

Euclidean formulation of discrete uniformization of the disk

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Thurston’s circle packing approximation of the Riemann Mapping (proven to give the Riemann Mapping in the limit by Rodin-Sullivan) is largely based on the theorem that any topological disk with a circle packing metric can be deformed into a circle packing metric in the disk with boundary circles internally tangent to the circle. The main proofs of the uniformization use hyperbolic volumes (Andreev) or hyperbolic circle packings (by Beardon and Stephenson). We reformulate these problems into a Euclidean context, which allows more general discrete conformal structures and boundary conditions. The main idea is to replace the disk with a double covered disk with one side forced to be a circle and the other forced to have interior curvature zero. The entire problem is reduced to finding a zero curvature structure. We also show that these curvatures arise naturally as curvature measures on generalized manifolds (manifolds with multiplicity) that extend the usual discrete Lipschitz-Killing curvatures on surfaces.

1. Introduction

The Riemann Mapping Theorem states that any simply connected, open, proper subset of \mathbb{C} can be conformally mapped to \mathbb{D} , and that this map is unique up to Möbius transformations that fix the disk. Thurston formulated a way to approximate the mapping guaranteed by the theorem (for precompact domains in \mathbb{C}), and his formulation was proven to converge to that mapping by Rodin and Sullivan [27]. Thurston’s discrete conformal mappings are based on mappings of circle packings, and central to the argument is a theorem that a circle packing of a simply connected, bounded domain can be deformed to a circle packing of \mathbb{D} while maintaining the tangency pattern but allowing the circle radii to change. We call such theorems *discrete*

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uniformization theorems. These theorems are based the existence of circle packings on spheres, the original being the Koebe theorem (rediscovered by Andreev and generalized by Thurston):

Theorem 1.1. *Let K be a triangulation of the 2-sphere that is not simplicially equivalent to the tetrahedron. Then there is a realization of K as a geodesic triangulation of the sphere with circles at the vertices such that the circles corresponding to two vertices on an edge are externally tangent to each other.*

The original proofs of this theorem and some of its generalizations, for instance where the circles have prescribed intersection angles or they appear on other surfaces, can be found in [1, 2, 19, 21, 24, 31].

Beardon and Stephenson [4] (see also [30]) were able to reprove the circle packing theorem via a variant of the Perron method. The Perron method is a method of solving a partial differential equation on the interior of the disk (the equation is that the curvature is zero) subject to the condition that the boundary circles are internally tangent to the boundary of the disk. The method relies on transferring the problem to a problem of packing hyperbolic circles. This idea is especially elegant because it turns the boundary condition to the condition that the boundary circles are horocycles, i.e., circles of infinite radius. Another advantage to the formulation of the problem as a problem in hyperbolic geometry is that the nonuniqueness of the Riemann Mapping Theorem falls out because Mobius transformations are isometries in hyperbolic space. Bowers and Stephenson were later able to extend the argument to other boundary conditions by considering curves of constant curvature [7]. In particular, they prove

Theorem 1.2 ([7], Proposition 1). *Let K be a triangulation of the disk and Φ be an assignment of values in $[0, \pi/2]$ to the edges of K . Given a proper choice of curvatures for boundary vertices, there exists an assignment of curvatures to interior vertices such that*

1. *there is a drawing of circles in the hyperbolic plane such that two circles corresponding to two vertices joined by an edge intersect at the angle prescribed by Φ ,*
2. *these edges can be given hyperbolic lengths based on the hyperbolic distances between circle centers, and the sum of the angles of these triangles at any interior vertex is equal to 2π , and*
3. *the circles corresponding to boundary vertices have the assigned curvatures.*

This theorem, for instance, handles the case that boundary circles intersect the unit circle (which is the infinity of the disk model of hyperbolic space) at right angles by specifying that the boundary circles are geodesics (curvature is zero).

The goal of the present work is to reformulate the problem of discrete uniformization theorems of disks in terms of Euclidean discrete conformal structures. In light of the facts that the hyperbolic formulation is especially elegant and that there has been recent progress on understanding hyperbolic discrete conformal structures in great generality (e.g., [5, 18, 20, 33]), one may question the advantage of reformulating in terms of Euclidean structures. There are several reasons why having a comprehensive Euclidean treatment could be useful. The primary advantage is that the Euclidean case can treat otherwise difficult cases such as the multiplicative discrete conformal structures studied in [22] and [29] with the boundary condition that the triangulation is internally tangent to the unit circle (note that [5] solves this problem by treating the boundary condition in a different way). The multiplicative structure is well motivated, and has close connections with the finite element Laplacian (see, e.g., [15]). Another advantage of the Euclidean background structure is that it is more intuitive, making it more accessible for applications than hyperbolic formulations. Finally, this formulation of a discrete uniformization theorem motivates the definitions at the end of this paper on manifolds with multiplicity, giving a notion of how one might try to use geometric flows on the interior of two disks glued together to find uniform structures such as the Riemann mapping transformation (note that Brendle has a slightly different formulation of a geometric flow to this end in [9]).

2. Discrete conformal structures and M-weighted points

Discrete conformal structures are generalizations of circle packings, circles with fixed intersection angle, circles with fixed inversive distance, and multiplicative structures studied in papers such as [8, 11, 20, 22, 24, 29, 31]. Discrete conformal structures were described from an axiomatic framework in [17]. In [18] it is shown that discrete conformal structures have the form $C_{\alpha,\eta}$, where the length of an edge $\ell_{ij} = \ell(v_i, v_j)$ is given by

$$\ell_{ij}^2 = \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i+f_j}$$

where the α and η are part of the conformal structure and the f choose the particular metric in the conformal class. Technically, the conformal structure

$C_{\alpha,\eta}$ is a map from a label f to a discrete metric; however, we will use $f \in C_{\alpha,\eta}$ to mean that f is a label that determines geometry using the conformal structure $C_{\alpha,\eta}$. Conformal structures were generalized to piecewise Euclidean (or polyhedral) manifolds in [19] by considering only the Delaunay triangulations and our case where the α are all zero.

Discrete conformal structures are associated to triangulated surfaces, but we will need to compare these geometric structures with triangulations of points in the plane that have certain decorations or weights. This is quite common in the literature on weighted Delaunay triangulations (e.g., [3, 13, 15]). We will define a generalization of weighted points in order to more carefully describe the setting for discrete conformal uniformization of Euclidean disks.

Definition 2.1. A weighted point (p, W) in the plane is a point $p \in \mathbb{R}^2$ together with a weight $W \in \mathbb{R}$. If $W \geq 0$, we denote the disk $D(p, W) = \{x \in \mathbb{R}^2 : |p - x|^2 \leq W\}$ and the circle $C(p, W) = \{x \in \mathbb{R}^2 : |p - x|^2 = W\}$.

If $W > 0$, then we can think of the weighted point as a disk centered at p with radius \sqrt{W} . If $W = 0$, we think of the weighted point simply as a point. When $W > 0$, there are several notions of “distance” between two weighted points (p_1, W_1) and (p_2, W_2) , including the distance between the points $|p_1 - p_2|$ and the Laguerre (power) distance $|p_1 - p_2|^2 - W_1 - W_2$. There are also notions of angle between two intersecting weighted points (circle intersection angles), inversive distance between nonintersecting weighted points, and other notions. In order to make these more precise and to generalize to when $W < 0$, we will expand the notion of a weighted point to an M-weighted point. The M-weighted points correspond to the caps in [32].

Definition 2.2. Given $\xi = (a, b, c, d)$ and $\zeta = (x, y, z, w)$ in \mathbb{R}^4 , the Minkowski product is

$$\xi * \zeta = ax + by + cz - dw.$$

We now define M-weighted points, which is short for Minkowski-weighted points.

Definition 2.3. Given a point $\xi = (\xi^1, \xi^2, \xi^3, \xi^4) \in \mathbb{R}^4 \setminus \{0\}$ with $\xi^3 < \xi^4$, we may define the map $p(\xi) = \left(\frac{\xi^1}{\xi^4 - \xi^3}, \frac{\xi^2}{\xi^4 - \xi^3} \right)$ and $W(\xi) = \frac{\xi * \xi}{(\xi^4 - \xi^3)^2}$. We call ξ an *M-weighted point*. An M-weighted point ξ has a corresponding weighted point $(p(\xi), W(\xi))$. We use \mathbb{R}_\perp^4 to denote the points $\xi \in \mathbb{R}^4$ satisfying $\xi^3 < \xi^4$, the set of possible *M-weighted points*.

Remark 2.4. We may extend this map to a map $\mathbb{R}^4 \setminus \{0\} \rightarrow S^2 \times S^1$ using the one point compactification formulations of S^2 and S^1 . The first component is essentially the Hopf map (if we restrict to unit vectors in \mathbb{R}^4).

The preimage of a weighted point (p, W) is a line of points in $\mathbb{R}^4 \setminus \{0\}$. Often we will choose the preimage that makes $\xi^4 - \xi^3 = 1$, so that for $((x, y), W)$ we choose

$$\xi = \left(x, y, \frac{1}{2}(x^2 + y^2 - W - 1), \frac{1}{2}(x^2 + y^2 - W + 1) \right).$$

Note that if we pull back the Minkowski product via rescaling to so that $\xi^4 - \xi^3 = 1$ we get the Klein model for hyperbolic 3-space, and we can consider (1-point compactified) Euclidean 2-space as the boundary at infinity of hyperbolic 3-space.

An M-weighted point can sometimes be interpreted as a disk or point, and this leads to geometric interpretations for the product $*$. The following proposition is easily verified, and described well in the work of Wilker [32]. Recall $C(p, W)$ and $D(p, W)$ as in Definition 2.1.

Proposition 1. *Let $\xi, \zeta \in \mathbb{R}_\perp^4$. The following are immediate consequences of Theorem 8 in [32].*

- *The unit disk corresponds to the point $(0, 0, 1, 0)$ (or any point $(0, 0, t, 0)$). We denote $U = (0, 0, 1, 0)$.*
- *We always have*

$$|p(\xi) - p(\zeta)|^2 = \left(\frac{\xi}{\xi^4 - \xi^3} - \frac{\zeta}{\zeta^4 - \zeta^3} \right) * \left(\frac{\xi}{\xi^4 - \xi^3} - \frac{\zeta}{\zeta^4 - \zeta^3} \right)$$

- *If $\xi * \xi > 0$ and $\zeta * \zeta > 0$ and $\xi * \zeta \leq 1$ then the circles $C(p(\xi), W(\xi))$ and $C(p(\zeta), W(\zeta))$ intersect with angle θ satisfying*

$$\cos \theta = \frac{\xi * \zeta}{\sqrt{\xi * \xi} \sqrt{\zeta * \zeta}}.$$

- *If $\xi * \xi > 0$ and $\zeta * \zeta > 0$ and $\xi * \zeta > 1$ then the disks $D(p(\xi), W(\xi))$ and $D(p(\zeta), W(\zeta))$ are disjoint and have inversive distance δ satisfying*

$$\cosh \delta = \frac{\xi * \zeta}{\sqrt{\xi * \xi} \sqrt{\zeta * \zeta}}.$$

- If $\xi * \xi = 0$ and $\zeta * \zeta > 0$ then the circle $C = C(p(\zeta), W(\zeta))$ and the point $p = p(\xi)$ satisfy

$$P_C(p) = |p - p(\zeta)|^2 - W(\zeta) = \frac{\xi * \zeta}{\sqrt{\zeta * \zeta}}.$$

where P_C is the power (see, e.g., [3]) In particular, $p \in C$ if and only if $\xi * \zeta = 0$.

- If $\xi * \xi = 0$ and $\zeta * \zeta = 0$ then

$$|p(\xi) - p(\zeta)|^2 = \frac{\xi * \zeta}{(\xi^4 - \xi^3)(\zeta^4 - \zeta^3)}$$

We can also consider triangulations in the plane by M-weighted points.

Definition 2.5. A *M-weighted triangulation* (T, ξ) of a planar region D is a triangulation T , together with a map

$$\xi : V(T) \rightarrow \mathbb{R}_{\perp}^4,$$

where we will use $V(T)$, $E(T)$, and $F(T)$ to denote the vertices, edges, and faces of T , respectively.

The Möbius group acts naturally on an M-weighted triangulation (see [32]).

Proposition 2. *The Möbius group acts on (T, ξ) by linear maps L such that for each v and w , $(L\xi_v) * (L\xi_w) = \xi_v * \xi_w$. There is a subgroup of the Möbius group corresponding to the Euclidean transformations of $p(\xi)$ that preserve the weights $W(\xi)$.*

The M-weighted points with the connectivity of the triangulation induce a discrete conformal structure. Note that throughout this paper, we will assume triangulations are proper, in the sense that each edge has two distinct vertices and that there is at most one edge connecting two vertices. Thus we can refer to an edge vw between vertices v and w without confusion. Much of the work is independent of this restriction, but the statements are much clearer in this form.

Proposition 3. *Let T be a triangulation of a plane region with vertex points $\{p_v\}_{v \in V(T)}$ in the plane. For each $v \in V(T)$ and $vw \in E(T)$ fix α_v and η_{vw} .*

There is a bijection between M -weighted triangulations (T, ξ) such that

$$\begin{aligned} \alpha_v &= \xi_v * \xi_v, \\ \eta_{vw} &= -\xi_v * \xi_w, \text{ and} \\ p(\xi_v) &= p_v \end{aligned}$$

and conformal structures $C_{\alpha, \eta}$ on T such that $\ell_{vw} = |p_v - p_w|$.

Proof. Given ξ , we recall that

$$\begin{aligned} |p_v - p_w|^2 &= \left(\frac{\xi_v}{(\xi_v^4 - \xi_v^3)} - \frac{\xi_w}{(\xi_w^4 - \xi_w^3)} \right) * \left(\frac{\xi_v}{(\xi_v^4 - \xi_v^3)} - \frac{\xi_w}{(\xi_w^4 - \xi_w^3)} \right) \\ &= \alpha_v (\xi_v^4 - \xi_v^3)^{-2} + \alpha_w (\xi_w^4 - \xi_w^3)^{-2} \\ &\quad + 2\eta_{vw} (\xi_v^4 - \xi_v^3)^{-1} (\xi_w^4 - \xi_w^3)^{-1} \end{aligned}$$

and so we find that ξ corresponds to $f_v = -\log(\xi_v^4 - \xi_v^3)$. Conversely, given $f \in C_{\alpha, \eta}$, we can take

$$\xi_v = \left(e^{f_v} p_v, \frac{e^{f_v}}{2} (|p_v|^2 - \alpha_v e^{2f_v} - 1), \frac{e^{f_v}}{2} (|p_v|^2 - \alpha_v e^{2f_v} + 1) \right).$$

□

3. Conformal uniformization of the Euclidean disk

3.1. Formulation of the problem

We will follow the notation in Stephenson's book [30]. Recall the following definition.

Definition 3.1. A combinatorial closed disk S is a simplicial 2-complex that triangulates a topological closed disk, i.e., S is finite, simply connected, and has nonempty boundary. We use $\overset{\circ}{S}$ to denote the interior of the disk and ∂S to denote the boundary.

We know that any geometric realization of the boundary of S must be homeomorphic to the circle. The notation is such that the vertices of S are partitioned into $V(\overset{\circ}{S})$ and $V(\partial S)$.

We can now state the problem of discrete conformal uniformization of a Euclidean disk.

Problem 1. Let $U \in \mathbb{R}^4$ denote the representation of the unit disk as an M -weighted point. Let S be a combinatorial closed disk and let $C_{\alpha,\eta}$ be a conformal structure and $\mu : V(\partial S) \rightarrow \mathbb{R}$. Then find $\xi : V(S) \rightarrow \mathbb{R}_{\perp}^4$ such that

$$\begin{aligned}\xi_v * \xi_v &= \alpha_v \\ \xi_v * \xi_w &= \eta_{vw}\end{aligned}$$

for all $v \in V(S)$ and all $vw \in E(S)$, and

$$\xi_v * U = (\xi_v^4 - \xi_v^3) \mu_v$$

for all $v \in V(\partial S)$.

While this formulation looks a bit mysterious, consider the following cases:

- If we choose a conformal structure such that $\alpha_v = 1$ for all vertices $v \in V(S)$, $\eta_e = 1$ for all $e \in E(S)$, and $\mu_v = -1$ for all $v \in V(\partial S)$, this corresponds to circle packings with boundary circles internally tangent to the disk (see [30] for a fairly comprehensive coverage of this well-studied case).
- If we choose a conformal structure such that $\alpha_v = 1$ for all vertices $v \in V(S)$, $\eta_e = 1$ for all $e \in E(S)$, and $\mu_v = 0$ for all $v \in V(\partial S)$, this corresponds to circle packings with boundary circles orthogonal to the unit disk.
- If we choose a conformal structure such that $\alpha_v = 0$ for all vertices $v \in V(S)$, $\eta_e > 0$ for all $e \in E(S)$, and $\mu_v = 0$ for all $v \in V(\partial S)$, this corresponds to triangulations with boundary points on the unit circle with the multiplicative conformal structure.

Particular cases of this problem have been solved by Andreev-Thurston [31], Beardon-Stephenson [4], and Bowers-Stephenson [7]. The reformulation for the multiplicative conformal structure with boundary vertices on the unit circle as also handled in [5] in a different way by prescribing boundary angles on a half plane.

3.2. Reformulation as a curvature problem

We return to the case of an abstract triangulated surface with a discrete conformal structure. Recall we can define curvature at an interior vertex as follows.

Definition 3.2. The curvature K_v at an interior vertex v is defined to be

$$K_v = 2\pi - \sum_f \theta(v < f)$$

where f runs over all faces and $\theta(v < f)$ is the angle at vertex v in face f (understood to be zero if the face f does not contain v).

We may use a monodromy theorem to see how triangulation labels giving zero curvature correspond to M-weighted points in the plane.

Theorem 3.3 (Monodromy). *If $f \in C_{\alpha,\eta}$ is a label such that $K_v = 0$ for all $v \in V(\mathring{S})$ there exist points $p : V(S) \rightarrow \mathbb{R}^2$ such that $\ell_{vv'} = |p(v) - p(v')|$ for any edge $e = vv'$. The map P is unique up to Euclidean isometry. In fact, there exist $\xi : V(S) \rightarrow \mathbb{R}_\perp^4$ such that $p(\xi) = p$ and $\xi_v * \xi_w = \alpha_v$ and $\xi_v * \xi_w = \eta_{vw}$ for edges all $vw \in E(S)$.*

Proof (sketch). We first place a face. This is unique up to Euclidean motion. We can now successively place neighboring faces in a uniquely determined place. We know that each vertex is placed since S is connected. One can now develop along chains (as in [30, Theorem 5.4]), and use the curvature is zero condition to show it is independent of the chain. The placement is unique once the first triangle is placed, and so is unique up to Euclidean isometry. The last statement follows from Proposition 3. \square

Definition 3.4. If $f \in C_{\alpha,\eta}$ is a label such that $K_v = 0$ for all $v \in V(\mathring{S})$ we call f a *flat label*.

Using the monodromy theorem, we can associate flat labels with triangulations of points in the plane. Together with Proposition 3, we get the following.

Corollary 1. *There is a one-to-one correspondence between the set of flat labels in $C_{\alpha,\eta}$ for a combinatorial closed disk S and the equivalence class of M-weighted triangulations (S, ξ) where ξ is considered up to Euclidean transformations (c.f., Proposition 2).*

It follows that Problem 1 can almost be reformulated into a problem of finding a flat label in a conformal class; however, we do not yet know how to describe the boundary condition. The most natural boundary condition for such a problem is to specify the weights f on the boundary (Dirichlet type condition) or specify the angle sum at the boundary (Neumann type boundary condition) as described in [5]. One might think that by considering

the disk U as an additional weighted point, one can consider this more like an equation on a closed manifold, and this is essentially what we will do. We will need to describe how to add the additional point as an augmentation.

Definition 3.5. An *augmented conformal combinatorial closed disk* \hat{S} is a simplicial complex determined by the following properties:

- There is a conformal combinatorially closed disk $S \subset \hat{S}$.
- There is a vertex $\hat{v} \in \hat{S} \setminus S$ such that $V(\hat{S}) = V(S) \cup \{\hat{v}\}$.
- For each $v \in \partial S$, there is an edge $\hat{e} = v\hat{v}$; furthermore, we have $E(\hat{S}) = E(S) \cup \{\hat{e} = v\hat{v} : v \in \partial S\}$.
- \hat{S} has a discrete conformal structure and a label in that conformal structure.

Note the following.

Proposition 4. *The vertices $V(\hat{S})$ are partitioned into $V(\overset{\circ}{S})$, $V(\partial S)$, and $\{\hat{v}\}$. The faces $F(\hat{S})$ are partitioned into $F(S)$ and $F(\hat{S} \setminus \overset{\circ}{S})$.*

In general, we will consider the augmentation of a conformal combinatorial closed disk to an augmented conformal combinatorially closed disk in order to impose boundary conditions on the determination of a conformal combinatorial closed disk with certain properties.

Note that \hat{S} is topologically a sphere, but we will not be considering it in this way, since geometrically it will be far from a sphere. In particular, the curvature of an augmented conformal combinatorial closed disk \hat{S} is slightly different than it would be for a sphere:

Definition 3.6. Let \hat{S} be an augmented conformal combinatorial closed disk. The curvatures $K : V(\hat{S}) \rightarrow \mathbb{R}$ are defined to be

$$K(v) = \begin{cases} 2\pi - \sum_{f \in F(S)} \theta(v < f) & \text{if } v \in V(\overset{\circ}{S}), \\ \sum_{f \in F(\hat{S} \setminus \overset{\circ}{S})} \theta(v < f) - \sum_{f \in F(S)} \theta(v < f) & \text{if } v \in V(\partial S), \\ \sum_{f \in F(\hat{S})} \theta(v < f) - 2\pi & \text{if } v = \hat{v}. \end{cases}$$

We say that \hat{S} is flat if $K(v) = 0$ for all $v \in V(\hat{S})$. If the geometry comes from a label in a conformal class, we call such a label a flat label.

The motivation for this definition, is that we want to consider \hat{S} as one disk folded on the other (see Appendix B). Note that zero curvature around

the boundary means that the complex folds over on itself perfectly. We may prove a restriction on the curvatures.

Proposition 5. *The curvatures satisfy*

$$\sum_{v \in V(\hat{S})} K(v) = 0.$$

Proof. This follows from the fact that the sum is equal to

$$\begin{aligned} \sum_{v \in V(\hat{S})} K(v) &= 2\pi \left| V(\hat{S}) \right| - \pi |F(S)| + \pi \left| F(\hat{S} \setminus \mathring{S}) \right| - 2\pi \\ &= 2\pi \left| V(\hat{S}) \right| - \pi |F(S)| + \pi |V(\partial S)| - 2\pi \\ &= 2\pi |V(S)| - \pi |F(S)| - \pi |E(\partial S)| - 2\pi \\ &= 2\pi \chi(S) + 2\pi |E(S)| - 3\pi |F(S)| - \pi |E(\partial S)| - 2\pi \\ &= 0, \end{aligned}$$

since $3F(S) = 2E(S) - E(\partial S)$ and $\chi(S) = 1$. □

Remark 3.7. If instead of this definition, one only used the usual curvature as angle deficit (first part of the curvature definition above) for every vertex, one would get $\sum_{v \in V(\hat{S})} K(v) = 4\pi$, following from the Euler characteristic of the sphere.

We will give some basic examples:

Example 3.8 (Circle packing with internally tangent boundary). Circle packing with internally tangent boundary. A circle packing arises from $\alpha_v = 1$ and $\eta_e = 1$ for vertices and edges in the disk. In order to get boundary circles to be internally tangent to a circle, we specify that $\alpha_{\hat{v}} = 1$ for the augmented vertex and $\eta_e = -1$ for any augmented edges. This assures that $\ell_{v\hat{v}} = e^{f_{\hat{v}}} - e^{f_v}$ as long as $f_{\hat{v}} \geq f_v$.

Example 3.9 (Circle packing orthogonal to boundary). In this case, we keep the circle packing for the disk, but must specify that $\alpha_{\hat{v}} = 1$ and $\eta_e = 0$ for any augmented edges.

Example 3.10 (Inscribed triangulation with multiplicative weights). Multiplicative weights arise when $\alpha_v = 0$ (and η_e are all fixed to positive numbers). If we specify this for all vertices of the disk, we can set $\alpha_{\hat{v}} = 1$ (or any other positive number) and specify that $\eta_e = 0$ for all augmented edges. This assures that for all augmented edges, $\ell_e = 1$, and so the boundary vertices

lie on the unit circle. This is particularly nice since if one can find a zero curvature solution, the triangulated disk is inscribed in the unit circle. In this setting, one needs to make sure that the triangle inequality is satisfied for any given choice of η 's and f 's.

We may now extend the monodromy theorem (Theorem 3.3) to augmented disks.

Theorem 3.11 (Monodromy). *If \vec{f} is a flat label, there exist points $P : V(\hat{S}) \rightarrow \mathbb{R}^2$ such that $\ell_{vv'} = |P(v) - P(v')|$ for any edge $e = vv'$. The map P is unique up to Euclidean isometry.*

Proof (sketch). In our setting, we first place \hat{v} at the origin, i.e., $P(\hat{v}) = 0$. Then place a face incident on \hat{v} , giving placement for the other two vertices. This is unique up to rotation. Now one can place the other faces around the vertex $P(\hat{v}) = 0$. Since the complex looks like a disk around \hat{v} and since the curvature at \hat{v} is zero, this will give a consistent choice of $P(v)$ for all $v \in V(\partial S)$. Now, given an edge in ∂S , we can now place the triangles with these edges, and we do this so that the triangles are folded back toward the center of the disk. We can place triangles in order around the vertex $v \in V(\partial S)$, and since the curvature $K(v) = 0$, we must have that the faces placed around the vertex give a consistent value for P . Now we can continue to place triangles inside, this time not folding back.

We know that each vertex is placed since S is connected. One can now develop along chains (as in [30, Theorem 5.4]), and use the curvature is zero condition to show it is independent of the chain. The placement is unique up to the placement of \hat{v} and rotation, which is clearly the same thing as being unique up to Euclidean isometry. \square

This allows us to reformulate Problem 1 into the following equivalent problem.

Problem 2. *Let S be a combinatorial closed disk and let $C_{\alpha,\eta}$ be a conformal structure and $\mu : V(\partial S) \rightarrow \mathbb{R}$. Let $\hat{\alpha}_v = \alpha_v$ if $v \in V(S)$ and $\hat{\alpha}_{\hat{v}} = 1$. Let $\hat{\eta}_e = \eta_e$ if $e \in E(S)$ and $\hat{\eta}_{\hat{v}v} = \mu_v$ for all $v \in \partial S$. Then find a flat label $f \in C_{\hat{\alpha},\hat{\eta}}(\hat{S})$.*

We have shown the following.

Theorem 3.12. *Problem 1 and Problem 2 are equivalent.*

Problems 1 and 2 are the direct generalizations of the problems described in [7]. In the latter, the formulation is done in hyperbolic geometry instead

of Euclidean, which has the advantage of allowing the assignment of curvatures (generalizing radii) of generalized circles (called cycles there) and formulating the problem as a boundary value problem where the boundary data is specified (a Dirichlet type problem); however, the case of $\alpha_v = 0$ does not appear to exist in that context. In addition, this formulation gives an alternative that uses primarily Euclidean instead of hyperbolic geometry. The formulation in [7] with the restrictions there allow one to prove existence and uniqueness (see also [24] and [28]), whereas the more general case of Problems 1 and 2 remain open.

4. The action of the Mobius group

Recall that Mobius transformations act on (\hat{S}, ξ) , a M -weighted triangulation of the augmented disk. Since, by Proposition 3 there is a correspondence with flat labels $f \in C_{\hat{\alpha}, \hat{\eta}}$, we see that Mobius transformations must act on the flat labels. In particular, taking a one parameter family of Mobius transformations through the identity, we get a deformation of f through flat labels, so it is not possible that the solution to Problems 1 and 2 is unique. However, we wish to show that these are the only deformations.

A first question is what are the possible deformations through flat labels.

Proposition 6. *The infinitesimal Mobius transformations induce the following variations of flat labels: Suppose p_i are the points representing the vertices. Then the variations are all of the form*

$$\delta f_i = 2(a, b) \cdot p_i + c,$$

where $(a, b) \in \mathbb{R}^2$ and $c \in \mathbb{R}$.

The proof is left for Appendix A. Note that by the work in [16], the derivative of the curvature map $f \rightarrow K$ takes the form

$$\frac{dK_v}{dt} = \Delta \frac{df_v}{dt} = \sum_{vw \in E(T)} \frac{\ell_{vw}^*}{\ell_{vw}} \left(\frac{df_w}{dt} - \frac{df_v}{dt} \right)$$

where ℓ_{vw}^* is the dual length of the edge vw as determined by the discrete conformal structure. Thus the Mobius transformations induce a kernel of the Laplacian operator Δ . It is interesting to observe that the kernel appears to correspond to linear functions analogously to the kernel of the smooth Euclidean Laplacian. We can see this directly as follows. Consider a vertex p_0 , with edges p_1, \dots, p_k adjacent to it. For each edge $p_0 p_i$, there is a dual edge

induced by the conformal structure, and all the edges around the vertex correspond to a cycle. Notice that if R denotes rotation by $\pi/2$, $\frac{\ell_{p_i p_0}^*}{\ell_{p_i p_0}} R(p_i - p_0)$ is a vector of length $\ell_{p_i p_0}^*$ in the direction of the dual edge. This sum is then zero because of it is the sum of vectors on a cycle. Thus,

$$0 = \Delta R p_i = R \Delta p_i.$$

5. Open problems

5.1. Uniqueness

We have already shown that the solutions to Problems 1 and 2 are invariant under Mobius transformations. We conjecture that these are the only such invariant deformations.

Conjecture 1. *Let $f \in C_{\hat{\alpha}, \hat{\eta}}$ be a flat label. Then the only deformations of f through flat labels correspond to Mobius transformations.*

We already know that the Mobius transformations form a three-dimensional family of deformations through flat labels, so we could try to use the implicit function theorem to show that there are no more. Instead of formulating the problem on labels, we can use M-weighted points, since there is a correspondence between M-weighted points and flat labels. Thus it will be sufficient to show that Mobius transformations are the only deformations of M-weighted points (\hat{S}, ξ) that fix $\xi_v * \xi_w = \alpha_v$ for all $v \in V(\hat{S})$ and $\xi_v * \xi_w = \eta_{vw}$ for $vw \in E(\hat{S})$.

Before reformulating the problem using the implicit function theorem, we count the dimensions. The number of constraints is $V(\hat{S}) + E(\hat{S})$. The number of variables is $4V(\hat{S})$ since there is a $\xi_v \in \mathbb{R}^4$ for each vertex. Since \hat{S} is topologically a triangulation of the sphere, we have $3F(\hat{S}) = 2E(\hat{S})$ and Euler's formula $V(\hat{S}) - E(\hat{S}) + F(\hat{S}) = 2$. It thus follows that

$$V(\hat{S}) + E(\hat{S}) = 4V(\hat{S}) - 6.$$

Since the Mobius transformations form a six dimensional set of deformations, if we can show that the differential of the map of the variables to the constraints is injective, then we are done by the implicit function theorem.

We can label the constraints first by the equations determined by vertices and then by the equations determined by edges. For the differential of the map, we can arrange the matrix to have the schematics as follows:

$$\begin{bmatrix} \xi_{v_1}^T & 0 & \cdots & 0 \\ 0 & \xi_{v_2}^T & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \xi_{v_N}^T \\ \xi_{v_2}^T & \xi_{v_1}^T & \cdots & 0 \\ 0 & 0 & \cdots & \\ \vdots & \xi_{v_k}^T & \cdots & \\ \xi_{v_j}^T & 0 & \cdots & \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

where the first $N = V(\hat{S})$ rows correspond to the vertices and the remaining rows correspond to the edges. In the first N rows, each row has one block of four potentially nonzero entries given by the row vector ξ_v^T , the transpose of the M-weighted point corresponding to vertex v . Below ξ_v^T are all the ξ_w^T such that $vw \in E(\hat{S})$. In the schematic above, it is assumed that $v_1v_2 \in E(\hat{S})$. The conjecture is now equivalent to showing that this matrix has full rank.

Another natural question is what happens when the curvatures are not zero. Brief numerical study suggests that the labels are rigid in this case.

Conjecture 2. *Under an appropriate assumption on the conformal structure, if the curvatures are all nonzero then the only deformations of the label that preserve the curvature scale all of the labels equally.*

5.2. Existence

The existence of circle packings of a disk was proven by Koebe [21], Andreev [1, 2], and a new proof was given by Marden-Rodin [24] in the spirit of work of Thurston [31]. A different proof was given by Beardon and Stephenson [4] (see also [6, 8, 30]) that uses a method similar to the Perron method in proving existence of solutions to elliptic PDE by considering the hyperbolic circle packing. This has the advantage to having a unique label (since Euclidean labels can be changed by elements of the Mobius group) and is possible partly because internally tangent circles are horocycles. In [8] it was shown that other boundary conditions (such as orthogonal intersection with the unit circle) can be realized in the hyperbolic background by considering

constant curvature curves instead of just circles. The main obstacles to using this method in our setting is that we do not have strict monotonicity of the curvatures due to the effect of the unusual boundary curvature definitions, and the space of solutions (satisfying triangle inequality) is not convex. One may be able to deal with the latter issue using the methods in [5] and [23], which extend locally convex functionals continuously by constants, allowing angles to be a continuous function of three lengths regardless of whether the triangle inequality is satisfied.

5.3. Computation

One way of finding solutions is suggested by the work of Chow and Luo in the setting of triangulated surfaces without boundary [11]. They propose a geometric flow of radii that is essentially a gradient flow of a convex functional. This idea was later improved by X. Gu, who suggested using Newton's method to do the computation (see [12]). In our setting, one can also try Newton's method to find solutions. Although we have not yet shown that the Jacobian matrix is nonsingular if the label is not flat, one can still apply Newton's method using a pseudoinverse instead of inverse as in [14]. This may be necessary, since we know that solutions do, in fact, have singular Jacobians due to Proposition 6.

The gradient flow idea does not work with this functional, which is clearly not convex; however, one could try other flows that do not arise as gradient flows. One particular choice is to take the following:

$$\begin{aligned}\frac{df_v}{dt} &= -K_v \text{ if } v \neq \hat{v}, \\ \frac{df_{\hat{v}}}{dt} &= K_{\hat{v}}.\end{aligned}$$

This flow has the advantage of locally looking like a heat flow on the curvature at the interior vertices and the augmented vertex, so that it is infinitesimally trying to make both flat; however, it is unclear what this is doing to the boundary vertices. Preliminary study of this flow numerically is promising, finding uniformizations relatively quickly. Figures 2, 3, and 4 show uniformizations computed using Mathematica's NDSolve with time up to 200 starting from the triangulation in Figure 1. It is still an open problem whether the convergence is exponential as is the case of combinatorial Ricci flow.

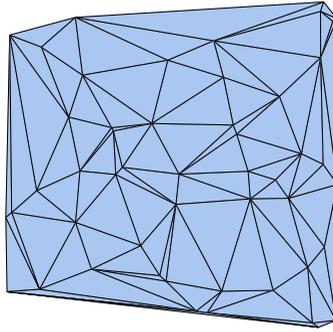


Figure 1: Original triangulation, generated as a random Delaunay triangulation.

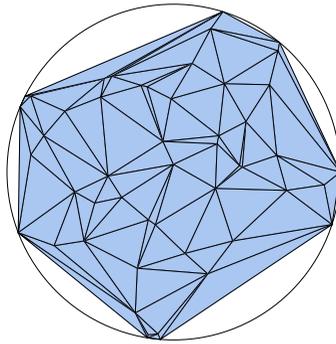


Figure 2: Multiplicative uniformization, with boundary vertices on unit circle.

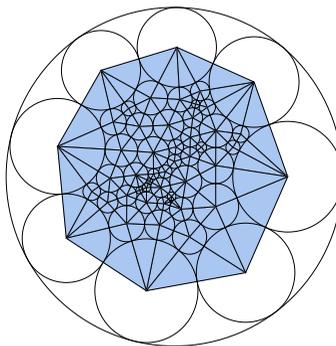


Figure 3: Circle packing uniformization, with boundary circles internally tangent to unit circle.

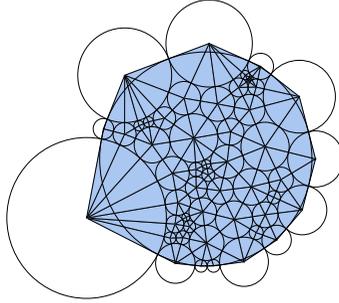


Figure 4: Circle packing uniformization, with boundary circles orthogonal to unit circle.

Appendix A. Mobius transformation computations

We consider small perturbations of the identity by Mobius transformations. A general perturbation of the identity is of the form

$$I + \varepsilon M$$

for small ε , where I is the identity matrix and

$$M = \begin{pmatrix} a & d & \ell & g \\ b & e & m & h \\ q & p & n & s \\ c & f & r & k \end{pmatrix}.$$

For $(I + \varepsilon M)$ to be a Mobius transformation, we must have

$$(I + \varepsilon M)^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} (I + \varepsilon M) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Looking at this up to $O(\varepsilon^2)$, we find that the family takes the form

$$\begin{pmatrix} 1 & \varepsilon r & -\varepsilon b & -\varepsilon a \\ -\varepsilon r & 1 & -\varepsilon d & -\varepsilon c \\ \varepsilon b & +\varepsilon d & 1 & \varepsilon t \\ -\varepsilon a & -\varepsilon c & \varepsilon t & 1 \end{pmatrix}$$

for real numbers a, b, c, d, t, r . To compute the derivative of the action of such a family of Möbius transformations, we compute

$$\xi = \begin{pmatrix} 1 & \varepsilon r & -\varepsilon b & -\varepsilon a \\ -\varepsilon r & 1 & -\varepsilon d & -\varepsilon c \\ \varepsilon b & +\varepsilon d & 1 & \varepsilon t \\ -\varepsilon a & -\varepsilon c & \varepsilon t & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ \frac{1}{2}(x^2 + y^2 - W - 1) \\ \frac{1}{2}(x^2 + y^2 - W + 1) \end{pmatrix}$$

and we get

$$\xi^4 - \xi^3 = 1 - \varepsilon((a + b)x + (c + d)y + t) + O(\varepsilon^2).$$

Since

$$\xi^4 - \xi^3 = e^{-f}$$

we must have that

$$\delta f = (a + b, c + d) \cdot (x, y) + t.$$

Appendix B. Curvature measures of polyhedral manifolds with multiplicities

The curvatures we used on the boundary of the disk in the augmented disk seem to be designed specifically for our purposes. In this section, we show how these are related to natural curvature measures; a good reference is [26]. An important property of curvature measures is the valuation property.

Definition B.1. A measure μ satisfies the *valuation* property if for measurable sets A and B , we have

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

The valuation property allows one to compute the measure from constituent parts. In particular, we can compute curvature measures of polyhedral surfaces that are not embedded, but that are made up of pieces that are each embedded. An alternative to having the need for the negative sign is to construct unions that have the correct multiplicity. In particular, if we assume $A \cup B$ is constructed by first gluing A and B . Since gluing A and B gives multiplicity 2 on $A \cap B$, we need to pull out a copy of $A \cap B$, or glue in a copy of $A \cap B$ with multiplicity -1 . Thus, the computation of the measure with the valuation property (in this case, curvature) is simply the sum of the gluings considered with multiplicities.

Example B.2. A flower of a vertex is defined to be the set of faces containing that vertex, which is said to be the center of the flower. We can compute the curvature at the center vertex in a flower by gluing together each constituent triangle around the flower. In order to keep multiplicity equal to one at all points we specify that the vertex has multiplicity 1, the (closed) edges have multiplicity -1 , and the triangles have multiplicity 1. In each triangle f_i with edges e_j , we can specify the curvature in terms of $K(v, v) = 2\pi$, $K(v, e_j) = -\pi$, $K(v, f_i) = \pi - \alpha_i$, where α_i is the interior angle at the vertex v of triangle f_i . These formulas arise from the tube formulas computing the change in area or volume of small balls around the triangle (isometrically embedded into Euclidean space of any dimension); see [26]. We can now sum to get the curvature at v to be

$$\begin{aligned} K(v) &= K(v, v) + \sum_{e_i > v} K(v, e_i) + \sum_{f_j > v} K(v, f_j) \\ &= 2\pi - \sum_{f_j > v} \alpha_j, \end{aligned}$$

which is the usual definition for curvature at a vertex.

In our setting, we have taken a triangulated disk, which can be given multiplicities as above in the interior, and attached another disk in the augmentation. The multiplicities of each of the simplices in the augmented disk should be as follows to allow for a total multiplicity of zero when the disks fit on top of each other (curvatures are all zero):

Location	Simplex type	Multiplicity
interior	vertex	1
boundary	vertex	0
augmented	vertex	-1
interior	edge	-1
boundary	edge	0
augmented	edge	1
interior	face	1
augmented	face	-1 .

Using these multiplicities and the usual definition of curvature measure at a vertex of a polyhedral manifold, one gets precisely the definitions given above for curvatures.

The advantage to this definition is the relation to tube formulas (see [26]). The valuation property allows one to extend such tube formulas to

more general types of surfaces as the one we consider here: a folder over disk. Here is a formulation of polyhedral manifold with multiplicities that allows for the proper generalization of tube formulas.

Definition B.3. A *polyhedral manifold with multiplicities* is a polyhedral manifold M together with a multiplicity function $\mu : \Sigma \rightarrow \mathbb{Z}$, where Σ is the collection of all simplices in M . If the dimension of M is two, we call M a surface.

Definition B.4. If $x \in M$, the *multiplicity* at x is defined as

$$\mu(x) = \sum_{\sigma > x} \mu(\sigma)$$

where the sum is over all simplices containing x . If there exists m such that $\mu(x) = m$ for all $x \in M$, we say the multiplicity of M is equal to m .

We note that a polyhedral surface as described in Example B.2 is a polyhedral manifold with multiplicity 1. We can now define curvature for a polyhedral surface with multiplicities.

Definition B.5. The *curvature* K_v at a vertex $v \in V(M)$ of a polyhedral surface with multiplicities (M, μ) is equal to

$$K_v = 2\pi\mu(v) + \sum_{e > v} \pi \mu(e) + \sum_{f > v} (\pi - \theta_{v < f}) \mu(f).$$

Example B.2 describes the relationship to the usual definition of curvature on a polyhedral manifold. These manifolds seem to be related to the theory of currents in geometric measure theory (see, e.g., [25]). In the case of zero curvature, it is clear that there is a map from the flat polyhedral manifold with multiplicities to a current (in \mathbb{R}^2 or \mathbb{R}^3 , for instance). In the case of augmented domain, the goal is to find a polyhedral manifold with multiplicities that maps to a current that is equivalent to the zero current (since multiplicities are such that the top and bottom parts of the domain sum to zero everywhere). In general, the existence of a current occurs on each hinge (a pair of simplices sharing a codimension 1 simplex, see [3], [13]). Hinges have bistellar flips, and certain geometric invariants do not change with bistellar flips (volume, the induced distance function, the curvature) while others do (conformal variations, Laplacians). The current associated to a hinge is a generalization of volume, and does not change with a bistellar flip.

Proposition 7. *Any hinge can be mapped to a current. Note that the image of a hinge and its bistellar flip are the same.*

Proof. For a hinge, if the two faces have the same multiplicity, we simply mapped to the unfolded hinge with that multiplicity. If the two have different multiplicities, we fold the hinge over and add the multiplicities on the overlap. \square

Remark B.6. The curvature measures here and their generalizations to higher dimensions have various names including Gauss-Bonnet curvatures and Lipschitz-Killing curvatures. These curvatures appear in essentially three places: tube formulas, kinematic formulas, and heat trace formulas. See, e.g., [10]. It would be interesting to better understand the relationship of these on polyhedral manifolds with multiplicities.

Remark B.7. Marden and Rodin used a similar construction in [24] to prove the Andreev-Thurston theorem on the sphere. If one makes the triangle have multiplicity -1 then one can replace their formulation of having a point $(2\pi/3, 2\pi/3, 2\pi/3, 0, \dots, 0)$ in the image of f with the existence of a flat label using our definition of curvature.

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