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Polymer Kinetic Theory Temperature Dependent Configurational Probability Diffusion Equations: Existence of Positive Solution Results

Ionel Sorin Ciuperca and Liviu Iulian Palade

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Abstract. A new configurational probability diffusion - CPD - equation that accounts for temperature dependent molecular dynamics for incompressible polymer fluids is introduced. We prove the existence of positive solutions for the corresponding variational formulation using Schauder's fixed point theory. The polymer fluid macromolecules are modeled as Finitely Extensible Nonlinear Elastic dumbbells, also known as FENE chains.

Contents

1. Introduction

Polymer fluid industrial flows are commonly subjected to strong temperature gradients: this is a feature of almost all processing technologies (see e.g. [**4**], [**27**], [**28**]). Even though this was known for a long time, studying non-isothermal flows is notoriously difficult at both modeling and mathematical levels and the issue remains largely an open question only seldom addressed.

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Kinetic theories ([**5**], [**19**], [**32**]) focus on for the dominant inter and intra molecular interactions responsible for the macroscopically observed rheological behavior. Gathering inspiration from the earlier work of Curtiss and Bird in [**18**], we obtained in [**16**] a more general configurational probability diffusion (CPD) equation based on temperature dependent polymer molecular dynamics, without the linearizations originally assumed in [**18**], an undertaking never considered before. Very important to recall here, CPD equations are the theory's backbone for they enter the definition of the fluid stress tensor, reason for which they are studied to this day, see, e.g. [**24**], [**21**], [**22**], [**20**], [**25**], [**12**], [**13**], [**14**], [**1**], [**2**], [**3**], [**11**], [**9**], [**26**], [**6**], [**30**], [**8**], [**15**], [**10**].

Specifically, we assumed the polymer chains to be modeled as FENE (*F*initely *E*xtensible *N*onlinear *E*lastic) dumbbells, a popular approximation of the physical reality. In the modeling calculations we assumed the temperatures at dumbbell extremities to be different from each other. This situation corresponds to a rather general case in which the heat flow also occurs along the chain. The fluid is assumed incompressible: the macroscopic velocity v satisfies $\nabla_x \cdot v = 0$. Let ψ, θ and Q denote the probability density, the temperature and the end-to-end vector, respectively. Define the Péclet number $\mathbf{Pe} = \dfrac{2\zeta VL}{k_B\theta_0}$ and the Deborah number $\mathbf{De} = \dfrac{\zeta V l_0^2}{4k_B\theta_0 L}$ as e.g. in [**24**] and [**16**] (see also [**17**] for several remarks). $\mathbf{De}$ can equivalently be introduced as $\mathbf{De} = \dfrac{\zeta V}{4HL}$, but since $H = H(\theta) = \dfrac{k_B}{l_0^2}\theta$, upon normalizing θ one recovers exactly the definition just presented. The corresponding initial boundary value problem reads

$$\begin{aligned} \frac{\partial \psi}{\partial t} + \nabla_x \cdot (v\psi) &- \frac{1}{\mathbf{Pe}} \nabla_x \cdot \{ [\nabla_x (\theta \ln \psi) + \nabla_Q ((Q \cdot \nabla_x \theta) \ln \psi)] \psi \} \\ &+ \nabla_Q \cdot (\kappa Q \psi) - \frac{1}{\mathbf{Pe}} \nabla_Q \cdot [\psi \nabla_x ((Q \cdot \nabla_x \theta) \ln \psi)] \\ &- \frac{1}{2\mathbf{De}} \nabla_Q \cdot (\theta \nabla_Q \psi) - \frac{1}{2\mathbf{De}} \nabla_Q \cdot \left(\frac{Q}{1 - \|Q\|^2/Q_0^2} \theta \psi \right) = 0 \end{aligned} \tag{1.1}$$

$$\psi|_{\partial \bar{\Sigma}_T \times (0,T)} = 0 \tag{1.2}$$

$$\psi(t=0) = \psi_0, \ \ \psi_0 \text{ given} \tag{1.3}$$

In [**16**] we proved the existence of positive solutions to the corresponding variational formulation of (1.1) using fixed point techniques.

In this paper we continue the aforementioned work and take on to studying a closely related CPD. Specifically, instead of assuming two different temperatures at the dumbbell's ends, we now assume the temperature field on the average and be given at the FENE dumbbell's barycenter. The subsequent calculations are identical in nature to those of [**16**]. The new CPD reads:

$$\begin{aligned} \frac{\partial \psi}{\partial t} &+ \nabla_x \cdot (v\psi) - \frac{1}{\mathbf{Pe}} \nabla_x \cdot [\nabla_x (\theta \ln \psi) \psi] + \nabla_Q \cdot (\kappa Q \psi) \\ &- \frac{1}{2\mathbf{De}} \nabla_Q \cdot (\theta \nabla_Q \psi) - \frac{1}{2\mathbf{De}} \nabla_Q \cdot \left(\frac{Q}{1 - \|Q\|^2/Q_0^2} \theta \psi \right) = 0 \end{aligned} \tag{1.4}$$

For commodity, a further simplification can be achieved by carrying out the re-scaling Q/Q_0. Relabeling the newly introduced variable again by Q for sake of simplicity of notations leads to

$$\begin{aligned} \frac{\partial \psi}{\partial t} &+ \nabla_x \cdot (v\psi) - d_1 \nabla_x \cdot [\nabla_x (\theta \ln \psi)\, \psi] + \nabla_Q \cdot (\kappa Q \psi) - d_2 \nabla_Q \cdot (\theta \nabla_Q \psi) \\ &- d_2 \delta \nabla_Q \cdot \left(\frac{Q}{1 - \|Q\|^2} \theta \psi \right) = 0 \end{aligned} \tag{1.5}$$

with $d_1 = \dfrac{1}{\mathbf{Pe}}$, $d_2 = \dfrac{1}{2Q_0^2 \mathbf{De}}$ and $\delta = Q_0^2$ being positive constants.

As an aside: if the temperature θ is set constant and the probability density ψ considered to be independent of the macroscopic (Eulerian) variable x, then both (1.1) and (1.4) simplify to the standard FENE diffusion equation (see equation 13.2-13 on page 62 in [**5**]) the steady state form of which was studied in [**12**].

Now, equation (1.5) is taken with the more specialized boundary conditions (1.6) and (1.7) - as compared to (1.2) - in order to also ensure ψ is normalized (see also [**25**]):

$$[v\psi - d_1 \nabla_x (\theta \ln \psi)\, \psi] \cdot \nu_x = 0,\ x \in \partial\Omega,\ Q \in B(0,1),\ t \in (0,T) \tag{1.6}$$

$$\left[\kappa Q \psi - d_2 \theta \nabla_Q \psi - d_2 \delta \theta \frac{2Q}{1 - \|Q\|^2} \psi \right] \cdot \nu_Q = 0,\ x \in \Omega,\ Q \in \partial B(0,1),\ t \in (0,T) \tag{1.7}$$

Of notice the above boundary conditions (1.6)-(1.7) render the study of solutions existence significantly more complicated than it is the case for the standard Dirichlet condition (1.2). In order to deal with the said complexity we need to use (2.7) (introduced in the next section) and to work within the framework of appropriately defined weighted Sobolev spaces V and H - fact leading to increased proofs technicality - as opposed to working with classical Sobolev spaces H_0^1 as done in [**16**].

The ensuing problem is proved to have positive solutions using Schauder's fixed point theory.

The paper is organized as follows:

- Section 2 is devoted to obtaining the variational formulation of the CPD equation under consideration
- Section 3 deals with the problem regularization
- In Section 4 we obtain estimates uniform in ϵ
- Section 5 gives the proof of the main and final existence result

2. Introducing the Problem and Its Variational Form

Let Ω be an open domain with smooth enough boundary $\partial\Omega \in \mathscr{C}^1$ and $B(0,1) \subset \mathbb{R}^d$, $d = 2$ or 3. Next $\Sigma := \Omega \times B(0,1)$, $x \in \Omega$, $Q \in B(0,1)$. Let $\Sigma_T := \Sigma \times [0,T)$, $T > 0$, and $\Omega_T := \Omega \times [0,T)$. Let $\kappa = \nabla_x v$ denote the macroscopic velocity gradient, and the usual summation convention over repeated indices applies, i.e. $\kappa Q = \kappa_{ij} Q_j$.

The problem is stated as follows: find $\psi : \Sigma_T \mapsto \mathbb{R}$, such that

$$\begin{aligned}\frac{\partial\psi}{\partial t} &+ \nabla_x\cdot(v\psi) - d_1\nabla_x\cdot\left[\nabla_x(\theta\ln\psi)\psi\right] + \nabla_Q\cdot(\kappa Q\psi) - d_2\nabla_Q\cdot(\theta\nabla_Q\psi)\\ &- d_2\delta\nabla_Q\cdot\left(\frac{2Q}{1-\|Q\|^2}\theta\psi\right) = 0,\ \forall(x,Q,t)\in\Sigma_T\end{aligned}\tag{2.1}$$

solution complying with the boundary conditions

$$\left[v\psi - d_1\nabla_x(\theta\ln\psi)\psi\right]\cdot\nu_x = 0,\ x\in\partial\Omega,\ Q\in B(0,1),\ t\in(0,T)\tag{2.2}$$

$$\left[\kappa Q\psi - d_2\theta\nabla_Q\psi - d_2\delta\theta\frac{2Q}{1-\|Q\|^2}\psi\right]\cdot\nu_Q = 0,\ x\in\Omega,\ Q\in\partial B(0,1),\ t\in(0,T)\tag{2.3}$$

and with the initial conditions

$$\psi(x,Q,t=0) = \psi_0(x,Q),\ \text{with } \psi_0:\Sigma\mapsto\mathbb{R} \text{ being given}\tag{2.4}$$

Since by a simple calculation one has

$$\nabla_x(\theta\ln\psi)\psi = \psi\ln\psi\nabla_x\theta + \theta\nabla_x\psi$$

then (2.2) takes the form

$$\begin{aligned}\frac{\partial\psi}{\partial t} &+ \nabla_x\cdot(v\psi) - d_1\nabla_x\cdot(\theta\nabla_x\psi) - d_1\nabla_x\cdot\left[\psi\ln\psi\nabla_x\theta\right]\\ &+ \nabla_Q\cdot(\kappa Q\psi) - d_2\nabla_Q\cdot(\theta\nabla_Q\psi) - d_2\delta\nabla_Q\cdot\left(\frac{2Q}{1-\|Q\|^2}\theta\psi\right) = 0,\\ &\forall(x,Q,t)\in\Sigma_T\end{aligned}\tag{2.5}$$

Also, (2.2) can be re-written as

$$\left[v\psi - d_1\theta\nabla_x\psi - d_1\psi\ln\psi\nabla_x\theta\right]\cdot\nu_x = 0\tag{2.6}$$

Observe now that

$$\begin{aligned}d_2\theta\nabla_Q\psi &+ d_2\theta\delta\frac{2Q}{1-\|Q\|^2}\psi = d_2\theta\left\{\nabla_Q\psi - \delta\left[\nabla_Q\ln\left(1-\|Q\|^2\right)\right]\psi\right\}\\ &= d_2\theta\nabla_Q\left(\psi e^{-\delta\ln\left(1-\|Q\|^2\right)}\right)e^{\delta\ln\left(1-\|Q\|^2\right)}\end{aligned}$$

which leads to

$$d_2\theta\nabla_Q\psi + d_2\theta\delta\frac{2Q}{1-\|Q\|^2}\psi = d_2\theta M\nabla_Q\left(\frac{\psi}{M}\right),\ M(Q) = (1-\|Q\|)^{\delta}\tag{2.7}$$

Assume in the following that

$$\delta > 1\tag{2.8}$$

We now take on to give the variational formulation of the problem (2.5), (2.6) and (2.3). Let

$$H := \left\{ \phi : \Sigma \mapsto \mathbb{R} : \int_\Sigma \frac{1}{M}\phi^2 \, \mathrm{d}x\mathrm{d}Q < \infty \right\}$$

endowed with the inner product $\langle \phi, \psi \rangle_H = \int_\Sigma \frac{1}{M}\phi\psi \, \mathrm{d}x\mathrm{d}Q$.

Let

$$V := \left\{ \phi : \Sigma \mapsto \mathbb{R} : \int_\Sigma \left[\frac{1}{M}\phi^2 + \frac{1}{M}\left|\nabla_x \phi\right|^2 + M\left|\nabla_Q\left(\frac{\phi}{M}\right)\right|^2 \right] \mathrm{d}x\mathrm{d}Q < \infty \right\}$$

endowed with the inner product

$$\langle \phi, \psi \rangle_V = \int_\Sigma \left[\frac{1}{M}\phi\psi + \frac{1}{M}\nabla_x\phi \cdot \nabla_x\psi + M\nabla_Q\left(\frac{\phi}{M}\right) \cdot \nabla_Q\left(\frac{\psi}{M}\right) \right] \mathrm{d}x\mathrm{d}Q$$

Both H and V are Hilbert spaces. Next, one multiplies (2.5) by $\frac{\phi}{M}$, $\phi \in V$, and using Stoke's Theorem and (2.7) one gets:

$$\begin{aligned}
&\int_\Sigma \frac{1}{M}\frac{\partial \psi}{\partial t}\phi - \int_\Sigma \frac{\psi}{M} v \cdot \nabla_x \phi + d_1 \int_\Sigma \frac{\theta}{M}\nabla_x\psi \cdot \nabla_x\phi \\
&+ d_1 \int_\Sigma \frac{1}{M}\nabla_x\theta\,(\ln\psi)\,\psi \cdot \nabla_x\phi \\
&+ \int_{\partial\Omega\times B(0,1)} \frac{1}{M}\left[v\psi - d_1\theta\nabla_x\psi - \nabla_x\theta\,(\ln\psi)\,\psi\right] \cdot \nu_x - \int_\Sigma \kappa Q\psi \cdot \nabla_Q\left(\frac{\phi}{M}\right) \\
&+ d_2 \int_\Sigma M\theta\nabla_Q\left(\frac{\psi}{M}\right) \cdot \nabla_Q\left(\frac{\phi}{M}\right) \\
&+ \int_{\Omega\times\partial B(0,1)} \frac{1}{M}\left[\kappa Q\psi - d_2 M\theta\nabla_Q\left(\frac{\psi}{M}\right)\right] \cdot \nu_Q = 0
\end{aligned}$$

The boundary terms in the above do vanish thanks to (2.3), (2.6) and (2.7). Therefore the variational formulation reads:

$$\begin{aligned}
\frac{d}{dt}\int_\Sigma \frac{1}{M}\psi\phi \quad &- \quad \int_\Sigma \frac{\psi}{M} v \cdot \nabla_x\phi + d_1 \int_\Sigma \frac{\theta}{M}\nabla_x\psi \cdot \nabla_x\phi \\
&+ \quad d_1 \int_\Sigma \frac{1}{M}\nabla_x\theta\,(\ln\psi)\,\psi \cdot \nabla_x\phi - \int_\Sigma \kappa Q\psi \cdot \nabla_Q\left(\frac{\phi}{M}\right) \\
&+ \quad d_2 \int_\Sigma M\theta\nabla_Q\left(\frac{\psi}{M}\right) \cdot \nabla_Q\left(\frac{\phi}{M}\right) = 0, \ \forall \phi \in V
\end{aligned} \tag{2.9}$$

Moreover, (2.5) and (2.4) can be recast in the form that is given in the following for better grasping: for any $\xi \in \mathscr{C}^1[0,T]$ such that $\xi(T) = 0$,

$$\begin{aligned}\int_{\Sigma_T} \frac{1}{M}\psi\phi\xi' \quad &- \quad \xi(0)\int_{\Sigma}\psi_0\phi - \int_{\Sigma_T}\frac{1}{M}\psi\xi v\cdot\nabla_x\phi + d_1\int_{\Sigma_T}\frac{1}{M}\theta\xi\nabla_x\psi\cdot\nabla_x\phi \\ &+ \quad d_1\int_{\Sigma_T}\frac{1}{M}\xi\nabla_x\theta\,(\ln\psi)\,\psi\cdot\nabla_x\phi - \int_{\Sigma_T}\xi\kappa Q\psi\cdot\nabla_Q\left(\frac{\psi}{M}\right) \\ &+ \quad d_2\int_{\Sigma_T} M\theta\nabla_Q\left(\frac{\psi}{M}\right)\cdot\nabla_Q\left(\frac{\phi}{M}\right)\xi = 0,\ \forall\phi\in V \end{aligned} \tag{2.10}$$

3. The Regularized Problem. Estimates, Positive Solutions Existence

3.1. Notations, Assumptions and Preliminary Results. Let $X_T := \left\{\phi \in L^2(0,T;V) : \frac{d\phi}{dt} \in L^2(0,T;V')\right\}$.

The inclusions $V \subset H \subset V'$ are continuous. The inclusion $V \subset H$ also being compact (see [**31**]), it implies the inclusion $X_T \subset L^2(0,T;H)$ is also compact. We make the following assumptions regarding the problem data:

- $v \in L^\infty(\Sigma_T)$, $\kappa \in L^\infty(\Sigma_T)$
- $\theta \in L^\infty\left(0,T;W^{1,\infty}(\Sigma)\right)$
  there exists $\theta_m > 0$ such that $\theta(x,t) \geqslant \theta_m$ a.e. for $(x,t) \in \Omega\times(0,T)$
- assume $\psi_0 \in H$

From hereon we employ the shorthand notation $B = B(0,1)$. Let $p > 2$ be such that $H^1\left(\mathbb{R}^d\right) \subset L^p\left(\mathbb{R}^d\right)$ holds true. Let $c_s > 0$ be such that $\|u\|_{L^p(\Omega)} \leqslant c_s\|u\|_{H^1(\Omega)}$, for all $u \in H^1(\Omega)$, and, $\|u\|_{L^p(B)} \leqslant c_s\|u\|_{H^1(B)}$, for all $u \in H^1(B)$.

Let the following Hilbert spaces be defined as:

$$H_B := \left\{\phi : B \mapsto \mathbb{R} : \int_B \frac{1}{M}\phi^2 \mathrm{d}Q < \infty\right\}$$

$$V_B := \left\{\phi : B \mapsto \mathbb{R} : \int_B \frac{1}{M}\phi^2 + M\left|\nabla_Q\left(\frac{\phi}{M}\right)\right|^2 \mathrm{d}Q < \infty\right\}$$

Using assumption (2.8) one can prove (see Theorem 6.2.5 in [**29**], equation (26) in [**25**], and Lemma 3.6 in [**7**]) the below recalled result:

PROPOSITION 3.1. *Let $\phi \in V_B$. Then $\frac{\phi}{\sqrt{M}\,(1-\|Q\|^2)} \in L^2(B)$, $\frac{\phi}{\sqrt{M}} \in L^p(B)$. Moreover, there exists $c_B > 0$ such that*

(i) $\left\|\frac{\phi}{\sqrt{M}\,(1-\|Q\|^2)}\right\|_{L^2(B)} \leqslant c_B\|\phi\|_{V_B}$, *for all* $\phi \in V_B$

(ii) $\left\|\frac{\phi}{\sqrt{M}}\right\|_{L^p(B)} \leqslant c_B\|\phi\|_{V_B}$, *for all* $\phi \in V_B$

Let $p_1 > 2$, $p_2 > 2$ be such that $1/p_1 + 1/p_2 = 1$ and $1/p + 1/p_2 = 1/2$. The following result is proved below:

PROPOSITION 3.2. *Let $\phi \in V$. Then $\frac{\phi}{\sqrt{M}} \in H^1(\Sigma)$, and for all $\phi \in V$ there exists a positive constant c_2 such that $\left\|\frac{\phi}{\sqrt{M}}\right\|_{H^1(\Sigma)} \leqslant c_2\|\phi\|_V$.*

PROOF. Capitalizing on the definition of the functional space V, one gets that $\dfrac{\phi}{\sqrt{M}} \in L^2(\Sigma)$ and $\dfrac{1}{\sqrt{M}}\nabla_x\phi \in L^2(\Sigma)$. Next,

$$\nabla_Q\left(\frac{\phi}{\sqrt{M}}\right) = \nabla_Q\left(\sqrt{M}\frac{\phi}{M}\right) = \sqrt{M}\nabla_Q\left(\frac{\phi}{M}\right) + \frac{\phi}{M}\nabla_Q\left(\sqrt{M}\right) \tag{3.1}$$

Since $\nabla_Q\left(\sqrt{M}\right) = -\delta Q\dfrac{\sqrt{M}}{1-\|Q\|^2}$, then $\dfrac{\phi}{M}\nabla_Q\left(\sqrt{M}\right) = -\delta Q\dfrac{\phi}{\sqrt{M}\left(1-\|Q\|^2\right)}$.
By observing that

$$\int_\Sigma \frac{\phi^2}{M\left(1-\|Q\|^2\right)^2} = \int_\Omega\int_B \frac{\phi^2}{M\left(1-\|Q\|^2\right)^2} \leqslant c_B\int_\Omega \|\phi\|^2_{V_B}\mathrm{d}x$$

and by Proposition 3.1 one gets $\dfrac{\phi}{M}\nabla_Q\left(\sqrt{M}\right) \in L^2(\Sigma)$.

□

Before ending this Section we give the proof of the following result.

PROPOSITION 3.3. *Let* $MX := \{M\phi,\ \phi \in X\}$. *Then* $M\mathscr{C}^\infty(\bar{\Sigma})$ *is dense in* V.

PROOF. Let first the following functional spaces be given by:

$$\tilde{V} := \left\{\tilde{\phi} : \Sigma \mapsto \mathbb{R} : \int_\Sigma M\left(\tilde{\phi}^2 + \left|\nabla_x\tilde{\phi}\right|^2 + \left|\nabla_Q\tilde{\phi}\right|^2\right)\mathrm{d}x\mathrm{d}Q < \infty\right\}$$

$$\tilde{V}_B := \left\{\tilde{\phi} : B \mapsto \mathbb{R} : \int_B M\left(\tilde{\phi}^2 + \left|\nabla_Q\tilde{\phi}\right|^2\right)\mathrm{d}Q < \infty\right\}$$

They are Hilbert spaces endowed with corresponding inner products. The mappings $\tilde{V} \ni \tilde{\phi} \mapsto M\tilde{\phi} \in V$, and $\tilde{V}_B \ni \tilde{\phi} \mapsto M\tilde{\phi} \in V_B$ are isomorphisms from $\tilde{V}$ to V, and from $\tilde{V}_B$ to V_B respectively.

In order to prove the announced result, we point out that it suffices to prove that $\mathscr{C}^\infty(\Sigma)$ is densely included into $\tilde{V}$. First, that $\mathscr{C}^\infty\left(\bar{B}\right)$ is densely included into $\tilde{V}_B$ is a consequence of part c of Theorem 3.2.2 on page 239 in [**31**]. Even though we capitalize on the aforementioned result, its proof will here be significantly modified in order to deliver the result we stated.

Let $\Sigma_0 = \mathbb{R}^d \times B(0,1) \subset \mathbb{R}^{2d}$ and let the functional space

$$\begin{aligned}\tilde{V}_0 := \Bigg\{ & \tilde{\phi} : \Sigma_0 \mapsto \mathbb{R} : \int_\Sigma \left(\tilde{\phi}^2 + \left|\nabla_x\tilde{\phi}\right|^2 + \left|\nabla_Q\tilde{\phi}\right|^2\right)\mathrm{d}x\mathrm{d}Q < \infty,\ \text{and } \tilde{\phi} = 0 \\ & \text{for } |x| \geqslant \alpha, \text{for some } \alpha > 0\Bigg\}\end{aligned}$$

We now introduce the extension operator P_{Σ_0} that sends $\tilde{\phi} \in \tilde{V}$ into $P_{\Sigma_0}\tilde{\phi} \in \tilde{V}_0$. There exists $c_{\Sigma_0} > 0$ such that, for all $\tilde{\phi} \in \tilde{V}$, $\left\|P_{\Sigma_0}\tilde{\phi}\right\| \leqslant c_{\Sigma_0}\left\|\tilde{\phi}\right\|_{\tilde{V}}$. This operator is introduced using a partition of unity with respect to the variable x (thus "ignoring" the variable Q).

We next prove that for any fixed $\tilde{\psi} \in \tilde{V}_0$ and for any $\epsilon > 0$, there exists $\tilde{\psi}_\epsilon \in \mathscr{C}^\infty\left(\bar{\Sigma}_0\right)$ such that $\left\|\tilde{\psi} - \tilde{\psi}_\epsilon\right\|_{\tilde{V}_0} \leqslant \epsilon$, a fact that ends the proof. To achieve this goal we draw inspiration from Theorem 3.2.2 of [**31**]. What is actually needed is to approximate by a $\mathscr{C}^\infty$ function a function $\tilde{\psi}(x, Q+q)$, $(x,Q) \in \Sigma_0$, where q

is taken in a cone contained in $\mathbb{R}^d$, with $\|q\|$ small enough, with the late function being defined on a space larger than Σ_0. Next, this function is convoluted with a suitably chosen mollifier function, and from now on the difference with Triebel's proof of [**31**] is that the convolution is carried out with respect to the two variables x and Q.

□

3.2. Problem Regularization. The regularization of (2.5) is performed by replacing $\ln\psi$ by a regularizing function $g_\epsilon(\psi)$, with $\epsilon > 0$ small enough. Let thus $g_\epsilon : \mathbb{R} \mapsto \mathbb{R}$ be given by:

$$g_\epsilon(\psi) = \begin{cases} \ln\left(\frac{1}{\epsilon}\right) & \text{if } z \geqslant \frac{1}{\epsilon} \\ \ln z & \text{if } \epsilon \leqslant z \leqslant \frac{1}{\epsilon} \\ \ln\epsilon & \text{if } \frac{1}{\ln\epsilon} \leqslant z \leqslant \epsilon \\ \frac{1}{z} & \text{if } z \leqslant \frac{1}{\ln\epsilon} \end{cases} \tag{3.2}$$

Denote $E_\epsilon(z) = z g_\epsilon(z)$, for any $z \in \mathbb{R}$. Let $E : 0, +\infty \mapsto \mathbb{R}$,

$$E(z) = \begin{cases} z\ln z & \text{if } z > 0 \\ 0 & \text{if } z = 0 \end{cases} \tag{3.3}$$

Consequently the regularized problem (2.9) will read

$$\begin{aligned} &\frac{d}{dt}\int_\Sigma \frac{1}{M}\psi_\epsilon\phi - \int_\Sigma \frac{1}{M}\psi_\epsilon v\cdot\nabla_x\phi + d_1\int_\Sigma \frac{1}{M}\nabla_x\psi_\epsilon\cdot\nabla_x\phi \\ &+d_1\int_\Sigma \frac{1}{M}\psi_\epsilon g_\epsilon(\psi_\epsilon)\nabla_x\theta\cdot\nabla_x\phi - \int_\Sigma \kappa Q\psi_\epsilon\cdot\nabla_Q\left(\frac{\phi}{M}\right) \\ &+d_2\int_\Sigma M\theta\nabla_Q\left(\frac{\psi_\epsilon}{M}\right)\cdot\nabla_Q\left(\frac{\phi}{M}\right) = 0, \text{ for all } \phi\in V \end{aligned} \tag{3.4}$$

to be considered together with (2.4).

Next we endeavor to prove the existence of a solution to (3.4) using the fixed point technique. Let the operator $S_\epsilon : L^2(0,T;H) \mapsto (0,T;H)$, $S_\epsilon\left(\tilde{\psi}_\epsilon\right) = \psi_\epsilon$, where ψ_ϵ is a solution to the equation:

$$\begin{aligned} &\frac{d}{dt}\int_\Sigma \frac{1}{M}\psi_\epsilon\phi - \int_\Sigma \frac{1}{M}\psi_\epsilon v\cdot\nabla_x\phi + d_1\int_\Sigma \frac{1}{M}\nabla_x\psi_\epsilon\cdot\nabla_x\phi \\ &+d_1\int_\Sigma \frac{1}{M}\psi_\epsilon g_\epsilon\left(\tilde{\psi}_\epsilon\right)\nabla_x\theta\cdot\nabla_x\phi - \int_\Sigma \kappa Q\psi_\epsilon\cdot\nabla_Q\left(\frac{\phi}{M}\right) \\ &+d_2\int_\Sigma M\theta\nabla_Q\left(\frac{\psi_\epsilon}{M}\right)\cdot\nabla_Q\left(\frac{\phi}{M}\right) = 0, \text{ for all } \phi\in V \end{aligned} \tag{3.5}$$

Now the problem of our focus reads

$$\frac{d}{dt}\langle\psi_\epsilon(t),\phi\rangle + a\left(t,\tilde{\psi}_\epsilon(t),\psi_\epsilon(t),\phi\right) = 0, \ \forall\phi\in V \tag{3.6}$$

with $a : (0,T] \times H \times V \mapsto \mathbb{R}$,

$$(t,r,u,w) \mapsto a(t,r,u,w) = \int_\Sigma \frac{1}{M} v(t,x) u \cdot \nabla_x w + d_1 \int_\Sigma \frac{1}{M} \theta(x,t) \nabla_x u \cdot \nabla_x w$$
$$+ d_1 \int_\Sigma \frac{1}{M} \nabla_x \theta(x,t) u g_\epsilon(r) \cdot \nabla_x w - \int_\Sigma \kappa Q u \cdot \nabla_Q \left(\frac{w}{M}\right)$$
$$+ d_2 \int_\Sigma M \theta(x,t) \nabla_Q \left(\frac{u}{M}\right) \cdot \nabla_Q \left(\frac{w}{M}\right)$$

Let $\tilde{\psi}_\epsilon \in L^2(\Sigma_T)$ be fixed. We now prove the following result:

LEMMA 3.1. *There exists a constant $c_1 > 0$ such that*

(i) $\left| a\left(t, \tilde{\psi}_\epsilon, u, w\right)\right| \leqslant c_1 \|u\|_V \|w\|_V, \ \forall u, w \in V$

There exist two constants $\alpha > 0$, $\beta \in \mathbb{R}$ such that

(ii) $a\left(t, \tilde{\psi}_\epsilon, u, u\right) + \beta \|u\|_H^2 \geqslant \alpha \|u\|_V^2, \ \forall u \in V$

PROOF. To prove (i), notice first that:

$$\left| \int_\Sigma \frac{1}{M} v u \cdot \nabla_x w \right| \leqslant \|v\|_{L^\infty(\Omega)} \int_\Sigma \left| \frac{u}{\sqrt{M}} \right| \left| \frac{\nabla_x w}{\sqrt{M}} \right| \leqslant \|v\|_{L^\infty(\Omega)} \|u\|_V \|w\|_V$$

$$\left| \int_\Sigma \kappa Q u \cdot \nabla_Q \left(\frac{w}{M}\right) \right| \leqslant \|\kappa\|_{L^\infty(\Omega)} \int_\Sigma \left| \frac{u}{\sqrt{M}} \right| \left| \sqrt{M} \nabla_Q \left(\frac{w}{M}\right) \right| \leqslant \|\kappa\|_{L^\infty(\Omega)} \|u\|_V \|w\|_V$$

The estimates for the remaining terms are obtainable in a similar nature hence the details are skipped.

Switching now to proving (ii), observe first that

$$\left| - \int_\Sigma \frac{1}{M} v u \cdot \nabla_x u + d_1 \int_\Sigma \frac{1}{M} u g_\epsilon \left(\tilde{\psi}_\epsilon\right) \nabla_x \theta \cdot \nabla_x u - \kappa Q u \cdot \nabla_Q \left(\frac{u}{M}\right) \right|$$
$$\leqslant \eta \|u\|_V^2 + c_\eta \|u\|_H^2, \ \forall \eta > 0$$

and the end of the proof follows. □

This allows to prove the existence and uniqueness of a solution $\psi_\epsilon \in L^2(X_T) \cap L^\infty(0,T;H)$ to (3.6) (see Theorem 4.1 on page 257 and Remark 4.3 on page 258 in [**23**]). This also shows the operator S_ϵ is well defined.

Consider now the dense and continuous injections $V \subset H \subset V'$. Let the operator A_ϵ be such that $A_\epsilon : (0,T) \times L^2(\Sigma) \times V \times V \mapsto V'$, $\langle A_\epsilon(t,r,u), w \rangle = a(t,r,u,w)$, for all $w \in V$ and for all $(t,r,u) \in (0,T) \times L^2(\Sigma) \times V$. Then the equation (3.6) takes the form

$$\frac{d}{dt}\psi_\epsilon + A_\epsilon \left(t, \tilde{\psi}_e, \psi_\epsilon\right) = 0 \tag{3.7}$$

Acting with this equality on ψ_e and making use of Lemma (3.1) leads to

$$\|\psi_\epsilon\|_{X_T} \leqslant c(\epsilon) \tag{3.8}$$

Let us now prove the following result:

Proposition 3.4. *S_ϵ continuously maps $L^2(0,T;H)$ on $L^2(0,T;H)$.*

Proof. Assume $\tilde{\psi} \in L^2(0,T;H)$. Let $\tilde{\psi}_k$ be a sequence strongly converging towards $\tilde{\psi}$ in $L^2(0,T;H)$. Let $\psi_{\epsilon,k} = S_\epsilon\left(\tilde{\psi}_k\right)$ and $\psi_\epsilon = S_\epsilon\left(\tilde{\psi}\right)$. We shall prove that $\psi_{\epsilon,k}$ strongly converges to ψ_ϵ in $L^2(0,T;H)$.

As in (3.8) one gets $\|\psi_{\epsilon,k}\| \leqslant 2c$. By compactness we get a sub-sequence (for simplicity equally denoted $\psi_{\epsilon,k}$) and a $\hat{\psi}_\epsilon \in X_T$ such that $\psi_{\epsilon,k} \xrightarrow[k\to+\infty]{L^2(0,T;H)} \hat{\psi}_\epsilon$ strongly. Next, we take the limit $k \to +\infty$ in the equation satisfied by $\psi_{\epsilon,k}$. Let us first focus on the limit

$$\int_{\Sigma_T} \frac{\psi_{\epsilon,k}}{\sqrt{M}} \nabla_x \theta g_\epsilon\left(\tilde{\psi}\right) \cdot \frac{1}{\sqrt{M}} \nabla_x \phi \xi(t) \xrightarrow[k\to+\infty]{\mathbb{R}} \int_{\Sigma_T} \frac{\psi_\epsilon}{\sqrt{M}} \nabla_x \theta g_\epsilon\left(\tilde{\psi}\right) \cdot \frac{1}{\sqrt{M}} \nabla_x \phi \xi(t)$$

that occurs for any $\xi \in \mathscr{C}^1(0,T)$. Actually, as $\frac{\tilde{\psi}_k}{\sqrt{M}} \xrightarrow[k\to+\infty]{L^2(\Sigma_T)} \frac{\tilde{\psi}}{\sqrt{M}}$, one deduces $\tilde{\psi}_k \xrightarrow[k\to+\infty]{L^2(\Sigma_T)} \tilde{\psi}$. This gives $g_\epsilon\left(\tilde{\psi}_k\right) \xrightarrow[k\to+\infty]{L^2(\Sigma_T)} g_\epsilon\left(\tilde{\psi}\right)$, because $g_\epsilon \in W^{1,\infty}(\mathbb{R})$. Consequently, $\hat{\psi}_\epsilon = S_\epsilon(\tilde{\psi})$, and we get the stated result by virtue of uniqueness. All the remaining limits can be calculated likewise.

Next, by making use of Schauder's fixed point theorem it follows the existence of a fixed point S_ϵ which is a solution of the regularized problem (3.4).

We shall always denote by ψ_ϵ this solution and we also have that $\psi_\epsilon \in X_T$ and ψ_ϵ satisfies (3.8). Therefore:

$$\frac{d}{dt}\psi_\epsilon + A_\epsilon\left(t, \psi_e, \psi_\epsilon\right) = 0 \tag{3.9}$$

□

4. Uniform Estimates

4.1. L^1 Norm Estimates. To begin with, let us first prove the following positive solutions result:

Lemma 4.1. *Assume $\psi_0 \geqslant 0$ and let ψ_ϵ be a solution to the problem* (3.9) *and* (2.4). *Then $\psi_\epsilon \geqslant 0$.*

Proof. We proceed as in [**16**]. Let $\psi_\epsilon = \psi_\epsilon^+ - \psi_\epsilon^-$, with $\psi_\epsilon^+,\ \psi_\epsilon^- \in L^2(0,T;V)$. Using (3.9) upon ψ_ϵ^- gives

$$\frac{1}{2}\frac{d}{dt}\left\|\psi_\epsilon^-\right\|_H^2 + a_\epsilon\left(t, \psi_\epsilon, \psi_\epsilon^+, \psi_\epsilon^-\right) = 0$$

and

$$\psi_\epsilon^-(t=0) = 0$$

Next, upon making use of the estimates in Lemma (3.1) we get $\psi_\epsilon^- = 0$, fact which ends the proof.

□

The following result gives the announced uniform estimate:

Lemma 4.2. *For a.e. $t \in (0,T)$, one has $\|\psi_\epsilon(t)\|_{L^1(\Sigma)} = \|\psi_0\|_{L^1(\Sigma)}$.*

PROOF. Set $M \in V$ into the variational formulation (3.4) for ψ_ϵ. We get

$$\frac{d}{dt}\int_\Sigma \psi_\epsilon = 0$$

which triggers

$$\int_\Sigma \psi_\epsilon(t) = \int_\Sigma \psi_0, \text{ for } t \in (0,T) \text{ a.e.}$$

The fact that ψ_ϵ has been proved to be positive leads to the above stated conclusion. Additionally we get the below uniform estimate

$$\|\psi_\epsilon(t)\|_{L^\infty(0,T;L^1(\Sigma))} = \|\psi_0\|_{L^1(\Sigma)}$$

□

4.2. Stronger Uniform Estimate. We here prove the following result:

LEMMA 4.3. *There exists* $c > 0$, *independent of* ϵ, *such that any solution* ψ_ϵ *to* (3.4) *and* (3.9) *satisfies*

$$\|\psi_\epsilon\|_{X_T} + \|\psi_\epsilon\|_{L^\infty(0,T;H)} \leqslant c$$

PROOF. Using (3.9) upon ψ_ϵ gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Sigma \frac{1}{M}\psi_\epsilon^2 + d_1\int_\Sigma \frac{\theta}{M}\left|\nabla_x\psi_\epsilon\right|^2 + d_2\int_\Sigma \theta M\left|\nabla_Q\left(\frac{\psi_\epsilon}{M}\right)\right|^2 \\
&\leqslant \int_\Sigma \left|\frac{\psi_\epsilon}{\sqrt{M}}v\cdot\frac{\nabla_x\psi_\epsilon}{\sqrt{M}}\right| + \int_\Sigma \left|\kappa Q\frac{\psi_\epsilon}{\sqrt{M}}\sqrt{M}\nabla_Q\left(\frac{\psi_\epsilon}{M}\right)\right| \\
&+d_1\int_\Sigma \frac{1}{M}\left|\nabla_x\theta\psi_\epsilon g_\epsilon\left(\psi_\epsilon\right)\cdot\nabla_x\phi\right| \qquad (4.1)
\end{aligned}$$

We now proceed to finding estimates for the terms appearing in the right hand side of (4.1). For any $\eta > 0$,

$$\int_\Sigma \left|\frac{\psi_\epsilon}{\sqrt{M}}v\cdot\frac{\nabla_x\psi_\epsilon}{\sqrt{M}}\right| \leqslant \eta\left\|\frac{\nabla_x\psi_\epsilon}{\sqrt{M}}\right\|^2_{L^2(\Sigma)} + \frac{1}{4\eta}\|v\|_{L^\infty(\Sigma_T)}\int_\Sigma \frac{1}{M}\psi_\epsilon^2 \qquad (4.2)$$

$$\begin{aligned}
\int_\Sigma \left|\kappa Q\frac{\psi_\epsilon}{\sqrt{M}}\sqrt{M}\nabla_Q\left(\frac{\psi_\epsilon}{M}\right)\right| &\leqslant \eta\left\|\sqrt{M}\nabla_Q\left(\frac{\psi_\epsilon}{M}\right)\right\|^2_{L^2(\Sigma)} \\
&+ \frac{1}{4\eta}\|\kappa\|_{L^\infty(\Sigma_T)}\int_\Sigma \frac{1}{M}\psi_\epsilon^2 \qquad (4.3)
\end{aligned}$$

We now focus on the last term in the right hand side of (4.1). We denote

$$I_\epsilon = d_1\int_\Sigma \frac{1}{M}\left|\nabla_x\theta\psi_\epsilon g_\epsilon\left(\psi_\epsilon\right)\cdot\nabla_x\psi_\epsilon\right|$$

First observe that we have

$$|g_\epsilon(z)| \leqslant |\ln(z)|, \ \forall z > 0$$

Next, since

$$\lim_{z\to 0_+} \frac{z\,|\ln(z)|}{z^{1-\delta_1}+z^{1+\delta_1}} = \lim_{z\to +\infty} \frac{z\,|\ln(z)|}{z^{1-\delta_1}+z^{1+\delta_1}} = 0,\ \forall \delta_1>0$$

it follows that, for any $\delta_1 > 0$, there exists a $c_{\delta_1} > 0$ such that

$$z\,|\ln(z)| \leqslant c_{\delta_1}\left(z^{1-\delta_1}+z^{1+\delta_1}\right),\ \forall z>0$$

We further obtain

$$|\psi_\epsilon \ln(\psi_\epsilon)| \leqslant C_1\left(\psi_\epsilon^{1-\delta_1}+\psi_\epsilon^{1+\delta_1}\right)$$

Therefore

$$|I_\epsilon| \leqslant c_1\|\theta\|_{L^\infty(0,T;W^{1,\infty}(\Sigma))}\left(I_{1\epsilon}+I_{2\epsilon}\right) \tag{4.4}$$

where

$$I_{1\epsilon} = \int_\Sigma \frac{1}{M}\psi_\epsilon^{1-\delta_1}\,\|\nabla_x\psi_\epsilon\|$$

$$I_{2\epsilon} = \int_\Sigma \frac{1}{M}\psi_\epsilon^{1+\delta_1}\,\|\nabla_x\psi_\epsilon\|$$

The following estimates hold true:

$$\begin{aligned} I_{1\epsilon} &= \int_\Omega\int_B \left(\frac{\psi_\epsilon}{\sqrt{M}\,(1-\|Q\|^2)}\right)^{1-\delta_1} \frac{\|\nabla_x\psi_\epsilon\|}{\sqrt{M}}\frac{\left(1-\|Q\|^2\right)^{1-\delta_1}}{M^{\delta_1/2}} \\ &\leqslant \int_\Omega \left\|\frac{\psi_\epsilon}{\sqrt{M}\,(1-\|Q\|^2}\right\|_{L^2(B)}^{1-\delta_1} \left\|\frac{\nabla_x\psi_\epsilon}{\sqrt{M}}\right\|_{L^2(B)} \left(\int_B \frac{\left(1-\|Q\|^2\right)^{2/\delta_1-2}}{M}\mathrm{d}Q\right)^{\delta_1/2} \end{aligned} \tag{4.5}$$

Consider $\delta_1 > 0$ small enough such that

$$\int_B \frac{\left(1-\|Q\|^2\right)^{2/\delta_1-2}}{M}\mathrm{d}Q < \infty$$

Using (i) of Proposition (3.1) one gets that

$$I_{1\epsilon} \leqslant c_2\|\psi_\epsilon\|_V^{2-\delta_1} \tag{4.6}$$

Next,

$$I_{2\epsilon} = \int_\Omega\int_B \frac{\psi_\epsilon^{1-\delta_1}}{M^{(1-\delta_1)/2}}\psi_\epsilon^{2\delta_1}\frac{\|\nabla_x\psi_\epsilon\|}{\sqrt{M}}\frac{1}{M^{\delta_1/2}}$$

with $\dfrac{\psi_\epsilon^{1-\delta_1}}{M^{(1-\delta_1)/2}}\psi_\epsilon^{2\delta_1} \in L^{p/(1-\delta_1)}$, $\psi_\epsilon^{2\delta_1} \in L^{1/(2\delta_1)}$, $\dfrac{\|\nabla_x\psi_\epsilon\|}{\sqrt{M}} \in L^2$, $\dfrac{1}{M^{\delta_1/2}} \in L^a$, where $\dfrac{1-\delta_1}{p}+2\delta_1+\dfrac{1}{2}+\dfrac{1}{a}=1$, and $p>2$ being given in Section (3.1). As $p>2$ one can find an $a>2$ so that the later equality is satisfied by choosing $\delta_1 >$ small enough. Therefore $a = \dfrac{2p}{p-2+2\delta_1-4p\delta_1}$. Clearly we can choose $\delta_1 > 0$ small enough such that $\dfrac{1}{M^{\delta_1/2}} \in L^a(B)$. It results that

$$I_{2\epsilon} \leqslant c \int_\Omega \left\| \frac{\psi_\epsilon}{\sqrt{M}} \right\|_{L^p(B)}^{1-\delta_1} \left\| \frac{\nabla_x \psi_\epsilon}{\sqrt{M}} \right\|_{L^2(B)} \left(\int_B \psi_\epsilon \right)^{2\delta_1} \mathrm{d}x$$

with

$$\left\| \frac{\psi_\epsilon}{\sqrt{M}} \right\|_{L^p(B)}^{1-\delta_1} \in L^{p/(1-\delta_1)}(\Omega),$$

$$\left\| \frac{\nabla_x \psi_\epsilon}{\sqrt{M}} \right\|_{L^2(B)} \in L^2(\Omega),$$

$$\left(\int_B \psi_\epsilon \right)^{2\delta_1} \mathrm{d}x \in L^{1/(2\delta_1)}(\Omega).$$

Using Lemma (4.2) we obtain

$$I_{2\epsilon} \leqslant c_3 \|\psi_0\|_{L^1(\Sigma)} \|\psi_\epsilon\|_V^{2-\delta_1} \tag{4.7}$$

Further on, from (4.4), (4.6) and (4.7), one sees that for any $\eta > 0$ there exists $c_\eta >$ such that

$$|I_\epsilon| \leqslant \eta \|\psi_\epsilon\|_V^2 + c_\eta \tag{4.8}$$

and from (4.1), (4.2), (4.3) and (4.8) and upon using Gronwall's inequality that

$$\int_\Sigma \frac{1}{M} \psi_\epsilon^2 \leqslant c_4(T), \ \forall t \in (0,T)$$

Next, upon integrating (4.1) for $t \in (0,T)$, it gives

$$\int_{\Sigma_T} \left[\frac{1}{M} |\nabla_x \psi_e|^2 + M \left| \nabla_Q \left(\frac{\psi_e}{M} \right) \right|^2 \right] \mathrm{d}x \mathrm{d}Q \mathrm{d}t \leqslant c_5(T)$$

Going back to (3.9) we see the proof is achieved.

□

5. Taking the Limit for $\epsilon \to 0$

Equations (3.4) and (2.4) say that for any $\xi \in \mathscr{C}^1(0,T)$, with $\xi(T) = 0$, one has

$$\begin{aligned} &- \int_{\Sigma_T} \frac{1}{M} \psi_\epsilon \phi \xi' - \int_\Sigma \frac{1}{M} \psi_0 \phi \xi(0) - \int_{\Sigma_T} \frac{\psi_\epsilon}{M} v \cdot \nabla_x \phi \xi + d_1 \int_{\Sigma_T} \frac{\theta}{M} \nabla_x \psi_\epsilon \cdot \nabla_x \phi \xi \\ &+ d_1 \int_{\Sigma_T} \frac{1}{M} \nabla_x \theta \psi_\epsilon g_\epsilon(\psi_\epsilon) \cdot \nabla_x \phi \xi - \int_{\Sigma_T} \kappa Q \psi_\epsilon \cdot \nabla_Q \left(\frac{\phi}{M} \right) \xi \\ &+ d_2 \int_\Sigma \theta M \nabla_Q \left(\frac{\psi}{M} \right) \cdot \nabla_Q \left(\frac{\phi}{M} \right) \xi = 0 \end{aligned} \tag{5.1}$$

Thanks to Lemma (4.3) it follows there exists $\psi \in X_T \cap L^\infty(0,T;H)$ such that, for a sub-sequence for convenience also denoted here ψ_ϵ, one has

$$\psi_\epsilon \xrightarrow[\epsilon \to 0]{L^2(0,T;V)} \psi \text{ weakly}$$

$$\psi_\epsilon \xrightarrow[\epsilon\to 0]{L^\infty(0,T;V)} \psi \text{ weakly} - *$$

We also have $\psi \geqslant 0$ and $\|\psi\|_{L^\infty(0,T;L^1(\Sigma))} = \|\psi_0\|_{L^1(\Sigma}$, thanks to Lemma (4.2). We now prove the following Theorem:

THEOREM 5.1 (**Main Result**). *The function ψ is a solution to* (2.10).

PROOF. Taking the limit $\epsilon \to 0$ in (5.1) involves standard procedures save for the following limit

$$\int_{\Sigma_T} \frac{1}{M}\nabla_x\theta\psi_\epsilon g_\epsilon(\psi_\epsilon)\cdot\nabla_x\phi\xi \xrightarrow[\epsilon\to 0]{\mathbb{R}} \int_{\Sigma_T}\frac{1}{M}\nabla_x\theta\psi\ln(\psi)\cdot\nabla_x\phi\xi \tag{5.2}$$

which is here addressed. In doing so we draw inspiration from our previous paper [**16**]. A density based reasoning is put to work: assume first that $\dfrac{\phi}{M} \in \mathscr{C}^\infty\left(\bar{\Sigma}\right)$. Up to a ϵ-related sub-sequence, by compactness one has the strong convergence $\psi_\epsilon \xrightarrow[\epsilon\to 0]{L^2(0,T;H)} \psi$. It follows the strong convergence $\dfrac{\psi_\epsilon}{\sqrt{M}} \xrightarrow[\epsilon\to 0]{L^2(\Sigma_T)} \dfrac{\psi}{\sqrt{M}}$. We next deduce, equally up to a ϵ-related sub-sequence, that

$$\frac{\psi_\epsilon}{\sqrt{M}}(x,Q,t) \xrightarrow[\epsilon\to 0]{\mathbb{R}} \frac{\psi}{\sqrt{M}}(x,Q,t),\ (x,Q,t)\in\Sigma_T \text{ a.e.} \tag{5.3}$$

and that there exists $h \in L^2(\Sigma_T)$, independent of ϵ, such that

$$\left|\frac{\psi_\epsilon}{\sqrt{M}}\right| \leqslant h,\ (x,Q,t)\in\Sigma_T \text{ a.e.} \tag{5.4}$$

From (5.3) it results that $\psi_\epsilon(x,Q,t) \xrightarrow[\epsilon\to 0]{L^2(\Sigma_T)} \psi(x,Q,t)$ for $(x,Q,t)\in\Sigma_T$ a.e..

On one hand, assume first that $\psi(x,Q,t) > 0$. Selecting $\epsilon > 0$ small enough one gets $g_\epsilon(\psi_\epsilon(x,Q,t)) = \ln(\psi_\epsilon(x,Q,t))$. Thus, since function E is continuous,

$$E_\epsilon(\psi_\epsilon(x,Q,t)) = E(\psi_\epsilon(x,Q,t)) \xrightarrow[\epsilon\to 0]{\mathbb{R}} E(\psi(x,Q,t))$$

Assume next $\psi(x,Q,t) = 0$. Given that $\psi_\epsilon \geqslant 0$ we have $|\psi_e g_\epsilon(\psi_e)| \leqslant |E(\psi_e)|$, thus triggering

$$E_\epsilon(\psi_\epsilon(x,Q,t)) \xrightarrow[\epsilon\to 0]{\mathbb{R}} 0 = E(\psi(x,Q,t)),\ \text{a.e. } (x,Q,t)\in\Sigma_T$$

We have thus proved that

$$E_\epsilon(\psi_\epsilon(x,Q,t)) \xrightarrow[\epsilon\to 0]{\mathbb{R}} E(\psi(x,Q,t)),\ \text{a.e. } (x,Q,t)\in\Sigma_T$$

On the other hand now, for any $\delta_2 > 0$ there exists $c(\delta_2) > 0$ such that

$$|E(\psi_\epsilon)| \leqslant c(\delta_2)\left[\psi_\epsilon^{1-\delta_2} + \psi_\epsilon^{1+\delta_2}\right]$$

Invoking (5.4) we get

$$|E_\epsilon(\psi_\epsilon)| \leqslant c(\delta_2)\left[\left(h\sqrt{M}\right)^{1-\delta_2} + \left(h\sqrt{M}\right)^{1+\delta_2}\right]$$

Observing that functions $\left(h\sqrt{M}\right)^{1-\delta_2}$ and $\left(h\sqrt{M}\right)^{1+\delta_2}$ belong to $L^1(\Sigma_T)$ (given that $h \in L^2(\Sigma_T)$), we obtain the expected result invoking Lebesgue's dominated convergence theorem. This ends the reasoning to proving the passing to the limit of (5.2) is lawful.

Then the limit function ψ solves (2.10) with ϕ such that $\dfrac{\phi}{M} \in \mathscr{C}^\infty\left(\bar{\Sigma}\right)$.

Let now $\phi \in V$ be fixed. Proposition (3.3) implies there exists a sequence $\phi_k \in V$, with $\dfrac{\phi_k}{M} \in \mathscr{C}^\infty\left(\bar{\Sigma}\right)$, such that $\phi_k \xrightarrow[k\to+\infty]{V} \phi$. Then ψ solves (2.10) with ϕ being replaced by ϕ_k. We now let $k \to +\infty$ in this late equation. All limits are easily evaluated save for the one below

$$\lim_{k\to+\infty} \int_{\Sigma_T} \frac{1}{M} \nabla_x \theta \psi \ln(\psi) \cdot \nabla_x \phi_k \xi = \int_{\Sigma_T} \frac{1}{M} \nabla_x \theta \psi \ln(\psi) \cdot \nabla_x \phi \xi \tag{5.5}$$

for which we give a detailed proof. We recall here that for any $\delta_2 > 0$, there exists a $c(\delta_2) >$ such that

$$|\psi \ln(\psi)| \leqslant c(\delta_2)\left(|\psi|^{1-\delta_2} + |\psi|^{1+\delta_2}\right)$$

It then suffices to prove that

$$\frac{\psi^{1-\delta_2}}{\sqrt{M}} \in L^2(\Sigma_T) \tag{5.6}$$

and that

$$\frac{\psi^{1+\delta_2}}{\sqrt{M}} \in L^2(\Sigma_T) \tag{5.7}$$

which would give the expected result.

Actually, observe that

$$\int_{\Sigma_T} \frac{1}{M} \psi^{2-2\delta_2} = \int_0^T \int_\Sigma \int_B \left(\frac{\psi}{\sqrt{M}(1-\|Q\|^2)}\right)^{2-2\delta_2} \frac{(1-\|Q\|^2)^{2-2\delta_2}}{M^{\delta_2}}$$

with $\left(\dfrac{\psi}{\sqrt{M}(1-\|Q\|^2)}\right)^{2-2\delta_2} \in L^{2/(2-2\delta_2)}(B)$; for $\delta_2 > 0$ small enough we have $\dfrac{(1-\|Q\|^2)^{2-2\delta_2}}{M^{\delta_2}} \in L^\infty(B)$.

Then, upon observing that $\left\|\dfrac{\psi}{\sqrt{M}(1-\|Q\|^2)}\right\|_{L^2(B)}^{2-2\delta_2} \in L^{2/(2-2\delta_2)}(\Omega)$,

$$\begin{aligned}
\int_{\Sigma_T} \frac{1}{M} \psi^{2-2\delta_2} &\leqslant c \int_0^T \int_\Omega \left\|\frac{\psi}{\sqrt{M}(1-\|Q\|^2)}\right\|_{L^2(B)}^{2-2\delta_2} \mathrm{d}x\mathrm{d}t \\
&\leqslant c \int_0^T \int_\Omega \|\psi\|_{V_B}^{2-2\delta_2} \mathrm{d}x\mathrm{d}t \\
&\leqslant c \int_0^T \|\psi\|_V^{2-2\delta_2} \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{5.8}$$

thus proving (5.6); use of Proposition (3.1) has been made in obtaining the above estimates.

To prove (5.7) we proceed as following:

$$\begin{aligned} \int_\Sigma \frac{1}{M}\psi^{2+2\delta_2} &= \int_0^T \int_\Sigma \frac{1}{M}\psi^2\psi^{2\delta_2} \mathrm{d}Q\mathrm{d}x\mathrm{d}t \leqslant \int_0^T \left\| \frac{\psi}{\sqrt{M}} \right\|^2_{L^{2/(1-2\delta_2)}(\Sigma)} \left(\int_\Sigma \psi \right)^{2\delta_2} \mathrm{d}t \\ &\leqslant \|\psi_0\|^{2\delta_2}_{L^1(\Sigma)} \int_0^T \left\| \frac{\psi}{\sqrt{M}} \right\|^2_{H^1(\Sigma)} \mathrm{d}t \qquad (5.9) \end{aligned}$$

where, in the above, we have used the fact that $H^1(\Sigma) \hookrightarrow L^{2/(1-2\delta_2)}(\Sigma)$ for δ_2 small enough. Also $\frac{1}{M}\psi^2 \in L^{1/(1-2\delta_2)}(\Sigma)$, $\psi^{2\delta_2} \in L^{1/(2\delta_2)}(\Sigma)$.

Now, invoking Proposition (3.2), we get the result and the proof is achieved. □

6. Conclusions

In this paper we have proved the existence of positive solutions to a new configurational probability diffusion equation of relevance for polymer fluid dynamics. The results contained here are a continuation of the work [**16**]. On the modeling level the most notable of all the features of the CPD is that in obtaining it use has been made of the temperature influence on the dominant molecular interactions. This equation is further to be studied together with a temperature diffusion equation, itself obtained on kinetic theory grounds. The results will be published in a following-up paper.

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Université Lyon 1, CNRS, Institut Camille Jordan UMR 5208. Bât Braconnier, 43 Boulevard du 11 Novembre 1918, F-69622, Villeurbanne, France.
Email address: ciuperca@math.univ-lyon.fr

INSA-Lyon, CNRS, Institut Camille Jordan UMR 5208 & Pôle de Mathématiques. Bât. Leonard de Vinci No. 401, 21 Avenue Jean Capelle, F-69621, Villeurbanne, France.
Email address: liviu-iulian.palade@insa-lyon.fr