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STEENROD OPERATIONS ON THE NEGATIVE CYCLIC HOMOLOGY OF THE SHC-COCHAIN ALGEBRAS

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Abstract

In this paper we prove that the Steenrod operations act naturally on the negative cyclic homology of a differential graded algebra A over the prime field $\mathbb{F}_p$ satisfying some extra conditions. When A denotes the singular cochains with coefficients in $\mathbb{F}_p$ of a 1-connected space X, these extra conditions are satisfied. The Jones isomorphism identifies these Steenrod operations with the usual ones on the S^1-equivariant cohomology of the free loop space on X with coefficients in $\mathbb{F}_p$. We conclude by performing some calculations on the negative cyclic homology.

1. Introduction

Since their construction by N. Steenrod [**22**], Steenrod operations have played a central role in homotopy theory and in representation theory. In the topological setting, Steenrod operations $\{P^i\}_{i\in\mathbb{N}}$ are stable natural transformations

$$P^i\colon H^*(-;\mathbb{F}_p)\to\begin{cases}H^{*+i}(-;\mathbb{F}_p) & \text{if } p=2\\ H^{*+(p-1)i}(-;\mathbb{F}_p) & \text{if } p \text{ is an odd prime}\end{cases}$$

where $H^*(-;\mathbb{F}_p)$ denotes the singular cohomology functor with coefficients in the prime field $\mathbb{F}_p$. When $p=2$, P^i is called an i-Steenrod square and usually denoted by Sq^i, while when p is an odd prime, P^i is called an i-Steenrod power. These transformations satisfy the following properties:

1. $P^0 = id$.
2. $P^i_{|_{H^k(-,\mathbb{F}_p)}} = 0$, (resp. ξ) if $\begin{cases} i>k & (\text{resp. } i=k)\ p=2\\ i>2k & (\text{resp. } i=2k)\ p \text{ is an odd prime.}\end{cases}$
3. $P^k(-\cup-)=\sum_{i+j=k}P^i-\cup P^j-$, (the Cartan formula).
4. The Adem relations (see [**16**], pages 129 and 367).

Here id (respectively 0) denotes the identity transformation (respectively the constant transformation whose value is 0) while ξ denotes the Frobenius transformation $x\mapsto$

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x^p. A. Dold [**4**] defined them in a more general context replacing the singular chains on a topological space by an arbitrary simplicial coalgebra. Later P. May [**15**] gave a purely algebraic construction of Steenrod operations which leads to the notion of E_∞-algebra, as recently developed by Mandell [**14**] in homotopy theory, and to the construction of Steenrod operations in other settings. For example, there exist Steenrod operations on

- the cohomology of a commutative $\mathbb{F}_p$-Hopf algebra due to A. Liulevicius [**12**], after the paper of Dold quoted above;
- the cohomology of a restricted p-Lie algebra due to P. May [**15**];
- the cohomology of non commutative p-differential forms due to M. Karoubi [**11**];
- the cyclic cohomology of a commutative $\mathbb{F}_p$-Hopf algebra due to M. Elhamdadi and Y. Gouda [**5**].

Let us recall here that if $\{V^i\}_{i\in\mathbb{N}}$ is a graded $\mathbb{F}_p$-vector space and if $T^c(sV)$ denotes the free coalgebra generated by the suspension of V, denoted sV, then:

- V is an A_∞-algebra if there exists a degree 1 coderivation D on $T^c(sV)$ such that $D\circ D = 0$ and $D_{|_{T^0(sV)}} = 0$.
- V is a B_∞-algebra if it is an A_∞-algebra and if there exists a product on $T^c(sV)$ such that $T^c(sV)$ is a differential graded Hopf algebra.
- V is a C_∞-algebra if it is an A_∞-algebra such that $T^c(sV)$ is a differential graded Hopf algebra for the shuffle product.

While it is possible to define the Hochschild homology (and the negative cyclic homology) of an A_∞-algebra [**9, 8**], the lack of associativity of these operadic algebras complicates explicit computations. Hopefully, strongly homotopy commutative algebras (*shc* for short), as introduced by H.J. Munkholm [**17**], will considerably simplify the above mentioned calculations. They are associative B_∞-algebras. Moreover, it is known that:

- The normalized singular cochain complex with coefficients in $\mathbb{F}_p$ of a connected space X, $C^*(X;\mathbb{F}_p)$, is a *shc*-algebra [**17**].
- The Hochschild homology of a *shc*-algebra A with coefficients in A, $HH_*(A;A)$, is a graded algebra [**20**].
- The negative cyclic homology of a *shc*-algebra A, $HC_*^-(A)$, is a graded algebra [**18**].
- Let X be a 1-connected space and LX be the free loop space. That is, $LX = \mathrm{Map}(S^1, X)$ is the space of continuous maps from S^1 to X endowed with the compact open topology. The Jones isomorphism $HH_*(C^*(X,\mathbb{F}_p); C^*(X,\mathbb{F}_p)) \to H^*(LX;\mathbb{F}_p)$ is a homomorphism of graded algebras [**20**].
- Let X be a 1-connected space and LX the free loop space. The Jones isomorphism $HC_*^-(C^*(X,\mathbb{F}_p)) \to H^*_{S^1}(LX;\mathbb{F}_p)$ is a homomorphism of graded algebras [**18**].

Here $H^*_{S^1}(LX;\mathbb{F}_p)$ denotes the S^1-equivariant cohomology of LX with coefficients in $\mathbb{F}_p$.

B. Ndombol and Jean-Claude Thomas [**21**] have introduced the notion of π-*shc*-algebra and have proved that

- The normalized singular cochain complex with coefficients in $\mathbb{F}_p$ of a connected space X, $C^*(X;\mathbb{F}_p)$, is a π-*shc*-algebra [**21**].
- There exist Steenrod operations on the Hochschild homology of a π-*shc*-algebra A with coefficients in A.
- Let X be a 1-connected space and LX be the free loop space.
  The Jones isomorphism $HH_*(C^*(X,\mathbb{F}_p);C^*(X,\mathbb{F}_p)) \to H^*(LX;\mathbb{F}_p)$ respects the Steenrod operations.

In this paper we complete the above result in proving:

Theorem 1.1. *Let $((A,d_A),\mu_A,\kappa_A)$ be a π-*shc *cochain algebra as in [**21**].*

1. *The negative cyclic homology of a differential graded π-*shc *algebra A with coefficients in A, $HC_*^-(A)$, has algebraic Steenrod operations.*
2. *Let X be a 1-connected space and LX be the free loop space.*
   *The Jones isomorphism $HC_*^-(C^*(X,\mathbb{F}_p)) \to H^*_{S^1}(LX;\mathbb{F}_p)$ respects the Steenrod operations. (See [**18, 10**].)*

The Steenrod operations, considered in our theorem, are defined at the chain level and satisfy the properties:

1. $P^0(1) = 1$ if 1 denotes the unit of the graded algebra $HC_*^-(A)$.
2. $P^i_{|_{HC_k^-(-,\mathbb{F}_p)}} = 0$, (resp. ξ) if $\begin{cases} i > k & \text{(resp. } i = k\text{) } p = 2 \\ i > 2k & \text{(resp. } i = 2k\text{) } p \text{ is an odd prime.} \end{cases}$
3. $P^k(- \cup -) = \sum_{i+j=k} P^i \cup P^j$, the Cartan formula.

Except if $A = C^*(X;\mathbb{F}_p)$, the Steenrod operations constructed by Ndombol-Thomas or those considered in part 1 of our theorem do not in general satisfy the Adem relations. An operadic construction as in [**2**] or [**1**] allows us to define an action of the large Steenrod algebra on the Hochschild homology of a E_∞-algebra. Such an action on the negative cyclic homology remains an open question. In these notes, quasi-isomorphism means a homomorphism which is an isomorphism in (co)homology.

The paper is organized as follows. Section 2 is a recollection of definitions. Part 1 (respectively Part 2) of Theorem 1.1 is proved in Section 3 (respectively Section 4). Recalling the π-*shc*-minimal model and explicit computations are the subjects of Section 5 and Section 6 respectively. This paper is a part of my thesis supervised by professors B. Ndombol and J. C. Thomas of Yaounde I University, Cameroon, and Angers University, France respectively.

Convention

Throughout this paper, we use the Kronecker convention: an object with lower negative graduation has upper non-negative graduation.

2. Preliminaries

Let π be any finite group and p a fixed prime. Throughout this paper, we work over the field $\mathbb{F}_p$ equipped with the trivial action of π. The ring group $\mathbb{F}_p[\pi]$ is an augmented algebra.

2.1. Algebraic Steenrod operations

The material involved here is contained in [**17**]. Let $\pi = \{1, \tau, \dots, \tau^{p-1}\}$ be the cyclic group of order p. Let $W \stackrel{\varepsilon_W}{\to} \mathbb{F}_p$ be a projective resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[\pi]$; that is $W = (W_i)_{i\geqslant 0}$; $W_i \stackrel{\delta_i}{\longrightarrow} W_{i-1}$; $W_0 \simeq \mathbb{F}_p[\pi]$, where each W_i is a right projective $\mathbb{F}_p[\pi]$-module and δ_i is π-linear. We choose $\mathbb{F}_p \stackrel{\eta_W}{\longrightarrow} W$ such that $\varepsilon_W \circ \eta_W = id_{\mathbb{F}_p}$. Necessarily $\eta_W \circ \varepsilon_W \simeq id_W$.

Let $A = \{A\}_{i\in\mathbb{Z}}$ be a differential graded algebra (not necessarily associative). We denote by $m_A^{(p)}$ (resp. $(Hm_A)^{(p)}$) the iterated product $a_1 \otimes a_2 \otimes \cdots \otimes a_p \mapsto a_1 \cdot (a_2(\cdots a_p))$ (resp. the iterated product induced on HA by $m_A^{(p)}$).

Identify $\tau \in \pi$ with the p-cycle $(p, 1, \dots, p-1)$ and assume that π acts trivially on A, thus π acts diagonally on $A^{\otimes p}$ and on $W \otimes A^{\times p}$.

If the natural map $H(W \otimes A^{\otimes p}) \stackrel{\cong}{\longrightarrow} H(A)^{\otimes p} \stackrel{(Hm_A)^{(p)}}{\longrightarrow} HA$ lifts to a π-linear chain map $\theta\colon W \otimes A^{\otimes p} \longrightarrow A$, then for any $i \in \mathbb{Z}$ and $x \in H^n A$, there exists a well defined class

$$P^i(x) \in \begin{cases} H^{n+i}(A) & \text{if } p = 2 \\ H^{n+2i(p-1)}(A) & \text{if } p > 2, \end{cases}$$

such that:

1. $$P^i(1_{HA}) = 0 \text{ if } i \neq 0.$$

2. $$\text{If } p = 2, \begin{cases} P^i(x) = 0 & \text{if } i > n \\ P^i(x) = x^2 & \text{if } i = n. \end{cases}$$

3. $$\text{If } p > 2, \begin{cases} P^i(x) = 0 & \text{if } 2i > n \\ P^i(x) = x^p & \text{if } n = 2i. \end{cases}$$

Moreover these classes do not depend on the choice of the resolution W nor η and are compatible with algebra homomorphisms commuting with structural map θ.

These operations do not in general satisfy $P^i(x) = 0$ if $i < 0$, $P^0(x) = x$, the Cartan formulas and the Adem relations.

2.1.1. Cartan formula

Let us consider the differential graded algebra $A = \{A^i\}_{i\in\mathbb{Z}}$ such that $A^i = 0$ for $i < 0$. A homogeneous map $W \stackrel{f}{\longrightarrow} A$ has a degree k if $f \in \mathrm{Hom}^k(W, A) = \prod_{i\geqslant 0} \mathrm{Hom}(W_i, A^{k-i}) = \bigoplus_{i=0}^k \mathrm{Hom}(W_i, A^{k-i})$.

The differential of f is $D(f) = d \circ f - (-1)^{|f|} f \circ \delta$; π acts on each $\mathrm{Hom}^k(W, A)$ by $(\sigma f)(w) = f(w\sigma)$; the evaluation map

$$\begin{aligned} \mathrm{Hom}(W, A) &\stackrel{ev_0}{\longrightarrow} A \\ f &\longmapsto ev_0(f) = f(e_0) \end{aligned}$$

is a homomorphism of chain complexes.

Let $W \xrightarrow{\psi_W} W \otimes W$ be any diagonal approximation and denote by m_A the product on the algebra A. We have the cup-product

$$\overset{k}{\mathrm{Hom}}(W,A) \otimes \overset{l}{\mathrm{Hom}}(W,A) \xrightarrow{\cup} \overset{k+l}{\mathrm{Hom}}(W,A)$$
$$f \otimes g \longmapsto f \cup g = m_A \circ (f \otimes g) \circ \psi_W$$

that defines a nonassociative differential graded algebra structure on $\mathrm{Hom}(W, A)$.

Proposition 2.1 ([**21**])**.** *If $A = \{A^i\}_{i\in\mathbb{Z}}$ is differential graded algebra such that $A^i = 0$ if $i < 0$ and θ the structural map as in 2.1, then:*

1. *The structural map $W \otimes A^{\otimes p} \xrightarrow{\theta} A$ induces a π-chain map*

$$A^{\otimes p} \xrightarrow{\tilde{\theta}} \mathrm{Hom}(W,A)$$
$$u \longmapsto \begin{cases} W & \xrightarrow{\tilde{\theta}(u)} A \\ w & \longmapsto \tilde{\theta}(u)(w) = (-1)^{|u||w|}\theta(w \otimes u) \end{cases}$$

   such that $ev_0 \circ \tilde{\theta} = m_A^{(p)}$ and $H(ev_0) \circ H(\tilde{\theta}) = H(m_A)^{(p)}$.

2. *If we assume that $H(\tilde{\theta})$ respects the products, the algebraic Steenrod operations defined by $\tilde{\theta}$ satisfy the Cartan formula $P^i(xy) = \sum_{j+i=i} P^j(x)P^k(y), x, y \in H^*A$.*

2.1.2. Review of the construction of Steenrod operations

We consider the standard small free resolution of $\pi = \{1, \tau, \dots, \tau^{p-1}\}$:

$$W = (W_i)_{i\geqslant 0}; \quad W_i = e_i\mathbb{F}_p[\pi]; \quad W_0 \simeq \mathbb{F}_p[\pi],$$
$$W_i \xrightarrow{\delta_i} W_{i-1}, \quad \delta(e_{2i+1}) = (1+\tau)e_{2i}, \quad \delta(e_{2i}) = (1+\tau+\cdots+\tau)e_{2i-1}, \quad [\mathbf{21}]$$

$$W \xrightarrow{\varepsilon_W} \mathbb{K}$$
$$e_i \longmapsto \varepsilon_W(e_i) = \begin{cases} 0 & \text{if } i \geqslant 0 \\ 1_{\mathbb{K}} & \text{if } i = 0. \end{cases}$$

Note that this standard free resolution equipped with its diagonal approximation ψ_W has a coalgebra structure.

Let $\theta_\pi \colon W \otimes_\pi A^{\otimes p} \longrightarrow A$ be the map induced by the structural map θ and denote by θ^* the homomorphism $H(\theta_\pi)$. Observe that any section ρ of a natural projection $A \cap kerd \longrightarrow H(A)$ lifts to a π-linear chain map $\rho \colon W \otimes (HA)^{\otimes p} \longrightarrow W \otimes A^{\otimes p}$ and thus to a chain map $\rho_\pi \colon W \otimes_\pi (HA)^{\otimes p} \longrightarrow W \otimes_\pi A^{\otimes p}$. Since W is a semifree $\mathbb{F}_p$-module in the sense of [**7**], $\rho^* = H(\rho_\pi)$ is an isomorphism. The algebraic Steenrod operations are defined as follows [**15**]; for $x \in H^n(A)$, each $e_k \otimes x^{\otimes p}$ is a cocycle in $W \otimes_\pi (HA)^{\otimes p}$.

If $p = 2$,

$$\begin{aligned} Sq^i(x) &= \theta^* \circ \rho^*(cl(e_{n-i} \otimes_\pi x^{\otimes p})) \\ &= cl(\theta_\pi \circ \rho_\pi(e_{n-i} \otimes_\pi x^{\otimes p})) \\ &= cl(\theta_\pi(e_{n-i} \otimes_\pi \rho_\pi(x^{\otimes p}))) \\ &= cl(\theta_\pi(e_{n-i} \otimes_\pi \rho_\pi(x)^{\otimes p})) \\ &= cl(\tilde{\theta}(\rho(x)^{\otimes p}))(e_{n-i}) \end{aligned}$$

and if p is odd,

$$\begin{aligned} P^i(x) &= (-1)^i \nu(n) \theta^* \circ \rho^*(cl(e_{(n-2i)(p-1)-1} \otimes x^{\otimes p})) \\ &= (-1)^i \nu(n) cl(\tilde{\theta}(\rho(x)^{\otimes p}))(e_{(n-2i)(p-1)}) \\ \beta P^i(x) &= (-1)^i \nu(n) cl(\tilde{\theta}(\rho(x)^{\otimes p}))(e_{(n-2i)(p-1)-1}) \end{aligned}$$

where $\nu(n) = (-1)^j ((\frac{p-1}{2})!)^\epsilon$ if $n = 2j + \epsilon$, $\epsilon = 0, 1$ and $\tilde{\theta}$ the π-chain map defined in Proposition 2.1.

2.1.3. Hochschild homology and negative cyclic homology
Here, Hochschild homology and negative cyclic homology are recalled.

Let DA and DC denote respectively the category of connected cochain algebras and the category of connected cochain coalgebras. The reduced bar and cobar construction are a pair of adjoint functors $B\colon DA \leftrightarrow DC\colon \Omega$ (see [**6**]). The generators of BA (resp. ΩC) are denoted $[a_1|a_2|\cdots|a_k] \in B_k A$ (resp. $\langle c_1|c_2|\cdots|c_l\rangle \in \Omega_l C$) and $[\,] = 1 \in B_0 A \simeq \mathbb{K}$ (resp. $\langle\rangle = 1 \in \Omega_0 C \simeq \mathbb{K}$).

The adjunction mentioned above yields for a cochain algebra (A, d_A), a natural quasi-isomorphism of cochain algebras $\alpha_A\colon \Omega BA \longrightarrow A$ [**17**]. The linear map $\iota_A\colon A \longrightarrow \Omega BA$ such that $\iota_A(1) = 1$, $\iota_A(a) = \langle [a] \rangle$, $a \in \bar{A}$ is a chain complex quasi-isomorphism. In any case, it satisfies $\alpha_A \circ \iota_A = id_A$, $id_{\Omega BA} - \iota_A \circ \alpha_A = d_{\Omega BA} \circ h + h \circ d_{\Omega BA}$ for some chain homotopy $h\colon \Omega BA \to \Omega BA$ such that $\alpha_A \circ h = 0$, $h \circ \iota_A = 0$, $h^2 = 0$.

Let (A, d_A) be a cochain algebra. Recall also that the normalized Hochschild chain complex of (A, d_A) is a graded vector space $\{\mathfrak{C}_k A\}_{k \geqslant 0}$, $\mathfrak{C}_k A = A \otimes B_k A$ where the generators of $\mathfrak{C}_k A$ are of the form $a_0[a_1|a_2|\ldots|a_k]$ if $k > 0$ and $a[\,]$ if $k = 0$. We set $\epsilon_i = |a_0| + |sa_1| + |sa_2| + \cdots + |sa_{i-1}|$, $i \geqslant 1$ and define the Hochschild differential $d = d^1 + d^2$ by

$$d^1(a_0[a_1|\cdots|a_k]) = d_A(a_0)[a_1|a_2|\cdots|a_k] - \sum_i = 1^k (-1)^\epsilon_i a_0[a_1|a_2|\cdots|d_A(a_i)|\cdots|a_k]$$

$$\begin{aligned} d^2(a_0[a_1|\cdots|a_k]) &= (-1)^{|a_0|} a_0 a_1 [a_2|\cdots|a_k] \\ &\quad + \sum_{i=2}^{k} a_0[a_1|a_2|\cdots|a_{i-1}a_i|\cdots|a_k] \\ &\quad - (-1)^{|sa_k|\epsilon_k} a_k a_0 [a_2|\cdots|a_k]. \end{aligned}$$

The Hochschild differential decreases the degree by one (see [**13**] or [**20**] for more details).

By definition,

$$HH_*A := H_*\mathfrak{C}A$$

is the **Hochschild homology** of the cochain algebra (A, d_A). It is clear that $\mathfrak{C}A$ is concentrated in non-negative total degrees. Hence so is HH_*A.

If $(A, d_A) = (N^*(X;\mathbb{K}), d_{N^*(X;\mathbb{K})})$ is the algebra of normalized singular cochains on the topological space X, then $\mathfrak{C}_*N^*(X;\mathbb{K})$ is the normalized Hochschild chain complex of X and $HH_*X := HH_*N^*(X;\mathbb{K})$ is the Hochschild homology of X.

For the cochain algebra (A, d_A), the Connes operator is the linear map

$$B\colon \mathfrak{C}_*A \to \mathfrak{C}_{*+1}A$$

defined by $Ba_0[a_1|\cdots|a_n] = \sum_{i=0}^{n}(-1)^{\epsilon_i}1[a_i|\cdots|a_n|a_0|\cdots|a_{i-1}]$, where $\epsilon_i = |a_0| + (|a_0| + |a_1| + \cdots + |a_{i-1}| + i)(|a_i| + \cdots + |a_n| + n - i + 1)$. Consider the polynomial algebra $\mathbb{K}[u]$ on the single generator u of upper degree $+2$ and form the complex $C_*^- A = \mathbb{K}[u] \otimes \mathfrak{C}_*A$ with differential $\mathfrak{D}$ defined by $\mathfrak{D}(u^l \otimes a_0[a_1|\cdots|a_n]) = u^l \otimes d(a_0[a_1|\cdots|a_n]) + u^{l+1} \otimes B(a_0[a_1|\cdots|a_n])$. The chain complex $C_*^- A$ is the negative cyclic chain complex of the cochain algebra (A, d_A) (see [**10**]). Let L and M be two graded $\mathbb{F}_p$-modules; $L\hat{\otimes}M$ will denote the tensor product defined by $(L\hat{\otimes}M)_n = \prod L_i \otimes M_{n-i}$. Generally for a differential graded algebra A, $C_*^- A = \mathbb{F}_p[u]\hat{\otimes}_{\mathbb{F}_p} A$. So, for example, an element of degree d is given by an infinite sum of the form $\sum u^i \otimes e_i$ where $e_i \in A_{d+2i}$ [**3**]. If A is positively graded, $C_*^- A = \mathbb{F}_p[u]\hat{\otimes}_{\mathbb{F}_p} A \cong \mathbb{F}_p[u]\otimes_{\mathbb{F}_p} A$, and its homology $HC^- A$ is the negative cyclic homology of (A, d_A).

Again, it is clear that $C_*^- A$ is concentrated in non-negative total degrees and so is $HC_*^- A$.

If $(A, d_A) = N^*(X;\mathbb{K})$ is the algebra of normalized singular cochains on the topological space X, then $C_*^- N^*(X;\mathbb{K})$ is the negative cyclic chain complex of X, and $HC^- X := HC_*^- N^*(X;\mathbb{K})$ the associated negative cyclic homology.

If $(A; d_A)$ is commutative (in the graded sense), then the multiplication $m_A\colon A \otimes A \to A$ is a homomorphism of DG-algebras. Thus the composite $\mathfrak{C}m_A \circ sh\colon \mathfrak{C}A \otimes \mathfrak{C}A \to \mathfrak{C}A$ defines a multiplication on $\mathfrak{C}A$ which makes it into a commutative algebra [**13**, 4.2.2], where $sh\colon \mathfrak{C}A \otimes \mathfrak{C}A \to \mathfrak{C}(A \otimes A)$ denotes the shuffle map.

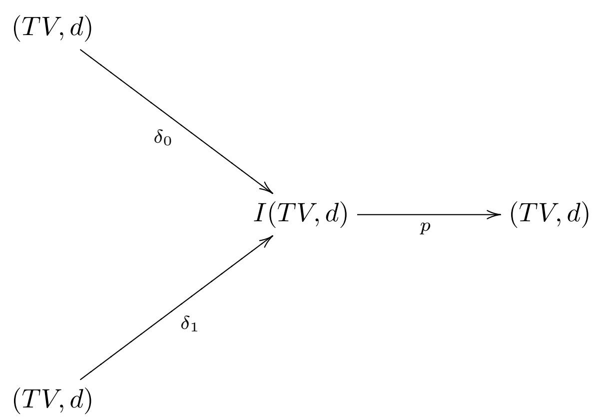
2.1.4. Homotopy

1. Recall that $f, g \in \mathrm{DA}(A, A')$ are homotopic in $\mathrm{D}A$ if there exists a linear map $h\colon A \to A'$ such that $f - g = d_{A'} \circ h + h \circ d_A$ and $h(xy) = h(x)g(y) + (-1)^{|x|}f(x)h(y)$ with $x,\ y \in A$. If $f, g \in \mathrm{DA}(A, A')$ are homotopic, we write $f \simeq_{DA} g$.
2. Let (A, d_A) (resp. (C, d_C)) be a differential graded algebra (resp. coalgebra). Let $T(C, A) = \{t \in \mathrm{Hom}^1(C, A)\colon Dt = t \cup t, t \circ \eta_C = 0 = \varepsilon_A \circ t\}$ be the twisting cochain space as in [**17**, 1.8], where D denotes the differential in $\mathrm{Hom}(C, A)$ and $\cup$ the usual cup-product on $\mathrm{Hom}(C, A)$.
   The universal twisting cochains $t, t' \in T(C, A)$ are homotopic in $TC(C, A)$ if there exists a linear map $h \in \mathrm{Hom}^1(C, A)$ such that $Dh = t \cup h - h \cup t$, $h \circ \eta_C = \eta_A$ and $\varepsilon A \circ h = \varepsilon_C$ and we write $t \simeq_T t'$ [**17**, 1.11].
3. Denote by π-DM the category whose objects are differential graded modules over $\mathbb{F}_p$ equipped with an action of the cyclic group π and whose morphisms are

$\mathbb{F}_p[\pi]$-linear. If the $\mathbb{F}_p[\pi]$-linear maps f and g are homotopic with a $\mathbb{F}_p[\pi]$-linear homotopy, we write $f \simeq_{\pi-DM} g$.
Denote by π-DA the subcategory whose objects are differential graded algebras over $\mathbb{F}_p$ equipped with an action of the cyclic group π and whose morphisms are linear morphisms of $\mathbb{F}_p[\pi]$-differential graded algebras. If the maps $f, g \in \pi$-DA are homotopic with a $\mathbb{F}_p[\pi]$-linear homotopy, we write $f \simeq_{\pi-DA} g$.

Lemma 2.2. *Let (TV, d_V) be a differential graded algebra and assume that a finite group π acts freely on V. Let (A, d_A) be a $\mathbb{F}_p[\pi]$-differential graded algebra and f, $g \in \pi$-DA(TV,A). if $f \simeq_{\pi-DA} g$ then $C^- f \simeq_{\pi-DM} C^- g$.*

Proof. We consider as in [**21**, Lemma A.6], the cylinder object $I(TV, d) := (T(V_0 \oplus V_1 \oplus sV))$ on (TV, d):



with $\delta_0(V) = V_0$, $\delta_1(V) = V_1$, $p(v_0) = p(v_0) = v$, $p(sv_0) = 0$, $D = d$ on V_0 and V_1, $Dsv = dSv$ where S is the unique (δ_0, δ_1)-derivation $S\colon TV \longrightarrow T(V_0 \oplus V_1 \oplus sV)$ extending the graded isomorphism $s\colon V \longrightarrow sV$. The free π-action on TV naturally extends to a free π-action on $I(TV, d)$ so that $I(TV, d)$ is a π-free algebra and the maps p, δ_0, δ_1 are π-equivariant quasi-isomorphisms of differential graded algebras. Moreover from the free π-action on the cylinder object $I(TV, d)$, we have a free π-action on the negative cyclic complex $C_*^- I(TV)$ of the cylinder object $I(TV)$ by the following rule:
For any $u^l \otimes x_0[x_1 \mid x_2 \mid \cdots \mid x_{n-1} \mid x_n] \in C_*^- I(TV)$ and $\sigma \in \pi$,

$$\sigma \cdot u^l \otimes x_0[x_1 \mid x_2 \mid \cdots \mid x_{n-1} \mid x_n] = u^l \otimes \sigma \cdot x_0[\sigma \cdot x_1 \mid \sigma \cdot x_2 \mid \cdots \mid \sigma \cdot x_{n-1} \mid \sigma \cdot x_n].$$

Thus $C_*^- I(TV)$ is an object of the category π-DM.
By definition, $f \simeq_{DA} g$ (resp. $f \simeq_{\pi-DA} g$) if and only if there exists $H \in DA(I(TV), A)$ (resp. $H \in \pi - DA(I(TV), A)$) such that $H \circ \delta_0 = f$ and $H \circ \delta_1 = g$.
If $H \in \pi$-DA(I(TV)) is a homotopy between f and g, then $C_*^- H$ is a π-linear homotopy between $C_*^- f$ and $C_*^- g$. Hence $C_*^- f \simeq_{\pi-DM} C_*^- g$. $\square$

2.1.5. A strongly homotopy commutative algebra (see [**17**, 4])
A strongly homotopy commutative algebra (*shc-algebra* for short) is a triple (A, d_A, μ_A) with $(A, d_A) \in \mathrm{Obj}DA$ and $\mu_A \in DA(\Omega B(A \otimes A), \Omega BA)$ satisfying

1. $\alpha_A \circ \mu_A \circ \iota_{A\otimes A} = m_A$, where m_A is the product on A;
2. $\alpha_A \circ \mu_A \circ \Omega B(id_A \otimes \eta_A) \circ \iota_A = \alpha_A \circ \mu_A \circ \Omega B(\eta_A \otimes id_A) \circ \iota_A = id_A$, where $\mathbb{K} \xrightarrow{\eta_A} A$ is the unit in A; i.e. η_A is the unit up to homotopy for μ_A;
3. $\mu_A \circ \Omega B(\alpha_A \otimes id_A) \circ \Omega B(\mu_A \otimes id_A) \circ \chi_{(A\otimes A)\otimes A} \simeq_{DA} \mu_A \circ \Omega B(id_A \otimes \alpha_A) \circ \Omega B(id_A \otimes \mu_A) \circ \chi_{A\otimes(A\otimes A)}$; i.e. μ_A is associative up to homotopy;
4. $\mu_A \circ \Omega BT \simeq_{DA} \mu_A$; where T denotes the interchange map $T(x \otimes y) = (-1)^{|x|\,|y|} y \otimes x$; i.e. μ_A is commutative up to homotopy.

The following natural homomorphisms of DG-algebras are defined in [**17**, 2.2];

$$\Omega B(\Omega B(A \otimes A) \otimes A) \xleftarrow{\chi_{(A\otimes A)\otimes A}} \Omega B(A \otimes A \otimes A) \xrightarrow{\chi_{A\otimes(A\otimes A)}} \Omega B(A \otimes \Omega B(A \otimes A)),$$

and satisfy

$$\alpha_{(A\otimes A)\otimes A} \circ \chi_{(A\otimes A)\otimes A} = \alpha_{A\otimes A\otimes A} = \alpha_{A\otimes(A\otimes A)} \circ \chi_{A\otimes(A\otimes A)}.$$

Consider A and A' in $\mathrm{Obj}DA$. The map $f \in DA(A, A')$ is a *shc*-map from (A, d_A, μ_A) to $(A', d'_A, \mu_{A'})$ if

1. $\alpha_{A'} \circ \Omega Bf \circ \iota_A = f$;
2. $\alpha_{A'} \circ \Omega Bf \circ \eta_{\Omega BA} = \eta_{A'}$;
3. $\Omega Bf \circ \mu_A \simeq_{DA} \mu'_A \circ \Omega B(f \otimes f)$.

As proved by [**17**], an example of *shc*-cochain algebra is the algebra $N^*(X; \mathbb{K})$ of normalized singular cochains of a topological space X.

On the other hand it is proved in [**20**] that if (A, d_A, μ_A) is a *shc*-cochain algebra, then

1. BA is a differential graded Hopf algebra and H^*BA is a commutative graded Hopf algebra;
2. $\mathfrak{C}A$ is a (non associative) graded algebra such that HH_*A is a commutative graded algebra;
3. if $f\colon (A, d_A, \mu_A) \to (A', d'_A, \mu_{A'})$ is a morphism of *shc*-cochain algebras, we have the commutative diagram $\begin{array}{ccccc} A & \xrightarrow{i} & \mathfrak{C}A & \xrightarrow{\rho} & BA \\ f\downarrow & & \downarrow \mathfrak{C}f & & \downarrow Bf \\ A' & \xrightarrow{i'} & \mathfrak{C}A' & \xrightarrow{\rho'} & BA' \end{array}$, where i, i', ρ, ρ' and $\mathfrak{C}f$ are homomorphisms of cochain algebras and Bf is a homomorphism of differential graded Hopf algebras.

Remark 2.3. We recall the following facts given in [**21**, A.2]:

1- If $((A, d_A), \mu_A)$ is a *shc* differential graded algebra, so is $\Omega B(A)$, with the *shc* structural map $\mu_{\Omega B(A)}$ given by the composite $\mu_{\Omega B(A)} = \theta_{\Omega B(A)} \circ \mu_A \circ \Omega B(\alpha_A \otimes \alpha_A)$, where $\theta_{\Omega B(A)} \in DA(\Omega B(A), \Omega B\Omega B(A))$ is the unique section of $\alpha_{\Omega B(A)} \in DA(\Omega B\Omega B(A), \Omega B(A))$ such that $\alpha_A \circ \alpha_{\Omega B(A)} \circ \theta_{\Omega B(A)} = \alpha_A$.

2- If $((A,d_A),\mu_A)$ is a *shc* differential graded algebra and $((W,\delta_W),\psi_W)$ a standard small free resolution of π equipped with its differential graded coalgebra structure, then $\mathrm{Hom}(W;A)$ is a *shc* differential graded algebra with the *shc* structural map $\mu_{\mathrm{Hom}(W;A)} \in DA(\Omega B([\mathrm{Hom}(W;A)]^{\otimes 2}), \Omega B([\mathrm{Hom}(W;A)]))$ defined by

$$\mu_{\mathrm{Hom}(W;A)} = \Omega B(\mathrm{Hom}(W, \alpha_A \circ \mu_A)) \circ \theta_{\mathrm{Hom}(W;A^{\otimes 2})} \circ \Omega B(\psi_A),$$

where $[\mathrm{Hom}(W;A)]^{\otimes 2} \xrightarrow{\psi_A} [\mathrm{Hom}(W;A^{\otimes 2})]$ the map defined by $\psi_A(f\otimes g) = (f\otimes g)\circ \psi_W$ is the homomorphism of differential graded algebras satisfying $\mathrm{Hom}(W,m_A)\circ\psi_A = \cup$ and $\theta_{\mathrm{Hom}(W,A^{\otimes 2})} \in DA(\Omega B\,\mathrm{Hom}(W,A\otimes A), \Omega B(\mathrm{Hom}$ $(W,\Omega B(A\otimes A))))$ the unique homomorphism of differential graded algebras such that

$$\mathrm{Hom}(W,\alpha_{A^{\otimes 2}}) \circ \alpha_{\mathrm{Hom}(W,\Omega B(A^{\otimes 2}))} \circ \theta_{\mathrm{Hom}(W,A^{\otimes 2})} = \alpha_{\mathrm{Hom}(W,A\otimes A)}.$$

2.1.6. *Shc*-equivalence and *shc*-formality [**20**, 5]
The *shc* cochain algebras (A,d_A,μ_A) and $(A',d_{A'},\mu_{A'})$ are said to be *shc*-equivalent $(A \simeq_{shc} A')$ if there exists a sequence of *shc* morphisms $A \leftarrow A_1 \to \cdots \to A'$ which are quasi-isomorphisms. One particular case of *shc*-equivalence is the *shc*-formality. Recall that the cohomology algebra of a *shc* cochain algebra is commutative. Every commutative cochain algebra is a *shc* algebra with *shc* structural map $\mu_A = \Omega B(m_A)\colon \Omega B(A \otimes A) \to \Omega BA$, where m_A is the product on A.

A *shc* cochain algebra A is *shc*-formal if it is *shc*-equivalent to its cohomology algebra H^*A.

2.1.7. A π-strongly homotopy commutative algebra; see [**21**, 1.6]
Let (A,d_A,μ_A) and $(A',d_{A'},\mu_{A'})$ be two *shc* differential graded algebras. There exists a natural homomorphism $\mu_{A\otimes A'} \in DA(\Omega B((A\otimes A')^{\otimes 2}); \Omega B(A\otimes A'))$ such that $(A \otimes A', d_{A\otimes A'}; \mu_{A\otimes A'})$ is a *shc* differential graded algebra.

In particular, if (A,d_A,μ_A) is a *shc* differential graded algebra, then for any $n \geqslant 2$, there exists a homomorphism of differential graded algebras called the *shc* iterated structural map $\Omega B(A^{\otimes n}) \xrightarrow{\mu_A^{(n)}} \Omega B(A)$ such that $\mu^{(2)} = \mu$ and $\alpha_A \circ \mu^{(n)} \circ i_{A^{\otimes n}} \simeq m_A^{(n)}$ (see [**21**, Lemma A.3].

A *shc*-algebra (A,d_A,μ_A) is a π-strongly homotopy commutative algebra (a π-*shc*-algebra for short) if there exists a map $\Omega B(A^{\otimes p}) \xrightarrow{\tilde\kappa_A} \mathrm{Hom}(W,A)$ that is a π-linear homomorphism of differential graded algebras such that

$$ev_0 \circ \tilde\kappa_A \simeq_{DA} \alpha_A \circ \mu_A^{(p)},$$

where p is the prime number characteristic of $\mathbb{F}_p$.

The action of S_p on $B(A^{\otimes p})$ (resp. $\Omega B(A^{\otimes p})$) is defined by the rule: For any $\sigma \in S_p$ $\sigma[a_1|a_2|\cdots|a_{p-1}|a_p] = [a_{\sigma(1)}|a_{\sigma(2)}|\cdots|a_{\sigma(p-1)}|a_{\sigma(p)}]$, $a_i \in A^{\otimes p}$ (resp. $\sigma < x_1|x_2|\cdots|x_{p-1}|x_p > = < \sigma x_1|\sigma x_2|\cdots|\sigma x_{p-1}|\sigma x_p >$, $x_i \in B(A^{\otimes p})$).

Recall that a strict π-*shc* homomorphism $((A,d_A),\mu_A,\tilde\kappa_A) \xrightarrow{f} ((A',d_{A'}),\mu_{A'},\tilde\kappa_{A'})$ is a strict *shc* homomorphism such that the following homomorphisms of differential graded algebras $\tilde\kappa_{A'} \circ \Omega B(f^{\otimes p})$ and $\mathrm{Hom}(W,f)\circ\tilde\kappa_A$ are π-linear homotopic.

3. Proof of the first part of Theorem 1.1

As explained in [**13**, 4.3], a (p,q)-cyclic shuffle is a permutation $\{\sigma(1),\dots,\sigma(p),\sigma(p+1),\dots,\sigma(p+q)\}$ in S_{p+q} obtained as follows: Perform a cyclic permutation of any order on the set $\{1,\dots,p\}$ and perform a cyclic permutation of any order on the set $\{p+1,\dots,p+q\}$. We shuffle the two results to obtain $\{\sigma(1),\dots\sigma(p),\sigma(p+1),\dots,\sigma(p+q)\}$ in S_{p+q}. The permutation obtained in that way is a cyclic shuffle if 1 appears before $p+1$; we denote by $\sum_{p,q}^{C}$ the set of (p,q)-cyclic shuffles.

A map $\perp\colon \mathfrak{C}_p(A,A)\otimes\mathfrak{C}_q(A,A)\to\mathfrak{C}_{p+q}(A^{\otimes 2},A^{\otimes 2})$ is defined by

$$a_0[a_1|a_2|\cdots|a_{p-1}|a_p]\perp b_0[b_1|b_2|\cdots|b_{q-1}|b_q] = \sum_{\sigma\in\sum_{(p,q)}^{C}}(-1)^{\varepsilon(\sigma)}a_0\otimes b_0[c_{\sigma(1)}|c_{\sigma(2)}|\cdots|c_{\sigma(p+q-1)}|c_{\sigma(p+q)}],$$

where $\varepsilon(\sigma)$ is the signature of σ and

$$c_{\sigma(i)}=\begin{cases}a_{\sigma(i)}\otimes 1 & \text{if } 1\leqslant i\leqslant p\\ 1\otimes b_{\sigma(i)-p} & \text{if } p+1\leqslant i\leqslant p+q.\end{cases}$$

The cyclic shuffle (see [**13**, 4.3.2]) is a linear map

$$\mathfrak{C}_*(A,A)\otimes\mathfrak{C}_{*'}(A,A)\xrightarrow{sh'}\mathfrak{C}_{*+*'+2}(A^{\otimes 2},A^{\otimes 2})$$

defined by

$$sh'(a_0[a_1|a_2|\cdots|a_{p-1}|a_p]\otimes b_0[b_1|b_2|\cdots|b_{q-1}|b_q]) = 1[a_0|a_1|a_2|\cdots|a_{p-1}|a_p]\perp 1[b_0|b_1|b_2|\cdots|b_{q-1}|b_q].$$

It is clear from the definition that if $a_0=1$ or $b_0=1$ then

$$sh'(a_0[a_1|a_2|\cdots|a_{p-1}|a_p]\otimes b_0[b_1|b_2|\cdots|b_{q-1}|b_q])=0.$$

Proposition 3.1. *(see [**13**, 4.3.7]) The following identities are satisfied:*

- $d\circ sh=sh\circ d$;
- $sh'\circ B=B\circ sh'$;
- $B\circ sh-sh\circ B+d\circ sh'-sh'\circ d=0$,

where d and B are the Hochschild differential and the Connes operator respectively. When $\mathbb{K}[u]\otimes\mathfrak{C}(A)\otimes\mathfrak{C}(A)$ and $\mathbb{K}[u]\otimes\mathfrak{C}(A\otimes A)$ are equipped with the obvious differentials denoted respectively by $\mathcal{D}$ and $\hat{\mathcal{D}}$, the linear map $Sh\colon\mathbb{K}[u]\otimes\mathfrak{C}(A)\otimes\mathfrak{C}(A)\longrightarrow\mathbb{K}[u]\otimes\mathfrak{C}(A\otimes A)$ defined by $Sh=id_{\mathbb{K}[u]}\otimes sh+u(id_K[u]\otimes sh')$ satisfies $\mathcal{D}\circ Sh=Sh\circ\hat{\mathcal{D}}$.

Let (A,d_A,μ_A) be a *shc*-algebra. The chain map $m_{C_*^-A}$ given by the composite

$$\mathbb{K}[u]\otimes\mathfrak{C}(A)\otimes\mathbb{K}[u]\otimes\mathfrak{C}(A)\xrightarrow{id\otimes T\otimes id}\mathbb{K}[u]\otimes\mathbb{K}[u]\otimes\mathfrak{C}(A)\otimes\mathfrak{C}(A)\xrightarrow{m_{\mathbb{K}[u]}\otimes id}$$
$$\mathbb{K}[u]\otimes\mathfrak{C}(A)\otimes\mathfrak{C}(A)\xrightarrow{Sh}\mathbb{K}[u]\otimes\mathfrak{C}(A\otimes A)\xrightarrow{S_{A\otimes A}}$$
$$\mathbb{K}[u]\otimes\mathfrak{C}(\Omega B(A\otimes A))\xrightarrow{C_*^-(\mu_A)}C_*^-(\Omega B(A))\xrightarrow{C_*^-(\alpha_A)}C_*^-(A)$$

defined a product on $C^-_*(A)$ associative up to homotopy; where $S_{A\otimes A}$ is a linear section induced by the surjective quasi-isomorphism $C^-_*(\Omega B(A\otimes A)) \xrightarrow{C^-_*(\alpha_{A\otimes A})} C^-_*(A\otimes A)$, T the interchange isomorphism and $m_{\mathbb{K}[u]}$ the product on $\mathbb{K}[u]$.

Precomposing $H_*(m_{C^-_*(A)})$ by the Künneth isomorphism yields an associative product on $HC^-(A)$. Together with this product, $HC^-(A)$ is an associative graded algebra (see [**18**]).

Proposition 3.2. *Let $((A;d_A))$ be a cochain algebra and $((W;\delta_W);\Psi_W)$ a standard free resolution of π equipped with its coassociative coalgebra structure. There exists a natural homomorphism of chain complexes ϕ_A such that the following diagram commutes.*

$$\begin{array}{ccc} C^-_*(\mathrm{Hom}(W,A)) & \xrightarrow{\phi_A} & \mathrm{Hom}(W;C^-_*(A)) \\ \Big\downarrow{\scriptstyle C^-_*(ev_0)} & \swarrow{\scriptstyle ev_0} & \\ C^-_*(A) & & \end{array}$$

Proof. Let us prove the existence of the map ϕ_A.

We define ϕ_A by

$$\begin{aligned} \phi_A\colon \mathbb{K}[u]\otimes\mathfrak{C}(\mathrm{Hom}(W;A)) &\longrightarrow \mathrm{Hom}(W;\mathbb{K}[u]\otimes\mathfrak{C}A) \\ u^l\otimes f_0[f1|f_2|\cdots|f_{k-1}|f_k] &\longmapsto \phi_A(u^l\otimes f_0[f1|f_2|\cdots|f_{k-1}|f_{k-1}]) \end{aligned}$$

such that

$$\begin{aligned} &\phi_A(u^l\otimes f_0[f1|f_2|\cdots|f_{k-1}|f_{k-1}]) \\ &\qquad = (Id\otimes Id\otimes s^{\otimes k})\circ(g(u^l)\otimes f_0 f1\otimes f_2\otimes\cdots\otimes f_{k-1}\otimes f_k)\circ\Psi_W^{(k+1)}, \end{aligned}$$

where $W \xrightarrow{\Psi_W^{(1)}:=\Psi_W} W\otimes W$; $W \xrightarrow{\Psi_W^{(k+1)}} W^{\otimes k+2}$ denotes the iterated diagonal, and g the map defined by

$$\begin{aligned} \mathbb{K}[u] &\xrightarrow{g} \mathrm{Hom}(W;\mathbb{K}[u]) \\ u^l &\longmapsto g(u^l) \end{aligned}$$

such that

$$\begin{aligned} g(u^l)\colon W &\longrightarrow \mathbb{K}[u] \\ e_i &\longmapsto g(u^l)(e_i) \quad = g(u^l)(\tau^j e_i) \\ &\qquad\qquad = \begin{cases} u^{l-k} & \text{if } i=2k,\ \ 0\leqslant k\leqslant l; \\ 0 & \text{if not.} \end{cases} \end{aligned}$$

Here we check in detail that ϕ_A commutes with the differentials.

Consider for this purpose $\bar{D}$, the differential in $C_*^-(\mathrm{Hom}(W;A))$ defined by

$$\bar{D}(u^l \otimes f_0[f_1|f_2|\cdots|f_{k-1}|f_k]) = u^l \otimes d_{\mathfrak{C}_*(\mathrm{Hom}(W;A))}(f_0[f_1|f_2|\cdots|f_{k-1}|f_{k-1}]) + u^{l+1} \otimes B(f_0[f_1|f_2|\cdots|f_{k-1}|f_{k-1}])$$

and $\tilde{D}$ the differential in $\mathrm{Hom}(W;C_*^-A)$ defined by $\tilde{D}(f) = D \circ f - (-1)^{|f|} f \circ \delta$ where D denotes the differential in C_*^-A; $f \in \mathrm{Hom}(W;C_*^-A)$.

We have to prove that $\tilde{D} \circ \phi_A = \phi_A \circ \bar{D}$.

$$\phi_A \circ \bar{D}(u^l \otimes f_0[f1|f_2|\cdots|f_{k-1}|f_k]) = \phi_A(u^l \otimes d_{\mathfrak{C}_*(\mathrm{Hom}(W;A))}(f_0[f1|f_2|\cdots|f_{k-1}|f_k]) + u^{l+1} \otimes B(f_0[f1|f_2|\cdots|f_{k-1}|f_k])).$$

From the definition of the Hochschild differential, one has

$$\begin{aligned}
&\phi_A(u^l \otimes d^1(f_0[f_1|\cdots|f_k])) \\
&= \phi_A[u^l \otimes df_0[f_1|\cdots|f_k] - \sum_{i=1}^{k}(-1)^{\varepsilon_i} u^l \otimes f_0[f_1|\cdots|d(f_i)|\cdots|f_k]] \\
&= \phi_A(u^l \otimes d(f_0)[f_1|\cdots|f_k]) - \sum_{i=1}^{k}(-1)^{\varepsilon_i}\phi_A(u^l \otimes f_0[f_1|\cdots|d(f_i)|\cdots|f_k]) \\
&= (Id \otimes Id \otimes s^{\otimes k}) \circ (g(u^l) \otimes df_0 \otimes f_1 \otimes \cdots \otimes f_k) \circ \Psi_W^{(k+2)} - \\
&\quad \sum_{i=1}^{k}(-1)^{\varepsilon_i}(Id \otimes Id \otimes s^{\otimes k})(g(u^l) \otimes f_0 \otimes f_1 \otimes \cdots \otimes df_i \otimes \cdots \otimes \\
&\quad f_k) \circ \Psi_W^{(k+2)} \\
&= (Id \otimes Id \otimes s^{\otimes k})(g(u^l) \otimes (d_A f_0 - (-1)^{|f_0|} f_0 \delta_W) \otimes f_1 \otimes \cdots \otimes f_k) \circ \Psi_W^{(k+2)} - \\
&\quad \sum_{i=1}^{k}(-1)^{\varepsilon_i}(Id \otimes Id \otimes s^{\otimes k})(g(u^l) \otimes f_0 \otimes f_1 \otimes (d_A f_i - (-1)^{|f_i|} f_i \circ \delta_W) \otimes \cdots \\
&\quad \otimes f_k) \circ \Psi_W^{(k+2)} \\
&= (Id \otimes Id \otimes s^{\otimes k})(Id \otimes d_A \otimes Id - Id \otimes Id \otimes d_{A^{\otimes k}}) \circ (g(u^l) \otimes f_0 \otimes f_1 \otimes \cdots \\
&\quad \otimes f_k) \circ \Psi_W^{(k+2)} - (-1)^{(\Sigma_0^k |f_i|)+k}(Id \otimes Id \otimes s^{\otimes k}) \circ (g(u^l) \otimes f_0 \otimes f_1 \otimes \cdots \otimes f_k) \\
&\quad \circ (Id \otimes \delta_{W^{\otimes k+1}}) \circ \Psi_W^{k+1} \\
&= (Id \otimes d_1) \circ (Id \otimes Id \otimes s^{\otimes k})(g(u^l) \otimes f_0 \otimes f_1 \otimes f_2 \otimes \cdots \otimes f_{k-1} \otimes f_k) \circ \Psi_W^{(k+2)} - \\
&\quad (-1)^{\varepsilon_k}(Id \otimes Id \otimes s^{\otimes k})(g(u^l) \otimes f_0 \otimes f_1 \otimes \cdots \otimes f_k) \circ (Id \otimes \delta_W) \circ \varphi_W^{k+1} \\
&= (Id \otimes d^1) \circ \phi_A(u^l \otimes f_0[f_1|\cdots|f_k]) - (-1)^{\varepsilon_k}\phi_A(u^l \otimes f_0[f_1|\cdots|f_k]) \circ \delta.
\end{aligned}$$

This result follows from the fact that $((W, \delta_W), \psi_W)$ is a differential graded coalgebra:

$$\begin{aligned}
&\phi_A(u^l \otimes d^2(f_0[f_1|f_2|\cdots|f_{k-1}|f_k])) \\
&= \phi_A((-1)^{|f_0|}u^l \otimes (f_0 \cup f_1)[f_2|f_3|\cdots|f_{k-1}|f_k] + \\
&\quad \sum_{i=2}^{k}(-1)^{\varepsilon_i}u^l \otimes f_0[f_1|f_2|\cdots|f_{i-1}\cup f_i|\cdots|f_{k-1}|f_k] - \\
&\quad (-1)^{(|f_k|+1)\varepsilon_k}u^l \otimes (f_k \cup f_0)[f_1|f_2|\cdots|f_{k-2}|f_{k-1}]) \\
&= \phi_A((-1)^{|f_0|}u^l \otimes m_A \circ (f_0 \otimes f_1) \circ \Psi_W[f_2|f_3|\cdots|f_{k-1}|f_k] + \\
&\quad \sum_{i=2}^{k}(-1)^{\varepsilon_i}u^l \otimes f_0[f_1|f_2|\cdots|m_A \circ (f_{i-1}\otimes f_i) \circ \Psi_W|\cdots|f_{k-1}|f_k] - \\
&\quad (-1)^{(|f_k|+1)\varepsilon_k}u^l \otimes m_A \circ (f_k \otimes f_0) \circ \psi_W\,[f_1|f_2|\cdots|f_{k-2}|f_{k-1}]) \\
&= (-1)^{|f_0|}(Id \otimes Id \otimes s^{\otimes(k-1)}) \circ [g(u^l) \otimes m_A \circ (f_0 \otimes f_1) \\
&\quad \circ \psi_W \otimes f_2 \otimes f_3 \otimes \cdots \otimes f_{k-1} \otimes f_k] \circ \Psi_W^{(k)} + \\
&\quad \sum_{i=2}^{k}(-1)^{\varepsilon_i}(Id \otimes Id \otimes s^{\otimes(k-1)})\circ \\
&\quad [g(u^l) \otimes f_0 \otimes f_1 \otimes \cdots \otimes m_A \circ (f_{i-1} \otimes f_i) \circ \Psi_W \otimes \cdots \otimes f_k] \circ \varphi_W^k - \\
&\quad (-1)^{(|f_k|+1)\varepsilon_k}(Id \otimes Id \otimes s^{\otimes(k-1)}) \\
&\quad \circ [g(u^l) \otimes m_A \circ (f_k \otimes f_0) \circ \Psi_W \otimes f_1 \otimes f_2 \otimes \cdots \otimes f_{k-2} \otimes f_{k-1}] \\
&= (Id \otimes Id \otimes s^{\otimes(k-1)}) \circ [(Id \otimes m_A \otimes Id) + \\
&\quad \sum_{2}^{k}(-1)^i(Id \otimes Id \otimes m_A \otimes Id) + (Id \otimes (m_A \otimes Id) \circ \sigma_k)]\circ \\
&\quad (Id \otimes Id \otimes s^{\otimes(k-1)})^{-1} \circ \phi_A(u^l \otimes f_0\,[f_1|f_2|\cdots|f_{k-1}|f_k]) \\
&= (Id \otimes d^2_{\mathfrak{C}(A)}) \circ \phi_A(u^l \otimes f_0\,[f_1|f_2|\cdots|f_{k-1}|f_k]).
\end{aligned}$$

We have the result from the definition of the cup-product on $\mathrm{Hom}(W, A)$ and the fact that $((W, \delta_W), \psi_W)$ is a differential graded coalgebra.

Let us verify that $\phi_A(u^{l+1} \otimes B(f_0[f1|f_2\cdots|f_{k-1}|f_k])) = (Id \otimes B) \circ \phi_A(u^{l+1} \otimes f_0[f1|f_2|\cdots|f_{k-1}|f_k])$.

Since

$$B(f_0[f1|f_2|\cdots|f_{k-1}|f_k]) = \sum_{i=0}^{k}(-1)^{\varepsilon_i}1[f_i|f_{i+1}|\cdots|f_k|f_1|\cdots|f_{i-2}|f_{i-1}]$$

with

$$\varepsilon_i = (\sum_{j=0}^{i-1}|f_j| + 1)(\sum_{j=i}^{k-i+1}|f_j| + k - i + 1),$$

then

$$
\begin{aligned}
&\phi_A(u^{l+1} \otimes B(f_0[f1|f_2|\cdots|f_{k-1}|f_k])) \\
&\quad = \phi_A(u^{l+1} \otimes \sum_{i=0}^{k}(-1)^{\varepsilon_i} 1[f_i|f_{i+1}|\cdots|f_k|f_1|\cdots|f_{i-2}|f_{i-1}]) \\
&\quad = \phi_A(\sum_{i=0}^{k}(-1)^{\varepsilon_i} u^{l+1} \otimes 1[f_i|f_{i+1}|\cdots|f_k|f_0|\cdots|f_{i-2}|f_{i-1}]) \\
&\quad = \sum_{i=0}^{k}(-1)^{\varepsilon_i} \phi_A(u^{l+1} \otimes 1[f_i|\cdots|f_k|f_0|\cdots|f_{i-1}]) \\
&\quad = \sum_{i=0}^{k}(-1)^{\varepsilon_i} (Id \otimes Id \otimes s^{\otimes(k+1)})\circ \\
&\qquad\quad [g(u^{l+1}) \otimes 1 \otimes f_i \otimes f_{i+1} \otimes \cdots \otimes f_k \otimes f_0 \otimes \cdots \otimes f_{i-1}] \circ \Psi_W^{(k+2)} \\
&\quad = \sum_{i=0}^{k}(-1)^{\varepsilon_i} g(u^{l+1}) \otimes (Id \otimes s^{\otimes(k+1)})\circ \\
&\qquad\quad [1 \otimes f_i \otimes f_{i+1} \otimes \cdots \otimes f_k \otimes f_0 \cdots \otimes f_{i-1}] \circ \Psi_W^{(k+1)} \\
&\quad = (g(u^{l+1}) \otimes \sum_{i=0}^{k}(-1)^{\varepsilon_i} (Id \otimes s^{\otimes(k+1)})\circ \\
&\qquad\quad [1 \otimes f_i \otimes f_{i+1} \otimes f_{i+2} \otimes \cdots \otimes f_k \otimes f_0 \cdots \otimes f_{i-2} \otimes f_{i-1}]) \circ \Psi_W^{(k+2)} \\
&\quad = g(u^{l+1}) \otimes B((Id \otimes s^{\otimes k}) \circ [f_0 \otimes f_1 \otimes f_2 \otimes \cdots \otimes f_k]) \circ \psi_W^{(k+2)} \\
&\quad = (Id \otimes B) \circ (g(u^{l+1}) \otimes (Id \otimes s^{\otimes k}) \otimes [f_0 \otimes f_1 \otimes \cdots \otimes f_k]) \circ \Psi_W^{(k+1)} \\
&\quad = (Id \otimes B) \circ \phi_A(u^{l+1} \otimes f_0\,[f_1|\cdots|f_{k-1}|f_k]).
\end{aligned}
$$

From the above calculations, we conclude that ϕ_A is a chain complex homomorphism such that the diagram above commutes. □

Lemma 3.3. *If* $(A, d_A, \mu_A, \tilde{\kappa}_A)$ *is a* π-shc-*algebra, then*

(i) $\tilde{\kappa}_A$ *is a strict homomorphism of* shc-*algebras; and*

(ii) $H_*(\phi_A)\colon HC_*^- \operatorname{Hom}(W, A) \longrightarrow H_*(\operatorname{Hom}(W, C^-A))$ *preserves the natural products.*

Proof. (i) was proved in [**21**, A.5].

In order to establish (ii), consider the following commutative diagrams A and B:

$$
\begin{array}{ccccc}
[C_*^- \operatorname{Hom}(W,A)]^{\otimes 2} & \xrightarrow{\phi_A^{\otimes 2}} & [\operatorname{Hom}(W;C_*^- A)]^{\otimes 2} & \xrightarrow[\psi_{C_*^- A}]{} & \operatorname{Hom}(W,(C_*^- A)^{\otimes 2}) \\
\Big\downarrow{\scriptstyle \overline{Sh}} & & (A) & & \Big\downarrow{\scriptstyle \operatorname{Hom}(W,\overline{Sh})} \\
C_*^-((\operatorname{Hom}(W,A))^{\otimes 2}) & \xrightarrow{C_*^-(\psi_A)} & C_*^- \operatorname{Hom}(W,A^{\otimes 2}) & \xrightarrow{\phi_{A^{\otimes 2}}} & \operatorname{Hom}(W;C_*^-(A^{\otimes 2})) \\
\Big\uparrow{\scriptstyle C_*^-(\alpha_{\operatorname{Hom}(W,A)^{\otimes 2}})} & & \Big\uparrow{\scriptstyle C_*^-(\alpha_{\operatorname{Hom}(W,A^{\otimes 2})})} & & \\
C_*^-(\Omega B(\operatorname{Hom}(W,A))^{\otimes 2}) & \xrightarrow{C_*^-(\Omega B\psi_A)} & C_*^-(\Omega B(\operatorname{Hom}(W,A^{\otimes 2}))) & &
\end{array}
$$

and

$$
\begin{array}{ccc}
C_*^-(\operatorname{Hom}(W,A^{\otimes 2})) & \xrightarrow{\phi_{A^{\otimes 2}}} & \operatorname{Hom}(W,C_*^-(A^{\otimes 2})) \\
\Big\uparrow{\scriptstyle C_*^-(\operatorname{Hom}(W,\alpha_{A^{\otimes 2}}))} & (B) & \Big\uparrow{\scriptstyle \operatorname{Hom}(W,C_*^-\alpha_{A^{\otimes 2}})} \\
C_*^-(\operatorname{Hom}(W,\Omega B(A^{\otimes 2}))) & \xrightarrow{\phi_{\Omega B(A^{\otimes 2})}} & \operatorname{Hom}(W,C_*^-(\Omega B(A^{\otimes 2}))).
\end{array}
$$

From F_3, F_8 and [**21**, Lemma A.4(b)], we deduce that there exist homomorphisms of differential graded algebras

1. $\Omega B(\operatorname{Hom}(W,A)) \xrightarrow{\theta'_A} \operatorname{Hom}(W,\Omega BA)$
2. $\operatorname{Hom}(W,A^{\otimes 2})) \xrightarrow{\theta'_{A^{\otimes 2}}} \operatorname{Hom}(W,\Omega BA^{\otimes 2})$
3. $\Omega B(\operatorname{Hom}(W,A^{\otimes 2})) \xrightarrow{\theta''_{A^{\otimes 2}}} \Omega B \operatorname{Hom}(W,\Omega BA^{\otimes 2})$

such that

$$
\operatorname{Hom}(W,\alpha_A)\circ\theta'_A = \alpha_{\operatorname{Hom}(W,A)},
$$
$$
\operatorname{Hom}(W,\alpha_{A^{\otimes 2}})\circ\alpha_{\operatorname{Hom}(W,\Omega B(A^{\otimes 2}))}\circ\theta''_{A^{\otimes 2}} = \alpha_{\operatorname{Hom}(W,A^{\otimes 2})}
$$

and

$$
\operatorname{Hom}(W,\alpha_{A^{\otimes 2}})\circ\theta'_{A^{\otimes 2}} = \alpha_{\operatorname{Hom}(W,A^{\otimes 2})},
$$

since $\mu_{\operatorname{Hom}(W,A)} = \Omega B \operatorname{Hom}(W,\alpha\circ\mu_A)\circ\theta_{\operatorname{Hom}(W,A^{\otimes 2})}\circ\Omega B\psi_A$, where $\theta_{\operatorname{Hom}(W,A^{\otimes 2})}$ denotes the unique homomorphism of differential graded algebras such that

$$
\operatorname{Hom}(W,\alpha_{A^{\otimes 2}})\circ\alpha_{\operatorname{Hom}(W,\Omega B(A^{\otimes 2}))}\circ\theta_{\operatorname{Hom}(W,A^{\otimes 2})} = \alpha_{\operatorname{Hom}(W,A\otimes A)}.
$$

Then we obtain the following commutative diagrams C and D:

$$\begin{array}{ccccc} C_*^-(\Omega B((\mathrm{Hom}(W,A))^{\otimes 2})) & \xrightarrow{C_*^-\Omega B(\psi_A)} & C_*^-\Omega B(\mathrm{Hom}(W,A^{\otimes 2})) & \xrightarrow{C_*^-\theta''_{A^{\otimes 2}}} & C_*^-\Omega B(\mathrm{Hom}(W,\Omega B(A^{\otimes 2}))) \\ \downarrow{\scriptstyle C_*^-(\mu_{\mathrm{Hom}(W,A)})} & & (C) & & \downarrow{\scriptstyle C_*^-\alpha_{\mathrm{Hom}(W,\Omega B(A))}\circ C_*^-(\Omega B\,\mathrm{Hom}(W,\mu_A))} \\ C_*^-(\Omega B\,\mathrm{Hom}(W,A)) & & \xrightarrow{\qquad C_*^-\theta'_A \qquad} & & C_*^-(\mathrm{Hom}(W,\Omega B(A))) \end{array}$$

and

$$\begin{array}{ccccc} C_*^-(\Omega B(\mathrm{Hom}(W,A^{\otimes 2}))) & \xrightarrow{C_*^-(\theta'_{A^{\otimes 2}})} & C_*^-(\mathrm{Hom}(W,\Omega B(A^{\otimes 2}))) & \xrightarrow{\phi_{\Omega B(A^{\otimes 2})}} & \mathrm{Hom}(W,C_*^-(\Omega B(A^{\otimes 2}))) \\ \downarrow{\scriptstyle C_*^-(\theta''_{A^{\otimes 2}})} & \nearrow{\scriptstyle C_*^-(\alpha_{\mathrm{Hom}(W,\Omega B(A^{\otimes 2}))})} & & & \\ C_*^-\Omega B(\mathrm{Hom}(W,\Omega B(A^{\otimes 2}))) & & (D) & & \downarrow{\scriptstyle \mathrm{Hom}(W,C_*^-\mu_A)} \\ \downarrow{\scriptstyle C_*^-\alpha_{\mathrm{Hom}(W,\Omega B(A))}\circ C_*^-\Omega B\,\mathrm{Hom}(W,\mu_A)} & & & & \\ C_*^-(\mathrm{Hom}(W,\Omega BA)) & & \xrightarrow{\qquad \phi_{\Omega BA} \qquad} & & \mathrm{Hom}(W,C_*^-\Omega B(A)) \end{array}$$

By choosing linear sections of $C^-\alpha_{(\mathrm{Hom}(W,A^{\otimes 2}))}$ (resp. $\mathrm{Hom}(W,C_*^-\alpha_{A_{\otimes 2}})$), one can define the product $m_{C_*^-(\mathrm{Hom}(W,A))}$ on $C_*^-(\mathrm{Hom}(W,A))$ as in 2.3 (resp. $m_{\mathrm{Hom}(W,C_*^-A)}$ on $\mathrm{Hom}(W,C_*^-A)$). Thus by gluing together diagrams A, B, C and D, we have the following commutative diagram:

$$\begin{array}{ccc} [C_*^-(\mathrm{Hom}(W,A))]^{\otimes 2} & \xrightarrow{\phi_A^{\otimes 2}} & [\mathrm{Hom}(W,C_*^-A)]^{\otimes 2} \\ \downarrow{\scriptstyle m_{C_*^-(\mathrm{Hom}(W,A))}} & & \downarrow{\scriptstyle m_{\mathrm{Hom}(W,C_*^-A)}} \\ C_*^-\,\mathrm{Hom}(W,A) & \xrightarrow{\phi_A} & \mathrm{Hom}(W,C_*^-A) \end{array}$$

which commutes up to linear homotopy. This proves that $H_*\phi_A$ preserves the natural products. □

3.1. End of the proof of the first part of Theorem 1.1

Let $((A,d_A),\mu,\tilde{\kappa}_A)$ be a π-*shc* algebra. Since for any $p \geqslant 2$, $\Omega B(A^{\otimes p})$ is a *shc* algebra with $\alpha_A \circ \mu_A^{(p)} \circ i_{A^{\otimes p}} = m_A^{(p)}$ (see 2.1.7), hence we deduce from the definition of the π-*shc* algebra that the following diagram commutes up to homotopy in the category π-DM:

$$\begin{array}{ccc} \Omega B(A^{\otimes p}) & \xrightarrow{\tilde{\kappa}_A} & \mathrm{Hom}(W;A) \\ \Big\downarrow{\scriptstyle \alpha_{A^{\otimes p}}} & (1) & \Big\downarrow{\scriptstyle ev_0} \\ A^{\otimes p} & \xrightarrow{m_A^{(p)}} & A. \end{array}$$

Let S_p be the symmetric group on $\{1, 2, \ldots, p\}$ and consider the action of S_p on $C_*^-(A^{\otimes p})$ (resp. on $C_*^-(\Omega B(A^{\otimes p}))$) defined by the following rule: $\sigma(u^l \otimes b_0[b_1|b_2|\cdots |b_{s-1}|b_s]) = u^l \otimes \sigma b_0[\sigma b_1|\sigma b_2|\cdots|\sigma b_{s-1}|\sigma b_s]$, $b_i \in A^{\otimes p}$ (resp. $b_i \in \Omega B(A^{\otimes p})$) so that $C_*^- A^{\otimes p}$, $C_*^- \Omega B(A^{\otimes p})$ are π-chain complexes and $\Omega B(A^{\otimes p}) \overset{\alpha_{A^{\otimes p}}}{\to} A^{\otimes p}$, $C_*^-(\Omega B(A^{\otimes p})) \overset{C_*^-(\alpha_{A^{\otimes p}})}{\to} C_*^-(A^{\otimes p})$ are quasi-isomorphisms of π-chain complexes.

Consider on the other hand the π-chain complex homomorphism $[C_*^- A]^{\otimes p} \overset{\bar{Sh}^{(p)}}{\to} C_*^-(A^{\otimes p})$, called a p-iterated cyclic shuffle map and defined by induction as follows: $\bar{Sh} = Sh \circ (m_{\mathbb{K}[u]} \otimes Id) \circ (id \otimes T \otimes id)$, $\bar{Sh}^{(2)} = \bar{Sh} \circ (\bar{Sh} \otimes id)$ and for all $p \geqslant 2$, $\bar{Sh}^{(p)} = \bar{Sh}^{(p)}(\bar{Sh}^{(p-1)} \otimes id)$ (where $Sh = (id \otimes sh) + u(id \otimes sh')$ denotes the cyclic shuffle map). Indeed we have for any

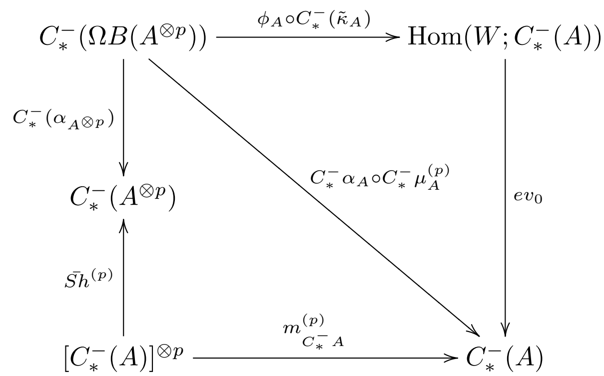
$$x = x_{n_1} \otimes x_{n_2} \otimes \cdots \otimes x_{n_{p-1}} \otimes x_{n_p} \in [C_*^- A]^{\otimes p},$$
$$(x_{n_i} = u^{l_{n_i}} \otimes a_0^{n_i}[a_1^{n_i}|a_2^{n_i}|\cdots|a_{s_i-1}^{n_i}|a_{s_i}^{n_i}] \in C_*^- A)_{i=1}^p$$

$$\begin{aligned} \bar{Sh}^{(p)}(x) &= u^{l_{n_1}+\cdots+l_{n_p}} \otimes (a_0^{n_1} \otimes \cdots \otimes a_0^{n_p}) \\ &\otimes \sum_\sigma (-1)^{\epsilon_\sigma}[a_1^{n_1} \otimes 1 \otimes \cdots \otimes 1|\cdots|a_{s_1}^{n_1} \otimes 1 \otimes \cdots \otimes 1|1 \otimes a_1^{n_2} \otimes 1 \otimes \cdots \otimes 1|\cdots \\ &\quad |1 \otimes a_{s_2}^{n_2} \otimes 1 \otimes \cdots \otimes 1|\cdots|1 \otimes 1 \otimes \cdots \otimes 1 \otimes a_1^{n_p}|\cdots|1 \otimes 1 \otimes \cdots \otimes 1 \otimes a_{s_p}^{n_p}] \\ &+ u^{l_{n_1}+\cdots+l_{n_p}+p} \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \\ &\otimes \sum_{\sigma'} (-1)^{\epsilon_{\sigma'}}[a_0^{n_1} \otimes 1 \otimes \cdots \otimes 1|\cdots|a_0^{n_p} \otimes 1 \otimes \cdots \otimes 1|1 \otimes a_1^{n_1} \otimes 1 \otimes \cdots \otimes 1|\cdots \\ &\qquad |1 \otimes a_{s_1}^{n_1} \otimes 1 \otimes \cdots \otimes 1|1 \otimes 1 \otimes a_1^{n_2} \otimes 1 \otimes \cdots \otimes 1|\cdots \\ &\qquad\qquad |1 \otimes a_{s_2}^{n_2} \otimes 1 \otimes \cdots \otimes 1|\cdots|1 \otimes 1 \otimes \cdots \otimes 1 \otimes a_{s_p}^{n_p}] \end{aligned}$$

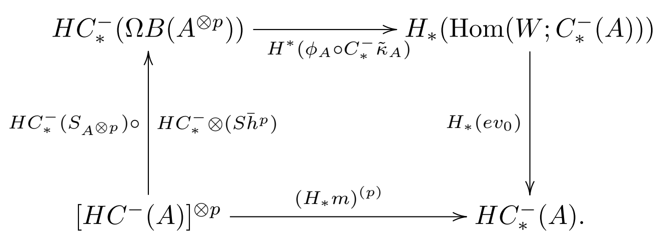
with σ a $(n_1, n_2, \ldots, n_p)$-shuffle and σ' a $(n_1, n_2, \ldots, n_p)$-cyclic shuffle. Since for every $\sigma \in S_p$, $x \in (C_*^-(A))^{\otimes p}$, $\sigma x = x_{\sigma(n_1)} \otimes x_{\sigma(n_2)} \otimes \cdots \otimes x_{\sigma({}_n p)}$, the sets of $(n_1, n_2, \ldots, n_p)$-shuffles and $(n_1, n_2, \cdots n_p)$-cyclic shuffles coincide respectively with the set of $(\sigma(n_1), \sigma(n_2), \ldots, \sigma({}_n p))$-shuffles and the set of $(\sigma(n_1), \sigma(n_2), \ldots, \sigma({}_n p))$-cyclic shuffles, then the p-iterated cyclic shuffle map $\bar{Sh}^{(p)}$ is π linear and we obtain the following sequence of π-quasi-isomorphisms:

$$C_*^-(\Omega B(A^{\otimes p})) \overset{C_*^-(\alpha_{A^{\otimes p}})}{\longrightarrow} C_*^-(A^{\otimes p}) \overset{\bar{Sh}^{(p)}}{\longleftarrow} [C_*^- A]^{\otimes p}.$$

Let ϕ_A be the map defined in the proof of Proposition 3.2. By applying the functor $C_*^-(-)$ to the diagram (1) above and using Lemma 2.2 and the definition of the product $m^{(p)}_{C_*^- A}$ on $C_*^- A$, we obtain the following the diagram commutative up to homotopy;



which induces the following commutative diagram in homology:



Thus $C_*^-(A)$ together with the structural map $\theta = \phi_A \circ C_*^-(\tilde{\kappa}_A)$ is a Dold quasi-algebra. From the lines of the proof of May[**15**, Proposition 2.3], this diagram defined algebraic Steenrod operations on $HC^-(A)$.

Following the same lines, Proposition 2.1 can still be used to prove the Cartan formulas [**21**].

4. Proof of the second part of Theorem 1.1

Let us begin this section by giving the proof of the following result due to Bitjong Ndombol and Jean-Claude Thomas [**21**, Theorem B].

Proposition 4.1. *Let X be a topological space. The algebra $N^*(X)$ is a natural π-*shc-*algebra.*

Proof. Here we replace the ground field $\mathbb{F}_p$ by $\mathbb{F}_p[\pi]$.

Munkholm established in [**17**, 4–7] that there exists a natural transformation $\mu_X : \Omega B(N^*(X) \otimes N^*(X)) \longrightarrow \Omega B(N^*(X)$ such that $\alpha_{N^*(X)} \circ \mu_X \circ i_{N^*(X) \otimes N^*(X)} = m_{N^*(X)}$, where $m_{N^*(X)}$ denotes the usual product on $N^*(X)$. Furthermore $(N^*(X)$, $\mu_X)$ is a *shc*-algebra.

Let Δ denote the topological diagonal. From [**21**, A.2], F_3, and the fact that the Eilenberg-Zilber map EZ is a trivialized extension, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\Omega B((N^*X)^{\otimes 2}) & \xrightarrow{\theta_{N^*(X)}} & \Omega B(N^*(X^{\times 2})) & \xrightarrow{\Omega B(N(\Delta))} & \Omega B(N^*X) \\
\Big\downarrow{\scriptstyle \alpha_{(N^*X)^{\otimes 2}}} & & \Big\downarrow{\scriptstyle \alpha_{N^*X^{\times 2}}} & & \Big\downarrow{\scriptstyle \alpha_{N^*X}} \\
(N^*X)^{\otimes 2} & \xleftarrow{EZ} & N^*(X^{\times 2}) & \xrightarrow{N^*(\Delta)} & N^*X,
\end{array}$$

from which we define the following homomorphism of differential graded algebras

$$\tilde{\mu}_X = \Omega B(N(\Delta)) \circ \theta_{N^*(X)}.$$

Moreover there exists a homomorphism of differential graded algebras

$$\Omega B(N^*X) \overset{\tilde{\theta}_{N^*X}}{\longrightarrow} \mathrm{Hom}(W, N^*X)$$

deduced from [**21**, Lemma A.4(a)] and F_3, which rise to the commutative diagram

$$\begin{array}{ccc}
\Omega B((N^*X)^{\otimes 2}) & \xrightarrow{\kappa_X} & \mathrm{Hom}(W, N^*X) \\
\Big\downarrow{\scriptstyle \tilde{\mu}_X} & \nearrow{\scriptstyle \tilde{\theta}_{N^*X}} & \Big\downarrow{\scriptstyle ev_0} \\
\Omega B(N^*X) & \xrightarrow{\alpha_{N^*(\Delta)}} & N^*X.
\end{array}$$

Thus we obtain a $\mathbb{F}_p[\pi]$-homomorphism of differential graded algebras κ_X defined by $\kappa_X = \theta_{N^*X} \circ \tilde{\mu}_X$ such that $ev_0 \circ \kappa_X = \alpha_{N^*(\Delta)} \circ \tilde{\mu}_X$. To end this proof it is enough to establish that $\tilde{\mu}_X \simeq_{DA} \mu_X$. We remark that $\tilde{\mu}_X \simeq_{DA} \mu_X$ if and only if $t_X \simeq_T \tilde{t}_X$ where $t_X, \tilde{t}_X \in T(B(N^*(X))^{\otimes 2}, \Omega B(N^*(X)))$ denote the universal twisting cochains associated respectively to μ_X and $\tilde{\mu}_X$ [**17**]. Finally, it is necessary to find $H \in \mathrm{Hom}^1(\Omega B((N^*X)^{\otimes 2}), \Omega BN^*X)$ such that $H\colon t_X \simeq_T \tilde{t}_X$. i.e., $DH = \tilde{t}_X \cup H - H \cup t_X$ and $H \circ \eta_{BN^*X^{\otimes p}} = \eta_{BN^*X}; \varepsilon_{\Omega BN^*X} \circ H = \varepsilon_{BN^*X}$. We define H by the following induction formula:

$$\begin{aligned} H &= h(\tilde{t}_X \cup H - H \cup t_X) + \eta_{\Omega BN^*X} \circ \varepsilon_{BN^*X^{\otimes p}} \\ H \circ i_0 &= H \circ \eta_{BN^*X^{\otimes p}} = \eta_{\Omega BN^*X}, \end{aligned} \tag{I}$$

where $h\colon 1_{\Omega BN^*X} \simeq i_{N^*X} \circ \alpha_{N^*X}$ and $i_0 = \eta_{B(N^*X)^{\otimes p}}$. Since

$$\tilde{t}_X \cup H = m_{\Omega B(N^*X)} \circ (\tilde{t}_X \otimes H) \circ \Delta_{B(N^*X)^{\otimes p}}$$

and

$$H \cup t_X = m_{\Omega B(N^*X)} \circ (H \otimes t_X) \circ \Delta_{B(N^*X)^{\otimes p}},$$

then

$$H = h \circ m_{\Omega B(N^*X)} \circ (\tilde{t}_X \otimes H - H \otimes t_X) \circ \Delta_{B(N^*X)^{\otimes p}} + \eta_{\Omega B(N^*X)} \circ \varepsilon_{\Omega B(N^*X)^{\otimes p}}.$$

Notice that

$$\varepsilon_{BN^*X^{\otimes p}} \circ \eta_{B(N^*X)^{\otimes p}} = Id_{\mathbb{K}};$$
$$\Delta_{B(N^*X)^{\otimes p}} \circ \eta_{B(N^*X)^{\otimes p}} = \eta_{B(N^*X)^{\otimes p}} \otimes \eta_{B(N^*X)^{\otimes p}}$$

and

$$\tilde{t}_X \circ \eta_{B(N^*X)^{\otimes p}} = t_X \circ \eta_{B(N^*X)^{\otimes p}} = 0,$$

so

$$\begin{aligned} H \circ i_0 = H \circ \eta_{B(N^*X)^{\otimes p}} &= h \circ m_{\Omega B(N^*X)} \circ (\tilde{t}_X \circ \eta_{B(N^*X)^{\otimes p}} \otimes H\eta_{B(N^*X)^{\otimes p}} \\ &- H\eta_{B(N^*X)^{\otimes p}} \otimes t_X \circ \eta_{B(N^*X)^{\otimes p}}) + \eta_{\Omega B(N^*X)} = \eta_{\Omega B(N^*X)}. \end{aligned}$$

More generally, suppose that $H \circ i_j$ can be written as (I) for $j \leqslant k$ and let us prove that $H \circ i_{k+1}$ can be written by formula (I) and $H \circ i_j$ ($j \leqslant k < k'$).

$$\begin{aligned} H \circ i_{k'} &= h \circ m_{\Omega B(N^*X)}(\tilde{t}_X \otimes H - H \otimes t_X) \circ \Delta_{B(N^*X)^{\otimes p}} \circ i_{k'} \\ &\quad + \eta_{\Omega B(N^*X)} \circ \varepsilon_{\Omega B(N^*X)^{\otimes p}} \circ i_{k'} \\ &= h \circ m_{\Omega B(N^*X)} \circ [\tilde{t}_X \otimes H - H \otimes t_X][\eta_{B(N^*X)^{\otimes p}} \otimes i_{k'} \\ &+ \sum_{\nu=1}^{k'} i_\nu \otimes i_{k'-\nu} + i_{k'} \otimes \eta_{B(N^*X)^{\otimes p}}] + \eta_{\Omega B(N^*X)} \circ \varepsilon_{\Omega B(N^*X)^{\otimes p}} \circ i_{k'} \end{aligned}$$

where

$$(s(IA))^{\otimes \nu} \xrightarrow{i_\nu} BA$$
$$(s(a_1) \otimes s(a_2) \otimes \cdots \otimes s(a_\nu)) \longmapsto i_\nu(s(a_1) \otimes s(a_2) \otimes \cdots \otimes s(a_\nu)) = [a_1|\cdots|a_\nu]$$

($A = (N^*X)^{\otimes p}$). In particular $i_{k\prime}(s(a)) = [a]$ and

$$\varepsilon_{N^*X}([a_1|\cdots|a_k]) = \begin{cases} 0 & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

and

$$\varepsilon_{N^*X^{\ \otimes p}} \circ i_k(sa_1 \otimes sa_2 \otimes \cdots \otimes sa_k) = \begin{cases} 0 & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

$\Delta_{B(N^*X)} \circ i_{k'} = \eta_{B(N^*X)^{\otimes p}} \otimes i_{k'} + \sum_{\nu=1}^{k'} i_\nu \otimes i_{k'-\nu} + i_{k'} \otimes \eta_{B(N^*X)^{\otimes p}}$.

Thus we deduce that

$$\begin{aligned} H \circ i_{k'} &= h \circ m_{\Omega B(N^*X)}(\tilde{t}_X \otimes H - H \otimes t_X)(i_0 \otimes i_{k'} + \sum_{\nu=1}^{k'-1} i_\nu \otimes i_{k'-\nu} + i_{k'} \otimes i_0) \\ &= h \circ m_{\Omega B(N^*X)}[\sum_{\nu=0}^{k\prime} (\tilde{t}_X \circ i_\nu) \otimes (H \circ i_{k'-\nu}) - \sum_{\nu=0}^{k\prime} H \circ i_\nu \otimes t_X \circ i_{k'-\nu}] \\ &= h \circ m_{\Omega B(N^*X)}[\sum_{\nu=1}^{k'} (\tilde{t}_X \circ i_\nu) \otimes (H \circ i_{k'-\nu}) - \sum_{\nu=0}^{k'-1} (H \circ i_\nu) \otimes (t_X \circ i_{k'-\nu})]. \end{aligned}$$

Let us verify that $DH = \tilde{t}_X \cup H - H \cup t_X$.

Since $DH = D[h(\tilde{t}_X \cup H - H \cup t_X)] + D(\eta_{\Omega B(N^*X)} \circ \varepsilon_{\Omega B(N^*X)^{\otimes p}}$ with

$$\begin{aligned} D(\eta_{\Omega B(N^*X)} \circ \varepsilon_{\Omega B(N^*X)^{\otimes p}}) &= \underbrace{d_{\Omega B(N^*X)}\eta_{\Omega B(N^*X)}}_{0}\varepsilon_{\Omega B(N^*X)^{\otimes p}} \\ &\quad - \eta_{\Omega B(N^*X)}\underbrace{\varepsilon_{\Omega B(N^*X)^{\otimes p}} d_{\Omega B(N^*X)^{\otimes p}}}_{0}, \\ &= 0 \end{aligned}$$

we also have $Dh = 1_{\Omega B(N^*X)} - i_{N^*X} \circ \alpha_{N^*X}$ and $Df = df - (-1)^{|f|} fd$.

Thus

$$\begin{aligned} DH = (Dh)[\tilde{t}_X \cup H - H \cup t_X] + h(D[\tilde{t}_X \cup H - H \cup t_X]) = \\ (1_{\Omega B(N^*X)} - i_{N^*X} \circ \alpha_{N^*X})[\tilde{t}_X \cup H - H \cup t_X] + h(D[\tilde{t}_X \cup H - H \cup t_X]). \end{aligned}$$

Since $\alpha_{N^*X} \circ h = 0$ and $D[\tilde{t}_X \cup H - H \cup t_X] = 0$, we deduce that $DH = \tilde{t}_X \cup H - H \cup t_X$, hence $\tilde{t}_X \simeq_T t_X$. □

4.1. End of the proof of the second part of the theorem

Consider the simplicial model K of the unit cycle S^1 and the cosimplicial model space $\underline{X}$ defined by $\underline{X} = \mathrm{Map}(K(n), X)$ whose geometric resolution $\| \underline{X} \|$ is homeomorphic to LX, [**20**, Part II-3]. In [**10**, Lemma 5.5, Proof of Theorem A and Theorem B], J.D.S. Jones has constructed the $\mathbb{F}_p[u]$-modules quasi-isomorphism $C_*^- N^*X \xrightarrow{\Psi} \mathbb{F}_p[u] \otimes N^*(\| X \|)$ which induces a graded algebra isomorphism $HC_*^- X \cong H_{S^1}^{-*}(LX, \mathbb{F}_p)$, [**18**]. Following the lines of [**21**], consider a π-*shc* differential graded algebra $(N^*X, \mu_X, \tilde{\kappa}_X)$ endowed with the structural map $\hat{\theta}_X = \phi_{N^*X} \circ C^- \tilde{\kappa}_X$ defining the algebraic Steenrod operations on $HC_*^-(N^*X)$ and $W \otimes (N^*(\| \underline{X} \|))^{\otimes p} \xrightarrow{\Gamma_X} N^*(\| \underline{X} \|)$ the structural map defining Steenrod operations on $N^*(\| \underline{X} \|))$, [**15**, 7.5]. Define the map γ_X as the composite

$$\begin{aligned} &W \otimes (\mathbb{F}_p[u] \otimes N^*(\| \underline{X} \|))^{\otimes p} \overset{id \otimes (id \otimes T \otimes id)^{\otimes p}}{\longrightarrow} \\ &\qquad W \otimes (\mathbb{F}_p[u])^{\otimes p} \otimes (N^*(\| \underline{X} \|))^{\otimes p} \overset{id \otimes m^{(p)}_{\mathbb{F}_p[u]} \otimes id}{\longrightarrow} \\ &W \otimes (\mathbb{F}_p[u]) \otimes (N^*(\| \underline{X} \|))^{\otimes p} \overset{T \otimes id}{\longrightarrow} (\mathbb{F}_p[u]) \otimes W \otimes (N^*(\| \underline{X} \|))^{\otimes p} \overset{id \otimes \Gamma_X}{\longrightarrow} \\ &\qquad \mathbb{F}_p[u] \otimes N^*(\| \underline{X} \|); \end{aligned}$$

γ_X is the structural map defining Steenrod operations on $(\mathbb{F}_p[u] \otimes N^*(\| \underline{X} \|))$ and inducing the chain map $\tilde{\gamma}_X$ (see Proposition 2.1). Consider the following diagram:

$$\begin{array}{ccccc} C_*^-(\Omega B(N^*X)^{\otimes p}) & \xrightarrow{C_*^-(\alpha_{(N^*X)^{\otimes p}})} & C_*^-((N^*X)^{\otimes p}) & \xleftarrow{\bar{Sh}^p} & (C_*^-((N^*X)))^{\otimes p} \\ \Big\downarrow {\scriptstyle \phi_{N^*X} \circ (C_*^- \tilde{\kappa}_X)} & (1) & \Big\downarrow {\scriptstyle \tau_X} & (2) & \Big\downarrow {\scriptstyle \Psi_X^{\otimes p}} \\ \mathrm{Hom}(W; C_*^-(N^*X)) & \xrightarrow{\mathrm{Hom}(W;\Psi_X)} & \mathrm{Hom}(W; \mathbb{K}[u] \otimes N^*(\|\underline{X}\|)) & \xleftarrow{\tilde{\gamma}_X} & (\mathbb{K}[u] \otimes N^*(\|\underline{X}\|))^{\otimes p} \end{array}$$

where the functors $X \longrightarrow C_*^-((N^*X)^{\otimes p})$ and $X \longrightarrow \mathrm{Hom}(W;\mathbb{K}[u]\otimes N^*(\| \underline{X} \|))$ defined on the category **Top** with models $\mathcal{M} = \{\mathbb{O}^n, \mathbb{O}^n = \bigvee_{p\geqslant 0}(\bigvee^{p+1}(\triangle^n \times \triangle^p)); n \in \mathbb{N}\}$ preserve the units and are respectively acyclic and corepresentable. We obtain from the equivariant cyclic model theorem (see [**21**, Appendix B] and [**18**]) that there exists a π-linear natural transformation $C_*^-((N^*X)^{\otimes p}) \xrightarrow{T_X} \mathrm{Hom}(W;\mathbb{K}[u]\otimes N^*(\| \underline{X} \|))$ such that $T \circ C_*^-(\alpha_{(N^*X)^{\otimes p}}) \simeq_\pi \mathrm{Hom}(W;\Psi_X)\circ \phi_{N^*X} \circ (C_*^- \tilde{\kappa}_X)$. Consequently the Jones isomorphism respects Steenrod operations.

5. π-*shc* models, [21, 3]

5.1. Minimal algebra

Let $V = \{V^i\}_{i\geqslant 1}$ be a graded vector space and let (TV, d_V) denotes the free differential graded algebra generated by V: $T^rV = V\otimes V \otimes \cdots \otimes V$ (r times) and $v_1 \cdot v_2 \cdots v_k \in (TV)^n$ if $\sum_{i=1}^k \mid v_i \mid = n$. The differential d_V on TV is the unique degree 1 derivation on TV defined by a given linear map $V \longrightarrow TV$ and such that $d_V \circ d_V = 0$. The differential $d_V : TV \longrightarrow TV$ decomposes as $d_V = d_0 + d_1 + \cdots$ with $d_k V \subset T^{k+1}V$. If we assume that $V^1 = 0$ and $d_0 = 0$ then (TV, d_V) is called a 1-connected minimal algebra. For any differential graded algebra (TU, d_U) such that $H^0(TU, d_U) = \mathbb{F}_p$ and $H^1(TU, d_U) = 0$, there exists a sequence of homomorphisms of differential graded algebras, $(TU, d_U) \xrightarrow{P_V} (TV, d_V) \xrightarrow{\varphi_V} (TU, d_U)$ where (TV, d_V) denotes a 1-connected minimal algebra, $\varphi_V \circ P_V \simeq_{DA} id$ and $P_V \circ \varphi_V = id$ such that $V \cong H(U, d_U)$ (see [**20**, 3.1] or [**21**, 6.4]). Moreover (TV, d_V) is unique up to isomorphism.

5.2. Minimal model of a product

Assume that (A, d_A) is a differential graded algebra such that $H^0(A) = \mathbb{F}_p$ and $H^1(A) = 0$ and let $(TU[n], d_{U[n]}) = \Omega((BA)^{\otimes n})$, $n \geqslant 1$. Following the discussion above, we obtain a sequence

$$(TU[n], d_{U[n]}) = \Omega((BA)^{\otimes n}) \xrightarrow{P_{V[n]}} (TV[n], d_{V[n]}) \xrightarrow{\varphi_{V[n]}} (TU[n], d_{U[n]})$$

with

$$\begin{aligned} V[n] &= s^{-1}(\overline{H((BA)^{\otimes n})}) \cong s^{-1}(\overline{(H(BA))^{\otimes n}}) = s^{-1}(\overline{(\mathbb{F}_p \oplus sV)^{\otimes n}}) \\ &\cong (\bigoplus_{k=1}^{n} ((\mathbb{F}_p)^{\otimes k-1} \otimes V \otimes (\mathbb{F}_p)^{\otimes n-k})) \oplus \cdots \oplus s^{-1}(sV \otimes sV \otimes \cdots \otimes sV). \end{aligned}$$

For $n = 1$, $V[1] = V = s^{-1}\overline{H(BA)}$ and the composite

$$\psi_V = \alpha_A \circ \varphi_V : (TV, d_V) \longrightarrow A$$

is a quasi-isomorphism. The algebra (TV, d_V) is called a 1-connected minimal model of A.

For $n \geqslant 2$, consider the homomorphism $q_{\widehat{V}} : (TV[n], d_{V[n]}) \longrightarrow (TV, d_V)^{\otimes n}$ defined by $q_{\widehat{V}}(y) = 1^{\otimes k-1} \otimes y \otimes 1^{\otimes n-k}$, if $y \in V_k := \mathbb{F}_p^{\otimes k-1} \otimes V \otimes \mathbb{F}_p^{\otimes n-k}$, $k \in \{1; 2; \ldots n\}$

and $q_{\widehat{V}}(y) = 0$ if $y \in V[n] - \bigoplus_{i=1}^{n} V_i$. The composite

$$(TV[n], d_{V[n]}) \xrightarrow{q_{\widehat{V}}} (TV, d_V)^{\otimes n} \overset{(\psi_V)^{\otimes n}}{\longrightarrow} A^{\otimes n}$$

is a quasi-isomorphism ([**20**]). Therefore $(TV[n], d_{V[n]})$ is a minimal model of $A^{\otimes n}$.

5.3. π-*shc* minimal models

For any $n > 1$, the cyclic group S_n acts on $\widehat{V} = V[n] \subset s^{-1}(H(BA))^{\otimes n}$. This action extends diagonally on $TV[n]$ so that $d_{V[n]}$ and the homomorphism $(\psi_V)^{\otimes n} \circ q_{V[n]}$ are S_n-linear. Since $\alpha_{A^{\otimes n}}$ is a S_n-equivariant quasi-isomorphism, we deduce from [**21**, Lemma 3.3] that the composite $(\psi_V)^{\otimes n} \circ q_{V[n]}$ lifts to a homomorphism of differential graded algebra $L\colon (TV[n]d_{V[n]}) \longrightarrow \Omega B(A^{\otimes n})$ which is S_n-equivariant and $\alpha_{A^{\otimes n}} \circ L = (\psi_V)^{\otimes n} \circ q_{V[n]}$.

Let $((A, d_A), \mu_A)$ be an augmented *shc*-algebra and assume that $H^0(A) = \mathbb{F}_p$ and $H^0(A) = 0$. Define the composite $\mu_V^{(n)} = P_V \circ \mu_A^{(n)} \circ L\colon (TV[n], d_{V[n]}) \longrightarrow (TV, d_V)$ such that $\mu_V^{(2)} = \mu_V\colon (TV[2], d_{V[2]}) \longrightarrow (TV, d_V)$. The triple $((TV, d_V), \mu_V)$ is called a *shc*-minimal model for $((A, d_A), \mu_A)$ [**20**, Section 6].

Let $((A, d_A), \mu_A, \tilde{\kappa}_A)$ be an augmented π-*shc*-algebra and assume that $H^0(A) = \mathbb{F}_p$ and $H^0(A) = 0$. Following [**21**, Lemma 3.3], the composite $\tilde{\kappa}_A \circ L$ lifts to S_p-equivariant homomorphism of algebras $\hat{\kappa}_A\colon (TV[p], d_{V[p]}) \longrightarrow \mathrm{Hom}(W, \Omega BA)$. Hence the composite $\tilde{\kappa}_V = \mathrm{Hom}(W, P_V) \circ \hat{\kappa}_A\colon (TV[p], d_{V[p]}) \longrightarrow \mathrm{Hom}(W, TV)$ is a S_p-equivariant homomorphism of algebras and the triple $((TV, d_V), \mu_V, \tilde{\kappa}_V)$ is called the π-*shc*-minimal model for the $((A, d_A), \mu_A, \tilde{\kappa}_A)$ [**21**, 3.4].

6. Examples

In this section, the characteristic of the field $\mathbb{F}_p$ is $p = 2$.

6.1. Projective space $\mathbb{C}P^\infty$

Let $X = \mathbb{C}P^\infty$, $H^*(X; \mathbb{K}) = \Lambda(x) = \mathbb{F}_2[x]$ with $|x| = 2$. Thus the graded algebras $HC_*^-(N^*\mathbb{C}P^\infty; \mathbb{F}_2)$ and $HC_*^-(\Lambda(x))$ are isomorphic. It also follows that the algebraic Steenrod operations on $HC_*^-(N^*\mathbb{C}P^\infty; \mathbb{F}_2)$ are computed by those on $HC_*^-(\Lambda(x))$.

Since $((\Lambda(x), 0), \mu_{\Lambda(x)}, \tilde{\kappa}_{\Lambda(x)})$ is a 1-connected π-*shc* differential graded algebra together with finite generated cohomology groups such that the *shc* structural map $\mu_{\Lambda(x)} = \Omega B((m_{\Lambda(x)}))$ and the π-*shc* structural map $\Omega B(\Lambda(x)^{\otimes 2}) \overset{\tilde{\kappa}_{\Lambda(x)}}{\to} \mathrm{Hom}(W; \Lambda(x))$ defined on the generic elements as follows:

$$\begin{aligned} &\text{For any } y = < c_1|c_2|\cdots|c_{r-1}|c_r >,\\ &c_i = [b_1^i|b_2^i|\cdots|b_{l_i-1}^i|b_{l_i}^i],\\ &b_j^i = \underbrace{(x \wedge x \wedge \cdots \wedge x)}_{m_j} \otimes \underbrace{(x \wedge x \wedge x \wedge \cdots \wedge x)}_{p_j}, \end{aligned}$$

we have $|b_j^i| = 2(m_j + p_j)$.

Thus $|y| = 2\alpha + q$, $\alpha = \sum_{i=1}^{r}\sum_{j}^{l_i}(m_j + p_j)$; $q = r - \sum_{i=1}^{r} l_i$ and $|\tilde{\kappa}_{\Lambda(x)}(y)| = 2\alpha + q$, and finally:

$$\tilde{\kappa}_{\Lambda(x)}(y)(e_k\tau) = \begin{cases} x^{\alpha-\beta} & \text{if } 2\beta = k - q \\ 0 & \text{if not.} \end{cases}$$

In particular, for any $y \in \{<[x\otimes 1]>, <[1\otimes x]>\}$, we obtain:

$$\tilde{\kappa}_{\Lambda(x)}(y)(e_k\tau) = \begin{cases} x & \text{if } k = 0 \\ 0 & \text{if } k > 0, \end{cases}$$

and for $y = <[x\otimes x]>$

$$\tilde{\kappa}_{\Lambda(x)}(y)(e_k\tau) = \begin{cases} x^2 & \text{if } k = 0 \\ x & \text{if } k = 2 \\ 0 & \text{if } k \notin \{0,\, 2\}\,. \end{cases}$$

Then $((\Lambda(x), 0), \mu_{\Lambda(x)}, \tilde{\kappa}_{\Lambda(x)})$ has a 1-connected π-*shc* minimal model $((TV, d_V), \mu_V, \widetilde{\kappa}_V)$ where $V = x\mathbb{F}_2$, $\widetilde{\kappa}_V \colon T\widehat{V} \longrightarrow \mathrm{Hom}(W, TV)$ is the map such that $\widehat{V} = V[2] := x'\mathbb{F}_2 \oplus x''\mathbb{F}_2 \oplus x'\mathbb{F}_2 \# x''\mathbb{F}_2$ and

$$\tilde{\kappa}_{\hat{V}}(x')(e_i) = \tilde{\kappa}_{\hat{V}}(x')(e_i\tau) = \tilde{\kappa}_{\hat{V}}(x'')(e_i) = \tilde{\kappa}_{\hat{V}}(x'')(e_i\tau) = \begin{cases} x & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

$$\tilde{\kappa}_{\hat{V}}(x'\sharp x'')(e_i) = \tilde{\kappa}_{\hat{V}}(x'\sharp x'')(e_i\tau) = \begin{cases} x & \text{if } i = 1 \\ 0 & \text{if not.} \end{cases}$$

Let $T\hat{V} \xrightarrow{q_{\hat{V}}} (T(x))^{\otimes 2}$ be the surjective quasi-isomorphism of differential graded algebras defined on the generic elements as follows: $q_{\hat{V}}(x') = x\otimes 1$; $q_{\hat{V}}(x'') = 1\otimes x$; and $q_{\hat{V}}(x'\sharp x'') = 0$, and inducing the chain complex quasi-isomorphism $C^-T\hat{V} \xrightarrow{C^-q_{\hat{V}}} C^-(T(x))^{\otimes 2}$. We have the following diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(W,A) & \overset{}{\dashleftarrow} & W\otimes_\pi A^{\otimes 2} & \dashrightarrow & \mathrm{Hom}(W,A) \\
\uparrow{\scriptstyle \mathrm{Hom}(W,\varphi_2)} & & \downarrow{\scriptstyle 1_W\otimes\Psi_1^{\otimes 2}} & & \uparrow{\scriptstyle \mathrm{Hom}(W,\varphi_2)} \\
\mathrm{Hom}(W, C^-T(x)) & & W\otimes_\pi [C^-T(x)]^{\otimes 2} & & \mathrm{Hom}(W, C^-T(x) \\
\uparrow{\scriptstyle \tilde{\theta}} & & \downarrow{\scriptstyle 1_W\otimes S\overline{h}} & & \uparrow{\scriptstyle \tilde{\theta}} \\
W\otimes C^-T\widehat{V} & \overset{S}{\dashleftarrow} & W\otimes_\pi C^-((T(x))^{\otimes 2}) & \xleftarrow{1_W\otimes C^-q_{\hat{V}}} & W\otimes C^-T\widehat{V}
\end{array}$$

where

$$\begin{aligned} A &= \frac{\mathbb{F}_p[u]\otimes[\mathbb{F}_p \oplus_{i=1}^{p-1}\mathbb{F}_p{<}z_r{>}]\otimes\Lambda(y)}{<u\otimes z_r, r\neq p-1>} \\ &\cong H^*_{S^1}(L\mathbb{CP}(\infty),\mathbb{F}_p) \\ &\cong HC^-_*(N^*(\mathbb{CP}(\infty),\mathbb{F}_p)) \\ &\cong HC^-_*(\Lambda(x)) \\ &\cong HC^-_*T(x), \end{aligned}$$

with $\mid u\mid = 2$, $\mid y\mid = 2p$, $z_r = 2r+1$ and $\mathbb{F}_p{<}z_r{>}$ the graded vector space generated by z_r(see [**19**, Theorem 2]). $\overline{Sh}$ is the homomorphism of chain complexes defined in Section 1, S a linear section of $1_W\otimes C^-q_{\hat{V}}$; the structural map $\tilde{\theta}$ is defined by $\tilde{\theta} = \phi_{\Lambda(x)}\circ C^-\tilde{\kappa}_{\hat{V}}$ and φ_2 is defined as follows: The linear differential map

$$\mathbb{K}[u]\otimes[\mathbb{K}\oplus(\oplus_{i=0}^{p-1}\mathbb{K}{<}z_r{>})]\otimes\Lambda(x)\xrightarrow{\bar{\varphi}}\mathbb{K}[u]\otimes\Lambda(x)\otimes\Lambda(sx)$$

factors through

$$A\longrightarrow\mathbb{K}[u]\otimes\Lambda(x)\otimes\Lambda(sx)$$

and

$$\mathbb{K}[u]\otimes\Lambda(x)\otimes\Lambda(sx)\xrightarrow{\psi}A$$

where φ is a differential graded algebras quasi-isomorphism defined in [**19**, 4.1] or [**18**, 4.6].

Consider the differential graded algebras quasi-isomorphisms

$$\mathbb{K}[u]\otimes\Lambda(x)\otimes\Gamma(sx)\ \underset{\bar{\theta}}{\overset{\tau}{\rightleftarrows}}\ \mathbb{K}[u]\otimes\mathfrak{C}(\Lambda(x)) = \mathcal{C}^-\Lambda(x)$$

(see [**18**, 4] or [**19**, 3.2]).

From this we define the chain complex homomorphisms

$$A\ \underset{\varphi_2}{\overset{\varphi_1}{\rightleftarrows}}\ C^-_*(\Lambda(x))$$

such that $\varphi_1 = \varphi\circ\tau$ and $\varphi_2 = \psi\circ\bar{\theta}$.

More precisely we have:

(i)

$$\varphi_1(u) = \tau\circ\bar{\varphi}(u) = \tau(u\otimes 1) = u\otimes 1[\,]$$

$$\varphi_1(y) = \tau\circ\bar{\varphi}(1\otimes y) = \tau(1\otimes x^p) = 1\otimes x^p[\,]$$

$$\forall r\in\{0,\ldots,p-1\},\quad \varphi_1(z_r) = \tau\circ\bar{\varphi}(z_r) = \tau(x^r\otimes sx) = 1\otimes x^r[x].$$

(ii)

$$\varphi_2(u^l\otimes x^n[x^k]) = \psi\circ\bar{\theta}(u^l\otimes x^n[x^k]) = k\psi(u^l\otimes x^{k+n-1}\otimes sx)$$

for $q > 0$

$$\varphi_2(u^l \otimes x^n[x^{k_1}|x^{k_2}|\cdots|x^{k_{q-1}}|x^{k_q}]) = 0.$$

$$\varphi_2(u^l \otimes x^n[\,]) = \psi \circ \bar{\theta}(u^l \otimes x^n[\,]) = \begin{cases} u^l \otimes y^k & \text{if } n = kp \\ 0 & \text{if } n \neq kp. \end{cases}$$

6.1.1. Finally, observe that S, the linear section of $1_W \otimes C^- q_{\hat{V}}$, is only defined in low degrees as follows:

1. $S(e_i \otimes u^l \otimes 1[\,]) = e_i \otimes u^l \otimes 1[\,]$
2. $S(e_i \otimes u^l \otimes (x^k \otimes 1)[\,]) = e_i \otimes u^l \otimes x'^k[\,]$
3. $S(e_i \otimes u^l \otimes (1 \otimes x^k)[\,]) = e_i \otimes u^l \otimes x''^k[\,]$
4. $S(e_i \otimes u^l \otimes (x^k \otimes x^{k'})[\,]) = e_i \otimes u^l \otimes (x'^k x''^{k'})[\,]$
5. $S(e_i \otimes u^l \otimes (x^k \otimes 1)[x^{k'} \otimes 1]) = e_i \otimes u^l \otimes x'^k[x'^{k'}]$
6. $S(e_i \otimes u^l \otimes (1 \otimes x^k)[1 \otimes x^{k'}]) = e_i \otimes u^l \otimes x''^k[x''^{k'}]$
7. $S(e_i \otimes u^l \otimes 1[x^k \otimes 1]) = e_i \otimes u^l \otimes 1[x'^k]$
8. $S(e_i \otimes u^l \otimes 1[1 \otimes x^k]) = e_i \otimes u^l \otimes 1[x''^k]$
9. $S(e_i \otimes u^l \otimes (x^k \otimes 1)[1 \otimes x^{k'}]) = e_i \otimes u^l \otimes x'^k[x''^{k'}]$
10. $S(e_i \otimes u^l \otimes (1 \otimes x^k)[x^{k'} \otimes 1]) = e_i \otimes u^l \otimes x''^k[x'^{k'}]$
11. $S(e_i \otimes u^l \otimes 1[1 \otimes x^k|1 \otimes x^{k'}]) = e_i \otimes u^l \otimes 1[x''^k|x''^{k'}]$
12. $S(e_i \otimes u^l \otimes 1[x^k \otimes 1|x^{k'} \otimes 1]) = e_i \otimes u^l \otimes 1[x'^k|x'^{k'}]$
13. $S(e_i \otimes u^l \otimes 1[1 \otimes x^k|x^{k'} \otimes 1]) = e_i \otimes u^l \otimes 1[x''^k|x'^{k'}]$
14. $S(e_i \otimes u^l \otimes 1[x^k \otimes 1|1 \otimes x^{k'}]) = e_i \otimes u^l \otimes 1[x'^k|x''^{k'}]$
15. $S(e_i \otimes u^l \otimes 1[x^k \otimes x^{k'}]) = e_i \otimes u^l \otimes 1[x'^k x''^{k'}]$ $(k > 1)$
16. $S(e_i \otimes u^l \otimes 1[x \otimes 1|1 \otimes x]) = e_i \otimes u^l \otimes 1[x'|x''] + e_i \otimes u^l \otimes 1[x'\sharp x'']$ $(k = 1)$
17. $S(e_i \otimes u^l \otimes 1[x^{k_1} \otimes 1|x^{k_2} \otimes 1|1 \otimes x^{k_3}|1 \otimes x^{k_4}]) = e_i \otimes u^l \otimes 1[x'^{k_1}|x'^{k_2}|x''^{k_3}|x''^{k_4}]$
18. $S(e_i \otimes u^l \otimes 1[1 \otimes x|x \otimes x|x \otimes 1]) = e_i \otimes u^l \otimes 1[x''|x'x''|x']$
19. $S(e_i \otimes u^l \otimes 1[x \otimes x|x \otimes 1|1 \otimes x]) = e_i \otimes u^l \otimes 1[x' \cdot x''|x'|x''] +$ $e_i \otimes u^l \otimes 1[x' \cdot x''|x'\sharp x'']$
20. $S(e_i \otimes u^l \otimes 1[x \otimes x|x \otimes 1|1 \otimes x]) = e_i \otimes u^l \otimes 1[x'|x''|x' \cdot x''] +$ $e_i \otimes u^l \otimes 1[x'\sharp x''|x' \cdot x'']$
21. $S(e_i \otimes u^l \otimes 1[x \otimes x|x \otimes 1|1 \otimes x]) = e_i \otimes u^l \otimes (x'^2 x'')[x''] +$ $e_i \otimes u^l \otimes (x'x'')(x'\sharp x'')[\,]$
22. $S(e_i \otimes u^l \otimes 1[x \otimes x|x \otimes 1|1 \otimes x]) = e_i \otimes u^l \otimes (x'x''^2)[x'] +$ $e_i \otimes u^l \otimes (x'\sharp x'')(x' \cdot x'')[\,]$.

6.1.2. Now we define algebraic Steenrod operations on $H^q(A)$ by the formula $Sq^i(x) = cl(\tilde{\theta}(S \circ (id_W \otimes \bar{S}h)(e_{n-i} \otimes x \otimes x)))$, $x \in H^*(A)$

1. For $x = u$,
   (a) $Sq^0(u) = cl[\varphi_2(u \otimes 1[\,])] = u = x$.
   (b) $Sq^1(u) = cl[\varphi_2(g(u^2)(e_1) \otimes \tilde{\kappa}_{\hat{V}}(1)(e_0)[\,] + g(u^2)(e_0) \otimes \tilde{\kappa}_{\hat{V}}(1)(e_1)[\,])] = 0$.
   (c) $Sq^2(u) = cl[\varphi_2(g(u^2)(e_0) \otimes \tilde{\kappa}_{\hat{V}}(1)(e_0)[\,])] + cl[\varphi_2(u^2 \otimes 1[\,])] = x^2$.

(d) for $i > 2$, $Sq^i(u) = 0$.

2. For $x = y$,

(a) $Sq^0(y) = cl[\varphi_2(g(1)(e_0) \otimes \tilde{\kappa}_{\hat{V}}(x'^2x''^2)(e_4)[\,])] = cl[\varphi_2(1 \otimes x^2[\,])] = 1 \otimes 1 \otimes y$.

(b) $Sq^1(y) = cl[\varphi_2(Id \otimes Id)(g(1) \otimes \tilde{\kappa}_{\hat{V}}(x'^2x''^2)) \circ \psi_W(e_3)] = 0$.

(c) $Sq^2(y) = cl[\varphi_2(Id \otimes Id)(g(1) \otimes \tilde{\kappa}_{\hat{V}}(x'^2x''^2)) \circ \psi_W(e_2)] = cl[\varphi_2(1 \otimes x^3[\,])] = 0$.

(d) $Sq^3(y) = cl[\varphi_2(Id \otimes Id)(g(1) \otimes \tilde{\kappa}_{\hat{V}}(x'^2x''^2)) \circ \psi_W(e_1)] = 0$.

(e) $Sq^4(y) = cl[\varphi_2(Id \otimes Id)(g(1) \otimes \tilde{\kappa}_{\hat{V}}(x'^2x''^2)) \circ \psi_W(e_0)] = cl[\varphi_2(1 \otimes x^4[\,])] = y^2$.

(f) for $i > 4 = 2p$, $Sq^i(y) = 0$.

3. For $x = z_r \in H^{2r+1}(A)$ with $0 \leqslant r \leqslant p-1$, since $p = 2$ then $r \in \{0, 1\}$.

(a) $x = z_0$

i.
$$\begin{aligned} Sq^0(z_0) = cl[\varphi_2((&Id \otimes Id \otimes s^{\otimes 2})(g(1) \otimes \tilde{\kappa}_{\hat{V}}(1) \otimes \tilde{\kappa}_{\hat{V}}(x') \otimes \tilde{\kappa}_{\hat{V}}(1) \otimes \\ &\tilde{\kappa}_{\hat{V}}(x'') \otimes \tilde{\kappa}_{\hat{V}}(x')))\psi_W^{(3)}(e_1) + \\ &Id \otimes Id \otimes s^{\otimes 2})(g(1) \otimes \tilde{\kappa}_{\hat{V}}(1) \otimes \tilde{\kappa}_{\hat{V}}(x'\sharp x''))\psi_W^{(2)}(e_1)] \\ =\ & Sq^0(z_0)cl[\varphi_2(1 \otimes 1[x])] \\ =\ & z_0. \end{aligned}$$

ii.
$$\begin{aligned} Sq^1(z_0) = cl[\varphi_2((&Id \otimes Id \otimes s^{\otimes 2})(g(1) \otimes \tilde{\kappa}_{\hat{V}}(1) \otimes \tilde{\kappa}_{\hat{V}}(x') \otimes \tilde{\kappa}_{\hat{V}}(x'') + \\ &g(1) \otimes \tilde{\kappa}_{\hat{V}}(1) \otimes \tilde{\kappa}_{\hat{V}}(x'') \otimes \tilde{\kappa}_{\hat{V}}(x')))\psi_W^{(3)}(e_0) + \\ &(Id \otimes Id \otimes s^{\otimes 2})(g(1) \otimes \tilde{\kappa}_{\hat{V}}(1) \otimes \tilde{\kappa}_{\hat{V}}(x'\sharp x''))\psi_W^{(2)}(e_0)] \\ = cl[&\varphi_2(2(1 \otimes 1[x|x]))] = z_0^2 = 0. \end{aligned}$$

iii. for $i > 1$, $Sq^i(z_0) = 0$.

(b) $x = z_1$.

i. $Sq^0(z_1) = z_1$.

ii.
$$\begin{aligned} Sq^1(z_1) = cl[\varphi_2((&Id \otimes Id \otimes s^{\otimes 2})(g(1) \otimes \tilde{\kappa}_{\hat{V}}(x'x'') \otimes \tilde{\kappa}_{\hat{V}}(x') \otimes \tilde{\kappa}_{\hat{V}}(x'') \\ &+ g(1) \otimes \tilde{\kappa}_{\hat{V}}(x'x'') \otimes \tilde{\kappa}_{\hat{V}}(x'') \otimes \tilde{\kappa}_{\hat{V}}(x')))\psi_W^{(3)}(e_2) + \\ &(Id \otimes Id \otimes s^{\otimes 2})(g(1) \otimes \tilde{\kappa}_{\hat{V}}(x'x'') \otimes \tilde{\kappa}_{\hat{V}}(x'\sharp x''))\psi_W^{(2)}(e_2)] \\ = cl[&\varphi_2(2(1 \otimes x[x|x]))] = 0. \end{aligned}$$

iii.
$$\begin{aligned} Sq^2(z_1) = cl[\varphi_2((&Id \otimes Id \otimes s^{\otimes 2})(g(1) \otimes \tilde{\kappa}_{\hat{V}}(x'x'') \otimes \tilde{\kappa}_{\hat{V}}(x') \otimes \tilde{\kappa}_{\hat{V}}(x'') \\ &+ g(1) \otimes \tilde{\kappa}_{\hat{V}}(x'x'') \otimes \tilde{\kappa}_{\hat{V}}(x'') \otimes \tilde{\kappa}_{\hat{V}}(x')))\psi_W^{(3)}(e_1) + \\ &(Id \otimes Id \otimes s^{\otimes 2})(g(1) \otimes \tilde{\kappa}_{\hat{V}}(x'x'') \otimes \tilde{\kappa}_{\hat{V}}(x'\sharp x''))\psi_W^{(2)}(e_1)] \\ = cl[&\varphi_2(1 \otimes x_2[x|x])] \\ = y &\otimes z_0. \end{aligned}$$

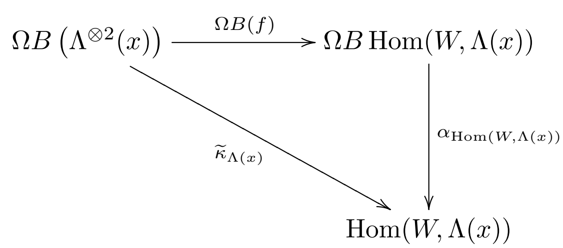
iv. $Sq^3(z_1) = cl[\varphi_2(2(1 \otimes x^2[x|x]))] = 0$.

v. for $i > 3$, $Sq^i(z_1) = 0$.

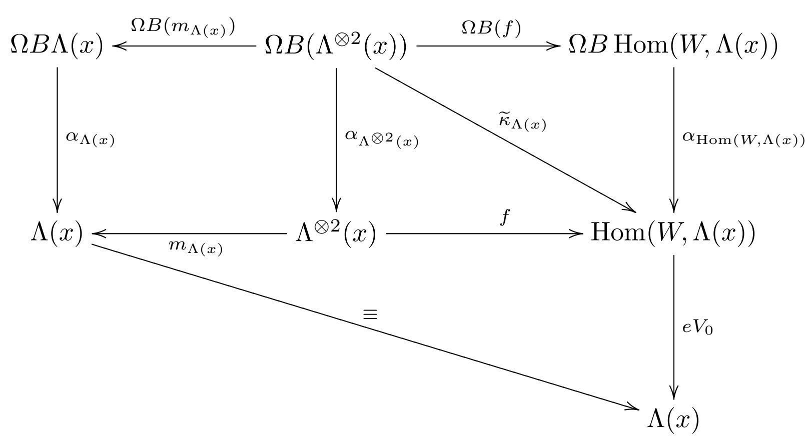
6.2. Odd sphere S^{2q+1}

As in the previous example, let $X = S^{2q+1}$, $H^*(X;\mathbb{K}) = \Lambda(x) = \mathbb{F}_2[x]$ with $|x| = 2q+1$. Thus the graded algebras $HC_*^-(N^*S^{2q+1};\mathbb{F}_2)$ and $HC_*^-(\Lambda(x))$ are isomorphic. It also follows that the algebraic Steenrod operations on $HC_*^-(N^*S^{2q+1};\mathbb{F}_2)$ are computed by those on $HC_*^-(\Lambda(x))$.

Since $((\Lambda(x),0),\mu_{\Lambda(x)},\tilde{\kappa}_{\Lambda(x)})$ is a 1-connected π-*shc* differential graded algebra together with finite generated cohomology groups such that the *shc* structural map $\mu_{\Lambda(x)} = \Omega B((m_{\Lambda(x)}))$ and the π-*shc* structural map $\Omega B(\Lambda(x)^{\otimes 2}) \overset{\tilde{\kappa}_{\Lambda(x)}}{\to} \mathrm{Hom}(W;\Lambda(x))$ are defined by the following diagram



which induces the diagram



where f is defined by

$$\begin{aligned} f\colon \Lambda^{\otimes 2}(x) &\longrightarrow \mathrm{Hom}(W,\Lambda(x)) \\ x\otimes x &\longmapsto f(x\otimes x) = f_{x\otimes x} \\ 1\otimes x &\longmapsto f(1\otimes x) = f_{1\otimes x} \\ x\otimes 1 &\longmapsto f(x\otimes 1) = f_{x\otimes 1} \end{aligned}$$

such that

$$f_{x\otimes x}(e_i) = f_{x\otimes x}(\tau e_i) = \begin{cases} x & \text{if } i = 2q+1 \\ 0 & \text{if } i \neq 2q+1 \end{cases}$$

$$f_{1\otimes x}(e_i) = f_{1\otimes x}(\tau e_i) = \begin{cases} x & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

$$f_{x\otimes 1}(e_i) = f_{x\otimes 1}(\tau e_i) = \begin{cases} x & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

$$f_1(e_i) = f_1(\tau e_i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0, \end{cases}$$

where f_1 is the unit of the differential graded algebra $\mathrm{Hom}(W, \Lambda(x))$.

Then $((\Lambda(x), 0), \mu_{\Lambda(x)}, \tilde{\kappa}_{\Lambda(x)})$ has a 1-connected π-*shc* minimal model $((TV, d_V), \mu_V, \widetilde{\kappa}_V)$ where $V = x\mathbb{F}_2$, $\widetilde{\kappa}_V \colon T\widehat{V} \longrightarrow \mathrm{Hom}(W, TV)$ a map such that $\widehat{V} = V[2] := V' \oplus V'' \oplus V'\#V''$ with $V' := s^{-1}(sV \otimes \mathbb{K}) = s^{-1}(H^+B\Lambda(x) \otimes \mathbb{K})$; $V'' = s^{-1}(\mathbb{K} \otimes sV) = s^{-1}(\mathbb{K} \otimes H^+B\Lambda(x))$; $V'\#V'' = s^{-1}(sV \otimes sV) = s^{-1}(H^+B\Lambda(x) \otimes H^+B\Lambda(x))$, where a'_{k_1}, a''_{k_2}, $a'_{k_1}\#a''_{k_2}$ are the respective generators of V', V'', $V'\#V''$;

$$a'_{k_1} := s^{-1}(sa_{k_1} \otimes 1); \quad a''_{k_2} := s^{-1}(1 \otimes sa_{k_2}); \quad a'_{k_1}\#a''_{k_2} := s^{-1}(sa_{k_1} \otimes sa_{k_2});$$

with $a_{k_1} := \underbrace{[x|\cdots|x]}_{k_1 times}$; $a_{k_2} := \underbrace{[x|\cdots|x]}_{k_2 times}$; $|a'_{k_1}| = 2qk_1 + 1$; $|a''_{k_2}| = 2qk_2 + 1$; $|a'_{k_1}\#a''_{k_2}| = 2q(k_1 + k_2) + 1$. And

$$\widetilde{\kappa}_{\widehat{V}}(a'_{k_1}) = \begin{cases} f_1 & \text{if } k_1 = 0 \\ f_{1\otimes x} & \text{if } k_1 = 1 \\ 0 & \text{if not.} \end{cases}$$

Since $\left|a'_{k_1}\right| = 2q+1; \widetilde{\kappa}_{\widehat{V}}(a'_{k_1})(\tau^j e_i) = \begin{cases} x & \text{if } i = 0 \\ 0 & \text{if } i \neq 0; \end{cases} \quad j \geqslant 0,$

$$\widetilde{\kappa}_{\widehat{V}}(a'_{k_1}) = \begin{cases} f_1 & \text{if } k_2 = 0 \\ f_{1\otimes x} & \text{if } k_2 = 1 \\ 0 & \text{if not;} \end{cases}$$

since $\left|a''_{k_2}\right| = 2q+1, \widetilde{\kappa}_{\widehat{V}}(a''_1)(\tau^j e_i) = \begin{cases} x & \text{if } i = 0 \\ 0 & \text{if } i \neq 0, \end{cases}$

$$\widetilde{\kappa}_{\widehat{V}}(a'_{k_1}\#a''_{k_2})(\tau^j e_i) = \begin{cases} x & \text{if } i = 2 \quad k_1 = k_2 = 1 \\ 0 & \text{if not.} \end{cases}$$

Let $T\hat{V} \xrightarrow{q_{\hat{V}}} (T(x))^{\otimes 2}$ be the surjective quasi-isomorphism of differential graded algebras defined on the generic elements as follows:

$$q_{\widehat{V}}(a'_{k_1}) = s^{-1}\underbrace{[x|\cdots|x]}_{k_1 times}\otimes 1; \quad q_{\widehat{V}}(a''_{k_2}) = 1 \otimes s^{-1}\underbrace{[x|\cdots|x]}_{k_2 times}; \quad q_{\widehat{V}}(a'_k\#a''_k) = 0.$$

and inducing the chain complex quasi-isomorphism $C^-T\hat{V} \overset{C^-q_{\hat{V}}}{\to} C^-(T(x))^{\otimes 2}$. We have the following diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(W,A) & \overset{}{\longleftarrow} & W\otimes_\pi A^{\otimes 2} & \longrightarrow & \mathrm{Hom}(W,A) \\
\uparrow{\scriptstyle \mathrm{Hom}(W,\varphi_2)} & & \downarrow{\scriptstyle 1_W\otimes \Psi_1^{\otimes 2}} & & \uparrow{\scriptstyle \mathrm{Hom}(W,\varphi_2)} \\
\mathrm{Hom}(W,C^-T(x)) & & W\otimes_\pi [C^-T(x)]^{\otimes 2} & & \mathrm{Hom}(W,C^-T(x) \\
\uparrow{\scriptstyle \tilde{\theta}} & & \downarrow{\scriptstyle 1_W\otimes S\overline{h}} & & \uparrow{\scriptstyle \tilde{\theta}} \\
W\otimes C^-T\widehat{V} & \overset{S}{\longleftarrow} & W\otimes_\pi C^-((T(x))^{\otimes 2}) & \overset{1_W\otimes C^-q_{\widehat{V}}}{\longleftarrow} & W\otimes C^-T\widehat{V},
\end{array}$$

where

$$A=\frac{\mathbb{K}[u]\otimes\Lambda(y)\otimes\Gamma(sx)}{\langle u\otimes\gamma^n(sx),\, \gamma^n(sx)\otimes y\rangle, n\neq kp}\cong H^{-*}(\mathbb{K}[u]\otimes\Lambda(x)\otimes\Gamma(sx))$$
$$\cong H^{-*}_{S^1}(LX,\mathbb{K})$$

with $|u|=2$; $\|x\|=2q+1$; $|y|=2qp+1$ [**19**], see [**19**, Theorem 2]. S is the section of $1_W\otimes C^-q_{\widehat{V}}$; $\widetilde{\theta}=\phi_{\Lambda(x)}\circ C^-\widetilde{\kappa}_{\widehat{V}}$ and φ_2 is the map defined as follows: Consider the differential algebra homomorphisms

$$\mathbb{K}[u]\otimes\Lambda(x)\otimes\Gamma(sx) \underset{\overline{\theta}}{\overset{\tau}{\rightleftarrows}} \mathbb{K}[u]\otimes\mathcal{C}(\Lambda(x))=\mathcal{C}^-\Lambda(x).$$

respectively defined by:

(i) $\tau(1\otimes x\otimes 1)=1\otimes x[\,]$; $\quad\tau(u^l\otimes 1\otimes 1)=u^l\otimes 1[\,]$
$\tau(1\otimes 1\otimes\gamma^k(sx))=1\otimes\underbrace{[x|\cdots|x]}_{(k)times}$; $\quad\tau(u^l\otimes x\otimes\gamma^k(sx))=u^l\otimes x\underbrace{[x|\cdots|x]}_{ktimes}$

(ii) $\overline{\theta}(u^l\otimes 1[\,])=u^l\otimes 1\otimes 1$; $\quad\overline{\theta}(1\otimes 1\underbrace{[x|\cdots|x]}_{(k)times})=1\otimes 1\otimes\gamma^k(sx)$
$\overline{\theta}(1\otimes x[\,])=1\otimes x\otimes 1$; $\quad\overline{\theta}(u^l\otimes x\underbrace{[x|\cdots|x]}_{(k)times})=u^l\otimes x\otimes\gamma^k(sx)$.

τ and $\overline{\theta}$ are homotopic equivalence algebras inverse to each other and $H(\overline{\theta})=H(\tau)^{-1}$. From this differential graded algebra quasi-isomorphism we deduce the following chain complex homomorphisms:

$$C^-(\Lambda(x)) \underset{\varphi_2}{\overset{\varphi_1}{\rightleftarrows}} A$$

$\varphi_1=\tau\circ\phi$ et $\varphi_2=\Psi\circ\overline{\theta}$. More precisely, we have:

1.
$$\varphi_1(u)=\tau\circ\varphi(u)=u\otimes 1[\,];$$
$$\varphi_1(y)=\tau\circ\varphi(y)=\tau(x\otimes\gamma^{p-1}(sx))=1\otimes x\underbrace{[x|\cdots|x]}_{(p-1)times};$$
$$\varphi_1(sx)=\tau\circ\phi(\gamma^1(sx))=\tau(\gamma^1(sx))=\tau(sx)=1\otimes 1[x]$$

2.
$$\varphi_2(u^l \otimes 1[\,]) = \Psi \circ \overline{\theta}(u^l \otimes 1 \otimes 1) = u^l \otimes 1 \otimes 1$$
$$\varphi_2(1 \otimes x[\,]) = \Psi \circ \overline{\theta}(1 \otimes x[\,]) = \Psi(1 \otimes x \otimes 1) = 0$$
$$\begin{aligned}\varphi_2(1 \otimes 1 \underbrace{[x|\cdots|x]}_{ntimes}) &= \Psi \circ \overline{\theta}(1 \otimes 1 \underbrace{[x|\cdots|x]}_{ntimes}) \\ &= \Psi(1 \otimes 1 \otimes \gamma^n(sx)) \\ &= 1 \otimes 1 \otimes \gamma^n(sx)\end{aligned}$$

$$\varphi_2(u^l \otimes x \underbrace{[x|\cdots|x]}_{ntimes})$$
$$= \Psi(u^l \otimes x \otimes \gamma^n(Sx)) = \begin{cases} (C_n^{p-1})^{-1} u^l \otimes y \otimes \gamma^{n-p+1}(sx) & \text{if } n+1 = kp \\ \\ 0 & \text{if } n+1 \neq kp \end{cases}$$

(see [**19**]).

6.2.1. Finally, we observe that S, the linear section of $1_W \otimes C^- q_{\hat{V}}$, is only defined in low degrees as follows:

1. $S(e_i \otimes u^l \otimes 1[\,]) = e_i \otimes u^l \otimes 1[\,]$
2. $S(e_i \otimes u^l \otimes 1[x \otimes 1|1 \otimes x]) = e_i \otimes u^l \otimes 1[a_1'|a_1''] + e_i \otimes u^l \otimes 1[a_1' \# a_1'']$
3. $S(e_i \otimes u^l \otimes 1[x \otimes 1]) = e_i \otimes u^l \otimes 1[a_1']$
4. $S(e_i \otimes u^l \otimes 1[1 \otimes x]) = e_i \otimes u^l \otimes 1[a_1'']$
5. $S(e_i \otimes u^l \otimes 1[x \otimes x]) = e_i \otimes u^l \otimes 1[a_1' \# a_1''] + e_i \otimes u^l \otimes 1[a_1' \cdot a_1'']$
6. $S(e_i \otimes u^l \otimes x \otimes 1[1 \otimes x]) = e_i \otimes u^l \otimes a_1'[a_1'']$
7. $S(e_i \otimes u^l \otimes 1 \otimes x[x \otimes 1]) = e_i \otimes u^l \otimes a_1''[a_1']$
8. $S(e_i \otimes u^l \otimes 1[1 \otimes x|x \otimes 1]) = e_i \otimes u^l \otimes 1[a_1''|a_1']$
9. $S(e_i \otimes u^l \otimes x \otimes x[x \otimes x]) = e_i \otimes u^l \otimes a_1' \cdot a_1''[a_1' \cdot a_1'']$
10. $S(e_i \otimes u^l \otimes x^2 \otimes x[1 \otimes x]) = e_i \otimes u^l \otimes a_1'^2 \cdot a_1''[a_1'']$
11. $S(e_i \otimes u^l \otimes x \otimes x^2[x \otimes 1]) = e_i \otimes u^l \otimes a_1' \cdot a_1''^2[a_1']$
12. $S(e_i \otimes u^l \otimes 1[x \otimes 1|1 \otimes x|x \otimes 1|1 \otimes x]) = e_i \otimes u^l \otimes 1[a_1'|a_1''|a_1'|a_1'']$
13. $S(e_i \otimes u^l \otimes 1[1 \otimes x|x \otimes 1|1 \otimes x|x \otimes 1]) = e_i \otimes u^l \otimes 1[a_1''|a_1'|a_1''|a_1']$
14. $S(e_i \otimes u^l \otimes 1[x \otimes x|x \otimes 1|1 \otimes x]) = e_i \otimes u^l \otimes 1[a_1' \cdot a_1''|a_1'|a_1'']$
15. $S(e_i \otimes u^l \otimes 1[1 \otimes x|x \otimes x|x \otimes 1]) = e_i \otimes u^l \otimes 1[a_1''|a_1' \cdot a_1''|a_1']$
16. $S(e_i \otimes u^l \otimes 1[x \otimes 1|1 \otimes x|x \otimes x]) = e_i \otimes u^l \otimes 1[a_1'|a_1''|a_1' \cdot a_1'']$
17. $S(e_i \otimes u^l \otimes x \otimes x[x \otimes 1|1 \otimes x]) = e_i \otimes u^l \otimes a_1' \cdot a_1''[a_1'|a_1'']$
18. $S(e_i \otimes 1 \otimes 1 \otimes x[x \otimes 1|1 \otimes x|x \otimes 1]) = e_i \otimes 1 \otimes a_1''[a_1'|a_1''|a_1']$
19. $S(e_i \otimes 1 \otimes x \otimes 1[1 \otimes x|x \otimes 1|1 \otimes x]) = e_i \otimes 1 \otimes a_1'[a_1''|a_1'|a_1''].$

6.2.2. Now we define algebraic Steenrod operations on $H^q(A)$ by the formula

$$Sq^i(x) = cl(\tilde{\theta}(S \circ (id_W \otimes \bar{S}h)(e_{n-i} \otimes x \otimes x))), \quad x \in H^*(A).$$

If $a \in H^q(A)$, $a \in \{u, sx, y\}$

1. $\underline{a = u}$
   (a) For $i = 0$, $Sq^0(u) = cl\,[\varphi_2(u \otimes 1[\ ])] = u$.
   (b) For $i = 1$, $Sq^1(u) = 0$.
   (c) For $i = 2$, $Sq^2(u \otimes 1 \otimes 1) = cl\,\left[\varphi_2(u^2 \otimes 1[\ ])\right] = u^2$.
   (d) For $i > 2$, $Sq^i(u \otimes 1 \otimes 1) = 0$.
2. $\underline{a = sx \in H^{2q}(A)}$
   (a) For $i = 0$, $Sq^0(sx) = cl\,[\varphi_2(1 \otimes 1[x])] = cl[\Psi \circ \theta(1 \otimes 1[x])] = sx$.
   (b) For $i \in \{1, \dots, 2q-1\}$, $Sq^i(a) = 0$.
   (c) For $i = 2q$,

$$\begin{aligned} Sq^{2q}(sx) &= cl\,[\varphi_2\,(2(1 \otimes 1[x|x]))] \\ &= cl\,\left[2\left(1 \otimes 1 \otimes \gamma^2 sx\right)\right] \\ &= cl\,\left(C_2^1\left(1 \otimes 1 \otimes \gamma^2 sx\right)\right) \\ &= (1 \otimes 1 \otimes \gamma sx)(1 \otimes 1 \otimes \gamma sx) \\ &= 0 \\ &= (1 \otimes 1 \otimes sx)^2. \end{aligned}$$

   (d) For $i > 2q$, $Sq^i(a) = 0$.
3. $\underline{a = y \in H^{2qp+1}(A)}$
   (a) For $i = 0$, $Sq^0(y) = cl[\varphi_2(1 \otimes x \otimes [x])] = y$.
   (b) For $i = 1$, $Sq^1(y) = 0$.
   (c) For $i = 2$, $Sq^2(a) = 0$.
   (d) For $i \in \{3, \dots, 4q\}$, $Sq^i(a) = 0$.
   (e) For $i = 4q+1$, $Sq^{4q+1}(a) = 0 = y^2$.
   (f) For $i > 4q+1$, $Sq^i(a) = 0$.

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