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A HOMOTOPY 2–GROUPOID FROM A FIBRATION

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Abstract

In this paper we give an elementary derivation of a 2–groupoid from a fibration. This extends a previous result for pointed fibrations due to Loday. Discussion is included as to the translation between 2–groupoids and cat^1–groupoids.

0. Introduction

In his 1982 paper [L], Loday noted that if $F \longrightarrow E \xrightarrow{f} B$ was a fibration sequence of pointed spaces then $\pi_1(F) \longrightarrow \pi_1(E)$ had the structure of a crossed module. If $A \longrightarrow X$ is an inclusion of pointed spaces then factoring it as

$$A \xrightarrow{\simeq} \bar{A} \xrightarrow{f} X$$

where f is a fibration and $\simeq$ a homotopy equivalence, then π_1 (fibre of f) $\cong \pi_2(X, A)$ and one obtains the other usual basic example of a crossed module

$$\pi_2(X, A) \longrightarrow \pi_1(A).$$

An elementary proof that this is a crossed module can be found in Hilton's book [H] and was originally due to J.H.C. Whitehead.

Again Loday considered the equivalence relation corresponding to $f : E \longrightarrow B$ namely

$$E \underset{B}{\times} E \quad \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} \quad E$$

and noted that on applying π_1 one obtained a cat^1–group

$$\pi_1(E \underset{B}{\times} E) \quad \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} \quad \pi_1(E)$$

which was seen to be the cat^1–group associated in the usual way to the crossed module $\pi_1(F) \longrightarrow \pi_1(E)$.

Since the crossed module of the pair (X, A) exists in a many pointed, groupoid version, it was clear there should be a groupoid version of the fibration result. The groupoid version of a cat^1–group is equivalent to a 2–groupoid, so we sought an elementary direct proof that there was a 2–groupoid that could be constructed from a (non pointed) fibration. The resulting 2–groupoid is, as one might expect, given by

$$\Pi_1(E \underset{B}{\times} E, \triangle E) \quad \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} \quad \Pi_1(E)$$

where $\triangle E \subseteq E \underset{B}{\times} E$ is the diagonal and $\Pi_1(E \underset{B}{\times} E, \triangle E)$ is the fundamental groupoid of

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relative homotopy classes of paths in $E \underset{B}{\times} E$, that start and end on the diagonal. The proof that this does form a 2–groupoid is fairly elementary but does need a new result on fibrations. Furthermore, the algebraic structure of the homotopy bigroupoid of a topological space developed in [HKK 2] has been exploited systematically. The pleasing aspect of the proof given here is, we feel, that it shows the geometry of what is going on in a very transparent way.

Recent work on related areas seems to have concentrated on the study of 2–groupoids or bigroupoids either associated to a single space, cf. work by Hardie, Kamps, Kieboom [HKK 1,2] or to a space X relative to subspaces $S \subseteq A \subseteq X$ as in the Whitehead 2–groupoid of Moerdijk–Svensson, [M, MS]. Viewing 2–groupoids as cofibred categorical groups, Cegarra and Fernández [CF] introduced a variant of the Moerdijk–Svensson construction for a longer filtration $S \subseteq B \subseteq A \subseteq X$. We will briefly compare these latter constructions with ours when the fibration is that obtained by factoring $A \longrightarrow X$ as a homotopy equivalence followed by a fibration as above. A 2–groupoid from a fibration, but dependent only on a single space (i.e. no filtration by subspaces) can be obtained by using the construction given in Steiner's paper, [St]. This does not seem to give the 'same' 2–groupoid as the Hardie–Kamps–Kieboom construction for trivial reasons, however a variant of the Steiner method does give a closely related 2–groupoid.

We should also mention relevant papers by Gilbert [G], Porter [P 2] and Bullejos, Cegarra and Duskin, [BCD]. Those papers suggest ways of describing higher dimensional analogues of our results. The exact formulation is still to be done however.

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1. 2–groupoids and related models

A 2–*groupoid* is a 2–category such that all 1–cells and all 2–cells are invertible. Thus it will be a groupoid–enriched category with invertible 1–cells, i.e. a groupoid–enriched groupoid. The 2–cells will have vertical and horizontal compositions obeying an interchange law.

One of our purposes here is to link the Loday [L] and Brown–Loday [BL 1,BL 2] results into the 2–groupoid setting, so as an illustrative example we define a cat^1–groupoid generalising the cat^1–groups of Loday [L] and show how to construct a 2–groupoid from it.

A *cat*^{1}*–group* (G, s, t) consists of a group G and two endomorphisms s, t of G such that

(i) $$ss = s,\ \ tt = t,\ \ st = t,\ \ ts = s,$$

and

(ii) $$[\operatorname{Ker} s, \operatorname{Ker} t] = 1$$

where $[\operatorname{Ker} s,\ \operatorname{Ker} t]$ denotes the subgroup generated by the commutators $xyx^{-1}y^{-1},\ x \in \operatorname{Ker} s,\ \ y \in \operatorname{Ker} t$.

Generalising this to a groupoid setting, take G to be a groupoid with object set, O, and s, t endomorphisms of G leaving objects fixed. Finally for (ii), the interpretation requires that 1 denotes the subgroupoid of G consisting just of the identities. The other conditions remain the same.

The conditions (i) above guarantee that the images of s and t coincide giving a subgroupoid, N, of G. Both Ker s and Ker t are subgroupoids of G consisting of disjoint families of subgroups of the vertex groups of G itself.

We can define the 2–groupoid **G** corresponding to (G, s, t) as follows. The objects of **G** are the objects of the groupoid G. The 1–cells of **G** are the arrows of the subgroupoid N, whilst the 2–cells are arbitrary arrows of G. Note this considers 1–cells as degenerate 2–cells, namely their own identity 2–cells. The composition of 1–cells is that within the groupoid N. The composition within G also gives the horizontal composition in **G**. The source and target 1–cells of a 2–cell, g, are given by $s(g)$ and $t(g)$ respectively and the vertical composition of g and h is defined if $s(h) = t(g)$ when we set $g{\circ}h = gt(g)^{-1}h$. It is simple to check $s(g{\circ}h) = s(g), t(g{\circ}h) = t(h)$. Finally the interchange law holds:

$$(gg'){\circ}(hh') = (g{\circ}h)(g'{\circ}h')$$

for g, g', h, h' such that the composites are defined. This is easily reduced to checking

$$g't(g')^{-1}t(g)^{-1}h = t(g)^{-1}hg't(g')^{-1}$$

which follows from $[\operatorname{Ker} s,\ \operatorname{Ker} t] = 1$ on noting that $t(g) = s(h)$.

We thus have a 2–groupoid derived from a cat^1–groupoid. The process is clearly reversible and sets up an equivalence between the categories of the two types of structures.

If O is the set of objects of a 2–groupoid **G**, then for any $S \subseteq$ O we can consider the full 2–subgroupoid of **G** with objects S. If moreover a subgroupoid of the 1–cells is given we can further restrict to obtain a new 2–groupoid.

If we have that **G** corresponds to a cat^1–groupoid (G, s, t) then we can define $M = \operatorname{Ker} s$ and N as before $\operatorname{Im} s$ and ∂ to be the restriction of t to M. The groupoid N then acts on M by conjugation within G and we get a crossed module (of groupoids)

$$\partial : M \longrightarrow N,$$

corresponding to the 2–groupoid **G**. Of course the usual semidirect product construction allows one to rebuild **G** from (M, N, ∂).

We should mention that 2–groupoids resp. cat^1–groupoids are also equivalent to double groupoids with thin structure and crossed modules over groupoids ([BS, Spe, SW, BH, P 1, Br]).

2. Tools from homotopical algebra

In this section we recall the necessary tools from 1– and 2–dimensional homotopical algebra. We shall make use of the algebra of the homotopy bigroupoid of a topological space developed in [HKK 2].

If X is a topological space and x and y points of X, then a *path*, $f : x \simeq y$, in X from x to y, is a map f from the unit interval $I = [0, 1]$ into X such that the *initial point* of f, $f(0)$, is

x and the *final point* of f, $f(1)$, is y. If $f : x \simeq y$ and $g : y \simeq z$ are paths in X, we denote by $g \bullet f : x \simeq z$ their *concatenation*, i.e.

(2.1) $\quad (g \bullet f)(s) = f(2s), \quad 0 \leqslant s \leqslant 1/2; \quad (g \bullet f)(s) = g(2s-1), \quad 1/2 \leqslant s \leqslant 1.$

The *constant path* at $x \in X$ will be denoted c_x. If f is a path in X, we denote by f^{-1} the path *reverse* to f, i.e. $f^{-1}(s) = f(1-s)$.

Let $f, f' : x \simeq y$ be paths. A *relative homotopy* $f_t : f \simeq f' : x \simeq y$ is a homotopy $f_t : f \simeq f'$ such that the initial and final points remain fixed during the homotopy.

By $\Pi_1 X$ we denote the *fundamental groupoid* of X. The morphisms, $[f]$, of $\Pi_1 X$, which we shall call 1–*tracks*, are classes of paths, f, in X, two paths f and f' belonging to the same class if the initial (resp. final) points of f and f' coincide and if there is a relative homotopy, f_t, between f and f'. Composition in $\Pi_1 X$, denoted $\bullet$, is induced by concatenation of paths. The class, $[c_x]$, of the constant path is the identity at $x \in X$, the class, $[f^{-1}]$, of the reverse path is inverse to $[f]$. In order to prove the groupoid properties of ΠX one makes use of certain canonical relative homotopies

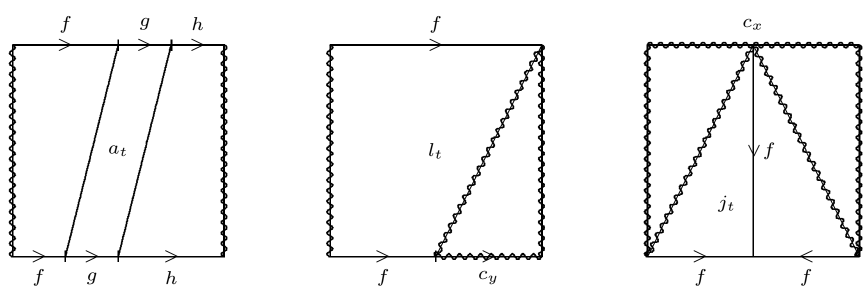
(2.2) $\quad a_t : h \bullet (g \bullet f) \simeq (h \bullet g) \bullet f : x \simeq u$ (*rescaling*)

(where $f : x \simeq y, \;\; g : y \simeq z, \;\; h : z \simeq u$ are paths in X)

(2.3) $\quad l_t : c_y \bullet f \simeq f : x \simeq y, \quad r_t : f \bullet c_x \simeq f : x \simeq y$ (*dilation*)

(2.4) $\quad j_t : f^{-1} \bullet f \simeq c_x : x \simeq x, \quad j'_t : f \bullet f^{-1} \simeq c_y : y \simeq y$ (*cancellation*)

which are indicated by the following figures.



Explicit formulae are given in [Spa, pp. 47, 48]. If A is a subset of X, the *fundamental groupoid of X based on the set A*, denoted by $\Pi_1(X, A)$, is defined to be the full subgroupoid of $\Pi_1 X$ with A as set of objects.

Let $f_t, f'_t : f \simeq f' : x \simeq y$ be two relative homotopies. We may consider f_t and f'_t themselves to be *relatively homotopic*, if they are homotopic via a homotopy $I \times I \times I \longrightarrow X$ which is constant on the boundary of $I \times I$. The relative homotopy class $\{f_t\}$ of f_t will be called a 2–*track*. In that case we will use the notation

$$\{f_t\} : f \Longrightarrow f' : x \simeq y \text{ or simply } \{f_t\} : f \Longrightarrow f'.$$

We are now in a position to recall the essential features of the *homotopy bigroupoid*, $\Pi_2 X$, of a topological space X as defined in [HKK 2].

(i) The objects (0–cells) of $\Pi_2(X)$ are the points of X.
(ii) The 1–morphisms (1–cells) of $\Pi_2(X)$ are the paths in X.
(iii) The 2–morphisms (2–cells) of $\Pi_2(X)$ are the 2–tracks.

(*Horizontal*) *composition* of 1–morphisms is given by concatenation of paths. Relative homotopies can be pasted vertically and horizontally in the obvious way by formulae of type (2.1). More precisely, let $f, f', f'' : x \simeq y$ be paths, let $f_t : f \simeq f'$, $f'_t : f' \simeq f''$ be relative homotopies we define the *vertical pasting* $f_t + f'_t : f \simeq f'' : x \simeq y$ to be the relative homotopy h_t such that

$$h_t = f_{2t}, \quad 0 \leqslant t \leqslant 1/2; \quad h_t = f'_{2t-1}, \quad 1/2 \leqslant t \leqslant 1.$$

If $f_t : f \simeq f' : x \simeq y$ and $g_t : g \simeq g' : y \simeq z$ are relative homotopies, then we obtain the *horizontal pasting* $g_t \bullet f_t : g \bullet f \simeq g' \bullet f' : x \simeq z$ by concatenation of the respective paths at each stage of the homotopy. Vertical and horizontal pasting of relative homotopies gives rise to *vertical* resp. *horizontal composition* of 2–tracks, denoted $+$ resp. $\bullet$. Vertical and horizontal composition are related by the *interchange law*: Let

$$\varphi : f \Longrightarrow f' : x \simeq y, \;\; \varphi' : f' \Longrightarrow f'' : x \simeq y, \;\; \psi : g \Longrightarrow g' : y \simeq z, \;\; \psi' : g' \Longrightarrow g'' : y \simeq z$$

be 2–tracks. Then we have that

$$(\psi + \psi') \bullet (\varphi + \varphi') = (\psi \bullet \varphi) + (\psi' \bullet \varphi').$$

For each pair (x, y) of points in X we have a groupoid $\Pi_2 X(x, y)$. The objects are the paths f, f' etc. from x to y. The morphisms are the 2–tracks $\{f_t\} : f \Longrightarrow f' : x \simeq y$. Composition is given by vertical composition, $+$, of 2–tracks. The identity element $0_f : f \Longrightarrow f$ is the 2–track of the constant homotopy at f. Inverses are obtained by reversing relative homotopies.

Composition of 1–morphisms is not strictly associative, but associative only up to coherent isomorphism. If $f : x \simeq y$, $g : y \simeq z$, $h : z \simeq u$ are paths, then the *associativity isomorphism*

$$\alpha : h \bullet (g \bullet f) \Longrightarrow (h \bullet g) \bullet f$$

is defined as 2–track of the rescaling homotopy a_t (2.2).

Identities and inverses exist up to coherent isomorphism. If $f : x \simeq y$ is a path, we have left and right identities

$$\lambda : c_y \bullet f \Longrightarrow f, \quad \rho : f \bullet c_x \Longrightarrow f$$

defined as 2–tracks of the respective dilation homotopies (2.3). Furthermore cancellation (2.4) gives rise to cancellation isomorphisms

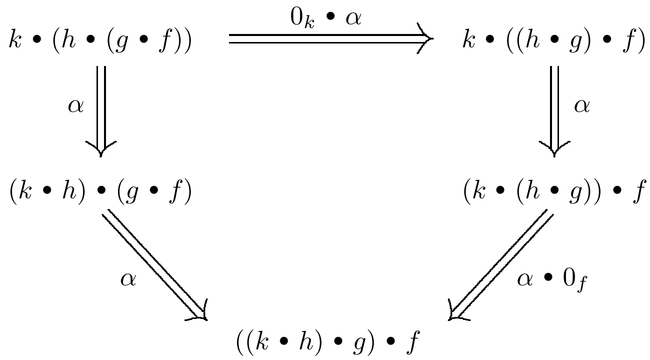
$$\iota : f^{-1} \bullet f \Longrightarrow c_x, \quad \iota' : f \bullet f^{-1} \Longrightarrow c_y.$$

The constraints $\alpha, \lambda, \rho, \iota, \iota'$ satisfy three coherence conditions (AC) (*associativity coherence*), (IC) (*identity coherence*) and (CC) (*cancellation coherence*) (see [HKK 2]).

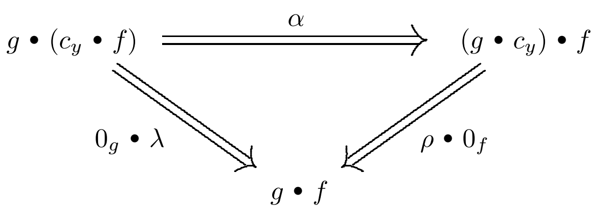
(AC) *Associativity coherence*: Let

$$f : x \simeq x', \quad g : x' \simeq x'', \quad h : x'' \simeq x''', \quad k : x''' \simeq x''''$$

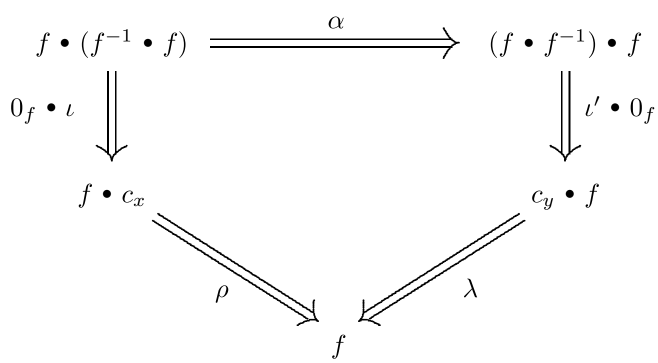
be paths in X. Then the following diagram commutes.



(IC) *Identity coherence*: Let $f : x \simeq y$, and $g : y \simeq z$ be paths in X. Then the following diagram commutes.



(CC) *Cancellation coherence*: Let $f : x \simeq y$ be a path in X. Then the following diagram commutes.



Finally, we note that the *Poincaré category* in the sense of [Bé,(7.1)] associated to the homotopy bigroupoid, $\Pi_2 X$, is the fundamental groupoid, $\Pi_1 X$, of X described above.

3. The construction of the homotopy 2–groupoid

3.1 Idea of construction

Let $p : E \longrightarrow B$ be a fibration and form the pullback $E \underset{B}{\times} E$ of p along itself.

$$E \underset{B}{\times} E = \{(x, y) \in E \times E \,|\, p(x) = p(y)\}$$

Note that $E \underset{B}{\times} E$ is just the equivalence relation in E derived from p. The equivalence relation gives rise to a groupoid in the usual way. The objects are the elements of E. If $(x,y) \in E \times_B E$ we have a unique morphism $(x,y) : x \longrightarrow y$ from x to y. The three structural maps are $s(x,y) = x$, $t(x,y) = y$ and the diagonal $\triangle : E \longrightarrow E \times_B E$, $\triangle(x) = (x,x)$. The groupoid composition is given by the formula

$$(x,y) + (y,z) = (x,z). \tag{3.1.1}$$

Form the fundamental groupoid $\Pi_1(E \times_B E, \triangle E)$ of $E \times_B E$ based on the diagonal $\triangle E$, so objects are elements (x,x) whilst an arrow from (x,x) to (y,y) is a 1–track (path class) $[f]$ where $f = (f_1, f_2)$, f_1, f_2 paths in E from x to y and $pf_1 = pf_2$. Next form $\Pi_1 E$. The structure maps $s, t, \triangle$ induce groupoid morphisms

$$\Pi_1(E \times_B E, \triangle E) \quad \begin{array}{c} \xrightarrow{s_*} \\ \xrightarrow{t_*} \\ \xleftarrow[\triangle_*]{} \end{array} \quad \Pi_1 E$$

This is over the set E of basepoints, i.e. these are groupoid maps that are the identity on objects. Now a 2–dimensional structure is in sight. The ingredients are in

- dimension 0 (0–cells, objects): the elements of E
- dimension 1 (1–cells, arrows): the morphisms of $\Pi_1 E$
- dimension 2 (2–cells): the morphisms of $\Pi_1(E \times_B E, \triangle E)$.

Horizontal composition of 1–cells resp. 2–cells is given by groupoid composition in $\Pi_1 E$ resp. $\Pi_1(E \times_B E, \triangle E)$. Vertical composition $+$ is induced by the groupoid composition given above (3.1.1) and will be made precise later.

Theorem. *With the above structural data, we get a 2–groupoid.*

3.2 The detailed structure

Denoting the overall structure by $G_2(\mathbf{E})$ where $\mathbf{E}$ is the fibration $p : E \longrightarrow B$, we have: The set of objects is just the set E. Let x, y be points of E. Then $G_2(\mathbf{E})(x,y)$ denotes the collection of 1–tracks (path classes) in $E \times_B E$ between (x,x) and (y,y). This has the structure of a category as follows. The set of objects is the set of 1–tracks (path classes) from x to y in E with source and target maps from $G_2(\mathbf{E})(x,y)$ given by:

$$\text{if } [f] = [(f_1, f_2)], \text{ then } s[f] = [f_1], \;\; t[f] = [f_2].$$

The identity $0_{[f_1]}$ at a 1–track $[f_1]$, $f_1 : x \simeq y$ in E, is $[(f_1, f_1)]$.
It remains to specify the composition. Suppose $[f] = [(f_1, f_2)]$, $[g] = [(g_1, g_2)]$ are in $G_2(\mathbf{E})(x,y)$ and $[f_2] = [g_1]$. The initial idea is that we should use a homotopy from f_2 to g_1, but the obvious way is not evidently independent of the choice of the homotopy. That method also fails to use the fact that we can change both f and g within their path classes. The method that works is the following variant of the vague idea sketched in section 3.1.

(i) Postcompose f with $(g_1 \bullet g_1^{-1}, g_1 \bullet g_1^{-1})$ to get

$$[f] = [(f_1, f_2)] = [((g_1 \bullet g_1^{-1}) \bullet f_1, (g_1 \bullet g_1^{-1}) \bullet f_2)] = [(g_1 \bullet (g_1^{-1} \bullet f_1), g_1 \bullet (g_1^{-1} \bullet f_2))].$$

(ii) Precompose g with $(g_1^{-1} \bullet f_2, g_1^{-1} \bullet f_2)$ which is null homotopic by the assumption, $[f_2] = [g_1]$, i.e. $t[f] = s[g]$; as this is done to both components of g,

$$[g] = [(g_1, g_2)] = [(g_1 \bullet (g_1^{-1} \bullet f_2), g_2 \bullet (g_1^{-1} \bullet f_2))].$$

Finally use the + composition of the equivalence relation to get

$$[f] + [g] = [(g_1 \bullet (g_1^{-1} \bullet f_1), g_2 \bullet (g_1^{-1} \bullet f_2))]. \tag{3.2.1}$$

Note that + is well defined and we have

$$s([f] + [g]) = [g_1] \bullet [g_1]^{-1} \bullet [f_1] = [f_1] = s[f]$$

$$t([f] + [g]) = [g_2] \bullet [g_1]^{-1} \bullet [f_2] = [g_2] = t[g].$$

For each triple of points $x, y, z \in E$ we can define a mapping

$$G_2(\mathbf{E})(x, y) \times G_2(\mathbf{E})(y, z) \xrightarrow{\bullet} G_2(\mathbf{E})(x, z),$$

$([f], [h]) \longmapsto [h] \bullet [f]$, making use of the groupoid composition $\bullet$ in $\Pi_1(E \underset{B}{\times} E, \triangle E)$ by

$$[h] \bullet [f] = [(h_1 \bullet f_1, h_2 \bullet f_2)].$$

Note that + can be expressed by $\bullet$ in the following way.

$$[f] + [g] = [(g_1, g_2)] \bullet [(g_1, g_1)]^{-1} \bullet [(f_1, f_2)] = [g] \bullet [(g_1, g_1)]^{-1} \bullet [f] = [g] \bullet 0_{[g_1]}^{-1} \bullet [f] \tag{3.2.2}$$

3.2.3. Lemma. *For any points $x, y \in E$, $G_2(\mathbf{E})(x, y)$ is a groupoid.*

Proof. Associativity is fairly easy. If $[f] + [g]$ and $[g] + [h]$ are defined then by (3.2.2)

$$([f] + [g]) + [h] = [h] \bullet [(h_1, h_1)]^{-1} \bullet [g] \bullet [(g_1, g_1)]^{-1} \bullet [f]$$

whilst

$$[f] + ([g] + [h]) = [h] \bullet [(h_1, h_1)]^{-1} \bullet [g] \bullet [((h_1 \bullet h_1^{-1}) \bullet g_1, (h_1 \bullet h_1^{-1}) \bullet g_1)]^{-1} \bullet [f]$$

which are clearly equal.
If $[f] = [(f_1, f_2)]$ then

$$0_{[f_1]} + [f] = [f] \bullet [(f_1, f_1)]^{-1} \bullet [(f_1, f_1)] = [f].$$

Similarly; $[f] + 0_{[f_2]} = [f]$. Hence $0_{[f_1]} = [(f_1, f_1)]$ serves as an identity.

That + has an inverse is not trivial. The obvious candidate for an inverse of $[f] = [(f_1, f_2)]$ is

$$-[f] = [(f_2, f_1)].$$

By (3.2.1) we obtain

$$[f] - [f] = [(f_1, f_2)] + [(f_2, f_1)] = [(f_2 \bullet (f_2^{-1} \bullet f_1), f_1 \bullet (f_2^{-1} \bullet f_2))].$$

Each individual part is homotopic to f_1, but the two homotopies are not the same, although linked by cancellation coherence. More precisely, we have the following composite 2–track between $f_2 \bullet (f_2^{-1} \bullet f_1)$ and $f_1 \bullet (f_2^{-1} \bullet f_2)$ involving the constraints $\alpha, \lambda, \rho, \iota, \iota'$ for the homotopy bigroupoid $\Pi_2 E$ of E.

$$f_2 \bullet (f_2^{-1} \bullet f_1) \overset{\alpha}{\Longrightarrow} (f_2 \bullet f_2^{-1}) \bullet f_1 \overset{\iota \bullet 0_{f_1}}{\Longrightarrow} c_y \bullet f_1 \overset{\lambda}{\Longrightarrow} f_1 \overset{-\rho}{\Longrightarrow} f_1 \bullet c_x \overset{-0_{f_1} \bullet \iota}{\Longrightarrow} f_1 \bullet (f_2^{-1} \bullet f_2)$$

Projecting down by $p : E \longrightarrow B$, by cancellation coherence (CC) we obtain the identity 2–track $0_{(pf_1) \bullet ((pf_1)^{-1} \bullet (pf_1))}$ at $(pf_1) \bullet ((pf_1)^{-1} \bullet (pf_1))$. (It is at this point that we get the 'pay off' for working in the setting of a bigroupoid where coherence has to be taken seriously.) We now invoke a corollary, (3.4.2), to a fibration lemma which we shall prove in section 3.4 This gives us a relative homotopy from $f_2 \bullet (f_2^{-1} \bullet f_1)$ to $f_1 \bullet (f_2^{-1} \bullet f_2)$ *over* B. Hence

$$[(f_2 \bullet (f_2^{-1} \bullet f_1), f_1 \bullet (f_2^{-1} \bullet f_2))] = [(f_1 \bullet (f_2^{-1} \bullet f_2), f_1 \bullet (f_2^{-1} \bullet f_2))] = 0_{[f_1] \bullet [f_2]^{-1} \bullet [f_2]} = 0_{[f_1]}.$$

This proves $[f] - [f] = 0_{[f_1]}$. Similarly $-[f] + [f] = 0_{[f_2]}$. This completes the proof of Lemma 3.2.3. □

3.3 The interchange law

The 1–dimensional substructure of $G_2(\mathbf{E})$ clearly being a groupoid the only remaining part of the verification that $G_2(\mathbf{E})$ is a 2–groupoid is to check the interchange law for the compositions

$$G_2(\mathbf{E})(x, y) \times G_2(\mathbf{E})(y, z) \overset{\bullet}{\longrightarrow} G_2(\mathbf{E})(x, z).$$

Suppose given $[f], [g] \in G_2(\mathbf{E})(x, y)$, $[h], [k] \in G_2(\mathbf{E})(y, z)$ so that $[f] + [g]$ and $[h] + [k]$ are defined, i.e. $[f_2] = [g_1]$ and $[h_2] = [k_1]$.
Then

$$([h] + [k]) \bullet ([f] + [g]) = [k] \bullet [(k_1, k_1)]^{-1} \bullet [h] \bullet [g] \bullet [(g_1, g_1)]^{-1} \bullet [f]$$

whilst

$$[h] \bullet [f] + [k] \bullet [g] = [k] \bullet [g] \bullet [(k_1 \bullet g_1, k_1 \bullet g_1)]^{-1} \bullet [h] \bullet [f],$$

so for equality it suffices to have that

$$(3.3.1) \quad [(k_1, k_1)]^{-1} \bullet [h] \bullet [g] \bullet [(g_1, g_1)]^{-1} = [((k_1^{-1} \bullet h_1) \bullet (g_1 \bullet g_1^{-1}), (k_1^{-1} \bullet h_2) \bullet (g_2 \bullet g_1^{-1}))]$$

and

$$(3.3.2) \quad [g] \bullet [(g_1, g_1)]^{-1} \bullet [(k_1, k_1)]^{-1} \bullet [h] = [((g_1 \bullet g_1^{-1}) \bullet (k_1^{-1} \bullet h_1), (g_2 \bullet g_1^{-1}) \bullet (k_1^{-1} \bullet h_2))]$$

are equal. In order to check this, first choose a relative homotopy $v_t : k_1^{-1} \bullet h_2 \simeq c_y$ giving rise to a 2–track in $\Pi_2 E$

$$\nu = \{v_t\} : k_1^{-1} \bullet h_2 \Longrightarrow c_y.$$

Then consider the composite 2–tracks in $\Pi_2 E$

$$\Phi : (k_1^{-1} \bullet h_1) \bullet (g_1 \bullet g_1^{-1}) \Longrightarrow (g_1 \bullet g_1^{-1}) \bullet (k_1^{-1} \bullet h_1)$$

resp.

$$\Psi : (k_1^{-1} \bullet h_2) \bullet (g_2 \bullet g_1^{-1}) \Longrightarrow (g_2 \bullet g_1^{-1}) \bullet (k_1^{-1} \bullet h_2)$$

given as follows.

$$\Phi : (k_1^{-1} \bullet h_1) \bullet (g_1 \bullet g_1^{-1}) \overset{0 \bullet \iota'}{\Longrightarrow} (k_1^{-1} \bullet h_1) \bullet c_y \overset{\rho}{\Longrightarrow} k_1^{-1} \bullet h_1 \overset{-\lambda}{\Longrightarrow} c_y \bullet (k_1^{-1} \bullet h_1) \overset{-\iota' \bullet 0}{\Longrightarrow} (g_1 \bullet g_1^{-1}) \bullet (k_1^{-1} \bullet h_1)$$

$$\Psi : (k_1^{-1}\bullet h_2)\bullet(g_2\bullet g_1^{-1}) \overset{\nu\bullet 0}{\Longrightarrow} c_y\bullet(g_2\bullet g_1^{-1}) \overset{\lambda}{\Longrightarrow} g_2\bullet g_1^{-1} \overset{-\rho}{\Longrightarrow} (g_2\bullet g_1^{-1})\bullet c_y \overset{-0\bullet\nu}{\Longrightarrow} (g_2\bullet g_1^{-1})\bullet(k_1^{-1}\bullet h_2)$$

Projecting down by $p : E \longrightarrow B$ we have that in $\Pi_2 B$

$$p\Phi = p\Psi. \tag{3.3.3}$$

The idea to prove (3.3.3) is to manipulate $p\Psi$ in such a way as to produce a situation where $p\nu$ and $-p\nu$ cancel. This can be achieved by factoring $p\Psi$ through c_{py} and making use of the naturality of ρ, λ and the fact that $\lambda = \rho : c_{py} \bullet c_{py} \Longrightarrow c_{py}$. The details are left to the reader. We now invoke the fibration lemma, (3.4.1), which will be proved in section 3.4. This gives us relative homotopies

$$\phi_t : (k_1^{-1} \bullet h_1) \bullet (g_1 \bullet g_1^{-1}) \simeq (g_1 \bullet g_1^{-1}) \bullet (k_1^{-1} \bullet h_1)$$

resp.

$$\psi_t : (k_1^{-1} \bullet h_2) \bullet (g_2 \bullet g_1^{-1}) \simeq (g_2 \bullet g_1^{-1}) \bullet (k_1^{-1} \bullet h_2)$$

such that $p\phi_t = p\psi_t$. This proves that the expressions in (3.3.1) and (3.3.2) are equal.

Thus the interchange law holds and we get

3.3.4. Theorem. *$G_2(\mathbf{E})$ is a 2–groupoid.*

Remark. In section 3.2 the existence of vertical inverses has been proved in a geometrical way. Alternatively, using the interchange law, one can prove that the vertical inverse is determined by the horizontal inverse in the following way (cf. [BCD], section 1).

$$-[(f_1, f_2)] = 0_{[f_1]} \bullet [(f_1, f_2)]^{-1} \bullet 0_{[f_2]} = [(f_2, f_1)]$$

3.4 A fibration lemma

In this section we state and prove a fibration lemma needed for the results of sections 3.2 and 3.3. We note that the fibration lemma and its corollary may be of interest in their own right.

3.4.1. Lemma. *Let $p : E \longrightarrow B$ be a fibration. If $f, f', g, g' : x \simeq y$ are paths in E such that $pf = pg$, $pf' = pg'$ and $f_t : f \simeq f'$, $g_t : g \simeq g'$ are relative homotopies such that the 2–tracks $\{pf_t\}, \{pg_t\} : pf \Longrightarrow pf'$ coincide, then there is a relative homotopy $f'_t : f \simeq f'$ such that $pf'_t = pg_t$.*

If we apply the lemma to the case $f' = g = g'$ and g_t the constant homotopy at g, then we obtain the following corollary.

3.4.2. Corollary. *Let $p : E \longrightarrow B$ be a fibration. If $f, f' : x \simeq y$ are paths in E such that $pf = pf'$ and $f_t : f \simeq f'$ is a relative homotopy such that $\{pf_t\} : pf \Longrightarrow pf$ is the identity 2–track 0_{pf} at pf, then there exists a relative homotopy $f'_t : f \simeq f'$ over B.*

Proof of Lemma 3.4.1. Let $\varphi : I \times I \times I \longrightarrow B$ be a relative homotopy between pf_t and pg_t, i.e. we have

$$\begin{array}{ll} \varphi(0, s, t) = pf_t(s), & \varphi(1, s, t) = pg_t(s) \\ \varphi(r, 0, t) = px, & \varphi(r, 1, t) = py \\ \varphi(r, s, 0) = pf(s), & \varphi(r, s, 1) = pf'(s). \end{array}$$

Since p is a fibration we can lift φ to a map $\Phi : I \times I \times I \longrightarrow E$ such that

$$\begin{array}{l} \Phi(0, s, t) = f_t(s),\ \ \Phi(r, 0, t) = x,\ \ \Phi(r, 1, t) = y \\ \Phi(r, s, 0) = f(s),\ \ \Phi(r, s, 1) = f'(s). \end{array}$$

Now define f'_t by $f'_t(s) = \Phi(1, s, t)$. $\square$

4. Examples and discussion

4.1 The 2–groupoid of a map

In [G], Gilbert discusses the case of a fibration induced from a map in the usual way, in order to get a cat^1–group from a '1–cube' of pointed spaces. Here, of course, we use spaces not pointed ones, but the theory is very similar.

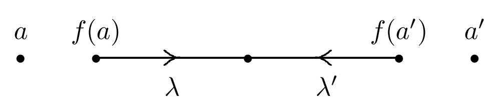
Suppose $f : A \longrightarrow X$ is a map of spaces, then one factorises it as $A \xrightarrow{\simeq} E_f \xrightarrow{p} X$ with p a fibration and $\simeq$ a homotopy equivalence. The construction is standard:

$$E_f = \{(a, \lambda) \mid \lambda : I \longrightarrow X, \lambda(0) = f(a)\} \subseteq A \times X^I$$

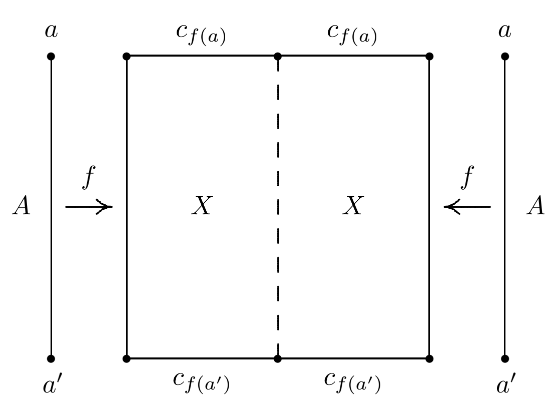
and $p(a, \lambda) = \lambda(1)$. We will want to consider A as a subspace of E_f via the embedding $a \longmapsto (a, c_{f(a)})$ where as before $c_{f(a)}$ is the constant path at $f(a)$ within X. This embedding is part of the deformation retraction giving $A \simeq E_f$. We will write $\mathbf{E}_f$ for this fibration and $G_2(\mathbf{E}_f, A)$ for the full subgroupoid

$$\Pi_1(E_f \underset{X}{\times} E_f, \triangle A) \quad \rightleftarrows \quad \Pi_1(E_f, A)$$

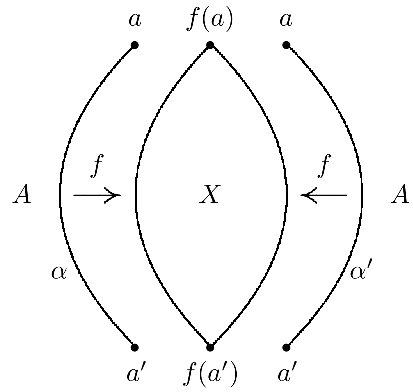
based at the subspace A. A point of $E_f \underset{X}{\times} E_f$ can be represented as



with $\lambda(1) = \lambda'(1)$ and so a path in $E_f \underset{X}{\times} E_f$ starting and finishing on $\triangle A \subseteq \triangle E_f$ can be represented by a diagram

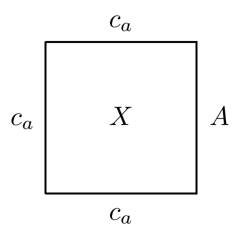


and hence by a lens shaped diagram



The homotopy relation is then given by relative homotopies through similar lens diagrams. One composition is via concatenation of lens diagrams in the direction of composing paths (i.e. vertically in the pictures we have given). The other is more complex and is where the 'pre' and 'post' composition trick comes in. We leave the reader to draw suitable diagrams!

If we restrict attention to the case where f is the inclusion of a subspace A and we pick those elements which start at c_a for some $a \in A$, then the diagram is



and when we take relative homotopy classes, we get a group isomorphic to $\pi_2(X, A, a)$. The intersection of this subgroup with the subgroup of those classes ending at c_a is then the 2–endomorphism group, $G_2(\mathbf{E}_f, c_a)$, and is clearly a quotient of $\pi_2(X, a)$. It will only be isomorphic to this second homotopy group if $\pi_2(A, a) \longrightarrow \pi_2(X, a)$ is trivial, since we are now in the usual situation of the long exact sequence of the pair (X, A)

$$\cdots \longrightarrow \pi_2(A, a) \longrightarrow \pi_2(X, a) \longrightarrow \pi_2(X, A, a) \longrightarrow \pi_1(A, a) \longrightarrow \pi_1(X, a)$$

with the crossed module corresponding to the vertex 2–group of $G_2(\mathbf{E}_f, A)$ at $a \in A$ being exactly the classical Whitehead crossed module of the pair mentioned in the introduction. In particular if X is a CW–complex and A is its 1–skeleton, we have the well known situation studied by Whitehead in the reduced case.

4.2 The vertex 2–group at a point in E

Let $\mathbf{E}$ be a general fibration $p : E \longrightarrow B$ giving the 2–groupoid $G_2(\mathbf{E})$. Pick $e \in E$ and consider the vertex 2–group at e. We have already seen this in a special case. We can consider $\mathbf{E}$ as a pointed fibration with $F = p^{-1}(p(e))$ as fibre. The group of 1–cells at e is clearly $\pi_1(E, e)$ and the splitting, induced from the diagonal $\triangle : E \longrightarrow E \underset{B}{\times} E$, gives $\pi_1(E \underset{B}{\times} E, (e, e)) \cong \mathrm{Ker}\ s \rtimes \pi_1(E, e)$. If $[(f_1, f_2)] \in \mathrm{Ker}\ s$ then $f_1 \simeq c_e$, so f_2 is relative homotopic to a path in the fibre, F, and vice versa.

The vertex 2–group at e is that corresponding to the crossed module

$$\pi_1(F, e) \longrightarrow \pi_1(E, e).$$

4.3 $G_2(X)$ and $G_2(\mathbf{E})$

It would be nice if given a Hausdorff space X we could give a fibration $\mathbf{E}$ so that $G_2(\mathbf{E})$ was isomorphic to the homotopy 2–groupoid $G_2(X)$ of [HKK 1]. We do not yet know how to do this, but the nearest we can get is to take the evaluation map from the geometric realisation of the 1–skeleton of the singular complex of X, that is $|\, sk_1\mathrm{Sing}(X)\,| \longrightarrow X$ and apply to it the methods of 4.1. This idea in the reduced case is due to Steiner [St]. It has the merit of being functorial in X but leads to a 2–groupoid which seems much bigger than we really want.

4.4 $G_2(\mathbf{E})$ and function spaces

Here we will work with compactly generated Hausdorff spaces so as to be able to consider function spaces Y^A etc. If $f : A \longrightarrow X$ is a cofibration and Y is a space then

$$Y^X \longrightarrow Y^A$$

is a fibration. A path in Y^X corresponds uniquely to a homotopy

$$X \times I \longrightarrow Y$$

and so paths in $Y^X \underset{Y^A}{\times} Y^X$ via this correspondence are pairs of such homotopies agreeing on the subspace $A \times I$. Using relative homotopy classes of such homotopies leads to a 2–groupoid

$$\Pi_1(Y;\, X \underset{A}{\sqcup} X) \quad \overrightarrow{\underleftarrow{\longrightarrow}} \quad \Pi_1(Y;\, X)$$

where $\Pi_1(Y;\, X \underset{A}{\sqcup} X)$ is $\Pi_1(Y^{X \underset{A}{\sqcup} X}, Y^X) \cong \Pi_1(Y^X \underset{Y^A}{\times} Y^X, Y^X)$. The discussion in [HKK 1] raises some interesting points (in section 3) for the possible uses of such machinery. This also relates to some constructions of Baues [Ba].

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