

## BROWN'S CRITERION IN BREDON HOMOLOGY

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### Abstract

We generalize Brown's criterion for homological finiteness properties to the setting of Bredon homology.

### 1. Introduction

Bredon cohomology has become an important algebraic tool for studying classifying spaces  $E_{\mathfrak{F}}G$  of discrete groups  $G$  with stabilisers in a given family  $\mathfrak{F}$  of subgroups of  $G$ . It has been defined for finite groups by Bredon [Bre67], and the definition has been extended to arbitrary groups and families of subgroups by Lück [Lüc89]. The basic idea in passing from classical cohomology to Bredon cohomology is to replace  $G$ , regarded as a small category, by the orbit category  $\mathcal{O}_{\mathfrak{F}}G$ .

More precisely, let  $G$  be a discrete group. By a *family* of subgroups of  $G$  we mean a non-empty set  $\mathfrak{F}$  of subgroups of  $G$  which is closed under conjugation. The transitive  $G$ -sets  $G/K$  with  $K \in \mathfrak{F}$  and  $G$ -maps between them form the *orbit category*  $\mathcal{O}_{\mathfrak{F}}G$ . A *right (left) Bredon module over  $\mathcal{O}_{\mathfrak{F}}G$*  is a contravariant (covariant) functor from  $\mathcal{O}_{\mathfrak{F}}G$  to the category  $\mathfrak{Ab}$  of abelian groups. A morphism of Bredon modules of the same variance is a natural transformation. Right (left) Bredon modules and the morphisms between them form a category that is denoted by  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  ( $\mathcal{O}_{\mathfrak{F}}G\text{-Mod}$ ).

$\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  and  $\mathcal{O}_{\mathfrak{F}}G\text{-Mod}$  are functor categories, and therefore they inherit many properties from the category  $\mathfrak{Ab}$ . Among others, they are abelian categories in which all small limits and colimits exist, they have enough projectives and there exists a notion of being finitely generated. Details are given in Section 2 below.

Let  $n \in \mathbb{N} \cup \{\infty\}$ . An  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  is said to be of *type  $\mathfrak{F}$ -FP $_n$*  if there exists a resolution

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

of  $M$  by projective  $\mathcal{O}_{\mathfrak{F}}G$ -modules such that  $P_k$  is finitely generated for every  $k \leq n$ . The trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}$  maps every object of  $\mathcal{O}_{\mathfrak{F}}G$  to  $\mathbb{Z}$  and every morphism of  $\mathcal{O}_{\mathfrak{F}}G$  to the identity. A group  $G$  is said to be of *type  $\mathfrak{F}$ -FP $_n$*  if  $\mathbb{Z}$  is of type  $\text{FP}_n$  as a right  $\mathcal{O}_{\mathfrak{F}}G$ -module.

In the special case that  $\mathfrak{F} = \{1\}$  consists only of the trivial group Bredon cohomology reduces to classical cohomology of groups. The finiteness properties  $\text{FP}_n$  in this case have been extensively studied. The classical proofs sometimes can be extended

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to also cover the case where  $\mathfrak{F} = \mathcal{FIN}$  is the family of finite subgroups. This is trivial for torsion free groups, but it is also true, for example, for hyperbolic groups, arithmetic groups, mapping class groups, and outer automorphism groups of finitely generated free groups; see [Lüc05, Sections 4.6–4.10]. On the other hand, Leary and Nucinkis [LN03] showed how much finiteness properties with respect to  $\mathcal{FIN}$  can differ from the classical ones.

The next family of interest is the family  $\mathcal{VC}$  of virtually cyclic subgroups. One result here is that for an elementary amenable group being of type  $\mathcal{VC}\text{-FP}_\infty$  is equivalent to being virtually cyclic [KMPN11].

In the classical setting, Brown’s Criterion [Bro87, Theorem 2.2] has been fruitful in the study of the properties  $\text{FP}_n$ . Our main result is a generalization of this criterion to the Bredon setting. In order to state it, some more definitions are needed.

A  $G$ -CW-complex  $X$  is a CW-complex on which  $G$  acts by cell-permuting homeomorphisms such that the stabilizer of a cell fixes that cell pointwise. We let  $\underline{C}_*(X)$  denote the Bredon cellular chain complex of  $X$ ; cf. [MV03, p. 11]. The *Bredon homology modules*  $\underline{H}_*(X)$  of  $X$  are defined to be the homology modules of the Bredon chain complex  $\underline{C}_*(X)$ . Evaluated at  $G/K \in \mathcal{O}_{\mathfrak{F}}G$  these Bredon modules give

$$\underline{H}_*(X)(G/K) = H_*(X^K),$$

where the right-hand side is the ordinary homology of the fixed point complex  $X^K$ . This definition is functorial. Analogously to the classical case, we define the *reduced Bredon homology modules*  $\tilde{\underline{H}}_*(X)$  to be the kernel of the morphism  $\underline{H}_*(X) \rightarrow \underline{H}_*(\text{pt.})$ , which is induced by the map from  $X$  to the singleton space. We say that  $X$  is  $\mathfrak{F}$ -acyclic up to dimension  $n$  if  $\tilde{\underline{H}}_k(X) = 0$  for every  $k \leq n$ . Note that being  $\mathfrak{F}$ -acyclic up to dimension  $-1$  is equivalent to the condition that  $X^K \neq \emptyset$  for every  $K \in \mathfrak{F}$ .

The following is completely analogous to Brown’s original article [Bro87]: Let  $n \in \mathbb{N}$ . A  $G$ -CW-complex  $X$  is said to be  $\mathfrak{F}$ - $n$ -good if the following two conditions hold:

1.  $X$  is  $\mathfrak{F}$ -acyclic up to dimension  $n - 1$ , and
2. For every cell  $\sigma$  of dimension  $\leq n$ , the family  $\mathfrak{F} \cap G_\sigma = \{K \cap G_\sigma \mid K \in \mathfrak{F}\}$  is contained in  $\mathfrak{F}$ , and the stabiliser  $G_\sigma$  of  $\sigma$  is of type  $(\mathfrak{F} \cap G_\sigma)\text{-FP}_{n-p}$ .

A filtration  $(X_\alpha)_{\alpha \in I}$  of a  $\Gamma$ -CW-complex  $X$  by  $\Gamma$ -invariant subcomplexes is said to be of *finite  $n$ -type* if the  $n$ -skeleta  $X_\alpha^{(n)}$  are cocompact for all  $\alpha \in I$ .

A directed system of Bredon modules  $(M_\alpha)_{\alpha \in I}$  is said to be *essentially trivial* if for every  $\alpha \in I$  there exists  $\beta \geq \alpha$  such that the homomorphism  $M_\alpha \rightarrow M_\beta$  is trivial.

**Main Theorem.** *Let  $G$  be a group and  $\mathfrak{F}$  a family of subgroups of  $G$ . Let  $X$  be an  $\mathfrak{F}$ - $n$ -good  $G$ -CW-complex and let  $(X_\alpha)_{\alpha \in I}$  be a filtration by  $G$ -invariant subcomplexes of finite  $n$ -type.*

*Then  $G$  is of type  $\mathfrak{F}\text{-FP}_n$  if and only if the directed system  $(\tilde{\underline{H}}_k(X_\alpha))_{\alpha \in I}$  of reduced Bredon homology modules is essentially trivial for all  $k < n$ .*

The importance of a directed system being essentially trivial stems from the following fact, which is the analogue of [Bro87, Lemma 2.1]:

**Observation 1.** *A directed system  $(M_\alpha)_{\alpha \in I}$  of  $\mathcal{O}_{\mathfrak{F}}G$ -modules is essentially trivial if and only if*

$$\varinjlim_{\alpha} \prod_{K \in \mathfrak{F}} \prod_{J_K} M_\alpha(G/K) = 0$$

for every family of cardinals  $(J_K)_{K \in \mathfrak{F}}$ .

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## 2. Basic definitions and results on Bredon modules

In this section we collect basic definitions and facts related to Bredon modules for further reference. Unless stated otherwise the results can be found in [Lüc89, pp. 162–169] or [MV03, pp. 7–27]. By a Bredon module we mean either a left or a right Bredon module unless the variance is explicitly mentioned.

Since the category of Bredon modules is a functor category, it follows that limits and colimits of Bredon modules are calculated componentwise. In particular, a sequence of Bredon modules  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is exact if and only if the corresponding sequence

$$0 \rightarrow M'(G/K) \rightarrow M(G/K) \rightarrow M''(G/K) \rightarrow 0$$

of abelian groups is exact for every  $G/K \in \mathcal{O}_{\mathfrak{F}}G$ .

For subgroups  $L$  and  $K$  of  $G$ , we denote by  $[G/L, G/K]_G$  the set of all  $G$ -maps  $G/L \rightarrow G/K$ . For a fixed subgroup  $K$  of  $G$ , we denote by  $\mathbb{Z}[-, G/K]_G$  the right  $\mathcal{O}_{\mathfrak{F}}G$ -module, which sends  $G/L \in \mathcal{O}_{\mathfrak{F}}G$  to the free abelian group  $\mathbb{Z}[G/L, G/K]_G$  on the basis  $[G/L, G/K]_G$ . The left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}[G/K, -]_G$  is defined analogously. The *free objects* in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  are now precisely the direct sums of  $\mathbb{Z}[-, G/K]_G$  with  $K \in \mathfrak{F}$ . Likewise, the *free objects* in  $\mathcal{O}_{\mathfrak{F}}G\text{-Mod}$  are the direct sums of the Bredon modules  $\mathbb{Z}[G/K, -]_G$  with  $K \in \mathfrak{F}$ . In either case a free Bredon module is *finitely generated* if the direct sum can be taken to be finite. An arbitrary Bredon module is finitely generated if it is the surjective image of a finitely generated free module.

For any two Bredon modules  $M$  and  $N$  of the same variance, the set of morphisms between them is denoted by  $\text{mor}_{\mathfrak{F}}(M, N)$ . It carries the structure of an abelian group. A Bredon module  $P$  is *projective* if the functor  $\text{mor}_{\mathfrak{F}}(P, -)$  is exact. This is the case if and only if  $P$  is a direct summand of a free Bredon module.

The categorical tensor product [Sch70, p. 45] gives rise to a *tensor product over*  $\mathfrak{F}$ . It assigns to a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$  and left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  an abelian group  $N \otimes_{\mathfrak{F}} M$ . The  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$  is said to be *flat* if the functor  $N \otimes_{\mathfrak{F}} -$  is exact. Every projective Bredon module is flat.

There also exists the *tensor product over  $\mathbb{Z}$* . For two Bredon modules  $M$  and  $N$  of the same variance, it is defined to be the Bredon module  $M \otimes N$ , which evaluated at any  $G/K \in \mathcal{O}_{\mathfrak{F}}G$ , is given by  $(M \otimes N)(G/K) = M(G/K) \otimes N(G/K)$ .

**Lemma 2.1.** *Let  $N$  be a right  $\mathcal{O}_{\mathfrak{F}}G$ -module and  $K \in \mathfrak{F}$ . There exists an isomorphism*

$$N \otimes_{\mathfrak{F}} \mathbb{Z}[G/K, -]_G \cong N(G/K),$$

which is natural in  $N$ .

This statement follows from a Yoneda type argument. See, for example, [MV03, p. 9].

If  $K$  is a subgroup of  $G$  such that  $\mathfrak{F} \cap K := \{L \cap K \mid L \in \mathfrak{F}\} \subset \mathfrak{F}$ , then there exists a functor

$$I_K: \mathcal{O}_{\mathfrak{F} \cap K}K \rightarrow \mathcal{O}_{\mathfrak{F}}G,$$

which sends  $K/L$  to  $G/L$  for every  $L \in \mathfrak{F} \cap K$ . The induction functor

$$\mathrm{Ind}_K^G: \mathcal{O}_{\mathfrak{F} \cap K}K \rightarrow \mathcal{O}_{\mathfrak{F}}G$$

and the restriction functor

$$\mathrm{Res}_K^G: \mathcal{O}_{\mathfrak{F}}G \rightarrow \mathcal{O}_{\mathfrak{F} \cap K}K$$

with respect to  $I_K$  are defined in [Lüc89, p. 166].

**Lemma 2.2.** *The functor  $\mathrm{Ind}_K^G$  preserves the properties of being finitely generated and being projective. Furthermore, it is an exact functor and  $\mathrm{Ind}_K^G \mathbb{Z} = \mathbb{Z}[-, G/K]_G$ .*

The first statement is from [Lüc89, p. 169]. The second statement consists of Lemma 2.9 and Lemma 2.7 in [Sym05, p. 268].

**Lemma 2.3.** *The functor  $\mathrm{Res}_K^G$  is exact and preserves being projective.*

For the first part of the above statement see [Lüc89, p. 169]; the remaining part is [MP02, Lemma 3.7]. We also need the following special case of Proposition 3.5 in [MP02]:

**Lemma 2.4.** *There exists a natural isomorphism*

$$(N \otimes \mathbb{Z}[-, G/K]_G) \otimes_{\mathfrak{F}} M \cong \mathrm{Res}_K^G N \otimes_{\mathfrak{F} \cap K} \mathrm{Res}_K^G M$$

for any right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$  and any left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$ .

If  $\Delta$  is a  $G$ -set, then we denote by  $\mathfrak{F}(\Delta)$  the set of stabilisers of  $\Delta$ . We denote by  $\mathbb{Z}[-, \Delta]_G$  the right  $\mathcal{O}_{\mathfrak{F}}G$ -module which sends  $G/K \in \mathcal{O}_{\mathfrak{F}}G$  to the free abelian group on the basis  $[G/K, \Delta]_G$ , which is by definition the set of all  $G$ -maps from  $G/K \rightarrow \Delta$ .

**Lemma 2.5.** *Let  $\Delta$  be a  $G$ -set such that  $\mathfrak{F} \cap K \subset \mathfrak{F}$  for every  $K \in \mathfrak{F}(\Delta)$ . For every projective right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $Q$ , the  $\mathcal{O}_{\mathfrak{F}}G$ -module  $Q \otimes \mathbb{Z}[-, \Delta]_G$  is flat.*

*Proof.* Since tensoring over  $\mathbb{Z}$  is an additive functor, it is enough to verify the claim in the case that  $\Delta = G/K$  for some  $K$ . Since  $Q$  is projective, it follows that the  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -module  $\mathrm{Res}_K^G Q$  is projective and hence flat. Furthermore,  $\mathrm{Res}_K^G$  is an exact

functor. Thus the functor, which sends any left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  to  $\text{Res}_K^G Q \otimes_{\mathfrak{F} \cap K} \text{Res}_K^G M$ , is exact. Hence, in light of the natural isomorphism of Lemma 2.4, it follows that tensoring  $Q \otimes \mathbb{Z}[-, G/K]_G$  over  $\mathfrak{F}$  is an exact functor, that is, the module  $Q \otimes \mathbb{Z}[-, G/K]_G$  is flat.  $\square$

For a left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$ , the left derived functors of  $-\otimes_{\mathfrak{F}} M$  are denoted by  $\text{Tor}_*^{\mathfrak{F}}(-, M)$ . The following is a key ingredient to our proof and can be found as Theorem 5.4 in [MPN11]:

**Proposition 2.6** (Bieri–Eckmann Criterion for Bredon homology). *Let  $N$  be a right  $\mathcal{O}_{\mathfrak{F}}G$ -module and let  $n \in \mathbb{N}$ . The following statements are equivalent:*

1.  $N$  is of type  $\mathfrak{F}\text{-FP}_n$ .
2. For every family  $(J_K)_{K \in \mathfrak{F}}$  of cardinals, the natural map

$$\text{Tor}_k^{\mathfrak{F}}\left(N, \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z}[G/K, -]_{G^i}\right) \rightarrow \prod_{K \in \mathfrak{F}} \prod_{J_K} \text{Tor}_k^{\mathfrak{F}}(N, \mathbb{Z}[G/K, -]_G)$$

is an isomorphism for  $k < n$  and an epimorphism for  $k = n$ .

Note that  $\text{Tor}_k^{\mathfrak{F}}(N, \mathbb{Z}[G/K, -]_G) = 0$  for every  $K \in \mathfrak{F}$  and  $k \geq 1$ . Thus the requirement in (ii) that the natural map is an epimorphism is automatically satisfied for  $k \geq 1$ .

The Bredon homology  $H_*^{\mathfrak{F}}(G; M)$  of  $G$  with coefficients in the left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  is defined to be  $\text{Tor}_*^{\mathfrak{F}}(\mathbb{Z}, M)$ . Analogous to the classical case [Bro82, p. 172], we define the equivariant Bredon homology  $H_*^{\mathfrak{F}}(X, M)$  of a  $G$ -CW-complex with coefficients in the left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  as follows; cf. [DPT12]. Let  $Q_*$  be a projective resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}$  by right  $\mathcal{O}_{\mathfrak{F}}G$ -modules. Then we have the bigraded complex

$$(\underline{C}_*(X) \otimes Q_{**}) \otimes_{\mathfrak{F}} M$$

of abelian groups. We define  $H_*^{\mathfrak{F}}(X, M)$  to be the homology of the total complex of this bicomplex. Note that  $H_*^{\mathfrak{F}}(G, M) = H_*^{\mathfrak{F}}(\text{pt.}, M)$ .

### 3. The case $n = 0$

In the classical case, being of type  $\text{FP}_0$  for a group is an empty condition. In the context of Bredon homology, this is not true any more. Kochloukova, Martínez-Pérez and Nucinkis [KMPN11, Lemma 2.3] have given a characterisation of when a group is of type  $\mathfrak{F}\text{-FP}_0$ :

**Proposition 3.1.** *A group  $G$  is of type  $\mathfrak{F}\text{-FP}_0$  if and only if there is a finite subset  $\mathfrak{F}_0$  of  $\mathfrak{F}$  such that every  $K \in \mathfrak{F}$  is subconjugate to some element of  $\mathfrak{F}_0$ , i.e., there is a  $g \in G$  and a  $L \in \mathfrak{F}_0$  such that  $K^g \leq L$ .*

Using this result, the case  $n = 0$  of the Main Theorem is readily verified:

*Proof of the Main Theorem for  $n = 0$ .* For one implication assume that the directed system  $(\tilde{H}_{-1}(X_\alpha))_{\alpha \in I}$  is essentially trivial. Then there is a  $\beta \in I$  such that  $X_\beta$  is

$\mathfrak{F}$ -acyclic up to dimension  $-1$ , i.e.,  $X^K$  is non-empty for every  $K \in \mathfrak{F}$ . By assumption,  $X_\beta^{(0)}$  is finite modulo  $G$ . The stabilizer  $G_x$  of every  $x \in X_\beta^{(0)}$  is of type  $(\mathfrak{F} \cap G_x)$ -FP $_0$ . Hence there is a finite subset  $\mathfrak{F}_{x,0}$  of  $\mathfrak{F} \cap G_x$  such that every  $K \in \mathfrak{F} \cap G_x$  is subconjugate to some element of  $\mathfrak{F}_{x,0}$ . Let  $\Sigma_0$  be a set of representatives for  $X_\beta^{(0)}$  modulo  $G$  and let

$$\mathfrak{F}_0 = \bigcup_{x \in \Sigma_0} \mathfrak{F}_{x,0},$$

which is a finite subset of  $\mathfrak{F}$ . If  $K \in \mathfrak{F}$  is arbitrary, then  $X_\beta^K$  is non-empty. Hence  $K$  fixes some point  $x$  of  $X_\beta^{(0)}$  and therefore is subconjugate to some element of  $\mathfrak{F}_{x,0}$ .

Conversely, assume that  $G$  is of type  $\mathfrak{F}$ -FP $_0$ . Let  $\mathfrak{F}_0 \subseteq \mathfrak{F}$  be finite such that every element of  $\mathfrak{F}$  is subconjugate to some element of  $\mathfrak{F}_0$ . For arbitrary  $\alpha \in I$  let  $\beta \geq \alpha$  be such that  $X_\beta$  contains a fixed point of each element of  $\mathfrak{F}_0$ . Let  $K \in \mathfrak{F}$  be arbitrary and  $K^g \leq L \in \mathfrak{F}_0$ . If  $x \in X_\beta$  is a fixed point of  $L$ , then  $g.x \in X_\beta$  is a fixed point of  $K$ .  $\square$

#### 4. Proof of the Main Theorem

The following proposition is contained in [DPT12] for the case  $n = \infty$ , and the proof is essentially the same. We reproduce it for convenience.

**Proposition 4.1.** *Let  $X$  be a  $G$ -CW-complex that is  $\mathfrak{F}$ -acyclic up to dimension  $n - 1$  and let  $M$  be a left  $\mathcal{O}_{\mathfrak{F}}G$ -module. Then the natural morphism*

$$H_k^{\mathfrak{F}}(X, M) \rightarrow H_k^{\mathfrak{F}}(G, M)$$

(induced by the projection of  $X$  to a point) is an isomorphism for  $k < n$ .

*Proof.* Let  $C_*$  be the chain complex of Bredon modules  $X$  and let  $Q_*$  be a projective resolution of  $\mathbb{Z}$ . By definition there is a spectral sequence

$$E_{pq}^1 = H_q((C_* \otimes Q_p) \otimes_{\mathfrak{F}} M) \Rightarrow H_k^{\mathfrak{F}}(X, M).$$

We claim that

$$C_n \otimes Q_p \rightarrow C_{n-1} \otimes Q_p \rightarrow \cdots \rightarrow C_0 \otimes Q_p \rightarrow Q_p \rightarrow 0 \quad (1)$$

is a partial flat resolution of Bredon modules. By acyclicity of  $X$  up to dimension  $n - 1$ , the sequence

$$C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_0 \rightarrow \mathbb{Z} \rightarrow 0$$

is exact. To see that (1) is exact we have to see that it is exact when evaluated at any orbit  $G/K$ . This is true because  $Q_p(G/K)$  is free abelian. Flatness follows from Lemma 2.5 because every  $C_p$  is of the form  $\mathbb{Z}[-, \Delta]_G$ .

It follows from (1) that  $E_{pq}^1 = \text{Tor}_q^{\mathfrak{F}}(Q_p, M)$  for  $q < n$ . Since  $Q_p$  is projective we

get

$$\mathrm{Tor}_q^{\mathfrak{F}}(Q_p, M) = \begin{cases} Q_p \otimes_{\mathfrak{F}} M & q = 0 \\ 0 & 0 < q < n. \end{cases}$$

But  $Q_* \otimes_{\mathfrak{F}} M$  can be used to compute  $H_*^{\mathfrak{F}}(G, M)$ ; therefore,

$$E_{pq}^2 = \begin{cases} H_p^{\mathfrak{F}}(G, M) & q = 0 \\ 0 & 0 < q < n. \end{cases}$$

Since the triangle  $p + q < n$  remains stable, this closes the proof.  $\square$

**Proposition 4.2.** *Let  $X$  be a  $G$ -CW-complex with cocompact  $n$ -skeleton. Assume that for every  $p$ -cell  $\sigma$  of  $X$ ,  $p \leq n$ , the following two conditions hold:  $\mathfrak{F} \cap G_\sigma \subset \mathfrak{F}$ , and  $G_\sigma$  is of type  $(\mathfrak{F} \cap G)_\sigma$ -FP $_{n-p}$ . Then for  $k \leq n$  and every family of cardinals  $(J_K)_{K \in \mathfrak{F}}$  there exists an isomorphism*

$$H_k^{\mathfrak{F}}\left(X, \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z}[G/K, -]_G\right) \rightarrow \prod_{K \in \mathfrak{F}} \prod_{J_K} \underline{H}_k(X)(G/K)$$

that is natural in  $X$ .

*Proof.* As in the previous proof, let  $C_* = \underline{C}_*(X)$  and let  $Q_* \rightarrow \underline{\mathbb{Z}}$  be a projective resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module.

There exists a spectral sequence converging to  $H_*^{\mathfrak{F}}(X, M)$  whose  $E^1$ -sheet is given by

$$E_{pq}^1 = H_q((C_p \otimes Q_*) \otimes_{\mathfrak{F}} M).$$

Since  $C_p(G/K)$  is a free abelian group for every  $K \in \mathfrak{F}$ , it follows that  $C_p \otimes Q_*$  is a resolution of  $C_p \otimes \underline{\mathbb{Z}} = C_p$ . Moreover, this resolution is flat by Lemma 2.5, and thus there exist isomorphisms

$$H_q((C_p \otimes Q_*) \otimes_{\mathfrak{F}} M) \cong \mathrm{Tor}_q^{\mathfrak{F}}(C_p, M),$$

which are natural in  $X$  and  $M$ .

Next we show that  $C_p$  is of type FP $_{n-p}$  for  $p \leq n$ . Note that the last statement of Lemma 2.2 implies

$$C_p \cong \prod_{\sigma \in \Sigma_p} \mathrm{Ind}_{G_\sigma}^G \underline{\mathbb{Z}},$$

where  $\Sigma_p$  is a set of representatives for the  $p$ -cells of  $X$  modulo  $G$ . Note also that  $\Sigma_p$  is finite. By assumption,  $\underline{\mathbb{Z}}$  is of type FP $_{n-p}$  as an  $\mathcal{O}_{\mathfrak{F} \cap G_\sigma} G_\sigma$ -module for every  $p$ -cell  $\sigma \in \Sigma_p$ . The claim now follows from Lemma 2.2.

Now take  $M$  to be  $\prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z}[G/K, -]_G$  and consider the spectral sequence above. Since  $C_p$  is of type FP $_{n-p}$ , the Bieri–Eckmann Criterion, Proposition 2.6,

implies that

$$E_{pq}^1 = \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathrm{Tor}_q^{\mathfrak{F}}(C_p, \mathbb{Z}[G/K, -]_G),$$

which is 0 for  $q > 0$ . The entry  $E_{p0}^1$  is naturally isomorphic to the module

$$\prod_{K \in \mathfrak{F}} \prod_{J_K} C_p \otimes_{\mathfrak{F}} \mathbb{Z}[G/K, -]_G,$$

and the differentials are induced by the differentials of the chain complex  $C_*$ . Therefore, one can read off from the second page of the spectral sequence that

$$\begin{aligned} H_k^{\mathfrak{F}}\left(X, \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z}[G/K, -]_G\right) &\cong \prod_{K \in \mathfrak{F}} \prod_{J_K} H_k(C_* \otimes_{\mathfrak{F}} \mathbb{Z}[G/K, -]_G) \\ &\cong \prod_{K \in \mathfrak{F}} \prod_{J_K} H_k(C_*(G/K)) \end{aligned}$$

for  $k < n$ , where the last isomorphism is the isomorphism from Lemma 2.1. But  $H_k(C_*(G/K))$  is just  $\underline{H}_k(X)(G/K)$ , and this concludes the proof.  $\square$

**Lemma 4.3.** *Let  $X$  be a  $G$ -CW-complex and let  $(X_\alpha)_{\alpha \in I}$  be a filtration of  $X$  by  $G$ -invariant subcomplexes. Then the inclusions  $X_\alpha \hookrightarrow X$  induce an isomorphism*

$$\varinjlim_{\alpha} H_*^{\mathfrak{F}}(X_\alpha, M) \rightarrow H_*^{\mathfrak{F}}(X, M)$$

for all left  $\mathcal{O}_{\mathfrak{F}}G$ -modules  $M$ .

*Proof.* This is due to the fact that  $\varinjlim_{\alpha}$  is a filtered colimit and, in particular, exact.  $\square$

*Proof of the Main Theorem.* Since we have already covered the case  $n = 0$ , we may and do assume that  $n \geq 1$ . By the Bieri–Eckmann Criterion,  $G$  is of type  $\mathfrak{F}$ -FP $_n$  if and only if for every family  $(J_K)_{K \in \mathfrak{F}}$  the natural map

$$\varphi: H_k^{\mathfrak{F}}\left(G, \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z}[G/K, -]_G\right) \rightarrow \prod_{K \in \mathfrak{F}} \prod_{J_K} H_k^{\mathfrak{F}}(G, \mathbb{Z}[G/K, -]_G)$$

is an isomorphism for  $0 \leq k < n$  and an epimorphism for  $k = n$ . Since the right-hand side is 0 for  $k > 0$  and since we are assuming that  $n \geq 1$ , the statement about the epimorphism is trivially satisfied.

Since the codomain of  $\varphi$  is trivial for  $k \geq 1$ , we first show that the domain of  $\varphi$  is also trivial for  $0 < k < n$  (which is a special case of the proof for  $k = 0$  below). We have the isomorphisms

$$H_k^{\mathfrak{F}}\left(G, \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z}[G/K, -]_G\right) \cong H_k^{\mathfrak{F}}\left(X, \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z}[G/K, -]_G\right)$$

from Proposition 4.1,

$$H_k^{\mathfrak{F}}\left(X, \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z}[G/K, -]_G\right) \cong \varinjlim_{\alpha} H_k^{\mathfrak{F}}\left(X_\alpha, \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z}[G/K, -]_G\right)$$



from Lemma 4.3 and

$$\varinjlim_{\alpha} H_k^{\mathfrak{F}}\left(X_{\alpha}, \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z}[G/K, -]_G\right) \cong \varinjlim_{\alpha} \prod_{K \in \mathfrak{F}} \prod_{J_K} \underline{H}_k(X_{\alpha})(G/K)$$

from Proposition 4.2. By Observation 1 this is trivial if and only if the system  $(\underline{H}_k(X_{\alpha}))_{\alpha \in I}$  of  $\mathcal{O}_{\mathfrak{F}}G$ -modules is essentially trivial.

For the remaining case  $k = 0$ , consider the following commuting diagram (where we dropped the index sets for readability):

$$\begin{array}{ccc} H_0^{\mathfrak{F}}\left(G, \prod \mathbb{Z}[G/K, -]_G\right) & \xrightarrow{\varphi} & \prod H_0^{\mathfrak{F}}\left(G, \mathbb{Z}[G/K, -]_G\right) \\ \uparrow & & \uparrow \\ H_0^{\mathfrak{F}}\left(X, \prod \mathbb{Z}[G/K, -]_G\right) & \longrightarrow & \prod H_0^{\mathfrak{F}}(\text{pt.}, \mathbb{Z}[G/K, -]_G) \\ \uparrow & & \uparrow \\ \varinjlim_{\alpha} H_0^{\mathfrak{F}}\left(X_{\alpha}, \prod \mathbb{Z}[G/K, -]_G\right) & \longrightarrow & \varinjlim_{\alpha} \prod H_0^{\mathfrak{F}}(\text{pt.}, \mathbb{Z}[G/K, -]_G) \\ \uparrow & & \uparrow \\ \varinjlim_{\alpha} \prod \underline{H}_0(X_{\alpha})(G/K) & \xrightarrow{\psi} & \varinjlim_{\alpha} \prod \underline{H}_0(\text{pt.})(G/K). \end{array}$$

The vertical arrows of the top square are isomorphisms by Proposition 4.1. The vertical arrows of the middle square are induced by the inclusions  $X_{\alpha} \hookrightarrow X$  and the identity on the one point space respectively; it follows from Lemma 4.3 that they are isomorphisms. Finally, the vertical arrows of the bottom square are the isomorphisms from Proposition 4.2.

Since all the vertical arrows in the diagram are isomorphisms, it follows that  $\varphi$  is an isomorphism if and only if  $\psi$  is an isomorphism. But  $\psi$  fits into the short exact sequence

$$0 \rightarrow \varinjlim_{\alpha} \prod_{K \in \mathfrak{F}} \prod_{J_K} \tilde{H}_0(X_{\alpha})(G/K) \rightarrow \varinjlim_{\alpha} \prod_{K \in \mathfrak{F}} \prod_{J_K} \underline{H}_0(X_{\alpha})(G/K) \xrightarrow{\psi} \varinjlim_{\alpha} \prod_{K \in \mathfrak{F}} \prod_{J_K} \mathbb{Z} \rightarrow 0,$$

and it follows from Observation 1 that  $\psi$  (and therefore  $\varphi$ ) is an isomorphism if and only if the system  $(\tilde{H}_0(X_{\alpha}))_{\alpha \in I}$  of  $\mathcal{O}_{\mathfrak{F}}G$ -modules is essentially trivial.  $\square$

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