hmidi-Keraani-Rousset [**12**] established the global well-posedness with the dissipative power by deeply developing the structures of the coupling system. motivated by this, Ye [**13**] showed the global-in-time existence of smooth solutions to the 2D Bénard equations with critical dissipation. For the dynamics problem of a Boussinesq approximated Bénard convection fluid evolving in 3D, Guo-Xie-Zeng [**14**] proved the exponential decay of solutions in the framework of high regularity by applying a flattening coordinate method. However, when we consider the density-dependent viscosity and the initial density allowing vacuum states, it seems bring some difficulties that the strong coupling between velocity and temperature under vacuum.

Recently, more attentions have attracted by density is not a constant. When the initial density allowing vacuum states and the thermal diffusivity tends to zero, Ye-Zhu [**15**] proved the global existence and the asymptotic behavior of the solutions for the initial boundary problem. Fan-Sun-Tang [**16**] showed the global strong solutions for 2D initial boundary value problem by using the bootstrap argument. Zhong [**17**] investigated the global existence and uniqueness of strong solutions to the 2D initial boundary value problem with general initial data including vacuum. Zhong [**18**] established a unique global strong solution for nonhomogeneous Bénard system with zero density at infinity and large initial data. While assuming that some small conditions hold true, Zhong [**19**] studied the global existence and exponential decay of strong solutions to the Cauchy problem of nonhomogeneous Bénard system in $\mathbb{R}^3$. However, the viscosity coefficient is independent of density in those above results.

Recently, when the viscosity coefficient $\mu(\rho)$ depends on the density ρ, He-Li-Lü [**20**] obtained the global existence and exponential stability of strong solutions in unbounded domains while the initial velocity $\|u_0\|_{\dot{H}^\beta}(\beta \in (\frac{1}{2}, 1])$ is suitably small. Our purpose is to study the global existence and uniqueness of strong solutions of 3D density-dependent viscosity Bénard system with vacuum. There is no need any smallness conditions on the initial density and here we only need the initial energy for the gradients of velocity and temperature are suitably small. Meanwhile, we also obtain the exponential decay rates of the spatial gradients for the velocity, temperature and the pressure.

We give some notations and conventions employed throughout the paper. For $R > 0$, set

$$B_R \triangleq \{x \in \mathbb{R}^3 \mid |x| < R\}, \qquad \int f dx \triangleq \int_{\mathbb{R}^3} f dx.$$

Moreover, for $1 \leq r < \infty$, $k \geq 1$, we denote the standard and Sobolve spaces as follows:

$$L^r = L^r(\mathbb{R}^3), \qquad W^{k,r} = W^{k,r}(\mathbb{R}^3), \qquad H^r = W^{k,2}.$$

Our main result stated as follows:

THEOREM 1.1. *Suppose the initial data* (ρ_0, u_0, θ_0) *satisfy that for any given numbers* $\bar{\rho} > 0$, $q \in (3, \infty)$, *and* $\beta \in (\frac{1}{2}, 1]$

$$\begin{cases} 0 \leq \rho_0 \leq \bar{\rho}, \rho_0 \in L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}, \nabla\mu(\rho_0) \in L^q, \\ u_0 \in H^1_{0,\sigma}, \theta_0 \in H^1_{0,\sigma}, \\ \mathrm{div} u_0 = 0; \end{cases} \tag{1.4}$$

there exists small positive constant ε_0 *depending only on* q, β, $\underline{\mu}$, $\bar{\mu}$, κ, $\|\rho_0\|_{L^{\frac{3}{2}}}$, $\bar{\rho}$, *and* $M \triangleq \|\nabla\mu(\rho_0)\|_{L^q}$ *such that if*

$$\|\nabla u_0\|^2_{L^2} + \|\nabla\theta_0\|^2_{L^2} \leq \varepsilon_0. \tag{1.5}$$

Then the problem $(1.1) - (1.3)$ *exists unique global strong solutions* (ρ, u, p, θ) *such that for any* $0 < \tau < T < \infty$ *and* $p \in [2, r_0)$ *with* $r_0 \triangleq \min\{6, q\}$,

$$\begin{cases} 0 \leq \rho \in C([0,T]; L^{\frac{3}{2}} \cap H^1), \nabla\mu(\rho) \in C([0,T]; L^q) \\ \nabla u \in L^\infty(0,T; L^2) \cap L^\infty(\tau,T; W^{1,r_0}) \cap C([\tau,T]; H^1 \cap W^{1,p}), \\ \nabla\theta \in L^\infty(0,T; L^2) \cap L^\infty(\tau,T; W^{1,r_0}) \cap C([\tau,T]; H^1 \cap W^{1,p}), \\ \nabla p \in L^\infty(\tau,T; L^{r_0}) \cap C([\tau,T]; L^p), \\ \nabla u_t \in L^\infty(\tau,T; L^2) \cap L^2(\tau,T; L^{r_0}), \\ \nabla\theta_t \in L^\infty(\tau,T; L^2) \cap L^2(\tau,T; L^{r_0}), \\ \sqrt{\rho}u_t, \sqrt{\rho}\theta_t \in L^2(0,T; L^2) \cap L^\infty(\tau,T; L^2). \end{cases} \tag{1.6}$$

Moreover, it holds that

$$\sup_{0\leq t\leq\infty} \|\nabla\rho\|_{L^2} \leq 2\|\nabla\rho_0\|_{L^2}, \qquad \sup_{0\leq t\leq\infty} \|\nabla\mu(\rho)\|_{L^q} \leq 2\|\nabla\mu(\rho_0)\|_{L^q}, \tag{1.7}$$

and for positive constant σ *depending only* $\|\rho_0\|_{L^{\frac{3}{2}}}$, $\underline{\mu}$, *and* κ, *such that, for all* $t \geq 1$,

$$\begin{cases} \|\nabla u(\cdot,t)\|_{H^1\cap W^{1,r_0}} + \|\nabla u_t(\cdot,t)\|_{L^2}^2 + \|p(\cdot,t)\|_{H^1\cap W^{1,r_0}} \leq Ce^{-\sigma t}, \\ \|\nabla\theta(\cdot,t)\|_{H^1\cap W^{1,r_0}} + \|\nabla\theta_t(\cdot,t)\|_{L^2}^2 \leq Ce^{-\sigma t}, \end{cases} \tag{1.8}$$

where C *depends only on* q, β, $\underline{\mu}$, $\bar{\mu}$, κ, M, $\bar{\rho}$, $\|\rho_0\|_{L^{\frac{3}{2}}}$, $\|\nabla u_0\|_{L^2}$, *and* $\|\nabla\theta_0\|_{L^2}$.

REMARK 1.2. It is all considered the viscosity coefficient is independent of density in [**18, 19**], motivated by [**20**], we improve their results and study the global existence and uniqueness of strong solutions of nonhomogeneous Bénard system with the viscosity coefficient $\mu(\rho)$ depends on the density ρ.

REMARK 1.3. There is only the condition (1.5) and without other additional small initial conditions in Theorem 1.1. Meanwhile, we also obtain the exponential decay-in-time properties (1.8) to the Cauchy problem.

According to Lemma 2.1, in order to prove Theorem 1.1, one needs to get global a priori estimates on strong solution to (1.1)-(1.3) in proper higher norms. Firstly, the key ingredient here is to get the time-independent $L^1(0,T;L^\infty(\mathbb{R}^3))$-norm of ∇u. we derive that the bound on $L^2(0,T;L^2(\mathbb{R}^3))$-norm of $e^{\frac{\sigma t}{2}}\nabla u$ and $L^2(0,T;L^2(\mathbb{R}^3))$-norm of $e^{\frac{\sigma t}{2}}\nabla\theta$ by applying the upper bounds on the density (3.3) and the Sobolev's inequality. In addition, we need to define a function $\zeta(t) \triangleq \min\{1,t\}$ to get the estimates on $L^\infty(0,\zeta(T);L^2(\mathbb{R}^3))-$norm of $t^{\frac{1-\beta}{2}}\sqrt{\rho}u_t$ and $L^\infty(\zeta(T),T;L^2(\mathbb{R}^3))-$norm of $e^{\frac{1}{2}\sigma t}\sqrt{\rho}u_t$, which avoids the singularity of $\|\sqrt{\rho}u_t\|_{L^2}^2$ at $t=0$ and the same as $\|\sqrt{\rho}\theta_t\|_{L^2}^2$. And then the most important thing is to estimate $H^1(\mathbb{R}^3)$)-norm of ∇u by Lemma 2.3 (see (3.15)) and $H^1(\mathbb{R}^3)$)-norm of $\nabla\theta$ by L^2-theory of the elliptic equation (see (3.20)), which solves the strong coupling of $u\cdot\nabla u$ and $u\cdot\nabla\theta$ caused by density. And further we get the estimates on $L^\infty(0,\zeta(T);L^2(\mathbb{R}^3))-$norm of $t^{\frac{2-\beta}{2}}\sqrt{\rho}u_t$ and $L^\infty(0,\zeta(T);L^2(\mathbb{R}^3))-$norm of $t^{\frac{2-\beta}{2}}\sqrt{\rho}\theta_t$, which make us succeed in bounding the $L^1(0,T;L^\infty(\mathbb{R}^3))$-norm of ∇u (see (3.57)). Then, with a priori estimates stated above, we are in a position to prove Proposition 3.1. Finally, we can extend the local solution to all time, and meanwhile obtain the exponential decay rates for the gradients of velocity, temperature field and pressure.

Next, we shall first state some known inequalities, which will be employed later in this paper. In the last section, we will give some priori estimates and prove the Theorem 1.1.

2. Preliminaries

We will recall some known facts and elementary inequalities which will be used frequently later. The following local existence of strong solutions whose proof can be found in [**26**].

LEMMA 2.1. *Assume that* (ρ_0, u_0, θ_0) *satisfies* (1.4). *Then there exists a small time* $T > 0$ *and a unique strong solution* (ρ, u, p, θ) *to the problem* (1.1) − (1.3) *in* $\mathbb{R}^3 \times (0,T)$ *satisfying* (1.6).

LEMMA 2.2. *(see* [**27**]*) (Gagliardo-Nirenberg). Let* v *belongs to* $L^q(\mathbb{R}^n)$, *and its derivatives of order* m, $\nabla^m v$, *belong to* $L^r(\mathbb{R}^n)$, $1 \leq q, r \leq \infty$. *Then for the* *derivatives $\nabla^j v$, $0 \le j < m$, we have*

$$\|\nabla^j v\|_{L^q(\mathbb{R}^n)} \le \tilde{C}\|\nabla^m v\|^{\alpha}_{L^r(\mathbb{R}^n)}\|v\|^{1-\alpha}_{L^q(\mathbb{R}^n)}, \tag{2.1}$$

where

$$\tfrac{1}{p} = \tfrac{j}{n} + \alpha(\tfrac{1}{r} - \tfrac{m}{n}) + (1-\alpha)\tfrac{1}{q},$$

for all a in the interval

$$\tfrac{j}{m} \le \alpha \le 1,$$

and the constant $\tilde{C}$ depends only on n, m, j, q, r, α.

To derive the estimates of the solutions, we need the following regularity on the Stokes equations, which plays a key role in the following priori estimates of (ρ, u, p, θ), we can also refer to ([**20**], Lemma 2.4).

Lemma 2.3. *For positive constants $\underline{\mu}$, $\bar{\mu}$ and $q \in (3,\infty)$, in addition to* (1.2)*. Then, if $F \in L^{\frac{6}{5}} \cap L^r$ with $r \in [\frac{2q}{q+2}, q]$, there exists a positive constant C depending only on $\underline{\mu}$, $\bar{\mu}$, r and q, such that the unique weak solution $(u,p) \in D^1_{0,\sigma} \times L^2$ satisfies the following Cauchy problem*

$$\begin{cases} -\operatorname{div}(2\mu(\rho)\nabla u) + \nabla p = F, & x \in \mathbb{R}^3, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3, \\ u(x) \to 0, & |x| \to \infty, \end{cases} \tag{2.2}$$

where $D^1_{0,\sigma} = \bar{C^{\infty}_{0,\sigma}}$ closure in the norm of D^1 and satisfies

$$\|\nabla u\|_{L^2} + \|p\|_{L^2} \le C\|F\|_{L^{\frac{6}{5}}}, \tag{2.3}$$

$$\|\nabla^2 u\|_{L^r} + \|\nabla p\|_{L^r} \le C\|F\|_{L^r} + C\|\nabla\mu(\rho)\|^{\frac{q(5r-6)}{2r(q-3)}}_{L^q}\|F\|_{L^{\frac{6}{5}}}. \tag{2.4}$$

Moreover, if $F = \operatorname{div} g$ with $g \in L^2 \cap L^{\tilde{r}}$ for some $\tilde{r} \in (\frac{6q}{q+6}, q]$, there exists a positive constant C depending only on $\underline{\mu}$, $\bar{\mu}$, q and $\tilde{r}$ such that the unique weak solution $(u,p) \in D^1_{0,\sigma} \times L^2$ to (2.2) *satisfies*

$$\|\nabla u\|_{L^2\cap L^{\tilde{r}}} + \|p\|_{L^2\cap L^{\tilde{r}}} \le C\|g\|_{L^2\cap L^{\tilde{r}}} + C\|\nabla\mu(\rho)\|^{\frac{3q(\tilde{r}-2)}{2\tilde{r}(q-3)}}_{L^q}\|g\|_{L^2}. \tag{2.5}$$

3. Convergence rate of the solution.

Before proving the result of Theorem 1.1, we aim to get the following a priori estimates on (ρ, u, p, θ).

proposition 3.1. *There exists a positive constant ε_0 depending only on q, β, $\underline{\mu}$, $\bar{\mu}$, κ, $\bar{\rho}$, $\|\rho_0\|_{L^{\frac{3}{2}}}$, and M, such that if (ρ, u, p, θ) is a smooth solution of* (1.1)-(1.3) *on $\mathbb{R}^3 \times (0,T]$ satisfying*

$$\sup_{t\in[0,T]}\|\nabla\mu(\rho)\|^2_{L^q} \le 4M, \quad \int_0^T \|\nabla u\|^4_{L^2} \le 2(\|\nabla u_0\|^2_{L^2} + \|\nabla\theta_0\|^2_{L^2}), \tag{3.1}$$

then the following estimates hold:

$$\sup_{t\in[0,T]}\|\nabla\mu(\rho)\|^2_{L^q} \le 2M, \quad \int_0^T \|\nabla u\|^4_{L^2} \le \|\nabla u_0\|^2_{L^2} + \|\nabla\theta_0\|^2_{L^2}, \tag{3.2}$$

provided that $\|\nabla u_0\|^2_{L^2} + \|\nabla\theta_0\|^2_{L^2} \le \varepsilon_0$.

LEMMA 3.2. *Let (ρ, u, p, θ) be a smooth solution to* (1.1)-(1.3) *satisfying* (3.1). *Then there exists a positive constant C depending only on q, $\underline{\mu}$, $\bar{\mu}$, κ, $\bar{\rho}$, $\|\rho_0\|_{L^{\frac{3}{2}}}$, $\|\nabla u_0\|_{L^2}^2$, and $\|\nabla\theta_0\|_{L^2}^2$ such that*

$$0 \le \rho \le \bar{\rho}, \quad \|\rho\|_{L^{\frac{3}{2}}} = \|\rho_0\|_{L^{\frac{3}{2}}}, \tag{3.3}$$

$$\begin{aligned}&\sup_{t\in[0,T]}(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2) + \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)dt\\ &\le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2),\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\sup_{t\in[0,T]} e^{\sigma t}(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2) + \int_0^T e^{\sigma t}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)dt\\ &\le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2).\end{aligned} \tag{3.5}$$

Proof. (3.3) can be obtained by Theorem 2.1 in [**28**]. Adding $(1.1)_2 \times u$ to $(1.1)_2 \times \theta$ and integrating it over $\mathbb{R}^3$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \rho(|u|^2 + \theta^2)dx + \int(\underline{\mu}|\nabla u|^2 + \kappa|\nabla\theta|^2)dx\\ &\le 2\int \rho\theta u\cdot e_3 dx\\ &\le \|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2.\end{aligned} \tag{3.6}$$

Next, applying Grönwall's inequality to the above inequality and along with Hölder's, Sobolev's inequality, we get

$$\begin{aligned}&\sup_{t\in[0,T]}(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2) + \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)dt\\ &\le \|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\sqrt{\rho_0}\theta_0\|_{L^2}^2\\ &\le \|\rho_0\|_{L^{\frac{3}{2}}}(\|u_0\|_{L^6}^2 + \|\theta_0\|_{L^6}^2)\\ &\le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2).\end{aligned} \tag{3.7}$$

It deduces from (3.3), Hölder's and Sobolev's inequality that

$$\begin{aligned}\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2 &\le \|\rho\|_{L^{\frac{3}{2}}}(\|u\|_{L^6}^2 + \|\theta\|_{L^6}^2)\\ &\le \frac{4}{3}\|\rho_0\|_{L^{\frac{3}{2}}}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)\\ &\le \sigma^{-1}(\underline{\mu}\|\nabla u\|_{L^2}^2 + \kappa\|\nabla\theta\|_{L^2}^2),\end{aligned} \tag{3.8}$$

where $\sigma^{-1} = \max\{\frac{4\|\rho_0\|_{L^{\frac{3}{2}}}}{3\underline{\mu}}, \frac{4\|\rho_0\|_{L^{\frac{3}{2}}}}{3\kappa}\}$. Then combining (3.8) and (3.6), we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2) + \sigma(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\\ &\le 2(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2),\end{aligned} \tag{3.9}$$

which with the help of Grönwall's inequality and (3.4) to yield (3.5). Finally, the proof of Lemma 3.2 is finished. □

LEMMA 3.3. *Let (ρ, u, p, θ) be a smooth solution to (1.1)-(1.3) satisfying (3.1). Then there exists a positive constant C depending only on q, β, $\underline{\mu}$, $\bar{\mu}$, κ, M, $\bar{\rho}$, $\|\rho_0\|_{L^{\frac{3}{2}}}$, $\|\nabla u_0\|_{L^2}^2$, and $\|\nabla\theta_0\|_{L^2}^2$ such that*

$$\sup_{t\in[0,T]} t^{1-\beta}\|\nabla u\|_{L^2}^2 + \int_0^T t^{1-\beta}\|\sqrt{\rho}u_t\|_{L^2}^2 dt \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2), \tag{3.10}$$

$$\sup_{t\in[0,T]} t^{1-\beta}\|\nabla \theta\|_{L^2}^2 + \int_0^T t^{1-\beta}\|\sqrt{\rho}\theta_t\|_{L^2}^2 dt \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2), \tag{3.11}$$

$$\begin{aligned}&\sup_{t\in[0,T]} e^{\sigma t}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \int_0^T e^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2)dt \\ &\le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2).\end{aligned} \tag{3.12}$$

Proof. First, multiplying $(1.1)_2$ by u_t and integrating the resulting equality over $\mathbb{R}^3$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\mu(\rho)\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 \\ &= -\int \rho u\cdot\nabla u\cdot u_t dx - \frac{1}{2}\int \mu(\rho)u\cdot\nabla|\nabla u|^2 dx + \int \rho\theta e_3\cdot u_t dx \\ &\le \bar{\rho}^{\frac{1}{2}}\|u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^3} + \frac{\bar{\mu}}{2}\|u\|_{L^6}\|\nabla u\|_{L^3}\|\nabla^2 u\|_{L^2} \\ &\quad + C\|\sqrt{\rho}\theta\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} \\ &\le C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}} + C\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{3}{2}} \\ &\quad + C\|\sqrt{\rho}\theta\|_{L^2}^2 + \frac{1}{6}\|\sqrt{\rho}u_t\|_{L^2}^2,\end{aligned} \tag{3.13}$$

where Sobolev's and Gagliardo-Nirenberg inequality were used in the last integral. Next, it follows from Lemma 2.3, (3.1), (3.3) and the Gagliardo-Nirenberg inequality that

$$\begin{aligned}\|\nabla u\|_{H^1} + \|p\|_{H^1} &\le C(\|\rho u_t + \rho u\cdot\nabla u + \rho\theta e_3\|_{L^2} + \|\rho u_t + \rho u\cdot\nabla u + \rho\theta e_3\|_{L^{\frac{6}{5}}}) \\ &\le C(\bar{\rho}^{\frac{1}{2}} + \|\rho\|_{L^{\frac{3}{2}}}^{\frac{1}{2}})(\|\sqrt{\rho}u_t\|_{L^2} + \bar{\rho}^{\frac{1}{2}}\|u\cdot\nabla u\|_{L^2} + \|\sqrt{\rho}\theta\|_{L^2}) \\ &\le C\|\sqrt{\rho}u_t\|_{L^2} + C\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2} + C\|\sqrt{\rho}\theta\|_{L^2} + \frac{1}{2}\|\nabla^2 u\|_{L^2}.\end{aligned} \tag{3.14}$$

It can be directly obtained

$$\|\nabla u\|_{H^1} + \|p\|_{H^1} \le C\|\sqrt{\rho}u_t\|_{L^2} + C\|\sqrt{\rho}\theta\|_{L^2} + C\|\nabla u\|_{L^2}^3. \tag{3.15}$$

Combining with (3.13) and Young's inequality, we get

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^4\|\nabla u\|_{L^2}^2 + C\|\sqrt{\rho}\theta\|_{L^2}^2. \tag{3.16}$$

Then, with the help of Grönwall's inequality, (3.1), and (3.4), we have

$$\sup_{t\in[0,T]} \|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2 dt \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2). \tag{3.17}$$

For any $\beta \in (\frac{1}{2}, 1)$, multiplying (3.16) by $t^{1-\beta}$ together with the Grönwall's inequality, (3.1), and (3.17), we obtain the following estimate

$$
\begin{aligned}
&\sup_{t\in[0,T]} t^{1-\beta}\|\nabla u\|_{L^2}^2 + \int_0^T t^{1-\beta}\|\sqrt{\rho}u_t\|_{L^2}^2 dt \\
&\quad \le C \sup_{t\in[0,T]} \|\nabla u\|_{L^2}^2 \int_0^T t^{-\beta} dt \\
&\quad \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla \theta_0\|_{L^2}^2),
\end{aligned} \tag{3.18}
$$

which combining with (3.17) it obtains (3.10). Next, multiplying $(1.1)_3$ by θ_t and integrating the resulting equality over $\mathbb{R}^3$, it shows

$$
\begin{aligned}
&\frac{\kappa}{2}\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2 \\
&\quad = -\int \rho u\cdot\nabla\theta\theta_t dx + \int \rho u\cdot e_3\theta_t dx \\
&\quad \le \bar{\rho}^{\frac{1}{2}}\|u\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta\|_{L^3} + \|\sqrt{\rho}u\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2} \\
&\quad \le C\|\nabla u\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}} + C\|\sqrt{\rho}u\|_{L^2}^2 + \frac{1}{4}\|\sqrt{\rho}\theta_t\|_{L^2}^2.
\end{aligned} \tag{3.19}
$$

It deduces from $(1.1)_3$, L^2-theory of the elliptic equation, (3.3), and (3.4) that

$$
\begin{aligned}
\|\nabla\theta\|_{H^1} &\le C(\|\rho\theta_t + \rho u\cdot\nabla\theta + \rho u\cdot e_3\|_{L^2} + \|\nabla\theta\|_{L^2}) \\
&\le C(\|\sqrt{\rho}\theta_t\|_{L^2} + \|u\|_{L^6}\|\nabla\theta\|_{L^3} + \|\sqrt{\rho}u\|_{L^2} + \|\nabla\theta\|_{L^2}) \\
&\le C(\|\sqrt{\rho}\theta_t\|_{L^2} + \|\nabla u\|_{L^2}^2\|\nabla\theta\|_{L^2}) + C\|\sqrt{\rho}u\|_{L^2} + \frac{1}{2}\|\nabla^2\theta\|_{L^2},
\end{aligned} \tag{3.20}
$$

which yields

$$
\|\nabla\theta\|_{H^1} \le C\|\sqrt{\rho}\theta_t\|_{L^2} + C\|\sqrt{\rho}u\|_{L^2} + C\|\nabla u\|_{L^2}^2\|\nabla\theta\|_{L^2}. \tag{3.21}
$$

Combining (3.19) with (3.21), we yield

$$
\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^4\|\nabla\theta\|_{L^2}^2 + C\|\sqrt{\rho}u\|_{L^2}^2. \tag{3.22}
$$

Then we use Grönwall's integrating, and (3.1) to get that

$$
\sup_{t\in[0,T]}\|\nabla\theta\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}\theta_t\|_{L^2}^2 dt \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2). \tag{3.23}
$$

Multiplying (3.19) by $t^{1-\beta}(\beta\in(\frac{1}{2},1))$ together with Grönwall's inequality, (3.21), (3.1), and (3.23), we arrive at

$$
\begin{aligned}
\sup_{t\in[0,T]} t^{1-\beta}\|\nabla\theta\|_{L^2}^2 + \int_0^T t^{1-\beta}\|\sqrt{\rho}\theta_t\|_{L^2}^2 dt &\le C\sup_{t\in[0,T]}\|\nabla\theta\|_{L^2}^2\int_0^T t^{-\beta}dt \\
&\le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2).
\end{aligned} \tag{3.24}
$$

Adding (3.16) to (3.19) and together with (3.21), we have

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2) \\
&\quad \le C(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)\|\nabla u\|_{L^2}^4 + C\|\sqrt{\rho}u\|_{L^2}^2 + C\|\sqrt{\rho}\theta\|_{L^2}^2,
\end{aligned} \tag{3.25}
$$

then which multiplying by $e^{\sigma t}$ together with Grönwall's inequality, (3.1) and (3.4) we get (3.12). Finally, we finish the proof of lemma 3.3.

LEMMA 3.4. *Let* (ρ, u, p, θ) *be a smooth solution to* (1.1)-(1.3) *satisfying* (3.1). *Then there exists a positive constant* C *depending only on* q, β, $\underline{\mu}$, $\bar{\mu}$, κ, M, $\bar{\rho}$, $\|\rho_0\|_{L^{\frac{3}{2}}}$, $\|\nabla u_0\|_{L^2}^2$, *and* $\|\nabla\theta_0\|_{L^2}^2$ *such that*

$$\sup_{t\in[0,\zeta(T)]} t^{2-\beta}\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^{\zeta(T)} t^{2-\beta}\|\nabla u_t\|_{L^2}^2 dt \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2), \tag{3.26}$$

$$\sup_{t\in[\zeta(T),T]} e^{\sigma t}\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_{\zeta(T)}^{T} e^{\sigma t}\|\nabla u_t\|_{L^2}^2 dt \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2), \tag{3.27}$$

$$\sup_{t\in[\zeta(T),T]} e^{\sigma t}(\|\nabla u\|_{H^1}^2 + \|P\|_{H^1}^2) \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2), \tag{3.28}$$

with $\zeta(t) \triangleq \min\{1, t\}$.

Proof: First, operating ∂_t to $(1.1)_2$, it has

$$\begin{aligned} &\rho u_{tt} + \rho u\cdot\nabla u_t - \operatorname{div}(\mu(\rho)\nabla u_t) + \nabla p \\ &\quad = -\rho_t u_t - (\rho u)_t\cdot\nabla u + \operatorname{div}(\mu(\rho)_t\nabla u) + (\rho\theta)_t e_3, \end{aligned} \tag{3.29}$$

then multiplying it by u_t, with the help of integration by parts and $(1.1)_4$, we yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2 + \underline{\mu}\|\nabla u_t\|_{L^2}^2 \\ &\quad = -2\int \rho u\cdot\nabla u_t\cdot u_t dx - \int \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)dx - \int \rho u_t\cdot\nabla u\cdot u_t dx \\ &\qquad + \int u\cdot\nabla\mu(\rho)\nabla u\cdot\nabla u_t dx + \int (\rho\theta)_t e_3\cdot u_t dx \\ &\quad \triangleq \sum_{i=1}^{5} I_i. \end{aligned} \tag{3.30}$$

Next, we will estimate $I_i (i = 1, 2, ..., 5)$ one by one as follows

$$\begin{aligned} |I_1| + |I_3| &\le \int \rho|u_t|(|u||\nabla u_t| + |u_t||\nabla u|)dx \\ &\le C\|\sqrt{\rho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2} + C\|\sqrt{\rho}u_t\|_{L^3}\|u_t\|_{L^6}\|\nabla u\|_{L^2} \\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|u_t\|_{L^6}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2} \\ &\le \frac{\underline{\mu}}{10}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4. \end{aligned} \tag{3.31}$$

Due to (3.15) and Young's inequality, it shows

$$
\begin{aligned}
|I_2| \leq & \bar{\rho} \int (|u||\nabla u|^2|u_t| + |u|^2|\nabla^2 u||u_t| + |u|^2|\nabla u||\nabla u_t|)dx \\
\leq & C\|u\|_{L^6}\|u_t\|_{L^6}(\|\nabla u\|_{L^6}\|\nabla u\|_{L^2} + \|u\|_{L^6}\|\nabla^2 u\|_{L^2}) \\
& + C\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2} \\
\leq & C\|\nabla u_t\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2}^2 \\
\leq & \frac{\mu}{10}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^2}^{10}. \qquad (3.32)
\end{aligned}
$$

With the help of (3.1), it gives

$$
\begin{aligned}
|I_4| & \leq C\|\nabla\mu(\rho)\|_{L^q}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^{\frac{2q}{q-2}}} \\
& \leq C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{q-3}{q}}\|\nabla^2 u\|_{L^2}^{\frac{3}{q}} \\
& \leq \frac{\mu}{10}\|\nabla u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3 + C\|\nabla u\|_{L^2}^4 \\
& \leq \frac{\mu}{10}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^3\|\nabla u\|_{L^2} + C\|\nabla u\|_{L^2}^{10}. \qquad (3.33)
\end{aligned}
$$

According to $(1.1)_1$, $(1.1)_2$, and Hölder's inequality, we have

$$
\begin{aligned}
|I_5| = & \int (\rho\theta_t u_t \cdot e_3 + \rho_t\theta e_3 \cdot u_t)dx \\
= & \int ((\rho u \cdot e_3 - \rho u \cdot \nabla\theta + \kappa\triangle\theta)u_t \cdot e_3 - u \cdot \nabla\rho\theta e_3 \cdot u_t)dx \\
\leq & \int (\rho\theta|u||\nabla u_t| + \kappa|\nabla\theta||\nabla u_t| + \rho|u||u_t|)dx \\
\leq & \bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u\|_{L^3}\|\theta\|_{L^6}\|\nabla u_t\|_{L^2} + C\|\nabla\theta\|_{L^2}\|\nabla u_t\|_{L^2} \\
& + C\|\sqrt{\rho}u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} \\
\leq & \bar{\rho}^{\frac{3}{4}}\|\sqrt{\rho}u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^6}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}\|\nabla u_t\|_{L^2} + \frac{\mu}{10}\|\nabla u_t\|_{L^2}^2 \\
& + C\|\nabla\theta\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2} \\
\leq & \frac{\mu}{5}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}. \qquad (3.34)
\end{aligned}
$$

Substituting (3.31)-(3.34) into (3.30) and along with (3.4), (3.17), (3.23), we have the following inequality

$$
\begin{aligned}
& \frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \\
\leq & C\|\sqrt{\rho}u_t\|_{L^2}^2(\|\nabla u\|_{L^2}^4 + \|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}) + C\|\nabla\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{10}. \qquad (3.35)
\end{aligned}
$$

Then multiplying (3.35) by $t^{2-\beta}$ and combining with Grönwall's inequality, (3.1), (3.4), and (3.10), we have

$$
\begin{aligned}
&\sup_{t\in[0,\zeta(T)]} t^{2-\beta}\|\sqrt{\rho}u_t\|_{L^2}^2+\int_0^{\zeta(T)} t^{\beta-1}\|\nabla u_t\|_{L^2}^2dt\\
&\quad\le C\exp\{\int_0^{\zeta(T)}(\|\nabla u\|_{L^2}^4+\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2})dt\}(\zeta(T)^{2-\beta}\int_0^{\zeta(T)}\|\nabla\theta\|_{L^2}^2dt\\
&\qquad+\int_0^{\zeta(T)}(t^{4\beta-3}+t^{1-\beta}\|\sqrt{\rho}u_t\|_{L^2}^2)dt)\\
&\quad\le C\exp\{\sup_{t\in[0,\zeta(T)]} t^{\frac{1-\beta}{2}}\|\nabla u\|_{L^2}(\int_0^{\zeta(T)} t^{1-\beta}\|\sqrt{\rho}u_t\|_{L^2}^2dt)^{\frac12}(\int_0^{\zeta(T)} t^{2-2\beta}dt)^{\frac12}\}\\
&\quad\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2).
\end{aligned}\tag{3.36}
$$

Multiplying (3.35) by $e^{\sigma t}$ and together with Grönwall's inequality, (3.1), (3.5), (3.12) and (3.26), we get

$$
\begin{aligned}
&\sup_{t\in[\zeta(T),T]} e^{\sigma t}\|\sqrt{\rho}u_t\|_{L^2}^2+\int_{\zeta(T)}^T e^{\sigma t}\|\nabla u_t\|_{L^2}^2dt\\
&\quad\le C\exp\{\int_{\zeta(T)}^T(\|\nabla u\|_{L^2}^4+\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2})dt\}(e^{\sigma t}\|\sqrt{\rho}u_t\|_{L^2}^2|_{t=\zeta(T)}\\
&\qquad+\int_{\zeta(T)}^T e^{\sigma t}\|\nabla\theta\|_{L^2}^2dt+\int_{\zeta(T)}^T e^{-4\sigma t}dt)\\
&\quad\le C\exp\{\sup_{t\in[\zeta(T),T]} e^{\frac{\sigma t}{2}}\|\nabla u\|_{L^2}(\int_{\zeta(T)}^T e^{\sigma t}\|\sqrt{\rho}u_t\|_{L^2}^2dt)^{\frac12}(\int_{\zeta(T)}^T e^{-2\sigma t})^{\frac12}dt\}\\
&\quad\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2).
\end{aligned}\tag{3.37}
$$

Finally, (3.28) is a direct consequence of (3.12) and (3.27). Lemma 3.4 is completed.

LEMMA 3.5. *Let (ρ,u,p,θ) be a smooth solution to* (1.1)-(1.3) *satisfying* (3.1). *Then there exists a positive constant C depending only on q, β, $\underline{\mu}$, $\bar{\mu}$, κ, M, $\bar{\rho}$, $\|\rho_0\|_{L^{\frac32}}$, $\|\nabla u_0\|_{L^2}^2$, and $\|\nabla\theta_0\|_{L^2}^2$ such that*

$$\sup_{t\in[0,\zeta(T)]} t^{2-\beta}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\int_0^{\zeta(T)} t^{2-\beta}\|\nabla\theta_t\|_{L^2}^2dt\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2),\tag{3.38}$$

$$\sup_{t\in[\zeta(T),T]} e^{\sigma t}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\int_{\zeta(T)}^T e^{\sigma t}\|\nabla\theta_t\|_{L^2}^2dt\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2).\tag{3.39}$$

Proof: First, operating ∂_t to $(1.1)_3$, we have

$$\rho\theta_{tt}+\rho u\cdot\nabla\theta_t-\kappa\triangle\theta_t=-\rho_t\theta_t-(\rho u)_t\cdot\nabla\theta+(\rho u)_te_3,\tag{3.40}$$

multiplying it by θ_t, and integration by parts when necessary, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\kappa\|\nabla\theta_t\|_{L^2}^2\\
&=-2\int\rho u\cdot\nabla\theta_t\theta_t dx-\int\rho u\cdot\nabla(u\cdot\nabla\theta\theta_t)dx\\
&\quad-\int\rho u_t\cdot\nabla\theta\theta_t dx+\int\theta_t(\rho u)_t e_3 dx\\
&\triangleq\sum_{i=1}^{4}J_i. \qquad (3.41)
\end{aligned}$$

Similar to $I_i(i=1,2,...,5)$, we estimate the four terms of J_i as follows

$$\begin{aligned}
|J_1|+|J_3|\leq&\int(\rho|u||\nabla\theta_t||\theta_t|+\rho|u_t||\nabla\theta||\theta_t|)dx\\
\leq&\bar{\rho}^{\frac{3}{4}}\|u\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^3}\|\nabla\theta_t\|_{L^2}+\bar{\rho}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}\|u_t\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^3}\\
\leq&C\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}}(\|\nabla u\|_{L^2}\|\nabla\theta_t\|_{L^2}+C\|\nabla\theta\|_{L^2}\|\nabla u_t\|_{L^2})\\
\leq&\frac{\kappa}{6}\|\nabla\theta_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\nabla u\|_{L^2}^4+C\|\nabla u_t\|_{L^2}^2\\
&+C\|\nabla\theta\|_{L^2}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2, \qquad (3.42)
\end{aligned}$$

$$\begin{aligned}
|J_2|\leq&\int\rho|u|(\theta_t|\nabla u||\nabla\theta|+\theta_t|u||\nabla^2\theta|+|\nabla\theta_t||u||\nabla\theta|)dx\\
\leq&C\|u\|_{L^6}\|\theta_t\|_{L^6}(\|\nabla\theta\|_{L^6}\|\nabla u\|_{L^2}+\|u\|_{L^6}\|\nabla^2\theta\|_{L^2})\\
&+\|u\|_{L^6}^2\|\nabla\theta\|_{L^6}\|\nabla\theta_t\|_{L^2}\\
\leq&C\|\nabla\theta_t\|_{L^2}\|\nabla^2\theta\|_{L^2}\|\nabla u\|_{L^2}^2\\
\leq&\frac{\kappa}{6}\|\nabla\theta_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\nabla u\|_{L^2}^4+\|\nabla\theta\|_{L^2}^2\|\nabla u\|_{L^2}^8, \qquad (3.43)
\end{aligned}$$

$$\begin{aligned}
|J_4|\leq&\int(\rho\theta_t|u_t|+\rho\theta_t|u||\nabla u|+\rho|u|^2|\nabla\theta_t|)dx\\
\leq&\|\rho\|_{\frac{3}{2}}^2\|\sqrt{\rho}\theta_t\|_{L^2}\|u_t\|_{L^6}+\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u\|_{L^3}\|\nabla u\|_{L^2}\|\theta_t\|_{L^6}\\
&+\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u\|_{L^3}\|u\|_{L^6}\|\nabla\theta_t\|_{L^2}\\
\leq&C\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla u_t\|_{L^2}+\bar{\rho}^{\frac{3}{4}}\|\sqrt{\rho}u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^6}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}\|\nabla u\|_{L^2}\\
\leq&\frac{\kappa}{6}\|\nabla\theta_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla u_t\|_{L^2}^2+C\|\sqrt{\rho}u\|_{L^2}\|\nabla u\|_{L^2}^3. \qquad (3.44)
\end{aligned}$$

Substituting (3.42)-(3.44) into (3.41) and along with (3.4), (3.17), (3.23), we obtain the following estimates,

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\\
&\leq C\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\nabla u\|_{L^2}^4+C\|\nabla u_t\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\|\nabla u\|_{L^2}^8. \qquad (3.45)
\end{aligned}$$

Then multiplying (3.45) by $t^{2-\beta}$ and combining with Grönwall's inequality, (3.1), (3.11), and (3.26), we have

$$\begin{aligned}&\sup_{t\in[0,\zeta(T)]} t^{2-\beta}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\int_0^{\zeta(T)} t^{2-\beta}\|\nabla\theta_t\|_{L^2}^2 dt\\ &\quad\le C\exp\{\int_0^{\zeta(T)}\|\nabla u\|_{L^2}^4 dt\}\int_0^{\zeta(T)}(t^{1-\beta}\|\sqrt{\rho}\theta_t\|_{L^2}^2\\ &\qquad+t^{2-\beta}\|\nabla u_t\|_{L^2}^2+t^{4\beta-3})dt\\ &\quad\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2).\end{aligned}\tag{3.46}$$

Multiplying (3.45) by $e^{\sigma t}$ and together with Grönwall's inequality, (3.1), (3.27) and (3.38), we get

$$\begin{aligned}&\sup_{t\in[\zeta(T),T]} e^{\sigma t}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\int_{\zeta(T)}^{T} e^{\sigma t}\|\nabla\theta_t\|_{L^2}^2 dt\\ &\quad\le C\exp\{\int_{\zeta(T)}^{T}\|\nabla u\|_{L^2}^4 dt\}(e^{\sigma t}\|\sqrt{\rho}\theta_t\|_{L^2}^2|_{t=\zeta(T)}\\ &\qquad+\int_{\zeta(T)}^{T}(e^{\sigma t}\|\nabla u_t\|_{L^2}^2+e^{-4\sigma t})dt)\\ &\quad\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2).\end{aligned}\tag{3.47}$$

Then, we finish the Lemma 3.5.

LEMMA 3.6. *Let (ρ,u,p,θ) be a smooth solution to* (1.1)-(1.3) *satisfying* (3.1). *Then there exists a positive constant C depending only on q, β, $\underline{\mu}$, $\bar{\mu}$, κ, M, $\bar{\rho}$, $\|\rho_0\|_{L^{\frac{3}{2}}}$, $\|\nabla u_0\|_{L^2}^2$, and $\|\nabla\theta_0\|_{L^2}^2$ such that*

$$\int_0^T\|\nabla u\|_{L^\infty}dt\le C(\|\nabla u_0\|_{L^2}+\|\nabla\theta_0\|_{L^2}).\tag{3.48}$$

Proof. It deduces from Lemma 2.3, the Hölder's inequality, the Young's inequality, (2.1), (3.3)-(3.4) and (3.14), for any $p\in[2,\min\{6,q\}]$

$$\begin{aligned}&\|\nabla^2 u\|_{L^p}+\|\nabla P\|_{L^p}\\ &\quad\le\|\rho u_t+\rho u\cdot\nabla u+\rho\theta e_3\|_{L^p\cap L^{\frac{6}{5}}}\\ &\quad\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+\|u\|_{L^6}\|\nabla u\|_{L^{\frac{6p}{6-p}}}+C\|\sqrt{\rho}\theta\|_{L^2}^{\frac{6-p}{2p}}\|\sqrt{\rho}\theta\|_{L^6}^{\frac{3p-6}{2p}}\\ &\qquad+C\|\rho\|_{L^{\frac{3}{2}}}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}+\bar{\rho}^{\frac{1}{2}}\|u\cdot\nabla u\|_{L^2}+C\|\rho\|_{L^{\frac{3}{2}}}\|\theta\|_{L^6}\\ &\quad\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+C\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{p}{5p-6}}\|\nabla^2 u\|_{L^2}^{\frac{4p-6}{5p-6}}\\ &\qquad+C\|\sqrt{\rho}\theta\|_{L^2}^{\frac{6-p}{2p}}\|\nabla\theta\|_{L^2}^{\frac{3p-6}{2p}}+C\|\sqrt{\rho}u_t\|_{L^2}+C\|u\cdot\nabla u\|_{L^2}+C\|\nabla\theta\|_{L^2}\\ &\quad\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+C\|\nabla u\|_{L^2}^{\frac{6p-6}{p}}+\frac{1}{2}\|\nabla^2 u\|_{L^p}+C\|\sqrt{\rho}u_t\|_{L^2}\\ &\qquad+C\|\nabla u\|_{L^2}^3+C\|\nabla\theta\|_{L^2}^2.\end{aligned}\tag{3.49}$$

Due to (3.17) and (3.23) , and setting $r \in (3, \min\{q, \frac{6}{3-2\beta}\})$, we get

$$\begin{aligned}\|\nabla u\|_{L^\infty} \leq & C\|\nabla^2 u\|_{L^r} + C\|\nabla u\|_{L^2}\\ \leq & C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}} + C\|\sqrt{\rho}u_t\|_{L^2} + C\|\nabla u\|_{L^2}\\ & + C\|\nabla u\|_{L^2}^{\frac{6r-6}{r}} + C\|\nabla\theta\|_{L^2}^2. \end{aligned} \tag{3.50}$$

Next, we can obtain from (3.1), (3.17), (3.10) and (3.26) that,

$$\begin{aligned}&\int_0^{\zeta(T)} \|\nabla u\|_{L^\infty} dt\\ \leq & C\sup_{t\in[0,\zeta(T)]}(t^{2-\beta}\|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{6-r}{r}}(\int_0^{\zeta(T)} t^{2-\beta}\|\nabla u_t\|_{L^2}^2 dt)^{\frac{3r-6}{4r}}\cdot\\ &(\int_0^{\zeta(T)} t^{\frac{2(\beta-2)}{r+6}}dt)^{\frac{r+6}{4r}} + C\sup_{t\in[0,\zeta(T)]} t^{\frac{2-\beta}{2}}\|\sqrt{\rho}u_t\|_{L^2}\int_0^{\zeta(T)} t^{-\frac{\beta-2}{2}}dt\\ &+ C\sup_{t\in[0,\zeta(T)]} t^{\frac{1-\beta}{2}}\|\nabla u\|_{L^2}\int_0^{\zeta(T)} t^{-\frac{\beta-1}{2}}dt + C\int_0^{\zeta(T)}\|\nabla\theta\|_{L^2}^2 dt\\ \leq & C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2). \end{aligned} \tag{3.51}$$

Then, according to (3.50), (3.1), (3.12) and (3.27), we show that

$$\begin{aligned}&\int_{\zeta(T)}^T \|\nabla u\|_{L^\infty} dt\\ \leq & C\sup_{t\in[\zeta(T),T]} e^{\frac{\sigma t}{2}}(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2})\int_{\zeta(T)}^T e^{-\frac{\sigma t}{2}}dt\\ &+ (\int_{\zeta(T)}^T e^{\sigma t}\|\nabla u_t\|_{L^2}^2 dt)^{\frac12}(\int_{\zeta(T)}^T e^{-\sigma t}dt)^{\frac12}\\ &+ \sup_{t\in[\zeta(T),T]}(e^{\sigma t}\|\nabla u\|_{L^2}^2)^3\int_{\zeta(T)}^T e^{-3\sigma t}dt + C\int_{\zeta(T)}^T\|\nabla\theta\|_{L^2}^2 dt\\ \leq & C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2). \end{aligned} \tag{3.52}$$

According to Lemmas 3.2 − 3.6, we turn to prove Proposition 3.1.

Proof of Proposition 3.1. First, we can deduce from $(1.1)_1$ that

$$(\mu(\rho))_t + u\cdot\nabla\mu(\rho) = 0, \tag{3.53}$$

then multiplying the above equality by $|\nabla\mu(\rho)|^{q-2}\cdot\nabla\mu(\rho)$, we have

$$\frac{d}{dt}\|\nabla\mu(\rho)\|_{L^q} \leq C\|\nabla u\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q}. \tag{3.54}$$

Together with Grönwall's inequality and (3.38) leads to

$$\begin{aligned}\sup_{t\in[0,T]}\|\nabla\mu(\rho)\|_{L^q} \leq & \|\nabla\mu(\rho_0)\|_{L^q}\exp\{C\int_0^T\|\nabla u\|_{L^\infty}dt\}\\ \leq & \|\nabla\mu(\rho_0)\|_{L^q}\exp\{C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2)\}\\ \leq & 2M, \end{aligned} \tag{3.55}$$

where $\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2 \leq \varepsilon_1 \triangleq C^{-1}\ln 2$. Next, it obtained from (3.10) and (3.12) that

$$\begin{aligned}\int_0^T \|\nabla u\|_{L^2}^4 \leq& \sup_{t\in[0,\zeta(T)]}(t^{1-\beta}\|\nabla u\|_{L^2}^2)^2\int_0^{\zeta(T)} t^{2\beta-2}dt\\&+\sup_{t\in[\zeta(T),T]}(e^{\sigma t}\|\nabla u\|_{L^2}^2)^2\int_{\zeta(T)}^T e^{-2\sigma t}dt\\ \leq& C(\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2)^2\\ \leq&\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2.\end{aligned}\tag{3.56}$$

where $\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2 \leq \varepsilon_2 \triangleq C^{-\frac{1}{2}}$. Then, choosing $\varepsilon_0 \triangleq \min\{1,\varepsilon_1,\varepsilon_2\}$, by virtue of (3.13) and (3.14), we can directly get (3.2).

LEMMA 3.7. *Let (ρ,u,p,θ) be a smooth solution to* (1.1)-(1.3) *satisfying* (3.1). *Then there exists a positive constant C depending only on q, β, $\underline{\mu}$, $\bar{\mu}$, κ, M, $\bar{\rho}$, $\|\rho_0\|_{L^{\frac{3}{2}}}$, $\|\nabla u_0\|_{L^2}^2$, and $\|\nabla\theta_0\|_{L^2}^2$ such that for $r_0 \triangleq \min\{6,q\}$,*

$$\begin{aligned}&\sup_{t\in[0,T]} e^{\sigma t}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\zeta\|\nabla u\|_{H^1}^2+\zeta\|p\|_{H^1}^2+\zeta\|\nabla\theta\|_{H^1}^2)\\&\quad+\int_0^T \zeta e^{\sigma t}(\|\nabla u\|_{H^1}^2+\|p\|_{H^1}^2+\|\nabla\theta\|_{H^1}^2+\zeta\|\nabla u\|_{W^{1,r_0}}^2+\zeta\|p\|_{W^{1,r_0}}^2\\&\quad+\zeta\|\nabla\theta\|_{W^{1,r_0}}^2)dt+\int_0^T \zeta e^{\sigma t}(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2)dt \leq C.\end{aligned}\tag{3.57}$$

Proof. Similar to (3.53) and (3.54), it implies that

$$\|\nabla\rho\|_{L^2\cap L^q} \leq 2\|\nabla\rho_0\|_{L^2\cap L^q}.\tag{3.58}$$

Multiplying (3.25) by $e^{\sigma t}$ and along with Grönwall's inequality, (3.5), and (3.2), we obtain

$$\sup_{t\in[0,T]} e^{\sigma t}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\int_0^T e^{\sigma t}(\sqrt{\rho}u_t\|_{L^2}^2+\sqrt{\rho}\theta_t\|_{L^2}^2)dt \leq C.\tag{3.59}$$

Due to (3.35) and (3.45), it implies that

$$\begin{aligned}&\frac{d}{dt}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)+\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\\ \leq& C(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^2.\end{aligned}\tag{3.60}$$

Together with Grönwall's inequality, (3.59), and (3.5), it gives

$$\sup_{t\in[0,T]} \zeta e^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)+\int_0^T \zeta e^{\sigma t}(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2)dt \leq C.\tag{3.61}$$

Then, with the help of (3.15), (3.21), and (3.61), it is easy to deduce that

$$\begin{aligned}&\sup_{t\in[0,T]} \zeta e^{\sigma t}(\|\nabla u\|_{H^1}^2+\|p\|_{H^1}^2+\|\nabla\theta\|_{H^1}^2)\\&\quad+\int_0^T e^{\sigma t}(\|\nabla u\|_{H^1}^2+\|p\|_{H^1}^2+\|\nabla\theta\|_{H^1}^2)dt \leq C.\end{aligned}\tag{3.62}$$

At the same time, using L^p-theory of the elliptic equation, and (3.3), we have for any $p\in[2,\min\{6,q\})$,

$$
\begin{aligned}
\|\nabla^2\theta\|_{L^p} \le& C\|\rho\theta_t+\rho u\cdot\nabla\theta+\rho u\cdot e_3\|_{L^p}\\
\le& C\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{6-p}{2p}}\|\sqrt{\rho}\theta_t\|_{L^6}^{\frac{3p-6}{2p}}+C\|u\|_{L^6}\|\nabla\theta\|_{L^{\frac{6p}{6-p}}}+\|\rho\|_{L^{\frac{6p}{6-p}}}\|u\|_{L^6}\\
\le& C\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla\theta_t\|_{L^2}^{\frac{3p-6}{2p}}+C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac{p}{5p-6}}\|\nabla^2\theta\|_{L^p}^{\frac{4p-6}{5p-6}}+C\|\nabla u\|_{L^2}\\
\le& C\|\sqrt{\rho}\theta_t\|_{L^2}+C\|\nabla\theta_t\|_{L^2}+C\|\nabla u\|_{L^2}^{\frac{5p-6}{p}}\|\nabla\theta\|_{L^2}\\
&+\frac{1}{2}\|\nabla^2\theta\|_{L^p}+C\|\nabla u\|_{L^2}. \qquad (3.63)
\end{aligned}
$$

Then, it holds that

$$
\|\nabla^2\theta\|_{L^p}\le C\|\sqrt{\rho}\theta_t\|_{L^2}+C\|\nabla\theta_t\|_{L^2}+C\|\nabla u\|_{L^2}+C\|\nabla\theta\|_{L^2}. \qquad (3.64)
$$

It follows from (3.15), (3.21) (3.39), (3.64), and (3.59) that for $r_0\triangleq\min\{6,q\}$,

$$
\begin{aligned}
&\|\nabla u\|_{W^{1,r_0}}+\|p\|_{W^{1,r_0}}+\|\nabla\theta\|_{W^{1,r_0}}\\
&\qquad\le C\|\nabla u_t\|_{L^2}+C\|\nabla\theta_t\|_{L^2}+C\|\nabla u\|_{L^2}+C\|\nabla\theta\|_{L^2}, \qquad (3.65)
\end{aligned}
$$

combining with (3.61) and (3.5), we get

$$
\int_0^T\zeta e^{\sigma t}(\|\nabla u\|_{W^{1,r_0}}+\|p\|_{W^{1,r_0}}+\|\nabla\theta\|_{W^{1,r_0}})dt\le C. \qquad (3.66)
$$

According to (3.59) and (3.66), we can yield (3.57). Finally, the proof of Lemma 3.5 is finished. □

Proof of Theorem 1.1. With the help of the a priori estimates in Lemma 3.2-Lemma 3.7, we can prove the Theorem 1.1, and the detailed proof are similar to [**22**]. Here, we omit it for simplicity.

Acknowledgments: The authors would like to thank the referee for valuable comments, and Li is supported by Beijing Natural Science Foundation (Grant No.Z180007).

References

[1] P.H. Rabinowitz, Existence and nonuniqueness of rectangular solutions of the Bénard problem. Arch. Rational Mech. Anal. **29** (1968), 32-57.
[2] T. Ma, S. Wang, Rayleigh Bénard convection: dynamics and structure in the physical space, Commun. Math. Sci. **5**(3) (2007), 553-574.
[3] H. Bénard, Les tourbillons cellulaires dans une nappe liquide, Rev. Gén. Sci. Pures Appl. **10**(1) (1901), 254-266.
[4] M.J. Block, Surface tension as the cause of Bénard cell and surface deformation in a liquid film, Nature **178** (1956), 650-651.
[5] P.G. Drazin, W.H. Reid, Hydrodynamic Stability, 2nd ed., Cambridge University Press, Cambridge, 2004.
[6] K. Iohara, Bénard-Marangoni convection with a deformable surface, J. Math. Kyoto Univ. **38** (1998), 255-270.
[7] C. Foias, O. Manley, R. Temam, Attractors for the Béenard problem: existence and physical bounds on their fractal dimension, Nonlinear Anal. TMA **11**(8) (1987), 939-967.
[8] A. Farhat, M. Jolly, E. Titi, Continuous data assimilation for the 2D Bénard convection through velocity measurements alone, Physica D **303** (2015), 59-66.

[9] R. Danchin, M. Paicu, Global well-posedness issues for the inviscid Boussinesq system with Yudovich's type data, Comm. Math. Phys. **290** (2009), 1-14.

[10] V. Yudovich, Non-stationary flows of an ideal incompressible fluid, Zh. Vychisl. Mat. Mat. Fiz. **3** (1963), 1032-1066.

[11] G. Wu, L. Xue, Global well-posedness for the 2D inviscid Bénard system with fractional diffusivity and Yudovich's type data. J. Differ. Equ. **253** (2012), 100-125.

[12] T. Hmidi, S. Keraani, F. Rousset, Global well-posedness for Euler-Boussinesq system with critical dissipation, Comm. Partial Differential Equations **36** (2011), 420-445.

[13] Z. Ye, Regularity criterion of the 2D Bénard equations with critical and supercritical dissipation. Nonlinear Anal. **156** (2017), 111-143.

[14] B. Guo, B. Xie, L. Zeng, Exponential decay of Bénard convection problem with surface tension. J. Differ. Equ. **267** (2019), 2261-2283.

[15] X. Ye, M.X. Zhu, The zero limit of thermal diffusivity for the 2D density-dependent Boussinesq equations with vacuum. Z Angew Math Mech. (2020), 100: e201900064.

[16] J. Fan, J. Sun, T. Tang, Uniform global strong solutions of the 2D density-dependent incompressible magnetic Bénard problem in a bounded domain. Comput. Math. Appl. **77**(2) (2019), 494-500.

[17] X. Zhong, Global strong solution of nonhomogeneous Bénard system with large initial data and vacuum in a bounded domain, Z. Anal. Anwend. **40** (2021), 153-166.

[18] X. Zhong, Global strong solution to the nonhomogeneous Bénard system with large initial data and vacuum. Results Math. (2021), 76: 27.

[19] X. Zhong, Global strong solution and exponential decay to the 3D Cauchy problem of nonhomogeneous Bénard system with vacuum. Acta Applicandae Mathematicae. (2021), 172: 8.

[20] C. He, J. Li, B.Q. Lü, Global well-posedness and exponential stability of 3D Navier-Stokes equations with density-dependent viscosity and vacuum in unbounded domains. Arch. Rational Mech. Anal. **239** (2021), 1809-1835.

[21] J. Bergh, J. Lofstrom, Interpolation Spaces: an Introduction. Springer, Berlin, 1976.

[22] X.D. Huang, Y. Wang, Global strong solution of 3D inhomogeneous Navier-Stokes equations with density-dependent viscosity. J. Differ. Equ. **259** (2015), 1606-1627.

[23] J.W. Zhang, Global well-posedness for the incompressible Navier-Stokes equations with density-dependent viscosity coefficient. J. Differ. Equ. **259** (2015), 1722-1742.

[24] D. Hoff, Compressible flow in a half-space with Navier boundary conditions. J. Math. Fluid Mech. **7**(3) (2005), 315-338.

[25] X.D. Huang, J. Li, Z.P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations. Commun. Pure Appl. Math. **65** (2012), 549-585.

[26] X. Zhong, Local strong solutions to the nonhomogeneous Bénard system with nonnegative density. Rocky Mountain J. Math. **50**(4) (2020), 1497-1516.

[27] L. Nirenberg, On elliptic partial differential equations, Estratto Ann. Sc. Norm. Super. Pisa (3) XIII (II) 1959.

[28] P.L. Lions, Mathematical Topics in Fluid Mechanics, Vol. I: Incompressible Models, Oxford University Press. Oxford 1996.

[29] Z. Liang, Local strong solution and blow-up criterion for the 2D nonhomogeneous incompressible fluids, J. Differential Equations. **258** (2015), 2633-2654.

Faculty of Science, Beijing University of Technology, Beijing 100124, P.R. China
Email address: `liumin4521@163.com (M.Liu), yli@bjut.edu.cn (Y.Li )`