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HIGHER HOMOTOPY COMMUTATIVITY IN LOCALIZED LIE GROUPS AND GAUGE GROUPS

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Abstract

The first aim of this paper is to study the p-local higher homotopy commutativity of Lie groups in the sense of Sugawara. The second aim is to apply this result to the p-local higher homotopy commutativity of gauge groups. Although the higher homotopy commutativity of Lie groups in the sense of Williams is already known, the higher homotopy commutativity in the sense of Sugawara is necessary for this application. The third aim is to resolve the 5-local higher homotopy noncommutativity problem of the exceptional Lie group G_2, which has been open for a long time.

1. Introduction

Let G be a compact connected Lie group. It is well known that the p-localization $G_{(p)}$ decomposes into a product of spaces such that the number of the factor spaces is not larger than the rank of G and the factor spaces become p-local spheres as p gets large enough. Then we can say that the homotopy type of $G_{(p)}$ becomes simpler as p gets larger. Now it is natural to ask how the multiplication of $G_{(p)}$ changes as p grows. McGibbon [**McG84**] determined the exact values of p such that $G_{(p)}$ is homotopy commutative. In particular, it turned out that $G_{(p)}$ becomes homotopy commutative if p gets large enough, so as far as we consider homotopy commutativity, we can say that the multiplication of $G_{(p)}$ becomes simpler as p grows. One way to refine McGibbon's work is to consider the higher homotopy commutativity, that is, to consider how high the homotopy commutativity of $G_{(p)}$ gets as p grows. Saumell [**Sau95**] went along this line to study the multiplication of $G_{(p)}$ and showed that the homotopy commutativity of $G_{(p)}$ gets higher as p grows.

There are two major definitions of higher homotopy commutativities; one is *Williams C_k-space* [**Wil69**] and the other is *Sugawara C_k-space* [**Sug61, McG89**]. The definition of Williams C_k-space is done by explicit conditions on higher homotopies parametrized by permutohedra, so it is somewhat intuitive. On the other hand, the definition of Sugawara C_k-space is rather obstruction theoretic, so it is more

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applicable to practical problems. There is an implication [**McG89**, Proposition 6]

$$\text{Sugawara } C_k\text{-space } \Rightarrow \text{Williams } C_k\text{-space}.$$

For the converse, there is no known implication of Williams C_k-space on Sugawara C_l-space in general even if $k \neq l$. The only known counterexample for the converse is the case when $k = \ell = \infty$ [**McG89**, Example 5].

In the above mentioned result of Saumell, the higher homotopy commutativity is chosen to be the one in the sense of Williams, so it does not imply the one in the sense of Sugawara. To state the results of McGibbon and Saumell, we need to recall the definition of the type of a Lie group. Given a compact connected Lie group G, the rational cohomology is the exterior algebra

$$H^*(G;\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_1,\dots,x_\ell)$$

by the Hopf theorem, where $x_i \in H^{2n_i-1}(G;\mathbb{Q})$ and $n_1 \leqslant \cdots \leqslant n_\ell$. We call the sequence of the numbers $\{n_1,\dots,n_\ell\}$ the *type* of the Lie group G.

Theorem 1.1 (McGibbon and Saumell)**.** *Given a compact connected simple Lie group G of type $\{n_1,\dots,n_\ell\}$, a prime p and an integer $k \geqslant 2$, the following assertions hold:*

1. *If $p > kn_\ell$, then $G_{(p)}$ is a Williams C_k-space.*
2. *If $p < kn_\ell$, then $G_{(p)}$ is not a Williams C_k-space, except in the case when (G,p,k) is $(\mathrm{Sp}(2),3,2)$ or $(\mathrm{G}_2,5,k)$ such that $k \leqslant 4$.*

The first aim of this paper is to refine McGibbon's result by considering the higher homotopy commutativity in the sense of Sugawara.

Theorem 1.2. *Let G be a compact connected Lie group G of type $\{n_1,\dots,n_\ell\}$, p a prime and k a positive integer. If $p > kn_\ell$, then the p-localization $G_{(p)}$ is a Sugawara C_k-space.*

In the proof, we analyze the A_k-type of G in the sense of Stasheff [**Sta63**]. The key property of G is that G has the p-local A_k-type of the product of spheres (Proposition 4.2).

Let $P \to B$ be a principal G-bundle. The *gauge group* $\mathcal{G}(P)$ of P is the topological group consisting of bundle maps $P \to P$ covering the identity on B. For the homotopy commutativity of gauge groups, little is known. For example, see [**CS95, KKT13**]. The second aim of this paper is to study the higher homotopy commutativity of gauge groups in both the sense of Sugawara and Williams by applying Theorem 1.2. We stress that the higher homotopy commutativity in the sense of Williams is not sufficient for this application. Let $EG \to BG$ be the universal bundle of G and $E_nG \to B_nG$ be the restriction over the n-th projective space $B_nG \subset BG$.

Theorem 1.3. *Let G be a compact connected simple Lie group of type $\{n_1,\dots,n_\ell\}$ and p a prime. Then, given positive integers n and k, the following assertions hold:*

1. *If $p > (n+k)n_\ell$, then $\mathcal{G}(E_nG)_{(p)}$ is a Sugawara C_k-space.*
2. *If $(n+1)n_\ell < p < (n+k)n_\ell$, then $\mathcal{G}(E_nG)_{(p)}$ is not a Williams C_k-space.*

Remark 1.4. Since the gauge group $\mathcal{G}(P)$ need not be connected, we define its p localization by $\mathcal{G}(P)_{(p)} = \Omega(B\mathcal{G}(P)_{(p)})$.

To prove this theorem, we introduce a new higher homotopy commutativity $C(k_1,\dots,k_r)$-*space* which is a generalization of $C(k,\ell)$-space [**KK10**]. This result proves the conjecture by the third author [**Tsu16**, Conjecture 7.8] for simple Lie groups. For general principal bundles, we show the following.

Theorem 1.5. *Let G be a compact connected simple Lie group of type $\{n_1,\dots,n_\ell\}$ and p a prime. Given a principal G-bundle P over a connected finite complex B, the p-localized gauge group $\mathcal{G}(P)_{(p)}$ is a Sugawara C_k-space if $p > (\operatorname{cat} B + k)n_\ell$.*

When B is a sphere, this criterion is not sharp. We also show the following better criterion which refines the result of Kishimoto–Kono–Theriault [**KKT13**].

Theorem 1.6. *Let G be a compact connected simple Lie group of type $\{n_1,\dots,n_\ell\}$ and p a prime. If $p \geqslant kn_\ell + n_i$, then the p-localized gauge group $\mathcal{G}(P)_{(p)}$ of any principal G-bundle P over S^{2n_i} is a Sugawara C_k-space.*

In Theorem 1.1 (2), there are exceptional cases for $\mathrm{Sp}(2)_{(3)}$ and $(\mathrm{G}_2)_{(5)}$. $\mathrm{Sp}(2)_{(3)}$ and $(\mathrm{G}_2)_{(5)}$ are known to be homotopy commutative [**McG84**]. But the remaining cases for $(\mathrm{G}_2)_{(5)}$ has been open. The third aim of this paper is to resolve this problem.

Theorem 1.7. *The localized Lie group $(\mathrm{G}_2)_{(5)}$ is not a Williams C_3-space.*

This result provides a counterexample to the conjecture about the higher homotopy commutativity of the $S^{2p+1}_{(p)}$-bundle $B_1(p)$ over $S^3_{(p)}$ by Hemmi [**Hem91**, p. 107].

This paper is organized as follows. In Section 2, we recall A_n-spaces and A_n-maps. In Section 3, we study the characterizations and properties of Sugawara C_k-spaces and $C(k_1,\dots,k_r)$-spaces. In Section 4, we investigate the A_k-types of localized compact connected simple Lie groups. Theorem 1.2 is also shown there. In Section 5, we recall the theory of gauge groups. In Section 6, we study the higher homotopy commutativity of gauge groups and prove Theorems 1.3, 1.5 and 1.6. In Section 7, we prove Theorem 1.7 by computing Chern characters.

2. Higher homotopy associativity

In this section, we recall the theory of higher homotopy associativity we need in this paper. Higher homotopy associativity is formulated by Stasheff [**Sta63**]. To describe it, we need the *associahedra* $\mathcal{K}_2,\mathcal{K}_3,\dots$. The i-th associahedron $\mathcal{K}_i$ is homeomorphic to the $(i-2)$-dimensional disk. The boundary sphere is exactly the union of the images of the boundary maps

$$\partial_k\colon \mathcal{K}_r \times \mathcal{K}_s \to \mathcal{K}_i$$

for $r+s-1=i$ and $1 \leqslant k \leqslant r$, each of which is an embedding into the boundary. The degeneracy maps

$$s_k\colon \mathcal{K}_i \to \mathcal{K}_{i-1}$$

for $1 \leqslant k \leqslant i$ are also defined. For details, see [**Sta63**].

Definition 2.1. Let G be a based space. Then a family of maps $\{m_i\colon \mathcal{K}_i \times G^{\times i} \to G\}_{i=2}^n$ is said to be an A_n-*form* on X if the following conditions are satisfied:

1. $m_2(*, x) = m_2(x, *) = x$,
2. $m_{r+s-1}(\partial_k(\rho, \sigma); x_1, \ldots, x_{r+s-1})$
$= m_r(\rho; x_1, \ldots, x_{k-1}, m_s(\sigma; x_k, \ldots, x_{k-s+1}), x_{k-s}, \ldots, x_{r+s-1})$,
3. $m_i(\rho; x_1, \ldots, x_i) = m_{i-1}(s_k\rho; x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_i)$ if $x_k = *$.

A pair $(G, \{m_i\})$ of a based space G and an A_n-form $\{m_i\}$ on it is called an *A_n-space.*

We also recall A_n-maps between A_n-spaces [**IM89**]. In the definition, we need the *multiplihedra* $\mathcal{J}_1, \mathcal{J}_2, \ldots$ The i-th multiplihedron is homeomorphic to the $(i-1)$-dimensional disk. The boundary sphere is exactly the union of the images of the boundary maps

$$\delta_k \colon \mathcal{J}_r \times \mathcal{K}_s \to \mathcal{J}_i$$

for $r + s - 1 = i$ and $1 \leqslant k \leqslant r$ and

$$\delta \colon \mathcal{K}_r \times \mathcal{J}_{s_1} \times \cdots \times \mathcal{J}_{s_r} \to \mathcal{J}_i$$

for $s_1 + \cdots + s_r = i$, each of which is an embedding into the boundary. The degeneracy maps

$$d_k \colon \mathcal{J}_i \to \mathcal{J}_{i-1}$$

for $1 \leqslant k \leqslant i$ are also defined. For details, see [**IM89**].

Definition 2.2. Let $(G, \{m_i\})$ and $(G', \{m'_i\})$ be A_n-spaces and $f \colon G \to G'$ a based map. Then a family of maps $\{f_i \colon \mathcal{J}_i \times G^{\times i} \to G'\}_{i=1}^n$ is said to be an *A_n-form* on f if the following conditions are satisfied:

1. $f_1 = f$,
2. $f_{r+s-1}(\delta_k(\rho, \sigma); x_1, \ldots, x_{r+s-1})$
$= f_r(\rho; x_1, \ldots, x_{k-1}, m_s(\sigma; x_k, \ldots, x_{k-s+1}), x_{k-s}, \ldots, x_{r+s-1})$,
3. $f_{s_1+\cdots+s_r}(\delta(\rho, \sigma_1, \ldots, \sigma_r); x_1, \ldots, x_{s_1+\cdots+s_r})$
$= m'_r(\rho; f_{s_1}(\sigma_1; x_1, \ldots, x_{s_1}), \ldots, f_{s_r}(\sigma_r; x_{s_1+\cdots+s_{r-1}+1}, \ldots, x_{s_1+\cdots+s_r}))$,
4. $f_i(\rho; x_1, \ldots, x_i) = f_{i-1}(d_k\rho; x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_i)$ if $x_k = *$.

A pair $(f, \{f_i\})$ of a based map f and an A_n-form $\{f_i\}$ on it is called an *A_n-map*. In particular, if the underlying map of an A_n-map is a homotopy equivalence, it is said to be an *A_n-equivalence.*

If $(f, \{f_i\})$ is an A_n-equivalence between non-degenerately based A_n-spaces G and H, then the homotopy inverse of f also admits an A_n-form [**IM89**]. The following lemma is not difficult to prove.

Lemma 2.3. *Let $(G, \{m_i\})$ and $(G', \{m'_i\})$ be A_n-spaces and $(f, \{f_i\}) \colon G \to G'$ an A_n-map. If $f' \colon G \to G'$ is a based map homotopic to f, then there is an A_n-form $\{f'_i\}$ on f' such that $(f', \{f'_i\})$ is homotopic to $(f, \{f_i\})$ as an A_n-map.*

If $(G, \{m_i^G\})$ and $(H, \{m_i^H\})$ are A_n-spaces, the product space $G \times H$ admits the *product A_n-form* $\{m_i^{G\times H}\}$ defined by

$$m_i^{G\times H}(\rho; (x_1, y_1), \ldots, (x_i, y_i)) = (m_i^G(\rho; x_1, \ldots, x_i), m_i^H(\rho; y_1, \ldots, y_i)).$$

We call $(G \times H, \{m_i^{G\times H}\}_i)$ the *product A_n-space* of $(G, \{m_i^G\})$ and $(H, \{m_i^H\})$.

Stasheff introduced [**Sta63**] the A_n*-structure* of an A_n-space, which is a kind of iterated Dold–Lashof construction or partial universal principal bundle. We reformulate it as follows.

Definition 2.4 (Stasheff)**.** Given a based space G, the following data is called an A_n*-structure* on G:

(i) a commutative ladder of based spaces

$$\begin{array}{ccccccc} E_0 & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} \\ \downarrow & & \downarrow & & & & \downarrow \\ B_0 & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_{n-1} \end{array}$$

where B_0 is contractible,

(ii) a homotopy equivalence $\eta\colon G \to E_0$,

(iii) a factorization $E_{i-1} \xrightarrow{h_0} D_{i-1} \xrightarrow{h} E_i$ through a contractible space D_{i-1} of the above map $E_{i-1} \to E_i$ for each i.

We say that the A_n-structure is *cofibrant* if the basepoint of G is non-degenerate, each h_0 is a cofibration and the induced map

$$B_{i-1} \cup_{E_{i-1}} D_{i-1} \to B_i$$

from the pushout is a homeomorphism. We say that the A_n-structure is *fibrant* if each map $E_i \to B_i$ is a fibration and each square in the condition (i) is a pullback.

Remark 2.5. While we used the terms *cofibrant* and *fibrant*, we do not insist on the existence of any model category structures of A_n-structures.

Definition 2.6. Given A_n-structures $\{E_i, B_i, D_i, \eta, h_0, h\}$, $\{E'_i, B'_i, D'_i, \eta', h'_0, h'\}$ of G, G' and a based map $f\colon G \to G'$, a family of maps

$$f^E\colon E_{i-1} \to E'_{i-1} \qquad f^B\colon B_{i-1} \to B'_{i-1} \qquad \text{and} \qquad f^D\colon D_{i-1} \to D'_{i-1}$$

is said to be an A_n*-structure* on f or a map between these A_n-structures if the following conditions are satisfied:

(i) these maps satisfy $f^E(E_i) \subset E'_i$, $f^B(B_i) \subset B'_i$ and $f^D(D_i) \subset D'_i$ for each i and the following diagram commutes:

$$\begin{array}{ccccccccc} E_{i-2} & \longrightarrow & D_{i-2} & \longrightarrow & E_{i-1} & \longrightarrow & B_{i-1} & \longleftarrow & B_{i-2} \\ \downarrow{\scriptstyle f^E} & & \downarrow{\scriptstyle f^D} & & \downarrow{\scriptstyle f^E} & & \downarrow{\scriptstyle f^B} & & \downarrow{\scriptstyle f^B} \\ E'_{i-2} & \longrightarrow & D'_{i-2} & \longrightarrow & E'_{i-1} & \longrightarrow & B'_{i-1} & \longleftarrow & B'_{i-2} \end{array}$$

(ii) the following diagram commutes up to homotopy:

$$\begin{array}{ccc} G & \xrightarrow{\eta} & E_0 \\ {\scriptstyle f}\downarrow & & \downarrow \\ G' & \xrightarrow{\eta'} & E'_0 \end{array}$$

If G is an A_n-space and the basepoint is non-degenerate, Stasheff [**Sta63**] constructed a cofibrant A_n-structure

$$\begin{array}{ccccccc} E_0G & \longrightarrow & E_1G & \longrightarrow & \cdots & \longrightarrow & E_{n-1}G \\ \downarrow & & \downarrow & & & & \downarrow \\ B_0G & \longrightarrow & B_1G & \longrightarrow & \cdots & \longrightarrow & B_{n-1}G \end{array}$$

as a variant of bar construction, where $B_0 = *$, each $B_{i-1} \to B_i$ is a closed cofibration, $E_0G = G$, $E_{i-1}G$ is contained in a contractible subset $D_{i-1}G$ of E_iG, D_0G is the reduced cone of G and each square is a pullback. We call it the *canonical A_n-structure* of G. The space E_iG has the homotopy type of the $(i+1)$-fold join $G^{*(i+1)}$ of G. The space B_iG is called the *i-th projective space*, where, in fact, the n-th projective space B_nG is also canonically defined as the mapping cone of $E_{n-1}G \to B_{n-1}G$. When $n = \infty$, the space $BG = \operatorname{colim}_n B_nG$ is the classifying space of G and $EG = \operatorname{colim}_n E_nG$ is contractible. We denote the canonical inclusion by $i_k\colon B_kG \to BG$. Note that each square is a homotopy pullback if G is looplike, where we say an A_n-space $(G, \{m_i\})$ $(n \geqslant 2)$ is *looplike* if the left and the right translations in $\pi_0(G)$ induced from m_2 are bijections. Moreover, if an A_n-map $G \to G'$ between A_n-spaces is given, then there is the canonical map between the canonical A_n-structures. This is obtained by Iwase–Mimura [**IM89**]. More explicit constructions of these A_n-structures can be found in [**Iwa**].

Example 2.7. If G is a non-degenerately based topological group, then the projection

$$EG \to BG$$

of the canonical A_∞-structure is a principal bundle. Thus it is fibrant.

Conversely, Stasheff [**Sta63**] also constructed an A_n-space from an A_n-structure.

Lemma 2.8. *Let $\{E_i, B_i, D_i, \eta, h_0, h\}$ be an A_n-structure of a based space G such that each square*

$$\begin{array}{ccc} E_{i-1} & \longrightarrow & E_i \\ \downarrow & & \downarrow \\ B_{i-1} & \longrightarrow & B_i \end{array}$$

is a homotopy pullback. Then, there exists a map from $\{E_i, B_i, D_i, \eta, h_0, h\}$ to a fibrant A_n-structure $\{\tilde{E}_i, B_i, \tilde{D}_i, \tilde{\eta}, \tilde{h}_0, \tilde{h}\}$ on G such that the underlying map is the identity on G.

Proof. One can find a commutative square

$$\begin{array}{ccc} E_{n-1} & \longrightarrow & \tilde{E}_{n-1} \\ \downarrow & & \downarrow \\ B_{n-1} & = & B_{n-1} \end{array}$$

such that $E_{n-1} \to \tilde{E}_{n-1}$ is a closed cofibration and a homotopy equivalence, and $\tilde{E}_{n-1} \to B_{n-1}$ is a fibration. Take $\tilde{E}_{i-1} \to B_{i-1}$ as the pullback of $\tilde{E}_{n-1} \to B_{n-1}$

along the map $B_{i-1} \to B_{n-1}$ and $\tilde{D}_{i-1}$ the pushout of $\tilde{E}_{i-1} \leftarrow E_{i-1} \to D_{i-1}$. By this construction, there are canonical maps $\tilde{E}_{i-1} \xrightarrow{\tilde{h}_0} \tilde{D}_{i-1} \xrightarrow{\tilde{h}} \tilde{E}_i$ and $\tilde{\eta}\colon G \to \tilde{E}_0$. It is easy to see that $\{\tilde{E}_i, B_i, \tilde{D}_i, \tilde{\eta}, \tilde{h}_0, \tilde{h}\}$ is the desired A_n-structure. □

We call it the *fibrant replacement* of an A_n-structure.

Proposition 2.9 (Stasheff)**.** *Given a fibrant A_n-structure $E = \{E_i, B_i, D_i, \eta, h_0, h\}$ of a non-degenerately based space G, there exist an A_n-form $\{m_i\}$ on G and a map from the canonical A_n-structure of $(G, \{m_i\})$ to E of which the underlying map is the identity on G. Moreover, such an A_n-space $(G, \{m_i\})$ is looplike.*

For maps between A_n-structures, Iwase–Mimura [**IM89**] proved the following proposition.

Proposition 2.10 (Iwase–Mimura)**.** *Let G and G' be non-degenerately based A_n-spaces and suppose G' is looplike. Denote the canonical A_n-structure of G by E and a fibrant replacement of the canonical A_n-structure of G' by $\tilde{E}'$. If a based map $f\colon G \to G'$ admits an A_n-structure $E \to \tilde{E}'$, then f admits an A_n-form.*

Combining with the fiber-cofiber argument, the following corollary follows.

Corollary 2.11. *Let G be a non-degenerately based A_n-space and G' be a non-degenerately based looplike A_∞-space. Then a based map $f\colon G \to G'$ admits an A_n-form if and only if the composite*

$$\Sigma G \xrightarrow{\Sigma f} \Sigma G' \xrightarrow{i_1} BG'$$

extends over the n-th projective space $B_n G$.

3. Higher homotopy commutativity

In this section, we study the properties and relations of higher homotopy commutativities.

3.1. A_n-structure on product A_n-space

The following A_n-structure is given by Iwase [**Iwa98**, Section 4].

Lemma 3.1. *Let G and H be non-degenerately based A_n-spaces. Define spaces $E_i(G,H)$, $B_i(G,H)$ and $D_i(G,H)$ by*

$$E_i(G,H) = \bigcup_{j_1+j_2=i} E_{j_1}G \times E_{j_2}H,$$
$$B_i(G,H) = \bigcup_{j_1+j_2=i} B_{j_1}G \times B_{j_2}H,$$
$$D_i(G,H) = \bigcup_{j_1+j_2=i} (D_{j_1}G \times E_{j_2}H \cup * \times D_{j_2}H).$$

Then the family $\{E_i(G,H), B_i(G,H), D_i(G,H)\}$ *is an* A_n*-structure of* $G \times H$*. Moreover, if* G *and* H *are looplike, the square*

$$\begin{array}{ccc} E_{i-1}(G,H) & \longrightarrow & E_i(G,H) \\ \downarrow & & \downarrow \\ B_{i-1}(G,H) & \longrightarrow & B_i(G,H) \end{array}$$

is a homotopy pullback for each i*.*

The following proposition plays an important role in the proof of our theorems.

Proposition 3.2. *Let* G *and* H *be non-degenerately based looplike* A_n*-spaces. Then there is a homotopy commutative diagram*

$$\begin{array}{ccccccc} \Sigma(G \times H) & \longrightarrow & B_2(G \times H) & \longrightarrow \cdots \longrightarrow & B_n(G \times H) \\ {\scriptstyle \Sigma p_1 + \Sigma p_2}\downarrow & & \downarrow & & \downarrow \\ \Sigma G \vee \Sigma H & \longrightarrow & B_2(G,H) & \longrightarrow \cdots \longrightarrow & B_n(G,H) \end{array}$$

where p_i *is the* i*-th projection and the addition is given by the suspension parameter of* $\Sigma(G \times H)$*.*

Proof. By Proposition 2.9, there is an A_n-form $\{m'_i\}$ on $G \times H$ such that there is a map between the associated canonical A_n-structure to the fibrant replacement $\{\tilde{E}_i(G,H), B_i(G,H), \tilde{D}_i(G,H)\}$. Since the projections from $\{E_i(G,H), B_i(G,H), D_i(G,H)\}$ to the canonical A_n-structures of G and H are A_n-structures on $p_1 \colon G \times H \to G$ and $p_2 \colon G \times H \to H$, the identity map $G \times H \to G \times H$ admits an A_n-form $\{f_i\}$ as an A_n-map from $(G \times H, \{m'_i\})$ to the product A_n-space $G \times H$. Then, since the pair $(\mathrm{id}, \{f_i\})$ is an A_n-equivalence from $(G \times H, \{m'_i\})$ to the product A_n-space $G \times H$, the identity map also admits an A_n-form as an A_n-map from the product A_n-space $G \times H$ to $(G \times H, \{m'_i\})$. Thus we have a map between the canonical A_n-structures of them of which the underlying map is the identity on $G \times H$. Moreover, since the composite

$$E_{n-1}(G,H) \to B_{n-1}(G,H) \to B_n(G,H)$$

is null-homotopic, the map $B_{n-1}(G \times H) \to B_n(G,H)$ extends over $B_n(G \times H)$. Hence we have a homotopy commutative ladder

$$\begin{array}{ccccccc} \Sigma(G \times H) & \longrightarrow & B_2(G \times H) & \longrightarrow \cdots \longrightarrow & B_n(G \times H) \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma G \vee \Sigma H & \longrightarrow & B_2(G,H) & \longrightarrow \cdots \longrightarrow & B_n(G,H) \end{array}$$

By observing the composite

$$D_0(G \times H) = C(G \times H) \to D_0(G,H) = CG \times H \cup * \times CH \to \Sigma G \vee \Sigma H,$$

we can see that the map $\Sigma(G \times H) \to \Sigma G \vee \Sigma H$ is homotopic to $\Sigma p_1 + \Sigma p_2$. □

3.2. Sugawara C_n-space

Let us recall the higher homotopy commutativity introduced by Sugawara [**Sug61**] for $n = \infty$ and generalized by McGibbon [**McG89**] for $n < \infty$.

Definition 3.3. An A_n-space G is said to be a *Sugawara C_n-space* if the multiplication

$$m_2\colon G \times G \to G$$

admits an A_n-form as an A_n-map which respects the product A_n-form on $G \times G$.

We give an obstruction theoretic characterization of a Sugawara C_n-space. A similar characterization is obtained by Hemmi–Kawamoto [**HK11**, Corollary 1.1].

Proposition 3.4. *A looplike A_∞-space G having the based homotopy type of a CW complex is a Sugawara C_n-space if and only if the composite*

$$\Sigma G \vee \Sigma G \to BG \vee BG \xrightarrow{\nabla} BG$$

of the wedge sum of the inclusions and the folding map extends over the space $B_n(G, G)$.

Proof. One can find a topological group G' and an A_∞-equivalence $G' \to G$. For example, take G' as the geometric realization of Kan's simplicial loop group on BG. An A_∞-equivalence induces a homotopy equivalence between the projective spaces. Then we may assume that G is a topological group.

By Corollary 2.11, if the multiplication $m\colon G \times G \to G$ is an A_n-map, there is a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma(G \times G) & \xrightarrow{\Sigma m} & \Sigma G \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_1} \\ B_n(G \times G) & \xrightarrow{\mu} & BG \end{array}$$

The projections $B(G \times G) \to BG$ induce a homotopy equivalence $B(G \times G) \to BG \times BG$. Considering the homotopy inverse, we have the factorizations

$$\begin{array}{ccccccccc} \Sigma G \vee \Sigma G & \longrightarrow & B_2(G,G) & \longrightarrow & \cdots & \longrightarrow & B_n(G,G) & \longrightarrow & BG \times BG \\ {\scriptstyle \text{inclusion}}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \simeq} \\ \Sigma(G \times G) & \longrightarrow & B_2(G \times G) & \longrightarrow & \cdots & \longrightarrow & B_n(G \times G) & \xrightarrow{i_n} & B(G \times G) \end{array}$$

since $B_i(G, G) = B_{i-1}(G, G) \cup_{E_{i-1}(G,G)} D_{i-1}(G, G)$ has the homotopy type of the mapping cone of $E_{i-1}(G, G) \to B_{i-1}(G, G)$. Thus the composite

$$B_n(G, G) \xrightarrow{\varphi} B_n(G \times G) \xrightarrow{\mu} BG$$

is restricted to a map homotopic to the wedge sum of the inclusions $\Sigma G \vee \Sigma G \to BG$.

Conversely, suppose that there is a map $f\colon B_n(G, G) \to BG$ which is an extension of the wedge sum of the inclusions $\Sigma G \vee \Sigma G \to BG$ and $n \geqslant 2$. By Proposition 3.2 and Corollary 2.11, there is a map $m'\colon G \times G \to G$ admitting an A_n-form such that m' restricts to the folding map $G \vee G \to G$. Since m' admits an A_2-form, the two maps

$$(x_1, x_2, y_1, y_2) \mapsto m'(m(x_1, x_2), m(y_1, y_2)),$$
$$(x_1, x_2, y_1, y_2) \mapsto m(m'(x_1, y_1), m'(x_2, y_2))$$

are homotopic. Then by the Eckmann–Hilton argument, m and m' are homotopic. Therefore, m also admits an A_n-form by Lemma 2.3. □

3.3. $C(k_1, \dots, k_r)$-space

For our applications to gauge groups, it is convenient to generalize $C(k, \ell)$-space [**KK10**] as follows.

Definition 3.5. A looplike A_∞-space G is said to be a $C(k_1, \dots, k_r)$-*space* ($r \geqslant 2$, $k_1, \dots, k_r \geqslant 1$) if the wedge sum of inclusions

$$\Sigma G \vee \cdots \vee \Sigma G \to BG$$

extends over the product $B_{k_1}G \times \cdots \times B_{k_r}G$.

As in [**Sau95**, Section 3], when $k_1 = \cdots = k_r = 1$, a $C(k_1, \dots, k_r)$-space is exactly a Williams C_r-space. When $k_1 = \cdots = k_r = \infty$, a $C(k_1, \dots, k_r)$-space is exactly a $C(\infty, \infty)$-space and hence a Sugawara C_∞-space. Hemmi–Kawamoto [**HK11**] proved that a Sugawara C_n-space is described by explicit higher homotopies using the resultohedra. Analogously, the authors guess that our new "commutativity" is also described by certain polytopes. But we do not try to do this in the present paper.

The relations with other higher commutativities is obtained as follows.

Proposition 3.6. *Let G be a looplike A_∞-space having the homotopy type of a CW complex and $r \geqslant 2$ and $k_1, \dots, k_r \geqslant 1$ be integers. Then the implications (i)⇒(ii)⇒(iii) hold for the following conditions:*

(i) G is a Sugawara $C_{k_1+\cdots+k_r}$-space,
(ii) G is a $C(k_1, \dots, k_r)$-space,
(iii) G is a Williams $C_{k_1+\cdots+k_r}$-space.

Proof. To prove the implication (i)⇒(ii), suppose G is a Sugawara $C_{k_1+\cdots+k_r}$-space. By Proposition 3.4, there is a map

$$F\colon B_{k_1+\cdots+k_r}(G, G) \to BG,$$

which restricts to the wedge sum of the inclusions $\Sigma G \vee \Sigma G \to BG$. Assume that we have a map $f_i\colon B_{k_1}G \times \cdots \times B_{k_i}G \to BG$ which is an extension of the wedge sum of the inclusions for $i < r$. Since $\mathrm{cat}(B_{k_1}G \times \cdots \times B_{k_i}G) \leqslant k_1 + \cdots + k_i$, f_i factors through $B_{k_1+\cdots+k_i}G$ up to homotopy. We also denote this factorization by f_i. Define a map g as the composite

$$B_{k_1+\cdots+k_i}G \times B_{k_{i+1}}G \xrightarrow{\text{inclusion}} B_{k_1+\cdots+k_r}(G, G) \xrightarrow{F} BG.$$

Then the composite

$$g \circ (f_i \times \mathrm{id})\colon (B_{k_1}G \times \cdots \times B_{k_i}G) \times B_{k_{i+1}}G \to BG$$

is an extension of the wedge sum of the inclusions. Thus by induction, G is a $C(k_1, \dots, k_r)$-space.

To prove the implication (ii)⇒(iii), suppose G is a $C(k_1, \dots, k_r)$-space. By Definition 3.5, there is a map

$$F'\colon B_{k_1}G \times \cdots \times B_{k_r}G \to BG,$$

which restricts to the wedge sum of the inclusions $(\Sigma G)^{\vee r} \to BG$. For each i, we see by induction that there is a map $h_i\colon (\Sigma G)^{\times k_i} \to B_{k_i}G$ such that the composite of h_i and the inclusion $B_{k_i}G \to BG$ restricts to the wedge sum of the inclusions $(\Sigma G)^{\vee k_i} \to BG$. Assume we have a map $h' \colon (\Sigma G)^{\times j} \to B_jG$ for some $j < k_i$ such that the composite of h' and the inclusion $B_jG \to BG$ restricts to the wedge sum of the inclusions $(\Sigma G)^{\vee j} \to BG$. Then the composite

$$(\Sigma G)^{\times j} \times \Sigma G \xrightarrow{h' \times \mathrm{id}} B_jG \times \Sigma G \xrightarrow{\mathrm{incl}} B_{k_i}G \times B_{k_{i'}}G \xrightarrow{\mathrm{incl}} B_{k_1}G \times \cdots \times B_{k_r}G \xrightarrow{F'} BG$$

is an extension of the wedge sum of the inclusions, where we can choose $i' \neq i$ since $r \geqslant 2$ and $k_{i'} \geqslant 1$. This extension factors through $B_{j+1}G$ since $\mathrm{cat}(\Sigma G)^{\times(j+1)} \leqslant j+1$. Then we obtain h_i by induction. Now the composite

$$(\Sigma G)^{\times k_1} \times \cdots \times (\Sigma G)^{\times k_r} \xrightarrow{h_1 \times \cdots \times h_r} B_{k_1}G \times \cdots \times B_{k_r}G \xrightarrow{F'} BG$$

is an extension of the wedge sum of the inclusions. This implies that G is a Williams C_k-space. □

4. Reduction of the projective space

The key technique in McGibbon [**McG84**] and Saumell's [**Sau95**] work is reducing the obstruction problem of ΣG to that of the wedge of spheres. For our problem, we reduce the projective space B_kG to some easier space. This is the aim of this section. It can be done by proving that G is A_k-equivalent to a product of spheres. This fact can be considered as a higher version of p-regularity. Once it is done, Theorem 1.2 immediately follows.

Let G be a compact connected Lie group of type $\{n_1, \ldots, n_\ell\}$. In this section, we localize spaces and maps at an odd prime $p \geqslant n_\ell$ and omit the symbol (p) like $G = G_{(p)}$. Then G is A_∞-equivalent to the product of compact connected simple Lie groups and a torus. To prove Theorem 1.2, it is sufficient to consider the case when G is simple. So we suppose G is simple.

First we determine the homotopy type of the projective spaces of spheres.

Lemma 4.1. *An odd dimensional sphere S^{2n-1} admits an A_{p-1}-form. The cohomology of the projective space B_kS^{2n-1} for $k < p$ is computed as*

$$H^*(B_kS^{2n-1}; \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[x]/(x^{k+1}),$$

where $x \in H^{2n}(B_kS^{2n-1}; \mathbb{Z}_{(p)})$. Moreover, B_kS^{2n-1} has the homotopy type of the CW complex

$$S^{2n} \cup e^{4n} \cup \cdots \cup e^{2kn},$$

where e^d denotes a d-dimensional (p-local) cell.

Proof. This follows from the fact that the homotopy fiber of the double suspension map

$$E^2 \colon S^{2n-1} \to \Omega^2 S^{2n+1}$$

is $(2pn-4)$-connected and $\Omega^2 S^{2n+1}$ is an A_∞-space. □

As is well-known, G has the (p-local) homotopy type of the product of spheres. Take generators $\epsilon_i \in \pi_{2n_i-1}(G)$ of the free part of the homotopy groups. Then the composite

$$S^{2n_1-1} \times \cdots \times S^{2n_\ell-1} \xrightarrow{\epsilon_1\times\cdots\times\epsilon_\ell} G \times \cdots \times G \xrightarrow{\text{multiplication}} G$$

is a homotopy equivalence. Consider a union of the product of projective spaces

$$B_k(S^{2n_1-1}, \dots, S^{2n_\ell-1}) = \bigcup_{j_1+\cdots+j_\ell=k} B_{j_1}S^{2n_1-1} \times \cdots \times B_{j_\ell}S^{2n_\ell-1}.$$

Proposition 4.2. *If $p > kn_\ell$ for some $k \geqslant 1$, then the above homotopy equivalence admits an A_k-form.*

Proof. Note that $\pi_i(BG) = 0$ for odd $i < 2p+1$ since $G \simeq S^{2n_1-1} \times \cdots \times S^{2n_\ell-1}$. Then, by Lemma 4.1, there are no obstructions to extending the map $B_1(S^{2n_1-1}, \dots, S^{2n_\ell-1}) \to BG$ over $B_k(S^{2n_1-1}, \dots, S^{2n_\ell-1})$. Hence by Proposition 3.2 and Corollary 2.11, the map $S^{2n_1-1} \times \cdots \times S^{2n_\ell-1} \to G$ admits an A_k-form. □

The following proposition is used to reduce the projective space B_kG to $B_k(S^{2n_1-1}, \dots, S^{2n_\ell-1})$.

Proposition 4.3. *There exists an A_∞-form on $S^{2n_1-1} \times \cdots \times S^{2n_\ell-1}$ such that the restricted A_k-form coincides with the product A_k-form and the above homotopy equivalence $S^{2n_1-1} \times \cdots \times S^{2n_\ell-1} \to G$ admits an A_∞-form.*

Proof. By Proposition 4.2, the homotopy equivalence $S^{2n_1-1} \times \cdots \times S^{2n_\ell-1} \to G$ admits an A_k-form with respect to the product A_k-form of $S^{2n_1-1} \times \cdots \times S^{2n_\ell-1}$. Since this map is a homotopy equivalence, one can observe that there are no obstructions to constructing A_∞-forms on the map and on $S^{2n_1-1} \times \cdots \times S^{2n_\ell-1}$. □

Let us denote the A_∞-space $S^{2n_1-1} \times \cdots \times S^{2n_\ell-1}$ equipped with the above A_∞-form by H.

Proof of Theorem 1.2. Let G be a compact connected simple Lie group of type $\{n_1, \dots, n_\ell\}$ and take a prime p and a positive integer k such as $p > kn_\ell$. Then, by Propositions 3.2, 3.4 and 4.3, G is a Sugawara C_k-space if the composite

$$B_1(S^{2n_1-1}, \dots, S^{2n_\ell-1}) \vee B_1(S^{2n_1-1}, \dots, S^{2n_\ell-1}) \to BH \vee BH \xrightarrow{\nabla} BH$$

extends over the union $\bigcup_{k_1+k_2=k} B_{k_1}(S^{2n_1-1}, \dots, S^{2n_\ell-1}) \times B_{k_2}(S^{2n_1-1}, \dots, S^{2n_\ell-1})$. Now it does by Lemma 4.1 since $\pi_i(BG) = 0$ for odd $i < 2p+1$ and $p > kn_\ell$. Thus Theorem 1.2 follows. □

5. Gauge groups

In this section, we recall the basic definitions and facts about gauge groups.

Definition 5.1. Given a principal G-bundle $P \to B$, a map $P \to P$ is said to be an *automorphism* if f is G-equivariant and induces the identity on B. The topological group consisting of automorphisms on P is denoted by $\mathcal{G}(P)$ and called the *gauge group*.

Let $P \to B$ be a principal G-bundle. The associated bundle

$$\operatorname{ad} P = (P \times G)/\sim$$

defined by the equivalence relation

$$(ug, x) \sim (u, gxg^{-1})$$

is called the *adjoint bundle* of P. It is naturally a fiberwise topological group. Thus the space of sections $\Gamma(\operatorname{ad} P)$ is a topological group. It is not difficult to see that $\Gamma(\operatorname{ad} P)$ is naturally isomorphic to $\mathcal{G}(P)$.

The weak homotopy type of the classifying space of a gauge group is studied by Gottlieb [**Got72**].

Proposition 5.2. *Let P be a principal G-bundle over a CW complex B, which is classified by a map $\alpha\colon B \to BG$. Then, the classifying space $B\mathcal{G}(P)$ is weakly homotopy equivalent to the path-component* $\operatorname{Map}(B, BG)_\alpha$ *of* $\operatorname{Map}(B, BG)$ *based at* $\alpha \in \operatorname{Map}(B, BG)$.

By [**HMR72**, Theorem 3.11, Chapter II], if a p-localization $\ell\colon X \to X_{(p)}$ of a nilpotent space X is given and B is a finite complex, the induced map $\operatorname{Map}(B, X)_f \to \operatorname{Map}(B, X_{(p)})_{\ell\circ f}$ between the path components containing f and $\ell \circ f$ respectively is also a p-localization for any $f\colon B \to X$. This implies the following corollary. We recall that even if $\mathcal{G}(P)$ is not path-connected, we define $\mathcal{G}(P)_{(p)}$ as $\Omega(B\mathcal{G}(P)_{(p)})$.

Corollary 5.3. *Suppose G is a path-connected topological group having the homotopy type of a CW complex. Let P be a principal G-bundle over a finite CW complex B, which is classified by a map $\alpha\colon B \to BG$. Then, the classifying space $B(\mathcal{G}(P)_{(p)})$ is weakly homotopy equivalent to the path-component* $\operatorname{Map}(B, BG_{(p)})_{\ell\circ\alpha}$, *where $\ell\colon BG \to BG_{(p)}$ is a p-localization.*

6. Proof of Theorems 1.3, 1.5 and 1.6

As in the theorems, let G be a compact connected simple Lie group of type $\{n_1, \dots, n_\ell\}$, p a prime and n, k positive integers. In this section, we again localize all spaces and maps at p and omit the localization symbol (p).

First we prove that $\mathcal{G}(E_nG)$ is a Sugawara C_k-space if $p > (n+k)n_\ell$. When $k = 1$, we have nothing to prove. Let us consider the case when $k \geqslant 2$. By Theorem 1.2, G is a $C(k, n)$-space. Then the wedge sum of the inclusions

$$\Sigma G \vee \Sigma G \to BG$$

extends over the product $B_kG \times B_nG$. Combining with [**KK10**, Corollary 1.7], this implies that the adjoint bundle $\operatorname{ad} E_nG$ is fiberwise A_k-equivalent to the trivial bundle $B_nG \times G$. For the notions of fiberwise A_n-theory we need here, see [**KK10**, Section 3]. Consider the following homotopy commutative diagram of fiberwise spaces:

$$\begin{array}{ccc}
\operatorname{ad} E_nG \times_{B_nG} \operatorname{ad} E_nG & \xrightarrow{\text{multiplication}} & \operatorname{ad} E_nG \\
\simeq\downarrow & & \downarrow\simeq \\
B_nG \times (G \times G) & \xrightarrow{\text{multiplication}} & B_nG \times G
\end{array}$$

where the vertical arrows are fiberwise A_k-equivalences. Since G is a Sugawara C_k-space, the bottom arrow is a fiberwise A_k-map. Thus we obtain the following lemma.

Lemma 6.1. *The adjoint bundle* $\operatorname{ad} E_nG$ *is a fiberwise Sugawara* C_k*-space, that is, the fiberwise multiplication*

$$\operatorname{ad} E_nG \times_{B_nG} \operatorname{ad} E_nG \to \operatorname{ad} E_nG$$

is a fiberwise A_k*-map.*

This implies that the multiplication map

$$\mathcal{G}(E_nG) \times \mathcal{G}(E_nG) \to \mathcal{G}(E_nG)$$

is an A_k-map. Hence $\mathcal{G}(E_nG)$ is a Sugawara C_k-space.

For a space B such that $\operatorname{cat} B = n$ and a principal G-bundle P over B, the classifying map $B \to BG$ factors through B_nG. Then by Lemma 6.1, the gauge group $\mathcal{G}(P)$ is a Sugawara C_k-space. This completes the proof of Theorem 1.5.

Next, we observe the non-commutativity of $\mathcal{G}(E_nG)$. We suppose $(n+1)n_\ell < p < (n+k)n_\ell$. Since $(n+1)n_\ell < p$, the wedge sum of the inclusions

$$\Sigma G \vee B_nG \to BG$$

extends over the product $\Sigma G \times B_nG$. Taking the adjoint, we obtain the map

$$f\colon \Sigma G \to \operatorname{Map}(B_nG, BG)_{i_n}.$$

Consider the extension problem of the map

$$(\Sigma G)^{\vee k} \xrightarrow{(f,\dots,f)} \operatorname{Map}(B_nG, BG)_{i_n}$$

over the product $(\Sigma G)^{\times k}$. If $\mathcal{G}(E_nG)$ is a Williams C_k-space, this extends. Taking the adjoint, we have the map

$$(\Sigma G)^{\vee k} \times B_nG \to BG,$$

which is an extension of the wedge sum of the inclusions $(\Sigma G)^{\vee k} \vee B_nG \to BG$. This does not extend over the product since G is not a $C(r_1, \dots, r_k, n)$-space for $r_1 = \cdots = r_k = 1$. Therefore, the gauge group $\mathcal{G}(E_nG)$ is not a Williams C_k-space.

Now the proof of Theorem 1.6 might be obvious. Let P be a principal G-bundle over S^{2n_i} classified by $\alpha\colon S^{2n_i} \to BG$ and $k \geqslant 2$ an integer satisfying $p \geqslant kn_\ell + n_i$. One can prove by the analogous argument that the wedge sum $S^{2n_i} \vee \Sigma G \to BG$ of α and the inclusion extends over the product $S^{2n_i} \times B_kG$. Then the adjoint bundle $\operatorname{ad} P$ is fiberwise A_k-equivalent to the trivial bundle $S^{2n_i} \times G$. Since G is a Sugawara C_k-space, then the fiberwise multiplication

$$\operatorname{ad} P \times_{S^{2n_i}} \operatorname{ad} P \to \operatorname{ad} P$$

is a fiberwise A_k-map. Therefore, the gauge group $\mathcal{G}(P)$ is a Sugawara C_k-space.

7. 5-local higher homotopy commutativity of G_2

In this section, we prove Theorem 1.7. Hereafter, we localize all spaces and maps at $p = 5$. McGibbon [**McG84**] proved that G_2 is homotopy commutative. But Saumell [**Sau95**] proved that G_2 is not a Williams C_5-space.

By the results in [**Ada69**, Lecture 4], there is a loop map

$$E\colon BU \to BU$$

characterized by the homotopy commutative diagrams

$$\begin{array}{ccc} BU & \xrightarrow{E} & BU \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ch}_n} \\ * & \longrightarrow & K(\mathbb{Q}, 2n) \end{array} \qquad \begin{array}{ccc} BU & \xrightarrow{E} & BU \\ \downarrow{\scriptstyle \mathrm{ch}_{4n-2}} & & \downarrow{\scriptstyle \mathrm{ch}_{4n-2}} \\ K(\mathbb{Q}, 8n-4) & = & K(\mathbb{Q}, 8n-4) \end{array}$$

where the left square holds for $n \not\equiv 2 \bmod 4$ and ch_n denotes the n-th universal Chern character. From this, we have $E^2 = E$. We consider a telescope

$$B' = \mathrm{hocolim}(B^2U \xrightarrow{BE} B^2U \xrightarrow{BE} \cdots)$$

and define a loop space

$$B = \Omega B'.$$

The canonical map $B^2U \to B'$ induces a loop map $\pi\colon BU \to B$. Note that B also has the homotopy type of a telescope:

$$B \simeq \mathrm{hocolim}(BU \xrightarrow{E} BU \xrightarrow{E} \cdots).$$

We can compute the cohomology group as

$$H^*(B; \mathbb{Z}_{(5)}) = \mathbb{Z}_{(5)}[z_4, z_{12}, z_{20}, z_{28}, \ldots],$$

such that $\pi^* z_{8n-4} = E^* c_{4n-2}$ for the Chern class $c_{4n-2} \in H^{8n-4}(BU; \mathbb{Z}_{(5)})$ by the Newton identities.

Lemma 7.1. *The following congruences modulo the ideal* $(c_k \mid k\colon$ *odd or* $k \geqslant 7) + ((c_2, c_6)^2 + (c_4))^2$ *hold, that is, the following congruences are modulo monomials containing* c_k *for odd* k *or* $k \geqslant 7$ *or* $c_2^p c_4^q c_6^r$ *for* $p + 2q + r = 4$*:*

$$E^*c_2 \equiv c_2, \qquad E^*c_4 \equiv \frac{1}{2}c_2^2, \qquad E^*c_6 \equiv c_6 - c_4c_2 + \frac{1}{2}c_2^3,$$
$$E^*c_8 \equiv c_6c_2, \qquad E^*c_{10} \equiv -c_6c_4 + \frac{3}{2}c_6c_2^2, \qquad E^*c_{12} \equiv \frac{1}{2}c_6^2,$$
$$E^*c_{14} \equiv \frac{3}{2}c_6^2c_2, \qquad E^*c_{16} \equiv 0, \qquad E^*c_{18} \equiv \frac{1}{2}c_6^3.$$

Proof. These congruences can be verified by the equalities

$$E^*c_{4n-2} = -\frac{1}{4n-2}((E^*c_{4n-4})s_2 + \cdots + (E^*c_4)s_{4n-6} + s_{4n-2}),$$
$$E^*c_{4n} = -\frac{1}{4n}((E^*c_{4n-2})s_2 + \cdots + (E^*c_2)s_{4n-2})$$

and Girard's formula

$$s_i = \sum_{r_1+2r_2+\cdots+ir_i=i} (-1)^{i+r_1+\cdots+r_i} \frac{i(r_1+\cdots+r_i-1)!}{r_1!\cdots r_i!} c_1^{r_1}\cdots c_i^{r_i}. \qquad \square$$

We also need the indecomposables as in the following lemma. The proof is similar to the previous lemma.

Lemma 7.2. *We have the congruence $E^*c_{4n-2} \equiv c_{4n-2}$ mod decomposables for any integer $n \geqslant 1$.*

Now we recall elementary properties of the exceptional Lie group G_2. The following diagram of inclusions commutes:

$$\begin{array}{ccccc} \mathrm{SU}(3) & \longrightarrow & \mathrm{G}_2 & & \\ {\scriptstyle \text{realification}}\downarrow & & \downarrow & \searrow^{\rho} & \\ \mathrm{Spin}(6) & \longrightarrow & \mathrm{Spin}(7) & \longrightarrow & \mathrm{SU}(7) \end{array}$$

where $\mathrm{Spin}(7)/\mathrm{G}_2 \cong S^7$. As in [**Wat85**, Section 4], the following proposition holds.

Proposition 7.3. *The cohomology of $B\mathrm{G}_2$ is computed as*

$$H^*(B\mathrm{G}_2; \mathbb{Z}_{(5)}) = \mathbb{Z}_{(5)}[y_4, y_{12}],$$

such that the following equality holds:

$$\rho^*(c_i) = \begin{cases} -y_{2i} & i = 2, 6, \\ \frac{1}{4}y_4^2 & i = 4, \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 7.4. It is claimed in [**HKO14**, Proposition 2.10] that $\rho^*(c_4) = 0$, and this is false as above. However, this is irrelevant to verifying the results of [**HKO14**].

It is well known that $\Sigma\mathbb{C}P^6$ has the homotopy type of the wedge sum $A \vee S^5 \vee S^7 \vee S^9$ where $A = S^3 \cup e^{11}$. The composite of the inclusions $A \to \Sigma\mathbb{C}P^6 \to \mathrm{SU}(7)$ lifts to $\mathrm{Spin}(7)$. Moreover, it lifts to G_2 since $\mathrm{Spin}(7)/\mathrm{G}_2 \cong S^7$.

Lemma 7.5. *The cohomology of A is computed as*

$$H^*(A; \mathbb{Z}_{(5)}) = \mathbb{Z}_{(5)}\langle x_3, x_{11}\rangle, \qquad x_3 \in H^3, \quad x_{11} \in H^{11},$$

where x_3 and x_{11} are the images of the cohomology suspensions $-\sigma(y_4)$ and $-\sigma(y_{12})$ under the induced map of $A \to G$, respectively. Moreover, the K-theory of A is computed as

$$\tilde{K}(\Sigma A; \mathbb{Z}_{(5)}) = \mathbb{Z}_{(5)}\langle g, h\rangle, \qquad \mathrm{ch}\, g = \Sigma x_3 + \frac{1}{5!}\Sigma x_{11}, \quad \mathrm{ch}\, h = \Sigma x_{11}.$$

Consider the wedge sum of the inclusions

$$\Sigma A \vee \Sigma A \vee \Sigma A \to B\mathrm{G}_2.$$

Since G_2 is homotopy commutative, this map extends over the fat wedge $T(\Sigma A, \Sigma A, \Sigma A)$. Our goal is to see the higher Whitehead product

$$\omega\colon \Sigma^2(A \wedge A \wedge A) \to B\mathrm{G}_2$$

is non-trivial. Our basic idea is the same as the calculation of Samelson products in quasi-p-regular Lie groups in [**HKMO18**]. Once this is proved, Theorem 1.7 follows from [**Sau95**, Theorem-Definition 3.1].

Let $j\colon B\mathrm{G}_2 \to B$ be the composite

$$B\mathrm{G}_2 \xrightarrow{B\rho} B\,\mathrm{SU}(7) \xrightarrow{\text{inclusion}} BU \xrightarrow{\pi} B$$

and W be the homotopy fiber of j.

Lemma 7.6. *The following equalities hold:*

$$j^*z_4 = -y_4, \qquad j^*z_{28} \equiv -\frac{3}{2}y_{12}^2y_4,$$
$$j^*z_{12} = -y_{12} - \frac{1}{4}y_4^3, \qquad j^*z_{36} \equiv -\frac{1}{2}y_{12}^3,$$
$$j^*z_{20} \equiv -\frac{5}{4}y_{12}y_4^2,$$

where $\equiv$ *means the congruence modulo* $(y_4, y_{12})^4$*, namely modulo the monomials* $y_4^py_{12}^q$ *for* $p+q \geqslant 4$.

Proof. This lemma immediately follows from Lemma 7.1 and Proposition 7.3. □

Lemma 7.7. *The cohomology of* W *is computed as*

$$H^*(W;\mathbb{Z}_{(5)}) = \mathbb{Z}_{(5)}\langle a_{19}, a_{27}, a_{35}\rangle \qquad \text{for } * < 43,$$

where the transgressions $\tau(a_{19}), \tau(a_{27}), \tau(a_{35})$ *with respect to the fibration* $W \to B\mathrm{G}_2 \to B$ *satisfy*

$$\tau(a_{19}) \equiv z_{20} - \tfrac{5}{4}z_{12}z_4^2 \qquad \text{mod } (z_4^5),$$
$$\tau(a_{27}) \equiv z_{28} - \tfrac{3}{2}z_{12}^2z_4 \qquad \text{mod } (z_{20}) + (z_4, z_{12})^4,$$
$$\tau(a_{35}) \equiv z_{36} - \tfrac{1}{2}z_{12}^3 \qquad \text{mod } (z_{20}, z_{28}) + (z_4, z_{12})^4,$$

where the middle congruence is modulo the monomials containing z_{20} *or* $z_4^pz_{12}^q$ *for* $p+q=4$ *and the bottom congruence is the modulo monomials containing* z_{20}, z_{28} *or* $z_4^pz_{12}^q$ *for* $p+q=4$. *Moreover, the images of* a_{19}, a_{27}, a_{35} *under the induced map of* $\Omega B \to W$ *are the cohomology suspensions* $\sigma(z_{20}), \sigma(z_{28}), \sigma(z_{36})$.

Proof. This follows from the computation of the cohomology Serre spectral sequence and Lemma 7.6. □

The map j induces the exact sequence

$$[\Sigma^2A^{\wedge 3}, \Omega B] \to [\Sigma^2A^{\wedge 3}, W] \to [\Sigma^2A^{\wedge 3}, B\mathrm{G}_2] \xrightarrow{j_*} [\Sigma^2A^{\wedge 3}, B].$$

Let us construct an appropriate lift of $\omega \in [\Sigma^2A^{\wedge 3}, B\mathrm{G}_2]$ to $[\Sigma^2A^{\wedge 3}, W]$.

Lemma 7.8. *The extension of the wedge sum of the inclusions* $\Sigma A \vee \Sigma A \vee \Sigma A \to B$ *over the fat wedge* $T(\Sigma A, \Sigma A, \Sigma A)$ *is unique up to homotopy.*

Proof. This follows from the fact that the homotopy groups $\pi_8(B)$, $\pi_{16}(B)$ and $\pi_{24}(B)$ are trivial. □

Define a map $\tilde{\mu}\colon (\Sigma A)^{\times 3} \to B$ by the composite

$$(\Sigma A)^{\times 3} \to (B\mathrm{G}_2)^{\times 3} \xrightarrow{j^{\times 3}} B^{\times 3} \xrightarrow{\text{multiplication}} B.$$

Then we obtain the homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma^2 A^{\wedge 3} & \longrightarrow & T(\Sigma A, \Sigma A, \Sigma A) & \longrightarrow & (\Sigma A)^{\times 3} \\ \downarrow{\scriptstyle \mu} & & \downarrow & & \downarrow{\scriptstyle \tilde{\mu}} \\ W & \longrightarrow & BG_2 & \xrightarrow{j} & B \end{array}$$

as follows. The map $T(\Sigma A, \Sigma A, \Sigma A) \to BG_2$ is an extension of the wedge sum of the inclusions $(\Sigma A)^{\vee 3} \to BG_2$. Such extension exists since G_2 is homotopy commutative. By Lemma 7.8, the right square commutes up to homotopy. The map $\mu\colon \Sigma^2 A^{\wedge 3} \to W$ is defined up to homotopy and the left square commutes since the top row is a cofiber sequence and the bottom row is a fiber sequence. For a precise description of the top cofiber sequence, see [**Por65**]. Here μ is the lifting of the map $\omega\colon \Sigma^2 A^{\wedge 3} \to BG_2$.

To check the non-triviality of μ, we first embed $[\Sigma^2 A^{\wedge 3}, W]$ to an easier module.

Lemma 7.9. *The map*

$$[\Sigma^2 A^{\wedge 3}, W] \to H^{19}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}) \oplus H^{27}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}) \oplus H^{35}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}) (\cong \mathbb{Z}_{(5)}^{\oplus 7})$$

defined by $f \mapsto (f^*(a_{19}), f^*(a_{27}), f^*(a_{35}))$ *is injective.*

Proof. First we note that the homotopy set $[\Sigma^2 A^{\wedge 3}, W]$ is isomorphic to the stable homotopy set $\{\Sigma^2 A^{\wedge 3}, W\}$ since W is 18-connected and $\dim \Sigma^2 A^{\wedge 3} = 35$. The rationalized map

$$[\Sigma^2 A^{\wedge 3}, W] \otimes \mathbb{Q} \to H^{19}(\Sigma^2 A^{\wedge 3}; \mathbb{Q}) \oplus H^{27}(\Sigma^2 A^{\wedge 3}; \mathbb{Q}) \oplus H^{35}(\Sigma^2 A^{\wedge 3}; \mathbb{Q})$$

is an isomorphism by Lemma 7.7. Then it is sufficient to show that $[\Sigma^2 A^{\wedge 3}, W]$ is a free $\mathbb{Z}_{(5)}$-module. The homotopy groups of W are computed as

$$\pi_i(W) = \begin{cases} \mathbb{Z}_{(5)} & i = 19, 27, 35, \\ 0 & i = 28, 36, \end{cases}$$

by the approximation by a CW complex $S^{19} \cup e^{27} \cup e^{35} \to W$ and the stable homotopy groups of spheres. Thus by a skeletal consideration, one can see that $[\Sigma^2 A^{\wedge 3}, W]$ is $\mathbb{Z}_{(5)}$-free. □

Next we compute the image of $\mu \in [\Sigma^2 A^{\wedge 3}, W]$ in $H^{19}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}) \oplus H^{27}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}) \oplus H^{35}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)})$.

Proposition 7.10. *The following equalities hold:*

$$\mu^*(a_{19}) = -\frac{3}{2} b_{19}, \qquad \mu^*(a_{27}) = -2 b_{27}, \qquad \mu^*(a_{35}) = -2 b_{35},$$

where b_{19}, b_{27}, b_{35} *are defined as*

$$\begin{aligned} b_{19} &= \Sigma^2 x_{11} \otimes x_3 \otimes x_3 + \Sigma^2 x_3 \otimes x_{11} \otimes x_3 + \Sigma^2 x_3 \otimes x_3 \otimes x_{11}, \\ b_{27} &= \Sigma^2 x_{11} \otimes x_{11} \otimes x_3 + \Sigma^2 x_{11} \otimes x_3 \otimes x_{11} + \Sigma^2 x_3 \otimes x_{11} \otimes x_{11}, \\ b_{35} &= \Sigma^2 x_{11} \otimes x_{11} \otimes x_{11}. \end{aligned}$$

Proof. The previous diagram induces the map of cofiber sequences

$$\begin{array}{ccccccc}
\Sigma^2 A^{\wedge 3} & \longrightarrow & T(\Sigma A, \Sigma A, \Sigma A) & \longrightarrow & C_1 & \xrightarrow{\partial} & \Sigma^3 A^{\wedge 3} \\
\downarrow{\scriptstyle \mu} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma\mu} \\
W & \longrightarrow & B\mathrm{G}_2 & \longrightarrow & C_2 & \longrightarrow & \Sigma W
\end{array}$$

and hence the next homotopy commutative square

$$\begin{array}{ccc}
C_1 & \xrightarrow{\simeq} & (\Sigma A)^{\times 3} \\
\downarrow & & \downarrow{\scriptstyle \tilde{\mu}} \\
C_2 & \longrightarrow & B
\end{array}$$

where the composite $C_1 \to (\Sigma A)^{\times 3} \to (\Sigma A)^{\wedge 3} \cong \Sigma^3 A^{\wedge 3}$ is homotopic to ∂.

For $i = 20, 28, 36$, we have the following commutative diagram:

$$\begin{array}{ccccc}
\tilde{H}^{i-1}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}) & \xrightarrow{\cong} & \tilde{H}^i(C_1; \mathbb{Z}_{(5)}) & \xleftarrow{\cong} & \tilde{H}^i((\Sigma A)^{\times 3}; \mathbb{Z}_{(5)}) \\
\uparrow{\scriptstyle \mu^*} & & \uparrow & & \uparrow{\scriptstyle \tilde{\mu}^*} \\
H^{i-1}(W; \mathbb{Z}_{(5)}) & \hookrightarrow & \tilde{H}^i(C_2; \mathbb{Z}_{(5)}) & \twoheadleftarrow & \tilde{H}^i(B; \mathbb{Z}_{(5)})
\end{array}$$

The injectivity and the surjectivity in the bottom row follows from the following diagram and the computation of the transgressions in Lemma 7.7, where the top row is exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{i-1}(W; \mathbb{Z}_{(5)}) & \hookrightarrow & \tilde{H}^i(C_2; \mathbb{Z}_{(5)}) & \twoheadrightarrow & \tilde{H}^i(B\mathrm{G}_2; \mathbb{Z}_{(5)}) & \longrightarrow & 0 \\
 & & & & \uparrow & & \| & & \\
 & & & & \tilde{H}^i(B; \mathbb{Z}_{(5)}) & \twoheadrightarrow & \tilde{H}^i(B\mathrm{G}_2; \mathbb{Z}_{(5)}) & &
\end{array}$$

Under the induced map of the multiplication $(BU)^{\times 3} \to BU$, the class $E^* c_n \in H^{2n}(BU; \mathbb{Z}_{(5)})$ is mapped to

$$\sum_{p+q+r=n} (E^* c_p) \times (E^* c_q) \times (E^* c_r)$$

by the Cartan formula for Chern classes. From this, we can compute

$$\tilde{\mu}^*(z_{20} - \frac{5}{4} z_{12} z_4^2) = -\frac{3}{2}(\Sigma x_3 \otimes \Sigma x_3 \otimes \Sigma x_{11} + \Sigma x_3 \otimes \Sigma x_{11} \otimes \Sigma x_3 + \Sigma x_{11} \otimes \Sigma x_3 \otimes \Sigma x_3),$$

$$\tilde{\mu}^*(z_{28} - \frac{3}{2} z_{12}^2 z_4) = -2(\Sigma x_3 \otimes \Sigma x_{11} \otimes \Sigma x_{11} + \Sigma x_{11} \otimes \Sigma x_3 \otimes \Sigma x_{11} + \Sigma x_{11} \otimes \Sigma x_{11} \otimes \Sigma x_3),$$

$$\tilde{\mu}^*(z_{36} - \frac{1}{2} z_{12}^3) = -\, 2 \Sigma x_{11} \otimes \Sigma x_{11} \otimes \Sigma x_{11}.$$

Then by the above diagram and Lemma 7.7, we obtain $\mu^*(a_{19}), \mu^*(a_{27}), \mu^*(a_{35})$ as above. □

Finally, we compute the image of the composite

$$\Phi \colon [\Sigma^2 A^{\wedge 3}, \Omega B] \to [\Sigma^2 A^{\wedge 3}, W] \to H^{19}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}) \oplus H^{27}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}) \oplus H^{35}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}).$$

Consider the commutative diagram

$$\begin{array}{ccccc} [\Sigma^3 A^{\wedge 3}, B] & \xrightarrow{\cong} & [\Sigma^2 A^{\wedge 3}, \Omega B] & \longrightarrow & [\Sigma^2 A^{\wedge 3}, W] \\ \downarrow{\scriptstyle z_i} & & \downarrow{\scriptstyle \sigma(z_i)} & & \downarrow{\scriptstyle a_{i-1}} \\ H^i(\Sigma^3 A^{\wedge 3}; \mathbb{Z}_{(5)}) & \xrightarrow{\cong} & H^{i-1}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}) & = & H^{i-1}(\Sigma^2 A^{\wedge 3}; \mathbb{Z}_{(5)}) \end{array}$$

for $i = 20, 28, 36$. Since $\pi_*\colon \tilde{K}(\Sigma^3 A^{\wedge 3}; \mathbb{Z}_{(5)}) \cong [\Sigma^3 A^{\wedge 3}, BU] \to [\Sigma^3 A^{\wedge 3}, B]$ is an isomorphism, the image of the left vertical arrow coincides with the image of the following map by Lemma 7.2:

$$(9!\,\mathrm{ch}_{10}, 13!\,\mathrm{ch}_{14}, 17!\,\mathrm{ch}_{18})\colon \tilde{K}(\Sigma^3 A^{\wedge 3}; \mathbb{Z}_{(5)}) \to H^{20}(\Sigma^3 A^{\wedge 3}; \mathbb{Z}_{(5)}) \oplus H^{28}(\Sigma^3 A^{\wedge 3}; \mathbb{Z}_{(5)}) \oplus H^{36}(\Sigma^3 A^{\wedge 3}; \mathbb{Z}_{(5)}).$$

Under the Künneth isomorphism

$$\tilde{K}(\Sigma^3 A^{\wedge 3}; \mathbb{Z}_{(5)}) \cong \tilde{K}(\Sigma A; \mathbb{Z}_{(5)})^{\otimes 3},$$

we can compute as

$$\begin{aligned} 9!\,\mathrm{ch}_{10}(g \otimes g \otimes g) &= 9!((\mathrm{ch}_2\, g)(\mathrm{ch}_6\, g)(\mathrm{ch}_6\, g) + (\mathrm{ch}_2\, g)(\mathrm{ch}_6\, g)(\mathrm{ch}_2\, g) + (\mathrm{ch}_2\, g)(\mathrm{ch}_2\, g)(\mathrm{ch}_6\, g)) \\ &= \frac{9!}{5!}(\Sigma x_{11} \otimes \Sigma x_3 \otimes \Sigma x_3 + \Sigma x_3 \otimes \Sigma x_{11} \otimes \Sigma x_3 + \Sigma x_3 \otimes \Sigma x_3 \otimes \Sigma x_{11}), \end{aligned}$$

$$\begin{aligned} 13!\,\mathrm{ch}_{14}(g \otimes g \otimes g) &= 13!((\mathrm{ch}_6\, g)(\mathrm{ch}_6\, g)(\mathrm{ch}_2\, g) + (\mathrm{ch}_6\, g)(\mathrm{ch}_2\, g)(\mathrm{ch}_6\, g) + (\mathrm{ch}_2\, g)(\mathrm{ch}_6\, g)(\mathrm{ch}_6\, g)) \\ &= \frac{13!}{5!5!}(\Sigma x_{11} \otimes \Sigma x_{11} \otimes \Sigma x_3 + \Sigma x_{11} \otimes \Sigma x_3 \otimes \Sigma x_{11} + \Sigma x_3 \otimes \Sigma x_{11} \otimes \Sigma x_{11}), \end{aligned}$$

$$\begin{aligned} 17!\,\mathrm{ch}_{18}(g \otimes g \otimes g) &= 17!(\mathrm{ch}_6\, g)(\mathrm{ch}_6\, g)(\mathrm{ch}_6\, g) \\ &= \frac{17!}{5!5!5!}\Sigma x_{11} \otimes \Sigma x_{11} \otimes \Sigma x_{11} \end{aligned}$$

by Lemma 7.5. Similarly, we have

$$\begin{aligned} 9!\,\mathrm{ch}_{10}(h \otimes g \otimes g) &= 9!\Sigma x_{11} \otimes \Sigma x_3 \otimes \Sigma x_3, \\ 13!\,\mathrm{ch}_{14}(h \otimes g \otimes g) &= \frac{13!}{5!}(\Sigma x_{11} \otimes \Sigma x_{11} \otimes \Sigma x_3 + \Sigma x_{11} \otimes \Sigma x_3 \otimes \Sigma x_{11}), \\ 17!\,\mathrm{ch}_{18}(h \otimes g \otimes g) &= \frac{17!}{5!5!}\Sigma x_{11} \otimes \Sigma x_{11} \otimes \Sigma x_{11}, \\ 9!\,\mathrm{ch}_{10}(h \otimes h \otimes g) &= 0, \\ 13!\,\mathrm{ch}_{14}(h \otimes h \otimes g) &= 13!\Sigma x_{11} \otimes \Sigma x_{11} \otimes \Sigma x_3, \\ 13!\,\mathrm{ch}_{18}(h \otimes h \otimes g) &= \frac{17!}{5!}\Sigma x_{11} \otimes \Sigma x_{11} \otimes \Sigma x_{11}, \\ 9!\,\mathrm{ch}_{10}(h \otimes h \otimes h) &= 0, \\ 13!\,\mathrm{ch}_{14}(h \otimes h \otimes h) &= 0, \\ 13!\,\mathrm{ch}_{18}(h \otimes h \otimes h) &= 17!\Sigma x_{11} \otimes \Sigma x_{11} \otimes \Sigma x_{11}. \end{aligned}$$

The other terms are analogous.

Proof of Theorem 1.7. Now suppose that $(\mu^*(a_{19}), \mu^*(a_{27}), \mu^*(a_{35}))$ is contained in the image of the map Φ. Then by Proposition 7.10 and the above computation, there exist $a, b, c, d \in \mathbb{Z}_{(5)}$ satisfying the following equations:

$$\begin{cases} \frac{9!}{5!}a + 9!b & = \frac{3}{2}, \\ \frac{13!}{5!5!}a + 2 \cdot \frac{13!}{5!}b + 13!c & = 2, \\ \frac{17!}{5!5!5!}a + 3 \cdot \frac{17!}{5!5!}b + 3 \cdot \frac{17!}{5!}c + 17!d & = 2. \end{cases}$$

But one can find by a slight computation that the denominator of d must be divisible by 125. This contradicts the fact that $d \in \mathbb{Z}_{(5)}$. Thus, the higher Whitehead product $\omega \colon \Sigma^2(A \wedge A \wedge A) \to B\mathrm{G}_2$ is nontrivial. Therefore G_2 is not a Williams C_3-space at $p = 5$. □

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