Homology, Homotopy and Applications, vol.1, No.5, 1999, pp.135–145 ISSN 1512–0139

K-THEORY OF AFFINE TORIC VARIETIES

JOSEPH GUBELADZE

(*communicated by Hvedri Inassaridze*)

Abstract

This is an updated and expanded version of my preprint #68 in the K-theory server at Urbana (which was an abstract of my talk at Vechta conference on commutative algebra, 1994). In §2 two conjectures on nilpontency of the 'monoid Frobenius action' on the K-theory of toric cones and on stabilizations of the corresponding K-groups are stated. Both of these conjectures are higher analogues of Anderson's conjecture and their proof would bring a rather complete understanding of K-theory of toric varieties/semigroup rings.

1. Survey of results

Homotopy properties of algebraic K-functors have always been among the central topics of algebraic K-theory which, unlike its topological counterpart, is *not* homotopy invariant.

Recall, that a functor F from the category of rings (or its certain subcategory, or schemes, S-schemes, etc) is called *homotopic* (on the corresponing category) if the natural homomorphisms of type $F(R) \to F(R[t])$ ($F(X \times \mathbb{A}^1) \to F(X)$, respectively) are all isomorphisms (t a variable). Also, a ring R is called F*-regular* if $F(R) = F(R[t])$.

The starting point here is the Grothendieck-Serre classical theorem that a regular ring is K_0-regular. This has been extended to K_1 by Bass-Heller-Swan and to all K_i by Quillen in his fundamental work [**Q1**].

In the unstable setting the same homotopy properties are at least no less interesting. The well known *Serre Problem* on freeness of projective modules over polynomial rings with coefficients in a field (equivalently, on triviality of algebraic vector bundles over affine spaces) is certainly a distinguished question in this diretion. The 20 years of unceased activity to resolve this question, that was rised in Serre's famous *Faisceaux algébriques cohérents* and has playd an essential rôle in creating algebraic K-theory, culminated in two independent confirmations in 1976 by Quillen and Suslin [**Q2**][**Su1**].

Affine *toric varieties* are natural generalizations of affine spaces. Originally, M. Demazure considered in [**D**] *complete smooth* toric varieties (in the context of maximal agebraic tori in Cremona groups). They can be characterized as equivariant (smooth) compactifications of algebraic tori (say, a projective space). The theory of general (normal) toric varieties was then developed in [**KKMS**]. Geometrically, toric varieties are exactly the normal varieties containing an open torus (*the embedded torus*) whose group structure extends to an action on the whole variety (we refer [F,O] for the background).

Affine toric varieties, which glue up to general toric varieties, are exactly prime spectra of *affine normal monomial* algebras. Moreover, the condition of the presence of a stable point under the torus action is equivalent to the condition of the absence of invertible non-trivial monomials. The latter constitute a class of intuitively contractible varieties, generalizing in a natural

Supported in part by TMR Project grant ERB FMRX CT-97-0107
Received 22 April 1999, revised 11 May 1999; published on 17 May 1999.
1991 Mathematics Subject Classification: Primary 19-02; Secondary 14M25
Key words and phrases: toric varieties, projective modules, algebraic K-theory, monoid rings

way affine spaces (for instance, $\mathrm{Spec}(k[X^3, X^2Y, XY^2, Y^3])$, $\mathrm{Spec}(k[X^3, Y^3, Z^3, XYZ])$). Correspondingly, Anderson's conjecture [**A**] states that projective modules over affine normal monomial subalgebras of polynomial algebras are free (k a field).

Observe that the only regular rings in Anderson's conjecture are polynomial algebras. Thus there is no immediate application of the aforementioned Grothendieck-Serre theorem even in the stable case. (Actually, one should apply its corollary that $K_0(R_0) = K_0(R)$ for a graded regular ring $R = R_0 \oplus R_1 \oplus \cdots$.)

In order to achieve maximal generality (both for the coefficients rings and the involved monomial structures) we now switch to the *monoid rings* setting.

All the considered below monoids M are assumed to be commutative, cancellative and, unless specified otherwise, torsion free (that is, torsion free in the group of differences). These conditions ammount to the injectivity of the natural mappings $M \to \mathrm{gp}(M) \to \mathbb{Q} \otimes \mathrm{gp}(M)$, where $\mathrm{gp}(M)$ is the corresponding group of differences. By the same token we exactly get the class of additive submonoids of rational vector spaces.

A monoid M is called *normal* if (writing additively) $nx \in M$ for $n \in \mathbb{N}$ and $x \in \mathrm{gp}(M)$ imply $x \in M$. M is called *seminormal* if the following implication holds

$$x \in \mathrm{gp}(M),\ 2x \in M \text{ and } 3x \in M \Rightarrow x \in M.$$

Observe that monoids are in general assumed neither finitely generated nor without non-trivial invertible elements.

Normal monoids are seminormal, but there are many seminormal non-normal monoids [**Gu1**].

It is well known that a monoid domain $R[M]$ is normal (seminormal) if and only if the domain R and the monoid M are normal (seminormal, respectively).

We say that M is c-divisible for some $c \in \mathbb{N}$ if for any $x \in M$ there exists $y \in M$ for which $cy = x$. Observe that a c-divisible monoid is always seminormal.

Later on $\mathbb{Z}_+$ will denote the additive monoid of nonnegative rational integers and $\mathbb{Q}_+$ that of nonnegative rationals.

All the considered rings are assumed to be commutative.

The following result in particular confirms Anderson's conjecture:

Theorem 1.1 ([Gu1]). *For any principal ideal domain (PID) R and any monoid M (maybe infinitely generated and with non-trivial units) the following conditions are equivalent:*

(a) $\mathrm{Pic}(R[M]) = 0$,

(b) $K_0(R[M]) = \mathbb{Z}$,

(c) *finitely generated projective $R[M]$-modules are all free,*

(d) *M is seminormal.*

Remark 1.2.

(a) M. Masuda, L. Moser-Jauslin and T. Petrie [**MMJP**] succeeded in establishing a positive answer to the *Equivariant Serre Problem* for reductive abelian groups (that every G-vector bundle over the representation space is trivial whenever G is abelian) by connecting it with the corresponding *Quotient Problem*, which in its turn reduces to the special case of Theorem 1.1.

(b) R. Laubenbacher and C. Woodburn have developed an algorithmic vesrion of Theorem 1.1 [**LW**].

For the stable case we have

Theorem 1.3 ([Gu2][Gu5]). *For any regular ring R and any monoid M we have $SK_0(R) = SK_0(R[M])$ and $K_{-i}(R) = K_{-i}(R[M]) = 0$ (Bass negative K-groups). The following conditions are equivalent:*

(a) $\mathrm{Pic}(R) = \mathrm{Pic}(R[M])$,

(b) $K_0(R) = K_0(R[M])$,

(c) *M is seminormal.*

The equality above concerning SK_0 is still valid for monoids M for which $\mathrm{gp}(M)_{tor}$ is a p-group (for some prime p) if in addition $p \cdot 1 = 0$ in R [**Gu5**].

R. G. Swan has deduced from [Gu1] the most general unstable result:

Theorem 1.4 ([Sw1]).

(a) *For any monoid M and a Dedekind domain R all finitely generated projective $R[M]$-modules are of type free⊕rank 1.*

(b) *For any affine regular domain R and a seminormal monoid M without nontrivial units all finitely generated projective $R[M]$-modules are extended from R.*

Remark 1.5. The claim (a) confirms a conjecture of P. Murthy. By Popescu's approximation theorem on regular domains containing a field [**Sw2**] we immediately obtain the generalization of the statement (b) to *arbitrary* regular rings just containing a subfield (we recall that the case when M is free corresponds to the *Bass-Quillen Conjecture*, proved for geometric case by H. Lindel, [**Lin**]).

The converse to Theorem 1.1 is provided by the following

Theorem 1.6 ([Gu2][Sw1]). *For a not necessarily torsion free monoid M the following conditions are equivalent:*

(a) $\mathrm{Pic}(R[M]) = 0$ *for all PID's R,*

(b) *M is torsion free and seminormal.*

Remark 1.7. No longer the torsion freeness of M follows from the triviality of the $\mathrm{Pic}(k[M])$ if instead of PID's k runs through *fields.* The monoid $\mathbb{Z}_+ \times \mathbb{Z}_2 \setminus \{(0,1)\}$ is such an example.

The situation changes radically when we consider higher K-groups:

Theorem 1.8 ([Gu6]). *For any K_2-regular ring R and any intermediate finitely generated monoid $\mathbb{Z}_+^n \subset M \subset \mathbb{Q}_+^n$, where n is an arbitrary natural number, the following conditions are equivalent:*

(a) $M \approx \mathbb{Z}_+^n$,

(b) *$R[M]$ is K_1-regular,*

(c) *M is seminormal and $SK_1(R) = SK_1(R[M])$,*

and, if in addition, $\Omega^1_{R/\mathbb{Z}} \neq 0$

(d) $SK_1(R) = SK_1(R[M])$.

Remark 1.9.

(a) In case $\Omega^1_{R/\mathbb{Z}} = 0$ there are 'exotic' examples of fields and non-seminormal monoids whose monoid algebras have trivial SK_1-groups. For instance, $SK_1(k[X^2, X^3]) = 0$ for any number field k [**Kr**]. However, according to the statement (b) the ring $k[X^2, X^3]$ is not SK_1-regular.

(b) First explicit examples of nontrivial elements in $SK_1(\mathbb{C}[M])$ for certain rank 2 monoids (i. e. $\mathrm{rank}(\mathrm{gp}(M) = 2)$) were constructed by V. Srinivas [**SR**]. Actually, as it follows from [**Gu6**], Theorem 1.8 is valid for essentially more wide class of finitely generated monoids than in the statement above. Moreover, the corresponding nontrivial elements in SK_1-groups are explicitly constructed. This theorem in particular implies that rings of type $R[M]$ for M as above are not K_i-regular for all $i > 0$. Therefore, the 'naive' higher analogue (the equalities of type $K_i(R) = K_i(R[M])$) of Anderson's conjecture fails badly in the class of finitely generated monoids.

(c) The method developed in [**Gu6**] does not allow one to show that the homogeneous coordinate ring of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is not K_1-regular. I could not show independently that this ring $k[UW, UX, VW, VX]$ (k a field) is in fact not K-regular. Hence the question: is the mentioned ring an example of a singular ring which is K_i-regular for all $i \in \mathbb{N}$? We recall that the question wether or not regularity and K-regularity are equivalent in the Noetherian setting is still unsettled (even for affine rings).

However, there is a wide class of *infinitely generated* monoids for which the K-regularity is restored. The next three results are in this direction.

Theorem 1.10 ([Gu2]). *Let $c > 1$ be a natural number and M a c-divisible monoid. Then:*

(a) *$SL_r(R[M]) = E_r(R[M])$ for all Euclidean domains R and for all natural numbers $r \geqslant 3$,*

(b) *$K_1(R) = K_1(R[M])$ for any regular ring R provided M has no nontrivial units.*

Theorem 1.11 ([M]). *Let M be a c-divisible monoid for some $c > 1$, having no non-trivial units. Then $K_2(R) = K_2(R[M])$ for any regular ring R.*

Theorem 1.12 ([Gu5]). *$K_i(R) = K_i(R[M])$ for any regular ring R provided $\mathbb{Z}_+^n \subset M \subset \mathbb{Q}_+^n$, $i, n \in \mathbb{N}$, and M is c divisible for some $c > 1$.*

Remark 1.13. The key ingredient in the proof of Theorem 1.12 is the Suslin-Wodzicki solution to the excision problem in algebraic K-theory [**SuW**]. The condition $\mathbb{Z}_+^n \subset M \subset \mathbb{Q}_+^n$ is equivalent (up to isomorphism) to the condition that the set $\Phi(M)$ (to be defined below) is a simplex. The generalization to arbitrary convex polytopes provides a natural generalization of Anderson's conjecture to higher K-groups. See §2 for details.

The following result concerns certain class of monoids – the monoids of Φ-*simplicial growth*. This class generalizes the class of intermediate monoids $\mathbb{Z}_+^n \subset M \subset \mathbb{Q}_+^n$ exactly in the same way as the class of simplicial growth convex polytopes generalizes the class of arbitrary simplices (of arbitrary dimensions). Here a finite convex polytope $P \subset \mathbb{R}^n$ is said to be of simplicial growth if there exists a sequence of convex polytopes

$$P_1 \subset P_2 \subset \ldots \subset P_m = P$$

(for some natural m) such that P_1 and the closures (in the Euclidean topology) of $P_i \setminus P_{i-1}$ ($i \in [2, m]$) are all simplices. The polytope itself, associated to a monoid M, is obtained by a hyperplane cross section of the cone spanned by M (in the real space $\mathrm{gp}(M) \otimes \mathbb{R}$), provided such exists. The mentioned polytope will be denoted by $\Phi(M)$. Of course, $\Phi(M)$ is defined up to projective equivalence, but the properties we deal with are invariant under this equivalence.

It is a classical fact of convex geometry that the aforementioned cross section exists for any finite, convex, pointed, polyhedral cone. On the other hand M spans such a cone in $\mathbb{R} \otimes \mathrm{gp}(M)$ if it is finitely generated and has no non-trivial units. More generally, the cross section exists if there is a monoid extension $N \subset M$ satisfying the conditions: N is finitely generated and without non-trivial units, and for any $x \in M$ some positive multiple of x is in N (the *integral extension* condition)

Theorem 1.14 ([Gu3][Gu4]). *Let R be a noetherian ring of finite Krull dimension d and M a monoid of Φ-simplicial growth. Then the group of elementary matrices $E_r(R[M])$ acts transitively on the set of unimodular r-rows $\mathrm{Um}_r(R[M])$ for all $r \geqslant \max(3, d+2)$.*

Remark 1.15. The classical case of this theorem (i. e. when $M = \mathbb{Z}_+^n$) is due to Suslin [**Su2**]. We remark that the case of monoids M of type $\mathbb{Z}_+^n \subset M \subset \mathbb{Q}_+^n$ is also nontrivial and that in order to involve all monoids one has to treat the monoids to whom correspond arbitrary finite convex polytopes. It should also be mentioned that Theorem 1.14 generalizes to the monoids for which $\mathrm{gp}(M)_{tor}$ is either cyclic or a p-group for some prime p (provided $p \cdot 1 = 0$ in R) [**Gu5**]. Theorem 1.14 evidently implies (for monoid rings) an improvement of Bass-Vaserstein general estimate for surjective K_1-stabilization. In view of results of Bhatwadekar-Lindel-Rao-Roy, concerning K_0-stabilizations for polynomial rings, we can hope for such an improvement of K_0-stabilizations for monoid rings (see §2 for details).

Basing on [**Gu2**] (and the technique developed by Lindel) G. Schabhüser proved that a weak form of the expected injective K_0-stabilization for a monoid ring (i. e. the corresponding cancellation property for projective modules) actually takes place when the monoid is c-divisible for some $c > 1$.

It turns out that we can invoke 'generalized Discrete Hodge Algebras' in the picture as follows.

Theorem 1.16 ([Gu5]).

(a) *For a ring R and a monoid M the equality $K_0(R) = K_0(R[M])$ ($SK_1(R) = SK_1(R[M])$) implies $K_0(R) = K_0(R[M]/RI)$*
($SK_1(R) = SK_1(R[M]/RI)$ respectively), where I is an arbitrary proper ideal of M.

(b) *The implications as in (a) hold for the corresponding unstable objects (meening the properties that projective modules are extended from R and $SL_r = E_r$).*

(c) For a ring R and a c-divisible monoid M ($c > 1$) the equality $K_i(R) = K_i(R[M])$ implies $K_i(R) = K_i(R[M]/RI)$, where $i \in \mathbb{N}$ and I is an arbitrary proper radical ideal of M (i. e. $\sqrt{I} = I$).

The special case of Theorem 1.16(b) for 'ordinary' Discrete Hodge Algebras (i. e. monomial quotients of polynomial algebras) is due to Vorst [**Vor**].

The proofs of these results involve the corresponding algebraic tools (Quillen's local-to-global principle, various generalizations of Horrocks' monic inversion theorem, symbols, excision in algebraic K-theory, special automorphisms of polynomial and monoid rings etc.) combined with purely convex geometrical constructions (relative interiors, homothetic transformations, special decompositions of polytopes etc.).

In view of the results presented above it is natural to ask whether one can always distinguish monoid rings corresponding to non-isomorphic monoids. The theorem below gives the (essentially final) positive answer to this question (posed in [**Gi**]).

Theorem 1.17 ([Gu8]). *Let M and N be finitely generated monoids and R be a ring. Then each of the following conditions implies $M \approx N$:*

(a) *M and N have no non-trivial units and $R[M] \approx R[N]$ as augmented R-algebras (with respect to the natural augmentations*
$R[M] \to R$ and $R[N] \to R$ mapping non-unit monomials to 0).

(b) *$R[M] \approx R[N]$ as R-algebras and M is normal.*

We remark that for rank two monoids the isomorphism problem is answered in the positive without any additional condition [**Gu7**].

2. Conjectures

Here we state two main conjectures on stable and unstable K-theory of semigroup rings. They include the results from §1 in a uniform way and, simultaneously, provide their final possible generalizations.

A c-divisible monoid is a filtered union of its c-divisible submonoids of finite rank ($c > 1$). In its turn any finite rank c-divisible monoid is a filtered limit of c-divisible hulls of finitely generated monoids, i. e. limits of diagrams of type

$$M \xrightarrow{\exp_c} M \xrightarrow{\exp_c} \cdots, \quad M \text{ finitely generated},$$

where $\exp_c(m) = m^c$ (writing the monoid structure multiplicatively). The crucial point here is Gordan Lemma saying the following: any finite rank monoid M without non-trivial units is finitely generated if (and only if) $\mathrm{gp}(M)$ is finitely generated and the cone $C(M) \subset \mathbb{R} \otimes \mathrm{gp}(M)$ spanned by M is finite polyhedral.

In view of these remarks on c-divisible monoids and Theorems 1.10, 1.11, 1.12 the equalities $K_i(R) = K_i(R[M])$ for fixed R, i, $c > 1$, and M running through all c-divisible monoids without non-trivial units, are equivalent to the equalities

$$K_i(R) = \lim_{\rightarrow} \mathcal{D}_{(R,i,c,M)},$$

with the same R, i, c, and M running through *finitely generated* monoids without non-trivial units, where $\mathcal{D}_{(R,i,c,M)}$ is the diagram

$$K_i(R[M]) \xrightarrow{K_i(R[\exp_c])} K_i(R[M]) \xrightarrow{K_i(R[\exp_c])} \cdots.$$

We, therefore, arrive at the following conjecture whose special cases are Theorems 1.10, 1.11 and 1.12. Consider $\mathbb{N}$ as a monoid with respect to the multiplicative structure. Then there is an action of $\mathbb{N}$ on the quotient groups $K_i(R[M])/K_i(R)$ defined by

$$c \mapsto K_i(R[\exp_c])/K_i(R).$$

We say that this action is *nilpotent* if for any $x \in K_i(R[M])/K_i(R)$ and $c \in \mathbb{N} \setminus \{1\}$ there is $n \in \mathbb{N}$ such that $c^n \cdot x = 0$

Conjecture 2.1. *For any index $i \in \mathbb{Z}_+$, any regular ring R and any monoid M without non-trivial units the multplicative action of $\mathbb{N}$ on $K_i(R[M])/K_i(R)$ is nilpotent.*

In other words, we say that the equalities $K_i(R) = K_i(R[M])$ hold always for c-divisible monoids M without non-trivial units. But we state the conjecture in terms of the multiplicative actions of $\mathbb{N}$ because of Remark 2.3(c,d) below.

By Theorems 1.1, 1.10, 1.11 and 1.12 this conjecture holds in the special cases when either $i \leqslant 2$ or M runs through intermediate monoids $\mathbb{Z}_+^n \subset M \subset \mathbb{Q}_+^n$ ($n \in \mathbb{N}$). A further support is provided by

Theorem 2.2 ([Gu5]).

(a) *Conjecture 2.1 is equivalent to the same nilpotency condition when M runs through the subclass of finitely generated normal monoids without non-trivial units.*

(b) *Conjecture 2.1 implies the analogous nilpotency property of the multiplicative action of $\mathbb{N}$ on the groups $K_i(R[M]/RI)/K_i(R)$, where $I \subset M$ is any ideal.*

(c) *Assume R is any ring and the multiplicative action of $\mathbb{N}$ on the groups $SK_0(R[M])$ are nilpotent, where M runs through monoids without non-trivial units. (This condition in particular implies $SK_0(R) = 0$.) Then $SK_0(R[M]) = 0$ for all monoids without non-trivial units.*

(d) *Let R be a ring. If*

$$\lim_{\rightarrow} \left(K_0(R[M]) \xrightarrow{K_0(R[\exp_c])} K_0(R[M]) \xrightarrow{K_0(R[\exp_c])} \cdots\right) = \mathbb{Z}$$

for all monoids M (maybe with non-trivial units) and for all natural numbers $c > 1$ then $K_0(R[M]) = \mathbb{Z}$ for all seminormal monoids M.

Remark 2.3.

(a) The claim (a) follows from the facts that c-divisible monoids are seminormal, that the 'interior' submonoid of a seminormal monoid is normal [Gu5, Lemma 1.4.3], and that excision in algebraic K-theory holds for monoid rings for c-divisible monoids [Gu5,§3.2]. The claim (b) is a corollary of Theorem 3.2.1 in [Gu5] – one only needs to notice that all the nilpotent images of monomials (elements of M) in $R[M]/IR$ map to 0 in the corresponding limit.

(b) The claims (c) and (d) are proved in [Gu5,§3.3]. They in particular show that Conjecture 2.1 is a natural higher K-analogue of Anderson's conjecture.

(c) B. Totaro has shown in his unpublished notes the following. Let M be a finitely generated normal monoid without non-trivial invertible elements, let k be a field of chracteristic 0, and let i be any natural index. Then any natural number $n > 1$ acts (in the sense of the aforementioned action) nilpotently on any finite-dimensional n-invariant subspace of $K_i(k[M])/K_i(k)$. It can be shown that $K_i(k[M])/K_i(k)$ is a k-linear subspace of the nil K-theory (the obstruction to homotopy invariance of K-theory) of $k[M]$. This nil-group is known to be a module over the big Witt ring $\mathrm{W}(k)$, hence a k-vector space. The precise statement is the existence of a grading

$$K_i(k[M])/K_i(k) \otimes \mathbb{Q} = \oplus_{j \geqslant 1} A_j$$

such that $n \cdot A_j \subset A_{nj}$. In particular, Conjecture 2.1 would follow for fields of characteristic 0 if $K_i(k[M])/K_i(k)$ was concentrated in only finitely many degrees.

(d) It is not excluded that there is even a uniform nilpotency degree of the mentioned action of $\mathbb{N}$ for R and i fixed. For instance, all non-trivial elements in SK_1-groups, constructed in [**Gu6**], die already by multiplying by any natural $c > 1$.

Next we suggest another possibility for extending Anderson's conjecture to higher K-theory, but this time in terms of unstable groups.

Serre's *Unimodular element* and Bass' *Cancellation* theorems assert respectively that a projective module P over a Noetherian ring R of finite Krull dimension d contains a unimodular element and satisfies the cancellation condition whenever $\mathrm{rank}(P) > d$ [Bass, Ch 4,§2,§3]. Recall that an element is called *unimodular* if it defines a direct summand of P isomorphic to R, and that P satisfies the *cancellation condition* if $R \oplus P \approx R \oplus Q$ implies $P \approx Q$ for any R-module Q.

This has been extended to polynomial rings over R in arbitrary number of variables. That is a projective module P over $R[X_1, \ldots, X_n]$ has a unimodular element and satisfies the cancellation condition provided $\mathrm{rank}(P) > d$ ([**BR**] and [**R**], respectively). Observe that the mentioned cancellation condition implies the freeness of stably free $k[X_1, \ldots, X_n]$-modules, while the presence of unimodular elements just coincides with the freeness of projective $k[X_1, \ldots, X_n]$-modules (k a field).

The mentioned results can equivalently be formulated in terms of K_0-stabilizations: existence of unimodular elements corresponds to the surjective stabilization and cancellativity corresponds to the injective sabilization. We in particular see that a Noetherian ring R and its polynomial extensions have the same bounds for K_0-stabilizations in terms of Krull dimension of R.

Analogous result for K_1-stabilizations was previously obtained in [**Su2**], where it is shown that the surjectice K_1-stabilization for polynomial rings $R[X_1, \ldots, X_n]$ occurs from max(2, $\dim(R) + 1$) and the injective K_1-stabilization takes place from $\max(3, \dim(R) + 2)$ (R as above). This means that

$$\mathrm{GL}_r(R[X_1, \ldots, X_n]) \to K_1(R[X_1, \ldots, X_n])$$

is surjective for $r \geqslant \max(2, \dim(R)+1)$ and

$$\mathrm{GL}_r(R[X_1,\dots,X_n])/E_r(R[X_1,\dots,X_n]) \to K_1(R[X_1,\dots,X_n])$$

is isomorphism for $r \geqslant \max(3, \dim(R)+2)$. Recall that in the classical case (i. e. when $n=0$) the estimates 'surjectivity from $r \geqslant \dim(R)+1$' and 'injectivity from $r \geqslant \dim(R)+2$' were obtained by Bass and Vasershtein (see [Bass, Ch.5,§4]).

Tulenbaev's result on K_2-stabilizations for polynomial extensions [**T**] is in the same vein. In view of the previous results of Dennis, van der Kallen, Vasershtein, Suslin-Tulenbaev the main result of [**T**] says that a ring and its polynomial algebras have essentially the same K_2-stabilizations.

The best stabilization estimates for higher K-groups were obtained by Suslin in [**Su3**]. It turns out that instead of Quillen's theory (defined with use of his '+ contruction') the naturaly expected stabilizations can be proved for Volodin's theory [**Vol**]. In particular, Suslin showed that the natural mappings

$$K^V_{i,r}(R) \to K^V_{i,r+1}(R)$$

are surjective for $r \geqslant \mathrm{s.\,r.}(R)+i-1$ and injective for $i \geqslant \mathrm{s.\,r.}(R)+i$ (the *Bass Conjecture*). This enabled him to deduce general stabilizations for Quillen's theory too (previously obtained by van der Kallen). Here the *stable range* $\mathrm{s.\,r.}(R)$ of a ring R is defined as the minimal natural number r such that for any unimodular row $(a_0, a_1, \dots, a_r) \in \mathrm{Um}_{r+1}(R)$ there are $b_1, \dots, b_r \in R$ for which the row $(a_1 + b_1 a_0, \dots, a_r + b_r a_0)$ is unimodular. It is known that $\mathrm{s.\,r.}(R) \leqslant \dim(R)+1$ for a Noetherian ring R.

What has been said above indicates that for a Noetherian ring R the polynomial algebra $R[X_1,\dots,X_n]$ has the following estimates for higher K_i-stabilizations:

$$K^V_{i,r}(R[X_1,\dots,X_n]) \to K^V_{i,r+1}(R[X_1,\dots,X_n]), \quad i \geqslant 1,$$

is surjective for $r \geqslant \max(i+1, \dim(R)+i)$ and injective for $r \geqslant \max(i+2, \dim(R)+i+1)$. So we venture to make the following general

Conjecture 2.4. *Let R be a Noetherian ring, M be any monoid and $i \in \mathbb{N}$.*

(a) *A projetive $R[M]$-module P contains a unimodular element and satisfies the cancellation condition if*

$$\mathrm{rank}(P) \geqslant \max(2, \dim(R)+1).$$

(b) *The natural mappings*

$$K^V_{i,r}(R[M]) \to K^V_{i,r+1}(R[M]), \quad i \geqslant 1,$$

are surjective for $r \geqslant \max(i+1, \dim(R)+i)$ *and injective for* $r \geqslant \max(i+2, \dim(R)+i+1)$.

The situation here is more difficult for higher K-groups than in Conjecture 2.1, because even the original special case of polynomial algebras is an open question.

Remark 2.5.

(a) The claim (b) is equivalent to the surjectivity of the K_0-stabilization from $\max(1, \dim(R))$ and injectivity of the K_0-stabilization from $\max(2, \dim(R)+1)$. Thus we have a uniform picture comprising Grothendieck group and higher K-groups. Swan's aforemntioned result (Theorem 1.4) is the first non-trivial support to Conjecture 2.4(a) beyond the rings in Theorem 1.1. The case of singular coefficient rings remains open even in dimension 1.

(b) The crucial step in establishing the cancellation property for projective modules is the transitivity of the elementary action on unimodular rows of the appropriate lengths. In particular, Conjecture 2.4(a) suggests that the action

$$E_r(R[M]) \times \mathrm{Um}_r(R[M]) \to \mathrm{Um}_r(R[M])$$

is transitive for arbitrary monoids whenever $r \geqslant \max(3, \dim(R) + 2)$. For monoids of Φ-simplicial growth this is so by Theorem 1.14.

(c) It is proved in [Gu5,§3.3] that the general case of the transitivity as in (b) would follow if the following was true. Assume a subset $\{x_1, \dots, x_d\} \subset M$ is a basis for $\mathrm{gp}(M) \approx \mathbb{Z}^d$. Then the ring extension

$$R[M] \subset R[M, x_1/m]$$

makes the bigger ring a free module over $R[M]$ of rank m. Here

$$[M, x_1/m] \subset \mathbb{Q} \otimes \mathrm{gp}(M)$$

is a submonoid generated by M and x_1/m (writing additively). Now assume $A \in \mathrm{GL}_r(R[M])$ (r as above) is a matrix such that

$$A \sim_{E_r(R[M,x_1/m])} \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

for some $B \in \mathrm{GL}_{r-1}(R[M, x_1/m])$. Then there is $s \in \mathbb{N}$ such that

$$A^{m^s} \sim_{E_r(R[M])} \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$$

for some $C \in \mathrm{GL}_{r-1}(R[M])$. This seems to be a plausible approach to the general case.

(d) One might expect for a more general fact than that in (c). Namely, assume we are given a ring extension $S \subset T$ making T a free S-module of rank m. Assume further $A \in \mathrm{GL}_r(S)$, $r \geqslant 3$, is a matrix such that

$$A \sim_{E_r(T)} \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

for some $B \in \mathrm{GL}_{r-1}(T)$. (Here we assume $r \geqslant 3$ because E_r is normal in GL_r for such r.) Then there is $s \in \mathbb{N}$ such that

$$A^{m^s} \sim_{E_r(S)} \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$$

for some $C \in \mathrm{GL}_{r-1}(S)$. Actually, we have even conjectured this in [**Gu5**] as a K_1-stabilization analogue of transfer maps for K_0. However, van der Kallen subsequently sent me his elegant topological counterexample to this conjecture. In particular, he shows that none of the positive powers of the matrix

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w \end{pmatrix}$$

from $\mathrm{GL}_4(S)$, $S = \mathbb{R}[w, x, y, z]/(w^2 + x^2 + y^2 + z^2 = 1)$, can be reduced to a matrix of type

$$\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

using elementary transformations over S. On the other hand the first row of this matrix is clearly equivalent to

$$(w - iy, -x - iz, 0, 0)$$

under the elementary actions over $T = \mathbb{C} \otimes S$.

References

A. D. F. Anderson, *Projective modules over subrings of $k[X,Y]$ generated by monomials*, Pacific J. Math. **79** (1978), 5–17.

Bass. H. Bass, *Algebraic K-Theory*, W. A. Benjamin, Inc., 1968.

BR. S.M. Bhatwadekar and A. Roy, *Some theorems about projetive modules over polyomial rings*, J. Algebra **86** (1984), 150–158.

D. M. Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 507-588.

F. W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, 1993.

Gi. R. Gilmer, *Commutative Semigroup Rings*, The University of Chicago Press, 1984.

Gu1. J. Gubeladze, *Anderson's conjecture and the maximal monoid class, over which projective modules are free*, Math. USSR Sbornik **63** (1989), 165–180.

Gu2. J. Gubeladze, *Classical algebraic K-theory of monoid algebras*, Lect. Notes Math. **1437** (1990), Springer, 36–94.

Gu3. J. Gubeladze, *The elementary action on unimodular rows over a monoid ring*, J. Algebra **148** (1992), 135–161.

Gu4. J. Gubeladze, *The elementary action on unimodular rows over a monoid ring II*, J. Algebra **155** (1993), 171–194.

Gu5. J. Gubeladze, *Geometric and algebraic representations of commutative cancellative monoids*, Proc. A. Razmadze Math. Inst., Georgian Ac. Sci. **113** (1995), 31–81.

Gu6. J. Gubeladze, *Nontriviality of $SK_1(R[M])$*, J. Pure Appl. Algebra **104** (1995), 169–190.

Gu7. J. Gubeladze, *The isomorphism problem for monoid rings of rank 2 monoids*, J. Pure Appl. Algebra **105** (1995), 17–51.

Gu8. J. Gubeladze, *The isomorphism problem for commutative monoid rings*, J. Pure Appl. Algebra **129** (1998), 35–65.

KKMS. G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal Embeddings I.* Lect. Notes Math. **339**, Springer, 1973.

Kr. M. I. Krusemeyer, *Fundamental groups, algebraic K-theory and a problem of Abhyankar*, Invent. Math. **19** (1973), 15–47.

LW. R. C. Laubenbacher and C. J. Woodburn, *An algorithm for the Quillen-Suslin theorem for monoid rings*, J. Pure Appl. Algebra **117-118** (1997), 395-429.

Lin. H. Lindel, *On Bass-Quillen conjecture concerning projective modules over polynomial rings*, Invent. Math. **65** (1981), 319–323.

MMJP. M. Masuda, L. Moser-Jauslin, and T. Petrie, *The equivariant Serre problem for abelian groups*, Topology **35** (1996), 329–334.

M. Z. Mushkudiani, *K_2 groups of monoid algebras over regular rings*, Proc. A. Razmadze Math. Inst., Georgian Ac. Sci. **113** (1995), 121–137.

O. T. Oda, *Convex Bodies and Algebraic Geometry (An Introduction to the Theory of Toric Varieties)*, Springer, 1988.

Q1. D. Quillen, *Higher algebraic K-theory I*, Lect. Notes Math. **341** (1973), 85–147.

Q2. D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.

R. R. A. Rao, *A question of H. Bass on the cancellation nature of large projective modules over polynomial rings*, Amer. J. Math. **110** (1988), 641–657.

SR. V. Srinivas, *K_1 of the cone over a curve*, J. Reine Angew. Math. **381** (1987), 37–50.

Su1. A. Suslin, *Projective modules over a polynomial ring are free*, Sov. Math. Dokl. **17** (1976), 1160–1164.

Su2. A. Suslin, *On the structure of a special linear group over polynomial rings*, Izv. Akad. Nauk SSSR, Ser. Mat. **41** (1997), 233–252 (Russian).

Su3. A. Suslin, *Stability in algebraic K-theory*, Lect. Notes Math. **966** (1982), 304–333.

SuW. A. Suslin and M. Wodzicki, *Excision in algebraic K-theory*, Ann. Math., II. Ser. **136** (1992), 51–122.

Sw1. R. G. Swan, *Gubeladze's proof of Anderson's conjecture*, Contemp. Math. **124**, 1992, 215–250.

Sw2. R. G. Swan, *Neron-Popescu desingularizations*, Preprint.

Sw3. R. G. Swan, *On seminormality*, J. Algebra **67** (1980), 210–229.

T. M. S. Tulenbaev, *The Steinberg group of a polynomial ring*, Mat. Sbornik **117** (1982), 131–143 (Russian).

Vol. I. A. Volodin, *Algebraic K-theory as extraordinary homology theory on the category of associative rings with unity*, Math. USSR Izv. **5** (1971), 859–887 (Russian).

Vor. T. Vorst, *The Serre problem for discrete Hodge algebras*, Math. Z. **184** (1983), 425–433.

Thanks are due to Hvedri Inassaridze for suggesting to write this expository paper, to Burt Totaro for his unpublished notes on Conjecture 2.1, and to Wilberd van der Kallen for communicating his topological example in Remark 2.5(d).

This article may be accessed via WWW at http://www.rmi.acnet.ge/hha/ or by anonymous ftp at ftp://ftp.rmi.acnet.ge/pub/hha/volumes/1999/n5/n5.(dvi,ps,dvi.gz,ps.gz)

Joseph Gubeladze `gubel@rmi.acnet.ge`

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia