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GLOBAL MODEL STRUCTURES FOR $*$-MODULES

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Abstract

We extend Schwede's work on the unstable global homotopy theory of orthogonal spaces and $\mathcal{L}$-spaces to the category of $*$-modules (i.e., unstable S-modules). We prove a theorem which transports model structures and their properties from $\mathcal{L}$-spaces to $*$-modules and show that the resulting global model structure for $*$-modules is monoidally Quillen equivalent to that of orthogonal spaces. As a consequence, there are induced Quillen equivalences between the associated model categories of monoids, which identify equivalent models for the global homotopy theory of A_∞-spaces.

1. Introduction

Global homotopy theory is equivariant homotopy theory with respect to compatible actions of the family of all compact Lie groups. Many equivariant spaces and spectra are defined in a uniform way for all such groups G, and the idea of organizing the full functoriality in G in a "global" object goes back to [**11**, Chapter II] and [**7**, §5]. Important examples of global spaces include global classifying spaces [**16**, Def. 1.1.27] and different global versions of the space BO [**16**, §2.4]; many of these admit the extra structure of a global monoid space. Moreover, all orbispaces in the sense of [**6**] provide examples of unstable global homotopy types, as explained in [**15**].

Schwede [**16, 15**] established various Quillen equivalent models for unstable global homotopy theory. Two of these are the categories $\mathcal{IU}$ and $\mathcal{LU}$ of *orthogonal spaces* and *$\mathcal{L}$-spaces*, respectively. The former is a category of diagram spaces indexed on real inner product spaces, the latter is the category of spaces equipped with continuous actions of the topological monoid $\mathcal{L}(1)$ of linear isometric embeddings $\mathbb{R}^\infty \to \mathbb{R}^\infty$ (i.e., the space of unary operations in the linear isometries operad).

The category of orthogonal spaces admits a "global" model structure that is compatible with the symmetric monoidal structure given by Day convolution. The model structure lifts to the associated category of monoids and thus models the unstable global homotopy theory of A_∞ spaces. The same is true for the category of *$\mathcal{L}$-spaces*, up to a small defect: The operadic box product $\boxtimes_{\mathcal{L}}$ only defines a "weak" symmetric monoidal product that is unital up to global equivalence. The full subcategory

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$\mathcal{M}_* \subseteq \mathcal{LU}$ of $*$-*modules* is spanned by objects such that the unital transformation is an isomorphism, and is thus symmetric monoidal in the usual sense. It is the unstable analogue of the category of S-modules [**5**].

Main results: Our first main result, Theorem 3.8, establishes a symmetric monoidal model structure on the category of $*$-modules with weak equivalences the underlying global equivalences of $\mathcal{L}$-spaces, together with explicit (weak) monoidal Quillen equivalences to both orthogonal spaces and $\mathcal{L}$-spaces. Theorem 3.8 is a direct consequence of a more general "Transport Theorem" (Theorem 5.1) that transports model structures on $\mathcal{L}$-spaces with global equivalences and well-behaved cofibrations to the category of $*$-modules. It was first proven in the author's (unpublished) Master's thesis [**2**]. A key step in the proof is that the category of $*$-modules can be identified with a category of algebras over a monad, which has previously been used in [**5, 1**] to construct non-equivariant model structures on S-modules and $*$-modules, respectively.

Our second main result, Theorem 4.6, lifts the above global model structure on $*$-modules to a model structure on the category of monoids in $*$-modules. It is Quillen equivalent to Schwede's global model structure on monoids in orthogonal spaces. In other words, monoids in orthogonal spaces and in $*$-modules form equivalent models for the global homotopy theory of A_∞-spaces.

The global model structure on orthogonal spaces also lifts to commutative monoids. It remains to be seen whether the analogous result is true for $*$-modules, i.e., whether $*$-modules model the global homotopy theory of E_∞ spaces.

Relation to other work: Orthogonal spaces, $\mathcal{L}$-spaces and $*$-modules are the unstable counterparts of the category of orthogonal spectra, the category of $\mathcal{L}$-spectra, and the category of S-modules, respectively. We refer to [**12**] for a discussion of non-equivariant model structures, the relationship with the classical unstable and stable homotopy categories and further references. Our main source for properties of $\mathcal{L}$-spaces and $*$-modules is the discussion by Blumberg, Cohen and Schlichtkrull in Section 4 of [**1**].

For a fixed group G, orthogonal spectra and S-modules with additional structure encoding the G-action have been studied equivariantly, see e.g., [**13, 9, 14**]. These additional data are not necessary in global homotopy theory. For each compact Lie group G, the G-equivariant homotopy groups of an ordinary orthogonal spectrum can be defined by evaluating only at G-representations. This idea gives rise to the global homotopy theory of orthogonal spectra and orthogonal spaces developed by Schwede in his monograph [**16**]. Schwede's work includes variants that don't take into account all compact Lie groups, but only a certain family of groups. Hausmann [**8**] gave an equivalent description in the case of all finite groups, using symmetric spectra as a model.

Organization: Section 2 provides background material on orthogonal spaces, $\mathcal{L}$-spaces and relevant functors. In Section 3, we discuss Schwede's global model structure for $\mathcal{L}$-spaces and our global model structure on $*$-modules, assuming the statement of the Transport Theorem (Theorem 5.1). We lift our model structure and Quillen equivalences to the level of monoids in Section 4. Finally, the proof of the Transport Theorem and other technical details are given in Section 5.

Conventions: We work over the category $\mathcal{U}$ of compactly generated weak Hausdorff spaces. A *model category* is a Quillen model category as defined in [**4**, Def. 3.3].

The definition does not require functorial factorizations. A *monoidal model category* satisfies the pushout product axiom and the unit axiom, see [10, Def. 4.2.6]. An *h-cofibration* in a model category tensored over $\mathcal{U}$ is a map that satisfies the homotopy extension property. In diagrams, the upper or left arrow of an adjunction is the left adjoint.

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2. Preliminaries

This section provides some background on the categories of $\mathcal{L}$-spaces, $*$-modules and orthogonal spaces, as well as on G-universes and universal subgroups. It does not contain any original results. The main sources are Schwede's preprint [15], his monograph [16] and the article [1] by Blumberg, Cohen and Schlichtkrull.

2.1. $\mathcal{L}$-spaces and $*$-modules

Let $\mathbf{L}(V,W)$ be the space of linear isometric embeddings $V \to W$ between two real inner product spaces of finite or countable dimension, topologized as a subspace of $\mathcal{U}(V,W)$. Write $\mathbb{R}^\infty := \bigoplus_{\mathbb{N}} \mathbb{R}$ for the standard inner product space of countable dimension. The *operad of linear isometric embeddings* $\mathcal{L}$ is given by spaces $\mathcal{L}(n) = \mathbf{L}((\mathbb{R}^\infty)^n, \mathbb{R}^\infty)$, and structure maps induced by direct sum and composition of maps. It is a (symmetric) E_∞-operad with Σ_n-actions via permutation of the n summands of $(\mathbb{R}^\infty)^n$. The space of unary operations $\mathcal{L}(1)$ is a topological monoid under composition, and we will study $\mathcal{L}(1)$-equivariant homotopy theory.

Definition 2.1. An *$\mathcal{L}$-space* is a space $X \in \mathcal{U}$ together with a continuous action from the monoid $\mathcal{L}(1)$. We write $\mathcal{LU}$ for the category of $\mathcal{L}$-spaces and $\mathcal{L}(1)$-equivariant maps.

The category $\mathcal{LU}$ is bicomplete where (co-)limits are taken in the category $\mathcal{U}$ of spaces and equipped with the respective (co-)limit action, because the forgetful functor to spaces has both adjoints. Moreover, $\mathcal{LU}$ is enriched, tensored and cotensored over $\mathcal{U}$.

The *box product* of $\mathcal{L}$-spaces X and Y is the balanced product

$$X \boxtimes_{\mathcal{L}} Y := \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (X \times Y)$$

with respect to the right $\mathcal{L}(1)^2$-action on $\mathcal{L}(2)$ given by precomposition on either summand of $\mathbb{R}^\infty \oplus \mathbb{R}^\infty$. The space $X \boxtimes_{\mathcal{L}} Y$ is an $\mathcal{L}$-space via the left $\mathcal{L}(1)$-action on $\mathcal{L}(2)$ given by postcomposition.

Lemma 2.2 (Hopkins' lemma, see [**5**], Lemma I.5.4)**.** *For all $m, n \geqslant 1$, the space $\mathcal{L}(m+n)$ is a split coequalizer of the diagram*

$$\mathcal{L}(2) \times \mathcal{L}(1)^2 \times (\mathcal{L}(m) \times \mathcal{L}(n)) \rightrightarrows \mathcal{L}(2) \times (\mathcal{L}(m) \times \mathcal{L}(n)),$$

hence $\mathcal{L}(m) \boxtimes_{\mathcal{L}} \mathcal{L}(n) \cong \mathcal{L}(m+n)$ as $\mathcal{L}$-spaces.

The box product admits coherent associativity and commutativity isomorphisms and a right adjoint $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(Y, -)$ for the functor $- \boxtimes_{\mathcal{L}} Y : \mathcal{LU} \to \mathcal{LU}$, see [**1**, Sect. 4.1] and [**2**, Def. 2.19]; cf. also [**5**, Sect. I.5]. We will give an explicit description of $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(Y, -)$ in Lemma 5.4 and record some of its properties in Proposition 5.5. There is a natural transformation

$$\lambda_{X,Y} : X \boxtimes_{\mathcal{L}} Y = \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (X \times Y) \to X \times Y,$$
$$[\psi_1 \oplus \psi_2, (x, y)] \mapsto (\psi_1 \cdot x, \psi_2 \cdot y),$$

which restricts to a *unital transformation*

$$\lambda_X : X \boxtimes_{\mathcal{L}} * = \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (X \times *) \to X,$$
$$[\psi_1 \oplus \psi_2, (x, *)] \mapsto \psi_1 \cdot x.$$

Here we used that each linear isometric embedding $\psi : \mathbb{R}^\infty \bigoplus \mathbb{R}^\infty \to \mathbb{R}^\infty$ is given as $\psi_1 \oplus \psi_2$, where the $\psi_i \in \mathcal{L}(1)$ have orthogonal images. Unfortunately, λ fails to be an isomorphism for all $\mathcal{L}$-spaces: For instance, all linear maps in the image of $\lambda_{\mathcal{L}(1)}$ have an infinite-dimensional orthogonal complement, hence it is not surjective.

However, λ_X is always a weak equivalence of underlying spaces, see [**1**, Sect. 4.1], and it satisfies an even stronger, equivariant notion of equivalence, see Proposition 3.2. In order to be able to refer to this situation, we make the following definition.

Definition 2.3. A relative category $(\mathcal{C}, \mathcal{W})$ is called a *weak (closed) symmetric monoidal category* if it is (closed) symmetric monoidal in the usual sense except that the left and right unital transformations are only required to lie in the class of weak equivalences $\mathcal{W}$, not necessarily in the class of isomorphisms.

Remark 2.4. Note that the usual definition of a monoid in a symmetric monoidal category in terms of two commutative diagrams still makes sense in a weak symmetric monoidal category. By a slight abuse of terminology, we will simply call such an object a "monoid" instead of a "weak monoid". Similarly, it makes sense to speak of monoidal functors between weak symmetric monoidal categories, and monoidal model structures on weak symmetric monoidal categories.

Definition 2.5. A *$*$-module* is an $\mathcal{L}$-space M such that λ_M is an isomorphism of $\mathcal{L}$-spaces.

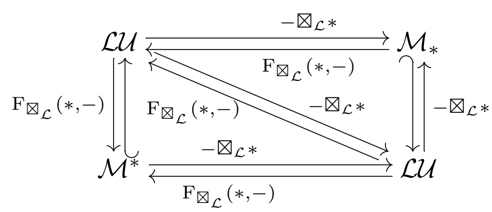
Surprisingly, the quotient $* \boxtimes_{\mathcal{L}} * = \mathcal{L}(2)/\mathcal{L}(1)^2$ is trivial, see [**5**, Lemma I.8.1]. Consequently, the functor $- \boxtimes_{\mathcal{L}} *$ on $\mathcal{L}$-spaces takes values in $*$-modules, and the box product restricts to a well-defined product on $\mathcal{M}_*$, which we denote by the same symbol $\boxtimes_{\mathcal{L}}$. So the category $\mathcal{LU}$ is a weak closed symmetric monoidal category, and then $\mathcal{M}_*$ is a symmetric monoidal category in the usual sense. The latter is also closed, as follows formally from Proposition 2.6 below.

Dually, we let $\mathcal{M}^*$ be the full subcategory of those $\mathcal{L}$-spaces such that the adjoint $\bar{\lambda}_Y : Y \to \mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, Y)$ is an isomorphism, and refer to its objects as *co-unital $\mathcal{L}$-spaces* or *co-$*$-modules.* The functor $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, -)$ on $\mathcal{LU}$ takes values in $\mathcal{M}^*$.

The following collection of statements from [**1**, Sect. 4.3] is an easy exercise in elementary category theory. It is the unstable analogue of a similar "mirror image" argument for S-modules, cf. [**5**, Sect. II.2].

Proposition 2.6. *The categories $\mathcal{M}_*$ and $\mathcal{M}^*$ of unital and co-unital $\mathcal{L}$-spaces, respectively, are "mirror image subcategories" in the following sense:*

a) All pairs of functors in the diagram below form adjunctions (where upper arrows and arrows on the left-hand side are left adjoints).



b) The subdiagrams of left-adjoint (respectively right-adjoint) functors commute up to natural equivalence.

c) The categories $\mathcal{M}_*$ and $\mathcal{M}^*$ are bicomplete. Colimits in $\mathcal{M}_*$ are created in $\mathcal{LU}$, limits are obtained by applying $-\boxtimes_{\mathcal{L}} *$ to limits in $\mathcal{LU}$; dually for $\mathcal{M}^*$.

d) The diagonal adjunction (co-)restricts to an equivalence of categories

$$\mathcal{M}^* \underset{\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-)}{\overset{-\boxtimes_{\mathcal{L}}*}{\rightleftarrows}} \mathcal{M}_*.$$

2.2. Completely universal subgroups

In this short section, we recall that every compact Lie group is isomorphic to an actual subgroup of $\mathcal{L}(1)$, a so-called *completely universal subgroup*. Thus, all compact Lie groups act simultaneously on each $\mathcal{L}$-space X; these actions are compatible in the sense that they all are restrictions of the same action of $\mathcal{L}(1)$ on X.

Definition 2.7. Let U_G be an orthogonal G-representation of countable dimension. We say that U_G is:

i) a *G-universe* if it contains a 1-dimensional trivial subrepresentation and has the property that for each finite-dimensional G-representation V that embeds into U_G, the representation $\bigoplus_{\mathbb{N}} V$ also embeds into U_G,

ii) a *complete G-universe* if it is a G-universe that contains one copy, and hence countably many copies of each irreducible G-representation.

Definition 2.8 ([**15**, Def. 1.4])**.** A compact subgroup $G \leqslant \mathcal{L}(1)$ is called *completely universal* if it admits the structure of a compact Lie group (necessarily unique, see [**3**, Prop. 3.12]) such that under the tautological action, $\mathbb{R}^\infty$ becomes a complete G-universe.

Remark 2.9. In [**15**], the completely universal subgroups are just called "universal subgroups", but we will use the more precise terminology.

Lemma 2.10 (cf. [**15**, Prop. 1.5])**.** *The equivalence classes of completely universal subgroups of $\mathcal{L}(1)$ under conjugation by invertible elements of $\mathcal{L}(1)$ biject with the isomorphism classes of compact Lie groups.*

In Section 3, we will introduce various notions of equivalences and fibrations detected on G-fixed points for all completely universal subgroups $G \leqslant \mathcal{L}(1)$.

2.3. Orthogonal spaces

Write $\mathcal{I}$ for the category of finite-dimensional real inner product spaces with morphisms the linear isometric embeddings. It is enriched over spaces, see the beginning of Subsection 2.1.

Definition 2.11. An *orthogonal space* is a continuous functor $Y: \mathcal{I} \to \mathcal{U}$. We write $\mathcal{IU}$ for the category of orthogonal spaces and natural transformations.

The category $\mathcal{IU}$ is bicomplete, with (co-)limits taken objectwise. Moreover, it is tensored and co-tensored over $\mathcal{U}$ where, for $Y \in \mathcal{IU}$, $A \in \mathcal{U}$, the tensor orthogonal space $Y \times A$ sends $V \in I$ to $(Y \times A)(V) := Y(V) \times A$. Equivalently, we can regard A as the constant orthogonal space with value A and form the product in $\mathcal{IU}$.

The category of orthogonal spaces is a closed symmetric monoidal category under the *box product*, which is the Day convolution product with respect to direct sum of vector spaces in $\mathcal{I}$ and the product in $\mathcal{U}$, see [**16**, Sect. 1.3, App. C] for further details. A unit is given by the constant one-point orthogonal space.

Following Schwede, we take *global equivalences* of orthogonal spaces to be those morphisms that, for each compact Lie group G, induce G-weak equivalences on homotopy colimits along G-representations. The precise definition is given in more elementary terms, cf. [**16**, Rem. 1.1.4].

Definition 2.12 ([**15**, Def. 3.4])**.** A morphism $f: X \to Y$ of orthogonal spaces is a *global equivalence* if for any compact Lie group G, any orthogonal G-representation V of finite dimension, any $k \geqslant 0$ and any commuting square

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{\alpha} & X(V)^G \\ {\scriptstyle \mathrm{incl}}\downarrow & & \downarrow{\scriptstyle f(V)^G} \\ D^k & \xrightarrow[\beta]{} & Y(V)^G \end{array}$$

there is a finite-dimensional G-representation W, a G-equivariant linear isometric embedding $\varphi: V \to W$ and a map $\lambda: D^k \to X(W)^G$ such that in the extended diagram

$$\begin{array}{ccccc} S^{k-1} & \xrightarrow{\alpha} & X(V)^G & \xrightarrow{X(\varphi)^G} & X(W)^G \\ {\scriptstyle \mathrm{incl}}\downarrow & & & \overset{\lambda}{\nearrow} & \downarrow{\scriptstyle f(W)^G} \\ D^k & \xrightarrow[\beta]{} & Y(V)^G & \xrightarrow[Y(\varphi)^G]{} & Y(W)^G \end{array}$$

the upper triangle commutes strictly and the lower triangle commutes up to homotopy relative to S^{k-1}.

Theorem 2.13 (Global model structures for orthogonal spaces, cf. [**16**, Thm. 1.2.21, Prop. 1.2.23])**.** *The global equivalences are part of two proper, topological, cofibrantly generated model structures on the category of orthogonal spaces, the* (absolute) global model structure $(\mathcal{IU})_{\mathrm{abs}}$ *and the* positive global model structure $(\mathcal{IU})_{\mathrm{pos}}$.

We omit the description of the (co-)fibrations in the two global model structures since we will not make use of them explicitly. As usual, the positive variant has a better behaviour with respect to commutative monoids, and for different reasons, it is also necessary for us to work with the positive global model structure throughout the paper, see Remark 3.7. The absolute model structure will only appear in Section 4.

Note that both global model structures are monoidal with respect to the box product of orthogonal spaces: The pushout product axiom is proven in [**16**, Prop. 1.4.12 iii), iv)], while the unit axiom follows from [**16**, Thm. 1.3.2 ii)].

The categories $\mathcal{IU}$ and $\mathcal{LU}$ can be connected by an adjoint pair of functors. By general theory, the right exact enriched functors (i.e., those which preserve colimits and tensors) $\mathcal{DU} \to \mathcal{C}$ out of a category of diagram spaces into a category $\mathcal{C}$ that is enriched and cocomplete in the enriched sense agree with the enriched functors $\mathcal{D}^{op} \to \mathcal{C}$ up to isomorphism of categories; see [**13**, Sect. I.2] and note that the results also apply in the unbased case.

Lind [**12**, Def. 8.2] defines a functor $\mathbb{Q}^* : \mathcal{I}^{op} \to \mathcal{LU}$ that sends V to $\mathcal{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty)$; it is strong symmetric monoidal by Lemma 2.2. The results of [**13**] then yield an adjunction

$$\mathcal{IU} \underset{\mathbb{Q}^\#}{\overset{\mathbb{Q}}{\rightleftarrows}} \mathcal{LU},$$

where the left adjoint $\mathbb{Q}$ is given as an enriched coend $\mathbb{Q}^* \otimes_{\mathcal{I}} (-)$ and the right adjoint is $\mathbb{Q}^\# X(V) = \mathcal{LU}(\mathbb{Q}^*(V), X)$. The first is strong, the latter lax symmetric monoidal.

The functor $\mathbb{Q}^* \colon \mathcal{I}^{op} \to \mathcal{LU}$ can be replaced by $\mathbb{Q}^*_* \colon V \mapsto \mathcal{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \boxtimes_{\mathcal{L}} *$. Then $\mathbb{Q}^*_*$ takes values in $*$-modules and yields an adjunction

$$\mathcal{IU} \underset{\mathbb{Q}^\#_*}{\overset{\mathbb{Q}_*}{\rightleftarrows}} \mathcal{M}_*,$$

defined in the same way as before. Again, the left adjoint is strong, the right adjoint lax symmetric monoidal. By [**12**, Lemma 8.6], this pair of functors agrees, up to natural equivalence, with the composition of adjunctions

$$\mathcal{IU} \underset{\mathbb{Q}^\#}{\overset{\mathbb{Q}}{\rightleftarrows}} \mathcal{LU} \underset{\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-)}{\overset{-\boxtimes_{\mathcal{L}} *}{\rightleftarrows}} \mathcal{M}_*.$$

Remark 2.14. There is another interesting choice of a functor $\mathcal{IU} \to \mathcal{LU}$. For an orthogonal space Y, the colimit $Y(\mathbb{R}^\infty) := \operatorname{colim}_V Y(V)$ taken over all finite-dimensional inner product subspaces $V \subseteq \mathbb{R}^\infty$ (or equivalently, all standard Euclidean spaces $\mathbb{R}^n$) has a canonical $\mathcal{L}$-space structure, see [**15**, Constr. 3.2]. The resulting functor $\mathbb{O} \colon \mathcal{IU} \to \mathcal{LU}$ is induced by $\mathbb{O}^* \colon \mathcal{I}^{op} \to \mathcal{LU}$ sending $V \in \mathcal{I}$ to $\mathbf{L}(V, \mathbb{R}^\infty)$, see [**12**, Lemma 9.6]. In unpublished work, Schwede proved that $\mathbb{O}$ is strong symmetric monoidal. It follows formally that its rightadjoint is a lax symmetric monoidal functor.

Any choice of a one-dimensional subspace of $\mathbb{R}^\infty$ defines a linear isometric embedding $V \to V \otimes \mathbb{R}^\infty$, hence a natural transformation $\xi^* \colon \mathbb{Q}^*(V) = \mathcal{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \to \mathcal{L}(V, \mathbb{R}^\infty) \cong \mathbb{O}^*(V)$ which in turn determines a natural map $\xi = \xi^* \otimes_I (-) \colon \mathbb{Q} \to \mathbb{O}$. The latter is symmetric monoidal; moreover, it is a global equivalence on cofibrant objects in the absolute model structure on orthogonal spaces, see [**15**, Prop. 3.7]. A precursor of the last statement was [**12**, Lemma 9.7].

3. Global model structures for $\mathcal{LU}$ and $\mathcal{M}_*$

We recall Schwede's model structures for $\mathcal{L}$-spaces from [**15**] and derive our first main result, Theorem 3.8. It establishes the global model structure for $*$-modules which is Quillen equivalent to orthogonal spaces via the functor $\mathbb{Q}_*$.

The following notions of equivalences and fibrations of $\mathcal{L}$-spaces will also be used for maps of $*$-modules by viewing them as maps in $\mathcal{LU}$.

Definition 3.1 ([**15**, Def. 1.6, Def. 1.8])**.** Let $\mathcal{C}^L$ denote the set of all compact Lie subgroups of $\mathcal{L}(1)$. A map $f\colon X \to Y$ of $\mathcal{L}$-spaces is called:

- a $\mathcal{C}^L$-*equivalence* (respectively $\mathcal{C}^L$-*fibration*) if the map $f^G\colon X^G \to Y^G$ is a weak homotopy equivalence (respectively Serre fibration) of spaces for all compact Lie subgroups $G \leqslant \mathcal{L}(1)$;
- a *global equivalence* if $f^G\colon X^G \to Y^G$ is a weak homotopy equivalence of spaces for all completely universal subgroups $G \leqslant \mathcal{L}(1)$;
- a *strong global equivalence* if the map f, considered as a map of G-spaces, is a G-equivariant homotopy equivalence for all completely universal subgroups $G \leqslant \mathcal{L}(1)$.

Proposition 3.2. *The natural map of $\mathcal{L}$-spaces $\lambda_{X,Y}\colon X \boxtimes_{\mathcal{L}} Y \to X \times Y$ is a strong global equivalence for all $X, Y \in \mathcal{LU}$. Consequently, so is the adjoint map $\bar{\lambda}_X\colon X \to \mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, X)$. Both functors $(-) \boxtimes_{\mathcal{L}} *$ and $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, -)$ preserve and reflect (strong) global equivalences. For all $Z \in \mathcal{LU}$, the functor $(-) \boxtimes_{\mathcal{L}} Z$ preserves (strong) global equivalences.*

Proof. The first part is [**15**, Thm. 1.21]. In combination with the 2-out-of-3 property and the following diagram, it implies the second statement; the third then follows immediately.

$$\begin{array}{ccc} X \boxtimes_{\mathcal{L}} * & \xrightarrow[\sim]{\lambda_X} & X \\ {\scriptstyle \bar{\lambda}_X \boxtimes_{\mathcal{L}} *}\downarrow \cong & & \downarrow {\scriptstyle \bar{\lambda}_X} \\ \mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, X) \boxtimes_{\mathcal{L}} * & \xrightarrow[\lambda_{\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,X)}]{\sim} & \mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, X) \end{array}$$

Now let $Z \in \mathcal{LU}$ be arbitrary. If f is a (strong) global equivalence, then so is $f \times Z$, hence also $f \boxtimes_{\mathcal{L}} Z$. □

Recall that for any $G \leqslant \mathcal{L}(1)$, the $\mathcal{L}$-space $\mathcal{L}(1)/G$ represents the fixed point functor $(-)^G\colon \mathcal{LU} \to \mathcal{U}$. The collection of fixed point functors associated to $G \in \mathcal{C}^L$ gives rise to the following model structure on $\mathcal{L}$-spaces.

Proposition 3.3 ($\mathcal{C}^L$-projective model structure for $\mathcal{L}$-spaces, [**15**, Prop. 1.11])**.** *There is a proper topological model structure $(\mathcal{LU})_{\mathcal{C}^L}$ on the category of $\mathcal{L}$-spaces with weak equivalences and fibrations the $\mathcal{C}^L$-equivalences and $\mathcal{C}^L$-fibrations. It is cofibrantly generated with sets of generating (acyclic) cofibrations obtained by tensoring the standard generating (acyclic) cofibrations for spaces $S^{n-1} \to D^n$ (respectively $D^n \times 0 \to D^n \times I$) with $\mathcal{L}$-spaces of the form $\mathcal{L}(1)/G$, where G runs through all compact Lie subgroups of $\mathcal{L}(1)$.*

The $\mathcal{C}^L$-projective model structure seems unlikely to be Quillen equivalent to $\mathcal{IU}$ with its positive global model structure, but one can perform a left Bousfield localization such that the weak equivalences become precisely the class of global equivalences. This detour is necessary in order to guarantee that the adjunction to orthogonal spaces becomes a Quillen adjunction. We refer the reader to [**15**, Section 1] for a detailed discussion of this Bousfield localization and an explicit description of the local objects.

Theorem 3.4 (Global model structure for $\mathcal{L}$-spaces, see [**15**, Thm. 1.20])**.** *There is a cofibrantly generated proper topological model structure $(\mathcal{LU})_{\mathrm{gl}}$ on the category of $\mathcal{L}$-spaces with weak equivalences the global equivalences and cofibrations as in $(\mathcal{LU})_{\mathcal{C}^L}$. Every $\mathcal{C}^L$-cofibration is an h-cofibration of $\mathcal{L}$-spaces and a closed embedding of underlying spaces.*

Proposition 3.5. *The model structure $(\mathcal{LU})_{\mathrm{gl}}$ is a monoidal model category.*

Proof. The pushout product axiom is proven in [**15**, Prop. 1.22], the unit axiom follows from Proposition 3.2. □

The global model structures for orthogonal spaces and $\mathcal{L}$-spaces model the same homotopy theory.

Theorem 3.6 ([**15**, Thm. 3.9])**.** *The adjunction*

$$(\mathcal{IU})_{\mathrm{pos}} \underset{}{\overset{\mathbb{Q}}{\rightleftarrows}} (\mathcal{LU})_{\mathrm{gl}}$$

is a Quillen equivalence.

Remark 3.7. The functor $\mathbb{Q}^{\#}$ is not a right Quillen functor anymore if we use the absolute model structure on orthogonal spaces instead: If $\mathbb{Q}^{\#}X$ is fibrant in the absolute model structure, then [**16**, Def. 1.2.12] (for G the trivial group and $V \to W$ the inclusion $0 \to \mathbb{R}$) implies that the inclusion of fixed points $X^{\mathcal{L}(1)} \to X$ must be a weak homotopy equivalence of spaces. It seems very unlikely that this could be true for all fibrant $\mathcal{L}$-spaces X.

Assuming the Transport Theorem (Theorem 5.1), we will now prove the following:

Theorem 3.8 (Global model structure for ∗-modules)**.** *There is a cofibrantly generated proper topological model structure on the category $\mathcal{M}_*$ of ∗-modules, the* global model structure $(\mathcal{M}_*)_{\mathrm{gl}}$. *Its weak equivalences are the global equivalences of underlying $\mathcal{L}$-spaces, its fibrations are detected by the functor $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-)\colon \mathcal{M}_* \to (\mathcal{LU})_{\mathrm{gl}}$. Let I and J be any sets of generating (acyclic) cofibrations for $(\mathcal{LU})_{\mathrm{gl}}$, then $I \boxtimes_{\mathcal{L}} *$ and $J \boxtimes_{\mathcal{L}} *$ are generating (acyclic) cofibrations for $(\mathcal{M}_*)_{\mathrm{gl}}$.*

*Moreover, the global model structure for $\mathcal{M}_*$ is monoidal and satisfies the monoid axiom [**17**, Def. 3.3] with respect to $\boxtimes_{\mathcal{L}}$. It fits into the following commutative (up to natural isomorphism) triangle of monoidal Quillen equivalences:*

$$\begin{array}{ccc} (\mathcal{IU})_{\mathrm{pos}} & \overset{\mathbb{Q}}{\rightleftarrows} & (\mathcal{LU})_{\mathrm{gl}} \\ & \searrow^{\mathbb{Q}_*} \nwarrow & \downarrow\uparrow \,\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-) \\ & & (\mathcal{M}_*)_{\mathrm{gl}} \end{array}$$

Proof. The global model structure obviously satisfies the requirements of the Transport Theorem (Theorem 5.1), which immediately implies the existence and properties of the model structure $(\mathcal{M}_*)_{\mathrm{gl}}$. It also proves that the vertical adjunction is a Quillen equivalence. The horizontal adjunction is a Quillen equivalence by Theorem 3.6, and we have already seen that all adjunctions are monoidal. □

Remark 3.9. There is a variant of Theorem 3.8 with respect to the functor $\mathbb{O}\colon \mathcal{IU} \to \mathcal{LU}$ introduced in Remark 2.14: It is possible to establish a model structure on $\mathcal{L}$-spaces with weak equivalences the global equivalences and such that $\mathbb{O}$ is a left Quillen equivalence with respect to the absolute global model structure on orthogonal spaces. This model structure also satisfies the hypotheses of the Transport Theorem, but is harder to work with as the cofibrations cannot only be characterized in terms of fixed points of group actions. It also lifts to monoids and the analogue of Theorem 4.6 holds. Details can be found in the author's (unpublished) Master's thesis [**2**].

Remark 3.10. The diagram in Theorem 3.8 can be extended to the right: By a version of Elmendorf's theorem, $(\mathcal{LU})_{\mathrm{gl}}$ is Quillen equivalent to a model category of "systems of global fixed point sets". As usual, these are diagram spaces indexed on the opposite of a suitable "global" orbit category. We refer to [**15**, Section 2] for details.

4. Monoids and modules in global homotopy theory

Monoids with respect to $\boxtimes_{\mathcal{L}}$ and their modules have been described non-equivariantly by Blumberg, Cohen and Schlichtkrull, see [**1**, Thm. 4.18]. We describe "global" analogues of their result and prove our second main result, Theorem 4.6.

Recall from Section 2 that $\mathcal{L}$ denotes the operad of linear isometric embeddings of $\mathbb{R}^\infty$. The following identifications are a consequence of Hopkins' Lemma 2.2.

Proposition 4.1 ([**1**, Prop. 4.7])**.** *The category of A_∞-spaces structured by $\mathcal{L}$ (considered as a non-symmetric operad) is isomorphic to the category of $\boxtimes_{\mathcal{L}}$-monoids in $\mathcal{LU}$. The category of E_∞-spaces structured by $\mathcal{L}$ (considered as a symmetric operad) is isomorphic to the category of commutative $\boxtimes_{\mathcal{L}}$-monoids in $\mathcal{LU}$.*

Corollary 4.2 ([**1**, Sect. 4.4])**.** *The $\boxtimes_{\mathcal{L}}$-monoids in $\mathcal{M}_*$ are those A_∞-spaces which are $*$-modules. The functor $-\boxtimes_{\mathcal{L}} *\colon \mathcal{LU} \to \mathcal{M}_*$ takes $\boxtimes_{\mathcal{L}}$-monoids in $\mathcal{LU}$ to $\boxtimes_{\mathcal{L}}$-monoids in $\mathcal{M}_*$ and the natural map $\lambda_X\colon X \boxtimes_{\mathcal{L}} * \to X$ is a map of $\boxtimes_{\mathcal{L}}$-monoids if X is a $\boxtimes_{\mathcal{L}}$-monoid. The analogous statement is true for commutative monoids and E_∞-spaces.*

In [**17**], Schwede and Shipley describe sufficient conditions for a cofibrantly generated monoidal model structure to lift to the associated categories of R-modules and R-algebras, respectively, where R is any (commutative) monoid. When applied to the global model structure on $*$-modules, this yields:

Theorem 4.3. *Consider the category of $*$-modules equipped with the global model structure and let R be a $\boxtimes_{\mathcal{L}}$-monoid in $\mathcal{M}_*$. Call a morphism of R-algebras a weak equivalence (respectively fibration) if it is a weak equivalence (respectively fibration) of underlying $*$-modules. With respect to these classes of morphisms, the following hold:*

1) The category of left R-modules is a cofibrantly generated model category.

2) If R is commutative, then the category of R-modules is a cofibrantly generated model category satisfying the pushout product axiom and the monoid axiom.

3) If R is commutative, then the category of R-algebras is a cofibrantly generated model category. If the source of a cofibration of R-algebras is cofibrant as an R-module, then the map is a cofibration of R-modules.

In all cases, sets of generating cofibrations and acyclic cofibrations are given by the images of generating sets for $\mathcal{M}_*$ under the free functor.

For $R = *$, the category of R-algebras is the category of $\boxtimes_{\mathcal{L}}$-monoids. It has a cofibrantly generated model structure by part 3) of the theorem.

Proof. We check the hypotheses of [**17**, Thm. 4.1]. As explained in [**17**, Rem. 4.2], the smallness assumption can be weakened; it then follows from the fact that the forgetful functors from R-modules and monoids, respectively, commute with filtered colimits, and from Lemma 5.6.

By part h) of Theorem 5.1, $(\mathcal{M}_*)_{\mathrm{gl}}$ satisfies the monoid axiom as defined in [**17**, Def. 3.3]. □

Theorem 4.4. *The analogue of Theorem 4.3 with respect to the monoidal model category $(\mathcal{LU})_{\mathrm{gl}}$ is true.*

Proof. A close inspection of the proof of [**17**, Thm. 4.1] shows that the first two statements do not require that the unital transformation is an isomorphism, so these hold because $(\mathcal{LU})_{\mathrm{gl}}$ satisfies the monoid axiom, see part h) of Theorem 5.1. The proof of the third statement makes use of the unital isomorphism in order to verify that all relative J_T-cell complexes are weak equivalences. We will give an alternative proof of this fact instead:

Here, $T\colon X \mapsto \coprod_{n\geqslant 0} X^{\boxtimes_{\mathcal{L}} n}$ is the composition of the free monoid functor with the forgetful functor, J is any set of generating acyclic cofibrations for $(\mathcal{LU})_{\mathrm{gl}}$, and J_T denotes its image under T. All maps in J are h-cofibrations (i.e., have the homotopy extension property) and global equivalences. For each $Z \in \mathcal{LU}$, the left adjoint functor $Z \boxtimes_{\mathcal{L}} (-)\colon \mathcal{LU} \to \mathcal{LU}$ preserves these properties by Proposition 3.2 and Lemma 5.2. Thus, for a map $j\colon A \to B$ in J and $n \geqslant 2$, we can write the n-th summand $j^{\boxtimes_{\mathcal{L}} n}$ of $T(j)$ as a composition

$$(j \boxtimes_{\mathcal{L}} A^{\boxtimes_{\mathcal{L}} (n-1)}) \circ (B \boxtimes_{\mathcal{L}} j \boxtimes_{\mathcal{L}} A^{\boxtimes_{\mathcal{L}} (n-2)}) \circ \ldots \circ (B^{\boxtimes_{\mathcal{L}} (n-1)} \boxtimes_{\mathcal{L}} j).$$

of maps which are both h-cofibrations and global equivalences. These properties are stable under composition and coproducts, hence $T(j)$ has both properties. Moreover, the class of h-cofibrations which are global equivalences is closed under cobase change and transfinite composition, thus each morphism in $J_T - cell$ is a global equivalence. Smallness is not an issue because all $\mathcal{L}$-spaces are small relative to closed embeddings (Lemma 5.6), and so relative to all images of cofibrations under T. □

Theorem 4.5. *The analogue of Theorem 4.3 with respect to the monoidal model categories $(\mathcal{IU})_{\mathrm{abs}}$ and $(\mathcal{IU})_{\mathrm{pos}}$ is true.*

Proof. Every acyclic cofibration in the positive global model structure on $\mathcal{IU}$ is an acyclic cofibration in the absolute global model structure. The latter satisfies the monoid axiom, see [**16**, Prop. 1.4.13], hence so does the former. Again, [**17**, Thm. 4.1] applies. □

We can now state our second main result.

Theorem 4.6 (Global model structure for monoids in $*$-modules)**.** *The triangle of monoidal Quillen equivalences from Theorem 3.8 gives rise to a triangle of Quillen equivalences between the respective model structures on categories of monoids.*

Proof. For all three categories, the forgetful functors from monoids preserve and reflect fibrations and weak equivalences. Thus for all three adjunctions from Theorem 3.8, the lifted right adjoints are always right Quillen functors, and it remains to show that they are Quillen equivalences.

The induced adjunction between monoids in $\mathcal{LU}$ and $\mathcal{M}_*$ is a Quillen equivalence because the functor $- \boxtimes_{\mathcal{L}} *$ preserves and reflects global equivalences and the counit $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, X) \boxtimes_{\mathcal{L}} * \to X$ is an isomorphism for all $X \in \mathcal{M}_*$, see Proposition 2.6.

Now consider the Quillen adjunction between monoids in $\mathcal{IU}$ and $\mathcal{LU}$. We will show that the derived adjunction is an equivalence of categories. More precisely, we will mimick parts of the proof of [**15**, Thm. 3.9] and show that:

(I) the derived right adjoint reflects isomorphisms, and

(II) the unit of the derived adjunction is a natural isomorphism.

Since we are working with model structures on monoids created by the forgetful functor, statement (I) immediately follows from fact (a) in the proof of [**15**, Thm. 3.9]. In order to prove statement (II), it suffices to show that for all positively cofibrant monoids M in $\mathcal{IU}$ and some (hence any) fibrant replacement $(-)_{f,mon}$ in the category of monoids in $\mathcal{LU}$, the underlying map of orthogonal spaces $M \to \mathbb{Q}^{\#}((\mathbb{Q}(M))_{f,mon})$ is a global equivalence. The monoidal unit in $\mathcal{IU}$ is absolutely cofibrant, hence the underlying orthogonal space of any positively cofibrant monoid M is absolutely cofibrant, see Theorem 4.5 and part 3) of Theorem 4.3. Now fact (b) in the proof of loc. cit. asserts that for all positively cofibrant orthogonal spaces A and some (hence any) fibrant replacement $(-)_f$ in $\mathcal{LU}$, the map of orthogonal spaces $A \to \mathbb{Q}^{\#}((\mathbb{Q}(A))_f)$ is a global equivalence. Moreover, the proof given in loc. cit. works without changes for absolutely cofibrant orthogonal spaces A. As any fibrant replacement of monoids in $\mathcal{LU}$ is also a fibrant replacement of the underlying $\mathcal{L}$-spaces, we see that our statement (II) follows.

Finally, the adjunction between monoids in $\mathcal{IU}$ and $\mathcal{M}_*$ is a Quillen equivalence as the composition of two Quillen equivalences. □

In light of Proposition 4.1, Theorem 4.6 states that there is an unambiguous global homotopy theory of A_∞-spaces. We don't know if this statement is true for E_∞-spaces: The positive global model structure $(\mathcal{IU})_{\mathrm{pos}}$ lifts to commutative monoids, see [**16**, Thm. 2.1.15], but it remains open whether the same holds for $(\mathcal{M}_*)_{\mathrm{gl}}$. The difficulty is in showing that the functor $(-)^{\boxtimes_{\mathcal{L}} n}/\Sigma_n$ takes acyclic cofibrations to global equivalences.

5. The Transport Theorem

We, finally, give a precise statement and proof of the Transport Theorem. Throughout this section, let F denote the functor $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, -)\colon \mathcal{LU} \to \mathcal{M}^*$ and let R be its right adjoint, the forgetful functor $\mathcal{M}^* \to \mathcal{LU}$.

Theorem 5.1 (Transport Theorem)**.** *Let $(\mathcal{LU})_a$ be any model structure on the category $\mathcal{LU}$ of $\mathcal{L}$-spaces such that:*

i) it is cofibrantly generated, with sets of generating cofibrations and acyclic cofibrations denoted by I and J, respectively

ii) all morphisms in I (and hence in J) are closed embeddings of underlying spaces

iii) the class $\mathcal{W}$ of weak equivalences contains all strong global equivalences (in the sense of Definition 3.1)

iv) the class of morphisms which are simultaneously weak equivalences and closed embeddings is closed under transfinite composition.

Then the category of $*$-modules $\mathcal{M}_*$ admits a model structure $(\mathcal{M}_*)_a$ satisfying the following properties:

a) It is cofibrantly generated, with sets of generating cofibrations and acyclic cofibrations given by $I \boxtimes_{\mathcal{L}} *$ and $J \boxtimes_{\mathcal{L}} *$, respectively.

b) The weak equivalences are precisely those morphisms of $*$-modules which are sent to $\mathcal{W}$ under the forgetful functor to $\mathcal{LU}$.

c) Fibrations are detected by the functor $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, -)\colon \mathcal{M}_* \to (\mathcal{LU})_a$.

d) The adjunction

$$(\mathcal{LU})_a \underset{\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-)}{\overset{-\boxtimes_{\mathcal{L}}*}{\rightleftarrows}} (\mathcal{M}_*)_a$$

is a Quillen equivalence.

Moreover:

e) If $(\mathcal{LU})_a$ is right proper, then so is $(\mathcal{M}_*)_a$. If $(\mathcal{LU})_a$ is a topological model category, then so is $(\mathcal{M}_*)_a$.

f) If $(\mathcal{LU})_a$ satisfies the pushout product axiom with respect to the box product, then so does $(\mathcal{M}_*)_a$.

Assume in addition that all elements of I are h-cofibrations in $\mathcal{LU}$ and $\mathcal{W}$ is a class of equivalences detected by a family of fixed point functors to spaces. Then:

g) Both $(\mathcal{LU})_a$ and $(\mathcal{M}_*)_a$ are left proper.

*h) Both $(\mathcal{LU})_a$ and $(\mathcal{M}_*)_a$ satisfy the unit axiom and monoid axiom [**17**, Def. 3.3].*

Before turning to the proof, we record some technical, but very useful results.

Lemma 5.2 ([**16**, Cor. A.30])**.** *Let $\mathcal{C}, \mathcal{C}'$ be two cocomplete categories which are enriched and tensored over spaces. Let $G\colon \mathcal{C} \to \mathcal{C}'$ be a continuous functor that preserves pushouts along h-cofibrations and commutes with taking tensors with the unit interval I. Then G takes h-cofibrations in $\mathcal{C}$ to h-cofibrations in $\mathcal{C}'$.*

Lemma 5.3 (Gluing lemma)**.** *Consider the following pushout diagram in $\mathcal{LU}$ or $\mathcal{M}_*$, where one of the maps f or g is an h-cofibration.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow[k]{} & W \end{array}$$

If f is a global equivalence, then so is k. The statement remains true if "global equivalence" is replaced with any class of weak equivalences detected by a family of fixed point functors to spaces.

Proof. Colimits in $\mathcal{M}_*$ are created in $\mathcal{LU}$. Since one of the legs of the pushout is an h-cofibration, it is a closed embedding of spaces. Thus, taking fixed points with respect to any closed subgroup of $\mathcal{L}(1)$ preserves the pushout. Moreover, by Lemma 5.2, taking fixed points sends h-cofibrations in $\mathcal{LU}$ (or $\mathcal{M}_*$, respectively) to h-cofibrations of spaces. Now the claim follows from the gluing lemma for h-cofibrations and weak homotopy equivalences in spaces. □

The next observation is obtained by composing several standard adjunctions.

Lemma 5.4. *The underlying $\mathcal{L}$-space of $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(Y,Z)$ is given by $\mathcal{LU}(Y,\mathcal{LU}(\mathcal{L}(2),Z))$ with actions as follows: The space of $\mathcal{L}(1)$-equivariant maps $\mathcal{LU}(\mathcal{L}(2),Z)$ is formed with respect to the left $\mathcal{L}(1)$-action on $\mathcal{L}(2)$ induced by post-composition of linear maps. This mapping space is an $\mathcal{L}$-space via the right $\mathcal{L}(1)$-action on $\mathcal{L}(2)$ induced by pre-composition on the second summand of $(\mathbb{R}^\infty)^2$. Finally, the $\mathcal{L}(1)$-action on $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(Y,Z)$ comes from the right $\mathcal{L}(1)$-action on $\mathcal{L}(2)$ induced by pre-composition on the first summand of $(\mathbb{R}^\infty)^2$.*

Proposition 5.5. *Let Y be any $\mathcal{L}$-space and consider the functor $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(Y,-)\colon \mathcal{LU}\to\mathcal{LU}$. Then the following hold:*

- *i) If f is a closed embedding, then so are $\mathcal{LU}(Y,f)$ and $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(Y,f)$.*
- *ii) The functor $\mathcal{LU}(\mathcal{L}(2),-)$ takes sequential colimits along closed embeddings to sequential colimits along closed embeddings.*
- *iii) The functor $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-)$ preserves sequential colimits along closed embeddings.*
- *iv) If $\mathcal{W}$ is a class of weak equivalences satisfying the assumptions of Theorem 5.1, then $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-)$ preserves and reflects $\mathcal{W}$.*

Proof. Ad i): The functor $\mathcal{LU}(Y,-)$ preserves closed embeddings because $\mathcal{LU}(Y,Z)$ is topologized as a closed subspace of $\mathcal{U}(Y,Z)$. The functor $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(Y,-)$ is a composition of $\mathcal{LU}(\mathcal{L}(2),-)$ and $\mathcal{LU}(Y,-)$.

Ad ii): Any choice of linear isometry $\mathbb{R}^\infty\cong(\mathbb{R}^\infty)^2$ induces an isomorphism of $\mathcal{L}$-spaces $\mathcal{L}(2)\cong\mathcal{L}(1)$, thus the underlying space of $\mathcal{LU}(\mathcal{L}(2),Z)$ is naturally isomorphic to Z. It follows that for any sequence of closed embeddings of $\mathcal{L}$-spaces

$$Z_0\to Z_1\to Z_2\to\dots,$$

the canonical map

$$\operatorname*{colim}_i \mathcal{LU}(\mathcal{L}(2),Z_i)\to\mathcal{LU}(\mathcal{L}(2),\operatorname*{colim}_i Z_i)$$

is a homeomorphism of spaces. Moreover, it is equivariant with respect to the $\mathcal{L}(1)$-action induced by precomposition on the first summand of $(\mathbb{R}^\infty)^2$.

Ad iii): By part ii) and Lemma 5.4, it suffices to show that $\mathcal{LU}(*,-)$ preserves sequential colimits along sequences of closed embeddings. This is true because it is just the fixed point functor $\mathcal{LU}(\mathcal{L}(1)/\mathcal{L}(1),-)\cong(-)^{\mathcal{L}(1)}$.

Ad iv): Let $f\colon X \to Y$ be in $\mathcal{W}$. In the diagram of $\mathcal{L}$-spaces

$$\begin{array}{ccc} X & \xrightarrow{\bar\lambda} & \mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, f)} \\ Y & \xrightarrow{\bar\lambda} & \mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, Y) \end{array}$$

both horizontal maps are strong global equivalences by Proposition 3.2. The strong global equivalences are contained in the class of weak equivalences $\mathcal{W}$ by assumption iii) of Theorem 5.1, thus $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, f)$ is a weak equivalence if and only if f is. □

Lemma 5.6. *All $\mathcal{L}$-spaces, co-$*$-modules and $*$-modules are small with respect to sequences of closed embeddings in the sense of [**10**, Def. 2.1.3].*

Proof. The forgetful functors $\mathcal{M}_* \to \mathcal{LU}$ and $\mathcal{LU} \to \mathcal{U}$ both have left adjoints, so colimits in either category can be formed in $\mathcal{U}$. Consequently, all $\mathcal{L}$-spaces and $*$-modules are small with respect to sequences of closed embeddings. Colimits in $\mathcal{M}^*$ are computed by applying $F = \mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, -)$ to a colimit formed in $\mathcal{LU}$. By Proposition 5.5, F preserves sequential colimits along closed embeddings, thus the smallness statement for $\mathcal{M}^*$ follows from the one for $\mathcal{LU}$. □

In order to prove Theorem 5.1, we construct an intermediate model structure $(\mathcal{M}^*)_a$ on co-$*$-modules, thus exploiting the fact that, up to equivalence of categories, $\mathcal{M}_*$ is a category of algebras over a well-behaved monad. This approach was used by Blumberg, Cohen and Schlichtkrull to transport their non-equivariant model structure in [**1**, Sect. 4.6], and goes back to [**5**]. Consider the following diagram:

$$\mathcal{LU} \underset{R}{\overset{F}{\rightleftarrows}} \mathcal{LU}\,[\mathbb{F}] \cong \mathcal{M}^* \underset{\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-)}{\overset{-\boxtimes_{\mathcal{L}} *}{\rightleftarrows}} \mathcal{M}_* \tag{5.7}$$

We have seen in Proposition 2.6 that the adjunction on the right-hand side is an equivalence of categories. The proof of the identification $\mathcal{LU}\,[\mathbb{F}] \cong \mathcal{M}^*$ is identical with the proof of [**5**, Prop. II.2.7], where $\mathbb{F}$ denotes the monad $\mathbb{F} = RF$ associated to the adjunction on the left-hand side.

The proof of Theorem 5.1 is built around a standard result which transports model structures along adjunctions and is sometimes referred to as “Kan's transfer theorem”. The formulation below is a slight variation of [**17**, Lemma 2.3]. Our condition (R3) is more general than that of Schwede-Shipley, but may be harder to verify in general. In the case of interest in this paper, it comes for free.

Theorem 5.8 (Lifting of model structures)**.** *Let $\mathcal{C}$ be a cofibrantly generated model category and I (respectively J) a set of generating (acyclic) cofibrations. Let T be a monad on $\mathcal{C}$ and denote by I_T and J_T the images of I and J, respectively, under the free T-algebra functor. Assume that:*

- *(R1) the domains of I_T and J_T are small relative to I_T-cell and J_T-cell, respectively*
- *(R2) every morphism in J_T-cell is sent to a weak equivalence in $\mathcal{C}$ under the forgetful functor*
- *(R3) the category $\mathcal{C}\,[T]$ of T-algebras is cocomplete.*

Then $\mathcal{C}\,[T]$ is a cofibrantly generated model category with generating sets of (acyclic) *cofibrations I_T (respectively J_T), and weak equivalences and fibrations detected by the forgetful functor to $\mathcal{C}$.*

Corollary 5.9. *Given a model category $(\mathcal{LU})_a$ as in Theorem 5.1, the category of co-$*$-modules admits a cofibrantly generated model structure $(\mathcal{M}^*)_a$ with weak equivalences and fibrations detected by the forgetful functor $R\colon \mathcal{M}^* \to (\mathcal{LU})_a$. Sets of generating cofibrations and acyclic cofibrations are given by $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, I)$ and $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, J)$, respectively.*

Proof. We verify the requirements of Theorem 5.8. All colimits exist since the forgetful functor to $\mathcal{L}$-spaces has a left adjoint. The smallness statement is a special case of Lemma 5.6. We now prove (R2): Let $j\colon A \to B$ be a morphism in J. Let Y be the pushout of the left-hand square of co-$*$-modules and let Y_0 be the pushout in the right-hand square of $\mathcal{L}$-spaces:

$$\begin{array}{ccc} F(A) & \longrightarrow & X \\ {\scriptstyle F(j)}\downarrow & & \downarrow{\scriptstyle g} \\ F(B) & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} A & \longrightarrow & RX \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle g_0} \\ B & \longrightarrow & Y_0 \end{array}$$

Under the functor F, the right-hand square is taken to the pushout square

$$\begin{array}{ccc} F(A) & \longrightarrow & (FR)(X) \cong X \\ {\scriptstyle F(j)}\downarrow & & \downarrow{\scriptstyle F(g_0)} \\ F(B) & \longrightarrow & F(Y_0), \end{array}$$

but $(FR)(X) \cong X$, hence $F(Y_0) \cong Y$ by uniqueness of the pushout, and the maps g and $F(g_0)$ agree under this isomorphism. The map j is an acyclic cofibration and a closed embedding by assumption. Both of these properties are stable under cobase change, hence g_0 is an acyclic cofibration and a closed embedding. Then by Proposition 5.5, the map $g \cong F(g_0)$ is a closed embedding and a weak equivalence.

Finally, we claim that the collections of maps that are simultaneously closed embeddings and weak equivalences is closed under transfinite composition in $\mathcal{M}^*$. This is true in $\mathcal{LU}$ by assumption iv) of Theorem 5.1, but colimits in $\mathcal{M}^*$ are not constructed in $\mathcal{LU}$. More precisely, they are obtained by applying F to a colimit formed in $\mathcal{LU}$. Since F preserves the class of weak equivalences by Proposition 5.5, the claim follows. Altogether, we have shown that all relative J_T-cell complexes are weak equivalences. □

We are now ready to give the

Proof of Theorem 5.1. The model structure $(\mathcal{M}^*)_a$ from Corollary 5.9 transports along the equivalence of categories

$$\mathcal{M}^* \underset{\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-)}{\overset{-\boxtimes_{\mathcal{L}}*}{\rightleftarrows}} \mathcal{M}_*$$

to a model structure $(\mathcal{M}_*)_a$ with weak equivalences and fibrations detected by the composite $R \circ \mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, -)\colon \mathcal{M}_* \to \mathcal{LU}$, which proves c). Sets of generating (acyclic) cofibrations are given by the images of I (resp. J) under $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*, -) \boxtimes_{\mathcal{L}} *\colon \mathcal{LU} \to \mathcal{M}_*$, which is naturally equivalent to the functor $(-) \boxtimes_{\mathcal{L}} *$ by Proposition 2.6, thus proving part a). Hypothesis iii) and Proposition 3.2 imply that $\mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-)$ preserves and reflects the weak equivalences $\mathcal{W}$; now b) follows immediately. In order to show d), it suffices to show that the left-hand adjunction in (5.7) is a Quillen equivalence. It is a Quillen adjunction by construction. It is a Quillen equivalence because RF preserves and reflects weak equivalences and because the unit $\bar{\lambda}\colon X \to RF(X)$ is a strong global equivalence, see Proposition 5.5 and Proposition 3.2, respectively.

Now we prove the enhancements e) through h):

Ad e): Assume that $(\mathcal{LU})_a$ is right proper. Then so is $(\mathcal{M}_*)_a$ since the right adjoint $R \circ \mathrm{F}_{\boxtimes_{\mathcal{L}}}(*,-)\colon \mathcal{M}_* \to \mathcal{LU}$ preserves pullbacks, and preserves and reflects weak equivalences and fibrations.

Now assume that $(\mathcal{LU})_a$ is topological. Let $f\colon X \to Y$ be a generating cofibration for $(\mathcal{LU})_a$ and $i\colon A \to B$ any cofibration in $\mathcal{U}$. By assumption, the pushout product

$$f \square i\colon P = Y \times A \cup_{X \times A} X \times B \to Y \times B$$

is again a cofibration in $(\mathcal{LU})_a$. The map $f \boxtimes_{\mathcal{L}} *$ is a generating cofibration in $(\mathcal{M}_*)_a$ whose pushout product with i is isomorphic to

$$(f \square i) \boxtimes_{\mathcal{L}} *\colon P \boxtimes_{\mathcal{L}} * \to (Y \times B) \boxtimes_{\mathcal{L}} *.$$

As $-\boxtimes_{\mathcal{L}} *\colon (\mathcal{LU})_a \to (\mathcal{M}_*)_a$ is a left Quillen functor, this map is a cofibration in $\mathcal{M}_*$. If f is a generating acyclic cofibration or i any acyclic cofibration, then $f \square i$ is an acyclic cofibration in $\mathcal{LU}$, hence so is $(f \boxtimes_{\mathcal{L}} *) \square i \cong (f \square i) \boxtimes_{\mathcal{L}} *$ in $\mathcal{M}_*$.

Ad f): There are natural isomorphisms

$$(X \boxtimes_{\mathcal{L}} *) \boxtimes_{\mathcal{L}} (X' \boxtimes_{\mathcal{L}} *) \cong (X \boxtimes_{\mathcal{L}} X') \boxtimes_{\mathcal{L}} *$$

for all $\mathcal{L}$-spaces X and X'. Similar reasoning as in the proof of g) then shows that for two generating cofibrations $f\colon A \to B$ and $f'\colon A' \to B'$ for $(\mathcal{LU})_a$, the pushout product of $f \boxtimes_{\mathcal{L}} *$ and $f' \boxtimes_{\mathcal{L}} *$ is isomorphic to $(f \square f') \boxtimes_{\mathcal{L}} *$, hence is a cofibration in $\mathcal{M}_*$, and acyclic if f or f' is a generating acyclic cofibration.

Ad g): Left properness follows immediately from Lemma 5.3.

Ad h): The box product is weakly equivalent to the categorical product by Proposition 3.2 and the assumption that any strong global equivalence is a weak equivalence in $(\mathcal{LU})_a$. As the weak equivalences are detected by fixed point functors, the functor $(-) \boxtimes_{\mathcal{L}} Z$ preserves weak equivalences, where $Z \in \mathcal{LU}$ is any $\mathcal{L}$-space. The unit axiom follows immediately.

Let $\mathcal{A}$ denote the class of morphisms $j \boxtimes_{\mathcal{L}} Z$ where j is an acyclic cofibration and $Z \in \mathcal{LU}$ is arbitrary. All cofibrations in $(\mathcal{LU})_a$ are h-cofibrations. As just observed, the functor $(-) \boxtimes_{\mathcal{L}} Z$ preserves weak equivalences. Because of Lemma 5.2, it always preserves h-cofibrations, too. Moreover, the class of weak equivalences which are h-cofibrations is stable under cobase changes (by Lemma 5.3), transfinite composition, and retracts. Thus, all relative $\mathcal{A}$-complexes are weak equivalences.

The same proof applies to $(\mathcal{M}_*)_a$. □

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