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AN ALGEBRAIC MODEL FOR RATIONAL G-SPECTRA OVER AN EXCEPTIONAL SUBGROUP

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Abstract

We give a simple algebraic model for rational G-spectra over an *exceptional* subgroup, for any compact Lie group G. Moreover, all our Quillen equivalences are symmetric monoidal, so as a corollary we obtain a monoidal algebraic model for rational G-spectra when G is finite. We also present a study of the relationship between induction-restriction-coinduction adjunctions and left Bousfield localizations at idempotents of the rational Burnside ring.

1. Introduction

Modelling the category of rational G-spectra

G-spectra are representing objects for cohomology theories designed to take symmetries of spaces into account. Rationalizing this category removes topological complexity, but leaves interesting equivariant behaviour. In order to understand this behaviour, we try to find a purely algebraic description of the category, i.e. an algebraic model category Quillen equivalent to the category of rational G-spectra, which we call *an algebraic model* for rational G-spectra. As a result, the homotopy category of an algebraic model is equivalent to the rational stable G-homotopy category. Having an actual zig-zag of Quillen equivalences between model categories ensures that the equivalence of homotopy categories is a triangulated equivalence and makes it easier to perform constructions and calculations on the algebraic side of the zig-zag. It also ensures that invariants such as homotopy types of mapping spaces or the algebraic K-theory of the subcategory of compact, cofibrant objects (see [**8**, Proposition 3.6]) are preserved.

For a compact Lie group G, the category of rational G-spectra is the category of G-spectra, but with the model structure that is a left Bousfield localization of the stable model structure at the rational sphere spectrum, see, for example, [**1**, Section 2.2]. Thus the weak equivalences are maps which become isomorphisms after applying the rational homotopy group functors, i.e. $\pi_*^H(-) \otimes \mathbb{Q}$ for all closed subgroups H in G.

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Existing work

It is expected that for any compact Lie group G, there exists an algebraic category $\mathcal{A}(G)$ which is Quillen equivalent to that of rational G-spectra.

There are many partial results and examples for specific Lie groups G for which an algebraic model has been given. An algebraic model for rational G-equivariant spectra for finite G is described in [**26**, Example 5.1.2]. It was shown in [**27**, Theorem 1.1] that rational (non-equivariant) spectra are monoidally Quillen equivalent to chain complexes of $\mathbb{Q}$-modules. An algebraic model for rational torus equivariant spectra was presented in [**12**], whereas a new approach in [**5**] gives a symmetric monoidal algebraic model for $SO(2)$. Also, an algebraic model for free rational G-spectra was given in [**13**] for any compact Lie group G.

However, there is no algebraic model known for the whole category of rational G-spectra for an arbitrary compact Lie group G. The present paper establishes the first part of a general result, providing a model for rational G-spectra over an exceptional subgroup (see Definition 2.1), for any compact Lie group G.

The approach to the algebraic model for rational G-spectra, where G is finite, in [**2**] relies on the equivariant version of [**9**, Chapter VIII, Theorem 2.2], which discusses localizations of commutative ring G-spectra. However, the equivariant version of this theorem has counterexamples, see [**22**]. The correct rephrasing of an equivariant localization theorem, and thus good foundations for [**2**] may need the work of Blumberg and Hill [**7**]. Thus, in its current foundations, [**2**] does not provide *monoidal* Quillen equivalences.

The method of this paper avoids such subtleties and thus presents a more immediate and easier proof of a zig-zag of symmetric monoidal Quillen equivalences in the case where G is a finite group.

Main result

We call a subgroup $H \leqslant G$ *exceptional* if $N_G H/H$ is finite and H can be completely separated from other subgroups of G in a sense that there is an idempotent $e_{(H)_G}$ in the rational Burnside ring $\mathsf{A}(G)_{\mathbb{Q}}$ corresponding to the conjugacy class of H in G and H does not contain a subgroup K such that H/K is a (non-trivial) torus (see Definition 2.1).

If H is an exceptional subgroup of G then the homotopy category of rational G-spectra with geometric isotropy H is a particularly nicely behaved part of the homotopy category of rational G-spectra, which in its structure resembles (or generalizes) rational Γ-spectra for finite Γ. It is modelled by the left Bousfield localization of rational orthogonal G-spectra at the idempotent $e_{(H)_G}$ and we call this localization a category of rational G-spectra over an exceptional subgroup H (see Definition 3.3).

Before we give the statement of the main result recall from [**25**] that a symmetric monoidal Quillen pair is a Quillen pair $L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$ between monoidal model categories, such that the right adjoint is lax symmetric monoidal and two conditions are satisfied:

- for any cofibrant objects A and B in $\mathcal{C}$ the comonoidal map $\widetilde{\phi}\colon L(A \otimes B) \longrightarrow L(A) \otimes L(B)$ is a weak equivalence in $\mathcal{D}$, where $\widetilde{\phi}$ is the adjoint of the composite

$$A \otimes B \xrightarrow{\eta_A \otimes \eta_B} RL(A) \otimes RL(B) \xrightarrow{\phi_{L(A),L(B)}} R(L(A) \otimes L(B))$$

(ϕ is a natural transformation, since R is a lax monoidal functor) and

- for some cofibrant replacement of the unit $q\colon \tilde{I}_{\mathcal{C}} \longrightarrow I_{\mathcal{C}}$ the composite map

$$L(\tilde{I}_{\mathcal{C}}) \xrightarrow{L(q)} L(I_{\mathcal{C}}) \xrightarrow{\tilde{\nu}} I_{\mathcal{D}}$$

is a weak equivalence ($\tilde{\nu}$ is the adjoint of ν, which exists since R is lax monoidal).

A *strong* symmetric monoidal Quillen pair is a symmetric monoidal Quillen pair, for which the above maps are isomorphisms. A (strong) symmetric monoidal Quillen equivalence is a (strong) symmetric monoidal Quillen pair which is also a Quillen equivalence.

Theorem 1.1. *Suppose G is any compact Lie group and H is an exceptional subgroup of G. Then there is a zig-zag of symmetric monoidal Quillen equivalences from rational G-spectra over H to*

$$\mathsf{Ch}(\mathbb{Q}[N_G H/H] - \mathrm{mod}),$$

with the projective model structure.

Many compact Lie groups G contain exceptional subgroups, for example all subgroups of a finite group are exceptional, all finite dihedral subgroups of $O(2)$ are exceptional and see Lemma 2.3 for some exceptional subgroups of $SO(3)$. When such a subgroup exists, it has a corresponding idempotent in the rational Burnside ring for G, and the category of rational G-spectra splits into a part over the exceptional subgroup and the part over the remaining subgroups. In this situation, any algebraic model for rational G-spectra will have a factor given by Theorem 1.1. In particular, the algebraic model for rational $SO(3)$-spectra has such a factor (see [**19**]). Hence this paper is a necessary step in finding an algebraic model for an arbitrary compact Lie group G.

If G is finite then every subgroup of G is exceptional and there are finitely many conjugacy classes of subgroups of G, so by the splitting result of [**3**] the category of rational G-spectra splits into a finite product (over conjugacy classes (H) of subgroups of G) of rational G-spectra over H. Thus our approach gives a new method of obtaining an algebraic model. What is more, this method gives a monoidal algebraic model for rational G-spectra for finite G, that is, all Quillen equivalences in the zig-zag preserve monoidal structure (see Section 5.4).

Corollary 1.2. *Suppose G is a finite group. Then there is a zig-zag of symmetric monoidal Quillen equivalences from rational G-spectra to*

$$\prod_{(H), H \leqslant G} \mathsf{Ch}(\mathbb{Q}[N_G H/H] - \mathrm{mod}),$$

with the (objectwise) projective model structure.

Results above were obtained using an analysis of the interplay between left Bousfield localizations at idempotents of the rational Burnside ring and the induction – restriction – coinduction adjunctions in Section 4. This analysis is similar in flavour to the one presented in [**14**] for the inflation-fixed point adjunction. The point is to recognize when these adjunctions become Quillen equivalences in situations that are of interest to us. Similar analysis is used to obtain an algebraic model for rational $SO(3)$-spectra in [**19**] and also for the toral part of rational G-spectra, for any compact Lie group G in [**4**].

Outline of the paper

This paper is structured as follows. In Section 2, we describe subgroups of a compact Lie group G and discuss some related idempotents in the rational Burnside ring of G. Section 3 recalls basic properties of orthogonal G-spectra that we will use later on. In Section 4, we link the different behaviour of subgroups of G with the left Bousfield localization and the induction-restriction-conduction adjunctions. This is the heart of the paper and it allows us to provide a zig-zag of symmetric monoidal Quillen equivalences in Section 5 that uses the inflation-fixed point adjunction which is strong symmetric monoidal, instead of using the "Morita equivalences" presented in [**26**].

Notation

We will stick to the convention of drawing the left adjoint above the right one in any adjoint pair.

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2. Subgroups of a Lie group G

Recall that for $H \leqslant G$, $N_G H = \{g \in G \mid gH = Hg\}$ is the normalizer of H in G. We use the notation $W = W_G H = N_G H/H$ for the Weyl group of H in G.

Suppose $\mathcal{F}(G)$ is a space of closed subgroups of G with finite index in their normalizer (i.e. all closed $H \leqslant G$ such that $N_G H/H$ is finite) considered with topology given by the Hausdorff metric. By the result of tom Dieck [**28**, 5.6.4, 5.9.13] there is an isomorphism of rings

$$\mathsf{A}(G) \otimes \mathbb{Q} \cong C(\mathcal{F}(G)/G, \mathbb{Q}),$$

where $\mathsf{A}(G) \otimes \mathbb{Q}$ denotes the rational Burnside ring of G and $C(\mathcal{F}(G)/G, \mathbb{Q})$ denotes the ring of continuous functions on the orbit space $\mathcal{F}(G)/G$ with values in discrete space $\mathbb{Q}$. From now on we will use notation $\mathsf{A}(G)_{\mathbb{Q}}$ for $\mathsf{A}(G) \otimes \mathbb{Q}$.

From the ring isomorphism above, it is clear that idempotents of the rational Burnside ring of G correspond to the characteristic functions on open and closed subspaces of the orbit space $\mathcal{F}(G)/G$ (or equivalently, to open and closed subspaces of $\mathcal{F}(G)/G$). Thus it makes sense to refer to an idempotent e_V, i.e. the one corresponding to the subspace V in $\mathcal{F}(G)/G$, provided that V is open and closed in $\mathcal{F}(G)/G$.

Every inclusion $i\colon H \longrightarrow G$ gives a ring homomorphism $i^*\colon \mathsf{A}(G)_\mathbb{Q} \longrightarrow \mathsf{A}(H)_\mathbb{Q}$. Generally, it is difficult to see what is the image of an idempotent under $i^*\colon C(\mathcal{F}(G)/G, \mathbb{Q}) \longrightarrow C(\mathcal{F}(H)/H, \mathbb{Q})$, because of the construction of spaces $\mathcal{F}(H)/H$ and $\mathcal{F}(G)/G$. Before even taking conjugacy classes into account, notice that a subgroup $K \leqslant H$ with finite index in the normalizer $N_H K$ does not have to have a finite index in the normalizer $N_G K$. Thus the map $i^*\colon C(\mathcal{F}(G)/G, \mathbb{Q}) \longrightarrow C(\mathcal{F}(H)/H, \mathbb{Q})$ is not always induced by a map from $\mathcal{F}(H)$ to $\mathcal{F}(G)$.

To understand what an image of an idempotent under i^* is, it is better to view idempotents as subspaces of the space of *all closed subgroups* of G as follows. Suppose $\mathrm{Sub_f}(G)$ is the topological space of all closed subgroups of G with the f-topology (see [**10**, Section 8] for details). For a closed subgroup $H \leqslant G$ and $\epsilon > 0$ we define a ball

$$O(H, \epsilon) = \{K \in \mathcal{F}(H) | d(H, K) < \epsilon\}$$

in $\mathcal{F}(H)$. Thus subgroups closed to H which have infinite Weyl groups are ignored, for example if H is a torus then $O(H, \epsilon)$ is a singleton. Given also a neighbourhood A of identity in G consider

$$O(H, \epsilon, A) = \cup_{a \in A} O(H, \epsilon)^a,$$

where $O(H, \epsilon)^a$ is the set of a-conjugates of elements of $O(H, \epsilon)$. We define f-topology to be generated by $O(H, \epsilon, A)$ as H, ϵ, A vary. See [**10**, Section 8] for more details and results concerning this topology. The f-topology gives a new way of understanding idempotents of the rational Burnside ring.

An idempotent in a rational Burnside ring $\mathsf{A}(G)_\mathbb{Q}$ corresponds to an open and closed, G-invariant subspace of $\mathrm{Sub_f}(G)$ which is a union of $\sim$-equivalence classes (where $\sim$ denotes the equivalence relation generated by $K \sim H$, where $K \sim H$ iff $K \leqslant H$ and H/K is a torus).

Now, if V is an open and closed G-invariant set in $\mathrm{Sub_f}(G)$ which is a union of $\sim$-equivalence classes, then $i^*(e_V) = e_{i^*V}$ where i^*V is the preimage of V under the inclusion on spaces of subgroups induced by i, i.e. $\mathrm{Sub_f}(H) \longrightarrow \mathrm{Sub_f}(G)$.

Definition 2.1. Suppose G is a compact Lie group. We say that a closed subgroup $H \leqslant G$ is *exceptional* in G if $W_G H$ is finite, there exist an idempotent $e_{(H)_G}$ in the rational Burnside ring of G corresponding to the conjugacy class of H in G (via the tom Dieck's isomorphism) and H does not contain any subgroup cotoral in H, where a subgroup $K \leqslant H$ is cotoral in H if H/K is a (non-trivial) torus.

If H is exceptional in G then $(H)_G$ (all the subgroups conjugate to H in G) is an open and closed G-invariant subspace of $\mathrm{Sub_f}(G)$, which already is a union of $\sim$-equivalence classes, since H does not contain any cotoral subgroup and $W_G H$ is finite. The other implication also holds; if there is an idempotent corresponding to $(H)_G$ in $\mathrm{Sub_f}(G)$, then H is an exceptional subgroup of G. Thus we could rephrase the definition in terms of the space $\mathrm{Sub_f}(G)$, but we decided to use the more familiar $\mathcal{F}(G)/G$ with the topology given by the Hausdorff metric.

It is easy to see that any subgroup of a finite group G is exceptional. In $O(2)$ only finite dihedral subgroups are exceptional; in particular, none of the finite cyclic subgroups is exceptional (since finite cyclic subgroups do not have idempotents in the rational Burnside ring of $O(2)$). The maximal torus $SO(2)$ in $O(2)$ has an idempotent in the rational Burnside ring of $O(2)$, however, it is not an exceptional subgroup, since it contains cotoral subgroups, for example the trivial one. In $SO(3)$ all finite dihedral subgroups are exceptional (except for D_2, which is conjugate to C_2 and therefore is a subgroup of a torus), but we have more: there are four more conjugacy classes of exceptional subgroups: A_4, Σ_4, A_5 and $SO(3)$, where A_4 denotes rotations of a tetrahedron, Σ_4 denotes rotations of a cube and A_5 denotes rotations of a dodecahedron, see [**19**].

If a trivial subgroup is exceptional in G then G has to be finite, since the normalizer of a trivial subgroup is the whole G, $W_G e = G$ and the condition that the Weil group is finite implies that G is a finite group.

We introduced the notion of an exceptional subgroup H because we will use the corresponding idempotent in the rational Burnside ring to split the category of rational G-spectra into the part over an exceptional subgroup H and its complement. In this paper we present the model for rational G-spectra over an exceptional subgroup H.

On the way towards the algebraic model different subgroups of G will behave slightly differently. This behaviour is closely related to the following

Definition 2.2. Suppose $K \leqslant H$ are closed subgroups of G such that K is exceptional in G. Suppose further that $i\colon H \longrightarrow G$ is an inclusion. We say that K is H-*good* in G if $i^*(e_{(K)_G}) = e_{(K)_H}$ and H-*bad* in G if it is not H-good, i.e. $i^*(e_{(K)_G}) \neq e_{(K)_H}$.

Notice that the above definition is all about subgroups conjugate to K in H and in G and their relation to each other. If $L \leqslant H$ is such that L is conjugate to K in H, then it is also true that L is conjugate to K in G. Thus if K is H-bad in G it just means that there exists $K' \leqslant H$ such that $(K')_G = (K)_G$ and $(K')_H \neq (K)_H$.

There is a definition of good and bad subgroups in [**11**, Definition 6.3], however, it was designed to capture different properties than our definition and thus they are not the same. As an example, if a trivial subgroup is exceptional in G it is always H-good in G for any $H \leqslant G$ according to our definition and H-bad according to Greenlees' definition (unless H is normal in G).

It is easy to see that any exceptional subgroup H in a compact Lie group G is H-good in G. In further analysis, we will consider the relationship between H and its normalizer, $N_G H$ in G, so we present some examples.

Lemma 2.3. *For exceptional subgroups in $G = SO(3)$ we have the following relation between H and its normalizer $N_G H$:*

1. *A_5 is A_5-good in $SO(3)$.*
2. *Σ_4 is Σ_4-good in $SO(3)$.*
3. *A_4 is Σ_4-good in $SO(3)$.*
4. *D_4 is Σ_4-bad in $SO(3)$.*

Proof. We only need to prove Parts 3 and 4, since any exceptional subgroup H in a compact Lie group G is H-good in G. Part 3 follows from the fact that there is one conjugacy class of A_4 in Σ_4, as there is just one subgroup of index 2 in Σ_4. Part 4 follows from the observation that there are two subgroups of order 4 in D_8 (so also in Σ_4) and they are conjugate by an element $g \in D_{16}$, which is the generating rotation by 45 degrees (thus $g \notin D_8$ and thus $g \notin \Sigma_4$). □

3. Orthogonal G-spectra, left Bousfield localization and splitting

There are many constructions of categories of spectra (G-spectra) equipped with model structures, such that the homotopy category is equivalent to the usual stable homotopy category of spectra (G-spectra, respectively). However, since we are interested in modelling the smash product as well, we choose to work with a model with a strictly associative, symmetric monoidal product compatible with model structure so that its homotopy category is equivalent to the usual stable homotopy category with the smash product known in algebraic topology.

When we work with non-equivariant spectra, there are several categories having this property, and we choose to work with the category of symmetric spectra defined in [**18**] and discussed in detail in [**24**]. We will use it briefly towards the end of Section 5.3. Whenever we are interested in modelling the category of G-equivariant cohomology theories we choose to work with the category of orthogonal G-spectra defined and described in [**21**], for which we use the notation $G\text{–}\mathrm{Sp}^{\mathcal{O}}$.

The construction of both categories is similar and we refer the reader to the papers above for details. The idea is to first construct a diagram of spaces (or simplicial sets) indexed by some fixed category, then to define a tensor product on the category of diagrams and choose a monoid S (sphere spectrum). Spectra are defined to be S-modules. Depending on the indexing category we get symmetric spectra or orthogonal G-spectra.

In this section, we recall briefly some properties of the category of orthogonal G-spectra after Chapter II of [**21**]. We stress that unless otherwise stated, we use the term of orthogonal G-spectra to implicitly mean the ones indexed on a complete G universe. By [**21**, Theorem 4.2.] there is a model structure on orthogonal G-spectra called the stable model structure where a map of orthogonal spectra $f\colon X \longrightarrow Y$ is a weak equivalence if it is a π_*-isomorphism (i.e. it is a π_*^H-isomorphism for all $H \leqslant G$). This model structure is cofibrantly generated, stable, monoidal, proper and cellular (see [**21**, Theorem III 4.2]).

What is more, we have a good way of checking that a map in $G\text{–}\mathrm{Sp}^{\mathcal{O}}$ is a weak equivalence. For any closed subgroup H in G, any orthogonal spectrum X and integers $p \geqslant 0$ and $q > 0$

$$[\Sigma^p S^0 \wedge G/H_+, X]^G \cong \pi_p^H(X) \quad [F_q S^0 \wedge G/H_+, X]^G \cong \pi_{-q}^H(X), \tag{3.1}$$

where the left hand sides denote morphisms in the homotopy category of $G\text{–}\mathrm{Sp}^{\mathcal{O}}$ and $F_q(-)$ is the left adjoint to the evaluation functor at $\mathbb{R}^q$, $Ev_{\mathbb{R}^q}(X) = X(\mathbb{R}^q)$ (for example $F_q(S^0)$ models S^{-q}).

There is one more property which makes the stable model structure on $G\text{–}\mathrm{Sp}^{\mathcal{O}}$ easy to work with, namely it has a set of compact generators. By [**16**, Definition 7.1.1] a homotopy category of a stable model category is triangulated and in this setting we can make the following definitions after [**26**, Definition 2.1.2].

Definition 3.1. Let $\mathcal{C}$ be a triangulated category with infinite coproducts. A full triangulated subcategory of $\mathcal{C}$ (with shift and triangles induced from $\mathcal{C}$) is called *localizing* if it is closed under coproducts in $\mathcal{C}$. A set $\mathcal{P}$ of objects of $\mathcal{C}$ is called a *set of generators* if the only localizing subcategory of $\mathcal{C}$ containing objects of $\mathcal{P}$ is the whole of $\mathcal{C}$. An object X in $\mathcal{C}$ is *compact* if for any family of objects $\{A_i\}_{i\in I}$ the canonical map

$$\bigoplus_{i\in I}[X, A_i]^{\mathcal{C}} \longrightarrow [X, \coprod_{i\in I} A_i]^{\mathcal{C}}$$

is an isomorphism. An object of a stable model category is called a compact generator if it is so when considered as an object of the homotopy category.

The set of suspensions and desuspensions of G/H_+, where H varies through all closed subgroups of G, is a set of compact generators in the stable model category G–$\mathrm{Sp}^{\mathcal{O}}$. Those objects are compact since homotopy groups commute with coproducts and it is clear from [**26**, Lemma 2.2.1] and (3.1) that this is a set of generators for G–$\mathrm{Sp}^{\mathcal{O}}$.

There is an easy-to-check condition for a Quillen adjunction between stable model categories with sets of compact generators to be a Quillen equivalence:

Lemma 3.2. *Suppose $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen pair between stable model categories with sets of compact generators, such that the right derived functor RU preserves coproducts (or equivalently, such that the left derived functor sends compact generators to compact objects). Then to know (F, U) is a Quillen equivalence it is enough to check that a derived unit and counit are weak equivalences for generators.*

Proof. The result follows from the fact that the homotopy category of a stable model category is a triangulated category. First notice that since the derived functor RU preserves coproducts, then the derived unit and counit are triangulated transformations. If the derived unit and counit conditions are satisfied for a set of objects $\mathcal{K}$ then they are also satisfied for every object in the localizing subcategory for $\mathcal{K}$. Since we assume that $\mathcal{K}$ consists of compact generators, the localizing subcategory for $\mathcal{K}$ is the whole category and the result follows. □

Our basic category to work with is the category G–$\mathrm{Sp}^{\mathcal{O}}$ of orthogonal G-spectra. However, in this paper we are interested only in the homotopy category of rational G-spectra with geometric isotropy in an exceptional subgroup H. Localization is our main tool to make the model category of G-spectra easier, so that it models exactly the part that we want. We obtain it by firstly rationalizing the stable model category of G-spectra using the localization at an object $S_{\mathbb{Q}}$, which is the rational sphere spectrum. Then, we localize it further to extract the behaviour of an exceptional subgroup. The notation used in the definition below is explained in the rest of this section.

Definition 3.3. Let $H \leqslant G$ be an exceptional subgroup of G. Then we call the model category $L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ *rational G-spectra over an exceptional subgroup H.*

By considering the geometric fixed point functors, it is clear that the homotopy category of $L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ is the homotopy category of rational G-spectra with geometric isotropy concentrated over the exceptional subgroup H (and its conjugates), which justifies the name.

For details on left Bousfield localization at an object we refer the reader to [**21**]. We recall the following result, which is [**21**, Chapter IV, Theorem 6.3].

Theorem 3.4. *Suppose E is a cofibrant object in G–$\mathrm{Sp}^{\mathcal{O}}$ or a cofibrant based G-space. Then there exists a new model structure on the category G–$\mathrm{Sp}^{\mathcal{O}}$, where a map $f\colon X \longrightarrow Y$ is*

- *a weak equivalence if it is an E-equivalence, i.e. $Id_E \wedge f\colon E \wedge X \longrightarrow E \wedge Y$ is a weak equivalence*
- *cofibration if it is a cofibration with respect to the stable model structure*
- *fibration if it has the right lifting property with respect to all trivial cofibrations.*

The E-fibrant objects Z are the underlying fibrant, E-local objects, i.e. $[f, Z]^G\colon [Y, Z]^G \longrightarrow [X, Z]^G$ is an isomorphism for all E-equivalences f. E-fibrant approximation gives Bousfield localization $\lambda\colon X \longrightarrow L_E X$ of X at E.

We use the notation $L_E(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ for the model category described above and will refer to it as a left Bousfield localization of the category of G-spectra at E.

Recall that, an E-equivalence between E-local objects is a weak equivalence (see [**15**, Theorems 3.2.13 and 3.2.14]). All our localizations are smashing (see [**23**] and [**17**]) because they are defined using idempotents of rational Burnside ring, thus they preserve compact generators (since the fibrant replacement preserves infinite coproducts).

As mentioned above, the first simplification of a category of G-spectra is rationalisation, i.e. localization at an object $S_{\mathbb{Q}}$, which is a rational sphere spectrum (the Moore spectrum for $\mathbb{Q}$, see, for example, [**3**, Definition 5.1]). This spectrum has the property that $\pi_*(X \wedge S_{\mathbb{Q}}) = \pi_*(X) \otimes \mathbb{Q}$. We refer to this model category as rational G-spectra.

The next step on the way towards the algebraic model is to split the category of rational G-spectra using idempotents of the rational Burnside ring $\mathsf{A}(G)_{\mathbb{Q}}$. We know that idempotents of the (rational) Burnside ring split the homotopy category of (rational) G-spectra. Barnes' result [**3**] allows us to perform a compatible splitting at the level of model categories. We want to use the idempotent $e_{(H)_G}$ corresponding to the (conjugacy class of the) exceptional subgroup H in G (see Definition 2.1) and the idempotent corresponding to its complement, $1 - e_{(H)_G}$. By [**3**, Theorem 4.4] this gives a monoidal Quillen equivalence.

Proposition 3.5. *There is a strong symmetric monoidal Quillen equivalence:*

$$\triangle\colon L_{S_{\mathbb{Q}}}(G - \mathrm{Sp}^{\mathcal{O}}) \rightleftarrows L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \times L_{(1-e_{(H)_G}) S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) : \Pi,$$

where the left adjoint is a diagonal functor, the right adjoint is a product and the product category on the right is considered with the objectwise model structure (a map (f_1, f_2) is a weak equivalence, a fibration or a cofibration if both factors f_i are).

From now on we will work only with the category $L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ as this is our model for rational G-spectra over an exceptional subgroup H. We use the name *H-equivalence* for a weak equivalences in the category $L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ and *H-fibrant replacement* for the fibrant replacement there. These names are motivated by the following

Lemma 3.6. *A map f between $e_{(H)_G} S_{\mathbb{Q}}$-local objects in $L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ is a weak equivalence iff $\pi_*^H(f)$ is an isomorphism.*

Proof. By definition, f is an H-equivalence iff $\pi_*^K(e_{(H)_G}S_{\mathbb{Q}} \wedge f)$ is an isomorphism for all closed subgroups $K \leqslant G$. This holds iff $\Phi^K(e_H S_{\mathbb{Q}} \wedge f)$ is a non-equivariant equivalence for all $K \leqslant G$. As geometric fixed point functor commutes with smash product that is equivalent to $\Phi^H(e_{(H)_G}S_{\mathbb{Q}} \wedge f)$ being a non-equivariant equivalence, i.e. $\pi_*(\Phi^H(e_{(H)_G}S_{\mathbb{Q}} \wedge f))$ being an isomorphism. Since f is a map between $e_{(H)_G}S_{\mathbb{Q}}$-local objects

$$\pi_*(\Phi^H(e_{(H)_G}S_{\mathbb{Q}} \wedge f)) \cong \pi_*^H(f),$$

which finishes the proof. □

4. Change-of-group functors and localizations using idempotents

Since later we will be interested in taking H-fixed points of G-spectra when H is not necessary normal in G, we need to pass to $N = N_G H$-spectra first. Suppose we have an inclusion $i\colon N \hookrightarrow G$ of a subgroup N in a group G. This gives a pair of adjoint functors at the level of orthogonal spectra (see, for example, [**21**, Section V.2]), namely induction, restriction and coinduction as below (the left adjoint is above the corresponding right adjoint):

$$\begin{tikzcd} G\text{–}\mathrm{Sp}^{\mathcal{O}} \arrow[r, "i^*"] & N - \mathrm{Sp}^{\mathcal{O}} \arrow[l, bend right, "G_+\wedge_N -"'] \arrow[l, bend left, "F_N(G_+,-)"] \end{tikzcd} .$$

We assume both categories of spectra are over complete universes and we slightly abuse the notation by not mentioning the change of universe functors. See [**21**, Section V.2] for details.

These two pairs of adjoint functors are Quillen pairs with respect to stable model structures, see [**21**, Chapter V, Propositions 2.3 and 2.4]. The restriction functor as a right adjoint is used for example when we want to take (both categorical and geometric) H-fixed points of G-spectra, where H is not a normal subgroup of G. The first step to take H-fixed points of G-spectra is to restrict to $N_G H$-spectra and then take H-fixed points. This is usually done in one go, since the restriction and H-fixed points are both right Quillen functors.

It is natural to ask when the pair of adjunctions above passes to the localized categories, in our case localized at $e_{(H)_G}S_{\mathbb{Q}}$ and $e_{(H)_N}S_{\mathbb{Q}}$, respectively. The answer is related to H being an N-good or bad subgroup in G. It turns out that the induction-restriction adjunction does not always induce a Quillen adjunction on the localized categories, unless H is N-good in G. However, the restriction-coinduction adjunction induces a Quillen adjunction on the localized categories, for all exceptional subgroups H. Before we discuss this particular adjunction we state a general result.

Lemma 4.1. *Suppose that $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$ is a Quillen adjunction of model categories where the left adjoint is strong monoidal. Suppose further that E is a cofibrant object* *in $\mathcal{C}$ and that both $L_E\mathcal{C}$ and $L_{F(E)}\mathcal{D}$ exist. Then*

$$F\colon L_E\mathcal{C} \rightleftarrows L_{F(E)}\mathcal{D} : R$$

is a strong monoidal Quillen adjunction. Moreover, if the original adjunction was a Quillen equivalence then the one induced on the level of localized categories is as well.

Proof. Since the localization did not change the cofibrations, the left adjoint F still preserves them. To show that it also preserves acyclic cofibrations, take an acyclic cofibration $f\colon X \longrightarrow Y$ in $L_E\mathcal{C}$. By definition $f \wedge Id_E$ is an acyclic cofibration in $\mathcal{C}$. Since F was a left Quillen functor before localization $F(f \wedge Id_E)$ is an acyclic cofibration in $\mathcal{D}$. As F was strong monoidal we have $F(f \wedge Id_E) \cong F(f) \wedge Id_{F(E)}$, so $F(f)$ is an acyclic cofibration in $L_{F(E)}\mathcal{D}$ which finishes the proof of the first part.

To prove the second part of the statement we use Part 2 from [**16**, Corollary 1.3.16]. Since F is strong monoidal and the original adjunction was a Quillen equivalence F reflects $F(E)$-equivalences between cofibrant objects. It remains to check that the derived counit is an $F(E)$-equivalence. $F(E)$-fibrant objects are fibrant in $\mathcal{D}$ and the cofibrant replacement functor remains unchanged by localization. Thus the claim follows from the fact that F, R was a Quillen equivalence before localizations. □

We will use this result in several cases for the following two adjoint pairs of orthogonal G-spectra. Notice that since both left adjoints are strong monoidal, the results below follow from Lemma 4.1.

Corollary 4.2. *Let $i\colon N \longrightarrow G$ denote the inclusion of a subgroup and let E be a cofibrant object in G–$\mathrm{Sp}^{\mathcal{O}}$. Then*

$$i^*\colon L_E(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \rightleftarrows L_{i^*(E)}(N\text{–}\mathrm{Sp}^{\mathcal{O}}) : F_N(G_+, -)$$

is a strong monoidal Quillen pair.

Corollary 4.3. *Let $\epsilon\colon N \longrightarrow W$ denote the projection of groups, where H is normal in N and $W = N/H$. Let E be a cofibrant object in W–$\mathrm{Sp}^{\mathcal{O}}$. Then*

$$\epsilon^*\colon L_E(W\text{–}\mathrm{Sp}^{\mathcal{O}}) \rightleftarrows L_{\epsilon^*(E)}(N\text{–}\mathrm{Sp}^{\mathcal{O}}) : (-)^H$$

is a strong monoidal Quillen pair.

The following two results describe the behaviour of the restriction-induction adjunction at the level of localized categories.

Proposition 4.4. *Suppose H is an exceptional subgroup of G which is $N = N_GH$-good in G. Then*

$$i^*\colon L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \leftrightarrows L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}}) : G_+ \wedge_N -$$

is a Quillen adjunction.

Proof. This was a Quillen adjunction before localization by [**21**, Chapter V, Proposition 2.3] so the left adjoint preserves cofibrations. It preserves acyclic cofibrations as $G_+ \wedge_N -$ preserved acyclic cofibrations before localization and we have a natural (in an N-spectrum X) isomorphism (see [**21**, Chapter V, Proposition 2.3]):

$$(G_+ \wedge_N X) \wedge e_{(H)_G} S_{\mathbb{Q}} \cong G_+ \wedge_N (X \wedge i^*(e_{(H)_G} S_{\mathbb{Q}})).$$

Note that, since H is N-good in G, $i^*(e_{(H)_G}) = e_{(H)_N}$, where the latter is the idempotent corresponding to $(H)_N$ in $\mathsf{A}(N)_{\mathbb{Q}}$.

$\square$

Proposition 4.5. *Suppose H is an exceptional subgroup of G which is $N = N_G H$-bad in G. Then*

$$i^* \colon L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \underset{\longrightarrow}{\longleftarrow} L_{e_{(H)_N} S_{\mathbb{Q}}}(N - \mathrm{Sp}^{\mathcal{O}}) : G_+ \wedge_N -$$

is not a Quillen adjunction.

Proof. It is enough to show that $G_+ \wedge_N -$ does not preserve acyclic cofibrations. Firstly, since H is N-bad in G there exists $H' \leqslant N$ such that $(H)_G = (H')_G$ and $(H)_N \neq (H')_N$.

Take f to be the inclusion into the coproduct $N/H_+ \longrightarrow N/H_+ \vee N/H'_+$. This is a weak equivalence in $L_{e_{(H)_N} S_{\mathbb{Q}}}(N - \mathrm{Sp}^{\mathcal{O}})$ since $\Phi^H(N/H_+) = \Phi^H(N/H_+ \vee N/H'_+)$. It is also a cofibration as a pushout of a cofibration $* \longrightarrow N/H_+$ along the map $* \longrightarrow N/H'_+$. Applying the left adjoint gives the inclusion $G_+ \wedge_N f \colon G/H_+ \longrightarrow G/H_+ \vee G/H'_+$. Now $\Phi^H(G/H_+ \vee G/H'_+) \cong N/H_+ \vee N/H'_+ \ncong N/H_+$ since H is N-bad by assumption and $(H)_G = (H')_G$. Thus $G_+ \wedge_N f$ is not a weak equivalence in $L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ which finishes the proof. $\square$

Proposition 4.5 shows that i^* is not always a right Quillen functor, when considered between categories localized at corresponding idempotents. However, it is a left Quillen functor and the restriction and function spectrum adjunction gives a Quillen adjunction under general conditions, which we proceed to discuss.

Lemma 4.6. *Suppose G is any compact Lie group, $i \colon N \longrightarrow G$ is an inclusion of a subgroup and V is an open and closed G-invariant set V in $\mathrm{Sub_f}(G)$ which is a union of $\sim$-equivalence classes (see Section 2). Then the adjunction*

$$i^* \colon L_{e_V S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \underset{\longleftarrow}{\longrightarrow} L_{e_{i^*V} S_{\mathbb{Q}}}(N - \mathrm{Sp}^{\mathcal{O}}) : F_N(G_+, -)$$

is a Quillen pair.

Proof. Before localizations on both sides this was a Quillen pair by [**21**, Chapter V, Proposition 2.4]. It is a Quillen pair after localization by Corollary 4.2, and the fact that i^* is strong symmetric monoidal. We use the notation i^*V for the preimage of V under the inclusion on spaces of subgroups induced by i, i.e. $\mathrm{Sub_f}(N) \longrightarrow \mathrm{Sub_f}(G)$, see Section 2. $\square$

We will repeatedly use the lemma above, mainly in situations where after further localization of the right hand side we will get a Quillen equivalence.

Corollary 4.7. *Suppose G is a compact Lie group and H is an exceptional subgroup of G. Then*

$$i^*\colon L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \rightleftarrows L_{e_{(H)_N}S_{\mathbb{Q}}}(N-\mathrm{Sp}^{\mathcal{O}}) : F_N(G_+,-)$$

is a Quillen adjunction.

Proof. For $N = N_GH$-good H the result follows from the fact that the idempotent on the right hand side satisfies $e_{(H)_N} = i^*(e_{(H)_G}) = e_{i^*((H)_G)}$. For N-bad H it is true since the left hand side is a further localization of $L_{e_{i^*((H)_G)}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$ at the idempotent $e_{(H)_N}$:

$$L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \underset{F_N(G_+,-)}{\overset{i^*}{\rightleftarrows}} L_{i^*(e_{(H)_G})S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}}) \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}}).$$

Note that since H is N-bad, $e_{(H)_N} \neq i^*(e_{(H)_G})$ and $e_{(H)_N} i^*(e_{(H)_G}) = e_{(H)_N}$. □

In the next two theorems we show that the Quillen adjunction above is, in fact, a Quillen equivalence.

Theorem 4.8. *Suppose $N = N_GH$ and H is an exceptional subgroup of G that is N-good. Then the adjunction*

$$i^*\colon L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \rightleftarrows L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}}) : F_N(G_+,-)$$

is a strong symmetric monoidal Quillen equivalence.

Proof. Firstly, if H is an N-good exceptional subgroup of G with an idempotent $e_{(H)_G}$ then $e_{(H)_N} = i^*(e_{(H)_G})$ in $\mathsf{A}(N)_{\mathbb{Q}}$.

The above is a Quillen adjunction by Corollary 4.7 and we claim that i^* preserves all H-equivalences. Suppose $f\colon X \longrightarrow Y$ is an H-equivalence in $L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$, i.e. $Id_{e_{(H)_G}S_{\mathbb{Q}}} \wedge f\colon e_{(H)_G}S_{\mathbb{Q}} \wedge X \longrightarrow e_{(H)_G}S_{\mathbb{Q}} \wedge Y$ is a π_*-isomorphism. As i^* is strong monoidal it follows that

$$i^*(Id_{e_{(H)_G}S_{\mathbb{Q}}} \wedge f) \cong Id_{i^*(e_{(H)_G}S_{\mathbb{Q}})} \wedge i^*(f) \simeq Id_{e_{(H)_N}S_{\mathbb{Q}}} \wedge i^*(f).$$

Since i^* preserves π_*-isomorphisms we can conclude.

To show this is a Quillen equivalence we will use Part 2 from [16, Corollary 1.3.16]. It is easy to see that i^* reflects H-equivalences using the fact it is strong monoidal and the isomorphism $[N/H_+, i^*(X)]^N_* \cong [G/H_+, X]^G_*$.

As i^* preserves all H-equivalences it is enough to check that for every fibrant $Y \in L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$ the counit map $\varepsilon_Y\colon i^*F_N(G_+,Y) \longrightarrow Y$ is an H-equivalence (in N-spectra), i.e. it is a π^H_*-isomorphism of N-spectra.

First we check that domain and codomain have isomorphic stable H homotopy groups:

$$\begin{aligned}\pi^H_*(i^*F_N(G_+,Y)) &\cong \pi^H_*(F_N(G_+,Y)) \cong [G/H+, F_N(G_+,Y)]^G_* \\ &\cong [i^*(G/H_+),Y]^N_* \cong [N/H_+,Y]^N_* \cong \pi^H_*(Y).\end{aligned}$$

The next-to-last isomorphism follows from the fact that the map $N/H_+ \longrightarrow G/H_+$ (induced by inclusion $N \longrightarrow G$) is an H-equivalence in N-spectra, i.e. a weak equivalence in $L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$.

By Lemma 3.2 to show that the restriction-coinduction adjunction is a Quillen equivalence it is enough to check the counit condition for a generator of $L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$. We will check it for the spectrum $i^*(\hat{f}G/H_+)$ (where $\hat{f}$ denotes fibrant replacement in $L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$). This is a compact generator for localized N-spectra (it is H-equivalent to N/H_+ and it is compact, since the localization is smashing). The stable H-homotopy groups of this generator are $\mathbb{Q}[W_GH]$ in degree 0 (where W_GH is the Weyl group for H in G, so, in particular, $\mathbb{Q}[W_GH]$ is a finite dimensional vector space by assumption that H is exceptional in G) and 0 in other degrees.

Now it is enough to show that $[N/H_+, \varepsilon_{i^*(\hat{f}G/H_+)}]_*^N$ is surjective. One of the triangle identities on $i^*(\hat{f}G/H_+)$ for the adjunction requires that the following diagram commutes

$$\begin{tikzcd}
i^*(\hat{f}G/H_+) \arrow[d, "i^*(\eta_{\hat{f}G/H_+})"'] \arrow[dr, "\mathrm{Id}"] & \\
i^*F_N(G_+, i^*(\hat{f}G/H_+)) \arrow[r, "\varepsilon_{i^*(\hat{f}G/H_+)}"'] & i^*(\hat{f}G/H_+).
\end{tikzcd}$$

Thus postcomposition with $\varepsilon_{i^*(\hat{f}G/H_+)}$ is surjective on the homotopy level. It follows that the counit map is an H-equivalence of N-spectra for every fibrant Y, which finishes the proof. $\square$

The argument above will not work in the context where i^* does not preserve fibrant objects and all weak equivalences. We used these two facts to know that we can work with the counit, rather than the derived counit and that $i^*(\hat{f}G/H_+)$ is a fibrant object in $L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$ and thus we can use the triangle identity in the last part of the proof (without having to fibrantly replace $i^*(\hat{f}G/H_+)$). We found the proof above amusing, so we decided to present it, even though the proof below can be applied also in the case where H is an exceptional N_GH-good subgroup of G.

Theorem 4.9. *Suppose H is an exceptional subgroup of G. Then the composite of adjunctions*

$$L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \underset{F_N(G_+,-)}{\overset{i_N^*}{\rightleftarrows}} L_{i^*(e_{(H)_G}S_{\mathbb{Q}})}(N\text{–}\mathrm{Sp}^{\mathcal{O}}) \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$$

is a strong symmetric monoidal Quillen equivalence, where $e_{(H)_N}$ denotes the idempotent of the rational Burnside ring $\mathsf{A}(N)_{\mathbb{Q}}$ corresponding to the characteristic function of $(H)_N$. Notice that if H is N-good then the middle model category and the right hand side model category are the same.

Proof. Firstly, if H is N-bad then $i_N^*(e_{(H)_G}S_{\mathbb{Q}}) \not\simeq e_{(H)_N}S_{\mathbb{Q}}$ are not equivalent (as localized N-spectra). The reason for that is that $(H)_G$ restricts to more than one conjugacy class of subgroups of N. That is why we need a further localization – we only want to consider $(H)_N$.

The composite above forms a Quillen adjunction by Corollary 4.7. We use Part 3 from [**16**, Corollary 1.3.16] to show that it is a Quillen equivalence. Observe that $F_N(G_+,-)$ preserves and reflects weak equivalences between fibrant objects by the following argument. Firstly, if f is a map between fibrant objects in $L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$ then it is a weak equivalence if and only if $[N/H_+,f]^N_*$ is an isomorphism by Lemma 3.6. Similarly, if f is a map between fibrant objects in $L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ then it is a weak equivalence if and only if $[G/H_+,f]^G_*$ is an isomorphism.

Let X be a fibrant object in $L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$. Then $F_N(G_+,X)$ is also fibrant in $L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ and thus we have the first isomorphism:

$$\begin{aligned}
&[G/H_+, e_{(H)_G}F_N(G_+,X)]^G_* \cong [G/H_+, F_N(G_+,X)]^G_* \\
&\qquad \cong [i^*_N(G/H_+), X]^N_* \cong [e_{(H)_N}i^*_N(G/H_+), e_{(H)_N}X]^N_* \cong [N/H_+, e_{(H)_N}X]^N_*.
\end{aligned}$$

The last isomorphism follows from the fact that $N/H_+ \longrightarrow e_{(H)_N}i^*_N(G/H_+)$ is an H-equivalence (since it is a π^H_*-isomorphism) and $e_{(H)_N}X$ is an $e_{(H)_N}S_{\mathbb{Q}}$-local object. Since all isomorphisms above are natural, we can conclude that $[N/H_+,f]^N_*$ is an isomorphism iff $[G/H_+,F_N(G_+,f)]^G_*$ is an isomorphism, so $F_N(G_+,-)$ preserves and reflects weak equivalences between fibrant objects.

Now we need to show that the derived unit is a weak equivalence on the cofibrant generator for $L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$, which is $e_{(H)_G}G/H_+$. This is

$$e_{(H)_G}G/H_+ \longrightarrow F_N(G_+, e_{(H)_N}i^*_N(e_{(H)_G}G/H_+)).$$

To check that this is a weak equivalence in $L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$, by Lemma 3.6 it is enough to check that on the homotopy level the induced map

$$[G/H_+, e_{(H)_G}G/H_+]^G_* \longrightarrow [G/H_+, F_N(G_+, e_{(H)_N}i^*_N(e_{(H)_G}G/H_+))]^G_*$$

is an isomorphism. This map fits into a commuting diagram:

$$\begin{tikzcd}
{[G/H_+, e_{(H)_G}G/H_+]^G_*} \arrow[d] \arrow[dr, "Li^*_N"] & \\
{[G/H_+, F_N(G_+, e_{(H)_N}i^*_N(e_{(H)_G}G/H_+))]^G_*} \arrow[r, "\cong"] & {[i^*_N G/H_+, e_{(H)_N}i^*_N(e_{(H)_G}G/H_+)]^N_*,}
\end{tikzcd}$$

where Li^*_N denotes the left derived functor of i^*_N.

Since the horizontal map is an isomorphism it is enough to show that Li^*_N is an isomorphism. This follows from the commutative diagram:

$$\begin{tikzcd}
 & & {[S^0, i^*(e_{(H)_N} i_N^*(e_{(H)_G} G/H_+))]_*^H} \\
 & {[G/H_+, e_{(H)_G} G/H_+]_*^G} \arrow[r, "\cong"] \arrow[d] \arrow[dl, bend right, "Li_N^*"'] & {[S^0, i_H^*(e_{(H)_G} G/H_+)]_*^H} \arrow[u, "\cong"] \\
 & {[i_N^* G/H_+, i_N^*(e_{(H)_G} G/H_+)]_*^N} \arrow[r, "j^*"] \arrow[d] & {[N/H_+, i_N^*(e_{(H)_G} G/H_+)]_*^N} \arrow[u, "\cong"] \arrow[d] \\
{[i_N^* G/H_+, e_{(H)_N} i_N^*(e_{(H)_G} G/H_+)]_*^N} \arrow[rr, "\cong", "j^*"'] & & {[N/H_+, e_{(H)_N} i_N^*(e_{(H)_G} G/H_+)]_*^N,} \arrow[uuu, bend right, "\cong"']
\end{tikzcd}$$

where $j\colon N/H_+ \longrightarrow i_N^* G/H_+$ is a weak equivalence in $L_{e_{(H)_N} S_{\mathbb{Q}}} N-\mathrm{Sp}^{\mathcal{O}}$ and i_H^* denotes the restriction functor from G-spectra to H-spectra and i^* denotes the restriction functor from N-spectra to H-spectra. The top right hand side isomorphism follows from the fact that restriction from G-spectra to N-spectra followed by restriction from N-spectra to H-spectra is the same as the restriction from G to H-spectra and the fact that

$$i^*(e_{(H)_N} i_N^*(e_{(H)_G} G/H_+)) \simeq i^*(i_N^*(e_{(H)_G} G/H_+)) \simeq i_H^*(e_{(H)_G} G/H_+)$$

in H-spectra. $\square$

5. A monoidal algebraic model for rational G-spectra over an exceptional subgroup

The category of rational G-spectra over an exceptional subgroup H is modelled by the left Bousfield localization at an idempotent $e_{(H)_G}$ corresponding to the conjugacy class of H in G. Thus from now on we will use the notation H for an exceptional subgroup of G, and we will work with the category $L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$.

To provide an algebraic model for the category $L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ we construct a zig-zag of Quillen equivalences as follows. First we move from the category $L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ to the category $L_{e_{(H)_N} S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$ using the restriction-coinduction adjunction. Recall that N denotes the normalizer $N_G H$.

The second step is to use the inflation-fixed point adjunction between $L_{e_{(H)_N} S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$ and $L_{e_1 S_{\mathbb{Q}}}(W\text{–}\mathrm{Sp}^{\mathcal{O}})$, where W denotes the Weyl group N/H. Recall that W is finite as H is an exceptional subgroup of G and e_1 denotes the idempotent in $\mathsf{A}(W)_{\mathbb{Q}}$ corresponding to the trivial subgroup.

Next we use the restriction of universe functor to pass from $L_{e_1 S_{\mathbb{Q}}}(W\text{–}\mathrm{Sp}^{\mathcal{O}})$ to the category $\mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}[W]$ of rational orthogonal spectra with W-action. Note that we could have combined the two steps above into one since both left adjoints point the same way, however, for the clarity of the arguments we decided to treat them separately.

Now we pass to symmetric spectra with W-action using the forgetful functor from orthogonal spectra. Next we move to $H\mathbb{Q}$-modules with W-action in symmetric spectra. From here we use the result of [**27**, Theorem 1.1] to get to $\mathsf{Ch}(\mathbb{Q})[W]$, the category of rational chain complexes with W-action, which is equivalent as a monoidal model category to $\mathsf{Ch}(\mathbb{Q}[W])$, the category of chain complexes of $\mathbb{Q}[W]$-modules with a projective model structure. That gives an algebraic model which is compatible with the monoidal product, i.e. the zig-zag of our Quillen equivalences induces a strong symmetric monoidal equivalence on the level of homotopy categories.

To illustrate the whole path we present a diagram which shows every step of this comparison. The reader may wish to refer to this diagram now, but the notation will be introduced as we proceed. Left Quillen functors are placed on the left. Recall that $N = N_G H$ and $W = W_G H = N_G H/H$.

$$\begin{tikzcd}
L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \arrow[d, shift right, "i^*"'] \\
L_{e_{(H)_N} S_{\mathbb{Q}}}(N - \mathrm{Sp}^{\mathcal{O}}) \arrow[u, shift right, "F_N(G_+,-)"'] \arrow[d, shift left, "(-)^H"] \\
L_{e_1 S_{\mathbb{Q}}}(W - \mathrm{Sp}^{\mathcal{O}}) \arrow[u, shift left, "\epsilon^*"] \arrow[d, shift left, "\mathrm{res}"] \\
\mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}}[W] \arrow[u, shift left, "\mathrm{I}_t^c"] \arrow[d, shift left, "\mathrm{Sing}\circ\mathbb{U}"] \\
Sp^{\Sigma}_{\mathbb{Q}}[W] \arrow[u, shift left, "\mathbb{P}\circ|-|"] \arrow[d, shift right, "H\mathbb{Q}\wedge -"'] \\
(H\mathbb{Q} - \mathrm{mod})[W] \arrow[u, shift right, "U"'] \arrow[d, leftrightarrow, "\text{zig-zag of Quillen equivalences}"] \\
\mathsf{Ch}(\mathbb{Q}[W])
\end{tikzcd}$$

5.1. The category $\mathsf{Ch}(\mathbb{Q}[W]$-mod)

Before we start describing the zig-zag of Quillen equivalences towards the algebraic model for rational G-spectra over an exceptional subgroup, we briefly describe the algebraic model. Suppose W is a finite group. In this section, we discuss the category of chain complexes of left $\mathbb{Q}[W]$-modules.

Firstly, this category may be equipped with the projective model structure, where weak equivalences are homology isomorphisms and fibrations are levelwise surjections. Cofibrations are levelwise split monomorphisms with cofibrant cokernel. This model structure is cofibrantly generated by [**16**, Section 2.3].

Note that $\mathbb{Q}[W]$ is not generally a commutative ring, however, it is a Hopf algebra with cocommutative coproduct given by

$$\Delta\colon \mathbb{Q}[W] \longrightarrow \mathbb{Q}[W] \otimes \mathbb{Q}[W] \quad , \quad g \mapsto g \otimes g.$$

This allows us to define an associative and commutative tensor product on $\mathsf{Ch}(\mathbb{Q}[W]$-mod), namely tensor over $\mathbb{Q}$, where the action on $X \otimes_{\mathbb{Q}} Y$ is diagonal. The unit is a chain complex with $\mathbb{Q}$ at the level 0 with trivial W-action and zeros everywhere else and it is cofibrant in the projective model structure. The monoidal product defined this way is closed, where the internal hom is given by an internal hom over $\mathbb{Q}$ with W-action given by conjugation, i.e. for $X, Y \in \mathbb{Q}[W]$-mod, $f \in Hom_{\mathbb{Q}}(X, Y)$ and $w \in W$ we define the action of w on f by $(w \circ f)(x) = wf(w^{-1}x)$.

This category is equivalent (as a monoidal model category) to the category of W-objects in a category of $\mathsf{Ch}(\mathbb{Q}\text{-mod})$ with the projective model structure, i.e. this is a model structure which is a transfer of the projective model structure on $\mathsf{Ch}(\mathbb{Q}\text{-mod})$ to the category of W objects there, using the forgetful functor as a right adjoint. We discuss the category of W-objects in a general category $\mathcal{C}$ in more detail in the next section.

It is shown in [**2**, Proposition 4.3] that $\mathsf{Ch}(\mathbb{Q}[W]\text{-mod})$ is a monoidal model category satisfying the monoid axiom.

5.2. W-objects in a category $\mathcal{C}$

In this section, we recall the category of W-objects in a category $\mathcal{C}$, where W is a finite group and $\mathcal{C}$ is any category. Material presented here will be used at the end of next section, where $\mathcal{C}$ will be for example a category of symmetric spectra or orthogonal spectra.

We denote the category of W-objects in a category $\mathcal{C}$ by $\mathcal{C}[W]$. We can think of $\mathcal{C}[W]$ as a category of functors from W, which is a one object category with $Hom(*,*) = W$ to $\mathcal{C}$, also known as $\mathcal{C}^W$. The inclusion j of a terminal category 1 into W gives two adjoint pairs (Lan_j, j^*) and (j^*, Ran_j). We will use notation $U_{\mathcal{C}}$ for j^*, since it is just a forgetful functor.

If $\mathcal{C}$ is a cofibrantly generated model category, then $\mathcal{C}[W]$ can be equipped with a model structure by applying transfer [**15**, Theorem 11.3.2] to the adjunction below:

$$\mathrm{Lan}_j \colon \mathcal{C} \rightleftarrows \mathcal{C}[W] : U_{\mathcal{C}}$$

where weak equivalences and fibrations in $\mathcal{C}[W]$ are these maps, that forget down to weak equivalences and fibrations in $\mathcal{C}$ (respectively).

Here Lan_j is the left Kan extension along j. Notice, that in this case it is sending X to a coproduct of X indexed by elements of W, with W acting by permuting the factors. It is a straightforward observation that $U_{\mathcal{C}}$ preserves colimits and cofibrations, as generating cofibrations in $\mathcal{C}[W]$ are just images of the generating cofibrations in $\mathcal{C}$ under Lan_j.

If $\mathcal{C}$ is a closed symmetric monoidal model category then $\mathcal{C}[W]$ is as well, by analogous observations to those in Section 5.1. Notice that the monoidal product on W-objects in $\mathcal{C}$ is the one from $\mathcal{C}$ with the diagonal W-action. Notice also that $U_{\mathcal{C}}$ is strong monoidal. It is enough to check the pushout-product axiom in $\mathcal{C}[W]$ for generating cofibrations and acyclic cofibrations, and since they are the images of the generating cofibrations and acyclic cofibrations (respectively) under Lan_j the pushout-product axiom follows from the one in $\mathcal{C}$. The unit axiom follows from the unit axiom in $\mathcal{C}$ and the fact that $U_{\mathcal{C}}$ preserves cofibrations.

For a finite group W, a Quillen equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ induces a Quillen equivalence between W-objects in $\mathcal{C}$ and $\mathcal{D}$.

Proposition 5.1. *Suppose*

$$F \colon \mathcal{C} \rightleftarrows \mathcal{D} : G$$

is a Quillen equivalence and W is a finite group. Then this adjunction induces a Quillen equivalence at the level of W-objects in $\mathcal{C}$ and $\mathcal{D}$ (with model structures transferred from that on $\mathcal{C}$ and $\mathcal{D}$, respectively). Moreover, if (F,G) is a monoidal Quillen *equivalence between monoidal model categories then it is so when induced to the level of W-objects in $\mathcal{C}$ and $\mathcal{D}$.*

Proof. We have the following diagram

$$\begin{tikzcd} \mathcal{C}[W] \arrow[r, shift left, "F_W"] \arrow[d, "U_{\mathcal{C}}"] & \mathcal{D}[W] \arrow[l, shift left, "G_W"] \arrow[d, "U_{\mathcal{D}}"] \\ \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D}, \arrow[l, shift left, "G"] \end{tikzcd}$$

where the functors $U_{\mathcal{C}}$ and $U_{\mathcal{D}}$ commute with both left and right adjoints. Moreover, $U_{\mathcal{C}}$ and $U_{\mathcal{D}}$ create weak equivalences and fibrations and they preserve cofibrant objects (they preserve cofibrations and initial objects) and fibrant objects. It is straightforward to check the condition from the definition of Quillen equivalence for the adjunction (F_W, G_W) using the above diagram.

For the monoidal consideration, recall that the monoidal product on W-objects in $\mathcal{C}$ is the one from $\mathcal{C}$ with the diagonal W-action. Moreover, $U_{\mathcal{C}}$ is strong monoidal. If (F, G) is a monoidal Quillen pair then it is again a diagram chase to show that (F_W, G_W) is also a monoidal Quillen pair. □

Now we are ready to establish the zig-zag of Quillen equivalences.

5.3. Monoidal comparison

Recall, that we want to provide a monoidal algebraic model for $L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–Sp}^{\mathcal{O}})$, where H is an exceptional subgroup of G.

At the beginning of this approach we would like to use the inflation-fixed point adjunction. However, as H is not necessary normal in G first we need to move to the category of orthogonal N-spectra, where $N = N_G H$. Notice that for our purpose this passage needs to be monoidal.

The inclusion of a subgroup $i\colon N \longrightarrow G$ induces two adjoint pairs between corresponding categories of orthogonal spectra that we discussed earlier. The first choice would be to work with the induction and restriction adjunction. However, in case of our localizations, this is not always a Quillen adjunction as we discussed in detail in Section 4. The restriction functor i^* is strong monoidal, so we choose to work with it as a left adjoint, where the right adjoint is the coinduction functor. We showed in Section 4 that this is always a strong monoidal Quillen adjunction for localizations at idempotents corresponding to conjugacy classes of exceptional subgroups. Thus Theorem 4.9 gives the first step of the zig-zag:

Theorem 4.9. (Recap) *Suppose H is an exceptional subgroup of G. Then the composite of adjunctions*

$$L_{e_{(H)_G}S_{\mathbb{Q}}}(G\text{–Sp}^{\mathcal{O}}) \underset{F_N(G_+,-)}{\overset{i^*}{\rightleftarrows}} L_{i^*(e_{(H)_G}S_{\mathbb{Q}})}(N\text{–Sp}^{\mathcal{O}}) \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} L_{e_{(H)_N}S_{\mathbb{Q}}}(N\text{–Sp}^{\mathcal{O}})$$

is a strong symmetric monoidal Quillen equivalence. Notice that if H is N-good then $i^*(e_{(H)_G}) = e_{(H)_N}$ and the right adjunction is trivial.

Now we use the inflation–fixed point adjunction. Recall that W below denotes the Weyl group $N_G H/H$ and by the assumption on H, it is finite. Moreover, there is a projection map $\epsilon\colon N \longrightarrow W$ which induces the left adjoint below.

Theorem 5.2. *The adjunction*

$$\epsilon^*\colon L_{e_1 S_{\mathbb{Q}}}(W\text{–}\mathrm{Sp}^{\mathcal{O}}) \rightleftarrows L_{e_{(H)_N} S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}}) : (-)^H$$

is a strong monoidal Quillen equivalence. Here e_1 is the idempotent of the rational Burnside ring $\mathsf{A}(W)_{\mathbb{Q}}$ corresponding to the trivial subgroup.

Proof. To prove this is a Quillen pair we refer to [**14**, Proposition 3.2] which states that (in notation adapted to our case):

$$\epsilon^*\colon (W\text{–}\mathrm{Sp}^{\mathcal{O}}) \rightleftarrows L_{\widetilde{E}[\not\supseteq H]}(N\text{–}\mathrm{Sp}^{\mathcal{O}}) : (-)^H$$

is a Quillen equivalence. Recall that $\widetilde{E}[\not\supseteq H]$ is a cofibre of a map $E[\not\supseteq H] \longrightarrow S^0$ where $[\not\supseteq H]$ denotes the family of subgroups of N not containing H (for definition of $E\mathcal{F}$, where $\mathcal{F}$ is a family of subgroups of G see, for example, [**20**, Chapter II, Definition 2.10]). Now we localize this result further at $e_1 S_{\mathbb{Q}}$ on the side of W-spectra and $e_{(H)_N} S_{\mathbb{Q}}$ on the side of N-spectra. Since $e_{(H)_N} S_{\mathbb{Q}} \simeq \epsilon^*(e_1 S_{\mathbb{Q}})$ in $L_{\widetilde{E}[\not\supseteq H]}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$, it follows from Lemma 4.1 and the fact that ϵ^* is strong monoidal that the resulting adjunction is a Quillen equivalence. The right hand side after this localization is just $L_{e_{(H)_N} S_{\mathbb{Q}}}(N\text{–}\mathrm{Sp}^{\mathcal{O}})$, since $e_{(H)_N} S_{\mathbb{Q}} \wedge \widetilde{E}[\not\supseteq H] \simeq e_{(H)_N} S_{\mathbb{Q}}$ (which can be checked using fixed points). □

Next we move from $L_{e_1 S_{\mathbb{Q}}}(W\text{–}\mathrm{Sp}^{\mathcal{O}})$ to $\mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}}[W]$ (where $\mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}} = L_{S_{\mathbb{Q}}}\mathrm{Sp}^{\mathcal{O}}$) using the restriction and extension of W-universe from the complete to the trivial one.

Lemma 5.3. *The adjunction*

$$\mathrm{I}_t^c\colon \mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}}[W] \rightleftarrows L_{e_1 S_{\mathbb{Q}}}(W\text{–}\mathrm{Sp}^{\mathcal{O}}) : \mathrm{I}_c^t = \mathrm{res}$$

is a strong monoidal Quillen equivalence. We use I_c^t to denote the restriction (denoted also res *above) from the complete W-universe to the trivial one. I_t^c denotes the extension from the trivial W-universe to the complete one.*

Proof. This adjunction is actually a composite of the following

$$\mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}}[W] \xleftrightarrow{\simeq \text{ of cat}} \text{free-}W\text{–}\mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}t} \underset{\mathrm{res}}{\overset{\mathrm{I}_t^c}{\rightleftarrows}} \text{free-}W\text{–}\mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} L_{e_1 S_{\mathbb{Q}}}(W\text{–}\mathrm{Sp}^{\mathcal{O}}).$$

Before we proceed to discussing the adjunctions and proving that they are Quillen equivalences let us first define the model category of free $W\text{–}\mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}}$. This is a special case of $\mathcal{F}$-model structure in [**21**, Chapter IV, Theorem 6.5] where we take the family $\mathcal{F} = \{1\}$, consisting just of the trivial subgroup of W. Thus recall that a free model structure on rational orthogonal W-spectra is defined as follows.

– A map f is a weak equivalence in free-$W\text{–}\mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}}$ iff $\pi^1_{*\mathbb{Q}}(f)$ is an isomorphism. 1 denotes the trivial subgroup in W (equivalently, f is a weak equivalence in free-$W\text{–}\mathcal{IS}_{\mathbb{Q}}$ iff $e_1 S_{\mathbb{Q}} \wedge f$ is a π_* rational isomorphism).

– A cofibration is a map obtained from the original generating cofibrations by restricting to the orbit W_+.
– Fibrations are defined via right lifting property.

Starting from the right hand side in the diagram above we have an identity adjunction between $L_{e_1 S_{\mathbb{Q}}}(W\text{–}\mathrm{Sp}^{\mathcal{O}})$ and a model category of free-$W\text{–}\mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}}$. This is a strong monoidal Quillen equivalence by [21, Chapter IV, Theorem 6.9].

The second adjunction is just the restriction and extension of universe adjunction from the complete universe (on the right) to the trivial one (on the left, indicated by a subindex t). This is a strong monoidal adjunction by [21, Chapter V, Theorem 1.5]. Now we note that the left adjoint preserves generating cofibrations and generating acyclic cofibrations since $\mathrm{I}_t^c F_V \cong F_V$ by [21, Chapter V, 1.4].

The right adjoint res preserves and reflects all weak equivalences since in both model structures they are defined as those maps which after forgetting to non-equivariant spectra are rational π_*-isomorphisms. The derived unit for the cofibrant generator W_+ (in this case categorical unit is also the derived unit) is an isomorphism which follows from [21, Chapter V, Theorem 1.5], and thus for any cofibrant object it is a weak equivalence. By Part 3 of [16, Corollary 1.3.16] this is a Quillen equivalence.

It remains to show that the equivalence of categories on the right is an equivalence of monoidal model categories. Note that weak equivalences on both sides are just non-equivariant rational π_*-isomorphisms and generating cofibrations in both model structures are the same. This is enough to deduce that these two model categories are the same, which finishes the proof. $\square$

We removed all difficulties coming from the equivariance with respect to a topological group. What is left now is a finite group action on rational orthogonal spectra.

To apply the result of [27, Theorem 1.1] and pass to the category of chain complexes we need to work with rational symmetric spectra in the form of $H\mathbb{Q}$-modules (where $H\mathbb{Q}$ is the Eilenberg-MacLane spectrum for $\mathbb{Q}$). We pass to this category using the next two results, which are direct corollaries of Proposition 5.1 and corresponding known results for spectra (see, for example, Section 7 in [25] and recall that $H\mathbb{Q}$ is weakly equivalent to $S_{\mathbb{Q}}$).

First we pass from rational orthogonal spectra with W-action to rational symmetric spectra with W-action using the composition of the forgetful functor and the functor induced by the singular complex functor:

Corollary 5.4. *The adjunction*

$$\mathbb{P} \circ |-| \colon Sp^{\Sigma}_{\mathbb{Q}}[W] \rightleftarrows \mathrm{Sp}^{\mathcal{O}}_{\mathbb{Q}}[W] : \mathrm{Sing} \circ \mathbb{U}$$

is a strong symmetric monoidal Quillen equivalence.

Next we move to $H\mathbb{Q}$-modules in symmetric spectra with W-action.

Corollary 5.5. *The adjunction*

$$H\mathbb{Q} \wedge - \colon Sp^{\Sigma}_{\mathbb{Q}}[W] \rightleftarrows (H\mathbb{Q}\text{-mod})[W] : U$$

is a strong symmetric monoidal Quillen equivalence. Here U denotes the forgetful functor and the model structure on $H\mathbb{Q}$-mod *is the one created from Sp^{Σ} by the right adjoint U.*

From here we use the result of [**27**, Theorem 1.1] for $R = \mathbb{Q}$ and Proposition 5.1 to get to $\mathsf{Ch}(\mathbb{Q})[W]$ with the projective model structure, which is equivalent as a monoidal model category to $\mathsf{Ch}(\mathbb{Q}[W])$ with the projective model structure (see Section 5.1).

Corollary 5.6. *There is a zig-zag of monoidal Quillen equivalences between the category* $(\mathrm{H}\mathbb{Q}\text{-mod})[W]$ *and the category* $\mathsf{Ch}(\mathbb{Q}[W])$ *with the projective model structure.*

We can summarize the results of this section in the theorem below.

Theorem 5.7. *There is a zig-zag of symmetric monoidal Quillen equivalences from* $L_{e_{(H)_G} S_\mathbb{Q}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ *to* $\mathsf{Ch}(\mathbb{Q}[W]\text{-mod})$ *with the projective model structure, where* $W = N_G H/H$.

An example of an application of the result above is to rational $SO(3)$-spectra over an exceptional subgroup, see [**19**]. In general, if a compact Lie group G has an exceptional subgroup, any algebraic model for rational G-spectra will split and will have a factor given by Theorem 5.7.

5.4. Finite G

If G is finite then every subgroup of G is exceptional and there are finitely many conjugacy classes of subgroups of G, thus by splitting result of [**3**, Theorem 4.4] and Proposition 3.5 the category of rational G-spectra splits as a finite product of categories, each localized at an idempotent corresponding to the conjugacy class of a subgroup of G.

Proposition 5.8. *[3, Theorem 4.4] Suppose G is a finite group. Then there is a strong symmetric monoidal Quillen equivalence:*

$$\triangle\colon L_{S_\mathbb{Q}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) \rightleftarrows \prod_{(H)_G, H\leqslant G} L_{e_{(H)_G} S_\mathbb{Q}}(G\text{–}\mathrm{Sp}^{\mathcal{O}}) :\Pi,$$

where the left adjoint is a diagonal functor, the right one is a product and the product category on the right is considered with the objectwise model structure.

This observation allows us to deduce the following

Corollary 5.9. *Suppose G is a finite group. Then there is a zig-zag of symmetric monoidal Quillen equivalences from* $L_{S_\mathbb{Q}}(G\text{–}\mathrm{Sp}^{\mathcal{O}})$ *to*

$$\prod_{(H)_G, H\leqslant G} \mathsf{Ch}(\mathbb{Q}[W_G H] : \mathrm{mod}).$$

Proof. This follows from Proposition 5.8 and Theorem 5.7. □

We remark that this is not a new result, as for a finite group G an algebraic model was given in [**26**] and monoidal consideration was presented in [**2**]. However, the use of localizations of commutative ring G-spectra in the proof of [**2**] (see [**2**, Lemma 5.10]) requires reformulations of equivariant version of [**9**, Chapter VIII, Theorem 2.2] and use of suitable foundations. Thus, [**2**] does not provide *monoidal* Quillen equivalences. Our proof avoids these issues.

The main difference between this approach and what appears in the literature is in replacing "Morita equivalence" used in [**26**] and in [**2**] by the inflation-fixed point adjunction. This became possible after analysing an interplay of induction-restriction-coinduction adjunctions with left Bousfield localizations in Section 4.

Remark 5.10. The result of Corollary 5.9 does not imply comparison of commutative algebras. The comparison of commutative algebras on both sides of the zig-zag requires more care. This is done in [**6**].

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