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A UNIVERSALITY THEOREM FOR VOEVODSKY'S ALGEBRAIC COBORDISM SPECTRUM

IVAN PANIN, KONSTANTIN PIMENOV AND OLIVER RÖNDIGS

(*communicated by J. F. Jardine*)

Abstract

An algebraic version of a theorem of Quillen is proved. More precisely, for a regular Noetherian scheme S of finite Krull dimension, we consider the motivic stable homotopy category $\mathrm{SH}(S)$ of $\mathbf{P}^1$-spectra, equipped with the symmetric monoidal structure described in [**7**]. The algebraic cobordism $\mathbf{P}^1$-spectrum MGL is considered as a commutative monoid equipped with a canonical orientation $th^{\mathrm{MGL}} \in \mathrm{MGL}^{2,1}(\mathrm{Th}(\mathcal{O}(-1)))$. For a commutative monoid E in the category $\mathrm{SH}(S)$, it is proved that the assignment $\varphi \mapsto \varphi(th^{\mathrm{MGL}})$ identifies the set of monoid homomorphisms $\varphi\colon \mathrm{MGL} \to E$ in the motivic stable homotopy category $\mathrm{SH}(S)$ with the set of all orientations of E. This result generalizes a result of G. Vezzosi in [**12**].

1. Introduction

Quillen proved in [**10**] that the formal group law associated to the complex cobordism spectrum MU is the universal one on the Lazard ring. As a consequence, the set of orientations on a commutative ring spectrum E in the stable homotopy category is in bijective correspondence with the set of homomorphisms of ring spectra from MU to E in the stable homotopy category. This result allowed a whole new approach to understanding the stable homotopy category, which is still actively pursued today.

On the algebraic side of things, there is a similar $\mathbf{P}^1$-ring spectrum MGL in the motivic stable homotopy category of a Noetherian finite-dimensional scheme S. The formal group law associated to MGL is not known to be the universal one, although unpublished work of Hopkins and Morel claims this if S is the spectrum of a field of characteristic zero. Nevertheless, the set of orientations on a $\mathbf{P}^1$-ring spectrum in the motivic stable homotopy category over S can be identified in the same fashion if S is regular.

Theorem 1.1. *Let S be a regular Noetherian finite-dimensional scheme, and let E be a commutative $\mathbf{P}^1$-ring spectrum over S. The set of orientations on E is in bijection with the set of homomorphisms of $\mathbf{P}^1$-ring spectra from* MGL *to E in the motivic stable homotopy category over S.*

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For a more detailed formulation, see Theorem 2.7. Our main motivation to write this paper was to prove the universality theorem 1.1 in a form convenient for its application in [**8**]. Theorem 1.1 has already been employed in [**1**] and [**11**]. In the special case where S is the spectrum of a field, Theorem 1.1 was stated originally in a slightly different form by G. Vezzosi in [**12**], although he ignored certain aspects of the multiplicative structure on MGL.

1.1. Preliminaries

We refer to [**7**, Appendix] for the basic terminology, notation, constructions, definitions and results. For the convenience of the reader we recall the basic definitions. Let S be a regular Noetherian scheme of finite Krull dimension. One may think of S being the spectrum of a field or the integers. Below we need to apply [**5**, Prop. 4.3.8], which is the basic reason to work with a regular base scheme S. Let $\mathcal{S}m/S$ be the category of smooth quasi-projective S-schemes, and let **sSet** be the category of simplicial sets. A *motivic space over* S is a functor

$$A\colon \mathcal{S}m/S^{\mathrm{op}} \to \mathbf{sSet}$$

(see [**7**, A.1.1]). The category of motivic spaces over S is denoted $\mathbf{M}(S)$. This definition of a motivic space is different from the one considered by Morel and Voevodsky in [**5**]; they consider only those simplicial presheaves which are sheaves in the Nisnevich topology on $\mathcal{S}m/S$. With our definition, the Thomason-Trobaugh K-theory functor obtained by using big vector bundles is a motivic space on the nose. It is not a simplicial Nisnevich sheaf. This is why we prefer to work with the above notion of "space".

We write $\mathrm{H}^{\mathrm{cm}}_{\bullet}(S)$ for the pointed motivic homotopy category and $\mathrm{SH}^{\mathrm{cm}}(S)$ for the stable motivic homotopy category over S as constructed in [**7**, A.3.9, A.5.6]. By [**7**, A.3.11, resp. A.5.6] there are canonical equivalences to $\mathrm{H}_{\bullet}(S)$ of [**5**], respectively, $\mathrm{SH}(S)$ of [**13**]. Both $\mathrm{H}^{\mathrm{cm}}_{\bullet}(S)$ and $\mathrm{SH}^{\mathrm{cm}}_{\bullet}(S)$ are equipped with closed symmetric monoidal structures such that the $\mathbf{P}^1$-suspension spectrum functor is a strict symmetric monoidal functor

$$\Sigma^{\infty}_{\mathbf{P}^1}\colon \mathrm{H}^{\mathrm{cm}}_{\bullet}(S) \to \mathrm{SH}^{\mathrm{cm}}(S).$$

Here $\mathbf{P}^1$ is considered as a motivic space pointed by $\infty \in \mathbf{P}^1$. The symmetric monoidal structure $(\wedge, \mathbb{I}_S = \Sigma^{\infty}_{\mathbf{P}^1} S_+)$ on the homotopy category $\mathrm{SH}^{\mathrm{cm}}(S)$ is constructed on the model category level by employing symmetric $\mathbf{P}^1$-spectra. It satisfies the properties required by Theorem 5.6 of Voevodsky's talk [**13**]. From now on we will usually omit the superscript $(-)^{\mathrm{cm}}$.

Every $\mathbf{P}^1$-spectrum $E = (E_0, E_1, \ldots)$ represents a cohomology theory on the category of pointed motivic spaces. Namely, for a pointed motivic space (A, a) set

$$E^{p,q}(A, a) = \mathrm{Hom}_{\mathrm{SH}_{\bullet}(S)}(\Sigma^{\infty}_{\mathbf{P}^1}(A, a), \Sigma^{p,q}(E))$$

and $E^{*,*}(A, a) = \oplus_{p,q} E^{p,q}(A, a)$. This definition extends to motivic spaces via the functor $A \mapsto A_+$ which adds a disjoint basepoint. That is, for a non-pointed motivic space A, set $E^{p,q}(A) = E^{p,q}(A_+, +)$ and $E^{*,*}(A) = \oplus_{p,q} E^{p,q}(A)$. Recall that there is a canonical element in $E^{2n,n}(E_n)$, denoted as $\Sigma^{\infty}_{\mathbf{P}^1} E_n(-n) \to E$. It is represented by the canonical map $(*, \ldots, *, E_n, E_n \wedge \mathbf{P}^1, \ldots) \to (E_0, E_1, \ldots, E_n, \ldots)$ of $\mathbf{P}^1$-spectra.

Every $X \in \mathcal{S}m/S$ defines a representable motivic space constant in the simplicial direction, taking an S-smooth scheme U to $\operatorname{Hom}_{\mathcal{S}m/S}(U, X)$. It is not possible in general to choose a basepoint for representable motivic spaces. So we regard S-smooth varieties as motivic spaces (non-pointed) and set

$$E^{p,q}(X) = E^{p,q}(X_+, +).$$

Given a $\mathbf{P}^1$-spectrum E we will reduce the double grading on the cohomology theory $E^{*,*}$ to a grading. Namely, set $E^m = \oplus_{m=p-2q} E^{p,q}$ and $E^* = \oplus_m E^m$.

To complete this section, we define a $\mathbf{P}^1$-*ring spectrum* to be a monoid (E, μ, e) in $(\mathrm{SH(S)}, \wedge, \mathbb{I}_S)$. A *commutative* $\mathbf{P}^1$-ring spectrum is a commutative monoid (E, μ, e) in $(\mathrm{SH(S)}, \wedge, \mathbb{I}_S)$. The cohomology theory E^* defined by a $\mathbf{P}^1$-ring spectrum is a ring cohomology theory. The cohomology theory E^* defined by a commutative $\mathbf{P}^1$-ring spectrum is a ring cohomology theory, however it is not necessarily graded commutative. The cohomology theory E^* defined by an oriented commutative $\mathbf{P}^1$-ring spectrum is a graded commutative ring cohomology theory, as will be explained in Subsection 1.3.

1.2. Oriented commutative ring spectra

Following Adams and Morel, we define an orientation of a commutative $\mathbf{P}^1$-ring spectrum. However we prefer to use Thom classes instead of Chern classes. Consider the pointed motivic space $\mathbf{P}^\infty = \operatorname{colim}_{n \geqslant 0} \mathbf{P}^n$ having basepoint $g_1 \colon S = \mathbf{P}^0 \hookrightarrow \mathbf{P}^\infty$.

The tautological "vector bundle" $\mathcal{T}(1) = \mathcal{O}_{\mathbf{P}^\infty}(-1)$ is also known as the Hopf bundle. It has zero section $z \colon \mathbf{P}^\infty \hookrightarrow \mathcal{T}(1)$. The fiber over the point $g_1 \in \mathbf{P}^\infty$ is $\mathbb{A}^1$. For a vector bundle V over a smooth S-scheme X, with zero section $z \colon X \hookrightarrow V$, its *Thom space* $\operatorname{Th}(V)$ is the Nisnevich sheaf associated to the presheaf

$$Y \mapsto V(Y)/\big(V \smallsetminus z(X)\big)(Y)$$

on the Nisnevich site $\mathcal{S}m/S$. In particular, $\operatorname{Th}(V)$ is a pointed motivic space in the sense of [**7**, Defn. A.1.1]. It coincides with Voevodsky's Thom space [**13**, p. 422], since $\operatorname{Th}(V)$ already is a Nisnevich sheaf. The Thom space of the Hopf bundle is then defined as the colimit $\operatorname{Th}(\mathcal{T}(1)) = \operatorname{colim}_{n \geqslant 0} \operatorname{Th}\big(\mathcal{O}_{\mathbf{P}^n}(-1)\big)$. Abbreviate $T = \operatorname{Th}(\mathbf{A}^1_S)$.

Let E be a commutative $\mathbf{P}^1$-ring spectrum. The unit gives rise to an element $1 \in E^{0,0}(S)$. Applying the $\mathbf{P}^1$-suspension isomorphism to that element we get an element $\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1, \infty)$. The canonical covering of $\mathbf{P}^1$ defines motivic weak equivalences

$$\mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1/\mathbf{A}^1 \xleftarrow{\sim} \mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\} = T \tag{1}$$

of pointed motivic spaces inducing isomorphisms

$$E(\mathbf{P}^1, \infty) \leftarrow E(\mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\}) \rightarrow E(T).$$

Let $\Sigma_T(1)$ be the image of $\Sigma_{\mathbf{P}^1}(1)$ in $E^{2,1}(T)$.

Definition 1.2. Let E be a commutative $\mathbf{P}^1$-ring spectrum. A *Thom orientation of* E is an element $th \in E^{2,1}(\operatorname{Th}(\mathcal{T}(1))$ such that its restriction to the Thom space of the fibre over the distinguished point coincides with the element $\Sigma_T(1) \in E^{2,1}(T)$. A *Chern orientation of* E is an element $c \in E^{2,1}(\mathbf{P}^\infty)$ such that $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1)$. *An orientation* of E is either a Thom orientation or a Chern orientation. One says that a Thom orientation th of E coincides with a Chern orientation c of E provided that $c = z^*(th)$, or equivalently the element th coincides with $th(\mathcal{O}(-1))$ given by (3) below.

Remark 1.3. The element th should be regarded as the Thom class of the tautological line bundle $\mathcal{T}(1) = \mathcal{O}(-1)$ over $\mathbf{P}^\infty$. The element c should be regarded as the Chern class of the tautological line bundle $\mathcal{T}(1) = \mathcal{O}(-1)$ over $\mathbf{P}^\infty$.

Example 1.4. The following orientations given below are relevant for our work. Here MGL denotes the $\mathbf{P}^1$-ring spectrum representing algebraic cobordism obtained below in Definition 2.4, and BGL denotes the $\mathbf{P}^1$-ring spectrum representing algebraic K-theory constructed in [**7**, Theorem 2.2.1].

- Let $u_1 \colon \Sigma^\infty_{\mathbf{P}^1} \mathrm{Th}(\mathcal{T}(1))(-1) \to \mathrm{MGL}$ be the canonical map of $\mathbf{P}^1$-spectra. Set $th^{\mathrm{MGL}} = u_1 \in \mathrm{MGL}^{2,1}(\mathrm{Th}(\mathcal{T}(1)))$. Since the equality
$$th^{\mathrm{MGL}}|_{\mathrm{Th}(\mathbf{1})} = \Sigma_T(1)$$
holds in $\mathrm{MGL}^{2,1}(\mathrm{Th}(\mathbf{1}))$, the class th^{MGL} is an orientation of MGL.
- Set $c^K = (-\beta) \cup ([\mathcal{O}] - [\mathcal{O}(1)]) \in \mathrm{BGL}^{2,1}(\mathbf{P}^\infty)$. The relation (11) from [**7**] shows that the class c^K is an orientation of BGL.

1.3. Certain properties of oriented $\mathbf{P}^1$-ring spectra

Let E be a commutative $\mathbf{P}^1$-ring spectrum and $E^{*,*}$ the cohomology theory it represents. For an element $\lambda \in \Gamma(S, \mathcal{O}_S^*)$, denote by Λ the morphism $\mathbf{P}^1 \to \mathbf{P}^1$ which maps $[x : y]$ to $[x : \lambda y]$. Let $\Lambda^* \colon E^{*,*}(\mathbf{P}^1, \infty) \to E^{*,*}(\mathbf{P}^1, \infty)$ be the pull-back map induced by Λ. Let $\Sigma_{\mathbf{P}^1} \colon E^{*,*}(S) \to E^{*+2,*+1}(\mathbf{P}^1, \infty)$ be the suspension isomorphism. Set
$$\epsilon = (\Sigma^{-1}_{\mathbf{P}^1} \circ (-1)^* \circ \Sigma_{\mathbf{P}^1})(1) \in E^{0,0}(S).$$
Clearly $\epsilon^2 = 1$. The following commuting rule is proved by Morel in [**4**]: for any $a \in E^{p,q}$ and $b \in E^{r,s}$ one has $a \cup b = (-1)^{ps} \epsilon^{qr} b \cup a$. Suppose that $\epsilon = 1$ for E. Define a Chern structure on $E^{*,*}$ as an assignment which associates to every $X \in \mathcal{S}m(S)$ and every line bundle L over X a class $c(L) \in E^{2,1}(X)$ such that

(1) the class $c(L)$ is natural,

(2) $c(\mathbf{1}) = 0$ (the class of a trivial bundle vanishes), and

(3) the set $\{1, c(\mathcal{O}(-1))\}$ is a basis of the two-sided $E^{*,*}(S)$-module $E^{*,*}(\mathbf{P}^1 \times S)$.

Given a Chern structure on $E^{*,*}$, one can state and prove the projective bundle theorem, construct a theory of Chern classes, and construct a theory of Thom classes by repeating literally the arguments and constructions from [**6**, Thm. 3.9, Thm. 3.27 and Proof of Thm. 3.35]. The resulting theory of Chern classes is uniquely defined by the property that for line bundles the classes c_1 and c coincide. The resulting theory of Thom classes is uniquely defined by the property that for every line bundle L with zero section z one has $z^*(th(L)) = c(L)$. We recall the construction of the theory of Thom classes at the end of this section.

Below, in this section, (E, th) is *an oriented commutative ring* $\mathbf{P}^1$*-spectrum*. The class $c = z^*(th) \in E^{2,1}(\mathbf{P}^\infty)$ is a Chern orientation of E by [**9**, Prop. 6.5.1]. Clearly the pull-back map $E^{*,*}(\mathbf{P}^2) \to E^{*,*}(\mathbf{P}^1)$ is surjective. We claim now that for any $\lambda \in \Gamma(S, \mathcal{O}_S^*)$ one has $\Lambda^* = \mathrm{id}$. In fact, to check this just repeat the arguments from [**2**, Proof of Lemma 1.6]. So $\epsilon = 1$ if E is orientable.

Now one can produce a Chern structure on $E^{*,*}$ as follows. The scheme S is regular. The functor isomorphism $\mathrm{Hom}_{\mathrm{H}_\bullet(S)}(-, \mathbf{P}^\infty) \to \mathrm{Pic}(-)$ on the category $\mathcal{S}m/S$, provided by [**5**, Thm. 4.3.8], sends the class of the identity map $\mathbf{P}^\infty \to \mathbf{P}^\infty$ to the class of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbf{P}^\infty$. For a line bundle L over $X \in \mathcal{S}m/S$, let $[L]$ be the class of L in the group $\mathrm{Pic}(X)$. Let $f_L \colon X_+ \to \mathbf{P}^\infty$ be a morphism in $\mathrm{H}_\bullet(S)$ corresponding to the class $[L]$ under the functor isomorphism above. For a line bundle L over $X \in \mathcal{S}m/S$, set $c(L) = f_L^*(c) \in E^{2,1}(X)$. Clearly, $c(\mathcal{O}(-1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ since $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1, \infty)$. With that Chern structure, $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ is an oriented ring cohomology theory in the sense of [**6**]. In particular, (BGL, c^K) defines an oriented ring cohomology theory on $\mathcal{S}m\mathcal{O}p$.

Combining the results given above, we obtain a theory of Thom classes

$$V \mapsto th(V) \in E^{2\mathrm{rank}(V),\mathrm{rank}(V)}(\mathrm{Th}(V))$$

on $E^{*,*}$. The latter means that the classes $th(V)$ are natural, multiplicative, and satisfy the Thom isomorphism property.

Theorem 1.5. *For a rank r vector bundle $p \colon V \to X$ on $X \in \mathcal{S}m/S$ with zero section $z \colon X \hookrightarrow V$, the map*

$$- \cup th(V) \colon E^{*,*}(X) \to E^{*+2r,*+r}\big(\mathrm{Th}(V)\big)$$

is an isomorphism of two-sided $E^{*,*}(X)$-modules, where $- \cup th(V)$ is written for the composition map $\big(- \cup th(V)\big) \circ p^*$.

Additionally we have a *normalization property*: $th(\mathbf{1}) = \Sigma_T(1) \in E^{2,1}(\mathrm{Th}(\mathbf{1}))$ as one can see from the relations (2) and (3) below. In fact,

$$\bar{th}(\mathbf{1}) = c(\mathcal{O}_{\mathbf{P}^1}(1)) = -c(\mathcal{O}_{\mathbf{P}^1}(-1)) = \Sigma_{\mathbf{P}^1}(1).$$

(The second relation here holds by [**6**, Lemma 3.6].) Thus $th(\mathbf{1}) = \Sigma_T(1)$.

Analogous to [**13**, p. 422], one obtains for vector bundles $V \to X$ and $W \to Y$ in $\mathcal{S}m/S$ a canonical map of pointed motivic spaces $\mathrm{Th}(V) \wedge \mathrm{Th}(W) \to \mathrm{Th}(V \times_S W)$, which is a motivic weak equivalence as defined in [**7**, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification. In the special case where $Y = S$ and $W = \mathbf{1}$ is the trivial line bundle, this motivic weak equivalence has the form $\mathrm{Th}(V) \wedge T \to Th(V \oplus \mathbf{1})$.

Corollary 1.6. *For $W = V \oplus \mathbf{1}$ consider the composite motivic weak equivalence*

$$\omega \colon \mathrm{Th}(V) \wedge \mathbf{P}^1 \to \mathrm{Th}(V) \wedge \mathbf{P}^1/\mathbf{A}^1 \leftarrow \mathrm{Th}(V) \wedge T \to \mathrm{Th}(W)$$

of pointed motivic spaces over S (see diagram (1) on page 213). The diagram

$$\begin{array}{ccc}
E^{*+2r,*+r}(\mathrm{Th}(V)) & \xrightarrow{\Sigma_{\mathbf{P}^1}} & E^{*+2r+2,*+r+1}(\mathrm{Th}(V)\wedge \mathbf{P}^1) \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \omega^*} \\
E^{*+2r,*+r}(\mathrm{Th}(V)) & \xrightarrow{\Sigma_T} & E^{*+2r+2,*+r+1}(\mathrm{Th}(W)) \\
\uparrow{\scriptstyle -\cup th(V)} & & \uparrow{\scriptstyle -\cup th(W)} \\
E^{*,*}(X) & \xrightarrow{\mathrm{id}} & E^{*,*}(X)
\end{array}$$

commutes.

Proof. The bottom square in this diagram commutes by the multiplicativity of Thom classes and the normalization property of the class $th(\mathbf{1})$. The top one commutes by definition. □

We conclude this section by recalling briefly how the associated theory of Thom classes is constructed. Given the Chern structure above, there is a unique theory of Chern classes $V \mapsto c_i(V) \in E^{2i,i}(X)$ such that for every line bundle L on X one has $c_1(L) = c(L)$. For a rank r vector bundle V over X consider the vector bundle $W := \mathbf{1} \oplus V$ and the associated projective space bundle $\mathbf{P}(W)$ of lines in W. Set

$$\bar{t}h(V) = c_r(p^*(V) \otimes \mathcal{O}_{\mathbf{P}(W)}(1)) \in E^{2r,r}(\mathbf{P}(W)). \tag{2}$$

It follows from [**6**, Cor. 3.18] that the support extension map

$$E^{2r,r}\big(\mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))\big) \to E^{2r,r}\big(\mathbf{P}(W)\big)$$

is injective and $\bar{t}h(E) \in E^{2r,r}\big(\mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))\big)$. Set

$$th(E) = j^*(\bar{t}h(E)) \in E^{2r,r}\big(\mathrm{Th}_X(V)\big), \tag{3}$$

where $j\colon \mathrm{Th}_X(V) \to \mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))$ is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding $V \hookrightarrow \mathbf{P}(W)$. The assignment V/X to $th(V)$ is a theory of Thom classes on $E^{*,*}$ (see the proof of [**6**, Thm. 3.35]). Moreover $th(\mathcal{O}(-1)) = th$ in $E^{2,1}(\mathbf{P}^\infty)$.

2. Cohomology of infinite Grassmannians

Let $\mathbf{Gr}(n, n+m)$ be the Grassmann scheme of n-dimensional linear subspaces of $\mathbf{A}^{n+m}_S$. The closed embedding $\mathbf{A}^{n+m}_S = \mathbf{A}^{n+m}_S \times \{0\} \hookrightarrow \mathbf{A}^{n+m+1}_S$ defines a closed embedding

$$\mathbf{Gr}(n, n+m) \hookrightarrow \mathbf{Gr}(n, n+m+1). \tag{4}$$

The tautological vector bundle is denoted $\mathcal{T}(n, n+m) \to \mathbf{Gr}(n, n+m)$. The closed embedding (4) is covered by a map $\mathcal{T}(n, n+m) \hookrightarrow \mathcal{T}(n, n+m+1)$ of vector bundles. Let $\mathbf{Gr}(n) = \mathrm{colim}_{m\geqslant 0}\, \mathbf{Gr}(n, n+m)$, $\mathcal{T}(n) = \mathrm{colim}_{m\geqslant 0}\, \mathcal{T}(n, n+m)$ and $\mathrm{Th}(\mathcal{T}(n)) = \mathrm{colim}_{m\geqslant 0}\, \mathrm{Th}(\mathcal{T}(n, n+m))$. These colimits are taken in the category of motivic spaces over S.

Remark 2.1. It is not difficult to prove that $E^{*,*}(\mathbf{Gr}(n, n+m))$ is multiplicatively generated by the Chern classes $c_i(\mathcal{T}(n, n+m))$ of the vector bundle $\mathcal{T}(n, n+m)$. This proves the surjectivity of the map $E^{*,*}(\mathbf{Gr}(n, n+m+1)) \to E^{*,*}(\mathbf{Gr}(n, n+m))$ and shows that the canonical map $E^{*,*}(\mathbf{Gr}(n)) \to \varprojlim E^{*,*}(\mathbf{Gr}(n, n+m))$ is an isomorphism. Thus for each i there exists a unique element $c_i = c_i(\mathcal{T}(n)) \in E^{2i,i}(\mathbf{Gr}(n))$, which for each m restricts to the element $c_i(\mathcal{T}(n, n+m))$ under the obvious pull-back map.

Theorem 2.2. *Let (E, c) be an oriented commutative $\mathbf{P}^1$-ring spectrum. Then*

$$E^{*,*}(\mathbf{Gr}(n)) = E^{*,*}(S)[[c_1, c_2, \ldots, c_n]]$$

is the formal power series ring, where $c_i := c_i(\mathcal{T}(n)) \in E^{2i,i}(\mathbf{Gr}(n))$ denotes the i-th Chern class of the tautological bundle $\mathcal{T}(n)$. The inclusion $\mathrm{inc}_n \colon \mathbf{Gr}(n) \hookrightarrow \mathbf{Gr}(n+1)$ satisfies $\mathrm{inc}_n^*(c_m) = c_m$ for $m < n+1$ and $\mathrm{inc}_n^*(c_{n+1}) = 0$.

Proof. The case $n = 1$ is well-known (see for instance [**6**, Thm. 3.9]). For a finite-dimensional vector space W and a positive integer m, let $\mathbf{F}(m, W)$ be the flag variety of flags $W_1 \subset W_2 \subset \cdots \subset W_m$ of linear subspaces of W such that the dimension of W_i is i. Let $\mathcal{T}^i(m, W)$ be the tautological rank i vector bundle on $\mathbf{F}(m, W)$.

Let $V = \mathbf{A}^\infty$ be an infinite-dimensional vector bundle over S and set $e = (1, 0, \ldots)$. Then V_n denotes the n-fold product of V, and $e_i^n \in V_n$ the vector $(0, \ldots, 0, e, 0, \ldots, 0)$ having e precisely at the i-th position. Let $F(m) = \mathrm{colim}_W\, \mathbf{F}(m, W)$ and let $\mathcal{T}^i(m) = \mathrm{colim}_W\, \mathcal{T}^i(m, W)$, where W runs over all finite-dimensional vector subspaces of V_n. Thus we have a flag $\mathcal{T}^1(m) \subset \mathcal{T}^2(m) \subset \cdots \subset \mathcal{T}^m(m)$ of vector bundles over $F(m)$. Set $L^i(m) = \mathcal{T}^i(m)/\mathcal{T}^{i-1}(m)$. It is a line bundle over $F(m)$.

Consider the morphism $p_m \colon F(m) \to F(m-1)$ which maps a flag $W_1 \subset W_2 \subset \cdots \subset W_m$ to the flag $W_1 \subset W_2 \subset \cdots \subset W_{m-1}$. If $W \subset V_n$ is a finite-dimensional vector subspace, then the restriction of $p_m \colon F(m) \to F(m-1)$ to $\mathbf{F}(m, W)$ is a projective space bundle over $\mathbf{F}(m-1, W)$. Thus there exists a tower of projective space bundles $F(m) \to F(m-1) \to \cdots \to F(1) = \mathbf{P}(V_n)$. The projective bundle theorem implies that

$$E^{*,*}(F(n)) = E^{*,*}(k)[[t_1, t_2, \ldots, t_n]]$$

(the formal power series in n variables), where $t_i = c(L^i(n))$ is the first Chern class of the line bundle $L^i(n)$ over $F(n)$.

Consider the morphism $q \colon F(n) \to \mathbf{Gr}(n)$, which sends the flag

$$W_1 \subset W_2 \subset \cdots \subset W_n$$

to the space W_n. It can be decomposed as a tower of projective space bundles. In particular, the pull-back map $q^* \colon E^{*,*}(\mathbf{Gr}(n)) \to E^{*,*}(F(n))$ is a monomorphism. It maps the class c_i to the symmetric polynomial

$$\sigma_i = t_1 t_2 \cdots t_i + \cdots + t_{n-i+1} \cdots t_{n-1} t_n.$$

So the image of q^* contains $E^{*,*}(k)[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$. It remains to check that the image of q^* is contained in $E^{*,*}(k)[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$. To do that consider another variety.

Namely, let V^0 be the n-dimensional subspace of V_n generated by the vectors e_i^n's. Let l_i^n be the line generated by the vector e_i^n. Let V_i^0 be a subspace of V^0 generated by all e_j^n's with $j \leqslant i$. So one has a flag $V_1^0 \subset V_2^0 \subset \cdots \subset V_n^0 = V^0$. We denote this flag F^0. For each vector subspace W in V_n containing V^0 consider three algebraic subgroups of the general linear group $\mathbb{GL}_W$. Namely, set

$$P_W = Stab(V^0),\ B_W = Stab(F^0),\ T_W = Stab(l_1^n, l_2^n, \dots, l_n^n).$$

The group T_W stabilizes each line l_i^n. Clearly, $T_W \subset B_W \subset P_W$ and $\mathbf{Gr}(n, W) = \mathbb{GL}_W/P_W$, $\mathbf{F}(n, W) = \mathbb{GL}_W/B_W$ Set $M(n, W) = \mathbb{GL}_W/T_W$. One has a tower of obvious morphisms

$$M(n, W) \xrightarrow{r_W} \mathbf{F}(n, W) \xrightarrow{q_W} \mathbf{Gr}(n, W).$$

Set $M(n) = \operatorname{colim}_W M(n, W)$, where W runs over all finite-dimensional subspaces W of V_n containing V^0. Now one has a tower of morphisms

$$M(n) \xrightarrow{r} F(n) \xrightarrow{q} \mathbf{Gr}(n).$$

The morphisms r_W can be decomposed in a tower of affine space bundles. Hence it induces an isomorphism on any cohomology theory. Choose a family

$$V_n^0 = W_0 \subset W_1 \subset W_2 \subset \cdots$$

of finite-dimensional subspaces of V_n such that $V_n = \cup W_i$. Then $F(n) = \cup \mathbf{F}(n, W_i)$ and $M(n) = \cup M(n, W_i)$. The short exact sequence

$$0 \to \varprojlim_{i \geqslant 0}{}^1 E^{*-1,*}\big(\mathbf{F}(n, W_i)\big) \to E^{*,*}\big(F(n)\big) \to \varprojlim_{i \geqslant 0} E^{*,*}\big(\mathbf{F}(n, W_i)\big) \to 0$$

as described in [**7**, Lemma A.34], and a similar sequence for $E^{*,*}$-groups of the spaces $M(n, W_i)$, show that the pull-back map

$$r^*\colon E^{*,*}(F(n)) \to E^{*,*}(M(n))$$

is an isomorphism. Permuting vectors e_i^n's yields an inclusion $\Sigma_n \subset \mathbb{GL}(V^0)$ of the symmetric group Σ_n in $\mathbb{GL}(V^0)$. The action of Σ_n by the conjugation on $\mathbb{GL}_W$ normalizes the subgroups T_W and P_W. Thus Σ_n acts as on $M(n)$ so on $\mathbf{Gr}(n)$ and the morphism $q \circ r\colon M(n) \to \mathbf{Gr}(n)$ respects this action. Note that the action of Σ_n on $\mathbf{Gr}(n)$ is trivial and the action of Σ_n on $E^{*,*}(M(n))$ permutes the variable $t_1, t_2, \dots, t_n$. Thus the image of $(q \circ r)^*$ is contained in $E^{*,*}(S)[[\sigma_1, \sigma_2, \dots, \sigma_n]]$. Whence the same holds for the image of q^*. The theorem is proven. □

The projection from the product $\mathbf{Gr}(m) \times \mathbf{Gr}(n)$, to the j-th factor is called p_j. For every integer $i \geqslant 0$ set $c_i' = p_1^*(c_i(\mathcal{T}(m)))$ and $c_i'' = p_2^*(c_i(\mathcal{T}(n)))$

Theorem 2.3. *Suppose E is an oriented commutative $\mathbf{P}^1$-ring spectrum. There is an isomorphism*

$$E^{*,*}\big((\mathbf{Gr}(m) \times \mathbf{Gr}(n))\big) = E^{*,*}(S)[[c_1', c_2', \dots, c_m', c_1'', c_2'', \dots, c_n'']],$$

where the right-hand side denotes the formal power series ring on c_i' and c_j'' with *coefficients in* $E^{*,*}(S)$*. The inclusion*

$$i_{m,n}\colon \mathbf{Gr}(m)\times\mathbf{Gr}(n)\hookrightarrow \mathbf{Gr}(m+1)\times\mathbf{Gr}(n+1)$$

satisfies

$$i_{m,n}^*(c'_r)=c'_r \quad \textit{for} \quad r<m+1, i_{m,n}^*(c'_{m+1})=0,$$

and

$$i_{m,n}^*(c''_r)=c''_r \quad \textit{for} \quad r<n+1, i_{m,n}^*(c''_{n+1})=0.$$

Proof. This follows as in the proof of Theorem 2.2. □

2.1. The symmetric ring spectrum representing algebraic cobordism

To give a construction of the symmetric $\mathbf{P}^1$-ring spectrum MGL, recall the external product of Thom spaces described in [**13**, p. 422]. For vector bundles $V\to X$ and $W\to Y$ in $\mathcal{S}m/S$, one obtains a canonical map of pointed motivic spaces $\mathrm{Th}(V)\wedge\mathrm{Th}(W)\to\mathrm{Th}(V\times_S W)$, which is a motivic weak equivalence as defined in [**7**, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification.

The algebraic cobordism spectrum appears naturally as a T-spectrum, not as a $\mathbf{P}^1$-spectrum. Hence we describe it as a symmetric T-ring spectrum and obtain a symmetric $\mathbf{P}^1$-ring spectrum (and in particular a $\mathbf{P}^1$-ring spectrum) by switching the suspension coordinate (see [**7**, A.6.9]). For $m,n\geqslant 0$, let $\mathcal{T}(n,mn)\to\mathbf{Gr}(n,mn)$ denote the tautological vector bundle over the Grassmann scheme of n-dimensional linear subspaces of $\mathbf{A}^{mn}_S=\mathbf{A}^m_S\times_S\cdots\times_S\mathbf{A}^m_S$. Permuting the copies of $\mathbf{A}^m_S$ induces a Σ_n-action on $\mathcal{T}(n,mn)$ and $\mathbf{Gr}(n,mn)$ such that the bundle projection is equivariant. The closed embedding $\mathbf{A}^m_S=\mathbf{A}^m_S\times\{0\}\hookrightarrow\mathbf{A}^{m+1}_S$ defines a closed Σ_n-equivariant embedding $\mathbf{Gr}(n,mn)\hookrightarrow\mathbf{Gr}(n,(m+1)n)$. In particular, $\mathbf{Gr}(n,mn)$ is pointed by $g_n\colon S=\mathbf{Gr}(n,n)\hookrightarrow\mathbf{Gr}(n,mn)$. The fiber of $\mathbf{Gr}(n,mn)$ over g_n is $\mathbf{A}^n_S$. Let $\mathbf{Gr}(n)$ be the colimit of the sequence

$$\mathbf{Gr}(n,n)\hookrightarrow\mathbf{Gr}(n,2n)\hookrightarrow\cdots\hookrightarrow\mathbf{Gr}(n,mn)\hookrightarrow\cdots$$

in the category of pointed motivic spaces over S. The pullback diagram

$$\begin{array}{ccc}\mathcal{T}(n,mn) & \longrightarrow & \mathcal{T}(n,(m+1)n)\\ \downarrow & & \downarrow\\ \mathbf{Gr}(n,mn) & \longrightarrow & \mathbf{Gr}(n,(m+1)n)\end{array}$$

induces a Σ_n-equivariant inclusion of Thom spaces

$$\mathrm{Th}(\mathcal{T}(n,mn))\hookrightarrow\mathrm{Th}(\mathcal{T}(n,(m+1)n)).$$

Let $\mathbb{MGL}_n$ denote the colimit of the resulting sequence

$$\mathbb{MGL}_n=\operatorname*{colim}_{m\geqslant n}\mathrm{Th}(\mathcal{T}(n,mn)) \tag{5}$$

with the induced Σ_n-action. There is a closed embedding

$$\mathbf{Gr}(n,mn)\times\mathbf{Gr}(p,mp)\hookrightarrow\mathbf{Gr}(n+p,m(n+p)), \tag{6}$$

which sends the linear subspaces $V \hookrightarrow \mathbf{A}^{mn}$ and $W \hookrightarrow \mathbf{A}^{mp}$ to the product subspace $V \times W \hookrightarrow \mathbf{A}^{mn} \times \mathbf{A}^{mp} = \mathbf{A}^{m(n+p)}$. In particular, (g_n, g_p) maps to g_{n+p}. The inclusion (6) is covered by a map of tautological vector bundles and thus gives a canonical map of Thom spaces

$$\mathrm{Th}(\mathcal{T}(n, mn)) \wedge \mathrm{Th}(\mathcal{T}(p, mp)) \to \mathrm{Th}(\mathcal{T}(n+p, m(n+p))), \tag{7}$$

which is compatible with the colimit (5). Furthermore, the map (7) is $\Sigma_n \times \Sigma_p$-equivariant, where the product acts on the target via the standard inclusion $\Sigma_n \times \Sigma_p \subseteq \Sigma_{n+p}$. After taking colimits, the result is a $\Sigma_n \times \Sigma_p$-equivariant map

$$\mu_{n,p} \colon \mathbb{MGL}_n \wedge \mathbb{MGL}_p \to \mathbb{MGL}_{n+p} \tag{8}$$

of pointed motivic spaces (see [**13**, p. 422]). The inclusion of the fiber $\mathbf{A}^p$ over g_p in $\mathcal{T}(p)$ induces an inclusion $\mathrm{Th}(\mathbb{A}^p) \subset \mathrm{Th}(\mathcal{T}(p)) = \mathbb{MGL}_p$. Precomposing it with the canonical Σ_p-equivariant map of pointed motivic spaces,

$$\mathrm{Th}(\mathbb{A}^1) \wedge \mathrm{Th}(\mathbb{A}^1) \wedge \cdots \wedge \mathrm{Th}(\mathbb{A}^1) \to \mathrm{Th}(\mathbb{A}^p)$$

defines a family of maps $e_p \colon (\Sigma_T^\infty S_+)_p = T^{\wedge p} \to \mathbb{MGL}_p$. Inserting it in the inclusion (8) yields $\Sigma_n \times \Sigma_p$-equivariant structure maps

$$\mathbb{MGL}_n \wedge \mathrm{Th}(\mathbb{A}^1) \wedge \mathrm{Th}(\mathbb{A}^1) \wedge \cdots \wedge \mathrm{Th}(\mathbb{A}^1) \to \mathbb{MGL}_{n+p} \tag{9}$$

of the symmetric T-spectrum $\mathbb{MGL}$. The family of $\Sigma_n \times \Sigma_p$-equivariant maps (8) form a commutative, associative and unital multiplication on the symmetric T-spectrum $\mathbb{MGL}$ (see [**3**, Sect. 4.3]). Regarded as a T-spectrum it coincides with Voevodsky's spectrum **MGL** described in [**13**, 6.3].

Let $\overline{T}$ be the Nisnevich sheaf associated to the presheaf $X \mapsto \mathbf{P}^1(X)/(\mathbf{P}^1 - \{0\})(X)$ on the Nisnevich site $\mathcal{S}m/S$. The canonical covering of $\mathbf{P}^1$ supplies an isomorphism

$$T = \mathrm{Th}(\mathbf{A}^1_S) \xrightarrow{\cong} \overline{T}$$

of pointed motivic spaces. This isomorphism induces an isomorphism $\mathbf{MSS}_T(S) \cong \mathbf{MSS}_{\overline{T}}(S)$ of the categories of symmetric T-spectra and symmetric $\overline{T}$-spectra. In particular, $\mathbb{MGL}$ may be regarded as a symmetric $\overline{T}$-spectrum by just changing the structure maps up to an isomorphism. Note that the isomorphism of categories respects both the symmetric monoidal structure and the model structure. The canonical projection $p \colon \mathbf{P}^1 \to \overline{T}$ is a motivic weak equivalence, because $\mathbf{A}^1$ is contractible. It induces a Quillen equivalence

$$\mathbf{MSS}(S) = \mathbf{MSS}_{\mathbf{P}^1}(S) \underset{p^*}{\overset{p_\sharp}{\rightleftarrows}} \mathbf{MSS}_{\overline{T}}(S)$$

when equipped with model structures as described in [**3**] (see [**7**, A.6.9]). The right adjoint p^* is very simple: it sends a symmetric $\overline{T}$-spectrum E to the symmetric $\mathbf{P}^1$-spectrum having terms $\big(p^*(E)\big)_n = E_n$ and structure maps

$$E_n \wedge \mathbf{P}^1 \xrightarrow{E_n \wedge p} E \wedge \overline{T} \xrightarrow{\text{structure map}} E_{n+1} \ .$$

In particular $\mathrm{MGL} := p^*\mathbb{MGL}$ is a symmetric $\mathbf{P}^1$-spectrum by just changing the structure maps. Since p^* is a lax symmetric monoidal functor, MGL is a commutative monoid in a canonical way. Finally, the identity is a left Quillen equivalence from the model category $\mathbf{MSS}^{\mathrm{cm}}(S)$ used in [**7**] to Jardine's model structure by the proof of [**7**, A.6.4]. Let $\gamma\colon \mathrm{Ho}(\mathbf{MSS}^{\mathrm{cm}}(S)) \to \mathrm{SH}(S)$ denote the equivalence obtained by regarding a symmetric $\mathbf{P}^1$-spectrum just as a $\mathbf{P}^1$-spectrum.

Definition 2.4. Let $(\mathrm{MGL}, \mu_{\mathrm{MGL}}, e_{\mathrm{MGL}})$ denote the commutative $\mathbf{P}^1$-ring spectrum, which is the image $\gamma(\mathrm{MGL})$ of the commutative symmetric $\mathbf{P}^1$-ring spectrum MGL in the motivic stable homotopy category $\mathrm{SH}(S)$.

2.2. Cohomology of the algebraic cobordism spectrum

Let (E, th) be an oriented commutative $\mathbf{P}^1$-ring spectrum and let $V \mapsto th(V)$ be the Thom classes theory given by equation (3). We will compute $E^{*,*}(\mathrm{MGL})$ and $E^{*,*}(\mathrm{MGL} \wedge \mathrm{MGL})$ in this short section.

By [**7**, Cor. 2.1.4], the group $E^{*,*}(\mathrm{MGL})$ fits into the short exact sequence

$$0 \to \varprojlim{}^1 E^{*+2i-1,*+i}(\mathrm{Th}(\mathcal{T}(i))) \to E^{*,*}(\mathrm{MGL}) \to \varprojlim E^{*+2i,*+i}(\mathrm{Th}(\mathcal{T}(i))) \to 0,$$

where the connecting maps in the tower are given by the top line of the commutative diagram

$$\begin{array}{ccccc}
E^{*+2i,*+i}(\mathrm{Th}(i)) & \xleftarrow{\Sigma^{-1}_{\mathbf{P}^1}} & E^{*+2i+2,*+i+1}(\mathrm{Th}(i)\wedge\mathbf{P}^1) & \xleftarrow{\sigma^*} & E^{*+2i+2,*+i+1}(\mathrm{Th}(i+1)) \\
\uparrow{\scriptstyle -\cup th(\mathcal{T}(i))} & & \uparrow{\scriptstyle \omega^*\circ(-\cup th(\mathcal{T}(i)\oplus\mathbf{1}))} & & \uparrow{\scriptstyle -\cup th(\mathcal{T}(i+1))} \\
E^{*,*}(\mathbf{Gr}(i)) & \xleftarrow{\mathrm{id}} & E^{*,*}(\mathbf{Gr}(i)) & \xleftarrow{\mathrm{inc}_i^*} & E^{*,*}(\mathbf{Gr}(i+1)).
\end{array} \tag{10}$$

Here $\omega\colon \mathrm{Th}(V) \wedge \mathbf{P}^1 \to Th(V \oplus \mathbf{1})$ is the canonical map described in Corollary 1.6 and $\sigma\colon \mathrm{Th}_i \wedge \mathbf{P}^1 \to \mathrm{Th}_{i+1}$ is the structure map of the $\mathbf{P}^1$-spectrum MGL. The pull-backs inc_i^* are all surjective by Theorem 1.5. So we proved the following

Lemma 2.5. *The canonical map*

$$E^{*,*}(\mathrm{MGL}) \to \varprojlim E^{*+2i,*+i}(\mathrm{Th}(\mathcal{T}(i))) = E^{*,*}(S)[[c_1, c_2, c_3, \ldots]]$$

is an isomorphism of two-sided $E^{*,*}(S)$-modules.

To compute $E^{*,*}(\mathrm{MGL} \wedge \mathrm{MGL})$, recall that the group $E^{*,*}(\mathrm{MGL} \wedge \mathrm{MGL})$ fits into the short exact sequence

$$\begin{aligned} 0 &\to \varprojlim{}^1 E^{*+4i-1,*+2i}(\mathrm{Th}(\mathcal{T}(i)) \wedge \mathrm{Th}(\mathcal{T}(i))) \to E^{*,*}(\mathrm{MGL} \wedge \mathrm{MGL}) \\ &\to \varprojlim E^{*+4i,*+2i}(\mathrm{Th}(\mathcal{T}(i)) \wedge \mathrm{Th}(\mathcal{T}(i))) \to 0 \end{aligned}$$

by [**7**, Cor. 2.1.5]. Note that since $\mathrm{Th}(\mathcal{T}(i)) \wedge \mathrm{Th}(\mathcal{T}(i)) \cong \mathrm{Th}(\mathcal{T}(i) \times \mathcal{T}(i))$, there is a Thom isomorphism $E^{*+4i-1,*+2i}(\mathrm{Th}(\mathcal{T}(i) \times \mathcal{T}(i))) \cong E^{*-1,*}(\mathbf{Gr}(i) \times \mathbf{Gr}(i))$ by Theorem 1.5. The $\varprojlim^1$-group is trivial because the connecting maps coincide with the pull-back maps

$$E^{*-1,*}(\mathbf{Gr}(i+1) \times \mathbf{Gr}(i+1)) \to E^{*-1,*}(\mathbf{Gr}(i) \times \mathbf{Gr}(i))$$

and these are surjective by Theorem 2.3. This implies the following

Lemma 2.6. *The canonical map*

$$E^{*,*}(\mathrm{MGL} \wedge \mathrm{MGL}) \to \varprojlim E^{*+4i,*+2i}(\mathrm{Th}(\mathcal{T}(i)) \wedge \mathrm{Th}(\mathcal{T}(i))) = E^{*,*}(S)[[c'_1, c''_1, c'_2, c''_2, \ldots]]$$

is an isomorphism of $E^{*,*}(S)$-modules. Here c'_i is the i-th Chern class coming from the first factor of $\mathbf{Gr} \times \mathbf{Gr}$ *and c''_i is the i-th Chern class coming from the second factor.*

2.3. A universality theorem for the algebraic cobordism spectrum

The complex cobordism spectrum, equipped with its natural orientation, is a universal oriented ring cohomology theory by Quillen's universality theorem [**10**]. In this section we prove a motivic version of Quillen's universality theorem. Over a field, the statement is contained already in [**12**]. Recall that the $\mathbf{P}^1$-ring spectrum MGL carries a canonical orientation th^{MGL} as defined in Example 1.4. It is the canonical map

$$th^{\mathrm{MGL}} \colon \Sigma^\infty_{\mathbf{P}^1} \mathrm{Th}(\mathcal{T}(1))(-1) \to \mathrm{MGL}$$

of $\mathbf{P}^1$-spectra.

Theorem 2.7 (Universality Theorem)**.** *Let E be a commutative $\mathbf{P}^1$-ring spectrum. The assignment*

$$\varphi \mapsto \varphi(th^{\mathrm{MGL}}) \in E^{2,1}(\mathrm{Th}(\mathcal{T}(1)))$$

identifies the set of homomorphisms

$$\varphi \colon \mathrm{MGL} \to E \tag{11}$$

of $\mathbf{P}^1$-ring spectra in the motivic stable homotopy category $\mathrm{SH}(S)$ with the set of orientations of E. The inverse bijection sends an orientation $th \in E^{2,1}(\mathrm{Th}(\mathcal{T}(1)))$ to the unique morphism

$$\varphi \in E^{0,0}(\mathrm{MGL}) = \mathrm{Hom}_{\mathrm{SH}(S)}(\mathrm{MGL}, E)$$

such that $u_i^*(\varphi) = th(\mathcal{T}(i)) \in E^{2i,i}(\mathrm{Th}(\mathcal{T}(i)))$, where $th(\mathcal{T}(i))$ is given by (3) and $u_i \colon \Sigma^\infty_{\mathbf{P}^1} \mathrm{Th}(\mathcal{T}(i))(-i) \to \mathrm{MGL}$ is the canonical map of $\mathbf{P}^1$-spectra.

Proof. Let $\varphi \colon \mathrm{MGL} \to E$ be a homomorphism of monoids in $\mathrm{SH}(S)$. The class $th := \varphi(th^{\mathrm{MGL}})$ is an orientation of E, because

$$\varphi(th)|_{Th(\mathbf{1})} = \varphi(th|_{Th(\mathbf{1})}) = \varphi(\Sigma_{\mathbf{P}^1}(1)) = \Sigma_{\mathbf{P}^1}(\varphi(1)) = \Sigma_{\mathbf{P}^1}(1).$$

Now suppose $th^E \in E^{2i,i}(\mathrm{Th}(\mathcal{O}(-1)))$ is an orientation of E. Let $V \mapsto th(V)$ be the Thom classes theory given by equation (3). We will construct a monoid homomorphism $\varphi \colon \mathrm{MGL} \to E$ in $\mathrm{SH}(S)$ such that $u_i^*(\varphi) = th(\mathcal{T}(i))$ and prove its uniqueness. To do so recall that the canonical map $E^{*,*}(\mathrm{MGL}) \to \varprojlim E^{*+2i,*+i}(\mathrm{Th}(\mathcal{T}(i)))$ is an isomorphism by Lemma 2.5. The connecting maps in the tower are given by the top line of diagram (10). The family of elements $th(\mathcal{T}(i))$ is an element in the $\varprojlim$-group because diagram (10) commutes. Thus there is a unique element $\varphi \in E^{0,0}(\mathrm{MGL})$ with $u_i^*(\varphi) = th(\mathcal{T}(i))$.

We claim that φ is a monoid homomorphism. To check that it respects the multiplicative structure, consider the diagram

$$\begin{array}{ccc} \Sigma^\infty_{\mathbf{P}^1}\mathrm{Th}(\mathfrak{T}(i))(-i)\wedge\Sigma^\infty_{\mathbf{P}^1}\mathrm{Th}(\mathfrak{T}(j))(-j) & \xrightarrow{\Sigma^\infty_{\mathbf{P}^1}(\mu_{i,j})(-i-j)} & \Sigma^\infty_{\mathbf{P}^1}\mathrm{Th}(\mathfrak{T}(i+j))(-i-j) \\ \downarrow{\scriptstyle u_i\wedge u_j} & & \downarrow{\scriptstyle u_{i+j}} \\ \mathrm{MGL}\wedge\mathrm{MGL} & \xrightarrow{\mu_{\mathrm{MGL}}} & \mathrm{MGL} \\ \downarrow{\scriptstyle \varphi\wedge\varphi} & & \downarrow{\scriptstyle \varphi} \\ E\wedge E & \xrightarrow{\mu_E} & E. \end{array}$$

Its enveloping square commutes in $\mathrm{SH}(S)$ by the chain of relations

$$\begin{aligned} \varphi\circ u_{i+j}\circ\Sigma^\infty_{\mathbf{P}^1}(\mu_{i,j})(-i-j) &= \mu^*_{i,j}(th(\mathfrak{T}(i+j))) = th(\mu^*_{i,j}(\mathfrak{T}(i+j))) = th(\mathfrak{T}(i)\times\mathfrak{T}(j)) \\ &= th(\mathfrak{T}(i))\times th(\mathfrak{T}(j)) = \mu_E(th(\mathfrak{T}(i))\wedge th(\mathfrak{T}(j))) \\ &= \mu_E\circ((\varphi\circ u_i)\wedge(\varphi\circ u_j)). \end{aligned}$$

The canonical map $E^{*,*}(\mathrm{MGL}\wedge\mathrm{MGL})\to\varprojlim E^{*+4i,*+2i}(\mathrm{Th}(\mathfrak{T}(i))\wedge\mathrm{Th}(\mathfrak{T}(i)))$ is an isomorphism by Lemma 2.6. Now the equality

$$\varphi\circ u_{i+i}\circ\Sigma^\infty_{\mathbf{P}^1}(\mu_{i,i})(-2i) = \mu_E\circ((\varphi\circ u_i)\wedge(\varphi\circ u_i))$$

shows that $\mu_E\circ(\varphi\wedge\varphi)=\varphi\circ\mu_{\mathrm{MGL}}$ in $\mathrm{SH}(S)$.

To prove the theorem it remains to check that the two assignments described in the theorem are inverse to each other. An orientation $th\in E^{2,1}(\mathrm{Th}(\mathcal{O}(-1)))$ induces a morphism φ such that for each i one has $\varphi\circ u_i = th(\mathfrak{T}_i)$. The new orientation $th' := \varphi(th^{\mathrm{MGL}})$ coincides with the original one, because of the chain of relations

$$th' = \varphi(th^{\mathrm{MGL}}) = \varphi(u_1) = \varphi\circ u_1 = th(\mathfrak{T}(1)) = th(\mathcal{O}(-1)) = th.$$

On the other hand a homomorphism φ of $\mathbf{P}^1$-ring spectra defines an orientation $th := \varphi(th^{\mathrm{MGL}})$ of E. The monoid homomorphism φ' we obtain then satisfies $u_i^*(\varphi') = th(\mathfrak{T}(i))$ for every $i\geqslant 0$. To check that $\varphi'=\varphi$, recall that MGL is oriented, so we may use Lemma 2.5 with $E=\mathrm{MGL}$ to deduce an isomorphism

$$\mathrm{MGL}^{*,*}(\mathrm{MGL})\to\varprojlim\mathrm{MGL}^{*+2i,*+i}(\mathrm{Th}(\mathfrak{T}(i))).$$

This isomorphism shows that the identity $\varphi'=\varphi$ will follow from the identities $u_i^*(\varphi') = u_i^*(\varphi)$ for every $i\geqslant 0$. Since $u_i^*(\varphi') = th(\mathfrak{T}_i)$ it remains to check the relation $u_i^*(\varphi) = th(\mathfrak{T}(i))$. It follows from the

Lemma 2.8. *There is an equality* $u_i = th^{\mathrm{MGL}}(\mathfrak{T}(i)\in\mathrm{MGL}^{2i,i}(\mathrm{Th}(\mathfrak{T}(i)))$.

In fact, $u_i^*(\varphi)=\varphi\circ u_i=\varphi(u_i)=\varphi(th^{\mathrm{MGL}}(\mathfrak{T}(i)))=th(\mathfrak{T}(i))$. The last equality in this chain of relations holds, because φ is a monoid homomorphism sending th^{MGL} to th. It remains to prove Lemma 2.8. We will do this in the case $i=2$. The general case can be proved similarly. The commutative diagram

$$\begin{array}{ccc} \Sigma^\infty_{\mathbf{P}^1}\mathrm{Th}(\mathfrak{T}(1))(-1)\wedge\Sigma^\infty_{\mathbf{P}^1}\mathrm{Th}(\mathfrak{T}(1))(-1) & \xrightarrow{\Sigma^\infty_{\mathbf{P}^1}(\mu_{1,1})(-2)} & \Sigma^\infty_{\mathbf{P}^1}\mathrm{Th}(\mathfrak{T}(2))(-2) \\ \downarrow{\scriptstyle u_1\wedge u_1} & & \downarrow{\scriptstyle u_2} \\ \mathrm{MGL}\wedge\mathrm{MGL} & \xrightarrow{\mu_{\mathrm{MGL}}} & \mathrm{MGL} \end{array}$$

in $\mathrm{SH}(k)$ implies that

$$\mu_{1,1}^*(u_2) = u_1 \times u_1 \in \mathrm{MGL}^{4,2}(\mathrm{Th}(\mathcal{T}(1)) \wedge \mathrm{Th}(\mathcal{T}(1))) = \mathrm{MGL}^{4,2}(\mathrm{Th}(\mathcal{T}(1) \times \mathcal{T}(1))).$$

The equalities

$$\begin{aligned} \mu_{1,1}^*(th^{\mathrm{MGL}}(\mathcal{T}(2))) &= th^{\mathrm{MGL}}(\mu_{1,1}^*(\mathcal{T}(2))) = th^{\mathrm{MGL}}(\mathcal{T}(1) \times \mathcal{T}(1)) \\ &= th^{\mathrm{MGL}}(\mathcal{T}(1)) \times th^{\mathrm{MGL}}(\mathcal{T}(1)) \end{aligned}$$

imply that it remains to prove the injectivity of the map $\mu_{1,1}^*$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{MGL}^{*,*}(\mathrm{Th}(\mathcal{T}(1) \times \mathcal{T}(1))) & \xleftarrow{\mu_{1,1}^*} & \mathrm{MGL}^{*,*}(\mathrm{Th}(\mathcal{T}(2))) \\ \text{Thom} \uparrow \cong & & \cong \uparrow \text{Thom} \\ \mathrm{MGL}^{*,*}(\mathbf{Gr}(1) \times \mathbf{Gr}(1)) & \xleftarrow{\nu_{1,1}^*} & \mathrm{MGL}^{*,*}(\mathbf{Gr}(2)), \end{array}$$

where the vertical arrows are the Thom isomorphisms from Theorem 1.5 and $\nu_{1,1} \colon \mathbf{Gr}(1) \times \mathbf{Gr}(1) \hookrightarrow \mathbf{Gr}(2)$ is the embedding described by equation (6). For an oriented commutative $\mathbf{P}^1$-ring spectrum (E, th), one has $E^{*,*}(\mathbf{Gr}(2)) = E^{*,*}(S)[[c_1, c_2]]$ (the formal power series on c_1, c_2) by Theorem 2.2. On the other hand,

$$E^{*,*}(\mathbf{Gr}(1) \times \mathbf{Gr}(1)) = E^{*,*}(S)[[t_1, t_2]]$$

(the formal power series on t_1, t_2) by Theorem 2.3 and the map $\nu_{1,1}^*$ sends c_1 to $t_1 + t_2$ and c_2 to $t_1 t_2$. Whence $\nu_{1,1}^*$ is injective. The proofs of Lemma 2.8 and of Theorem 2.7 are complete. □

3. Universality of MGL and formal group laws

In this section the universal property of the $\mathbf{P}^1$-spectrum MGL will be described in terms of formal group laws. Fix a commutative $\mathbf{P}^1$-ring spectrum E and a homomorphism $\varphi \colon \mathrm{MGL} \to E$ of $\mathbf{P}^1$-ring spectra over S. Let z be the zero section of the line bundle $\mathcal{O}(-1)$ over $\mathbf{P}^\infty$. Then $c^{\mathrm{MGL}} = z^*(th^{\mathrm{MGL}})$ is a Chern orientation of MGL and $c = \varphi(c^{\mathrm{MGL}})$ is a Chern orientation of E. The Chern orientation c defines in the standard way a formal group law F over the commutative ring $E^{2*,*}(S)$ (see for instance [**6**, Defn. 3.39] and set $F := F^-$, where F^- is the formal group law corresponding to the class $c(\mathcal{O}(-1))$).

If $\varphi_{\mathrm{new}} \colon \mathrm{MGL} \to E$ is another homomorphism of $\mathbf{P}^1$-ring spectra, then the element $c_{\mathrm{new}} := \varphi_{\mathrm{new}}(c^{\mathrm{MGL}}) \in E^{2,1}(\mathbf{P}^\infty)$ defines another formal group law F_{new}. Moreover it defines a unique formal power series $\Phi(t) \in E^{2*,*}(S)$ such that $c_{\mathrm{new}} = \Phi(c)$. It is straightforward to check that $\Phi(t)$ is of the form $t + b_1 t^2 + b_2 t^3 + \cdots$ with $b_i \in E^{-2i,-i}(S)$ and $\Phi(F(t_1, t_2)) = F_{\mathrm{new}}(\Phi(t_1), \Phi(t_2))$. In other words, $\Phi(t)$ is an isomorphism $F \to F_{\mathrm{new}}$ of formal group laws.

Theorem 3.1. *Let (E, c) be an oriented commutative $\mathbf{P}^1$-ring spectrum over S. The assignment $\varphi_{\mathrm{new}} \mapsto (F_{\mathrm{new}}, \Phi(t))$ is a bijection from the set of all homomorphisms $\mathrm{MGL} \to E$ of $\mathbf{P}^1$-ring spectra in $\mathrm{SH}(S)$ to the set of all pairs $\big(F'(t_1, t_2), \Psi(t)\big)$, where F' is a formal group law over the ring $E^{2*,*}(S)$ and $\Psi(t) \colon F(t_1, t_2) \to F'(t_1, t_2)$ is an isomorphism of formal group laws as above.*

Proof. Consider the set of all formal power series $\Psi(t) \in E^{2*,*}(S)[[t]]$ of the form described above. This set forms a group under the substitution of the power series: $(\Psi_2 \circ \Psi_1)(t) := \Phi_2(\Psi_1(t))$. The series t is the unit of this group. For a series Ψ in this group we will write Ψ^{-1} for its inverse.

By straightforward calculation one may check that the assignments

$$(F'(t_1,t_2), \Psi(t)) \mapsto \Psi(t) \text{ and } \Psi(t) \mapsto (\Psi(F(\Psi^{-1}(t_1), \Psi^{-1}(t_2))), \Psi(t))$$

are mutually inverse bijections of the set of all pairs from the theorem with the set of all formal power series $\Psi(t) \in E^{2*,*}(S)[[t]]$ such that $\Psi(t) = t + b_1t^2 + b_2t^3 + \cdots$, with $b_i \in E^{-2i,-i}(S)$ for all i. Secondly, note that the set of all formal power series $\Psi(t) \in E^{2*,*}(S)[[t]]$ such that $\Psi(t) = t + b_1t^2 + b_2t^3 + \cdots$ with $b_i \in E^{-2i,-i}(S)$ is in a bijective correspondence with the set of all Chern orientations $c' \in E^{2,1}(\mathbf{P}^\infty)$ of E. Namely, a formal power series $\Psi(t)$ as above maps to the Chern orientation $\Psi(c) \in E^{2,1}(\mathbf{P}^\infty)$. Given a Chern orientation c' of E, let $\Psi(t) \in E^{2*,*}(S)[[t]]$ be the unique formal power series such that $c' = \Psi(c)$. This supplies two mutually inverse bijections.

To prove the theorem it remains to check that the assignment $\varphi_{\mathrm{new}} \mapsto \varphi_{\mathrm{new}}(c^{\mathrm{MGL}})$ is a bijection of the set of all homomorphisms $\mathrm{MGL} \to E$ of $\mathbf{P}^1$-ring spectra with the set of all Chern orientations of E.

To do that, recall that the assignment $\varphi_{\mathrm{new}} \mapsto \varphi_{\mathrm{new}}(th^{\mathrm{MGL}})$ is a bijection of the set of all ring morphisms $\varphi_{\mathrm{new}}\colon \mathrm{MGL} \to E$ with the set of all Thom orientations of E (see Theorem 2.7). As well the set of Thom orientations of E is in in bijection with the set of Chern orientations via the assignment $th \mapsto z^*(th)$ (see [**6**, Thm. 3.5]). Clearly $z^*(\varphi_{\mathrm{new}}(th^{\mathrm{MGL}})) = \varphi_{\mathrm{new}}(c^{\mathrm{MGL}})$. Thus the assignment $\varphi_{\mathrm{new}} \mapsto \varphi_{\mathrm{new}}(c^{\mathrm{MGL}})$ is indeed a bijection, which completes the proof. □

Remark 3.2. The bijection inverse to $\varphi_{\mathrm{new}} \mapsto (F_{\mathrm{new}}, \Phi(t))$ is given as follows. Take $c_{\mathrm{new}} := \Phi(c)$, construct a Thom classes theory using formulas (2) and (3), and let $\varphi\colon \mathrm{MGL} \to E$ be the unique homomorphism of $\mathbf{P}^1$-ring spectra such that for every n the composition $\Sigma^\infty_{\mathbf{P}^1} \mathrm{Th}(\mathcal{T}(n))(-n) \xrightarrow{u_n} \mathrm{MGL} \xrightarrow{\varphi} E$ coincides with the Thom class $th(\mathcal{T}(n))$ of the bundle $\mathcal{T}(n)$ (here u_n is the canonical morphism from Theorem 2.7).

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Ivan Panin `paniniv@gmail.com`

Universität Bielefeld, SFB 701, Bielefeld, Germany, and Steklov Institute of Mathematics at St. Petersburg, Russia

Konstantin Pimenov `kip302002@yahoo.com`

Steklov Institute of Mathematics at St. Petersburg, Russia

Oliver Röndigs `oroendig@math.uni-osnabrueck.de`

Institut für Mathematik, Universität Osnabrück, Osnabrück, Germany