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HOMOLOGY OPERATIONS IN SYMMETRIC HOMOLOGY

SHAUN V. AULT

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Abstract

The symmetric homology of a unital associative algebra A over a commutative ground ring k, denoted $HS_*(A)$, is defined using derived functors and the symmetric bar construction of Fiedorowicz. In this paper we show that $HS_*(A)$ admits homology operations and a Pontryagin product structure making $HS_*(A)$ an associative commutative graded algebra. This is done by finding an explicit E_∞ structure on the standard chain groups that compute symmetric homology.

1. Introduction

The purpose of this paper is to define an E_∞ structure on the standard chain groups that compute symmetric homology of a unital associative algebra. The construction makes use of the fact that the symmetric category ΔS_+ (that is, ΔS with an initial object appended) is permutative, a property not shared by the simplicial category Δ nor the cyclic category ΔC, even if initial objects are appended. Such structure may facilitate computations of symmetric homology, which in turn may shed light on related functor homology theories.

The notion of symmetric homology was introduced under the broader context of crossed simplicial groups (CSGs) by Fiedorowicz and Loday in [**13**]. Some important properties and results were developed in the preprints of Fiedorowicz [**12**] and Ault and Fiedorowicz [**2**], as well as in the author's thesis, a portion of which has been published [**1**]. Symmetric homology can be thought of as an analog to cyclic homology, in which the symmetric groups play the role that the cyclic groups do in the latter. The usefulness of cyclic (co)homology in noncommutative geometry and K-theory is well established (see, for example, [**9, 6, 7**]). It becomes natural to examine generalizations such as symmetric homology in order to better understand cyclic homology itself. Moreover, these generalizations are important in their own right. For example, there are interesting links between symmetric homology and Γ-homology and related theories through the identification of ΔS with the category of *noncommutative sets*, $\mathcal{F}(as)$ (see §2.3). Furthermore, symmetric homology is related to stable homotopy theory in the following way: if G is a group, the symmetric homology of the group ring $k[G]$ is isomorphic to $H_*(\Omega\Omega^\infty S^\infty(BG); k)$ [**12, 1**].

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This paper, together with [**1**], is intended to supplant the unpublished preprints of Fiedorowicz [**12**] and Ault and Fiedorowicz [**2**].

1.1. Symmetric homology

We begin by recalling some of the notations and definitions regarding symmetric homology found in [**1**]. Let A be a unital associative algebra over a commutative ground ring k, and let k-**Mod** be the category of (left) k-modules. Let ΔS be the category whose objects are the sets $[n] = \{0, 1, 2, \ldots, n\}$ for $n \geqslant 0$ and whose morphisms $[n] \to [m]$ are pairs (ϕ, γ) such that ϕ is a nondecreasing set map $[n] \to [m]$ (that is, $\phi \in \Delta([n], [m])$), and $\gamma \in \Sigma_{n+1}^{\mathrm{op}}$ (the opposite of the symmetric group). The category ΔS is the structure category of the *symmetric CSG* [**13, 18**]. Briefly, a CSG is a sequence of groups $\{G_n\}_{n\geqslant 0}$ together with a structure category ΔG such that

- ΔG contains the simplicial category Δ as subcategory,
- $\mathrm{Aut}_{\Delta G}([n]) = G_n^{\mathrm{op}}$, and
- Each morphism of ΔG has unique decomposition into $\phi \circ \gamma$, which we denote by the pair (ϕ, γ), with $\phi \in \Delta$ and $\gamma \in G_n$ for some n.

Composition in ΔG is defined by $(\phi, \gamma) \circ (\psi, \delta) = (\phi \circ \psi^\gamma, \gamma^\psi \cdot \delta)$ for the appropriate morphisms ψ^γ of Δ and $\gamma^\psi \in G^{\mathrm{op}}$. As implied by the notation, a single dot $(\cdot)$ is used for multiplication in G^{op}; however, it is convenient to regard the group elements as living in G so that we typically do the multiplication "the right way" when writing the morphism: $(\phi, \gamma) \circ (\psi, \delta) = (\phi\psi^\gamma, \delta\gamma^\psi)$. See [**13, 1**] for more details and notational conventions. Observe that both Δ and the cyclic category ΔC are examples of structure categories of CSGs, the former having trivial automorphism groups and the latter having $\mathrm{Aut}_{\Delta C}([n]) = C_{n+1}^{\mathrm{op}} = C_{n+1}$, the cyclic group of order $n+1$. Using an appropriate bar construction, one may define a homology theory associated to a CSG. Indeed, the cyclic bar construction of Loday [**18**], a contravariant functor $B_*^{cyc}A \colon \Delta C \to k$-**Mod**, defines the cyclic homology of the algebra A via $HC_*(A) = \mathrm{Tor}_*^{\Delta C^{\mathrm{op}}}(\underline{k}, B_*^{cyc}A) = \mathrm{Tor}_*^{\Delta C}(B_*^{cyc}A, \underline{k})$, where $\underline{k}$ is the trivial cyclic k-module, that is, $\underline{k}[n] = k$ for all $n \geqslant 0$ and $\underline{k}\alpha = \mathrm{id}$ for all morphisms α. Fiedorowicz and Loday [**13**] found that any definition of symmetric homology using a contravariant bar construction results in a trivial theory—that is, if M is a ΔS^{op}-module, then $\mathrm{Tor}_*^{\Delta S}(M, \underline{k}) = H_*(M)$, the homology of the underlying simplicial module. On the other hand, Fiedorowicz [**12**] discovered that a *covariant* bar construction, rather than a contravariant one, yields an interesting nontrivial theory of symmetric homology;

$$B_*^{sym}A \colon \Delta S \to k\text{-}\mathbf{Mod},$$
$$B_*^{sym}A[n] = B_n^{sym}A \overset{def}{=} A^{\otimes(n+1)}.$$

The functor $B_*^{sym}A$ is referred to as $C^{sym}(A)$ in [**18**]. It is sufficient to define $B_*^{sym}A$ on $\gamma \in \Sigma_{n+1}^{\mathrm{op}}$ and $\phi \in \Delta([n], [m])$. We often refer to the morphism $B_*^{sym}A\alpha$ as *evaluation at α*.

$$\begin{aligned} B_*^{sym}A\gamma \colon A^{\otimes(n+1)} &\longrightarrow A^{\otimes(n+1)} \\ a_0 \otimes a_1 \otimes \cdots \otimes a_n &\mapsto a_{\gamma(0)} \otimes a_{\gamma(1)} \otimes \cdots \otimes a_{\gamma(n)}. \end{aligned}$$

$$\begin{aligned} B_*^{sym}A\phi \colon A^{\otimes(n+1)} &\longrightarrow A^{\otimes(m+1)} \\ a_0 \otimes a_1 \otimes \cdots \otimes a_n &\mapsto b_0 \otimes b_1 \otimes \cdots \otimes b_m, \end{aligned}$$

where

$$b_i = \prod_{a_j \in \phi^{-1}(i)} a_j \quad \text{(product taken in order of increasing indices).}$$

We define the *symmetric homology* of any ΔS-module M by $HS_*(M) = \mathrm{Tor}_*^{\Delta S}(\underline{k}, M)$, in which $\underline{k}$ is the trivial ΔS^{op}-module. For any unital associative algebra A, $B_*^{sym}A$ is a ΔS-module, and so we define the symmetric homology of A as follows:

$$HS_*(A) \stackrel{def}{=} \mathrm{Tor}_*^{\Delta S}\left(\underline{k}, B_*^{sym}A\right).$$

It is advantageous to enlarge ΔS by adding an initial object $[-1] \in \Delta S_+$. Define the extended symmetric bar construction, $B_*^{sym_+}A$, by $B_n^{sym_+}A = B_n^{sym}A$ for $n \geqslant 0$ and $B_{-1}^{sym_+}A = k$. Evaluation at the unique morphism $[-1] \to [n]$ sends $1 \in k$ to $1^{\otimes(n+1)}$. The author has shown [**1**] that symmetric homology also can be computed using ΔS_+. In (1), $\underline{k}$ is the trivial ΔS_+^{op}-module:

$$HS_*(A) \cong \mathrm{Tor}_*^{\Delta S_+}\left(\underline{k}, B_*^{sym_+}A\right). \tag{1}$$

We may use a standard resolution based on under-categories to compute the Tor groups. Recall, for a small category $\mathscr{C}$ there is a contravariant functor $- \setminus \mathscr{C}$ from $\mathscr{C}$ to **Cat** (the category of small categories), which takes an object c to the under-category $c \setminus \mathscr{C}$; in other words, $- \setminus \mathscr{C}$ is a $\mathscr{C}^{\mathrm{op}}$-category. Using the notation $N\mathscr{C}$ for the *nerve* of a small category $\mathscr{C}$, and the useful notation of Gabriel and Zisman [**14**], a simplicial k-module whose homology is exactly $HS_*(A)$ is written and defined as follows:

$$C_*(\Delta S_+, B_*^{sym_+}A) \stackrel{def}{=} k\left[N(- \setminus \Delta S_+)\right] \otimes_{\Delta S_+} B_*^{sym_+}A. \tag{2}$$

That is,

$$HS_*(A) \cong H_*(C_*(\Delta S_+, B_*^{sym_+}A)).$$

2. Preliminaries

2.1. Notational conventions

With an eye towards readability, we use the following notational conventions:

1. *Tuple of n items:* $\mathbf{m} \stackrel{def}{=} (m_1, m_2, \ldots, m_n)$. Each element m_i may be a number or an element of some set as context dictates. The number of elements, n, is suppressed in the notation, though it will always be clear what n is by context.
2. *"Single-variable" function applied to a tuple:* If $f\colon M \to N$ and $\mathbf{m} \in M^n$, then
$$f(\mathbf{m}) \stackrel{def}{=} (f(m_1), f(m_2), \ldots, f(m_n)) \in N^n.$$
3. *"Multi-variable" function applied to a tuple:* If $f\colon M^p \to N^q$, then we simply write the image of $\mathbf{m} \in M^p$ under f as $f(\mathbf{m}) \in N^q$.
4. *Permutation applied to a tuple:* If $\sigma \in \Sigma_n$, then
$$\sigma\mathbf{m} \stackrel{def}{=} (m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)}).$$
This convention ensures that Σ_n acts on the *left* of $\mathbf{m}$.

5. *Block permutation:* If $\sigma \in \Sigma_n$, then $\sigma_{\mathbf{k}} = \sigma_{k_1,\ldots,k_n} \in \Sigma_{k_1+\cdots+k_n}$ represents the *block transformation* of blocks of sizes $k_1, k_2, \ldots k_n$ (where each $k_i \in \mathbb{N} \cup \{0\}$). For example, $(1,2)_{2,3} = (1,4,2,5,3)$.
6. *Inter-block permutation:* If $\sigma_i \in \Sigma_{k_i}$ for each $i = 1, 2, \ldots, n$, then $\sigma_1 \oplus \cdots \oplus \sigma_n \in \Sigma_{k_1+\cdots+k_n}$ represents the permutation of block i by σ_i while retaining the original order of the blocks. For example, $(1,2) \oplus (1,2,3) = (1,2)(3,4,5)$.
7. *Products of tuples:* Suppose c_i are objects of a category $\mathscr{C}$ with an associative binary operation $\odot$. Then $\mathbf{c}^{\odot} \stackrel{def}{=} c_1 \odot c_2 \odot \cdots \odot c_n$. Moreover, if $\sigma \in \Sigma_n$, then $\sigma\mathbf{c}^{\odot} \stackrel{def}{=} c_{\sigma^{-1}(1)} \odot \cdots \odot c_{\sigma^{-1}(n)}$.
8. If there is a specified left action of Σ_n on a set X, then the notation $\sigma \bullet x$ denotes the image of $x \in X$ under the action of $\sigma \in \Sigma_n$. The same notation is used for right actions, only written the opposite way around: $x \bullet \sigma$. This notation is chosen so that there is a clear distinction between the similar notations $\sigma\mathbf{c}^{\odot}$ and $\sigma \bullet \mathbf{c}^{\odot}$.

2.2. Monoid algebras and the functor $\mathcal{T}$

Let **Mon** be the category of monoids and monoid homomorphisms (here, we mean *ordinary monoids in sets*). For a given monoid M, we define the extended symmetric bar construction

$$B^{sym_+}M \colon \Delta S_+ \to \mathbf{Mon},$$
$$B^{sym_+}M[n] \stackrel{def}{=} M^{n+1}.$$

Here, M^n is the cartesian product of n copies of M, and $M^0 = \{()\}$, a set containing just the empty tuple. Now let us define a similar notation as that of (2) for simplicial monoids. If F is a ΔS_+-monoid (*i.e.,* $F : \Delta S_+ \to \mathbf{Mon}$ is a functor) then let

$$C(\Delta S_+, F) \stackrel{def}{=} N(- \setminus \Delta S_+) \times_{\Delta S_+} F.$$

We define the symmetric homology (with coefficients in k) of the monoid M by

$$HS_*(M) \stackrel{def}{=} H_*(k[C(\Delta S_+, B^{sym_+}M)]).$$

See [**1**, §5.2], for more details on the symmetric bar construction for monoids.

Remark 2.1. We will generally use the notation $\langle f, \mathbf{m}\rangle$ in place of $B_*^{sym_+}Mf(\mathbf{m})$ to denote *evaluation* of f at $\mathbf{m}$. Because $B_*^{sym_+}M$ is functorial, the evaluation map satisfies the useful property

$$\langle fg, \mathbf{m}\rangle = \big\langle f, \langle g, \mathbf{m}\rangle\big\rangle. \tag{3}$$

Similarly, for $a_0 \otimes \cdots \otimes a_n \in A^{\otimes(n+1)}$, we may write $\langle f, a_0 \otimes \cdots \otimes a_n\rangle$ in place of $B_*^{sym_+}Af(a_0 \otimes \cdots \otimes a_n)$.

Let $\mathcal{T}$ be the functor from **Mon** to the category of small categories defined by sending a monoid M to the category $\mathcal{T}M$ whose objects are finite sequences of elements of M, including the empty sequence, $()$. Morphisms of $\mathcal{T}M$ consist of pairs $(f, \mathbf{m})$ such that $\mathbf{m} = (m_1, \ldots, m_p) \in M^p$ and $f \colon [p-1] \to [q-1]$ is a morphism of ΔS_+. The source and target of such a pair are $\mathbf{m}$ and $\langle f, \mathbf{m}\rangle$, respectively. When the source and target are clear, we simply use f to denote the morphism. The functor

$\mathcal{T}$ sends a monoid morphism $\psi\colon M \to N$ to the functor $\mathcal{T}\psi\colon \mathcal{T}M \to \mathcal{T}N$ that maps $\mathbf{m} \in M^p$ to $\psi(\mathbf{m}) \in N^p$.

Lemma 2.2. *$\mathcal{T}M$ is a permutative category.*

Proof. Define the product on objects by concatenation:

$$(m_1, \ldots, m_p) \odot (n_1, \ldots, n_q) \stackrel{def}{=} (m_1, \ldots, m_p, n_1, \ldots, n_q).$$

Since ΔS_+ is permutative [**1**], we can use the product of ΔS_+ to define products of morphisms in $\mathcal{T}M$. Associativity is strict, since it is induced by the associativity of $\odot$ in ΔS_+. The empty sequence, (), is a strict unit. The symmetry transformation is defined on objects by block transposition. □

2.3. The category of noncommutative sets

There is an interpretation of the morphisms of ΔS_+ as formal tensors, which provides an interesting connection to the category $\mathcal{F}(as)$, the category of *noncommutative sets* [**23, 24, 26**]. The objects of $\mathcal{F}(as)$ are the finite sets $\underline{m} = \{1, 2, 3, \ldots, m\}$ for $m \geqslant 1$. A morphism λ of $\mathcal{F}(as)$ is a set map $\underline{\lambda}\colon \underline{m} \to \underline{n}$ together with a specified total ordering $<_\lambda$ on each preimage set $\underline{\lambda}^{-1}(i)$, $1 \leqslant i \leqslant n$.

Let $X = \{x_0, x_1, x_2, x_3, \ldots\}$ be a set of formal indeterminates, and consider the free monoid, $X^\star$, generated by X. Define the *tensor representation* of a morphism $f \in \Delta S_+([n],[m])$ as the image of $(x_0, x_1, \ldots, x_n)$ under $B^{sym_+}X^\star f$. Typically, a morphism whose tensor representation is $(y_0, y_1, \ldots, y_m)$ (in which each y_i is a possibly empty monomial in the indeterminates x_j) will be written $y_0 \otimes y_1 \otimes \cdots \otimes y_m$, hence the terminology. The correspondence sending a morphism to its tensor representation is one-to-one by uniqueness of decomposition of ΔS_+ morphisms into a Δ_+ morphism, which determines the number of factors in each monomial y_i, and a permutation, which determines the total order of the indices.

Example 2.3. Let $\phi \in \Delta_+([2],[1])$ be the map sending $i \mapsto i$ for $i = 0, 1$, and $2 \mapsto 1$. Let $\gamma = (0, 1, 2) \in \Sigma_3$. The tensor represenatation of (ϕ, γ) is (x_1, x_2x_0), or $x_1 \otimes x_2x_0$.

Tensor notation provides the link to $\mathcal{F}(as)$.

Proposition 2.4. *There is an isomorphism of categories $F\colon \Delta S \to \mathcal{F}(as)$.*

Proof. The functor F takes $[n]$ to $\underline{n+1}$ for each $n \geqslant 0$. Let $f\colon [n] \to [m]$ be a morphism in ΔS and write $f = (y_0, y_1, \ldots, y_m)$ in tensor notation. Then $F(f) = \lambda$, where $\underline{\lambda}$ is the set function such that $\underline{\lambda}(j) = i \Leftrightarrow x_{j-1}$ appears as a factor in y_{i-1}, while the total ordering $<_\lambda$ on $\underline{\lambda}^{-1}(i)$ is induced by the ordering of factors in y_{i-1}, that is, if $y_{i-1} = x_{j_1-1}x_{j_2-1}\cdots x_{j_k-1}$, then $j_1 <_\lambda < j_2 <_\lambda \cdots <_\lambda j_k$. Bijectivity of F is clear, and verifying that F is indeed a functor is left to the reader. □

Remark 2.5. If we denote by $\mathcal{F}(as)_+$ the category $\mathcal{F}(as)$ enlarged by the initial object $\underline{0} = \emptyset$, then Prop. 2.4 implies $\Delta S_+ \cong \mathcal{F}(as)_+$.

There are tantalizing links among symmetric homology, cyclic homology and the so-called Γ-homology theories of Alan Robinson and Sarah Whitehouse and related $\Gamma(as)$ and $\mathcal{F}(as)$ homologies, theories that have been much studied recently [**27,**

28, 29, 26, 24]. Pirashvili and Richter [**24**] identify the cyclic homology of any $\mathcal{F}(as)$-module G with $\mathrm{Tor}_*^{\mathcal{F}(as)}(b, G)$, where b is the cokernel of a certain map of $\mathcal{F}(as)^{\mathrm{op}}$-modules. We shall interpret this statement using ΔS presently. Define for each $m \geqslant 0$ the projective ΔS^{op}-module

$$(\Delta S)_m \stackrel{def}{=} k\left[\Delta S(-, [m])\right], \tag{4}$$

so in particular, for any $n \geqslant 0$, $(\Delta S)_m([n])$ is the free k-module generated by the set $\Delta S([n], [m])$. The covariant version $(\Delta S)^m$ is defined analogously, but we have no need for it in this paper. In light of Prop. 2.4, we may interpret Pirashvili and Richter's result thus: $HC_*(G) \cong \mathrm{Tor}_*^{\Delta S}(b, G)$, where b fits into the exact sequence

$$(\Delta S)_1 \xrightarrow{\eta} (\Delta S)_0 \longrightarrow b \longrightarrow 0,$$

and η is defined on morphisms $f \colon [m] \to [1]$ by $\eta(f) = x_0 x_1 \circ f - x_1 x_0 \circ f$. When $G = B_*^{sym} A$, one finds the *cyclic homology of the symmetric bar construction*, $HC_*(B_*^{sym} A)$, which coincides with the cyclic homology of A, as Loday's *cyclic bar construction* [**18**], $B_*^{cyc} A$, is the restriction of $B_*^{sym} A$ under the inclusion of categories, $\Delta C \hookrightarrow \Delta S$, and the duality isomorphism, $\Delta C^{\mathrm{op}} \cong \Delta C$. Indeed, we have a chain of isomorphisms,

$$HC_*(A) = HC_*(B^{cyc} A) \cong HC_*(B_*^{sym} A) \cong \mathrm{Tor}_*^{\Delta S}(b, B_*^{sym} A).$$

2.4. Homotopy-everything operads

Let $\mathbf{S}$ be the *symmetric groupoid*, which has as objects $\underline{n}$ for $n \geqslant 0$ and whose only morphisms are the permutations $\sigma \colon \underline{n} \to \underline{n}$. Thus, $\mathbf{S}^{\mathrm{op}} \cong \mathrm{Aut}\Delta S_+$, via the map $\sigma \mapsto (\mathrm{id}, \sigma)$. Here, $\mathrm{Aut}\mathscr{C}$ is the subcategory of $\mathscr{C}$ containing the same objects and only the automorphisms of $\mathscr{C}$. Therefore, any ΔS_+ object is naturally an $\mathbf{S}^{\mathrm{op}}$ object. Present in the early work of Boardman and Vogt, and developed later by May and others, is the concept of *homotopy-everything*, or E_∞, operad [**5, 21, 19**]. As our operads will be defined in various categories, not just topological spaces, it is important to clearly define certain concepts. Let $\mathscr{C}$ be a small symmetric monoidal category with unit object $\mathbf{1}$. Suppose there is a model structure [**25, 16**] on $\mathscr{C}$ (although we only need the notion of equivalences, not (co)fibrations). We define an E_∞ operad in $\mathscr{C}$ to be a functor $\mathscr{P} \colon \mathbf{S}^{\mathrm{op}} \to \mathscr{C}$, with structure maps satisfying the standard commutative diagrams of an operad, such that each component $\mathscr{P}(n)$ is equivalent (in the model structure) to $\mathbf{1}$. We also require the symmetric group action on each $\mathscr{P}(n)$ to be free. We are primarily interested in operads in the category of small categories ($\mathbf{Cat}$), whose model structure is induced by the nerve functor, and in simplicial sets ($\mathbf{Set}^{\Delta^{\mathrm{op}}}$), simplicial k-modules (k-$\mathbf{Mod}^{\Delta^{\mathrm{op}}}$), and non-negatively-graded k-complexes ($\mathbf{Ch}_\bullet^+$)—each with the standard model structure.

Example 2.6. May's *little* ∞*-cubes operad* $\mathcal{C}_\infty$ is E_∞ in the category of topological spaces.

Example 2.7. Let $\mathscr{D}_{\mathbf{Cat}}$ denote the operad in $\mathbf{Cat}$ defined by $\mathscr{D}_{\mathbf{Cat}}(m) = E\Sigma_m$. That is, the objects of $\mathscr{D}_{\mathbf{Cat}}(m)$ are the elements of the symmetric group on m letters, and for each pair of objects (σ, τ), there is a unique morphism $\tau\sigma^{-1}$ from σ to τ. The structure map in $\mathscr{D}_{\mathbf{Cat}}$ is the family of functors $\mathscr{D}_{\mathbf{Cat}}(m) \times \mathscr{D}_{\mathbf{Cat}}(k_1) \times \cdots \times$

$\mathscr{D}_{\mathbf{Cat}}(k_m) \longrightarrow \mathscr{D}_{\mathbf{Cat}}(k)$, where $k = \sum_i k_i$, defined on objects by:

$$(\sigma, \tau_1, \ldots, \tau_m) \mapsto \sigma_{\mathbf{k}} \cdot (\tau_1 \oplus \cdots \oplus \tau_m).$$

The action of Σ_m^{op} on objects of $\mathscr{D}_{\mathbf{Cat}}(m)$ is given by right multiplication of group elements. Since each $E\Sigma_m$ has free Σ_m action and is a contractible category, the operad is E_∞.

Remark 2.8. The notation $\mathscr{D}_{\mathbf{Cat}}$ is related to the notation used in May [**21, 22**]. May uses $\widetilde{\Sigma}_m$ for $\mathscr{D}_{\mathbf{Cat}}(m)$ and defines the related operad $\mathscr{D}$ in the category of spaces as the geometric realization of the nerve of $\widetilde{\Sigma}$. The nerve of $\mathscr{D}_{\mathbf{Cat}}$ is generally known in the literature as the *Barratt–Eccles operad* (see [**3**], where the notation for $N\mathscr{D}_{\mathbf{Cat}}$ is Γ, not to be confused with the Γ of Γ-homology!). We denote by $\mathscr{D}_{\mathbf{Mod}}$ the associated E_∞ operad in the category of simplicial k-modules defined by $\mathscr{D}_{\mathbf{Mod}}(m) = E_*\Sigma_m$ (the standard bar resolution of k by free $k[\Sigma_m]$-modules), and the Moore complex (that is, the complex of normalized chains [**15**]) of $\mathscr{D}_{\mathbf{Mod}}(m)$ by $\mathscr{D}_{\mathbf{Ch}_\bullet^+}(m)$.

2.5. Operad-algebras

By *operad-algebra*, we mean an algebra over an operad in the usual sense (as in [**19**, II.1.4]), in which the algebra lies in the same underlying category as the operad acting on it. As an example, if $\mathscr{C}$ is a permutative category, then $B\mathscr{C}$ is naturally an E_∞-space [**22**] (that is, an E_∞ algebra in **Top**). In fact, $\mathscr{C}$ is itself an E_∞ algebra in **Cat**. It is useful to regard a permutative $\mathscr{C}$ explicitly as $\mathscr{D}_{\mathbf{Cat}}$-algebra according to the structure map θ of diagram (5). Here, $f_i \colon C_i \to D_i$ for each $i = 1, 2, \ldots, m$, and the map $T_{\tau\sigma^{-1}}$ permutes the components according to the permutation $\tau\sigma^{-1}$ using the symmetry transformation and strict associativity of the monoidal product $\odot$ of $\mathscr{C}$.

$$\begin{array}{ccc}
(\sigma, C_1, \ldots, C_m) & \xrightarrow{\theta} & \sigma\mathbf{C}^{\odot} \\
\Big\downarrow{\scriptstyle \tau\sigma^{-1}\times\mathbf{f}} & & \Big\downarrow{\scriptstyle \sigma\mathbf{f}^{\odot}} \\
 & & \sigma\mathbf{D}^{\odot} \\
 & & {\scriptstyle\cong}\Big\downarrow{\scriptstyle T_{\tau\sigma^{-1}}} \\
(\tau, D_1, \ldots, D_m) & \xrightarrow{\theta} & \tau\mathbf{D}^{\odot}
\end{array} \tag{5}$$

3. Operad structure within symmetric homology

3.1. Monoid algebras

In order to produce an E_∞ structure for the simplicial module that computes symmetric homology, we first have to work at the level of monoids and simplicial sets.

Lemma 3.1. *Let M be a monoid. $C(\Delta S_+, B^{sym_+}M)$ has the structure of E_∞ algebra in the category of simplicial sets.*

Proof. Consider $\mathcal{T}M$, as in §2.2. A typical i-simplex of $N\mathcal{T}M$ has the form

$$\langle f_i \cdots f_2 f_1, \mathbf{m}\rangle \xleftarrow{f_i} \cdots \xleftarrow{f_3} \langle f_2 f_1, \mathbf{m}\rangle \xleftarrow{f_2} \langle f_1, \mathbf{m}\rangle \xleftarrow{f_1} \mathbf{m}, \tag{6}$$

in which $\mathbf{m} = (m_0, m_1, \ldots, m_n)$. Expression (6) can be rewritten uniquely as an element of M^{n+1} together with an element (i-simplex) of $N\Delta S_+$,

$$\left([n_i] \xleftarrow{f_i} \cdots \xleftarrow{f_3} [n_2] \xleftarrow{f_2} [n_1] \xleftarrow{f_1} [n]\,,\, \mathbf{m}\right), \tag{7}$$

which in turn is uniquely identified with an element of $C(\Delta S_+, B^{sym_+}M)$,

$$\left([n_i] \xleftarrow{f_i} \cdots \xleftarrow{f_3} [n_2] \xleftarrow{f_2} [n_1] \xleftarrow{f_1} [n] \xleftarrow{\mathrm{id}} [n]\,,\, \mathbf{m}\right). \tag{8}$$

Thus, (6)–(8) define a map $L_M \colon N\mathcal{T}M \to C(\Delta S_+, B^{sym_+}M)$.

On the other hand, a typical element of $C(\Delta S_+, B^{sym_+}M)$ may not *a priori* have an identity morphism $[n] \to [n]$ as the "incoming morphism," but by using ΔS_+-equivariance, we can always express the element in the desired form:

$$\left([n_i] \xleftarrow{f_i} \cdots \xleftarrow{f_2} [n_1] \xleftarrow{f_1} [n] \xleftarrow{f_0} [n']\,,\, \mathbf{m}'\right) = \left([n_i] \xleftarrow{f_i} \cdots \xleftarrow{f_2} [n_1] \xleftarrow{f_1} [n] \xleftarrow{\mathrm{id}} [n]\,,\, \langle f_0, \mathbf{m}'\rangle\right). \tag{9}$$

This element is identified with the following i-simplex of $N\mathcal{T}M$:

$$\langle f_i \cdots f_0, \mathbf{m}'\rangle \xleftarrow{f_i} \cdots \xleftarrow{f_2} \langle f_1 f_0, \mathbf{m}'\rangle \xleftarrow{f_1} \langle f_0, \mathbf{m}'\rangle. \tag{10}$$

Thus, (9)–(10) define a map $R_M \colon C(\Delta S_+, B^{sym_+}M) \to N\mathcal{T}M$.

Clearly, L_M and R_M are simplicial maps that are inverse of one another and the isomorphism follows:

$$C(\Delta S_+, B^{sym_+}M) \cong N\mathcal{T}M.$$

By Lemma 2.2, $\mathcal{T}M$ is permutative. Since the nerve functor N is symmetric monoidal, the $\mathscr{D}_{\mathbf{Cat}}$-algebra structure of diagram (5) is induced to the level of simplicial sets. This implies that $N\mathcal{T}M$, and hence also $C(\Delta S_+, B^{sym_+}M)$, is an E_∞ algebra. ☐

Remark 3.2. The fact that neither Δ_+ nor ΔC_+ are permutative categories implies that the proof of Lemma 3.1 does not extend to simplicial or cyclic homology. However, it is interesting to see that in certain special cases, there does seem to be way to define a Dyer–Lashoff structure on cyclic homology [**4**].

Remark 3.3. The proof of Lemma 3.1 unfortunately does not extend directly to arbitrary algebras. Indeed this is a serious obstruction to Theorem 8 of [**2**]! Presently, we do not have a way to prove that $HS_*(A) \cong H_*(B(D, T, A))$ (as [**2**] claims), where D is the monad associated to the operad $\mathscr{D}_{\mathbf{Mod}}$ and T is the functor that takes a k-module to its tensor algebra. The author suspects that the isomorphism is false for arbitrary algebras.

3.2. A structure map

For each $m \geqslant 0$, set

$$\mathcal{F}(m) \stackrel{def}{=} [m-1] \setminus \Delta S_+. \tag{11}$$

Equation (11) defines $\mathcal{F}$ as a ΔS_+^{op} category via $[n] \mapsto \mathcal{F}(n+1)$, and hence also as an $\mathbf{S}$ category. Precompose the duality functor $\mathbf{S}^{\mathrm{op}} \to \mathbf{S}$ sending $\sigma \mapsto \sigma^{-1}$ to define $\mathcal{F}$ as an $\mathbf{S}^{\mathrm{op}}$ category. Because of the many "reversals" wrapped up in this definition, it is important to show the details. For each $m \geqslant 0$, there is a *right* Σ_m action on $\mathcal{F}(m)$,

$$(\phi, \gamma) \bullet \sigma \overset{def}{=} (\phi, \gamma) \circ (\mathrm{id}, \sigma^{-1}) \tag{12}$$
$$= (\phi, \sigma^{-1}\gamma). \tag{13}$$

There is also a left Σ_n action on each set $\Delta S_+([m-1], [n-1])$:

$$\begin{aligned} \tau \bullet (\phi, \gamma) &\overset{def}{=} (\mathrm{id}, \tau^{-1}) \circ (\phi, \gamma) \\ &= (\phi^{(\tau^{-1})}, \gamma(\tau^{-1})^{\phi}), \end{aligned}$$

and the two actions commute in the sense that

$$\tau \bullet (f \bullet \sigma) = (\tau \bullet f) \bullet \sigma.$$

Let $m, j_1, j_2, \ldots, j_m \geqslant 0$ and $j = \sum j_s$. Assume morphisms f_i, g_i of ΔS_+, for $1 \leqslant i \leqslant m$, have specified sources and targets: $[j_i - 1] \overset{f_i}{\to} [p_i - 1] \overset{g_i}{\to} [q_i - 1]$. Define a family of maps,

$$\mu = \mu_{m, j_1, \ldots, j_m} : \mathscr{D}_{\mathbf{Cat}}(m) \times \prod_{s=1}^{m} \mathcal{F}(j_s) \longrightarrow \mathcal{F}(j),$$

on objects by

$$\mu(\sigma, f_1, f_2, \ldots, f_m) \overset{def}{=} \sigma_{\mathbf{p}} \bullet \mathbf{f}^{\odot}. \tag{14}$$

Define μ on morphisms by the following diagram

$$\begin{array}{ccc} (\sigma, f_1, \ldots, f_m) & \xrightarrow{\mu} & \sigma_{\mathbf{p}} \bullet \mathbf{f}^{\odot} \\ \Big\downarrow \tau\sigma^{-1} \times \mathbf{g} & & \Big\downarrow (\sigma_{\mathbf{p}})^{-1} \\ & & \mathbf{f}^{\odot} \\ & & \Big\downarrow \mathbf{g}^{\odot} \\ & & \mathbf{g}\mathbf{f}^{\odot} \\ & & \Big\downarrow \tau_{\mathbf{q}} \\ (\tau, g_1 f_1, \ldots, g_m f_m) & \xrightarrow{\mu} & \tau_{\mathbf{q}} \bullet \mathbf{g}\mathbf{f}^{\odot} \end{array}$$

The effect is simply "untwisting" by block permutation, applying the morphisms g_i in the natural order, then "retwisting" by the appropriate block permutation. Functoriality of μ is clear. We show in § 3.3 that the maps μ define a left operad-module structure (over $\mathscr{D}_{\mathbf{Cat}}$) on $\mathcal{F}$ (the reader is referred to [**19**] for the definition of operad-module).

3.3. Operad-module structure of $\mathcal{F}$.

Consider the set of formal indeterminates $X = \{x_1, x_2, \ldots\}$ and the free monoid $X^\star$ as defined in §2.3. In this section, we prove that $\mathcal{T}X^\star$ is isomorphic, as a category, to a certain category built from $\mathcal{F}$. We then use this isomorphism to prove that $\mathcal{F}$ admits the structure of operad-module over $\mathscr{D}_{\mathbf{Cat}}$.

Since $\mathcal{F}$ is an $\mathbf{S}^{\mathrm{op}}$ object, there is a right action of the symmetric group Σ_m on $\mathcal{F}(m)$ for each $m \geqslant 0$; recall Eqns. (12)–(13). There is a left action of Σ_m on X^m by permutation, $\sigma \bullet \mathbf{x} = \sigma\mathbf{x} = \big(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(m)}\big)$. Thus, the fibered product of categories can be formed, $\mathcal{F}(m) \times_{\Sigma_m} X^m$. Here, X is taken to be a discrete category.

For a set $\{\mathscr{C}_i\}$ of small categories whose object sets are pairwise disjoint, we use the notation $\bigcup_i \mathscr{C}_i$ to represent the category whose object set is $\bigcup_i \mathrm{Ob}\mathscr{C}_i$ and whose morphisms only those morphisms in $\mathrm{Mor}\mathscr{C}_i$ for each i. This is, of course, a particular realization of the coproduct of a set of small categories.

Lemma 3.4. *There is an isomorphism of categories*

$$e\colon \bigcup_{m \geqslant 0} (\mathcal{F}(m) \times_{\Sigma_m} X^m) \to \mathcal{T}X^\star$$

via the evaluation map e defined by $e(f, \mathbf{x}) \overset{def}{=} \langle f, \mathbf{x}\rangle$.

Proof. We must show that the evaluation functor

$$\begin{aligned} \mathcal{F}(m) \times X^m &\to \mathcal{T}X^\star \\ (f, \mathbf{x}) &\mapsto \langle f, \mathbf{x}\rangle \end{aligned}$$

factors through the canonical projection $\mathcal{F}(m) \times X^m \to \mathcal{F}(m) \times_{\Sigma_m} X^m$. Let f be a ΔS_+ morphism and write $f = \phi \circ \gamma$ with ϕ a morphism of Δ_+ and $\gamma \in \Sigma_m^{\mathrm{op}}$. By unique factorization in ΔS_+, the pair is unique to f. Let $\mathbf{x} \in X^m$, and let $\sigma \in \Sigma_m$. Observe that Property (3) is used to “transfer” the permutation from the left to the right:

$$\begin{aligned} \langle f \bullet \sigma, \mathbf{x}\rangle &= \langle \phi \circ \sigma^{-1}\gamma, \mathbf{x}\rangle \\ &= \big\langle \phi, \langle \sigma^{-1}\gamma, \mathbf{x}\rangle\big\rangle \\ &= \langle \phi, \gamma^{-1}\sigma\mathbf{x}\rangle \\ &= \big\langle \phi, \langle \gamma, \sigma\mathbf{x}\rangle\big\rangle \\ &= \langle \phi \circ \gamma, \sigma\mathbf{x}\rangle \\ &= \langle f, \sigma \bullet \mathbf{x}\rangle. \end{aligned}$$

There is also a map (on objects) in the reverse direction, defined by

$$(y_1, y_2, \ldots, y_n) \mapsto (\phi, x_{i_1}, x_{i_2}, \ldots, x_{i_m}),$$

where each y_i is a possibly empty monomial in the indeterminates x_j, such that $y_1 y_2 \cdots y_n = x_{i_1} x_{i_2} \cdots x_{i_m} \in X^\star$, and ϕ is the Δ_+ morphism such that $\phi(j-1) = j' - 1 \Leftrightarrow x_{i_j}$ appears as a factor in $y_{j'}$. Whereas the map e has the effect of multiplying certain groups of indeterminates together, the reverse map *factors* the monomials completely, which can be done uniquely since $X^\star$ is a free monoid. The two maps are inverse to one another, making e bijective on objects.

We have yet to define e on morphisms. Observe that since X^m is discrete, the morphisms of $\mathcal{F}(m) \times_{\Sigma_m} X^m$ all have the form $g \times \mathrm{id}_{x_{i_1}} \times \cdots \times \mathrm{id}_{x_{i_m}}$. The functor e simply maps this morphism to g as interpreted in $\mathcal{T}X^\star$, as the following commutative diagram illustrates:

$$\begin{array}{ccc} (f,\mathbf{x}) & \xrightarrow{e} & \langle f,\mathbf{x}\rangle \\ {\scriptstyle g\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle g} \\ (gf,\mathbf{x}) & \xrightarrow{e} & \langle gf,\mathbf{x}\rangle . \end{array}$$

It is straightforward to check that e is fully faithful, and so e is an isomorphism of categories as claimed. □

For the remainder of this section, we prove that the family of maps μ defined in §3.2 give $\mathcal{F}$ the structure of an operad-module over $\mathscr{D}_{\mathbf{Cat}}$. Fix integers $m, j_1, j_2, \cdots, j_m \geqslant 0$, and let $j = \sum j_s$. In this section $\mathbf{x} = (x_1, x_2, \ldots, x_j)$. We will need to partition $\mathbf{x}$ into chunks of sizes $j_1, j_2, \ldots, j_s$. To that end, define for each s,

$$\mathbf{x}_1 = (x_1, \ldots, x_{j_1}),$$
$$\mathbf{x}_s = (x_{j_1+\cdots+j_{s-1}+1}, \ldots, x_{j_1+\cdots+j_s}), \quad \text{for } s > 1.$$

For each number $s = 1, 2, \ldots, m$, let a_s be the inclusion functor

$$a_s \colon \mathcal{F}(j_s) \longrightarrow \mathcal{F}(j_s) \times_{\Sigma_{j_s}} X^{j_s},$$
$$f \mapsto (f, \mathbf{x}_s).$$

We also require a similar functor,

$$a \colon \mathcal{F}(j) \longrightarrow \mathcal{F}(j) \times_{\Sigma_j} X^j,$$
$$f \mapsto (f, \mathbf{x})\,.$$

Consider the functor (in which $\mathbf{a} = a_1 \times \cdots \times a_m$)

$$\tilde{a} \overset{def}{=} \mathrm{id} \times \mathbf{a} \colon \mathscr{D}_{\mathbf{Cat}}(m) \times \prod_{s=1}^{m} \mathcal{F}(j_s) \longrightarrow \mathscr{D}_{\mathbf{Cat}}(m) \times \prod_{s=1}^{m} \left(\mathcal{F}(j_s) \times_{\Sigma_{j_s}} X^{j_s}\right). \tag{15}$$

For any number $i \geqslant 0$, let b_i be the inclusion of categories:

$$b_i \colon \mathcal{F}(i) \times_{\Sigma_i} X^i \longrightarrow \bigcup_{i \geqslant 0} \mathcal{F}(i) \times_{\Sigma_i} X^i.$$

Define $\tilde{b}$ analogously to (15):

$$\tilde{b} \overset{def}{=} \mathrm{id} \times \mathbf{b} \colon \mathscr{D}_{\mathbf{Cat}}(m) \times \prod_{s=1}^{m} \left(\mathcal{F}(j_s) \times_{\Sigma_{j_s}} X^{j_s}\right) \longrightarrow \mathscr{D}_{\mathbf{Cat}}(m) \times \left[\bigcup_{i \geqslant 0} \left(\mathcal{F}(i) \times_{\Sigma_i} X^i\right)\right]^m.$$

Now, by Lemma 3.4, there is an isomorphism

$$\tilde{e} \overset{def}{=} \mathrm{id} \times e^m \colon \mathscr{D}_{\mathbf{Cat}}(m) \times \left[\bigcup_{i \geqslant 0} \left(\mathcal{F}(i) \times_{\Sigma_i} X^i\right)\right]^m \xrightarrow{\cong} \mathscr{D}_{\mathbf{Cat}}(m) \times (\mathcal{T}X^\star)^m\,.$$

Consider the following diagram. The top row is the map μ of Eq. (14), and the bottom row is the operad-algebra structure map for $\mathcal{T}X^\star$, which comes from the $\mathscr{D}_{\mathbf{Cat}}$-algebra structure of this permutative category (see Lemma 2.2).

$$
\begin{array}{ccc}
\mathscr{D}_{\mathbf{Cat}}(m) \times \prod_{s=1}^{m} \mathcal{F}(j_s) & \xrightarrow{\quad \mu \quad} & \mathcal{F}(j) \\
\downarrow \tilde{a} & & \downarrow a \\
\mathscr{D}_{\mathbf{Cat}}(m) \times \prod_{s=1}^{m} \big(\mathcal{F}(j_s) \times_{\Sigma_{j_s}} X^{j_s}\big) & & \mathcal{F}(j) \times_{\Sigma_j} X^j \\
\downarrow \tilde{b} & & \downarrow b_j \\
\mathscr{D}_{\mathbf{Cat}}(m) \times \left[\bigcup_{i \geqslant 0} \big(\mathcal{F}(i) \times_{\Sigma_i} X^i\big) \right]^m & & \bigcup_{i \geqslant 0} \big(\mathcal{F}(i) \times_{\Sigma_i} X^i\big) \\
\downarrow \tilde{e} & & \downarrow e \\
\mathscr{D}_{\mathbf{Cat}}(m) \times (\mathcal{T}X^\star)^m & \xrightarrow{\quad \theta \quad} & \mathcal{T}X^\star
\end{array}
\tag{16}
$$

Diagram (16) commutes if we can show that $\theta\tilde{e}\tilde{b}\tilde{a} = eb_j a\mu$. Let $w = (\sigma, f_1, \ldots, f_m) \in \mathscr{D}_{\mathbf{Cat}}(m) \times \prod_{s=1}^{m} \mathcal{F}(j_s)$ be arbitrary. First follow the element w down the left column and across the bottom of diagram (16).

$$
\begin{array}{lcl}
(\sigma, f_1, \ldots, f_m) & & \\
\downarrow \tilde{b}\tilde{a} & & \\
(\sigma, (f_1, \mathbf{x}_1), \ldots, (f_m, \mathbf{x}_m)) & & \\
\downarrow \tilde{e} & & \\
(\sigma, \langle f_1, \mathbf{x}_1\rangle, \ldots, \langle f_m, \mathbf{x}_m\rangle) & \xrightarrow{\ \theta\ } & \langle f_{\sigma^{-1}(1)}, \mathbf{x}_{\sigma^{-1}(1)}\rangle \odot \cdots \odot \langle f_{\sigma^{-1}(m)}, \mathbf{x}_{\sigma^{-1}(m)}\rangle.
\end{array}
$$

Now follow the element w across the top and down the right column of diagram (16). Assume the codomain of f_i is $[p_i - 1]$ for each $i \leqslant m$.

$$
\begin{array}{ccc}
(\sigma, f_1, \ldots, f_m) & \xrightarrow{\ \mu\ } & \sigma_{\mathbf{p}} \bullet \mathbf{f}^{\odot} \\
 & & \downarrow b_j a \\
 & & (\sigma_{\mathbf{p}} \bullet \mathbf{f}^{\odot}, \mathbf{x}) \\
 & & \downarrow e \\
 & & \langle \sigma_{\mathbf{p}} \bullet \mathbf{f}^{\odot}, \mathbf{x}\rangle.
\end{array}
\tag{17}
$$

Now since $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$, the bottom right element in diagram (17) may be simplified thus:

$$
\begin{aligned}
\langle \sigma_{\mathbf{p}} \bullet \mathbf{f}^{\odot}, \mathbf{x}\rangle &= \big\langle (\mathrm{id}, \sigma_{\mathbf{p}}^{-1}) \circ \mathbf{f}^{\odot}, (\mathbf{x}_1, \ldots, \mathbf{x}_m)\big\rangle \\
&= \big\langle (\mathrm{id}, \sigma_{\mathbf{p}}^{-1}) \langle \mathbf{f}^{\odot}, (\mathbf{x}_1, \ldots, \mathbf{x}_m)\rangle\big\rangle \\
&= \big\langle (\mathrm{id}, \sigma_{\mathbf{p}}^{-1}), \langle f_1, \mathbf{x}_1\rangle \odot \cdots \odot \langle f_m, \mathbf{x}_m\rangle\big\rangle \\
&= \langle f_{\sigma^{-1}(1)}, \mathbf{x}_{\sigma^{-1}(1)}\rangle \odot \cdots \odot \langle f_{\sigma^{-1}(m)}, \mathbf{x}_{\sigma^{-1}(m)}\rangle.
\end{aligned}
$$

Using diagram (16), we find that μ is an operad-module structure map. Associativity is induced by the associativity condition of the algebra structure map θ (because both eb_ja and $\tilde{e}\tilde{b}\tilde{a}$ are injective). It is trivial to verify the left unit condition (note that there is no corresponding right unit condition in an operad-module structure). We include the routine check that verifies the equivariance condition on the level of objects. Assume $f_i \in \mathcal{F}(j_i)$ (for $1 \leqslant i \leqslant m$) have specified sources and targets, $[j_i - 1] \xrightarrow{f_i} [p_i - 1]$. Recall, the symmetric group acts on the *right*.

Equivariance A:

$$\begin{array}{ccc}
(\sigma, \mathbf{f}) & \xmapsto{\mathrm{id} \times T_\tau} & (\sigma, \tau\mathbf{f}) \\
\downarrow{\scriptstyle \tau\times\mathrm{id}} & & \downarrow{\scriptstyle \mu} \\
(\sigma\tau, \mathbf{f}) & & \sigma_{\tau\mathbf{p}} \bullet \tau\mathbf{f}^{\odot} \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \tau_{\mathbf{j}}} \\
(\sigma\tau)_{\mathbf{p}} \bullet \mathbf{f}^{\odot} & = & \sigma_{\tau\mathbf{p}} \bullet \tau\mathbf{f}^{\odot} \bullet \tau_{\mathbf{j}}
\end{array}$$

Equivariance B:

$$\begin{array}{ccc}
(\sigma, \mathbf{f}) & \xmapsto{\mu} & \sigma_{\mathbf{p}} \bullet \mathbf{f}^{\odot} \\
\downarrow{\scriptstyle \mathrm{id}\times\tau_1\times\cdots\times\tau_m} & & \\
(\sigma, f_1 \bullet \tau_1, \dots, f_m \bullet \tau_m) & & \downarrow{\scriptstyle \tau_1\oplus\cdots\oplus\tau_m} \\
\downarrow{\scriptstyle \mu} & & \\
\sigma_{\mathbf{p}} \bullet ((f_1 \bullet \tau_1) \odot \cdots \odot (f_m \bullet \tau_m)) & = & \sigma_{\mathbf{p}} \bullet \mathbf{f}^{\odot} \bullet (\tau_1 \oplus \cdots \oplus \tau_m)
\end{array}$$

Remark 3.5. It can be verified that $\mathcal{F}$ is in fact a pseudo-operad. The details are left to the reader, as this result will not be used in the present paper. Recall from [**19**] that a pseudo-operad is a "non-unitary" operad. The structure maps are defined by the composition

$$\mathcal{F}(m) \times \prod_{s=1}^{m} \mathcal{F}(j_s) \xrightarrow{\pi\times\mathrm{id}} \mathscr{D}_{\mathbf{Cat}}(m) \times \prod_{s=1}^{m} \mathcal{F}(j_s) \xrightarrow{\mu} \mathcal{F}(j_1 + \cdots + j_m),$$

where $\pi \colon \mathcal{F}(m) \to \mathscr{D}_{\mathbf{Cat}}(m)$ is the projection functor defined by $\pi(\phi, \gamma) = \gamma^{-1}$. Indeed, π defines an isomorphism of the subcategory $\mathrm{Aut}\,([m-1] \setminus \Delta S_+)$ onto $\mathscr{D}_{\mathbf{Cat}}(m)$. Note that $\mathcal{F}$ is not a full operad, since it fails the right-unit condition.

We shall denote the associated simplicial k-module $\widetilde{\mathcal{F}} \overset{def}{=} k[N(- \setminus \Delta S_+)]$.

Corollary 3.6. *There is a $\mathscr{D}_{\mathbf{Mod}}$-module structure on $\widetilde{\mathcal{F}}$.*

Proof. The $\mathscr{D}_{\mathbf{Cat}}$-module structure of $\mathcal{F}$ gets induced via the chain of symmetric monoidal functors

$$\mathbf{Cat} \xrightarrow{N} \mathbf{Set}^{\Delta^{\rm op}} \xrightarrow{k[-]} k\text{-}\mathbf{Mod}^{\Delta^{\rm op}}.$$

□

3.4. Operad-algebra structure

In this subsection we use the operad-module structure defined in §3.3 to induce a related operad-algebra structure. Let us first recall a fact of operad theory:

Proposition 3.7. *Suppose $(\mathscr{C}, \oplus, \odot)$ is a cocomplete distributive symmetric monoidal category, $\mathscr{P}$ is an operad in $\mathscr{C}$, $\mathscr{L}$ is a left $\mathscr{P}$-module, and $Z \in \mathrm{Obj}\mathscr{C}$. Then*

$$\mathscr{L}\langle Z\rangle \stackrel{def}{=} \bigoplus_{m\geqslant 0} \mathscr{L}(m) \odot_{\Sigma_m} Z^{\odot m}$$

admits the structure of a $\mathscr{P}$-algebra.

Remark 3.8. The notation $\mathscr{L}\langle Z\rangle$ appears in Kapranov and Manin [**17**] (where they use it in the category of vector spaces). The concept is also present in [**19**] as the *Schur functor* of an operad ([**19**, Def 1.24]).

Lemma 3.9. *The simplicial k-module $\widetilde{\mathcal{F}} \otimes_{\mathrm{Aut}\Delta S_+} B_*^{sym_+}A$ admits the structure of an E_∞ algebra.*

Proof. One may identify

$$\widetilde{\mathcal{F}} \otimes_{\mathrm{Aut}\Delta S_+} B_*^{sym_+}A = \bigoplus_{n\geqslant 0} \widetilde{\mathcal{F}}(n) \otimes_{\Sigma_n} A^{\otimes n} = \widetilde{\mathcal{F}}\langle A\rangle.$$

The result then follows directly from Cor. 3.6 and Prop. 3.7. □

In what follows, denote $CA_* \stackrel{def}{=} C_*(\Delta S_+, B_*^{sym_+}A)$, the simplicial k-module defined in Eqn. (2). Note that

$$CA_* = \widetilde{\mathcal{F}} \otimes_{\Delta S_+} B_*^{sym_+}A.$$

The inclusion $\mathrm{Aut}\Delta S_+ \hookrightarrow \Delta S_+$ induces a quotient map $Q\colon \widetilde{F}\langle A\rangle \to CA_*$.

Lemma 3.10. *The $\mathscr{D}_{\mathbf{Mod}}$-algebra structure on $\widetilde{\mathcal{F}} \otimes_{\mathrm{Aut}\Delta S_+} B_*^{sym_+}A$ induces a $\mathscr{D}_{\mathbf{Mod}}$-algebra structure on CA_*, which implies that CA_* is an E_∞ algebra in the category of simplicial k-modules.*

Proof. Let ν be the structure map implied by Lemma 3.9 (which is ultimately induced by the structure map μ of § 3.2):

$$\nu\colon \mathscr{D}_{\mathbf{Mod}}(n) \otimes_{\Sigma_n} \left(\widetilde{\mathcal{F}}\langle A\rangle\right)^{\otimes n} \longrightarrow \widetilde{\mathcal{F}}\langle A\rangle.$$

We will show that ν remains well defined upon passing to the quotient, as illustrated in the following diagram:

$$\begin{array}{ccc} \mathscr{D}_{\mathbf{Mod}}(n) \otimes_{\Sigma_n} \widetilde{\mathcal{F}}\langle A\rangle^{\otimes n} & \xrightarrow{\nu} & \widetilde{\mathcal{F}}\langle A\rangle \\ \big\downarrow{\scriptstyle \mathrm{id}\otimes Q^{\otimes n}} & & \big\downarrow{\scriptstyle Q} \\ \mathscr{D}_{\mathbf{Mod}}(n) \otimes_{\Sigma_n} (CA_*)^{\otimes n} & \xrightarrow{\nu} & CA_* \end{array} \tag{18}$$

It suffices to check that the structure is well defined in degree 0, because the face and degeneracy maps are induced by compositions and evaluations of ΔS_+ morphisms. A generator of $\mathscr{D}_{\mathbf{Mod}}(n) \otimes_{\Sigma_n} (CA_*)^{\otimes n}$ in degree 0 has the following form:

$$\sigma \otimes (g_1 f_1 \otimes V_1) \otimes \cdots \otimes (g_n f_n \otimes V_n)\,, \tag{19}$$

where $\sigma \in \Sigma_n$, f_i, g_i $(1 \leqslant i \leqslant n)$ are morphisms of ΔS_+ with specified sources and targets, $[m_i - 1] \xrightarrow{f_i} [p_i - 1] \xrightarrow{g_i} [q_i - 1]$, and $V_i \in A^{\otimes m_i}$. The map ν sends the element (19) to $(\sigma_{\mathbf{q}} \bullet \mathbf{g}\mathbf{f}^{\odot}) \otimes (V_1 \otimes \cdots \otimes V_n)$. On the other hand, element (19) is equal (under ΔS_+-equivariance) to

$$\sigma \otimes (g_1 \otimes \langle f_1, V_1\rangle) \otimes \cdots \otimes (g_n \otimes \langle f_n, V_n\rangle), \tag{20}$$

and ν sends (20) to

$$\begin{aligned} (\sigma_{\mathbf{q}} \bullet \mathbf{g}^{\odot}) \otimes (\langle f_1, V_1\rangle \otimes \cdots \otimes \langle f_n, V_n\rangle) &= (\sigma_{\mathbf{q}} \bullet \mathbf{g}^{\odot}) \otimes \langle \mathbf{f}^{\odot}, V_1 \otimes \cdots \otimes V_n\rangle \\ &= (\sigma_{\mathbf{q}} \bullet \mathbf{g}\mathbf{f}^{\odot}) \otimes (V_1 \otimes \cdots \otimes V_n). \end{aligned}$$

□

Theorem 3.11. *When the ground ring $k = \mathbb{F}_p$ for a prime p, symmetric homology $HS_*(A)$ admits Dyer–Lashof homology operations.*

Proof. This is an immediate result of Lemma 3.10 and the fact that $HS_*(A)$ is the homology of CA_*. The reader is referred to Dyer and Lashof [**10**], May [**20**], or Chapter I of [**8**] for details on constructing the operations on any E_∞ algebra. □

4. Product structure

4.1. Pontryagin product

There is a well-defined graded-commutative product on the graded k-module, $\{HS_i(A)\}_{i\geqslant 0}$.

Theorem 4.1. *$HS_*(A)$ admits a Pontryagin product, giving it the structure of associative, graded commutative algebra.*

Proof. This follows directly from Lemma 3.10. The product is defined by

$$\begin{gathered} (CA_*) \otimes (CA_*) \hookrightarrow \mathscr{D}_{\mathbf{Mod}}(2) \otimes_{\Sigma_2} (CA_*)^{\otimes 2} \xrightarrow{\nu} (CA_*) \\ x \otimes y \mapsto c \otimes (x \otimes y) \mapsto \nu(c \otimes (x \otimes y)), \end{gathered}$$

where ν is defined in diagram (18) and $c \in \mathscr{D}_{\mathbf{Mod}}(2)$ is a generator as a free k-module. □

Corollary 4.2. *Let A be a unital associative k-algebra. If the ideal generated by the commutator submodule is equal to the entire algebra (i.e. $([A,A]) = A$), then $HS_*(A)$ is trivial in all degrees.*

Proof. $HS_0(A) = A/([A,A])$, so $HS_0(A)$ is trivial. Now for any $x \in HS_q(A)$, we have $x = 1 \cdot x = 0 \cdot x$. □

Remark 4.3. It was pointed out in [**1**] that symmetric homology fails to preserve Morita equivalence. Corollary 4.2 shows the failure in a big way: $HS_*(M_n(A))$ is trivial if $n > 1$.

Proposition 4.4. *When restricted to $HS_0(A) \otimes HS_0(A) \to HS_0(A)$, the Pontryagin product is the standard algebra multiplication map $A/([A,A]) \otimes A/([A,A]) \to A/([A,A])$.*

Proof. Examine the first few terms of the sequence, $0 \leftarrow CA_0 \stackrel{d_1}{\leftarrow} CA_1$. It is straightforward to verify that d_1 collapses the generators in degree 0 to those of the form $([0] \leftarrow [0]) \otimes a$ via the iterated multiplication map $A^{\otimes n} \to A$. □

4.2. Explicit $HS_0(A)$-module structure of $HS_1(A)$

The main result of this subsection is a concrete computation of the Pontryagin product $HS_0(A) \otimes HS_1(A) \to HS_1(A)$. We shall need to induce the $\mathscr{D}_{\mathbf{Mod}}$-algebra structure of CA_* to the level of complexes in order to transfer the E_∞ structure across a chain equivalence. This step is trivial, as the "chains" functor of the Dold–Kan correspondence is lax monoidal. However, we must remember to use the shuffle map when making computations at the chain level. Let $\overline{CA}_\bullet$ denote the Moore complex of CA_*.

Lemma 4.5. *The $\mathscr{D}_{\mathbf{Mod}}$-algebra structure on CA_* induces a $\mathscr{D}_{\mathbf{Ch}_\bullet^+}$-algebra structure on $\overline{CA}_\bullet$.*

We shall also need some machinery from [**1**, §§10–11]. For each $n \geqslant -1$, define the projective ΔS_+^{op}-module $(\Delta S_+)_n$ as in (4). The following sequence is a partial resolution of $\underline{k}$ by projective ΔS_+^{op}-modules:

$$\underline{k} \xleftarrow{\epsilon} (\Delta S_+)_0 \xleftarrow{\delta} (\Delta S_+)_2 \xleftarrow{(\alpha,\beta)} (\Delta S_+)_3 \oplus (\Delta S_+)_0, \tag{21}$$

in which $\epsilon(f) = 1$ for any morphism $f\colon [n] \to [0]$, $\delta(f) = (x_0x_1x_2) \circ f - (x_2x_1x_0) \circ f$,

$$\alpha(f) = (x_0x_1 \otimes x_2 \otimes x_3) \circ f + (x_3 \otimes x_2x_0 \otimes x_1) \circ f + (x_1x_2x_0 \otimes 1 \otimes x_3) \circ f \\ +(x_3 \otimes x_1x_2 \otimes x_0) \circ f,$$

and $\beta(f) = (1 \otimes x_0 \otimes 1) \circ f$. Thus, there is a small partial chain complex that computes $HS_i(A)$ for $i = 0, 1$,

$$0 \longleftarrow A \xleftarrow{\partial_1} A^{\otimes 3} \xleftarrow{\partial_2} A^{\otimes 4} \oplus A, \tag{22}$$

in which

$$\partial_1(a \otimes b \otimes c) = abc - cba$$
$$\partial_2(a \otimes b \otimes c \otimes d, e) = ab \otimes c \otimes d + d \otimes ca \otimes b + bca \otimes 1 \otimes d + d \otimes bc \otimes a + 1 \otimes e \otimes 1.$$

If $a \in A$, denote by $[a]$ the corresponding element of $HS_0(A)$, and if $a \otimes b \otimes c \in A^{\otimes 3}$, denote by $[a \otimes b \otimes c]$ the corresponding element of $HS_1(A)$.

Our first goal is to set up an explicit equivalence between the partial complex (22) and $\overline{CA}_\bullet$, at least up to degree 1, and then use the equivalence to give a concrete formula for the product structure. In diagram (23), the differential d is induced from the simplicial face maps. Below, we define and discuss the maps F_i and G_i for $i = 0, 1$.

$$\begin{array}{ccccccc} 0 & \longleftarrow & A & \xleftarrow{\partial_1} & A^{\otimes 3} & \xleftarrow{\partial_2} & A^{\otimes 4} \oplus A \\ & & {\scriptstyle G_0}\uparrow\downarrow{\scriptstyle F_0} & & {\scriptstyle G_1}\uparrow\downarrow{\scriptstyle F_1} & & \\ 0 & \longleftarrow & \overline{CA}_0 & \xleftarrow{d_1} & \overline{CA}_1 & \xleftarrow{d_2} & \overline{CA}_2 \end{array} \tag{23}$$

For each $m \geqslant -1$, let $\pi_m \colon [m] \to [0]$ be the unique order-perserving ΔS_+ morphism, and $\rho_m \colon [m] \to [0]$ be the unique order-reversing ΔS_+ morphism. For convenience, let $\mathbf{a} = a_0 \otimes \cdots \otimes a_n$ stand for an arbitrary element of $A^{\otimes(n+1)}$. We define the maps F_0 and G_0 as follows:

$$F_0(a) \overset{def}{=} \left([0] \xleftarrow{\mathrm{id}} [0]\right) \otimes a$$

$$G_0\left(\left([m] \xleftarrow{f} [n]\right) \otimes \mathbf{a}\right) \overset{def}{=} \langle \pi_m f, \mathbf{a} \rangle.$$

Observe that $G_0F_0(a) = a$. To show $F_0G_0 \simeq \mathrm{id}$, define a homotopy map:

$$h_0 \colon \overline{CA}_0 \to \overline{CA}_1$$

$$\left([m] \xleftarrow{f} [n]\right) \otimes \mathbf{a} \mapsto \left([0] \xleftarrow{\pi_m} [m] \xleftarrow{f} [n]\right) \otimes \mathbf{a}$$

Observe that

$$\begin{aligned} d_1 h_0\left(\left([m] \xleftarrow{f} [n]\right) \otimes \mathbf{a}\right) &= \left(\left([0] \xleftarrow{\pi_m f} [n]\right) \otimes \mathbf{a}\right) - \left(\left([m] \xleftarrow{f} [n]\right) \otimes \mathbf{a}\right) \\ &= \left(\left([0] \xleftarrow{\mathrm{id}} [0]\right) \otimes \langle \pi_m f, \mathbf{a} \rangle\right) - \left(\left([m] \xleftarrow{f} [n]\right) \otimes \mathbf{a}\right) \\ &= (F_0G_0 - \mathrm{id})\left(\left([m] \xleftarrow{f} [n]\right) \otimes \mathbf{a}\right). \end{aligned}$$

Next, define F_1:

$$F_1(a \otimes b \otimes c) \overset{def}{=} \left[\left([0] \xleftarrow{\pi_2} [2] \xleftarrow{\mathrm{id}} [2]\right) - \left([0] \xleftarrow{\rho_2} [2] \xleftarrow{\mathrm{id}} [2]\right)\right] \otimes (a \otimes b \otimes c).$$

The maps F_0, F_1 are compatible with the differentials, as illustrated by a diagram-chase:

$$\begin{array}{ccc} abc - bca & \xleftarrow{\quad\partial_1\quad} & a \otimes b \otimes c \\ & & \downarrow {\scriptstyle F_1} \\ \downarrow {\scriptstyle F_0} & & \left[\left([0] \xleftarrow{\pi_2} [2] \xleftarrow{\mathrm{id}} [2]\right) - \left([0] \xleftarrow{\rho_2} [2] \xleftarrow{\mathrm{id}} [2]\right)\right] \otimes (a \otimes b \otimes c) \\ & & {\scriptstyle d_1} \downarrow \\ \left([0] \xleftarrow{\mathrm{id}} [0]\right) \otimes (abc - bca) & = & \left[\left([0] \xleftarrow{\pi_2} [0]\right) - \left([0] \xleftarrow{\rho_2} [0]\right)\right] \otimes (a \otimes b \otimes c) \end{array}$$

Defining G_1 is a bit trickier. For each $n \geqslant 0$, construct a quiver $\widetilde{\mathscr{G}}_n$ as follows: The vertices of $\widetilde{\mathscr{G}_n}$ are permutations of $\{0, 1, \ldots, n\}$. The edges of $\widetilde{\mathscr{G}_n}$ are in one-to-one correspondence with the elements of $\Delta S_+([n], [2])$. For any $f\colon [n] \to [m]$, write $f = (\phi(f), \gamma(f))$ for the unique ΔS_+ factorization. Now each f labels an edge in $\widetilde{\mathscr{G}_n}$ whose source is the permuation $\gamma(f) = \gamma(\pi_2 f)$ and whose target is $\gamma(\rho_2 f)$. For example, in $\widetilde{\mathscr{G}_5}$, the morphism $x_3x_1 \otimes x_4 \otimes x_0x_5$ (written in tensor notation) labels an edge from vertex "31405" to vertex "05431." Let $\mathscr{G}_n$ be a maximal subtree of $\widetilde{\mathscr{G}_n}$. Note that $\mathscr{G}_n$ is connected, which is a result of the fact that $k \leftarrow (\Delta S_+)_0([n]) \leftarrow (\Delta S_+)_2([n])$ is exact for all $n \geqslant 0$ (see (21) and [**1**, Lemma 79]). The purpose of $\mathscr{G}_n$ is to record ways in which one permutation may be converted to any other by way of block permutations of no more than three blocks at a time.

Example 4.6. $\mathscr{G}_2$ may be chosen to be the graph on vertices 01 and 10 with a single edge $01 \to 10$ labeled by $x_0 \otimes x_1 \otimes 1$. See Figures 1 and 2 for further examples (for brevity in the diagrams, we may write morphisms of ΔS_+ in tensor notation using the symbols $a = x_0$, $b = x_1$, $c = x_2$, $d = x_3$, etc.).

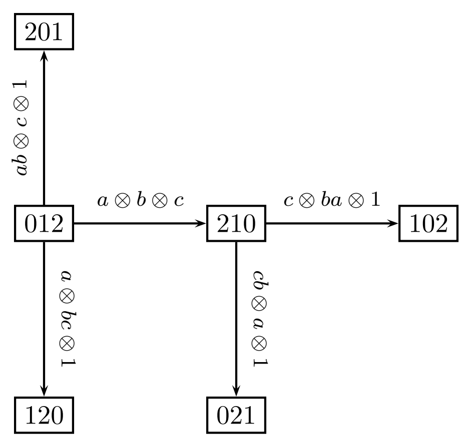

Figure 1: One possible choice of $\mathscr{G}_2$

Consider a typical element $([p] \xleftarrow{g} [m] \xleftarrow{f} [n]) \otimes \mathbf{a} \in \overline{CA}_2$. There is a unique path from $\gamma(gf)$ to $\gamma(f)$ in $\mathscr{G}_n$. Let $\mathrm{Path}(gf, f)$ be the set of edge labels, each taken to be positive or negative depending on the direction of the arrow as one proceeds from $\gamma(gf)$ to $\gamma(f)$ in the tree (positive if with the arrow; negative if against it). If $\gamma(gf) = \gamma(f)$, then $\mathrm{Path}(gf, f) = \emptyset$. Define $G_1\colon \overline{CA}_1 \to A^{\otimes 3}$ thus:

$$G_1\left(([p] \xleftarrow{g} [m] \xleftarrow{f} [n]) \otimes \mathbf{a}\right) = \sum_{e \in \mathrm{Path}(gf,f)} \langle e, \mathbf{a} \rangle.$$

Note that the choice of maximal subtree $\mathscr{G}_n$ for each n must be made once and not changed, as different choices for subtree will affect the definition of G_1.

Example 4.7. Let $f = 1 \otimes x_2 \otimes x_0 \otimes 1 \otimes x_1\colon [2] \to [4]$ and $g = x_3 \otimes x_2x_0 \otimes 1 \otimes x_1x_4\colon [4] \to [3]$. Then $gf = 1 \otimes x_0 \otimes 1 \otimes x_2x_1$, and $\gamma(gf) = 021$ is the "start" node, while

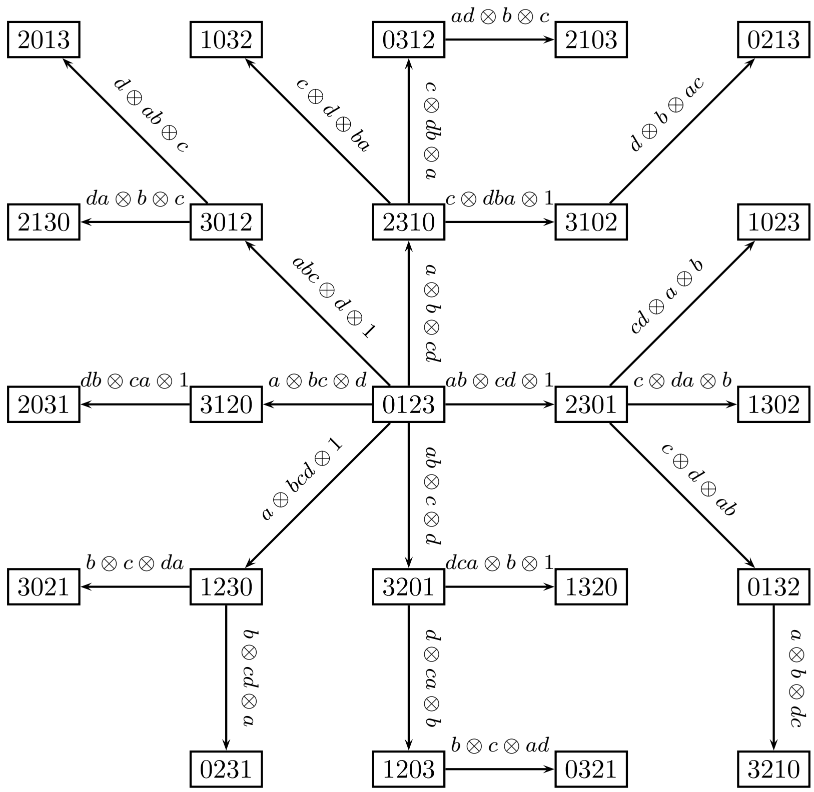

Figure 2: One possible choice of $\mathscr{G}_3$

$\gamma(f) = 201$ is the "end" node. We use Figure 1 to determine the path.

$$G_1\left(([3] \stackrel{g}{\leftarrow} [4] \stackrel{f}{\leftarrow} [2]) \otimes (a \otimes b \otimes c)\right) = -cb \otimes a \otimes 1 - a \otimes b \otimes c + ab \otimes c \otimes 1.$$

The maps G_0, G_1 are also compatible with the differentials, as we verify below,

$$G_0 d_1\left(([p] \stackrel{g}{\leftarrow} [m] \stackrel{f}{\leftarrow} [n]) \otimes \mathbf{a}\right) = \langle \pi_p g f, \mathbf{a}\rangle - \langle \pi_m f, \mathbf{a}\rangle.$$

$$\begin{aligned}\partial_1 G_1\left(([p] \stackrel{g}{\leftarrow} [m] \stackrel{f}{\leftarrow} [n]) \otimes \mathbf{a}\right) &= \partial_1\left(\sum_e \langle e, \mathbf{a}\rangle\right) \\ &= \sum_e \left(\langle \pi_2 e, \mathbf{a}\rangle - \langle \rho_2 e, \mathbf{a}\rangle\right).\end{aligned}$$

The sum telescopes so that only the start and end vertices of the path remain: $\langle \pi_p g f, \mathbf{a}\rangle - \langle \pi_m f, \mathbf{a}\rangle$.

Using $\mathscr{G}_2$ as in Figure 1, we find that $G_1 F_1 = \mathrm{id}$. The verification is provided below (here, $\mathbf{a} = a \otimes b \otimes c$, and observe that $\gamma(\rho_2) =$ "210" in Figure 1, so $\mathrm{Path}(\rho_2, \mathrm{id}) = \{-(a \otimes b \otimes c)\}$).

$$\begin{aligned}
G_1F_1(\mathbf{a}) &= G_1\left(\left([0] \overset{\pi_2}{\leftarrow} [2] \overset{\mathrm{id}}{\leftarrow} [2]\right) \otimes \mathbf{a}\right) - G_1\left(\left([0] \overset{\rho_2}{\leftarrow} [2] \overset{\mathrm{id}}{\leftarrow} [2]\right) \otimes \mathbf{a}\right)\\
&= \sum_{e \in \mathrm{Path}(\pi_2, \mathrm{id})} \langle e, \mathbf{a}\rangle - \sum_{e \in \mathrm{Path}(\rho_2, \mathrm{id})} \langle e, \mathbf{a}\rangle\\
&= 0 - (-(a \otimes b \otimes c))\\
&= \mathbf{a}.
\end{aligned}$$

Finally, we set up a homotopy map $h_1 \colon \overline{CA}_1 \to \overline{CA}_2$ to show that $F_1G_1 \simeq \mathrm{id}$.

$$\overline{CA}_0 \underset{d_1}{\overset{h_0}{\rightleftarrows}} \overline{CA}_1 \ (\circlearrowleft F_1G_1) \underset{d_2}{\overset{h_1}{\rightleftarrows}} \overline{CA}_2$$

At this point, it is helpful to use an abbreviated notation for i-chains of $\overline{CA}_\bullet$:

$$(g_i, \ldots, g_1, f, \mathbf{a}) \overset{def}{=} ([m_i] \overset{g_i}{\leftarrow} \cdots \overset{g_1}{\leftarrow} [m_1] \overset{f}{\leftarrow} [n]) \otimes \mathbf{a}. \tag{24}$$

Of course, ΔS_+-equivariance still applies; in particular, the element (24) is equal to $(g_i, \ldots, g_1, \mathrm{id}, \langle f, \mathbf{a}\rangle)$. Let $(g, f, \mathbf{a}) \in \overline{CA}_1$ be as above, that is, $[p] \overset{g}{\leftarrow} [m] \overset{f}{\leftarrow} [n]$ is a sequence of ΔS_+ morphisms.

$$F_1G_1(g, f, \mathbf{a}) = \sum_{e \in \mathrm{Path}(gf, f)} \left[(\pi_2, \mathrm{id}, \langle e, \mathbf{a}\rangle) - (\rho_2, \mathrm{id}, \langle e, \mathbf{a}\rangle)\right] \tag{25}$$

$$= \sum_{e \in \mathrm{Path}(gf, f)} \left[(\pi_2, e, \mathbf{a}) - (\rho_2, e, \mathbf{a})\right]. \tag{26}$$

$$h_0d_1(g, f, \mathbf{a}) = (\pi_p, gf, \mathbf{a}) - (\pi_m, f, \mathbf{a}). \tag{27}$$

We define h_1 by

$$\begin{aligned}
h_1 \colon (g, f, \mathbf{a}) \mapsto\ & (\pi_m, f, \mathrm{id}, \mathbf{a}) - (\pi_p g, f, \mathrm{id}, \mathbf{a}) - (\pi_p, g, f, \mathbf{a})\\
& + \sum_{e \in \mathrm{Path}(gf, f)} \left[(\pi_2, e, \mathrm{id}, \mathbf{a}) - (\rho_2, e, \mathrm{id}, \mathbf{a})\right].
\end{aligned} \tag{28}$$

Then a tedious but straighforward calculation shows that $F_1G_1 - \mathrm{id} = d_2h_1 + h_0d_1$. Some details are shown below, as d_2 is applied to the various terms that comprise the right-hand side of (28).

$$d_2 \colon (\pi_m, f, \mathrm{id}, \mathbf{a}) \mapsto (\pi_m, f, \mathbf{a}) - (\pi_m f, \mathrm{id}, \mathbf{a}) + (f, \mathrm{id}, \mathbf{a}). \tag{29}$$

$$d_2 \colon (\pi_p g, f, \mathrm{id}, \mathbf{a}) \mapsto (\pi_p g, f, \mathbf{a}) - (\pi_p gf, \mathrm{id}, \mathbf{a}) + (f, \mathrm{id}, \mathbf{a}). \tag{30}$$

$$d_2 \colon (\pi_p, g, f, \mathbf{a}) \mapsto (\pi_p, gf, \mathbf{a}) - (\pi_p g, f, \mathbf{a}) + (g, f, \mathbf{a}). \tag{31}$$

$$d_2 \colon (\pi_2, e, \mathrm{id}, \mathbf{a}) \mapsto (\pi_2, e, \mathbf{a}) - (\pi_2 e, \mathrm{id}, \mathbf{a}) + (e, \mathrm{id}, \mathbf{a}). \tag{32}$$

$$d_2 \colon (\rho_2, e, \mathrm{id}, \mathbf{a}) \mapsto (\rho_2, e, \mathbf{a}) - (\rho_2 e, \mathrm{id}, \mathbf{a}) + (e, \mathrm{id}, \mathbf{a}). \tag{33}$$

In view of (28) and (29)–(33), and after many cancellations,

$$d_2h_1(g, f, \mathbf{a}) = (\pi_m, f, \mathbf{a}) - (\pi_p, gf, \mathbf{a}) - (g, f, \mathbf{a}) + \sum_e \left[(\pi_2, e, \mathbf{a}) - (\rho_2, e, \mathbf{a})\right],$$

which is the same as (cf. Eqns. (26) and (27))

$$(-h_0d_1 - \mathrm{id} + F_1G_1)(g, f, \mathbf{a}).$$

Proposition 4.8. *For a unital associative algebra A over commutative ground ring k, $HS_1(A)$ is a left $HS_0(A)$-module, via*

$$[a] \bullet [b \otimes c \otimes d] = [ab \otimes c \otimes d] - [b \otimes ca \otimes d] + [b \otimes c \otimes ad].$$

Moreover, there is a right module structure,

$$[b \otimes c \otimes d] \bullet [a] = [ba \otimes c \otimes d] - [b \otimes ac \otimes d] + [b \otimes c \otimes da],$$

and the two actions agree in the sense that $[a] \bullet [b \otimes c \otimes d] = [b \otimes c \otimes d] \bullet [a]$.

Remark 4.9. This module structure was first discovered on the chain level before the Pontryagin product was discovered. Below is the explicit derivation using Theorem 4.1.

Proof. Let $w, x, y, z \in A$, so that w represents a 0-chain and $x \otimes y \otimes z$ represents a 1-chain in the partial sequence (22) used to compute $HS_*(A)$. Consider $\mathrm{id}_{\Sigma_2} \in \mathscr{D}_{\mathbf{Ch}_\bullet^+}(2)$, and let F_* and G_* be the chain equivalences developed above. Note that in line (34), morphisms of ΔS_+ are written in tensor notation.

$$\begin{aligned}
&\mathrm{id}_{\Sigma_2} \otimes (a) \otimes (b \otimes c \otimes d) \\
&\overset{\mathrm{id}\otimes F_*^{\otimes 2}}{\mapsto} \mathrm{id}_{\Sigma_2} \otimes (\mathrm{id}_{[0]}, a) \otimes \big((\pi_2, \mathrm{id}_{[2]}, b \otimes c \otimes d) - (\rho_2, \mathrm{id}_{[2]}, b \otimes c \otimes d)\big) \\
&\overset{\nu}{\mapsto} \big[(\mathrm{id}_{[0]} \odot \pi_2, \mathrm{id}_{[3]}) - (\mathrm{id}_{[0]} \odot \rho_2, \mathrm{id}_{[3]})\big] \otimes (a \otimes b \otimes c \otimes d) \\
&= \big[(x_0 \otimes x_1x_2x_3, \mathrm{id}_{[3]}) - (x_0 \otimes x_3x_2x_1, \mathrm{id}_{[3]})\big] \otimes (a \otimes b \otimes c \otimes d) \qquad (34) \\
&\overset{G_1}{\mapsto} b \otimes c \otimes ad + d \otimes ca \otimes b + ab \otimes c \otimes d
\end{aligned}$$

Finally, using the *sign relation* (see [**1**, §10]), we have equality in $HS_1(A)$:

$$[b \otimes c \otimes ad] + [d \otimes ca \otimes b] + [ab \otimes c \otimes d] = [ab \otimes c \otimes d] - [b \otimes ca \otimes d] + [b \otimes c \otimes ad].$$

The product $HS_1(A) \otimes HS_0(A) \to HS_1(A)$ can be found explicitly in a similar manner. The fact that the two products agree follows from the observation that their difference is a boundary. □

Remark 4.10. Theoretically, if the resolution (21) could be extended further, then one could extend the maps F_i and G_i to higher degrees in order to study the product structure of $HS_*(A)$. However, this tedious "nuts-and-bolts" approach does not seem to offer best ratio of payoff in exchange for the work put in.

4.3. Computed results

Using `GAP`, the following explicit computations of the $HS_0(A)$-module structure on $HS_1(A)$ were made for some $\mathbb{Z}$-algebras. Note in each case below, $HS_0(A) = A$ since A is commutative.

A	$HS_1(A \mid \mathbb{Z})$	$HS_0(A)$–module structure
$\mathbb{Z}[t]/(t^2)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	Generated by u with $2u = 0$
$\mathbb{Z}[t]/(t^3)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	Generated by u with $2u = 0$ and $t^2u = 0$
$\mathbb{Z}[t]/(t^4)$	$(\mathbb{Z}/2\mathbb{Z})^4$	Generated by u with $2u = 0$
$\mathbb{Z}[C_2]$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	Generated by u with $2u = 0$
$\mathbb{Z}[C_3]$	0	
$\mathbb{Z}[C_4]$	$(\mathbb{Z}/2\mathbb{Z})^4$	Generated by u with $2u = 0$
$\mathbb{Z}[C_5]$	0	

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Shaun V. Ault svault@valdosta.edu

Department of Mathematics and Computer Science, Valdosta State University, 1500 N. Patterson St. Valdosta, Georgia, 31698, USA.