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New Einstein metrics on the Lie group SO(n) which are not naturally reductive

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We obtain new invariant Einstein metrics on the compact Lie groups SO(n) ($n \geq 7$) which are not naturally reductive. This is achieved by imposing certain symmetry assumptions in the set of all left-invariant metrics on SO(n) and by computing the Ricci tensor for such metrics. The Einstein metrics are obtained as solutions of systems polynomial equations, which we manipulate by symbolic computations using Gröbner bases.

1. Introduction

A Riemannian manifold (M, g) is called Einstein if it has constant Ricci curvature, i.e. $\mathrm{Ric}_g = \lambda \cdot g$ for some $\lambda \in \mathbb{R}$. For results on Einstein manifolds before 1987 we refer to the book by A. Besse [5]. The two articles [19], [20] of M. Wang contain results up to 1999 and 2013 respectively. General existence results are difficult to obtain and some methods are described in [6], [7] and [21]. For homogeneous spaces G/K the problem is to find and classify all G-invariant Einstein metrics. The problem is even more difficult for the case of a Lie group, where we need to find (or prove existence) of left-invariant Einstein metrics. Even for the compact Lie groups SU(3) and SU(2) $\times$ SU(2) the number of left-invariant Einstein metrics is still unknown.

In the present paper we investigate left-invariant Einstein metrics on the compact Lie group SO(n). We recall that a Riemannian metric on a Lie group is called *left-invariant* (resp. right-invariant) if for all $a \in G$ the left translations $L_a : G \to G$, $L_a(x) = ax$ (resp. the right translations $R_a(x) = xa$) are isometries. A Riemannian metric which is both

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left-invariant and right-invariant is called *bi-invariant*. It is known that a compact and semisimple Lie group equipped with a bi-invariant metric is Einstein.

In the work [9], J. E. D'Atri and W. Ziller found a large number of left-invariant Einstein metrics, which are naturally reductive, on the compact Lie groups $\mathrm{SU}(n), \mathrm{SO}(n)$ and $\mathrm{Sp}(n)$. In the same article they posed the question whether there exist left-invariant Einstein metrics on compact Lie groups which are not naturally reductive (we refer to Section 4 for the precise definition of the natural reductivity).

Some contributions to this problem are the following: In [14] K. Mori obtained non naturally reductive Einstein metrics on the Lie group $\mathrm{SU}(n)$ for $n \geq 6$, and in [2] the first two authors and K. Mori proved existence of non naturally reductive Einstein metrics on the compact Lie groups $\mathrm{SO}(n)$ $(n \geq 11)$, $\mathrm{Sp}(n)$ $(n \geq 3)$, E_6, E_7 and E_8. In [8] Z. Chen and K. Liang found three naturally reductive and one non naturally reductive Einstein metric on the compact Lie group F_4. Also, in [4] the authors obtained new left-invariant Einstein metrics on the symplectic group $\mathrm{Sp}(n)$ $(n \geq 3)$, and in [10] I. Chrysikos and the second author obtained non naturally reductive Einstein metrics on the exceptional Lie groups E_6, E_7, E_8, F_4 and G_2. We also mention the works [11], [18] by G. W. Gibbons, H. Lü and C. N. Pope where they discussed left-invariant Einstein metrics on the Lie groups $\mathrm{SO}(n)$, G_2 and $\mathrm{SU}(3)$ which are however naturally reductive, as well as [15] by A. H. Mujtaba, who obtained certain classes of left-invariant metrics on $\mathrm{SU}(n)$, which were previously found in [13] and [9].

The aim of the present work is to obtain left-invariant Einstein metrics on the compact Lie groups $\mathrm{SO}(n)$ $(n \geq 7)$ which are not naturally reductive. The Einstein metrics obtained here are different from the ones obtained in [2]. The idea behind our approach is to consider an appropriate subgroup K and a corresponding homogeneous space G/K whose isotropy representation decomposes into $\mathrm{Ad}(K)$-irreducible and non equivalent summands. Then the tangent space $\mathfrak{g}$ of G decomposes, via the submersion $G \to G/K$ with fiber K, into a direct sum of non equivalent $\mathrm{Ad}(K)$-modules. By taking into account the diffeomorphism $G/\{e\} \cong (G \times K)/\mathrm{diag}(K)$ we consider left-invariant metrics on G which are determined by diagonal $\mathrm{Ad}(K)$-invariant scalar products on $\mathfrak{g}$, which in turn enables us to use well known formulas for the Ricci curvature (e.g. [5, Corollary 7.38, p. 184], [17, Lemma 1.1, p. 52]).

More precisely, for the group $\mathrm{SO}(n)$ we write $n = k_1 + k_2 + k_3$, (k_1, k_2, k_3 positive integers), and we consider the closed subgroup $K = \mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3)$. This determines the homogeneous space $G/K = \mathrm{SO}(k_1 +$

$k_2 + k_3)/(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3))$, which is an example of a *generalized Wallach space* according to [16]. For $k_1 \geq k_2 \geq k_3 \geq 2$ with $k_i \neq k_j$ the isotropy representation $\mathfrak{m} = \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}$ of G/K does not contain equivalent summands, thus we consider left-invariant metrics determined by $\mathrm{Ad}(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3))$-invariant scalar products of the form

$$\langle\ ,\ \rangle = \begin{array}{l} x_1\,(-B)|_{\mathfrak{so}(k_1)} + x_2\,(-B)|_{\mathfrak{so}(k_2)} + x_3\,(-B)|_{\mathfrak{so}(k_3)} \\ +x_{12}\,(-B)|_{\mathfrak{m}_{12}} + x_{13}\,(-B)|_{\mathfrak{m}_{13}} + x_{23}\,(-B)|_{\mathfrak{m}_{23}} \end{array}, \quad x_i, x_{ij} > 0.$$

If $k_3 = 1$, $k_1 \geq k_2 \geq 2$ we omit the variable x_3, and if $k_2 = k_3 = 1$, $k_1 \geq 3$ we omit the variables x_2 and x_3. In the last case the submodules $\mathfrak{m}_{12}$ and $\mathfrak{m}_{13}$ are equivalent and we treat this separately in Section 7. We note that the difference with the work [2] is that there, the compact Lie group G had been viewed as the total space over a generalized flag manifold G/K, whose isotropy representation recomposes into two isotropy summands.

By using the main theorems of D'Atri and Ziller ([9, Theorem 1, p. 9, and Theorem 3, p. 14]) we obtain conditions on the positive variables $x_1, x_2, x_3, x_{12}, x_{13}$ and x_{23} so that the above metric is naturally reductive. For all possible partitions of $n = k_1 + k_2 + k_3$ we write the components of the Ricci tensor and then use methods of symbolic computation to prove existence of positive solutions for the systems of algebraic equations obtained by the Einstein equation. For the case of the Lie groups SO(5) and SO(6) our method gives only naturally reductive metrics. This is explained in Section 7 and see also Table 1.

The main result is the following:

Theorem 1.1. *The compact simple Lie groups* $\mathrm{SO}(n)$ $(n \geq 7)$ *admit left-invariant Einstein metrics which are not naturally reductive.*

The paper is organized as follows: In Section 2 we recall a formula for the Ricci tensor for homogeneous spaces given in [17]. In Section 3 we describe the left-invariant metrics which will be considered in this work, and in Section 4 we give conditions under which such left-invariant metrics on $\mathrm{SO}(n)$ are naturally reductive with respect to $\mathrm{SO}(n) \times L$, for some closed subgroup L of $\mathrm{SO}(n)$. In Section 5 we obtain explicit formulas for the Ricci tensor of left-invariant metrics. In Section 6 we investigate the solutions of the Einstein equation by using Gröbner bases. We have used computer programs in Mathematica and Maple to get Gröbner bases and to obtain solutions numerically. Then Theorem 1.1 follows from Propositions 6.1, 6.2 and 6.3. Finally, in Section 7 we prove that the compact Lie groups $\mathrm{SO}(n)$ admit only naturally reductive left-invariant Einstein metrics determined by $\mathrm{Ad}(\mathrm{SO}(n-2))$-invariant scalar products (here $k_1 = n-2, k_2 = k_3 = 1$). This case is of special interest, because the decomposition $\mathfrak{so}(n) = \mathfrak{so}(k_1) \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}$ of the tangent space of $\mathrm{SO}(n)$ contains equivalent $\mathrm{Ad}(\mathrm{SO}(n-2))$-submodules. Hence, we need to confirm that for the $\mathrm{Ad}(\mathrm{SO}(n-2))$-invariant scalar products under consideration the Ricci tensor is diagonal.

2. The Ricci tensor for reductive homogeneous spaces

In this section we recall an expression for the Ricci tensor for an G-invariant Riemannian metric on a reductive homogeneous space whose isotropy representation is decomposed into a sum of non equivalent irreducible summands.

Let G be a compact semisimple Lie group, K a connected closed subgroup of G and let $\mathfrak{g}$ and $\mathfrak{k}$ be the corresponding Lie algebras. The Killing form B of $\mathfrak{g}$ is negative definite, so we can define an $\mathrm{Ad}(G)$-invariant inner product $-B$ on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$ with respect to $-B$ so that $[\,\mathfrak{k}, \mathfrak{m}\,] \subset \mathfrak{m}$ and $\mathfrak{m} \cong T_o(G/K)$. We assume that $\mathfrak{m}$ admits a decomposition into mutually non equivalent irreducible $\mathrm{Ad}(K)$-submodules as

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q. \tag{1}$$

Then any G-invariant metric on G/K can be expressed as

$$\langle\ ,\ \rangle = x_1(-B)|_{\mathfrak{m}_1} + \cdots + x_q(-B)|_{\mathfrak{m}_q}, \tag{2}$$

for positive real numbers $(x_1, \ldots, x_q) \in \mathbb{R}^q_+$. Note that G-invariant symmetric covariant 2-tensors on G/K are of the same form as the Riemannian metrics (although they are not necessarily positive definite). In particular, the Ricci tensor r of a G-invariant Riemannian metric on G/K is of the same form as (2), that is

$$r = y_1(-B)|_{\mathfrak{m}_1} + \cdots + y_q(-B)|_{\mathfrak{m}_q},$$

for some real numbers $y_1, \ldots, y_q$.

Let $\{e_\alpha\}$ be a $(-B)$-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e. $e_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$. We put $A^\gamma_{\alpha\beta} = -B\left([e_\alpha, e_\beta], e_\gamma\right)$ so that $[e_\alpha, e_\beta] = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$ and set $\left[{k \atop ij}\right] = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i,\ e_\beta \in \mathfrak{m}_j,\ e_\gamma \in \mathfrak{m}_k$ (cf. [21]). Then the positive numbers $\left[{k \atop ij}\right]$ are independent of the $(-B)$-orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, and $\left[{k \atop ij}\right] = \left[{k \atop ji}\right] = \left[{j \atop ki}\right]$.

Let $d_k = \dim \mathfrak{m}_k$. Then we have the following:

Lemma 2.1. ([17]) *The components $r_1, \dots, r_q$ of the Ricci tensor r of the metric $\langle\ ,\ \rangle$ of the form (2) on G/K are given by*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k}\sum_{j,i}\frac{x_k}{x_j x_i}\begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k}\sum_{j,i}\frac{x_j}{x_k x_i}\begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 1,\ \dots, q), \tag{3}$$

where the sum is taken over $i, j = 1, \dots, q$.

Since by assumption the submodules $\mathfrak{m}_i, \mathfrak{m}_j$ in the decomposition (1) are mutually non equivalent for any $i \neq j$, it is $r(\mathfrak{m}_i, \mathfrak{m}_j) = 0$. If $\mathfrak{m}_i \cong \mathfrak{m}_j$ for some $i \neq j$ then we need to check whether $r(\mathfrak{m}_i, \mathfrak{m}_j) = 0$. This is not an easy task in general. Once the condition $r(\mathfrak{m}_i, \mathfrak{m}_j) = 0$ is confirmed we can use Lemma 2.1. Then G-invariant Einstein metrics on $M = G/K$ are exactly the positive real solutions $g = (x_1, \dots, x_q) \in \mathbb{R}^q_+$ of the polynomial system $\{r_1 = \lambda,\ r_2 = \lambda,\ \dots,\ r_q = \lambda\}$, where $\lambda \in \mathbb{R}_+$ is the Einstein constant.

3. A class of left-invariant metrics on $\mathrm{SO}(n) = \mathrm{SO}(k_1 + k_2 + k_3)$

We will describe a decomposition of the tangent space of the Lie group SO(n) which will be convenient for our study. We consider the closed subgroup $K = \mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3)$ of $G = \mathrm{SO}(k_1 + k_2 + k_3)$ $(k_1 \geq k_2 \geq k_3 \geq 2)$, where the embedding of K in G is diagonal, and the fibration

$$\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3) \to \mathrm{SO}(k_1+k_2+k_3) \to \frac{\mathrm{SO}(k_1 + k_2 + k_3)}{\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3)}.$$

The base space of the above fibration is an example of a *generalized Wallach space* (cf. [16]). Then the tangent space $\mathfrak{so}(k_1 + k_2 + k_3)$ of the orthogonal group $G = \mathrm{SO}(k_1 + k_2 + k_3)$ can be written as a direct sum of two $\mathrm{Ad}(K)$-invariant modules, the horizontal space $\mathfrak{m} \cong T_o(G/K)$ and the vertical space $\mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{so}(k_3)$, i.e.

$$\mathfrak{so}(k_1 + k_2 + k_3) = \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{so}(k_3) \oplus \mathfrak{m}. \tag{4}$$

The tangent space $\mathfrak{m}$ of G/K is given by $\mathfrak{k}^{\perp}$ in $\mathfrak{g} = \mathfrak{so}(k_1 + k_2 + k_3)$ with respect to $-B$. If we denote by $M(p, q)$ the set of all $p \times q$ real matrices, then we see that $\mathfrak{m}$ is given by

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & A_{12} & A_{13} \\ -{}^tA_{12} & 0 & A_{23} \\ -{}^tA_{13} & -{}^tA_{23} & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} A_{12} \in M(k_1, k_2), \\ A_{13} \in M(k_1, k_3), \\ A_{23} \in M(k_2, k_3) \end{array} \right\} \tag{5}$$

and we have

$$
(6)\qquad
\begin{aligned}
&\mathfrak{m}_{12} = \begin{pmatrix} 0 & A_{12} & 0 \\ -{}^tA_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathfrak{m}_{13} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ -{}^tA_{13} & 0 & 0 \end{pmatrix}, \\
&\mathfrak{m}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & -{}^tA_{23} & 0 \end{pmatrix}.
\end{aligned}
$$

Note that the action of $\mathrm{Ad}(k)$ $(k \in K)$ on $\mathfrak{m}$ is given by

$$
(7)\qquad
\begin{aligned}
&\mathrm{Ad}(k) \begin{pmatrix} 0 & A_{12} & A_{13} \\ -{}^tA_{12} & 0 & A_{23} \\ -{}^tA_{13} & -{}^tA_{23} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & {}^th_1 A_{12} h_2 & {}^th_1 A_{13} h_3 \\ -{}^th_2 {}^tA_{12} h_1 & 0 & {}^th_2 A_{23} h_3 \\ -{}^th_3 {}^tA_{13} h_1 & -{}^th_3 {}^tA_{23} h_2 & 0 \end{pmatrix},
\end{aligned}
$$

where $\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \in K$. The subspaces $\mathfrak{m}_{12}$, $\mathfrak{m}_{13}$ and $\mathfrak{m}_{23}$ are irreducible $\mathrm{Ad}(K)$-submodules whose dimensions are $\dim \mathfrak{m}_{12} = k_1k_2$, $\dim \mathfrak{m}_{13} = k_1k_3$ and $\dim \mathfrak{m}_{23} = k_2k_3$. They are given as $(-B)$-orthogonal complements of $\mathfrak{so}(k_i) \oplus \mathfrak{so}(k_j)$ in $\mathfrak{so}(k_i + k_j)$ $(1 \leq i < j \leq 3)$, respectively.

Note that the irreducible submodules $\mathfrak{m}_{ij}$ are mutually non equivalent, so any G-invariant metric on the base space G/K is determined by an $\mathrm{Ad}(K)$-invariant scalar product $x_{12}(-B)|_{\mathfrak{m}_{12}} + x_{13}(-B)|_{\mathfrak{m}_{13}} + x_{23}(-B)|_{\mathfrak{m}_{23}}$. We also set $\mathfrak{m}_1 = \mathfrak{so}(k_1), \mathfrak{m}_2 = \mathfrak{so}(k_2)$ and $\mathfrak{m}_3 = \mathfrak{so}(k_3)$. Therefore, decomposition (4) of the tangent space of the orthogonal group $G = \mathrm{SO}(k_1 + k_2 + k_3)$ takes the form

$$
(8)\qquad \mathfrak{so}(k_1 + k_2 + k_3) = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}.
$$

Let E_{ab} be the $n \times n$ matrix with 1 at the (ab)-entry and 0 elsewhere. Then the set $\mathcal{B} = \{e_{ab} = E_{ab} - E_{ba} : 1 \leq a < b \leq n\}$ constitutes a $(-B)$-orthonormal basis of $\mathfrak{so}(n)$. Note that $e_{ba} = -e_{ab}$, thus we have the following:

Lemma 3.1. *If all four indices are distinct, then the Lie brakets in $\mathcal{B}$ are zero. Otherwise, $[e_{ab}, e_{bc}] = e_{ac}$, where a, b, c are distinct.*

By using Lemma 3.1 we obtain:

Lemma 3.2. *The submodules in the decomposition* (8) *satisfy the following bracket relations:*

$$\begin{array}{lll}
[\mathfrak{m}_1,\mathfrak{m}_1]=\mathfrak{m}_1, & [\mathfrak{m}_2,\mathfrak{m}_2]=\mathfrak{m}_2, & [\mathfrak{m}_3,\mathfrak{m}_3]=\mathfrak{m}_3,\\
[\mathfrak{m}_1,\mathfrak{m}_{12}]=\mathfrak{m}_{12}, & [\mathfrak{m}_1,\mathfrak{m}_{13}]=\mathfrak{m}_{13}, & [\mathfrak{m}_2,\mathfrak{m}_{12}]=\mathfrak{m}_{12},\\
[\mathfrak{m}_2,\mathfrak{m}_{23}]=\mathfrak{m}_{23}, & [\mathfrak{m}_3,\mathfrak{m}_{13}]=\mathfrak{m}_{13}, & [\mathfrak{m}_3,\mathfrak{m}_{23}]=\mathfrak{m}_{23},\\
[\mathfrak{m}_{12},\mathfrak{m}_{23}]=\mathfrak{m}_{13}, & [\mathfrak{m}_{13},\mathfrak{m}_{23}]=\mathfrak{m}_{12}, & [\mathfrak{m}_{12},\mathfrak{m}_{13}]=\mathfrak{m}_{23}\\
[\mathfrak{m}_{12},\mathfrak{m}_{12}]=\mathfrak{m}_1\oplus\mathfrak{m}_2, & [\mathfrak{m}_{13},\mathfrak{m}_{13}]=\mathfrak{m}_1\oplus\mathfrak{m}_3, & [\mathfrak{m}_{23},\mathfrak{m}_{23}]=\mathfrak{m}_2\oplus\mathfrak{m}_3.
\end{array}$$

Therefore, we see that the only non zero triplets (up to permutation of indices) are

$$\begin{gathered}\begin{bmatrix}1\\11\end{bmatrix},\begin{bmatrix}2\\22\end{bmatrix},\begin{bmatrix}3\\33\end{bmatrix},\begin{bmatrix}(12)\\1(12)\end{bmatrix},\begin{bmatrix}(13)\\1(13)\end{bmatrix},\begin{bmatrix}(12)\\2(12)\end{bmatrix},\\ \begin{bmatrix}(23)\\2(23)\end{bmatrix},\begin{bmatrix}(13)\\3(13)\end{bmatrix},\begin{bmatrix}(23)\\3(23)\end{bmatrix},\begin{bmatrix}(13)\\(12)(23)\end{bmatrix},\end{gathered}\tag{9}$$

where $\begin{bmatrix}i\\ii\end{bmatrix}$ is non zero only for $k_i\geq 3$ $(i=1,2,3)$.

Now we take into account the diffeomorphism

$$G/\{e\}\cong(G\times\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))/\mathrm{diag}(\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))$$

and consider left-invariant metrics on G which are determined by the $\mathrm{Ad}(\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))$-invariant scalar products on $\mathfrak{so}(k_1+k_2+k_3)$ given by

$$\begin{aligned}\langle\ ,\ \rangle\ &=\ x_1\,(-B)|_{\mathfrak{so}(k_1)}+x_2\,(-B)|_{\mathfrak{so}(k_2)}+x_3\,(-B)|_{\mathfrak{so}(k_3)}\\ &\quad +x_{12}\,(-B)|_{\mathfrak{m}_{12}}+x_{13}\,(-B)|_{\mathfrak{m}_{13}}+x_{23}\,(-B)|_{\mathfrak{m}_{23}}.\end{aligned}\tag{10}$$

For $k_3=1$ we also consider left-invariant metrics on G which are determined by the $\mathrm{Ad}(\mathrm{SO}(k_1)\times\mathrm{SO}(k_2))$-invariant scalar products on $\mathfrak{so}(k_1+k_2+1)$ of the form

$$\begin{aligned}\langle\ ,\ \rangle\ &=\ x_1\,(-B)|_{\mathfrak{so}(k_1)}+x_2\,(-B)|_{\mathfrak{so}(k_2)}\\ &\quad +x_{12}\,(-B)|_{\mathfrak{m}_{12}}+x_{13}\,(-B)|_{\mathfrak{m}_{13}}+x_{23}\,(-B)|_{\mathfrak{m}_{23}}.\end{aligned}\tag{11}$$

Finally, for $k_1=n-2$ and $k_2=k_3=1$ we consider left-invariant metrics on G which are determined by the $\mathrm{Ad}(\mathrm{SO}(n-2))$-invariant scalar products on $\mathfrak{so}(n)$ of the form

$$\begin{aligned}\langle\ ,\ \rangle\ &=\ x_1\,(-B)|_{\mathfrak{so}(n-2)}\\ &\quad +x_{12}\,(-B)|_{\mathfrak{m}_{12}}+x_{13}\,(-B)|_{\mathfrak{m}_{13}}+x_{23}\,(-B)|_{\mathfrak{m}_{23}}.\end{aligned}\tag{12}$$

For the scalar products (11) the only non zero triplets are

$$\begin{bmatrix}1\\11\end{bmatrix}, \begin{bmatrix}2\\22\end{bmatrix}, \begin{bmatrix}(12)\\1(12)\end{bmatrix}, \begin{bmatrix}(13)\\1(13)\end{bmatrix}, \begin{bmatrix}(12)\\2(12)\end{bmatrix}, \begin{bmatrix}(23)\\2(23)\end{bmatrix}, \begin{bmatrix}(13)\\(12)(23)\end{bmatrix},$$

and for the scalar products (12) the only non zero triplets are

$$\begin{bmatrix}1\\11\end{bmatrix}, \begin{bmatrix}(12)\\1(12)\end{bmatrix}, \begin{bmatrix}(13)\\1(13)\end{bmatrix}, \begin{bmatrix}(13)\\(12)(23)\end{bmatrix}.$$

4. Naturally reductive metrics on the compact Lie groups $\mathbf{SO}(n)$

A Riemannian homogeneous space $(M = G/H, g)$ with reductive complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ is called *naturally reductive* if

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle = 0 \quad \text{for all } X, Y, Z \in \mathfrak{m}.$$

Here $\langle\ ,\ \rangle$ denotes the inner product on $\mathfrak{m}$ induced from the Riemannian metric g. Classical examples of naturally reductive homogeneous spaces include irreducible symmetric spaces, isotropy irreducible homogeneous manifolds, and Lie groups with bi-invariant metrics. In general it is not always easy to decide if a given homogeneous Riemannian manifold is naturally reductive, since one has to consider all possible transitive actions of subgroups G of the isometry group of (M, g).

In [9] D'Atri and Ziller had investigated naturally reductive metrics among left-invariant metrics on compact Lie groups and gave a complete classification in the case of simple Lie groups. Let G be a compact, connected semisimple Lie group, L a closed subgroup of G and let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{l}$ the subalgebra corresponding to L. We denote by Q the negative of the Killing form of $\mathfrak{g}$. Then Q is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$.

Let $\mathfrak{m}$ be an orthogonal complement of $\mathfrak{l}$ with respect to Q. Then we have

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}, \quad \mathrm{Ad}(L)\mathfrak{m} \subset \mathfrak{m}.$$

Let $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \cdots \oplus l_p$ be a decomposition of $\mathfrak{l}$ into ideals, where $\mathfrak{l}_0$ is the center of $\mathfrak{l}$ and $\mathfrak{l}_i$ $(i = 1, \ldots, p)$ are simple ideals of $\mathfrak{l}$. Let $A_0|_{\mathfrak{l}_0}$ be an arbitrary metric on $\mathfrak{l}_0$.

Theorem 4.1. ([9, Theorem 1, p. 9, and Theorem 3, p. 14]) *Under the notations above a left-invariant metric on G of the form*

(13) $\langle\ ,\ \rangle = x\cdot Q|_{\mathfrak{m}} + A_0|_{\mathfrak{l}_0} + u_1\cdot Q|_{\mathfrak{l}_1} + \cdots + u_p\cdot Q|_{\mathfrak{l}_p},\ (x, u_1, \ldots, u_p > 0)$

is naturally reductive with respect to $G\times L$*, where* $G\times L$ *acts on* G *by* $(g,l)y = gyl^{-1}$*.*

Moreover, if a left-invariant metric $\langle\ ,\ \rangle$ *on a compact simple Lie group* G *is naturally reductive, then there is a closed subgroup* L *of* G *and the metric* $\langle\ ,\ \rangle$ *is given by the form* (13)*.*

For the Lie group SO(n), we consider left-invariant metrics determined by the Ad(SO(k_1) × SO(k_2) × SO(k_3))-invariant scalar products of the form (10) where $n = k_1 + k_2 + k_3$. Recall that $K = \mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3)$ with Lie algebra $\mathfrak{k} = \mathfrak{so}(k_1)\oplus\mathfrak{so}(k_2)\oplus\mathfrak{so}(k_3)$.

Proposition 4.2. *If a left invariant metric* $\langle\ ,\ \rangle$ *of the form* (10) *on* SO(n) *is naturally reductive with respect to* $\mathrm{SO}(n)\times L$ *for some closed subgroup* L *of* SO(n)*, then one of the following holds:*

1) $x_1 = x_2 = x_{12}$, $x_{13} = x_{23}$, 2) $x_2 = x_3 = x_{23}$, $x_{12} = x_{13}$, 3) $x_1 = x_3 = x_{13}$, $x_{12} = x_{23}$, 4) $x_{12} = x_{13} = x_{23}$.

Conversely, if one of the conditions 1), 2), 3), 4) *is satisfied, then the metric* $\langle\ ,\ \rangle$ *of the form* (10) *is naturally reductive with respect to* $\mathrm{SO}(n)\times L$ *for some closed subgroup* L *of* SO(n)*.*

Proof. Let $\mathfrak{l}$ be the Lie algebra of L. Then we have either $\mathfrak{l}\subset\mathfrak{k}$ or $\mathfrak{l}\not\subset\mathfrak{k}$. First we consider the case of $\mathfrak{l}\not\subset\mathfrak{k}$. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{l}$ and $\mathfrak{k}$. Since $\mathfrak{so}(k_1+k_2+k_3) = \mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3\oplus\mathfrak{m}_{12}\oplus\mathfrak{m}_{13}\oplus\mathfrak{m}_{23}$ is an irreducible decomposition as Ad(K)-modules, we see that the Lie algebra $\mathfrak{h}$ contains at least one of $\mathfrak{m}_{12}$, $\mathfrak{m}_{13}$, $\mathfrak{m}_{23}$. We first consider the case that $\mathfrak{h}$ contains $\mathfrak{m}_{12}$. Note that $[\mathfrak{m}_{12},\mathfrak{m}_{12}] = \mathfrak{m}_1\oplus\mathfrak{m}_2$ and $\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_{12}$ is a subalgebra $\mathfrak{so}(k_1+k_2)$. Thus we see that $\mathfrak{h}$ contains $\mathfrak{so}(k_1+k_2)\oplus\mathfrak{so}(k_3)$. If $\mathfrak{h} = \mathfrak{so}(k_1+k_2)\oplus\mathfrak{so}(k_3)$, then we obtain an irreducible decomposition $\mathfrak{so}(k_1+k_2+k_3) = \mathfrak{h}\oplus\mathfrak{n}$, where $\mathfrak{n} = \mathfrak{m}_{13}\oplus\mathfrak{m}_{23}$. Hence, the metric $\langle\ ,\ \rangle$ of the form (10) satisfies $x_1 = x_2 = x_{12}$ and $x_{13} = x_{23}$, so we obtain case 1). Cases 2) and 3) are obtained by a similar way.

Now we consider the case $\mathfrak{l}\subset\mathfrak{k}$. Since the orthogonal complement $\mathfrak{l}^\perp$ of $\mathfrak{l}$ with respect to $-B$ contains the orthogonal complement $\mathfrak{k}^\perp$ of $\mathfrak{k}$, we see that $\mathfrak{l}^\perp\supset\mathfrak{m}_{12}\oplus\mathfrak{m}_{13}\oplus\mathfrak{m}_{23}$. Since the invariant metric $\langle\ ,\ \rangle$ is naturally reductive with respect to $G\times L$, it follows that $x_{12} = x_{13} = x_{23}$ by Theorem 4.1. The converse is a direct consequence of Theorem 4.1. □

For the Ad(SO(k_1) × SO(k_2))-invariant scalar products of the form (11) the above proposition reduces to the following:

Proposition 4.3. *If a left invariant metric* $\langle\ ,\ \rangle$ *of the form* (11) *on* $\mathrm{SO}(n)$ *is naturally reductive with respect to* $\mathrm{SO}(n) \times L$ *for some closed subgroup* L *of* $\mathrm{SO}(n)$*, then one of the following holds:*

1) $x_1 = x_2 = x_{12}$, $x_{13} = x_{23}$, 2) $x_2 = x_{23}$, $x_{12} = x_{13}$, 3) $x_1 = x_{13}$, $x_{12} = x_{23}$, 4) $x_{12} = x_{13} = x_{23}$.

Conversely, if one of the conditions 1), 2), 3), 4) *is satisfied, then the metric* $\langle\ ,\ \rangle$ *of the form* (11) *is naturally reductive with respect to* $\mathrm{SO}(n)\times L$ *for some closed subgroup* L *of* $\mathrm{SO}(n)$.

Finally, for the $\mathrm{Ad}(\mathrm{SO}(k_1))$-invariant scalar products of the form (12) we have the following:

Proposition 4.4. *If a left invariant metric* $\langle\ ,\ \rangle$ *of the form* (12) *on* $\mathrm{SO}(n)$ *is naturally reductive with respect to* $\mathrm{SO}(n) \times L$ *for some closed subgroup* L *of* $\mathrm{SO}(n)$*, then one of the following holds:*

1) $x_1 = x_{12}$, $x_{13} = x_{23}$, 2) $x_1 = x_{13}$, $x_{12} = x_{23}$, 3) $x_{12} = x_{13}$.

Conversely, if one of the conditions 1), 2), 3) *is satisfied, then the metric* $\langle\ ,\ \rangle$ *of the form* (12) *is naturally reductive with respect to* $\mathrm{SO}(n) \times L$ *for some closed subgroup* L *of* $\mathrm{SO}(n)$.

5. The Ricci tensor for a class of left-invariant metrics on $\mathrm{SO}(n) = \mathrm{SO}(k_1 + k_2 + k_3)$

We will compute the Ricci tensor for the left-invariant metrics on $\mathrm{SO}(n) = \mathrm{SO}(k_1+k_2+k_3)$, determined by the $\mathrm{Ad}(\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))$-invariant scalar products of the form (10). We use Lemma 2.1 taking into account (9), and we obtain the following:

Proposition 5.1. *The components of the Ricci tensor* r *for the left-invariant metric* $\langle\ ,\ \rangle$ *on* G *defined by* (10) *are given as follows:*

$$
\begin{aligned}
r_1 &= \frac{1}{2x_1} + \frac{1}{4d_1}\left(\begin{bmatrix}1\\11\end{bmatrix}\frac{1}{x_1} + \begin{bmatrix}1\\(12)(12)\end{bmatrix}\frac{x_1}{{x_{12}}^2} + \begin{bmatrix}1\\(13)(13)\end{bmatrix}\frac{x_1}{{x_{13}}^2}\right)\\
&-\frac{1}{2d_1}\left(\begin{bmatrix}1\\11\end{bmatrix}\frac{1}{x_1} + \begin{bmatrix}(12)\\1(12)\end{bmatrix}\frac{1}{x_1} + \begin{bmatrix}(13)\\1(13)\end{bmatrix}\frac{1}{x_1}\right),\\
r_2 &= \frac{1}{2x_2} + \frac{1}{4d_2}\left(\begin{bmatrix}2\\22\end{bmatrix}\frac{1}{x_2} + \begin{bmatrix}2\\(12)(12)\end{bmatrix}\frac{x_2}{{x_{12}}^2} + \begin{bmatrix}2\\(23)(23)\end{bmatrix}\frac{x_2}{{x_{23}}^2}\right)\\
&-\frac{1}{2d_2}\left(\begin{bmatrix}2\\22\end{bmatrix}\frac{1}{x_2} + \begin{bmatrix}(12)\\2(12)\end{bmatrix}\frac{1}{x_2} + \begin{bmatrix}(23)\\2(23)\end{bmatrix}\frac{1}{x_2}\right),\\
r_3 &= \frac{1}{2x_3} + \frac{1}{4d_3}\left(\begin{bmatrix}3\\33\end{bmatrix}\frac{1}{x_3} + \begin{bmatrix}3\\(13)(13)\end{bmatrix}\frac{x_3}{{x_{13}}^2} + \begin{bmatrix}3\\(23)(23)\end{bmatrix}\frac{x_3}{{x_{23}}^2}\right)
\end{aligned}
$$

$$-\frac{1}{2d_3}\left(\begin{bmatrix}3\\33\end{bmatrix}\frac{1}{x_3}+\begin{bmatrix}(13)\\3(13)\end{bmatrix}\frac{1}{x_3}+\begin{bmatrix}(23)\\3(23)\end{bmatrix}\frac{1}{x_3}\right),$$

$$r_{12}=\frac{1}{2x_{12}}+\frac{1}{4d_{12}}\left(\begin{bmatrix}(12)\\1(12)\end{bmatrix}\frac{1}{x_1}\times 2+\begin{bmatrix}(12)\\2(12)\end{bmatrix}\frac{1}{x_2}\times 2+\begin{bmatrix}(12)\\(13)(23)\end{bmatrix}\frac{x_{12}}{x_{13}x_{23}}\times 2\right)$$

$$(14)\quad -\frac{1}{2d_{12}}\left(\begin{bmatrix}1\\(12)(12)\end{bmatrix}\frac{x_1}{{x_{12}}^2}+\begin{bmatrix}(12)\\(12)1\end{bmatrix}\frac{1}{x_1}+\begin{bmatrix}2\\(12)(12)\end{bmatrix}\frac{x_2}{{x_{12}}^2}+\begin{bmatrix}(12)\\(12)2\end{bmatrix}\frac{1}{x_2}\right.$$

$$\left.+\begin{bmatrix}(13)\\(12)(23)\end{bmatrix}\frac{x_{13}}{x_{12}x_{23}}+\begin{bmatrix}(23)\\(12)(13)\end{bmatrix}\frac{x_{23}}{x_{12}x_{13}}\right),$$

$$r_{13}=\frac{1}{2x_{13}}+\frac{1}{4d_{13}}\left(\begin{bmatrix}(13)\\1(13)\end{bmatrix}\frac{1}{x_1}\times 2+\begin{bmatrix}(13)\\2(13)\end{bmatrix}\frac{1}{x_2}\times 2+\begin{bmatrix}(13)\\(12)(23)\end{bmatrix}\frac{x_{13}}{x_{12}x_{23}}\times 2\right)$$

$$-\frac{1}{2d_{13}}\left(\begin{bmatrix}1\\(13)(13)\end{bmatrix}\frac{x_1}{{x_{13}}^2}+\begin{bmatrix}(13)\\(13)1\end{bmatrix}\frac{1}{x_1}+\begin{bmatrix}3\\(13)(13)\end{bmatrix}\frac{x_3}{{x_{13}}^2}+\begin{bmatrix}(13)\\(13)3\end{bmatrix}\frac{1}{x_3}\right.$$

$$\left.+\begin{bmatrix}(12)\\(13)(23)\end{bmatrix}\frac{x_{12}}{x_{13}x_{23}}+\begin{bmatrix}(23)\\(13)(12)\end{bmatrix}\frac{x_{23}}{x_{13}x_{12}}\right),$$

$$r_{23}=\frac{1}{2x_{23}}+\frac{1}{4d_{23}}\left(\begin{bmatrix}(23)\\2(23)\end{bmatrix}\frac{1}{x_2}\times 2+\begin{bmatrix}(23)\\3(23)\end{bmatrix}\frac{1}{x_3}\times 2+\begin{bmatrix}(23)\\(12)(13)\end{bmatrix}\frac{x_{23}}{x_{12}x_{13}}\times 2\right)$$

$$-\frac{1}{2d_{23}}\left(\begin{bmatrix}2\\(23)(23)\end{bmatrix}\frac{x_2}{{x_{23}}^2}+\begin{bmatrix}(23)\\(23)2\end{bmatrix}\frac{1}{x_2}+\begin{bmatrix}3\\(23)(23)\end{bmatrix}\frac{x_3}{{x_{23}}^2}+\begin{bmatrix}(23)\\(23)3\end{bmatrix}\frac{1}{x_3}\right.$$

$$\left.+\begin{bmatrix}(12)\\(23)(13)\end{bmatrix}\frac{x_{12}}{x_{23}x_{13}}+\begin{bmatrix}(13)\\(23)(12)\end{bmatrix}\frac{x_{13}}{x_{23}x_{12}}\right),$$

where $n=k_1+k_2+k_3$.

We recall the following lemma from [1] (a detailed proof was given in [3]).

Lemma 5.2. ([1, Lemma 4.2]) *For* $a,b,c=1,2,3$ *and* $(a-b)(b-c)(c-a)\neq 0$ *the following relations hold:*

$$(15)\quad \begin{bmatrix}a\\aa\end{bmatrix}=\frac{k_a(k_a-1)(k_a-2)}{2(n-2)},\quad \begin{bmatrix}a\\(ab)(ab)\end{bmatrix}=\frac{k_ak_b(k_a-1)}{2(n-2)},$$

$$\begin{bmatrix}(ac)\\(ab)(bc)\end{bmatrix}=\frac{k_ak_bk_c}{2(n-2)}.$$

By using the above lemma, we can now obtain the components of the Ricci tensor for the metrics we are considering in this work.

Proposition 5.3. *The components of the Ricci tensor* r *for the left-invariant metric* $\langle\ ,\ \rangle$ *on* G *defined by* (10) *are given as follows:*

$$
\begin{aligned}
r_1 &= \frac{k_1-2}{4(n-2)x_1} + \frac{1}{4(n-2)}\left(k_2\frac{x_1}{{x_{12}}^2} + k_3\frac{x_1}{{x_{13}}^2}\right),\\
r_2 &= \frac{k_2-2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(k_1\frac{x_2}{{x_{12}}^2} + k_3\frac{x_2}{{x_{23}}^2}\right),\\
r_3 &= \frac{k_3-2}{4(n-2)x_3} + \frac{1}{4(n-2)}\left(k_1\frac{x_3}{{x_{13}}^2} + k_2\frac{x_3}{{x_{23}}^2}\right),\\
r_{12} &= \frac{1}{2x_{12}} + \frac{k_3}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right)\\
&\quad -\frac{1}{4(n-2)}\left(\frac{(k_1-1)x_1}{{x_{12}}^2} + \frac{(k_2-1)x_2}{{x_{12}}^2}\right),\\
r_{13} &= \frac{1}{2x_{13}} + \frac{k_2}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right)\\
&\quad -\frac{1}{4(n-2)}\left(\frac{(k_1-1)x_1}{{x_{13}}^2} + \frac{(k_3-1)x_3}{{x_{13}}^2}\right),\\
r_{23} &= \frac{1}{2x_{23}} + \frac{k_1}{4(n-2)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right)\\
&\quad -\frac{1}{4(n-2)}\left(\frac{(k_2-1)x_2}{{x_{23}}^2} + \frac{(k_3-1)x_3}{{x_{23}}^2}\right),
\end{aligned}
\tag{16}
$$

where $n = k_1 + k_2 + k_3$.

For the $\mathrm{Ad}(\mathrm{SO}(k_1)\times\mathrm{SO}(k_2))$-invariant scalar products on G of the form (11), Proposition 5.3 reduces to

Proposition 5.4. *The components of the Ricci tensor* r *for the left-invariant metric* $\langle\ ,\ \rangle$ *on* G *defined by* (11) *are given as follows:*

$$
\begin{aligned}
r_1 &= \frac{k_1-2}{4(n-2)x_1} + \frac{1}{4(n-2)}\left(k_2\frac{x_1}{{x_{12}}^2} + \frac{x_1}{{x_{13}}^2}\right),\\
r_2 &= \frac{k_2-2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(k_1\frac{x_2}{{x_{12}}^2} + \frac{x_2}{{x_{23}}^2}\right),\\
r_{12} &= \frac{1}{2x_{12}} + \frac{1}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right)\\
&\quad -\frac{1}{4(n-2)}\left(\frac{(k_1-1)x_1}{{x_{12}}^2} + \frac{(k_2-1)x_2}{{x_{12}}^2}\right),\\
r_{13} &= \frac{1}{2x_{13}} + \frac{1}{4(n-2)}\left(k_2\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{(k_1-1)x_1}{{x_{13}}^2}\right),\\
r_{23} &= \frac{1}{2x_{23}} + \frac{1}{4(n-2)}\left(k_1\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) - \frac{(k_2-1)x_2}{{x_{23}}^2}\right),
\end{aligned}
\tag{17}
$$

where $n = k_1 + k_2 + 1$.

Finally, for the Ad(SO(k_1))-invariant scalar products on G of the form (12), Proposition 5.3 reduces to

Proposition 5.5. *The components of the Ricci tensor r for the left-invariant metric $\langle\ ,\ \rangle$ on G defined by* (12) *are given as follows:*

$$
\begin{aligned}
r_1 &= \frac{n-4}{4(n-2)x_1} + \frac{1}{4(n-2)}\left(\frac{x_1}{{x_{12}}^2} + \frac{x_1}{{x_{13}}^2}\right),\\
r_{12} &= \frac{1}{2x_{12}} + \frac{1}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}} - \frac{(n-3)x_1}{{x_{12}}^2}\right),\\
r_{13} &= \frac{1}{2x_{13}} + \frac{1}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}} - \frac{(n-3)x_1}{{x_{13}}^2}\right),\\
r_{23} &= \frac{1}{2x_{23}} + \frac{1}{4}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right),
\end{aligned}
\tag{18}
$$

where $n = k_1 + 2$.

6. Left-invariant Einstein metrics on SO(n)

In the present section we provide detailed proofs on how to obtain left-invariant Einstein metrics which are not naturally reductive for the Lie groups SO(7), SO(8) and SO(n), $n \geq 9$. For SO(7) and SO(8) we also describe left-invariant Einstein metrics which are naturally reductive. The other compact Lie groups SO(n) = SO($k_1 + k_2 + k_3$) for $n \geq 7$ and for all possible Ad(SO(k_1) $\times$ SO(k_2) $\times$ SO(k_3))-invariant scalar products of the forms (10) and (11) can be treated in an analogous manner and we omit the proofs. We summarise all the results at the end of Section 7. Here, we also provide information about solving or proving existence of solutions of algebraic systems of equations. These solutions correspond to Einstein metrics which are not naturally reductive.

Proposition 6.1. *The Lie group* SO(7) *admits at least one left-invariant Einstein metric determined by the* Ad(SO(3) $\times$ SO(3))*-invariant scalar product of the form* (11)*, which is not naturally reductive.*

Proof. This is the case $k_1 = k_2 = 3$ and $k_3 = 1$. From Proposition 5.4, we see that the components of the Ricci tensor r for the invariant metric are given by

$$
\begin{aligned}
r_1 &= \frac{1}{20x_1}+\frac{1}{20}\left(3\frac{x_1}{{x_{12}}^2}+\frac{x_1}{{x_{13}}^2}\right), \quad r_2 = \frac{1}{20x_2}+\frac{1}{20}\left(3\frac{x_2}{{x_{12}}^2}+\frac{x_2}{{x_{23}}^2}\right),\\
r_{12} &= \frac{1}{2x_{12}}+\frac{1}{20}\left(\frac{x_{12}}{x_{13}x_{23}}-\frac{x_{13}}{x_{12}x_{23}}-\frac{x_{23}}{x_{12}x_{13}}\right)-\frac{1}{10}\left(\frac{x_1}{{x_{12}}^2}+\frac{x_2}{{x_{12}}^2}\right),\\
r_{23} &= \frac{1}{2x_{23}}+\frac{3}{20}\left(\frac{x_{23}}{x_{13}x_{12}}-\frac{x_{13}}{x_{12}x_{23}}-\frac{x_{12}}{x_{23}x_{13}}\right)-\frac{1}{10}\frac{x_2}{{x_{23}}^2},\\
r_{13} &= \frac{1}{2x_{13}}+\frac{3}{20}\left(\frac{x_{13}}{x_{12}x_{23}}-\frac{x_{12}}{x_{13}x_{23}}-\frac{x_{23}}{x_{12}x_{13}}\right)-\frac{1}{10}\frac{x_1}{{x_{13}}^2}.
\end{aligned} \tag{19}
$$

We consider the system of equations

$$
r_1 = r_2, \ r_2 = r_{12}, \ r_{12} = r_{23}, \ r_{23} = r_{13}. \tag{20}
$$

Then finding Einstein metrics of the form (11) reduces to finding the positive solutions of system (20), and we normalize our equations by putting $x_{23} = 1$. Then we obtain the system of equations:

$$
\begin{aligned}
g_1 &= {x_1}^2{x_{12}}^2x_2+3{x_1}^2{x_{13}}^2x_2-x_1{x_{12}}^2{x_{13}}^2{x_2}^2-x_1{x_{12}}^2{x_{13}}^2\\
&\quad -3x_1{x_{13}}^2{x_2}^2+{x_{12}}^2{x_{13}}^2x_2=0,\\
g_2 &= 2x_1x_{13}x_2-{x_{12}}^3x_2+{x_{12}}^2x_{13}{x_2}^2+{x_{12}}^2x_{13}+x_{12}{x_{13}}^2x_2\\
&\quad -10x_{12}x_{13}x_2+x_{12}x_2+5x_{13}{x_2}^2=0,\\
g_3 &= -x_1x_{13}+2{x_{12}}^3+{x_{12}}^2x_{13}x_2-5{x_{12}}^2x_{13}+x_{12}{x_{13}}^2\\
&\quad +5x_{12}x_{13}-2x_{12}-x_{13}x_2=0,\\
g_4 &= x_1x_{12}-x_{12}{x_{13}}^2x_2+5x_{12}{x_{13}}^2-5x_{12}x_{13}-3{x_{13}}^3+3x_{13}=0.
\end{aligned} \tag{21}
$$

We consider a polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_{12}, x_{13}]$ and an ideal I generated by $\{g_1, g_2, g_3, g_4, z\, x_1\, x_2\, x_{12}\, x_{13} - 1\}$ to find non zero solutions of equations (21). We take a lexicographic order $>$ with $z > x_1 > x_2 > x_{12} > x_{13}$ for a monomial ordering on R. Then by the aid of computer programs in Mathematica or Maple, we see that a Gröbner basis for the ideal I contains the polynomial

$$
(x_{13}-1)\left(6{x_{13}}^3-44{x_{13}}^2+90x_{13}-45\right)\left(45{x_{13}}^3-90{x_{13}}^2+44x_{13}-6\right)\,h_1(x_{13}),
$$

where $h_1(x_{13})$ is a polynomial of x_{13} given by

$$
\begin{aligned}
h_1(x_{13}) &= 9078544800000{x_{13}}^{24}-87978150000000{x_{13}}^{23}+416122213455000{x_{13}}^{22}\\
&-1222223075437500{x_{13}}^{21}+2532878590309970{x_{13}}^{20}-4171390831990050{x_{13}}^{19}\\
&+5900094406718764{x_{13}}^{18}-7070644584919459{x_{13}}^{17}+6230617318198202{x_{13}}^{16}\\
&-4091340309226802{x_{13}}^{15}+1722695469975774{x_{13}}^{14}+983550542994755{x_{13}}^{13}\\
&-2624020500593532{x_{13}}^{12}+983550542994755{x_{13}}^{11}+1722695469975774{x_{13}}^{10}\\
&-4091340309226802{x_{13}}^{9}+6230617318198202{x_{13}}^{8}-7070644584919459{x_{13}}^{7}
\end{aligned}
$$

$$+5900094406718764x_{13}^6 - 4171390831990050x_{13}^5 + 2532878590309970x_{13}^4$$
$$-1222223075437500x_{13}^3 + 416122213455000x_{13}^2 - 87978150000000x_{13}$$
$$+9078544800000.$$

By solving the equation $h_1(x_{13}) = 0$ numerically by using computer programs in Mathematica or Maple, we obtain *two* positive solutions $x_{13} = a_{13}$ and $x_{13} = b_{13}$ which are given approximately as $a_{13} \approx 0.4254295$, $b_{13} \approx 2.350565$. We also see that the Gröbner basis for the ideal I contains the polynomials

$$x_{12} - w_{12}(x_{13}), \quad x_1 - w_1(x_{13}), \quad x_2 - w_2(x_{13}),$$

where $w_{12}(x_{13})$, $w_1(x_{13})$ and $w_2(x_{13})$ are polynomials of x_{13} with rational coefficients. By substituting the values a_{13} and b_{13} for x_{13} into $w_{12}(x_{13})$, $w_1(x_{13})$ and $w_2(x_{13})$, we obtain two positive solutions of the system of equations $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$ approximately as

$$(x_{13}, x_{12}, x_1, x_2) \approx (0.4254295,\ 0.9312204,\ 0.1200109,\ 0.1122291),$$
$$(x_{13}, x_{12}, x_1, x_2) \approx (2.350565,\ 2.188895,\ 0.2638018,\ 0.2820935).$$

We substitute these values into the system (19) together with $x_{23} = 1$. Then we obtain that $r_1 = r_2 = r_{12} = r_{23} = r_{13} \approx 0.470542$ and $r_1 = r_2 = r_{12} = r_{23} = r_{13} \approx 0.200182$ respectively. We multiply these solutions by a scale factor and we obtain the two solutions

$$(x_1, x_2, x_{12}, x_{23}, x_{13}) \approx (0.0564701,\ 0.0528085,\ 0.438178,\ 0.470542,\ 0.20018),$$
$$(x_1, x_2, x_{12}, x_{23}, x_{13}) \approx (0.0528085,\ 0.0564701,\ 0.438178,\ 0.20018,\ 0.470542).$$

for the system of equations

$$r_1 = r_2 = r_{12} = r_{23} = r_{13} = 1,$$

so we see that *these two solutions are isometric*. Note that this metric is not naturally reductive from Proposition 4.3.

Now we consider the case

$$(x_{13} - 1)\left(6x_{13}^3 - 44x_{13}^2 + 90x_{13} - 45\right)\left(45x_{13}^3 - 90x_{13}^2 + 44x_{13} - 6\right) = 0.$$

We consider the ideals J_1 generated by $\{6x_{13}^3 - 44x_{13}^2 + 90x_{13} - 45, g_1, g_2, g_3, g_4, z\,x_1\,x_2\,x_{12}\,x_{13} - 1\}$, J_2 generated by $\{45x_{13}^3 - 90x_{13}^2 + 44x_{13} - 6, g_1, g_2, g_3, g_4, z\,x_1\,x_2\,x_{12}\,x_{13} - 1\}$ and J_3 generated by $\{x_{13} - 1, g_1, g_2, g_3, g_4, z\,x_1\,x_2 \times x_{12}\,x_{13} - 1\}$ of the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_{12}, x_{13}]$.

We take a lexicographic order $>$ with $z > x_1 > x_2 > x_{12} > x_{13}$ for a monomial ordering on $R = \mathbb{Q}[z, x_1, x_2, x_{12}, x_{13}]$. Then, by the aid of computer again, we see that a Gröbner basis for the ideal J_1 is given by

$$\{6{x_{13}}^3 - 44{x_{13}}^2 + 90x_{13} - 45, x_{12} - x_{13}, x_2 - 1, x_1 + {x_{13}}^2 - 5x_{13} + 3,\\ -804{x_{13}}^2 + 5284x_{13} + 405z - 8112\}.$$

By solving the equation $6{x_{13}}^3 - 44{x_{13}}^2 + 90x_{13} - 45 = 0$ numerically by using computer programs in Mathematica or Maple, we obtain three positive solutions of the system of equations $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, 6{x_{13}}^3 - 44{x_{13}}^2 + 90x_{13} - 45 = 0\}$ approximately as

$$(x_{13}, x_{12}, x_1, x_2, x_{23}) \approx (4.16278,\ 4.16278,\ 0.485171,\ 1,\ 1),$$
$$(x_{13}, x_{12}, x_1, x_2, x_{23}) \approx (2.42874,\ 2.42874,\ 3.24492,\ 1,\ 1),$$
$$(x_{13}, x_{12}, x_1, x_2, x_{23}) \approx (0.741818,\ 0.741818,\ 0.158797,\ 1,\ 1).$$

We substitute these values into the system (19). Then we obtain that $r_1 = r_2 = r_{12} = r_{23} = r_{13} \approx 0.108656$, $r_1 = r_2 = r_{12} = r_{23} = r_{13} \approx 0.125429$ and $r_1 = r_2 = r_{12} = r_{23} = r_{13} \approx 0.372581$ respectively. We multiply these solutions by a scale factor and we obtain three solutions

$$(x_{13}, x_{12}, x_1, x_2, x_{23})\\ \approx (0.452311,\ 0.452311,\ 0.0527168,\ 0.108656,\ 0.108656),$$
$$(x_{13}, x_{12}, x_1, x_2, x_{23})\\ \approx (0.304634,\ 0.304634,\ 0.407007,\ 0.125429,\ 0.125429),$$
$$(x_{13}, x_{12}, x_1, x_2, x_{23})\\ \approx (0.276388,\ 0.276388,\ 0.0591647,\ 0.372581,\ 0.372581).$$

for the system of equations

$$r_1 = r_2 = r_{12} = r_{23} = r_{13} = 1,$$

We also see that a Gröbner basis for the ideal J_2 is given by

$$\{45{x_{13}}^3 - 90{x_{13}}^2 + 44x_{13} - 6, x_{12} - 1, 45{x_{13}}^2 - 72x_{13} + 6x_2 + 14,\\ x_1 - x_{13}, -4545{x_{13}}^2 + 8010x_{13} + 6z - 2554\}.$$

By solving the equation $45{x_{13}}^3 - 90{x_{13}}^2 + 44x_{13} - 6 = 0$ numerically by the same method, we obtain three positive solutions of the system of equations $\{g_1 = 0,\ g_2 = 0,\ g_3 = 0,\ g_4 = 0,\ 45{x_{13}}^3 - 90{x_{13}}^2 + 44x_{13} - 6 = 0\}$ approximately as

$$(x_{13}, x_{12}, x_1, x_2, x_{23}) \approx (0.240224,\ 1,\ 0.240224,\ 0.11655,\ 1),$$

$$(x_{13}, x_{12}, x_1, x_2, x_{23}) \approx (0.411737,\ 1,\ 0.411737,\ 1.33605,\ 1),$$
$$(x_{13}, x_{12}, x_1, x_2, x_{23}) \approx (1.34804,\ 1,\ 1.34804,\ 0.214064,\ 1).$$

We substitute these values into the system (19). Then we obtain that $r_1 = r_2 = r_{12} = r_{23} = r_{13} \approx 0.452311$, $r_1 = r_2 = r_{12} = r_{23} = r_{13} \approx 0.304634$ and $r_1 = r_2 = r_{12} = r_{23} = r_{13} \approx 0.276388$ respectively. We multiply these solutions by a scale factor and we obtain three solutions

$$\begin{aligned}&(x_1, x_2, x_{12}, x_{23}, x_{13})\\ &\quad\approx (0.108656,\ 0.0527168,\ 0.452311,\ 0.452311,\ 0.108656),\\ &(x_1, x_2, x_{12}, x_{23}, x_{13})\\ &\quad\approx (0.125429,\ 0.407007,\ 0.304634,\ 0.304634,\ 0.125429),\\ &(x_1, x_2, x_{12}, x_{23}, x_{13})\\ &\quad\approx (0.372581,\ 0.0591647,\ 0.276388,\ 0.276388,\ 0.372581).\end{aligned}$$

for the system of equations

$$r_1 = r_2 = r_{12} = r_{23} = r_{13} = 1.$$

Note that, from these two cases, we obtain *three Einstein metrics* up to isometry.

Now we consider the case of ideal J_3. We see that a Gröbner basis for the ideal J_3 is given by

$$\begin{aligned}&\{x_{13} - 1, (x_{12} - 1)(3x_{12} - 2)\left(2{x_{12}}^4 - 5{x_{12}}^3 + 19{x_{12}}^2 - 35x_{12} + 26\right),\\ &-24{x_{12}}^5 + 70{x_{12}}^4 - 267{x_{12}}^3 + 550{x_{12}}^2 - 522x_{12} + 63x_2 + 130,\\ &63x_1 - 24{x_{12}}^5 + 70{x_{12}}^4 - 267{x_{12}}^3 + 550{x_{12}}^2 - 522x_{12} + 130,\\ &-569934{x_{12}}^5 + 2980005{x_{12}}^4 - 8034374{x_{12}}^3 + 22826670{x_{12}}^2\\ &-25019231x_{12} + 246064z + 7570800\}.\end{aligned}$$

By solving the equation $2{x_{12}}^4 - 5{x_{12}}^3 + 19{x_{12}}^2 - 35x_{12} + 26 = 0$ numerically, we see that there are no real solutions. For $x_{12} = 1$, we obtain that $x_1 = x_2 = x_{13} = x_{23} = 1$, that is, the metric is bi-invariant. For $x_{12} = 2/3$, we obtain that $x_1 = x_2 = 2/3$ and $x_{13} = x_{23} = 1$, hence we obtain *two Einstein metrics* up to isometry.

From Proposition 4.3 it follows that the five above metrics obtained are all naturally reductive with respect to $\mathrm{SO}(7) \times L$, where L is a closed subgroup of SO(7), which is either $\mathrm{SO}(3) \times \mathrm{SO}(4)$, SO(6) or SO(7). □

Proposition 6.2. *The Lie group* SO(8) *admits at least two* (*non isometric*) *left-invariant Einstein metrics determined by the* $\mathrm{Ad}(\mathrm{SO}(4) \times \mathrm{SO}(3))$-*invariant scalar products of the form* (11), *which are not naturally reductive.*

Proof. This is the case $k_1 = 4, k_2 = 3$ and $k_3 = 1$. From Proposition 5.4, we see that the components of the Ricci tensor r for the invariant metric are given by

$$
\begin{aligned}
r_1 &= \frac{1}{12x_1} + \frac{1}{24}\left(3\frac{x_1}{{x_{12}}^2} + \frac{x_1}{{x_{13}}^2}\right), \quad r_2 = \frac{1}{24x_2} + \frac{1}{24}\left(4\frac{x_2}{{x_{12}}^2} + \frac{x_2}{{x_{23}}^2}\right),\\
r_{12} &= \frac{1}{2x_{12}} + \frac{1}{24}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{1}{24}\left(3\frac{x_1}{{x_{12}}^2} + 2\frac{x_2}{{x_{12}}^2}\right),\\
r_{23} &= \frac{1}{2x_{23}} + \frac{1}{6}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) - \frac{1}{12}\frac{x_2}{{x_{23}}^2},\\
r_{13} &= \frac{1}{2x_{13}} + \frac{1}{8}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{1}{8}\frac{x_1}{{x_{13}}^2}.
\end{aligned} \tag{22}
$$

We consider the system of equations

$$
r_1 = r_2, \ r_2 = r_{12}, \ r_{12} = r_{23}, \ r_{23} = r_{13}. \tag{23}
$$

Then finding Einstein metrics of the form (11) reduces to finding the positive solutions of system (23), and we normalize our equations by putting $x_{23} = 1$. Then we have the system of equations:

$$
\begin{aligned}
g_1 &= {x_1}^2{x_{12}}^2x_2 + 3{x_1}^2{x_{13}}^2x_2 - x_1{x_{12}}^2{x_{13}}^2{x_2}^2 - x_1{x_{12}}^2{x_{13}}^2\\
&\quad -4x_1{x_{13}}^2{x_2}^2 + 2{x_{12}}^2{x_{13}}^2x_2 = 0,\\
g_2 &= 3x_1x_{13}x_2 - {x_{12}}^3x_2 + {x_{12}}^2x_{13}{x_2}^2 + {x_{12}}^2x_{13} + x_{12}{x_{13}}^2x_2\\
&\quad -12x_{12}x_{13}x_2 + x_{12}x_2 + 6x_{13}{x_2}^2 = 0,\\
g_3 &= -3x_1x_{13} + 5{x_{12}}^3 + 2{x_{12}}^2x_{13}x_2 - 12{x_{12}}^2x_{13} + 3x_{12}{x_{13}}^2\\
&\quad +12x_{12}x_{13} - 5x_{12} - 2x_{13}x_2 = 0,\\
g_4 &= 3x_1x_{12} - {x_{12}}^2x_{13} - 2x_{12}{x_{13}}^2x_2 + 12x_{12}{x_{13}}^2 - 12x_{12}x_{13}\\
&\quad -7{x_{13}}^3 + 7x_{13} = 0.
\end{aligned} \tag{24}
$$

We consider a polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_{12}, x_{13}]$ and an ideal I generated by $\{g_1, g_2, g_3, g_4, z\,x_1\,x_2\,x_{12}\,x_{13} - 1\}$ to find non zero solutions of equations (24). We take a lexicographic order $>$ with $z > x_1 > x_2 > x_{12} > x_{13}$ for a monomial ordering on R. Then, by the aid of computer programs in Mathematica or Maple, we see that a Gröbner basis for the ideal I contains the polynomial

$$
(x_{13} - 5)(x_{13} - 1)\left(7{x_{13}}^2 - 24x_{13} + 14\right)\left(287{x_{13}}^3 - 625{x_{13}}^2 + 369x_{13} - 63\right)\, h_2(x_{13}),
$$

where $h_2(x_{13})$ is a polynomial of x_{13} given by

$$
\begin{aligned}
h_2(x_{13}) &= 542677550714848967040 0{x_{13}}^{28} - 8516118509262297787392 0{x_{13}}^{27}\\
&+643415930216926223949312{x_{13}}^{26} - 3054548385819855899001216{x_{13}}^{25}
\end{aligned}
$$

$$
\begin{aligned}
&+1017914049977712110066480 0{x_{13}}^{24} - 2558514736241665583523638 4{x_{13}}^{23} \\
&+5138042632407905915036427 2{x_{13}}^{22} - 8593418550466308717324904 8{x_{13}}^{21} \\
&+12035244791842130228956886 3{x_{13}}^{20} - 13693837238491096464926080 2{x_{13}}^{19} \\
&+12126841737945933546116745 7{x_{13}}^{18} - 7848377311891246781833359 0{x_{13}}^{17} \\
&+3204867998088819580765828 6{x_{13}}^{16} - 2103708121401859244766285 0{x_{13}}^{15} \\
&+9656772440390654525134860 4{x_{13}}^{14} - 27967382221378985947064352 0{x_{13}}^{13} \\
&+52783303504690297847933138 7{x_{13}}^{12} - 76963204586639064727452364 2{x_{13}}^{11} \\
&+93752173331693402178039747 3{x_{13}}^{10} - 97331891532932832916556237 4{x_{13}}^{9} \\
&+86490759963422406346244841 6{x_{13}}^{8} - 6644135450846553035188369 50{x_{13}}^{7} \\
&+44217554367433907041804197 0{x_{13}}^{6} - 2492829325849831748573597 64{x_{13}}^{5} \\
&+11423398141252539597870792 0{x_{13}}^{4} - 4047428102312746965023910 0{x_{13}}^{3} \\
&+1038232072105877913402600 0{x_{13}}^{2} - 1735984447231701886065000x_{13} \\
&+146138820428187141975000.
\end{aligned}
$$

By solving the equation $h_2(x_{13}) = 0$ numerically, we obtain *two* positive solutions $x_{13} = a_{13}$ and $x_{13} = b_{13}$ which are given approximately as $a_{13} \approx 0.48183112$, $b_{13} \approx 2.7966957$. We also see that the Gröbner basis for the ideal I contains the polynomials

$$x_{12} - w_{12}(x_{13}), \quad x_1 - w_1(x_{13}), \quad x_2 - w_2(x_{13}),$$

where $w_{12}(x_{13})$, $w_1(x_{13})$ and $w_2(x_{13})$ are polynomials of x_{13} with rational coefficients. By substituting the values a_{13} and b_{13} for x_{13} into $w_{12}(x_{13})$, $w_1(x_{13})$ and $w_2(x_{13})$, we obtain two positive solutions of the system of equations $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$ approximately as

$$
\begin{aligned}
&(x_{13}, x_{12}, x_1, x_2) \approx (0.48183112, 0.90692827, 0.20686292, 0.092856189), \\
&(x_{13}, x_{12}, x_1, x_2) \approx (2.7966957, 2.6698577, 0.54677135, 0.28461374).
\end{aligned}
$$

We substitute these values into the system (22) together with $x_{23} = 1$. Then we obtain that $r_1 = r_2 = r_{12} = r_{23} = r_{13} \approx 0.47140698$ and $r_1 = r_2 = r_{12} = r_{23} = r_{13} \approx 0.16491085$ respectively. We multiply these solutions by a scale factor and we obtain the two solutions

$$
\begin{aligned}
&(x_1, x_2, x_{12}, x_{23}, x_{13}) \\
&\quad \approx (0.097516624,\ 0.043773055,\ 0.42753231,\ 0.47140698,\ 0.22713855), \\
&(x_1, x_2, x_{12}, x_{23}, x_{13}) \\
&\quad \approx (0.090168527,\ 0.046935893,\ 0.44028850,\ 0.16491085,\ 0.46120545).
\end{aligned}
$$

for the system of equations

$$r_1 = r_2 = r_{12} = r_{23} = r_{13} = 1,$$

which *are not isometric*. Note that due to Proposition 4.3. these metrics are not naturally reductive.

Now we consider the case

$$(x_{13}-5)(x_{13}-1)\left(7{x_{13}}^2-24x_{13}+14\right)$$
$$\times\left(287{x_{13}}^3-625{x_{13}}^2+369x_{13}-63\right)=0.$$

We consider ideals J_1 generated by $\{287{x_{13}}^3-625{x_{13}}^2+369x_{13}-63, g_1, g_2, g_3, g_4, z\,x_1\,x_2\,x_{12}\,x_{13}-1\}$, J_2 generated by $\{7{x_{13}}^2-24x_{13}+14, g_1, g_2, g_3, g_4, z\,x_1\,x_2\,x_{12}\,x_{13}-1\}$, J_3 generated by $\{x_{13}-5, g_1, g_2, g_3, g_4, z\,x_1\,x_2\,x_{12}\times x_{13}-1\}$ and J_4 generated by $\{x_{13}-1, g_1, g_2, g_3, g_4, z\,x_1\,x_2\,x_{12}\,x_{13}-1\}$ of the polynomial ring $R=\mathbb{Q}[z,x_1,x_2,x_{12},x_{13}]$.

We take a lexicographic order $>$ with $z>x_1>x_2>x_{12}>x_{13}$ for a monomial ordering on $R=\mathbb{Q}[z,x_1,x_2,x_{12},x_{13}]$. Then, by the aid of computer, we see that a Gröbner basis for the ideal J_1 is given by

$$\{287{x_{13}}^3-625{x_{13}}^2+369x_{13}-63, 117-478x_{13}+287{x_{13}}^2+42x_2,$$
$$x_{12}-1, x_1-x_{13}, -123201+323882x_{13}-173635{x_{13}}^2+378z\}.$$

By solving the equation $287{x_{13}}^3-625{x_{13}}^2+369x_{13}-63=0$ numerically, we obtain *three* positive solutions of the system of equations $\{g_1=0, g_2=0, g_3=0, g_4=0, 287{x_{13}}^3-625{x_{13}}^2+369x_{13}-63=0\}$. Since the solutions satisfy $x_1=x_{13}$, $x_{12}=x_{23}=1$, then Proposition 4.3 implies that the metrics obtained are naturally reductive with respect to $\mathrm{SO}(8)\times(\mathrm{SO}(5)\times\mathrm{SO}(3))$.

Similarly, we see that a Gröbner basis for the ideal J_2 is given by

$$\{14-24x_{13}+7{x_{13}}^2, x_{12}-x_{13}, -1+x_2, 7+7x_1-12x_{13},$$
$$-43009+15960x_{13}+4802z\}.$$

By solving the equation $14-24x_{13}+7{x_{13}}^2=0$, we obtain *two* positive solutions of the system of equations $\{g_1=0,\ g_2=0,\ g_3=0,\ g_4=0,\ 14-24x_{13}+7{x_{13}}^2=0\}$. Since the solutions satisfy $x_2=x_{23}=1$, $x_{12}=x_{13}=1$, then Proposition 4.3 implies that the metrics obtained are naturally reductive with respect to $\mathrm{SO}(8)\times(\mathrm{SO}(4)\times\mathrm{SO}(4))$. (It is possible to check that *these two metrics are isometric*).

Similarly, we see that a Gröbner basis for the ideal J_3 is given by

$$\{-5+x_{13}, -5+x_{12}, -1+x_2, -1+x_1, -1+25z\}$$

and we obtain a *unique* positive solution of the system of equations $\{g_1=0,\ g_2=0,\ g_3=0,\ g_4=0,\ x_{13}=5\}$. From Proposition 4.3 we see that the metric obtained is naturally reductive with respect to $\mathrm{SO}(8) \times (\mathrm{SO}(4) \times \mathrm{SO}(4))$.

Finally, we see that a Gröbner basis for the ideal J_4 is given by

$$\{-1 + x_{13}, (-1 + x_{12})(-5 + 7x_{12}), -x_{12} + x_2, x_1 - x_{12}, \\ -888 + 763x_{12} + 125z\},$$

and we obtain *two* positive solutions of the system of equations $\{g_1 = 0,\ g_2 = 0,\ g_3 = 0,\ g_4 = 0,\ x_{13} = 1\}$. One of the solutions gives the bi-invariant metric $x_1 = x_2 = x_{12} = x_{13} = x_{23} = 1$ and the other $x_1 = x_2 = x_{12} = 5/7, x_{13} = x_{23} = 1$ gives naturally reductive metric with respect to $\mathrm{SO}(8) \times \mathrm{SO}(7)$ from Proposition 4.3. □

Proposition 6.3. *For any $n \geq 9$, the Lie group* SO(n) *admits at least one left-invariant Einstein metric determined by the* $\mathrm{Ad}(\mathrm{SO}(n-6) \times \mathrm{SO}(3) \times \mathrm{SO}(3))$*-invariant scalar product of the form* (10)*, which is not naturally reductive.*

Proof. We consider the system of equations

$$r_1 = r_2, \quad r_2 = r_3, \quad r_3 = r_{12}, \quad r_{12} = r_{13}, \quad r_{13} = r_{23}. \tag{25}$$

Then finding Einstein metrics of the form (10) reduces to finding positive solutions of system (25).

We put $k_2 = k_3 = 3$ and consider our equations by putting

$$x_{12} = x_{13} = 1, \quad x_2 = x_3.$$

Then the system of equations (25) reduces to the system of equations:

$$\begin{aligned} g_1 &= -nx_1{x_2}^2{x_{23}}^2 + nx_2{x_{23}}^2 + 6{x_1}^2x_2{x_{23}}^2 \\ &\quad +6x_1{x_2}^2{x_{23}}^2 - 3x_1{x_2}^2 - x_1{x_{23}}^2 - 8x_2{x_{23}}^2 = 0, \\ g_2 &= nx_1x_2{x_{23}}^2 + n{x_2}^2{x_{23}}^2 - 2nx_2{x_{23}}^2 - 7x_1x_2{x_{23}}^2 \\ &\quad -4{x_2}^2{x_{23}}^2 + 3{x_2}^2 + 3x_2{x_{23}}^3 + 4x_2{x_{23}}^2 + {x_{23}}^2 = 0, \\ g_3 &= -nx_1{x_{23}}^2 - n{x_{23}}^3 + 2n{x_{23}}^2 + 7x_1{x_{23}}^2 \\ &\quad -2x_2{x_{23}}^2 + 4x_2 + 3{x_{23}}^3 - 4{x_{23}}^2 - 8x_{23} = 0. \end{aligned} \tag{26}$$

We consider a polynomial ring $R = \mathbb{Q}[z, x_2, x_1, x_{23}]$ and an ideal I generated by $\{g_1,\ g_2,\ g_3,\ z(x_2 - x_{23})\,x_1\,x_{23}\,x_2 - 1\}$ to find non-zero solutions of equations (26) with $x_2 \neq x_{23}$. We take a lexicographic order $>$ with $z > x_2 > x_1 > x_{23}$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomials

$\{h(x_{23}), p_1(x_{23}, x_1), p_2(x_{23}, x_2)\}$, where $h(x_{23})$ is a polynomial of x_{23} given by

$$
\begin{aligned}
h(x_{23}) &= (n-6)^2(n-3)\left(n^2-7n+24\right){x_{23}}^8 \\
&-2(n-6)^2(n-2)\left(n^2-n+6\right){x_{23}}^7 \\
&+(n-6)\left(n^4+26n^3-269n^2+686n-516\right){x_{23}}^6 \\
&-44(n-6)(n-3)(n-2)(n+2){x_{23}}^5 \\
&+\left(14n^4+273n^3-3034n^2+5687n+1164\right){x_{23}}^4 \\
&-2(n-2)\left(157n^2-157n-2778\right){x_{23}}^3 \\
&+\left(49n^3+1658n^2-6539n+836\right){x_{23}}^2 \\
&-728(n-2)(n+5)x_{23}+2704(n-1),
\end{aligned}
$$

$p_1(x_{23}, x_1)$ is a polynomial of x_{23} and x_1 given by

$$
\begin{aligned}
p_1(x_{23}, x_1) &= 8(2n-5)\left(n^2-7n+27\right)x_1 \\
&+(n-6)^3(n-3)\left(n^2-7n+24\right){x_{23}}^7 \\
&-2(n-6)^3(n-2)\left(n^2-n+6\right){x_{23}}^6 \\
&+(n-6)^2\left(n^4+19n^3-199n^2+371n-12\right){x_{23}}^5 \\
&-6(n-6)^2(n-2)\left(5n^2-5n-58\right){x_{23}}^4 \\
&+(n-6)\left(7n^4+140n^3-1641n^2+3090n+1248\right){x_{23}}^3 \\
&-104(n-6)^2(n-2)(n+5){x_{23}}^2 \\
&+8\left(48n^3-625n^2+2305n-1719\right)x_{23}
\end{aligned}
$$

and $p_2(x_{23}, x_2)$ is a polynomial of x_{23} and x_2 given by

$$
\begin{aligned}
p_2(x_{23}, x_2) &= -(n-6)^2(n-3)\left(n^2-7n+24\right)\left(2n^2-14n+15\right){x_{23}}^7 \\
&+2(n-6)^2(n-2)\left(n^2-n+6\right)\left(2n^2-14n+15\right){x_{23}}^6 \\
&-(n-6)(2n^6+25n^5-666n^4+3955n^3-8860n^2+7452n-2124){x_{23}}^5 \\
&+2(n-6)(n-2)\left(31n^4-248n^3+127n^2+2142n-2448\right){x_{23}}^4 \\
&-(15n^6+194n^5-5442n^4+33531n^3-73361n^2+38979n+18396){x_{23}}^3 \\
&+2(n-2)\left(119n^4-952n^3-1276n^2+21873n-28098\right){x_{23}}^2 \\
&-\left(7n^5+849n^4-11830n^3+53569n^2-79135n+24552\right)x_{23} \\
&+52(n-7)(n-1)\left(n^2-7n+27\right)x_2+624(n-2)\left(n^2-7n+27\right).
\end{aligned}
$$

Thus we see that, if there exists a real root $x_{23} = \alpha_{23}$ of $h(x_{23}) = 0$, then there are a real solution $x_1 = \alpha_1$ of $p_1(\alpha_{23}, x_1) = 0$ and a real solution $x_2 = \alpha_2$ of $p_2(\alpha_{23}, x_2) = 0$.

Now we have $h(0) = 2704(n-1) > 0$ for $n > 1$, $h(2) = 4(16n^5 - 424n^4 + 4625n^3 - 25470n^2 + 70193n - 77128) = 4(16(n-6)^5 + 56(n-6)^4 + 209(n-6)^3 + 756(n-6)^2 + 1397(n-6) + 1022) > 0$ for $n \geq 6$ and $h(1) = -2(n-9)(n-1)n^2 < 0$ for $n > 9$. Note that for $n = 9$ $h(6/5) = -1751152/390625 < 0$. Thus we see that the equation $h(x_{23}) = 0$ has *two* positive roots $x_{23} = \alpha_{23}, \beta_{23}$ with $0 < \alpha_{23} < 1 < \beta_{23} < 2$ for $n > 9$. For $n = 9$ we have roots $x_{23} = 1, \beta_{23}$ with $6/5 < \beta_{23} < 2$.

Let $\gamma = \alpha_{23}$ or β_{23}. We have to show that the real solutions $x_1 = \alpha_1$ of $p_1(\gamma, x_1) = 0$ and $x_2 = \alpha_2$ of $p_2(\gamma, x_2) = 0$ are positive. To this end, we take a lexicographic order $>$ with $z > x_2 > x_{23} > x_1$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_1(x_1)$ of x_1 given by

$$\begin{aligned}
&h_1(x_1) = 4(n-1)\left(n^2-7n+24\right)\left(n^2-7n+27\right)^2 {x_1}^8 - 16(n-2)\left(n^2-7n+27\right) \times \\
&(2n^4 - 28n^3 + 170n^2 - 504n + 549){x_1}^7 + (112n^7 - 2728n^6 + 29992n^5 - 192017n^4 \\
&+761574n^3 - 1849727n^2 + 2498826n - 1434888){x_1}^6 - 4(n-2)(56n^6 - 1348n^5 \\
&+13792n^4 - 77805n^3 + 254449n^2 - 453225n + 344070){x_1}^5 + (280n^7 - 7880n^6 \\
&+93373n^5 - 609014n^4 + 2369548n^3 - 5476199n^2 + 6921202n - 3695208){x_1}^4 \\
&-2(n-8)(n-2)\left(112n^5 - 2304n^4 + 18506n^3 - 73480n^2 + 144545n - 109506\right){x_1}^3 \\
&+2(n-8)^2\left(56n^5 - 972n^4 + 6475n^3 - 20866n^2 + 32361n - 18921\right){x_1}^2 \\
&-2(n-8)^3(n-2)(2n-5)\left(8n^2 - 64n + 117\right)x_1 + (n-8)^4(n-3)(2n-5)^2.
\end{aligned}$$

Now we have

$$\begin{aligned}
&h_1(x_1) = 4(n-1)\left(n^2-7n+24\right)\left(n^2-7n+27\right)^2 {x_1}^8 - 16(n-2)\left(n^2-7n+27\right) \times \\
&(2(n-8)^4 + 36(n-8)^3 + 266(n-8)^2 + 936(n-8) + 1253){x_1}^7 + (112(n-8)^7 \\
&+3544(n-8)^6 + 49576(n-8)^5 + 395823(n-8)^4 + 1933510(n-8)^3 \\
&+5714577(n-8)^2 + 9285018(n-8) + 6127496){x_1}^6 - 4(n-2)(56(n-8)^6 \\
&+1340(n-8)^5 + 13632(n-8)^4 + 74259(n-8)^3 + 222137(n-8)^2 + 328423(n-8) \\
&+167678){x_1}^5 + (280(n-8)^7 + 7800(n-8)^6 + 91453(n-8)^5 + 578706(n-8)^4 \\
&+2089420(n-8)^3 + 4129977(n-8)^2 + 3804802(n-8) + 1039208){x_1}^4 \\
&-2(n-8)(n-2)(112(n-8)^5 + 2176(n-8)^4 + 16458(n-8)^3 + 59368(n-8)^2 \\
&+97185(n-8) + 5203){x_1}^3 + 2(n-8)^2(56(n-8)^5 + 1268(n-8)^4 + 11211(n-8)^3 \\
&+48006(n-8)^2 + 97929(n-8) + 73439){x_1}^2 - 2(n-8)^3(n-2)(2n-5)(8(n-8)^2 \\
&+64(n-8) + 117)x_1 + (n-8)^4(n-3)(2n-5)^2.
\end{aligned}$$

Thus we see that, for $n \geq 9$, the coefficients of the polynomial $h_1(x_1)$ are positive for even degree terms and negative for odd degree terms, so if the equation $h_1(x_1) = 0$ has real solutions then these are all positive.

We also take a lexicographic order $>$ with $z > x_1 > x_{23} > x_2$ for a monomial ordering on R. Then, by the aid of computer, we see that a

Gröbner basis for the ideal I contains the polynomial $h_2(x_2)$ of x_2 given by

$$h_2(x_2) = 64(n-6)^2(2n-5)^2\left(n^2-7n+27\right)^2 {x_2}^8 - 896(n-6)^2(n-2)(2n-5)\times \left(n^2-7n+27\right)\left(n^2-n+12\right){x_2}^7 + 4(n-6)(1176n^7 - 14228n^6 + 100368n^5 -730649n^4 + 4440678n^3 - 18369941n^2 + 41390868n - 33209244){x_2}^6 - 8(n-6)\times (n-2)(686n^6 - 2200n^5 + 33593n^4 - 489642n^3 + 2433897n^2 - 7853838n +19276848){x_2}^5 + (2401n^8 - 2114n^7 + 85477n^6 - 3433940n^5 + 22264067n^4 -66085822n^3 + 304096111n^2 - 1233542964n + 1558955520){x_2}^4 - 2(n-2)\times (4949n^6 - 15874n^5 + 114730n^4 - 3099532n^3 + 11930753n^2 + 23543310n -121637952){x_2}^3 + 3(5099n^6 - 34656n^5 + 78010n^4 - 1041692n^3 + 8171395n^2 -21025340n + 15585312){x_2}^2 - 104(n-2)(101n^4 - 661n^3 + 743n^2 + 5001n -15768)x_2 + 2704(n-3)(n-1)\left(n^2-7n+24\right).$$

Now we have

$$h_2(x_2) = 64(n-6)^2(2n-5)^2\left(n^2-7n+27\right)^2 {x_2}^8 - 896(n-6)^2(n-2)(2n-5)\times \left(n^2-7n+27\right)\left(n^2-n+12\right){x_2}^7 + 4(n-6)(1176(n-8)^7 + 51628(n-8)^6 +997968(n-8)^5 + 10699111(n-8)^4 + 68192070(n-8)^3 + 256591483(n-8)^2 +519880260(n-8) + 428454852){x_2}^6 - 8(n-6)(n-2)(686(n-8)^6 + 30728(n-8)^5 +604153(n-8)^4 + 6201974(n-8)^3 + 34466041(n-8)^2 + 95692802(n-8) +106856960){x_2}^5 + (2401(n-8)^8 + 151550(n-8)^7 + 4269685(n-8)^6 +66669212(n-8)^5 + 617496227(n-8)^4 + 3426718370(n-8)^3 +11106086431(n-8)^2 + 19421585724(n-8) + 14243093536){x_2}^4 -2(n-2)(4949(n-8)^6 + 221678(n-8)^5 + 4230810(n-8)^4 + 41090228(n-8)^3 +204389985(n-8)^2 + 502205726(n-8) + 490441840){x_2}^3 + 3(5099(n-8)^6 +210096(n-8)^5 + 3586810(n-8)^4 + 31488548(n-8)^3 + 148970467(n-8)^2 +362225908(n-8) + 357598976){x_2}^2 - 104(n-2)(101(n-8)^4 + 2571(n-8)^3 +23663(n-8)^2 + 96825(n-8) + 147056)x_2 + 2704(n-3)(n-1)\left(n^2-7n+24\right).$$

Thus we see that for $n \geq 8$ the coefficients of the polynomial $h_2(x_2)$ are positive for even degree terms and negative for odd degree terms, so if the equation $h_2(x_2) = 0$ has real solutions then these are all positive. Since the solutions satisfy the property $x_2 \neq x_{23}$, $x_{23} \neq 1$ and $x_{12} = x_{13} = 1$, then Proposition 4.2 implies that the metrics obtained are not naturally reductive □

From the above Propositions 6.1, 6.2 and 6.3 we obtain Theorem 1.1.

7. The Lie groups SO$((n-2)+1+1)$

In the present section we consider the scalar products (12) on the Lie groups $\mathrm{SO}(k_1+k_2+k_3)$, $k_1 = n-2, k_2 = k_3 = 1$, and prove that for $n \geq 5$ we obtain only naturally reductive Einstein metrics. In this case decomposition (8) becomes

$$\mathfrak{so}((n-2)+1+1) = \mathfrak{m}_1 \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23},$$

where $\mathfrak{m}_{12}$, $\mathfrak{m}_{13}$ are equivalent as $\mathrm{Ad}(\mathrm{SO}(n-2))$-submodules.

Lemma 7.1. *For the metric* (12) *it is*

$$r(\mathfrak{m}_{12}, \mathfrak{m}_{13}) = (0).$$

Proof. We use the formula for the Ricci curvature in [5, Corollary 7.38, p. 184] adapted for the case of a Lie group. Then using polarization we obtain that

$$\begin{aligned} r(X,Y) &= -\tfrac{1}{2}\textstyle\sum_j \langle [X,X_j],[Y,X_j]\rangle - \tfrac{1}{2}B(X,Y) \\ &\quad + \tfrac{1}{4}\textstyle\sum_{i,j}\langle [X_i,X_j],X\rangle\langle [X_i,X_j],Y\rangle, \end{aligned} \tag{27}$$

where $\{X_i\}$ is an orthonormal basis of $\mathfrak{so}(n)$ with respect to the metric (12). For any $X \in \mathfrak{m}_{12}, Y \in \mathfrak{m}_{13}$ we need to show that $r(X,Y) = 0$. A computation using (6) shows that $\mathfrak{m}_{12}, \mathfrak{m}_{13}$ are orthogonal with respect to $-B$, so the second term in (27) vanishes.

We claim that (i) $\langle [X,X_j],[Y,X_j]\rangle = 0$ and (ii) $\langle [X_i,X_j],X\rangle\langle [X_i,X_j],Y\rangle = 0$. Indeed, using the orthonormal basis $\{e_{ab} : 1 \le a < b \le n\}$ of $\mathfrak{so}(n)$ introduced in Section 3 we let $X = e_{i,n-1}$ $(1 \le i \le n-2)$, $Y = e_{jn}$ $(1 \le j \le n-2)$.

To prove claim (i) let $X_j = e_{ab}$ $(a < b)$. By using Lemma 3.1 it follows that

$$[X,X_j] = [e_{i,n-1}, e_{ab}] = \begin{cases} \pm e_{ib}, & a = n-1 \\ \pm e_{b,n-1}, & a = i \\ \pm e_{ia}, & b = n-1 \\ \pm e_{a,n-1}, & b = i \end{cases}$$

and

$$[Y,X_j] = [e_{jn}, e_{ab}] = \begin{cases} \pm e_{jb}, & a = n \\ \pm e_{bn}, & a = j \\ \pm e_{an}, & b = j \\ \pm e_{ja}, & b = n. \end{cases}$$

By using these expressions, we can check that $-B([X, X_j], [Y, X_j]) = 0$, hence, for the metric (12) it is also $\langle [X, X_j], [Y, X_j] \rangle = 0$. To prove claim (ii) we let $X_i = e_{ab}$ $(a < b)$, $X_j = e_{cd}$ $(c < d)$. By using Lemma 3.1 and obtain that

$$[X_i, X_j] = [e_{ab}, e_{cd}] = \begin{cases} \pm e_{ad}, & b = c \\ \pm e_{ac}, & b = d \\ \pm e_{bd}, & a = c \\ \pm e_{bc}, & a = d. \end{cases}$$

The conclusion then follows by a similar argument as for the proof of claim (i). □

Proposition 7.2. *For any $n \geq 5$, the only left-invariant Einstein metrics on the Lie group $\mathrm{SO}(n)$, determined by the $\mathrm{Ad}(\mathrm{SO}(n-2))$-invariant scalar products of the form (12), are naturally reductive metrics.*

Proof. The proof involves manipulations of polynomials using Gröbner bases, which are quite extensive to be presented in their complete form. Since the metrics obtained are only naturally reductive, we will only sketch the ideas behind these computations.

We use the components for the Ricci tensor in Proposition 5.5 and consider the system of equations

$$r_1 = r_{12}, \ r_{12} = r_{23}, \ r_{23} = r_{13}. \tag{28}$$

Then finding Einstein metrics of the form (12) reduces to finding the positive solutions of system (28), and we normalize our equations by putting $x_{23} = 1$. Then we have the system of equations:

$$\begin{aligned} &g_1 = n{x_1}^2{x_{13}}^2 - 2nx_1x_{12}{x_{13}}^2 + n{x_{12}}^2{x_{13}}^2 + {x_1}^2{x_{12}}^2 - 2{x_1}^2{x_{13}}^2 \\ &-x_1{x_{12}}^3x_{13} + x_1x_{12}{x_{13}}^3 + 4x_1x_{12}{x_{13}}^2 + x_1x_{12}x_{13} - 4{x_{12}}^2{x_{13}}^2 = 0, \\ &g_2 = -nx_1x_{13} + n{x_{12}}^3 - 2n{x_{12}}^2x_{13} + nx_{12}{x_{13}}^2 + 2nx_{12}x_{13} - nx_{12} \\ &-{x_{12}}^3 + 4{x_{12}}^2x_{13} - 3x_{12}{x_{13}}^2 - 4x_{12}x_{13} + x_{12} = 0, \\ &g_3 = nx_1x_{12} - n{x_{12}}^2x_{13} + 2nx_{12}{x_{13}}^2 - 2nx_{12}x_{13} - n{x_{13}}^3 + nx_{13} \\ &-3x_1x_{12} + 3{x_{12}}^2x_{13} - 4x_{12}{x_{13}}^2 + 4x_{12}x_{13} + {x_{13}}^3 - x_{13} = 0. \end{aligned} \tag{29}$$

We consider a polynomial ring $R = \mathbb{Q}[z, x_1, x_{12}, x_{13}]$ and an ideal I generated by $\{g_1, \, g_2, \, g_3, \, z\, x_1\, x_{12}\, x_{13} - 1\}$ to find non zero solutions of equations (29). We take a lexicographic order $>$ with $z > x_{13} > x_{12} > x_1$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial

$$(x_1-1)\left((n-1)x_1-(n-3)\right)\times \\ \left((n^3-2n^2+n-4)x_1-(n-4)(n-1)^2\right)A(x_1), \tag{30}$$

where

$$A(x_1)=16(n-3)^2(n-2)^3(n-1){x_1}^3-4(n-3)(n-2)^2\times \\ \left(5n^3-18n^2+21n-4\right){x_1}^2+4(n-2)^2(n-1)\left(n^4-27n^2+70n-32\right)x_1 \\ -(n-4)(n-3)(n-1)^2n\left(n^3-6n^2+25n-32\right)$$

is a polynomial in x_1 of degree 3. We divide our study in the following cases:

Case (a) $A(x_1)=0$. We claim that the system of equations (29) has no real solutions in this case.
We consider the ideal J generated by $\{g_1, g_2, g_3, A(x_1),\ z\,x_1\,x_{12}\,x_{13}-1\}$ of the polynomial ring $R=\mathbb{Q}[z, x_1, x_{12}, x_{13}]$. We take a lexicographic order $>$ with $z>x_{13}>x_{12}>x_1$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal J contains the polynomial

$$p(x_1,x_{12})=4(n-2)^2(n-1)(n^4-10n^3+37n^2-32n+16){x_{12}}^2 \\ -2(n-2)((n^4+2n^3-3n^2-36n+64)(n-1)^2+2(n-2)(2n^5-25n^4 \\ +93n^3-143n^2+93n-28)x_1+8(n-3)(n-2)^2(n+1)(n-1){x_1}^2)x_{12} \\ +(n-1)(4(n-3)(n-2)^2(n-1)(n^2-3n+4){x_1}^2-8(n-2)(3n^4 \\ -17n^3+31n^2-21n+8)x_1+(n-3)(n-1)(n^3-6n^2+25n-32)n^2).$$

We view the above polynomial as a polynomial of the variable x_{12}, that is

$$p(x_1,x_{12})=\tilde{p}(x_{12})=4(n-2)^2(n-1)f_2(n){x_{12}}^2+f_1(x_1,n)x_{12}+f_0(x_1,n),$$

where $f_2(n)=(n-4)^4+6(n-4)^3+13(n-4)^2+40(n-4)+96>0$ for $n\geq 4$. We will show that, for the roots of the equation $A(x_1)=0$, the polynomial $\tilde{p}(x_{12})$ has no real roots for x_{12}. Indeed, the discriminant $D_{\tilde{p}(x_{12})}$ of the polynomial $\tilde{p}(x_{12})$ has the form

$$D_{\tilde{p}(x_{12})}=f_1^2-16(n-2)^2(n-1)f_2f_0=-4(n-2)^2\cdot\tilde{q}(x_1),$$

where $\tilde{q}(x_1)$ is a polynomial in x_1 of degree 4. We need to show that $D_{\tilde{p}(x_{12})}<0$ whenever $x_1=\alpha$, with $A(\alpha)=0$ and $\alpha>0$ (since we are interested to

Riemannian metrics). By dividing the polynomial $\tilde{q}(x_1)$ by $A(x_1)$ we obtain that

$$\tilde{q}(x_1) = A(x_1)B(x_1) + q(x_1),$$

where $q(x_1)$ is a polynomial of degree 2 given by

$$q(x_1) = 4(n-1)f_2(n)(a_0(n) + a_1(n)x_1 + a_2(n){x_1}^2).$$

Then we need to show that $q(x_1) > 0$ or that

$$r(x_1) \equiv a_0(n) + a_1(n)x_1 + a_2(n){x_1}^2 > 0,$$

where the coefficients of the polynomial $r(x_1)$ are explicitly given as

$$\begin{aligned} a_0(n) &= (n-1)(5n^5 - 38n^4 + 113n^3 - 196n^2 + 192n - 64), \\ a_1(n) &= (n-2)(n^6 - 15n^5 + 69n^4 - 141n^3 + 150n^2 - 104n + 32), \\ a_2(n) &= 4(n-3)(n-2)^2(n-1)(n^2 - 3n + 4) > 0 \quad \text{for } n \geq 4. \end{aligned}$$

If $n \geq 9$ we expand the polynomials $a_0(n), a_1(n)$ as

$$\begin{aligned} a_0(n) &= 5(n-9)^6 + 227(n-9)^5 + 4291(n-9)^4 + 43197(n-9)^3 \\ &\quad +244036(n-9)^2 + 732812(n-9) + 912736, \\ a_1(n) &= (n-9)^7 + 46(n-9)^6 + 882(n-9)^5 + 9036(n-9)^4 \\ &\quad +52353(n-9)^3 + 164350(n-9)^2 + 229164(n-9) + 48104, \end{aligned}$$

hence, it follows that $r(x_1) > 0$.

If $5 \leq n < 9$ we consider the discriminant $D_{r(x_1)}$ of the polynomial $r(x_1)$ and this is given by

$$\begin{aligned} D_{r(x_1)} &= (n-2)^3(n^{11} - 28n^{10} + 227n^9 - 650n^8 - 761n^7 + 11240n^6 \\ &\quad -38635n^5 + 75262n^4 - 93472n^3 + 72352n^2 - 31232n + 5632). \end{aligned}$$

It is easy to see that for $n = 5, 6, 7, 8$ it is $D_{r(x_1)} < 0$ so $r(x_1) > 0$. Therefore $D_{\tilde{p}(x_{12})} < 0$ for all $n \geq 5$ and this completes the proof of Case (a).

Case (b) $A(x_1) \neq 0$. In this case we obtain only naturally reductive Einstein metrics.
Indeed, from equation (30) we obtain the solutions

$$x_1 = 1, \qquad x_1 = \frac{n-3}{n-1}, \qquad x_1 = \frac{(n-4)(n-1)^2}{n^3 - 2n^2 + n - 4}.$$

Table 1: Numbers of new non naturally reductive and naturally reductive left-invariant Einstein metrics on the Lie group $\mathrm{SO}(n) = \mathrm{SO}(k_1 + k_2 + k_3)$ up to isometry. These are $\mathrm{Ad}(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3))$-invariant scalar products of the forms (10), (11) or (12)

$\mathrm{SO}(k_1+k_2+k_3)$	(k_1,k_2,k_3)	Non naturally reductive	Naturally reductive
SO(5)	(3, 1, 1)	0	3
	(2, 2, 1)	0	3
SO(6)	(4, 1, 1)	0	3
	(3, 2, 1)	0	5
	(2, 2, 2)	0	2
SO(7)	(5, 1, 1)	0	3
	(4, 2, 1)	0	6
	(3, 3, 1)	**1**	5
	(3, 2, 2)	0	5
SO(8)	(6, 1, 1)	0	3
	(5, 2, 1)	0	6
	(4, 3, 1)	**2**	7
	(4, 2, 2)	0	5
	(3, 3, 2)	**1**	5
SO(9)	(7, 1, 1)	0	3
	(6, 2, 1)	0	6
	(5, 3, 1)	**2**	8
	(5, 2, 2)	0	5
	(4, 3, 2)	**2**	8
	(4, 4, 1)	**2**	5
	(3, 3, 3)	**2**	5

By substituting the above solutions to the system (29) and computing Gröbner bases for this system, we obtain the solutions

$$(x_1, x_{12}, x_{13}, x_{23}) = (1,1,1,1), \quad (x_1, x_{12}, x_{13}, x_{23}) = \left(\frac{n-3}{n-1}, 1, \frac{n-3}{n-1}, 1\right),$$
$$(x_1, x_{12}, x_{13}, x_{23})$$
$$= \left(\frac{(n-4)(n-1)^2}{n^3-2n^2+n-4}, \frac{(n-1)(n^2-3n+4)}{n^3-2n^2+n-4}, \frac{(n-1)(n^2-3n+4)}{n^3-2n^2+n-4}, 1\right).$$

From Proposition 4.4 it follows that the above metrics are naturally reductive with respect to $\mathrm{SO}(n) \times L$, where L is $\mathrm{SO}(n)$, $\mathrm{SO}(n-1)$ or $\mathrm{SO}(n-2) \times \mathrm{SO}(2)$ respectively. □

By working in a similar manner as in the above proofs, we can obtain Table 1 for the Lie groups $\mathrm{SO}(n) = \mathrm{SO}(k_1+k_2+k_3)$, which lists the numbers of the new non naturally reductive and naturally reductive left-invariant Einstein metrics of the forms (10), (11) or (12), up to isometry. The table also contains results for which, due to space limitations, we do not provide explicit calculations.

Conclusion

We obtained new left-invariant Einstein metrics on the compact Lie group $\mathrm{SO}(n)$ $(n \geq 7)$ which are not naturally reductive. We built such metrics by considering the submersion $\mathrm{SO}(k_1+k_2+k_3) \to \mathrm{SO}(k_1+k_2+k_3)/(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3))$ over a generalized Wallach space, and then making certain symmetry assumptions for left-invariant metrics. It is well known that the Einstein equation reduces to a polynomial system of equations. However, it is difficult to solve such systems (or even prove existence of positive solutions) for the general case of $\mathrm{SO}(n)$, since their coefficients depend on k_1, k_2 and k_3. A major part of the present paper is devoted to the study of such systems, and to show existence of positive solutions for small values of k_2 and k_3, by using Gröbner bases techniques.

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