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ITERATED DOUBLES OF THE JOKER AND THEIR REALISABILITY

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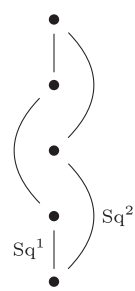
(*communicated by Donald M. Davis*)

Abstract

Let $\mathcal{A}(1)^*$ be the subHopf algebra of the mod 2 Steenrod algebra $\mathcal{A}^*$ generated by Sq^1 and Sq^2. The *Joker* is the cyclic $\mathcal{A}(1)^*$-module $\mathcal{A}(1)^*/\mathcal{A}(1)^*\{\mathrm{Sq}^3\}$ which plays a special rôle in the study of $\mathcal{A}(1)^*$-modules. We discuss realisations of the Joker both as an $\mathcal{A}^*$-module and as the cohomology of a spectrum. We also consider analogous $\mathcal{A}(n)^*$-modules for $n \geqslant 2$ and prove realisability results (both stable and unstable) for $n = 2, 3$ and non-realisability results for $n \geqslant 4$.

Introduction

The cyclic $\mathcal{A}(1)^*$-module $\mathcal{A}(1)^*/\mathcal{A}(1)^*\{\mathrm{Sq}^3\}$, commonly known as the *Joker*, was shown by Adams and Priddy [**2**] to give rise to a torsion summand in the Picard group of invertible stable $\mathcal{A}(1)^*$-modules. Here is a representation of the Joker where a vertical line indicates the action of Sq^1 and a curved line indicates the action of Sq^2.



There are various choices for the grading, but for topological reasons we use the one where the lowest degree of a nontrivial element is 0. More details about the Joker and its homological algebra can be found in [**5**, Appendix A.8]. For a recent result which highlights the special significance of the Joker see [**4**]. Incidentally, the use of the name Joker appears to be due to Frank Adams, although the earliest published occurrence that we are aware of is in [**10**]; it may be based on the similarity of the diagram above to a traditional jester's hat.

The mathematics in this paper owes much to the insights and inspiration of ***Michael Barratt*** *and* ***Mark Mahowald*** *and I would like to dedicate it to their memory.*
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The $\mathcal{A}(1)^*$-module structure of the Joker extends in two ways to an $\mathcal{A}^*$-module structure determined by whether Sq^4 acts non-trivially or not between the bottom and top degrees. The resulting $\mathcal{A}^*$-modules are linear duals of each other. We will show that both can be realised as cohomology of finite CW spectra which are Spanier-Whitehead dual using a construction we learnt from Peter Eccles, however, it also appeared in Mike Hopkins' Oxford PhD thesis but seems not to be otherwise published. We will show in Theorem 5.1 that these can be realised as cohomology of spaces with bottom cells of degrees 2 and 4, respectively.

Using a construction based on doubling, we introduce higher versions of the Joker defined as cyclic $\mathcal{A}(n)^*$-modules and show that these can be realised as cohomology of spectra precisely when $n \leqslant 3$. Most cases of the non-realisability result can be verified by a direct application of Adams' result on Hopf invariant 1, however, in one case we resort to a more delicate argument using the precise form of his factorisation of Sq^{2^r} for $r \geqslant 4$, so we give a proof which applies in all cases. In the cases where we can realise these modules, our constructions depend on the existence of triple Toda brackets containing the first three elements of Kervaire invariant 1, i.e., η^2, ν^2, σ^2. Finally, we consider unstable realisations and show that for $n = 1, 2$ we can indeed realise optimal unstable versions of the higher Joker modules; the techniques used involve modifying naturally occurring spaces by mapping into Eilenberg-Mac Lane spaces and certain spaces in the spectra $k\mathrm{O}$ and tmf, thus giving alternatives to the stable constructions above.

For the convenience of the reader, we include a brief appendix in which some connectivity results on infinite loop spaces are given; this material is standard but we were unable to locate convenient references.

We also make numerous references to calculational results obtained using the Adams spectral sequence. The reader can find relevant charts in the earlier `arxiv` versions of this paper

https://arxiv.org/abs/1710.02974

but at the request of the Editor we have omitted most of these. They were obtained using Bob Bruner's programmes available at the following address.

http://www.math.wayne.edu/∼rrb/cohom/index.html

Acknowledgments

I would like to thank the following for helpful comments and insights: Bob Bruner and John Rognes (from whom I learnt an enormous amount about working with the Steenrod algebra), Don Davis, Peter Eccles (who showed me how to use Toda brackets to construct complexes efficiently and so initiated the work described) and Grant Walker.

Conventions & notations

Throughout we work locally at the prime 2.

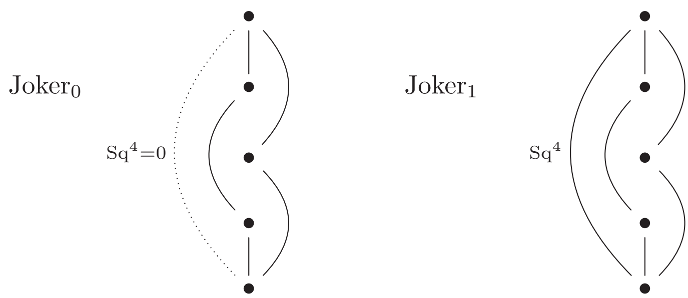
To avoid excessive display of gradings we will often suppress cohomological degrees and write V for a cohomologically graded vector space V^*; in particular, we will often write $\mathcal{A}$ for the Steenrod algebra. The linear dual of V is $\mathrm{D}V$ where $(\mathrm{D}V)^k = \mathrm{Hom}_{\mathbb{F}_2}(V^{-k}, \mathbb{F}_2)$, and we write $V[m]$ for graded vector space with $(V[m])^k = V^{k-m}$, so for the cohomology of a spectrum X, $H^*(\Sigma^m X) = H^*(X)[m]$.

For a connected graded algebra $\mathcal{B}^*$ we will write $\mathcal{B}^+$ for its positive degree part.

We will denote the Spanier-Whitehead dual of a spectrum X by DX.

1. $\mathcal{A}$-module structures on the Joker and duality

The Joker has two possible $\mathcal{A}$-module structures corresponding to the choice of action of Sq^4 between the top and bottom degrees. The resulting $\mathcal{A}$-modules Joker_0^* and Joker_1^* are displayed in the following diagrams in which the shorter edges represent non-trivial Sq^1 and Sq^2 actions.



Recall that for a left $\mathcal{A}$-module M, the $\mathbb{F}_2$-linear dual $\mathrm{D}M$ is naturally a *right* $\mathcal{A}$-module where for $f \in \mathrm{D}M$, $\theta \in \mathcal{A}$ and $x \in M$,

$$(f \cdot \theta)(x) = f(\theta x).$$

There is an associated *left* module structure given by

$$(\theta \cdot f)(x) = (f \cdot \chi\theta)(x) = f(\chi\theta x),$$

where $\chi\colon \mathcal{A} \to \mathcal{A}$ is the antipode. For a finite CW complex spectrum Z, as a left $\mathcal{A}$-module the cohomology of the Spanier-Whitehead dual DZ satisfies

$$H^*(DZ) \cong \mathrm{D}(H^*(Z)).$$

Since the following relations hold in $\mathcal{A}$,

$$\begin{aligned} &\mathrm{Sq}^3 = \mathrm{Sq}^1\mathrm{Sq}^2, \quad \mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1, \\ &\chi\mathrm{Sq}^1 = \mathrm{Sq}^1, \qquad \chi\mathrm{Sq}^2 = \mathrm{Sq}^2, \\ &\chi\mathrm{Sq}^4 = \mathrm{Sq}^4 + \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 = \mathrm{Sq}^4 + \mathrm{Sq}^2\mathrm{Sq}^2, \end{aligned}$$

it follows that Joker_0 and Joker_1 are dual up to a degree shift, i.e.,

$$\mathrm{Joker}_1 \cong \mathrm{D}\mathrm{Joker}_0[4]. \tag{1.1}$$

2. Doubling and higher versions of the Joker

Doubling for $\mathcal{A}$ and $\mathcal{A}(n)$ are discussed in Margolis [**14**, Section 15.3] (for a particularly relevant result on modules see Theorem 31). We give a brief description and, in particular, explain what happens under iterated doubling.

The dual of $\mathcal{A}(n)$ is the quotient Hopf algebra

$$\begin{aligned}\mathcal{A}(n)_* &= \mathcal{A}_*/(\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \ldots) \\ &= \mathbb{F}_2[\overline{\zeta_1}, \overline{\zeta_2}, \ldots, \overline{\zeta_{n+1}}]/(\overline{\zeta_1}^{2^{n+1}}, \overline{\zeta_2}^{2^n}, \ldots, \overline{\zeta_{n+1}}^2),\end{aligned}$$

where $\overline{(-)}$ indicates residue class. The dual of the normal exterior subHopf algebra $\mathcal{E}(n) \subseteq \mathcal{A}(n)$ generated by the Milnor primitives P_t^0 $(1 \leqslant t \leqslant n+1)$ is the quotient exterior algebra

$$\mathcal{E}(n)_* = \mathcal{A}_*/(\zeta_1^2, \zeta_2^2, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \zeta_{n+3}, \ldots) \cong \mathcal{A}(n)_*/(\overline{\zeta_1}^2, \overline{\zeta_2}^2, \ldots, \overline{\zeta_{n+1}}^2).$$

The dual of the quotient Hopf algebra

$$\mathcal{A}(n)/\!/\mathcal{E}(n) = \mathcal{A}(n) \otimes_{\mathcal{E}(n)} \mathbb{F}_2 \cong \mathcal{A}(n)/\mathcal{A}(n)\mathcal{E}(n)^+$$

is

$$\mathcal{A}(n)_* \square_{\mathcal{E}(n)_*} \mathbb{F}_2 = \mathbb{F}_2[\overline{\zeta_1}^2, \overline{\zeta_2}^2, \ldots, \overline{\zeta_{n+1}}^2]/(\overline{\zeta_1}^{2^{n+1}}, \overline{\zeta_2}^{2^n}, \ldots, \overline{\zeta_{n+1}}^2) \subseteq \mathcal{A}(n)_*.$$

There is an external Frobenius homomorphism $\mathbf{f}\colon \mathcal{A}(n)_* \to \mathcal{A}(n+1)_*$ which factors through the dual of $\mathcal{A}(n)/\!/\mathcal{E}(n)$ and satisfies $\mathbf{f}(\overline{\zeta_r}) = \overline{\zeta_r}^2$.

$$\begin{tikzcd} \mathcal{A}(n)_* \arrow[r] \arrow[dr, "\mathbf{f}"'] & \mathcal{A}(n+1)_* \square_{\mathcal{E}(n+1)_*} \mathbb{F}_2 \arrow[d] \\ & \mathcal{A}(n+1)_* \end{tikzcd}$$

More generally there are iterations $\mathbf{f}^{(k)}\colon \mathcal{A}(n)_* \to \mathcal{A}(n+k)_*$ where $k \geqslant 0$, so that $\mathbf{f}^{(0)} = \mathbf{f}$ and $\mathbf{f}^{(k)}(\overline{\zeta_r}) = \overline{\zeta_r}^{2^k}$.

$$\begin{tikzcd} \mathcal{A}(n)_* \arrow[r] \arrow[dr, "\mathbf{f}^{(k)}"'] & \mathcal{A}(n+k)_* \square_{\mathcal{E}(n+k)_*} \mathbb{F}_2 \arrow[d] \\ & \mathcal{A}(n+k)_* \end{tikzcd}$$

Each $\mathbf{f}^{(k)}$ is clearly a Hopf algebra homomorphism and there is a dual Verschiebung Hopf algebra homomorphism $\mathbf{v}^{(k)}\colon \mathcal{A}(n+k) \to \mathcal{A}(n)$.

$$\begin{tikzcd} \mathcal{A}(n) & \mathcal{A}(n+k)/\!/\mathcal{E}(n+k) \arrow[l] \\ & \mathcal{A}(n+k) \arrow[u] \arrow[ul, "\mathbf{v}^{(k)}"] \end{tikzcd}$$

Since

$$\mathbf{f}^{(k)}(\overline{\xi_1^{i_1} \cdots \xi_\ell^{i_\ell}}) = \overline{\xi_1^{2^k i_1} \cdots \xi_\ell^{2^k i_\ell}}$$

the effect of $\mathbf{v}^{(k)}$ is easily seen using the Milnor basis dual to the monomial basis in the elements $\overline{\xi_r} = \chi\overline{\zeta_r}$ and we have

$$\mathbf{v}^{(k)}(\mathrm{Sq}(j_1, \ldots, j_\ell)) = \begin{cases} \mathrm{Sq}(j'_1, \ldots, j'_\ell) & \text{if } j_r = 2^k j'_r \text{ for all } r, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{Sq}(j) = \mathrm{Sq}^j$,

$$\mathbf{v}^{(k)}(\mathrm{Sq}^j) = \begin{cases} \mathrm{Sq}^{j'} & \text{if } j = 2^k j', \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for $1 \leqslant t \leqslant n+1$ the elements $\mathbf{v}^{(k)}(\mathrm{P}^s_t) \in \mathcal{A}(n)$ are given by

$$\mathbf{v}^{(k)}(\mathrm{P}^s_t) = \mathbf{v}^{(k)}(\mathrm{Sq}(\overbrace{0,\dots,0,2^s}^{t})) = \begin{cases} \mathrm{P}^{s-k}_t & \text{if } s \geqslant k, \\ 0 & \text{otherwise.} \end{cases}$$

Let ${}_{\mathcal{A}(n)}\mathcal{M}^{(r)}$ be the full subcategory of ${}_{\mathcal{A}(n)}\mathcal{M}$ consisting of modules concentrated in degrees divisible by 2^r; for example ${}_{\mathcal{A}(n)}\mathcal{M}^{(0)} = {}_{\mathcal{A}(n)}\mathcal{M}$, and ${}_{\mathcal{A}(n)}\mathcal{M}^{(1)} = {}_{\mathcal{A}(n)}\mathcal{M}^{\mathrm{ev}}$. The Verschiebung $\mathbf{v}^{(k)}$ together with the quotient homomorphism $\pi\colon \mathcal{A}(n+k) \to \mathcal{A}(n+k)/\!/\mathcal{E}(n+k)$ induces restriction functors between categories of left modules fitting into the following commutative diagram.

$$\begin{tikzcd} {}_{\mathcal{A}(n)}\mathcal{M} \arrow[r, "(\mathbf{v}^{(k)})^*"] \arrow[dr, "\Delta^{(k)}"'] & {}_{\mathcal{A}(n+k)/\!/\mathcal{E}(n+k)}\mathcal{M}^{(k)} \arrow[d, "\pi^*"] \\ & {}_{\mathcal{A}(n+k)}\mathcal{M}^{(k)} \end{tikzcd} \tag{2.1}$$

Here $\Delta^{(k)}$ multiplies degrees by 2^k and it is a monoidal functor since $\mathbf{v}^{(k)}$ and π are both homomorphisms of Hopf algebras.

By [**14**, Theorem 15.3.31] $\Delta^{(1)}$ is an isomorphism of categories, but when $k > 1$ this is not true. However, this can be corrected by replacing $\mathcal{A}(n+k)/\!/\mathcal{E}(n+k)$ by the quotient of $\mathcal{A}(n+k)$ by the ideal generated by a larger set of the elements P^s_t. Let

$$\mathcal{E}(n+k,k) = \mathbb{F}_2(\mathrm{P}^s_t : 1 \leqslant t \leqslant n+k+1,\ 0 \leqslant s < k) \subseteq \mathcal{A}(n+k)$$

and consider the normal quotient

$$\mathcal{A}(n+k)/\!/\mathcal{E}(n+k,k) = \mathcal{A}(n+k)/\mathcal{A}(n+k)\mathcal{E}(n+k,k)^+ \cong \mathcal{A}(n+k) \otimes_{\mathcal{E}(n+k,k)} \mathbb{F}_2.$$

Then (2.1) can be replaced by

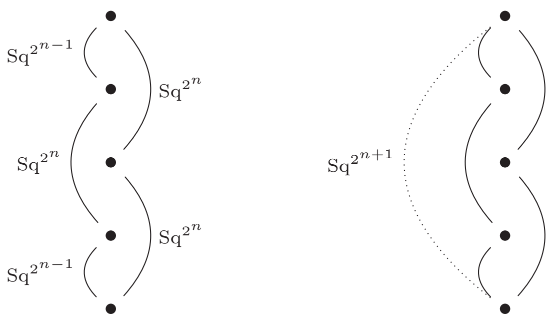
$$\begin{tikzcd} {}_{\mathcal{A}(n)}\mathcal{M} \arrow[r, "(\mathbf{v}^{(k)})^*", "\cong"'] \arrow[dr, "\Delta^{(k)}"'] & {}_{\mathcal{A}(n+k)/\!/\mathcal{E}(n+k,k)}\mathcal{M}^{(k)} \arrow[d, "\pi^*", "\cong"'] \\ & {}_{\mathcal{A}(n+k)}\mathcal{M}^{(k)} \end{tikzcd}$$

so that the proof of Margolis still applies to show that this $\Delta^{(k)}$ is an isomorphism of categories.

We remark that $\Delta^{(k)}$ does not induce a functor on stable module categories since it does not preserve projective modules.

For each $n \geqslant 2$, iterated doubling gives a generalisation of the Joker to a cyclic $\mathcal{A}(n)$-module $\mathrm{Joker}(n) = \Delta^{(n-1)}(\mathrm{Joker})$. The actions of $\mathrm{Sq}^{2^{n-1}}$ and Sq^{2^n} on $\mathrm{Joker}(n)$

are shown below.



There are two extensions to $\mathcal{A}$-module structures, each determined by a choice of action by $\mathrm{Sq}^{2^{n+1}}$ from the top to the bottom degree, and we denote the resulting $\mathcal{A}$-modules by $\mathrm{Joker}(n)_0$ and $\mathrm{Joker}(n)_1$ depending on whether $\mathrm{Sq}^{2^{n+1}}$ acts trivially or not. It is straightforward to verify that the action of $\chi\,\mathrm{Sq}^{2^{n+1}}$ on $\mathrm{Joker}(n)_0$ is non-trivial and

$$\mathrm{Joker}(n)_1 \cong \mathrm{DJoker}(n)_0[2^{n+1}].$$

As an $\mathcal{A}(n)$-module, $\mathrm{Joker}(n)$ is finitely presented. For example, $\mathrm{Joker}(1)$ has minimal presentation

$$0 \leftarrow \mathrm{Joker}(1) \leftarrow \mathcal{A}(1) \leftarrow \mathcal{A}(1)[3],$$

while

$$\mathrm{Joker}(2) = \mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{P}^0_1, \mathrm{P}^0_2, \mathrm{P}^0_3, \mathrm{Sq}^6\} = \mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{P}^0_1, \mathrm{P}^0_2, \mathrm{Sq}^6\},$$

so it has a minimal presentation

$$0 \leftarrow \mathrm{Joker}(2) \leftarrow \mathcal{A}(2) \leftarrow \mathcal{A}(2)[1] \oplus \mathcal{A}(2)[3] \oplus \mathcal{A}(2)[6].$$

Finally,

$$\mathrm{Joker}(3) = \mathcal{A}(3)/\mathcal{A}(3)\{\mathrm{P}^0_1, \mathrm{P}^0_2, \mathrm{P}^0_3, \mathrm{P}^1_1, \mathrm{P}^1_2, \mathrm{Sq}^{12}\} = \mathcal{A}(3)/\mathcal{A}(3)\{\mathrm{P}^0_1, \mathrm{P}^1_1, \mathrm{P}^1_2, \mathrm{Sq}^{12}\}$$

and there is a minimal presentation

$$0 \leftarrow \mathrm{Joker}(3) \leftarrow \mathcal{A}(3) \leftarrow \mathcal{A}(3)[1] \oplus \mathcal{A}(3)[2] \oplus \mathcal{A}(3)[6] \oplus \mathcal{A}(3)[12].$$

Of course the $\mathcal{A}$-modules $\mathrm{Joker}(n)_0$ and $\mathrm{Joker}(n)_1$ are not finitely presented. These presentations can be extended further using the `Sage` package of Mike Catanzaro and Bob Bruner which can be found at

http://www.math.wayne.edu/~mike/mods/

and is documented in [**6**].

3. Some recollections on Toda brackets

For ease of reference, we recall some basic ideas about triple Toda brackets in homotopy theory. A classic source for the basic ideas is the book of Mosher and

Tangora [**16**] and Toda's seminal work [**17**] provides a more exhaustive account, while Cohen [**7**] gives a different treatment, also discussed by Whitehead [**19**].

Let

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

be a sequence of maps (of based spaces or spectra) and assume that gf and hg are null homotopic. The mapping sequence for g extends to a commutative diagram of solid arrows

$$\begin{tikzcd} & & \Sigma W \arrow[d, dashed, "f_\flat"'] \arrow[dr, "\Sigma f"] & & \\ X \arrow[r, "g"] & Y \arrow[r, "j"] \arrow[dr, "h"'] & C(g) \arrow[r, "k"] \arrow[d, dashed, "h^\sharp"] & \Sigma X \arrow[r, "\Sigma g"] & \Sigma Y \\ & & Z & & \end{tikzcd}$$

and the composition $h^\sharp f_\flat \colon \Sigma W \to Z$ represents the Toda bracket $\langle f, g, h\rangle$. Of course this element is not necessarily well defined up to homotopy: the choices in $f_\flat$ and $h^\sharp$ contribute indeterminacy subgroups $h_*[\Sigma W, Y]$ and $(\Sigma f)^*[\Sigma X, Y]$ and when W is a suspension or a spectrum

$$\operatorname{indet}\langle f, g, h\rangle = h_*[\Sigma W, Y] + (\Sigma f)^*[\Sigma X, Y],$$

and

$$\langle f, g, h\rangle = h^\sharp f_\flat + h_*[\Sigma W, Y] + (\Sigma f)^*[\Sigma X, Y],$$

for some given choice of $f_\flat$ and $h^\sharp$.

Here are some important examples of such Toda brackets in the stable homotopy groups of spheres $\pi_*(S)$ where $S = S^0_{(2)}$ is the 2-local sphere spectrum. As usual, we identify $\theta \in \pi_n(S)$ with $\Sigma^k\theta \in \pi_{n+k}(\Sigma^k S) \cong \pi_{n+k}(S^k)$.

$$\langle 2, \eta, 2\rangle = \{\eta^2\}, \tag{3.1a}$$
$$\langle \eta, \nu, \eta\rangle = \{\nu^2\}, \tag{3.1b}$$
$$\langle \nu, \sigma, \nu\rangle = \{\sigma^2\}. \tag{3.1c}$$

Of course these elements $\theta_1 = \eta^2$, $\theta_2 = \nu^2$ and $\theta_3 = \sigma^2$ are the first elements of Kervaire invariant 1.

Proof/justification. Using the Peterson-Stein formula [**16**] or performing calculations with Massey products in $\mathrm{Ext}_{\mathcal{A}}$, it is straightforward to see that these brackets contain the stated elements; alternatively see [**17**, Corollary 3.7]. Also,

$$\operatorname{indet}\langle 2, \eta, 2\rangle = 2\pi_1(S) = 0,$$
$$\operatorname{indet}\langle \eta, \nu, \eta\rangle = \eta\pi_5(S) = 0,$$

since $\pi_1(S) \cong \mathbb{Z}_2$ and $\pi_5(S) = 0$. To see that $\operatorname{indet}\langle \nu, \sigma, \nu\rangle = 0$, we need to consider the Adams spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(S)$$

in degree 14. Although there is an element $h_2 \mathrm{P} h_2 \in \mathrm{E}_2^{6,20}$ this is killed by the differential d_3, so $\nu\pi_{11}(S) = 0$. $\square$

4. Constructing Joker spectra

The main idea for this construction was explained to us by Peter Eccles, and it also appears in the unpublished Oxford PhD thesis of Mike Hopkins [**11**] (see Section 1.7). We will make use of the well-known Toda bracket $\langle 2, \eta, 2\rangle = \{\eta^2\} \subseteq \pi_2(S^0)$.

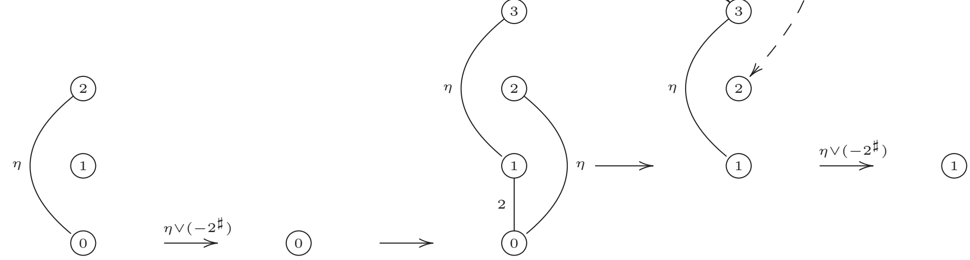
Let $2^\sharp \colon C(\eta) \to S^0$ extend 2 on the bottom cell; there is no indeterminacy in this choice because the non-trivial element $\eta^2 \in \pi_2(S^0)$ is in the image of the map $\pi_2(S^1) \to \pi_2(S^0)$ induced by $\eta\colon S^2 \to S^1$ on domains. Let $2_\flat \colon S^2 \to C(\eta)$ be the coextension of the degree 2 map onto the top cell; again there is no indeterminacy in this choice since the non-trivial element $\eta^2 \in \pi_2(S^0)$ is in the image of the map $\pi_2(S^1) \to \pi_2(S^0)$ induced by $\eta\colon S^1 \to S^0$ on codomains.

$$\begin{tikzcd}
 & & S^2 \arrow[dll, dashed, "\eta"'] \arrow[dl, dashed, "\eta^2"] \arrow[d, "2_\flat"] \arrow[dr, "2"] & & \\
S^1 \arrow[r, "\eta"'] & S^0 \arrow[r] \arrow[dr, "2"'] & C(\eta) \arrow[r] \arrow[d, "2^\sharp"] & S^2 \arrow[r, "\eta"] \arrow[dl, dashed, "\eta^2"'] & S^1 \arrow[dll, dashed, "\eta"] \\
 & & S^0 & &
\end{tikzcd}$$

By (3.1a) the composition $\eta^2 - 2^\sharp \circ 2_\flat$ in the following diagram is null homotopic.

$$\begin{tikzcd}
S^2 \arrow[r, "\eta\vee 2_\flat"'] \arrow[rr, bend left, "\eta^2-2^\sharp\circ 2_\flat"] & S^1 \vee C(\eta) \arrow[r, "\eta\vee(-2^\sharp)"'] & S^0
\end{tikzcd}$$

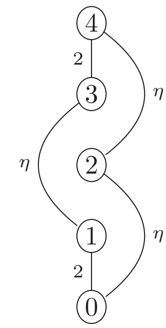
The mapping sequences for $\eta \vee 2^\sharp$ and $\eta \vee 2_\flat$ together yield the following diagram of solid and dashed arrows.



This shows the existence of a map θ which is well-defined up to indeterminacy which lies in the image of

$$\pi_3(S^0)/[\eta\pi_3(S^1) + 2\pi_3(S^0)] = \pi_3(S^0)/2\pi_3(S^0) \cong \mathbb{Z}_2,$$

so there are two choices of such a map θ up to homotopy. Because of the η component on the 2-sphere, the mapping cone of θ has the form



and its cohomology is the Joker $\mathcal{A}(1)$-module. The $\mathcal{A}$-module structure has a Sq^4 action between degrees 0 and 4 and this could be zero or non-zero. Each of these possibilities can occur, depending on which of the two of choices for θ is made. Putting all this together with the algebraic identity (1.1) we obtain the following.

Theorem 4.1. *There are two equivalence classes of finite* 2*-local CW spectra,* J_0 *and* J_1*, whose cohomology realise the* $\mathcal{A}$*-modules* Joker_0 *and* Joker_1*. Up to suspension,* J_0 *and* J_1 *are Spanier-Whitehead dual, i.e.,*

$$DJ_0 \simeq \Sigma^{-4} J_1.$$

Up to degree 12, the Adams E_2-terms for such Joker spectra are almost identical, differing only by an h_0 multiplication in the 6-column and having no non-trivial differentials.

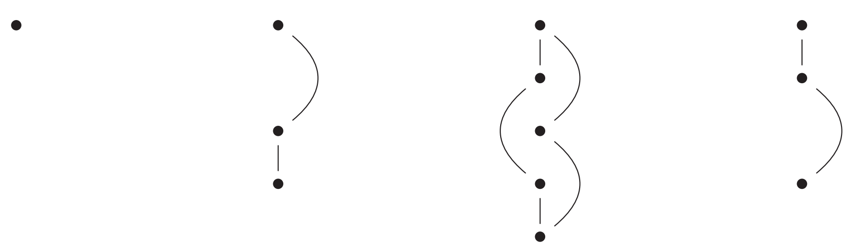
Here is a useful consequence of the existence of such Joker spectra; we assume this was known to Mark Mahowald but have not been able to locate an explicit statement on the existence of Joker spectra in his published work – however, see Remark 4.3 and also [**11**, Section 1.7].

Corollary 4.2. *The* (-1)*-connected cover of* $k\mathrm{O}$ *satisfies*

$$k\mathrm{O} \wedge \Sigma^2 J_0 \sim k\mathrm{O}\langle 2\rangle \sim k\mathrm{O} \wedge \Sigma^2 J_1.$$

It is well known that the other spectra which appear in the Whitehead tower of $k\mathrm{O}$ can all be defined in terms of $k\mathrm{O}$-module spectra of the form $k\mathrm{O} \wedge W$ where the $\mathcal{A}(1)^*$-module $H^*(W)$ has one of the following forms.

$H^*(W_0)$ $H^*(W_1)$ $H^*(W_2)$ $H^*(W_4)$

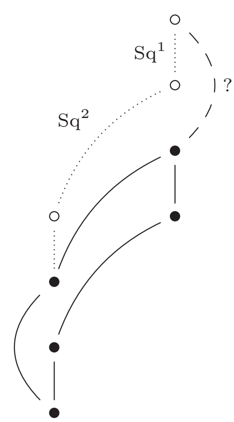


In general, when $r = 0, 1, 2, 4$ and $m \geqslant 0$,

$$k\mathrm{O}\langle 8m + r\rangle \sim k\mathrm{O} \wedge \Sigma^{8m} W_r.$$

For more details see [**12**, **13**].

Remark 4.3. A spectrum whose cohomology agrees with $\mathcal{A}(1)$ as an $\mathcal{A}(1)$-module (referred to as a 'space' in [**13**, Remark 1.6]) can be constructed using our Joker spectra. The following construction makes use of detailed information on homotopy groups that can be read off from Adams spectral sequence charts. Starting with J being either of J_0 or J_1 we find that there is a generator u of $\pi_2(J) \cong \mathbb{Z}_{(2)}$ in Adams filtration 1 (this is a manifestation of v_1 and has degree 2 on the 2-cell). Also, $\eta u = 0$ so u extends to a map $S^2 \cup_\eta e^4 \to J$. As $\pi_4(J) = 0$, this also extends to a map $f\colon S^2 \cup_\eta e^4 \cup_2 e^3 \to J$ where the attaching map of the 3-cell is yet another avatar of v_1. The cohomology of the mapping cone $C(f)$ has basis elements in the same degrees as $\mathcal{A}(1)$ and all but one Steenrod operation (indicated by the dashed line below) are clear from the above description.



The relation $\mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1$ shows that this is indeed a Sq^2 and therefore as $\mathcal{A}(1)$-modules, $H^*(C(f)) \cong \mathcal{A}(1)$.

Of course, the action of $\mathcal{A}$ on $H^*(J)$ extends to one on $H^*(C(f))$, thus giving at least two different $\mathcal{A}$-module structures on $\mathcal{A}(1)$. In [**9**, Theorem 1.4], Davis and Mahowald gave a different construction realising all four of the possible $\mathcal{A}$-module structures known to exist.

For small n we can realise $\mathrm{Joker}(n)_0$ and $\mathrm{Joker}(n)_1$ as the cohomology of spectra.

Theorem 4.4. *For $n = 2, 3$ there are finite 2-local CW spectra, $J(n)_0$ and $J(n)_1$, whose cohomology restricts to $\mathcal{A}(n)$-modules isomorphic to* $\mathrm{Joker}(n)$. *These realise the two $\mathcal{A}$-module structures extending the two dual $\mathcal{A}(n)$-module structures. Up to suspension, $J(n)_1$ can be taken to be the Spanier-Whitehead dual of $J(n)_0$.*

Proof. The approach of Section 4 also works using the Toda brackets $\langle \eta, \nu, \eta\rangle$ and $\langle \nu, \sigma, \nu\rangle$ given in (3.1).

The case $n = 2$:

Using ideas and notation from Section 3 we can form a map

$$S^3 \vee C(\Sigma\nu) \xrightarrow{\nu\vee(-\eta^\sharp)} S^0,$$

whose mapping cone fits into a cofibre sequence

$$S^0 \to C(\nu \vee (-\eta^\sharp)) \to S^4 \vee C(\Sigma^2\nu) \to S^1.$$

The map

$$S^7 \xrightarrow{\Sigma^4\nu\vee(\Sigma^5\eta)_\flat} S^4 \vee C(\Sigma^2\nu)$$

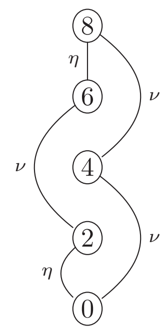
projects to

$$(\Sigma\nu^2 - \eta^\sharp(\Sigma^5\eta)_\flat)\colon S^7 \to S^1,$$

which is null homotopic as $\langle \eta, \nu, \eta\rangle = \{\nu^2\}$. Hence $\Sigma^4\nu \vee (\Sigma^5\eta)_\flat$ factors through a map

$$\theta'\colon S^7 \to C(\nu \vee (-\eta^\sharp)),$$

whose mapping cone has the following form.



The indeterminacy in θ' is

$$\pi_7(S^0)/[\nu\pi_4(S^0) + \eta\pi_6(S^0)] = \pi_7(S^0)/\{0\} \cong \mathbb{Z}_8.$$

The case $n = 3$:

A similar argument works and we obtain the desired spectrum as the mapping cone of a map

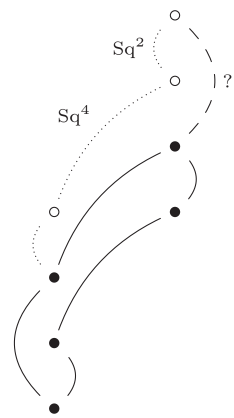
$$\theta''\colon S^{15} \to C(\sigma \vee (-\nu^\sharp)).$$

The indeterminacy in θ'' is

$$\pi_{15}(S^0)/[\sigma\pi_8(S^0) + \nu\pi_{12}(S^0)] = \pi_{15}(S^0)/\{0\} \cong \mathbb{Z}_{32}. \qquad \square$$

Remark 4.5. In similar fashion to the construction of a realisation of $\mathcal{A}(1)$ described in Remark 4.3, we can use either of the spectra $J(2)_0$ or $J(2)_1$ to build a spectrum whose cohomology realises the double $\Delta\mathcal{A}(1)$. In [**15**], such a spectrum is denoted $D\mathcal{A}(1)$, but this clashes with standard notation for Spanier-Whitehead duals so we avoid using it here. We sketch the details, making use of the information that can be read off of Adams spectral sequence charts.

Choose $J(2)$ to be either $J(2)_0$ or $J(2)_1$. We start with a map $S^5 \to J(2)$ realising the generator of $\pi_5(J(2)) \cong \mathbb{Z}_2$ (this has Adams filtration 1); since $\nu\pi_5(J(2)) = \{0\}$, this extends to a map $S^5 \cup_\nu e^9 \to J(2)$. As $\pi_{10}(J(2)) = \{0\}$ there is an extension to a map $g\colon S^5 \cup_\nu e^9 \cup_\eta e^{11} \to J(2)$ and the cohomology of its mapping cone has the following form where short/long lines indicate $\mathrm{Sq}^2/\mathrm{Sq}^4$ actions.



Using the type (B) Wall relation [**18**] we see that $\mathrm{Sq}^4\mathrm{Sq}^4 + \mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^2$ acts trivially on Joker(2) and so the dashed line must be a non-trivial Sq^4 action. Therefore $H^*(C(g)) \cong \Delta\mathcal{A}(1)$ as $\mathcal{A}(2)$-modules. Of course there are two possible $\mathcal{A}$ actions depending on which choice of $J(2)$ we make giving different Sq^{16} actions.

An attempt at a direct analogue of the preceding argument for the next case runs into difficulties as $\pi_{18}(J(3)) \neq \{0\}$ when $J(3)$ is either of $J(3)_0$ or $J(3)_1$.

In the other direction we have some non-existence results.

Theorem 4.6. *For $n \geqslant 4$, there is no finite 2-local CW spectrum whose cohomology restricts to an $\mathcal{A}(n)$-module isomorphic to* Joker(n).

When $n \geqslant 5$ such a spectrum would violate Adams' Hopf invariant 1 theorem because of the large gap between the two elements of lowest degrees. However, in all cases we can use the precise statement of the following crucial result on the factorisation of primary operations. Here X is a connective spectrum and we explain the notation after the statement.

Theorem 4.7 (Adams [**1**, Theorem 4.6.1])**.** *Let $k \geqslant 3$ and suppose that $u \in H^m(X)$ for $m > 0$ satisfies $\mathrm{Sq}^{2^r} u = 0$ for $0 \leqslant r \leqslant k$. Then*

$$\mathrm{Sq}^{2^{k+1}} u \equiv \sum_{\substack{0 \leqslant i \leqslant j \leqslant k \\ j \neq i+1}} \alpha_{i,j,k}\Phi_{i,j} u \quad \text{(mod indeterminacy)}.$$

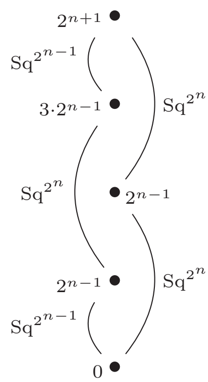
In this result, the secondary operation $\Phi_{i,j}$ has degree $2^i + 2^j - 1$, and the primary operation $\alpha_{i,j,k} \in \mathcal{A}$ has degree $2^{k+1} - 2^i - 2^j + 1$. The indeterminacy is the sum of the indeterminacies of all the $\Phi_{i,j}$ appearing and has form

$$\sum_{\substack{0\leqslant i\leqslant j\leqslant k\\ j\neq i+1}} \alpha_{i,j,k} Q^*(X;i,j),$$

for certain subgroups $Q^*(X;i,j) \subseteq H^*(X)$. Finally, for each pair i,j occurring,

$$1 \leqslant \deg \alpha_{i,j,k} \leqslant 2^{k+1} - 1.$$

Proof of Theorem 4.6. Let $n \geqslant 4$ and suppose that a Joker spectrum J exists for this n.



Consider the non-zero element u in degree 2^{n-1}. Taking $k = n-1$, we can apply Theorem 4.7. Carefully examining the possible terms in the sum we find that they are all 0, and similarly so is the indeterminacy. The conclusion is that $\mathrm{Sq}^{2^n} u = 0$, contradicting the assumptions on J. $\square$

5. Some unstable realisations

Now we turn to the question of unstable realisations, i.e., as the cohomology of spaces. If X is a 2-local space whose cohomology $\widetilde{H}^*(X)$ is isomorphic to $\mathrm{Joker}^*[n]$ as an $\mathcal{A}(1)$-module then $n \geqslant 2$ since Sq^2 acts non-trivially on the bottom generator. Similarly, realising the $\mathcal{A}$-module $\mathrm{Joker}_1[n]$ unstably requires that $n \geqslant 4$.

Theorem 5.1. *There are finite 2-local CW complexes X_2 and X_4 such that as $\mathcal{A}$-modules,*

$$\widetilde{H}^*(X_2) \cong \mathrm{Joker}_0^*[2], \quad \widetilde{H}^*(X_4) \cong \mathrm{Joker}_1^*[4].$$

Proof. Corollary 4.2 suggests looking for an unstable realisation of the Joker in the space $\underline{k\mathrm{O}\langle 2\rangle}_0 = B\mathrm{SO}$. However, the cohomology of this is too large in low degrees, instead we look at $B\mathrm{SO}(3)$.

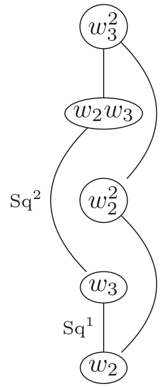
Recall the Wu formula

$$\mathrm{Sq}^r w_m = w_r w_m + \sum_{1\leqslant i\leqslant r} \binom{r-m}{i} w_{r-i} w_{m+i}.$$

Using this, in $H^*(B\mathrm{SO}(3)) = \mathbb{F}_2[w_2, w_3]$ we obtain

$$\mathrm{Sq}^1 w_2 = w_3, \quad \mathrm{Sq}^2 w_3 = 0.$$

Thus we obtain a copy of the Joker in the $\mathcal{A}(1)$-module $H^*(B\mathrm{SO}(3))$.



However, $H^6(B\mathrm{SO}(3)) = \mathbb{F}_2\{w_3^2, w_2^3\}$, so we next remove the additional generator by considering the fibre of the map classifying w_2^3,

$$B\mathrm{SO}(3) \xrightarrow{w_2^3} K(\mathbb{F}_2, 6),$$

which we will denote by $B\mathrm{SO}(3)\{w_2^3\}$. Calculating $H^*(B\mathrm{SO}(3)\{w_2^3\})$ using the Serre spectral sequence or the Eilenberg-Moore spectral sequence we find that when $k \leqslant 6$,

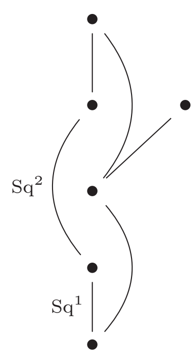
$$H^k(B\mathrm{SO}(3)\{w_2^3\}) \cong (\mathbb{F}_2[w_2, w_3]/(w_2^3))^k,$$

so taking the 6-skeleton of a minimal CW realisation (in the sense of [**3**, Section 3] for example) we obtain an isomorphism of $\mathcal{A}(1)$-modules

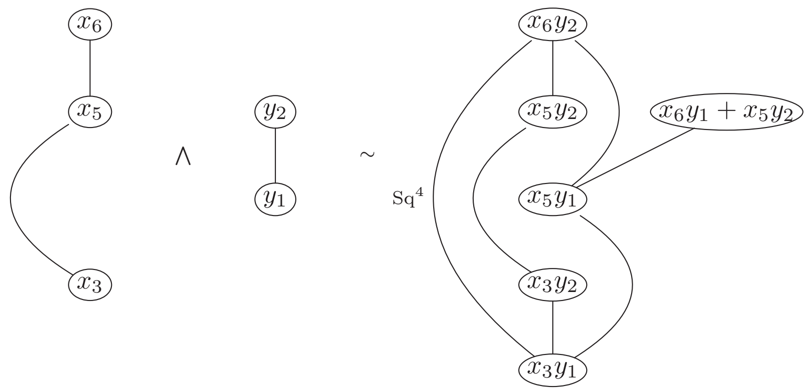
$$H^*(B\mathrm{SO}(3)\{w_2^3\}^{[6]}) \cong \mathrm{Joker}_0[2].$$

To realise $\mathrm{Joker}_1[4]$, we start with an unstable complex $S^3 \cup_{\eta_3} e^5 \cup_2 e^6$ which exists since the suspension of the Hopf map $S^3 \to S^2$ gives an element $\eta_3 \in \pi_4(S^3)$ of order 2, see [**17**, Chapter V]. Smashing with the Moore space $S^1 \cup_2 e^2$ we obtain a CW complex X' whose cohomology as an $\mathcal{A}(1)$-module realises the 4-fold suspensions of the 'whiskered Joker' module

$$\mathrm{Joker}' = \mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\} = \mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^2\,\mathrm{Sq}^3\}.$$



Labelling cells and cohomology generators in the obvious way, X' has the following cell diagram.



Notice that as well as the Sq^1 and Sq^2 actions we also have $\mathrm{Sq}^4(x_3y_1) = x_6y_2$ so this agrees with $\mathrm{Joker}'_1[4]$ as an $\mathcal{A}$-module.

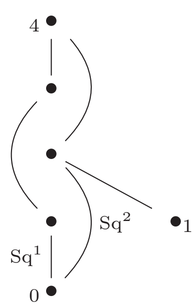
We will begin by showing that there is a factorisation

$$\begin{tikzcd} X' \arrow[r] \arrow[rr, bend left] & \underline{k\mathrm{O}_7} \arrow[r] & K(\mathbb{F}_2, 7) \end{tikzcd}$$

of the map classifying $x_6y_1 + x_5y_2 \in H^7(X')$. We will do this by producing a map of spectra $\Sigma^\infty X' \to \Sigma^7 k\mathrm{O}$ by dualising a map

$$S^1 \to k\mathrm{O} \wedge \Sigma^8 D\Sigma^\infty X' \sim k\mathrm{O} \wedge W'',$$

where D denotes Spanier-Whitehead dual and W'' is a CW spectrum whose cohomology realises the dual whiskered Joker $\mathcal{A}(1)$-module $\mathrm{Joker}''_1 = \mathrm{DJoker}'_1$ shown in the following diagram.



Since we are interested in elements of $\pi_1(k\mathrm{O} \wedge W'')$, we need to consider the $t - s = 1$ column in the Adams spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(k\mathrm{O} \wedge W''), \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(H^*(W''), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(k\mathrm{O} \wedge W'')$$

and a portion of the E_2-term is shown in Figure 1. As the generator in $\mathrm{E}_2^{0,1}$ cannot support a differential there is a non-trivial element of $\pi_1(k\mathrm{O} \wedge W'')$ detected in the zero line by the only $\mathcal{A}(1)$-indecomposable element of $H^1(W'')$. Hence there is a dual element of $k\mathrm{O}^7(X')$ with the desired properties.

Now take a minimal CW complex equivalent to the fibre of the map $X' \to \underline{k\mathrm{O}_7}$ and let X_4 be its 8-skeleton. By a straightforward calculation with either of the

Serre or Eilenberg-Moore spectral sequences and making use of the $k\mathrm{O}$ results of Examples A.3, we find that $H^*(X)$ realises the $\mathcal{A}$-module $\mathrm{Joker}_1[4]$. □

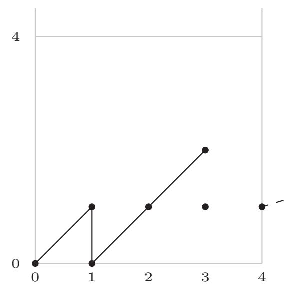


Figure 1: $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathrm{Joker}'', \mathbb{F}_2)$: $0 \leqslant s \leqslant 4$ and $0 \leqslant t - s \leqslant 4$.

Theorem 5.2. *There are finite 2-local CW complexes Y_4 and Y_8 such that as $\mathcal{A}$-modules,*

$$\widetilde{H}^*(Y_4) \cong \mathrm{Joker}(2)^*_0[4], \quad \widetilde{H}^*(Y_8) \cong \mathrm{Joker}(2)^*_1[8].$$

Proof. A similar construction to that of X_2 starting with $B\mathrm{SU}(3)$ leads to an unstable realisation of $\mathrm{Joker}(2)_0[4]$.

We will realise $\mathrm{Joker}(2)_1[8]$ using a similar approach to that for X_4. By Toda [**17**, Proposition 5.8], $\pi_{10}(S^6)$ is trivial, so the suspensions of the Hopf maps give elements $\eta_9 \in \pi_{10}(S^9)$ and $\nu_6 \in \pi_9(S^6)$ which satisfy

$$0 = \nu_6 \circ \eta_9 \in \pi_{10}(S^6).$$

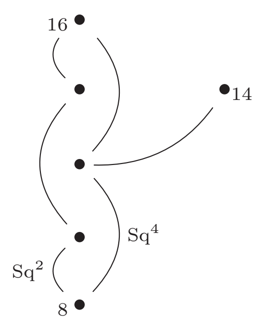
Hence we can form $S^5 \cup_{\nu_5} e^9 \cup_{\eta_9} e^{11}$ and $S^3 \cup_{\eta_3} e^5$. By smashing these together we obtain a CW complex

$$Y' = (S^5 \cup_{\nu_5} e^9 \cup_{\eta_9} e^{11}) \wedge (S^3 \cup_{\eta_3} e^5),$$

whose cohomology realises the $\mathcal{A}$-module with non-trivial Sq^8-action and is the 8-fold suspension of the whiskered double Joker cyclic $\mathcal{A}(2)$-module

$$\mathrm{Joker}(2)'_1 = \mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{P}^0_1, \mathrm{P}^1_1, \mathrm{P}^1_2, \mathrm{Sq}^4\,\mathrm{Sq}^6\}.$$

$\mathrm{Joker}(2)'_1[8]$

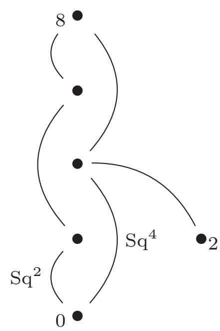


We would like to define a map $Y' \to \underline{\mathrm{tmf}}_{14}$ so that the cohomology class in $H^{14}(\underline{\mathrm{tmf}}_{14})$

carried on the bottom cell is mapped to $\mathrm{Sq}^2\,\mathrm{Sq}^4\,y_8$ by the induced homomorphism, where $y_8 \in H^8(Y)$ is the generator. Such a map corresponds to a map of spectra $\Sigma^\infty Y' \to \Sigma^{14}\mathrm{tmf}$ or equivalently a map

$$S^0 \to \Sigma^{14}(D\Sigma^\infty Y') \wedge \mathrm{tmf} \sim \Sigma^{-2}Z'' \wedge \mathrm{tmf},$$

where Z'' is a CW spectrum whose cohomology realises the other whiskered double Joker $\mathcal{A}(2)$-module $\mathrm{Joker}(2)''_1$ shown in the following diagram.



Since we are interested in elements of $\pi_2(\mathrm{tmf} \wedge Z'')$, we need to consider the $t - s = 2$ column in the Adams spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(\mathrm{tmf} \wedge Z''), \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{A}(2)}^{s,t}(H^*(Z''), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathrm{tmf} \wedge Z'')$$

and a portion of the E_2-term is shown in Figure 2. As the generator in $\mathrm{E}_2^{0,2}$ cannot support a differential this shows that there is a suitable element of $\pi_2(\mathrm{tmf} \wedge Z'')$ and hence of $\mathrm{tmf}^{14}(Y')$.

Now consider the fibre of the above map $Y' \to \underline{\mathrm{tmf}}_{14}$. By a spectral sequence calculation and making use of the tmf results of Examples A.3 we see that its cohomology agrees with $\mathrm{Joker}(2)_1[8]$ up to degree 21. The 16-skeleton of a minimal CW realisation of this fibre is a CW complex Y_8 whose cohomology as an $\mathcal{A}$-module agrees with $\mathrm{Joker}(2)_1[8]$. $\square$

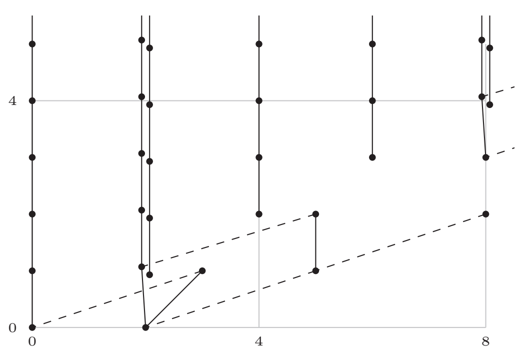


Figure 2: $\mathrm{Ext}_{\mathcal{A}(2)}^{s,t}(\mathrm{Joker}(2)'', \mathbb{F}_2)$: $0 \leqslant s \leqslant 5$ and $0 \leqslant t - s \leqslant 8$.

It is unclear how to realise $\mathrm{Joker}(3)_0[8]$ since there is no obvious analogue of $B\mathrm{SO}(3)$ and $B\mathrm{SU}(3)$ which appears relevant. Similarly, our argument for X_4 and Y_8 has no obvious generalisation since because of the non-existence of suitable elements of Hopf invariant 1 there is no spectrum playing an analogous rôle to $k\mathrm{O}$ and tmf in the last steps.

Concluding remarks

The appearance of the elements of Kervaire invariant 1 in our realisations of Joker modules raises the question of whether there other $\mathcal{A}(n)$-modules which admit realisations when θ_n exists, i.e., when $n = 4$, 5 and possibly 6. In particular, by [**20**, Theorem 5.2],

$$\{\theta_4\} = \langle 2, \sigma^2 + \kappa, 2\sigma, \sigma\rangle,$$

while older work of Barratt, Mahowald and Tangora, and Kochman shows that

$$\{\theta_4\} = \langle 2, \sigma^2, 2, \sigma^2\rangle = \langle 2, \sigma^2, \sigma^2, 2\rangle = \langle 2\sigma, \sigma, 2\sigma, \sigma\rangle = \langle 2, \sigma^2, 2\sigma, \sigma\rangle.$$

These suggest the intriguing possibility that appropriate constructions associated with such 4-fold Toda brackets might lead to realisation results for some interesting $\mathcal{A}(4)$-modules.

Appendix A. Some connectivity results

Let p be a prime and let $f\colon X \to Y$ be a map between two finite type p-local connective spectra or spaces which are simply connected or at least have abelian fundamental groups.

Recall that f is called an *n-equivalence* if $f_*\colon \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$ and an epimorphism if $k = n$; this is equivalent to the mapping cone C_f being n-connected. It is well-known that the following are also equivalent conditions:

- $f_*\colon H_k(X; \mathbb{Z}_{(p)}) \to H_k(Y; \mathbb{Z}_{(p)})$ is an isomorphism if $k < n$ and an epimorphism if $k = n$.
- $f_*\colon H_k(X; \mathbb{F}_p) \to H_k(Y; \mathbb{F}_p)$ is an isomorphism if $k < n$ and an epimorphism if $k = n$.

The next result relates connectivity information for spectra and their associated infinite loop spaces. Although such results are undoubtedly standard we are not aware of convenient references and we use them to establish Examples A.3.

We will denote the m-th space in a spectrum X by $\underline{X}_m = \Omega^\infty\Sigma^m X$. If $f\colon X \to Y$ is a map of (-1)-connected spectra then for each $m \geqslant 0$ there is an induced infinite loop map $f_m\colon \underline{X}_m \to \underline{Y}_m$.

Lemma A.1. *Let $f\colon X \to Y$ be an n-equivalence. Then for each $m \geqslant 1$, $f_m\colon \underline{X}_m \to \underline{Y}_m$ is an $m+n$-equivalence, hence $(f_m)_*\colon H_k(\underline{X}_m; \mathbb{F}_p) \to H_k(\underline{Y}_m; \mathbb{F}_p)$ is an isomorphism if $k < m+n$ and an epimorphism if $k = m+n$.*

Here is a sample application; we only state this for the prime 2, but a similar result also holds for odd primes.

Corollary A.2. *Take $p = 2$ and let X be a (-1)-connected spectrum and suppose that $\pi_0(X)$ is a cyclic $\mathbb{Z}_{(2)}$-module with generator given by a map $j\colon S^0 \to X$. If j is an n-equivalence then for each $m > n$, and $m < k \leqslant m + n$,*

$$H_k(\underline{X}_m; \mathbb{F}_2) = 0.$$

Proof. Recall that for $m \geqslant 1$, the homology of $\underline{S^0}_m = \mathrm{Q}S^m$ is given by

$$H_*(\mathrm{Q}S^m; \mathbb{F}_2) = \mathbb{F}_2[\mathrm{Q}^I x_m : I \text{ admissible, } \mathrm{exc}(I) > m],$$

where $x_m \in H_m(\mathrm{Q}S^m; \mathbb{F}_2)$. Thus the three elements of lowest positive degree are x_m, x_m^2 and $\mathrm{Q}^{2m+1}x_m$ in degrees m, $2m > m + n$ and $2m + 1 > m + n$ respectively.

The infinite loop map j_m induces an algebra homomorphism $(j_m)_*$ over the Dyer-Lashof algebra. By assumption on j,

$$(j_m)_*\colon H_k(\mathrm{Q}S^m; \mathbb{F}_2) \to H_k(\underline{X}_m; \mathbb{F}_2)$$

is an isomorphism when $k < m + n$ and an epimorphism when $k = m + n$. □

Thus the lowest degree non-zero element of $H_*(\underline{X}_m; \mathbb{F}_2)$ not in the image of $(j_m)_*$ occurs in some degree $k_0 \geqslant m + n + 1$ where $k_0 - m$ is also the smallest degree for which

$$\mathrm{coker}[j_*\colon H_k(S^0; \mathbb{F}_2) \to H_k(X; \mathbb{F}_2)] \neq 0.$$

Examples A.3. We set $H_*(-) = H_*(-; \mathbb{F}_2)$.

Recall that

$$H_*(k\mathrm{O}) = \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots] \subseteq \mathcal{A}_*, \quad H_*(\mathrm{tmf}) = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \ldots] \subseteq \mathcal{A}_*.$$

Thus on $H_*(-)$ the units $j^{k\mathrm{O}}\colon S \to k\mathrm{O}$ and $j^{\mathrm{tmf}}\colon S \to \mathrm{tmf}$ induce homomorphisms whose cokernels $\mathrm{coker}\, j_*^{k\mathrm{O}}$ and $\mathrm{coker}\, j_*^{k\mathrm{O}}$ have non-zero elements of lowest degrees 4 and 8 respectively. Thus $j^{k\mathrm{O}}$ is a 3-equivalence and j^{tmf} is a 7-equivalence.

For $4 \leqslant m < k \leqslant m + 3$,

$$H_k(\underline{k\mathrm{O}}_m) = 0,$$

while for $8 \leqslant m < k \leqslant m + 7$,

$$H_k(\underline{\mathrm{tmf}}_m) = 0.$$

The results for $k\mathrm{O}$ can also be deduced from work of Dena Cowen Morton [**8**] but as far as we know the algebra structure of the Hopf ring for tmf has not been determined.

References

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